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MULTIVARIABLE NEWTON-PUISEUX THEOREM FOR GENERALISED QUASIANALYTIC CLASSES

by Tamara SERVI (*)

ABSTRACT. — We show how to solve explicitly an equation satisfied by a real function belonging to certain general quasianalytic classes. More precisely, we show that if $f(x_1, \ldots, x_m, y)$ belongs to such a class, then the solutions $y = \varphi(x_1, \ldots, x_m)$ of the equation $f = 0$ in a neighbourhood of the origin can be expressed, piecewise, as finite compositions of functions in the class, taking n^{th} roots and quotients. Examples of the classes under consideration are the collection of convergent generalised power series, a class of functions which contains some Dulac Transition Maps of real analytic planar vector fields, quasianalytic Denjoy-Carleman classes and the collection of multisummable series.

RÉSUMÉ. — Nous montrons comment résoudre explicitement une équation satisfaite par une fonction réelle appartenant à certaines classes quasianalytiques générales. Plus précisément, nous montrons que si $f(x_1, \ldots, x_m, y)$ appartient à une telle classe, alors les solutions $y = \varphi(x_1, \ldots, x_m)$ de l'équation $f = 0$ au voisinage de l'origine peuvent être exprimées par morceaux comme des compositions finies de fonctions dans la classe, de racines n-ièmes et de quotients. Parmi les exemples de telles classes figurent les séries généralisées convergentes, une classe de fonctions qui contient certaines applications de transition de Dulac de champs de vecteurs analytiques du plan réel, les classes quasianalytiques de Denjoy-Carleman et la collection des séries multisommables.

1. Introduction

The Newton-Puiseux Theorem states that, if $f(x, y)$ is an analytic germ in two variables, then the solutions $y = \varphi(x)$ of the equation $f = 0$ can be expanded as Puiseux series that are convergent in a neighbourhood of the origin (see for example [2]). A multivariable version of this result in the real case states that, if $f(x_1, \ldots, x_m, y)$ is a real analytic germ, then,

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after a finite sequence of blow-ups with centre a real analytic manifold, the solutions $y = \varphi(x_1, \ldots, x_m)$ of the equation $f = 0$ are analytic in a neighbourhood of the origin (see for example [14, Theorem 4.1]). An equivalent formulation states that the solutions $y = \varphi(x_1, \ldots, x_m)$ in a neighbourhood of the origin are obtained, piecewise, as finite compositions of analytic functions, taking n^{th} roots and quotients (see for example [5, Corollary 2.15] and [12, Theorem 1]).

Here we extend this result to functions belonging to a *generalised quasi-analytic class* (see Definition 2.6). Roughly, a generalised quasianalytic class is a collection of algebras of continuous real-valued functions together with an *injective* $\mathbb{R}$-algebra morphism $\mathcal{T}$ which, given the germ at zero f of a function in the collection, associates to f a formal power series $\mathcal{T}(f)$ with natural or real exponents. Given a generalised quasianalytic class, we already have a local uniformisation result [18, 21, 17] which allows to parametrise the zero set of a function in the class. Our aim here is to refine this procedure, in the spirit of the elimination result in [3], in the following way: given a function $f(x, y)$ in the class under consideration, we provide a uniformisation algorithm which "respects" the variable y and hence allows to solve the equation $f = 0$ with respect to y.

Examples of generalised quasianalytic classes are the following (see Remark 2.7).

Example A. — Let $M = (M_0, M_1, \ldots)$ be an increasing sequence of positive real numbers (with $M_0 \geqslant 1$) and $B \subseteq \mathbb{R}^m$ be a compact box. We assume that M is strongly log-convex and we consider the Denjoy-Carleman algebra of functions $\mathcal{C}_B(M)$ defined in [18]. This is an algebra of functions $f : B \to \mathbb{R}$ which each extend to a $\mathcal{C}^\infty$ function on some open neighbourhood $U \supseteq B$ and whose derivatives satisfy a certain type of bounds depending on M (see [18, p. 751]). The functions in $\mathcal{C}_B(M)$ are not analytic in general, however, if $\sum_{i \in \mathbb{N}} \frac{M_i}{M_{i+1}} = \infty$, then $\mathcal{C}_B(M)$ is *quasianalytic*, i.e. for every $x \in B$, the algebra morphism which associates to $f \in \mathcal{C}_B(M)$ its (divergent) Taylor expansion at x is injective. The quasianalytic Denjoy-Carleman class $\mathcal{C}(M)$ is the union of the collection $\{\mathcal{C}_B(M) : \ m \in \mathbb{N}, \ B \subseteq \mathbb{R}^m \text{ compact box}\}$.

Example B. — Let $H = (H_1, \ldots, H_r) : (0, \varepsilon) \to \mathbb{R}^r$ be a $\mathcal{C}^\infty$ solution of a system of first order singular analytic differential equations of the form $x^{p+1} y'(x) = A(x, y)$, where A is real analytic in a neighbourhood of $0 \in \mathbb{R}^{p+1}$, satisfying conditions a) and b) in [16, p. 413], and $A(0, 0) = 0$. Suppose furthermore that H admits an asymptotic expansion for $x \to 0^+$ as in [16, 2.2]. As in [16, Section 3], we let $\mathcal{A}_H$ be the smallest collections of real germs containing the germ at zero of the H_i and closed under composition, monomial division and taking implicit functions. A function f, defined on an open set $U \subseteq \mathbb{R}^m$, is said to be *$\mathcal{A}_H$-analytic* if for every $a \in U$ there exists a germ $\varphi_a(x) \in \mathcal{A}_H$ such that the germ of $f(x)$ at a is equal to the germ $\varphi_a(x-a)$. It is proven in [16] that the collection of all $\mathcal{A}_H$-analytic functions forms a quasianalytic class of $\mathcal{C}^\infty$ functions.

Example C. — A (formal) generalised power series in m variables $X = (X_1, \ldots, X_m)$ is a series $F(X) = \sum_\alpha c_\alpha X^\alpha$ such that $\alpha \in [0,\infty)^m$, $c_\alpha \in \mathbb{R}$ and there are well-ordered subsets $S_1, \ldots, S_m \subseteq [0,\infty)$ such that the support of F is contained in $S_1 \times \ldots \times S_m$ (see [6]). The series F is convergent if there is a polyradius $r = (r_1, \ldots, r_m) \in (0,\infty)^m$ such that $\sum_\alpha |c_\alpha| r^\alpha < \infty$. A convergent generalised power series gives rise to a real-valued function $F(x) = \sum c_\alpha x^\alpha \in \mathbb{R}\{x^*\}_r$, which is continuous on $[0,r_1) \times \ldots \times [0,r_m)$ and analytic on the interior of its domain. We denote by $\mathbb{R}[\![X^*]\!]$ the algebra of all formal generalised power series and consider the algebra $\mathbb{R}\{x^*\} = \bigcup_{r\in(0,\infty)^m} \mathbb{R}\{x^*\}_r$ of all convergent generalised power series. Examples of convergent generalised power series are the function $\zeta(-\log x) = \sum_{n=1}^\infty x^{\log n}$ (where ζ is the Riemann zeta function) and the solution $f(x) = \sum_{n,i=0}^\infty \frac{1}{2^i} x^{2+n-\frac{1}{2^i}}$ of the functional equation $(1-x)f(x) = x + \frac{1}{2}x(1-\sqrt{x})f(\sqrt{x})$.

Example D. — For $R = (R_1, \ldots, R_m) \in (0,\infty)^m$ a polyradius, we consider the algebra $\mathcal{G}(R)$ of functions defined in [7, Definition 2.20] by means of sums of multisummable formal series in the real direction. Its elements are $\mathcal{C}^\infty$ functions defined on $[0,R_1] \times \ldots \times [0,R_m]$ and their derivatives satisfy a Gevrey condition. By a known result in multisummability theory, these algebras satisfy the following quasianalyticity condition: the morphism, which associates to the germ at zero of a function in $\mathcal{G}(R)$ its (divergent) Taylor expansion at the origin, is injective (see [7, Proposition 2.18]). We let $\mathcal{G}$ be the union of the collection $\{\mathcal{G}(R) :\ m \in \mathbb{N},\ R \in (0,\infty)^m\}$. This collection contains the function $\psi(x)$ appearing in Binet's second formula, i.e. such that $\log\Gamma(x) = \left(x - \frac{1}{2}\right)\log(x) + \frac{1}{2}\log(2\pi) + \psi\left(\frac{1}{x}\right)$, where Γ is Euler's Gamma function (see [7, Example 8.1]).

Example E. — For $r \in (0,\infty)^{m+n}$ a polyradius, we consider the algebra $\mathcal{Q}_{m,n,r}$ defined in [11, Definition 7.1]. Its elements are continuous real-valued functions which have a holomorphic extension to some "quadratic domain" $U \subseteq \mathbf{L}^{m+n}$, where $\mathbf{L}$ is the Riemann surface of the logarithm. One can define a morphism T which associates to the germ f of a function in $\mathcal{Q}_{m,n,r}$ an *asymptotic expansion* $T(f) \in \mathbb{R}[\![X^*]\!]$. It is shown in [11, Proposition 2.8], using results of Ilyashenko's in [10], that the morphism T is injective (quasianalyticity). We let $\mathcal{Q}$ be the collection $\left\{\mathcal{Q}_{m,n,r}:\ m,n\in\mathbb{N},\ r\in(0,\infty)^{m+n}\right\}$. The motivation for looking at this type of algebras is that they contain the Dulac transition maps of real analytic planar vector fields in a neighbourhood of hyperbolic non-resonant singular points.

Before stating our main result, we need to give a definition.

Definition 1.1. — *Let $\mathcal{A}$ be a collection of real-valued functions. An $\mathcal{A}$-*term *is defined inductively as follows. An $\mathcal{A}$-term of depth zero is an element of $\mathcal{A}$. Let $x=(x_1,\dots,x_m)$. A function $f(x)$ is an $\mathcal{A}$-term of depth $\leqslant k$ if there exist $m\in\mathbb{N},\ g\in\mathcal{A}$ and $\mathcal{A}$-terms $t_1(x),\dots,t_m(x)$ of depth $\leqslant k-1$ such that $\mathrm{Im}(t_1)\times\ldots\times\mathrm{Im}(t_m)\subseteq\mathrm{dom}(g)$ and $f(x)=g(t_1(x),\dots,t_m(x))$.*

*A connected set $C\subseteq\mathbb{R}^m$ is an $\mathcal{A}$-*base *if there are a polyradius $r\in(0,\infty)^m$ and $\mathcal{A}$-terms $t_0,t_1,\dots,t_q$ defined on $(0,r_1)\times\ldots\times(0,r_m)$, such that*

$$C=\{x\in(0,r_1)\times\ldots\times(0,r_m):\ t_0(x)=0,\ t_1(x)>0,\ \dots,t_q(x)>0\}.$$

*A set $D\subseteq\mathbb{R}^{m+1}$ is an $\mathcal{A}$-*cell *if there are an $\mathcal{A}$-base $C\subseteq\mathbb{R}^m$ and terms $t_1(x),t_2(x)$ in m variables such that D is of either of the following forms:*

$$\{(x,y):\ x\in C,\ y=t_1(x)\},\ \{(x,y):\ x\in C,\ t_1(x)<y\},$$
$$\{(x,y):\ x\in C,\ y<t_2(x)\},\ \{(x,y):\ x\in C,\ t_1(x)<y<t_2(x)\}.$$

*If $A\subseteq W\subseteq\mathbb{R}^{m+1}$, then an $\mathcal{A}$-*cell decomposition of W compatible with A *is a finite partition of W into $\mathcal{A}$-cells such that every $\mathcal{A}$-cell in the partition is either contained in A or disjoint from A.*

We consider the functions

$$(\cdot)^{-1}:x\mapsto\begin{cases}\frac{1}{x} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}\quad\text{and}\quad\sqrt[p]{\cdot}:x\mapsto\begin{cases}\sqrt[p]{x} & \text{if } x>0\\ 0 & \text{if } x\leqslant 0\end{cases}\ (\text{for all } p\in\mathbb{N}).$$

We can now state our main result.

Main Theorem. — *Let $\mathcal{C}$ be a generalised quasianalytic class, as in Definition 2.6. Let $\mathcal{A}=\mathcal{C}\cup\left\{(\cdot)^{-1}\right\}\cup\left\{\sqrt[p]{\cdot}:\ p\in\mathbb{N}\right\}$ and $x=(x_1,\dots,x_m)$. Let y be a single variable and let $f(x,y)\in\mathcal{C}$. Then there exist a neighbourhood $W\subseteq\mathbb{R}^{m+1}$ of the origin and an $\mathcal{A}$-cell decomposition of $W\cap\mathrm{dom}(f)$ which is compatible with the set $\{(x,y)\in W\cap\mathrm{dom}(f):\ f(x,y)=0\}$.*

The Main Theorem immediately implies that the solutions of the equation $f(x,y)=0$ have the form $\varphi:C\to\mathbb{R}$, where $C\subseteq\mathbb{R}^m$ is an $\mathcal{A}$-base and $\varphi(x)$ is an $\mathcal{A}$-term.

We now briefly illustrate the strategy of proof. In analogy with the real analytic case, we define a class of blow-up transformations adapted to the functions under consideration. We show that, after a finite sequence of such transformations, the germ at zero of f is normal crossing.

We stress that the monomialisation algorithm we exhibit here differs from the ones in [18, 21, 1]. In fact, the transformations we use *respect* the variable y in the following way: if $\rho : \mathbb{R}^{m+1} \ni (x', y') \mapsto (x, y) \in \mathbb{R}^{m+1}$ is one of such transformations and the Main Theorem holds for $f \circ \rho\,(x', y')$, then it also holds for $f\,(x, y)$. Moreover, such transformations are bijective outside a set of small dimension and the components of the inverse map, when defined, are $\mathcal{A}$-terms.

It is worth pointing out that our algorithm does not use the Weierstrass Preparation Theorem, since this theorem does not always hold in generalised quasianalytic classes (see for example [15]).

The desingularisation procedure which allows to reduce to the case when f is normal crossing exploits the fundamental property of quasianalyticity, which allows to deduce the wanted result for f from a formal monomialisation algorithm for the series $\mathcal{T}\,(f)$.

The Main Theorem could also be deduced from a general quantifier elimination result in [17]. However, the solving process described in [17] is not algorithmic, since it uses a highly nonconstructive result, namely an o-minimal Preparation Theorem in [8]. Here instead we deduce the explicit form of the solutions of $f = 0$ solely from the analysis of the Newton polyhedron of $\mathcal{T}\,(f)$.

Although all known generalised quasianalytic classes generate o-minimal structures (see [4] for the definition and basic properties of o-minimal structures), the proof of our main result does not use o-minimality.

2. Generalised quasianalytic classes

In this section we establish our setting.

We recall the definition and main properties of generalised power series (see [6] for more details).

Let $m \in \mathbb{N}$. A set $S \subset [0, \infty)^m$ is called *good* if S is contained in a cartesian product $S_1 \times \ldots \times S_m$ of well ordered subsets of $[0, \infty)$. If S is a good set, define $S_{\min}$ as the set of minimal elements of S with respect to the following order: let $s = (s_1, \ldots, s_m)\,,\ s' = (s'_1, \ldots, s'_m) \in S$; then $s \leqslant s'$ iff $s_i \leqslant s'_i$ for all $i = 1, \ldots, m$. By [6, Lemma 4.2], $S_{\min}$ is finite.

A *formal generalised power series* has the form

$$F(X)=\sum_{\alpha} c_\alpha X^\alpha,$$

where $\alpha=(\alpha_1,\dots,\alpha_m)\in[0,\infty)^m$, $c_\alpha\in\mathbb{R}$ and X^α denotes the formal monomial $X_1^{\alpha_1}\cdot\ldots\cdot X_m^{\alpha_m}$, and the *support of* F $\mathrm{Supp}\,(F):=\{\alpha:\ c_\alpha\neq 0\}$ is a good set. These series are added the usual way and form an $\mathbb{R}$-algebra denoted by $\mathbb{R}[\![X^*]\!]$.

Let $\mathcal{G}\subseteq\mathbb{R}[\![X^*]\!]$ be a family of series such that the *total support* $\mathrm{Supp}(\mathcal{G}):=\bigcup_{F\in\mathcal{G}}\mathrm{Supp}\,(F)$ is a good set. Then $\mathrm{Supp}\,(\mathcal{G})_{\min}$ is finite and we denote by $\mathcal{G}_{\min}:=\{X^\alpha:\ \alpha\in\mathrm{Supp}\,(\mathcal{G})_{\min}\}$ the *set of minimal monomials* of $\mathcal{G}$.

Let $m,n\in\mathbb{N}$ and $(X,Y)=(X_1,\dots,X_m,Y_1,\dots,Y_n)$. We define $\mathbb{R}[\![X^*,Y]\!]$ as the subring of $\mathbb{R}[\![(X,Y)^*]\!]$ consisting of those series F such that $\mathrm{Supp}(F)\subset[0,\infty)^m\times\mathbb{N}^n$. Since $\mathbb{R}[\![X^*,Y]\!]\subseteq\mathbb{R}[\![X^*]\!][\![Y]\!]$, we say that the variables X are *generalised* and that the variables Y are *standard*.

Definition 2.1. — *For every $m,n\in\mathbb{N}$ and polyradius $r=(s_1,\dots,s_m,t_1,\dots,t_n)\in(0,\infty)^{m+n}$, we let $\mathcal{C}_{m,n,r}$ be an algebra of real functions, which are defined and continuous on the set*

$$I_{m,n,r}:=[0,s_1)\times\ldots\times[0,s_m)\times(-t_1,t_1)\times\ldots\times(-t_n,t_n)\,,$$

and $\mathcal{C}^1$ on $\mathring{I}_{m,n,r}$. We denote by $x=(x_1,\dots,x_m)$ the generalised *variables and by $y=(y_1,\dots,y_n)$ the* standard *variables. We require that the algebras $\mathcal{C}_{m,n,r}$ satisfy the following list of conditions:*

- *The coordinate functions of $\mathbb{R}^{m+n}$ are in $\mathcal{C}_{m,n,r}$.*
- *If $r'\leqslant r$ (i.e. if $s_i'\leqslant s_i$ for all $i=1,\dots,m$ and $t_j'\leqslant t_j$ for all $j=1,\dots,n$) and $f\in\mathcal{C}_{m,n,r}$, then $f\restriction I_{m,n,r'}\in\mathcal{C}_{m,n,r'}$.*
- *If $f\in\mathcal{C}_{m,n,r}$ then there exists $r'>r$ and $g\in\mathcal{C}_{m,n,r'}$ such that $g\restriction I_{m,n,r}=f$.*
- *Let $k,l\in\mathbb{N}$, $s_1',\dots,s_k',t_1',\dots,t_l'\in(0,\infty)$ and*

$$r'=(s_1,\dots,s_m,s_1',\dots,s_k',t_1,\dots,t_n,t_1',\dots,t_l').$$

  Then $\mathcal{C}_{m,n,r}\subset\mathcal{C}_{m+k,n+l,r'}$ in the sense that if $f\in\mathcal{C}_{m,n,r}$ then the function $F:I_{m+k,n+l,r'}\to\mathbb{R}$ defined by

$$F\left(x_1,\dots,x_m,x_1',\dots,x_k',y_1,\dots,y_n,y_1',\dots,y_l'\right)=f\left(x_1,\dots,x_m,y_1,\dots,y_n\right)$$

  is in $\mathcal{C}_{m+k,n+l,r'}$.
- *$\mathcal{C}_{m,n,r}\subset\mathcal{C}_{m+n,0,r}$, in the sense that if $f\in\mathcal{C}_{m,n,r}$ then $f\restriction I_{m+n,0,r}\in\mathcal{C}_{m+n,0,r}$.*

Definition 2.2. — *We denote by $\mathcal{C}_{m,n}$ the algebra of germs at the origin of the elements of $\mathcal{C}_{m,n,r}$, for r a polyradius in $(0,\infty)^{m+n}$. We say* *that* $\{\mathcal{C}_{m,n}:\ m,n\in\mathbb{N}\}$ *is a collection of* quasianalytic algebras of germs *if, for all* $m,n\in\mathbb{N}$*, there exists an* injective $\mathbb{R}$*-algebra morphism*

$$\mathcal{T}_{m,n}:\mathcal{C}_{m,n}\to\mathbb{R}[\![X^*,Y]\!],$$

where $X=(X_1,\ldots,X_m)=\mathcal{T}(x)$, $Y=(Y_1,\ldots,Y_n)=\mathcal{T}(y)$*. Moreover, for all* $m'\geqslant m$, $n'\geqslant n$ *we require that the morphism* $\mathcal{T}_{m',n'}$ *extend* $\mathcal{T}_{m,n}$*, hence, from now on we will write* $\mathcal{T}$ *for* $\mathcal{T}_{m,n}$*.*

A number $\alpha\in[0,\infty)$ *is an* admissible exponent *if there are* $m,n\in\mathbb{N}$, $f\in\mathcal{C}_{m,n}$, $\beta\in Supp\,(\mathcal{T}(f))\subset\mathbb{R}^m\times\mathbb{N}^n$ *such that* α *is a component of* β*. If* $\mathbb{A}$ *is the set of all admissible exponents and* $\mathbb{A}\neq\mathbb{N}$*, then we let* $\mathbb{K}$ *be the set of nonnegative elements of the field generated by* $\mathbb{A}$*. Otherwise, we set* $\mathbb{K}=\mathbb{A}=\mathbb{N}$*.*

We require the collection $\{\mathcal{C}_{m,n}:\ m,n\in\mathbb{N}\}$ to be closed under certain operations, which we now define.

Definition 2.3. — *Let* $m,n\in\mathbb{N}$, $(x,y)=(x_1,\ldots,x_m,y_1,\ldots,y_n)$*. For* $m',n'\in\mathbb{N}$ *with* $m'+n'=m+n$*, we set* $(x',y')=(x'_1,\ldots,x'_{m'},y'_1,\ldots,y'_{n'})$*. Let* r,r' *be polyradii in* $\mathbb{R}^{m+n}$*. An* elementary transformation *is a map* $I_{m',n',r'}\ni(x',y')\mapsto(x,y)\in I_{m,n,r}$ *of either of the following forms.*

- *A* ramification*: let* $m=m',n=n'$, $\gamma\in\mathbb{K}^{>0}$ *and* $1\leqslant i\leqslant m$*, and set*

$$r_i^\gamma(x',y')=(x,y),\quad\text{where}\begin{cases}x_k=x'_k & 1\leqslant k\leqslant m,\ k\neq i\\ x_i=x_i'^{\gamma} & \\ y_k=y_k & 1\leqslant k\leqslant n\end{cases}.$$

- *A* Tschirnhausen translation*: let* $m=m',n=n'$ *and* $H\in\mathcal{C}_{m,n-1,s}$ *(where* $s\in(0,\infty)^{m+n-1}$ *is a polyradius), with* $H(0)=0$*, and set*

$$\tau_H(x',y')=(x,y),\ \text{where}\begin{cases}x_k=x'_k & 1\leqslant k\leqslant m\\ y_n=y'_n+H\left(x',y'_1,\ldots,y'_{n-1}\right) & \\ y_k=y'_k & 1\leqslant k\leqslant n-1\end{cases}.$$

- *A* linear transformation*: let* $m=m',n=n'$, $1\leqslant i\leqslant n$ and $c=(c_1,\ldots,c_{i-1})\in\mathbb{R}^{i-1}$*, and set*

$$L_{i,c}(x',y')=(x,y),\quad\text{where}\begin{cases}x_k=x'_k & 1\leqslant k\leqslant m\\ y_k=y'_k & i\leqslant k\leqslant n\\ y_k=y'_k+c_ky'_i & 1\leqslant k<i\end{cases}.$$

- *A* blow-up chart, *i.e. either of the following maps:*

– *for* $1 \leqslant j < i \leqslant m$ *and* $\lambda \in (0,\infty)$, *let* $m' = m-1$ *and* $n' = n+1$ *and set*

$$\pi^{\lambda}_{i,j}\left(x',y'\right) = (x,y), \quad \text{where } \begin{cases} x_k = x'_k & 1 \leqslant k < i \\ x_i = x'_j\left(\lambda + y'_1\right) & \\ x_k = x'_{k-1} & i < k \leqslant m \\ y_k = y'_{k+1} & 1 \leqslant k \leqslant n \end{cases};$$

– *for* $1 \leqslant j, i \leqslant m$, *with* $j \neq i$, *let* $m' = m$ *and* $n' = n$, *and set*

$$\pi^{0}_{i,j}\left(x',y'\right) = (x,y), \quad \text{where } \begin{cases} x_k = x'_k & 1 \leqslant k \leqslant m,\ k \neq i \\ x_i = x'_j x'_i & \\ y_k = y'_k & 1 \leqslant k \leqslant n \end{cases}$$

and $\pi^{\infty}_{i,j} = \pi^{0}_{j,i}$;

– *for* $1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m$ *and* $\lambda \in \mathbb{R}$, *let* $m' = m$ *and* $n' = n$, *and set*

$$\pi^{\lambda}_{m+i,j}\left(x',y'\right) = (x,y), \quad \text{where } \begin{cases} x_k = x'_k & 1 \leqslant k \leqslant m \\ y_i = x'_j\left(\lambda + y'_i\right) & \\ y_k = y'_k & 1 \leqslant k \leqslant n,\ k \neq i \end{cases};$$

– *for* $1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m$, *let* $m' = m+1$ *and* $n' = n-1$, *and set*

$$\pi^{\pm\infty}_{m+i,j}\left(x',y'\right) = (x,y), \quad \text{where } \begin{cases} x_k = x'_k & 1 \leqslant k \leqslant m,\ k \neq j \\ x_j = x'_{m+1} x'_j & \\ y_k = y'_k & 1 \leqslant k < i \\ y_i = \pm x'_{m+1} & \\ y_k = y'_{k-1} & i < k \leqslant n \end{cases}.$$

- *A* reflection*: let* $m' = m+1$, $n' = n-1$ *and* $1 \leqslant i \leqslant n$, *and set*

$$\sigma^{\pm}_{m+i}\left(x',y'\right) = (x,y), \quad \text{where } \begin{cases} x_k = x'_k & 1 \leqslant k \leqslant m \\ y_k = y'_k & 1 \leqslant k < i \\ y_i = \pm x'_{m+1} & \\ y_k = y'_{k-1} & i < k \leqslant n \end{cases}.$$

It is not difficult to see that an elementary transformation $(x',y') \mapsto (x,y)$ induces an injective $\mathbb{R}$-algebra homomorphism $\mathbb{R}[\![X^*,Y]\!] \mapsto \mathbb{R}[\![X'^*,Y']\!]$ by composition (where we replace H by $\mathcal{T}(H)$ in the Tschirnhausen translation).

ASSUMPTION 2.4. — *We require that the family of algebras of germs* $\{\mathcal{C}_{m,n}\colon\ m,n \in \mathbb{N}\}$ *satisfy the following closure and compatibility conditions with the morphism* $\mathcal{T}$*:*

(1) Monomials, permutations and setting a variable equal to zero. *For every* $\alpha \in \mathbb{K}$ *and* $i \in \{1,\dots,m\}$*, the germ* $x_i \mapsto x_i^{\alpha}$ *is in* $\mathcal{C}_{i,0}$ *and* $\mathcal{T}(x_i^{\alpha}) = X_i^{\alpha}$*. Moreover,* $\mathcal{C}_{m,n}$ *is closed under permutations of the generalised variables, under permutation of the standard variables, under setting any one variable equal to zero, and the morphism* $\mathcal{T}$ *commutes with these operations.*

(2) Monomial division. *Let* $f \in \mathcal{C}_{m,n}$ *and suppose that there exist* $\alpha \in \mathbb{K}$*,* $p \in \mathbb{N}$ *and* $G \in \mathbb{R}[\![X^*,Y]\!]$ *such that* $\mathcal{T}(f)(X,Y) = X_i^{\alpha}Y_j^{p}G(X,Y)$*, for some* $i \in \{1,\dots,m\}$ *and* $j \in \{1,\dots,n\}$*. Then there exists* $g \in \mathcal{C}_{m,n}$ *such that* $f(x,y) = x_i^{\alpha}y_j^{p}g(x,y)$*. It follows that* $\mathcal{T}(g) = G$*.*

(3) Elementary transformations. *Let* $f \in \mathcal{C}_{m,n}$ *and* $\nu : I_{m',n',r'} \to I_{m,n,r}$ *be an elementary transformation. Then the germ of* $f \circ \nu$ *belongs to* $\mathcal{C}_{m',n'}$ and $\mathcal{T}(f \circ \nu) = \mathcal{T}(f) \circ \nu$*.*

Notice that, thanks to the closure under monomial division and under linear transformations (which is an instance of Condition 3), $\mathcal{C}_{m,n}$ *is closed under taking partial derivatives with respect to any of the standard variables. In fact, if* $f \in \mathcal{C}_{m,n}$*, then the germ of* $\frac{\partial f}{\partial y_n}$ *is obtained as the germ of* $\frac{f(x,y_1,\dots,y_{n-1},y_n+w)-f(x,y)}{w}$*.*

(4) Implicit functions in the standard variables. *Let* $f \in \mathcal{C}_{m,n}$ *and suppose that* $\frac{\partial f}{\partial y_n}(0)$ *is nonzero. Then there exists* $g \in \mathcal{C}_{m,n-1}$ *such that* $f(x,y_1,\dots,y_{n-1},g(x,y_1,\dots,y_{n-1})) = 0$*. It follows that*

$$\mathcal{T}(f)(X,Y_1,\dots,Y_{n-1},\mathcal{T}(g)(X,Y_1,\dots,Y_{n-1})) = 0.$$

(5) Truncation. *Let* $f \in \mathcal{C}_{m,n}$*. Write*

$$\mathcal{T}(f) = \sum_{\alpha\in[0,\infty)} a_{\alpha}(X_1,\dots,X_{m-1},Y)X_m^{\alpha}$$

and let $\alpha_0 \in [0,\infty)$*. Then there exists* $g \in \mathcal{C}_{m,n}$ *such that* $\mathcal{T}(g) = \sum_{\alpha<\alpha_0} a_{\alpha}X_m^{\alpha}$*.*

Remark 2.5. — As a consequence of the first three conditions in 2.4, it is easy to see that $\mathcal{T}(f)(0,Y)$ is the Taylor expansion of $f(0,y)$ with respect to y. Moreover, Condition 5 follows automatically from the previous conditions if X_m is a standard variable. Finally, by the binomial formula and Condition 4, if $U \in \mathcal{C}_{m,n}$ is a *unit* (i.e. an invertible element) and $\alpha \in \mathbb{K}$, then $U^{\pm\alpha} \in \mathcal{C}_{m,n}$.

Definition 2.6. — *A collection of real functions*

$$\mathcal{C} = \bigcup \left\{ \mathcal{C}_{m,n,r} : \ m, n \in \mathbb{N}, \ r \in (0, \infty)^{m+n} \right\}$$

is a generalised quasianalytic class *if the algebras $\mathcal{C}_{m,n,r}$ satisfy the properties in 2.1 and the algebras of germs $\mathcal{C}_{m,n}$ are quasianalytic (see Definition 2.2) and satisfy the conditions in 2.4.*

Remark 2.7. — The Main Theorem applies to all the classes mentioned in the introduction, where the morphism $\mathcal{T}$ is the Taylor expansion at zero in cases a), b) and d), the identity in case c) and the asymptotic expansion $f \mapsto T(f)$ in case e). In fact, quasianalyticity is tautological in case c), it is proven in [16] in case b) and it follows by classical theorems in cases a), d) and e) (see [19, 20, 10]). Moreover, the closure and compatibility conditions in 2.4 are verified by construction in case b). They are proven in [18, Section 3] for case a), in [6, Sections 5,6] for case c), in [7, Sections 4,5] for case d) and finally in [11, Sections 5,6] for case e). In particular, in cases a), b) and e) the set $\mathbb{A}$ of admissible exponents is $\mathbb{N}$, so Condition 5 (truncation) in 2.4 is void. In case c) Condition 5 is clearly satisfied and in case e) it is a consequence of [11, Proposition 5.6]. Notice that in cases c) and e) the functions $x \mapsto \sqrt[p]{x}$ ($p \in \mathbb{N}$) already belong to the collection $\mathcal{C}$.

3. Strategy of proof of the Main Theorem

The key step for the proof of the Main Theorem is a monomialisation algorithm which respects a given variable. The monomialisation tools are the elementary transformations defined in Definition 2.3, the use of which we now describe.

Definition 3.1. — *Let $k \geqslant 1$ and for all $i \in \{1, \ldots, k\}$ let*

$$\nu_i : (x'_{(i)}, y'_{(i)}) \mapsto (x_{(i)}, y_{(i)})$$

be an elementary transformation, where $x'_{(i)}$ is an m'_i-tuple, $y'_{(i)}$ is an n'_i-tuple, $x_{(i)}$ is an m_i-tuple and $y_{(i)}$ is an n_i-tuple, with $m'_i + n'_i = m_i + n_i$. If $k = 1$ or if $k > 1$ and $m_i = m'_{i-1}$ for all $i = 1, \ldots, k$, then we say that $\rho := \nu_1 \circ \ldots \circ \nu_k$ is an admissible transformation.

An elementary family *is either of the following collections of elementary transformations: $\{r_i^{\gamma}\}$ (for some $1 \leqslant i \leqslant m$), $\left\{\sigma^+_{m+i}, \sigma^-_{m+i}\right\}$ (for some $1 \leqslant i \leqslant n$), $\{\tau_H\}$, $\{L_{i,c}\}$ (for some $1 \leqslant i \leqslant n$), $\left\{\pi^{\lambda}_{i,j} : \ \lambda \in [0, \infty]\right\}$ (for some $1 \leqslant i, j \leqslant m$), or $\left\{\pi^{\lambda}_{m+i,j} : \ \lambda \in \mathbb{R} \cup \{\pm\infty\}\right\}$ (for some $1 \leqslant i \leqslant n, \ 1 \leqslant j \leqslant m$). An* admissible family *is defined inductively. An admissible family of length*

1 *is an elementary family. An admissible family* $\mathcal{F}$ *of length* $\leqslant q$ *is obtained from an elementary family* $\mathcal{F}_0$ *in the following way: for all* $\nu \in \mathcal{F}_0$*, let* $\mathcal{F}_\nu$ *be an admissible family of length* $\leqslant q-1$ *such that* $\forall \rho' \in \mathcal{F}_\nu,\ \nu \circ \rho'$ *is an admissible transformation and define* $\mathcal{F} = \{\nu \circ \rho' :\ \nu \in \mathcal{F}_0,\ \rho' \in \mathcal{F}_\nu\}$.

Finally, we say that a series $F \in \mathbb{R}[\![X^*, Y]\!]$ *has a certain property* P after admissible family *if there exists ad admissible family* $\mathcal{F}$ *such that for every* $\rho \in \mathcal{F}$ *the series* $F \circ \rho\,(X', Y')$ *has the property* P*. The same notation extends to elements of* $\mathcal{C}$.

We fix a generalised quasianalytic class $\mathcal{C}$ and we let $\widehat{\mathcal{C}}_{m,n}$ be the image of $\mathcal{C}_{m,n}$ under the morphism $\mathcal{T}$ and $\widehat{\mathcal{C}} = \bigcup \widehat{\mathcal{C}}_{m,n}$. It follows from the conditions in 2.4 that, if $\rho : I_{m',n',r'} \ni (x', y') \mapsto (x, y) \in I_{m,n,r}$ is an admissible transformation and $F\,(X, Y) \in \widehat{\mathcal{C}}_{m,n}$, then $F\,(X', Y') \in \widehat{\mathcal{C}}_{m',n'}$.

Moreover, it is easy to verify that if $\mathcal{G} \subseteq \mathbb{R}[\![X^*, Y]\!]$ is a collection with good total support, then the collection $\{F \circ \rho :\ F \in \mathcal{G}\}$ has good total support. For example, let $F \in \mathbb{R}[\![X^*, Y]\!]$ and $H \in \mathbb{R}[\![X^*, Y_1, \ldots, Y_{n-1}]\!]$; suppose $\mathrm{Supp}\,(F) \subseteq S_1 \times \ldots \times S_m \times \mathbb{N}^n$ and $\mathrm{Supp}\,(H) \subseteq S_1' \times \ldots \times S_m' \times \mathbb{N}^{n-1}$, where $S_i, S_i' \subset [0, \infty)$ are well ordered sets. Then we have $\mathrm{Supp}\,(F \circ L_{i,c}) \subseteq S_1 \times \ldots \times S_m \times \mathbb{N}^n$ and $\mathrm{Supp}\,(F \circ \tau_H) \subseteq \tilde{S}_1 \times \ldots \times \tilde{S}_m \times \mathbb{N}^n$, with $\tilde{S}_k = \{a + lb :\ a \in S_k,\ b \in S_k',\ l \in \mathbb{N}\}$. Moreover, $\mathrm{Supp}\,(F \circ r_i^\gamma) \subseteq \tilde{S}_1 \times \ldots \times \tilde{S}_m \times \mathbb{N}^n$, with $\tilde{S}_i = \{\gamma a :\ a \in S_i\}$ and $\tilde{S}_k = S_k$ for $k \neq i$. Finally, for $1 \leqslant i, j \leqslant m$ with $i \neq j$, we have $\mathrm{Supp}\left(F \circ \pi_{i,j}^0\right) \subseteq \tilde{S}_1 \times \ldots \times \tilde{S}_m \times \mathbb{N}^n$, with $\tilde{S}_j = \{a + b :\ a \in S_j,\ b \in S_i\}$ and $\tilde{S}_k = S_k$ for $k \neq j$. The argument for the other types of blow-up transformation and for reflections is similar.

Definition 3.2. — *A series* $F \in \widehat{\mathcal{C}}_{m,n}$ *is* normal *if there are* $\alpha \in [0, \infty)^m$*,* $\beta \in \mathbb{N}^n$ *and a unit* $U \in \left(\widehat{\mathcal{C}}_{m,n}\right)^\times$ *such that* $F\,(X, Y) = X^\alpha Y^\beta U\,(X, Y)$.

Notation 3.3. — Throughout this section, we let $m, n \in \mathbb{N}$, $(x, y) = (x_1, \ldots, x_m, y_1, \ldots, y_n)$ and z be a single variable. We let $\mathcal{C}_{m,n,1}$ be either $\mathcal{C}_{m,n+1}$ (i.e. z is considered as a standard variable) or $\mathcal{C}_{m+1,n}$ (i.e. z is considered as a generalised variable). The same convention applies to the formal variables X, Y, Z and to $\widehat{\mathcal{C}}$.

Let $f\,(x, y, z) \in \mathcal{C}_{m,n,1}$. Our first aim is to show that, after a family of admissible transformations "respecting" Z, the series $\mathcal{T}\,(f)\,(X, Y, Z)$ is normal. This motivates the next definition.

Definition 3.4. — *Let* $\nu : I_{m',n'+1,r'} \ni (x', y', z') \mapsto (x, y, z) \in I_{m,n+1,r}$ *be an elementary transformation. Let* ν_0, r_0', r_0 *denote the first* $m+n$ *components of* ν, r', r *respectively. We say that* ν respects the variable z *if* ν_0 *does not depend on* z'*. Hence* $\nu_0 : I_{m',n',r_0'} \ni (x', y') \mapsto (x, y) \in I_{m,n,r_0}$

is an elementary transformation. Analogously, we extend this definition to the case when z' and/or z are generalised variables by requiring that the components of ν which correspond to the variables (x, y) depend only on (x', y') and not on z'.

Lemma 3.5. — *Suppose that ν respects z, as in the above definition. Then there exists a set $S \subseteq I_{m',n',r'_0}$ (which is either empty or the zeroset of some variable) such that the maps $\nu \upharpoonright I_{m',n'+1,r'} \setminus (S \times \mathbb{R})$ and $\nu_0 \upharpoonright I_{m',n',r'_0} \setminus S$ are bijections onto their image and for all $(x', y') \in I_{m',n',r'_0} \setminus S$ the map $z' \mapsto z = \nu_{m+n+1}(x', y', z')$ is a monotonic bijection onto its image. Moreover, the components of the inverse maps $(x, y) \mapsto (x', y')$ and $(x', y', z) \mapsto z'$ are $\mathcal{A}$-terms. Finally, if $S \neq \emptyset$ then ν is a blow-up chart and $\nu(S \times \mathbb{R})$ is the common zeroset of two variables.*

Proof. — We only give the details for $\nu : (x', y', z') \mapsto (x', y', x'_1(\lambda + z'))$, for some $\lambda \in \mathbb{R}$. In this case, ν_0 is the identity map, $S = \{x'_1 = 0\}$ and $\nu(S \times \mathbb{R}) = \{x_1 = z = 0\}$. For all $(x', y') \notin S$, the inverse function $z \mapsto z' = \frac{z}{x'_1} - \lambda$ is an $\mathcal{A}$-term. □

Definition 3.6. — *We say that an admissible family $\mathcal{F}$ of transformations $(x', y', z') \mapsto (x, y, z)$* respects *$z$ if all the elementary transformations appearing in $\mathcal{F}$ respect z (with the obvious convention that if, for example, $\mathcal{F} \ni \rho = \nu_1 \circ \nu_2 : (x', y', z') \mapsto (x'', y'', z'') \mapsto (x, y, z)$, then ν_1 respects z and ν_2 respects z''). We say that $\mathcal{F}$* almost respects *z if for all $\rho = \nu_1 \circ \ldots \circ \nu_k$ the elementary transformations $\nu_1, \ldots, \nu_{k-1}$ respect z and either ν_k respects z or ν_k is a blow-up chart at infinity involving z and some other variable (i.e. ν_k is either $\pi^{\infty}_{m+1,j}$ or $\pi^{\pm\infty}_{m+n+1,j}$, for some $j \in \{1, \ldots, m\}$).*

We prove the following monomialisation result.

Theorem 3.7. — *Let $F(X, Y, Z) \in \widehat{\mathcal{C}}_{m,n,1}$. Then, after admissible family almost respecting Z, we have that F is normal.*

Before proving the above theorem, we show how it implies the Main Theorem. Since we want to keep track of standard and generalised variables, we will change the notation and prove the Main Theorem for a germ $f(x, y, z) \in \mathcal{C}_{m,n,1}$, where y is now an n-tuple of variables and z is a single variable.

Proof of the Main Theorem. — Let $f(x, y, z) \in \mathcal{C}_{m,n,1}$. By Theorem 3.7 and the quasianalyticity property, after some admissible family almost respecting z, the germ of f is normal (i.e. it is the product of a monomial by a unit of $\mathcal{C}$). The proof is by induction on the pairs (d, l), where $d = m+n+1$

is the total number of variables and l is the minimal length of an admissible monomialising family for f.

If $d=0$ or $l=0$ then there is nothing to prove. So we may suppose $d,l>0$.

Let $\mathcal{F}$ be a monomialising family for f of length l. Note that, for every $\rho\in\mathcal{F}$, we may partition the domain of ρ (which is either of the form $I_{m_\rho+1,n_\rho,r_\rho}$ or $I_{m_\rho,n_\rho+1,r_\rho}$, for some m_ρ,n_ρ such that $m_\rho+n_\rho=m+n$) into a finite union of sub-quadrants $Q_{\rho,j}$ (i.e. sets of the form $B_1\times\ldots\times B_{m+n+1}$, where B_i is either $\{0\}$, or $(-r_{\rho,i},0)$, or $(0,r_{\rho,i})$) such that $f\circ\rho$ has constant sign on $Q_{\rho,j}$. By a classical compactness argument (see for example [6, p. 4406]), there exists a finite subfamily $\mathcal{F}_0\subseteq\mathcal{F}$ and an open neighbourhood $W\subseteq\mathbb{R}^{m+n+1}$ of the origin such that $W\cap\mathrm{dom}\,(f)=\bigcup_{\rho\in\mathcal{F}_0}\bigcup_{j\leqslant J}\rho\,(Q_{\rho,j})$, for some $J\in\mathbb{N}$. Notice that, if A,B are $\mathcal{A}$-cells, then $A\cap B$ and $A\setminus B$ are finite disjoint unions of $\mathcal{A}$-cells.

Let $\mathcal{F}_1$ be an elementary family and $\mathcal{F}_2$ be an admissible family of length $<l$ such that for every $\rho\in\mathcal{F}_0$ there exist $\nu_\rho\in\mathcal{F}_1$ and $\rho'\in\mathcal{F}_2$ such that $\rho=\nu_\rho\circ\rho'$. Notice that $\mathcal{F}_2$ necessarily almost respects z. We will first consider the admissible transformations such that ν_ρ respects z. Let S_ρ be the singular set of ν_ρ defined in Lemma 3.5. If $S_\rho\neq\emptyset$, then the set $T_\rho=\nu_\rho\,(S_\rho\times\mathbb{R})$ is the common zeroset of two variables. By Condition 1 in 2.4, the germ of $f\restriction T_\rho$ belongs to the collection $\mathcal{C}$ and depends on less than d variables. Hence the inductive hypothesis holds and the theorem is proved for $f\restriction T_\rho$. Notice that, by 2.1, the complement in $\mathrm{dom}\,(f)$ of the union of all T_ρ such that ν_ρ respects z can be partitioned into a finite union of domains $I\subseteq\mathrm{dom}\,(f)$ such that, possibly up to some reflection, the germ of $f\restriction I$ belongs to the collection $\mathcal{C}$. It therefore suffices to prove the theorem for $f\restriction I$.

If ν_ρ is either $\pi^{\infty}_{m+1,j}$ or $\pi^{\pm\infty}_{m+n+1,j}$, then necessarily $\rho=\nu_\rho$ and clearly for every sub-quadrant Q the set $\nu_\rho\,(Q)$ is an $\mathcal{A}$-cell.

Otherwise, ν_ρ respects z. We rename $\nu_\rho=\nu$ and $S_\rho=S$. In order to avoid a cumbersome notation, we will only treat the case, as in Definition 3.4, of the form $\nu:(x',y',z')\mapsto(\nu_0\,(x',y')\,,\nu_{m+n+1}\,(x',y',z'))$, i.e. where both z' and z are standard variables (the other cases can be treated analogously). By induction on l, the theorem applies to $f\circ\nu\restriction\mathrm{dom}\,(\nu)\setminus(S\times\mathbb{R})$. Let A be one of the $\mathcal{A}$-cells obtained thus. Without loss of generality, we may suppose that A is of the form $\{(x',y',z'):\ (x',y')\in C,\ z'*t\,(x',y')\}$, where $*\in\{=,<\}$, C is an $\mathcal{A}$-base and t is an $\mathcal{A}$-term. Using the fact that ν_0 is invertible and the map $z'\mapsto z=\nu_{m+n+1}\,(x',y',z')$ is monotonic, we obtain that $\nu\,(A)=\left\{(x,y,z):(x,y)\in\nu_0\,(C)\,,\ z*\nu_{m+n+1}\left(\nu_0^{-1}\,(x,y)\,,t\left(\nu_0^{-1}\,(x,y)\right)\right)\right\}$, and it is easy to see that $\nu_0(C)$ is an $\mathcal{A}$-base. Since f has constant sign on $\nu(A)$, this concludes the proof of the theorem. □

4. Proof of Theorem 3.7

Let $(X,Y) = (X_1,\dots,X_m,Y_1,\dots,Y_n)$ and $F(X,Y,Z) \in \widehat{\mathcal{C}}_{m,n,1}$. The proof of Theorem 3.7 is by induction on $m+n$, the case $m+n=0$ being trivial. Throughout the proof we will use the following easy consequence of the inductive hypothesis (see [18, Lemma 2.2] and [1, Lemma 4.7]; the proof for the case of standard variables extends trivially to the case of mixed variables).

Assumption 4.1 (Inductive Hypothesis). — *Let $G_1(X,Y)$, … , $G_s(X,Y) \in \widehat{\mathcal{C}}_{m,n}$. Then, after admissible family, the G_i are normal and linearly ordered by division.*

The first stage of the proof consists in giving a suitable presentation of F with respect to Z.

Definition 4.2. — *We say that $F \in \widehat{\mathcal{C}}_{m,n,1}$ admits a finite presentation of order d if there are $\alpha_1 > \dots > \alpha_d \in \mathbb{K}$, $H_1,\dots,H_d \in \widehat{\mathcal{C}}_{m,n}$, which are normal, and units $U_1,\dots,U_d \in \left(\widehat{\mathcal{C}}_{m,n,1}\right)^\times$ such that $F(X,Y,Z) = H_1(X,Y)\,G(X,Y,Z)$, where*

$$\begin{aligned} G(X,Y,Z) =& Z^{\alpha_1}U_1(X,Y,Z) + H_2(X,Y)\,Z^{\alpha_2}U_2(X,Y,Z) + \dots \\ &+ H_d(X,Y)\,Z^{\alpha_d}U_d(X,Y,Z). \end{aligned}$$

Proposition 4.3. — *Suppose that the Inductive Hypothesis 4.1 holds. Then F admits a finite presentation of some order $d \in \mathbb{N}$, after admissible family respecting the variable Z (in fact, the admissible transformations required act as the identity on Z).*

The ring $\mathbb{R}[\![X^*,Y]\!]$ is clearly not Noetherian. However, the next lemma provides a finiteness property which is enough for our purposes. The proof takes inspiration from [9, Theorem 6.3.3].

Lemma 4.4. — *Let $\mathcal{G} = \{F_\alpha(X,Y) :\ \alpha \in A\} \subseteq \widehat{\mathcal{C}}_{m,n}$ be a family with good total support. Then,*

a) *after admissible family, there are $\beta \in [0,\infty)^m$ and a collection $\{G_\alpha(X,Y)\colon\ \alpha \in A\} \subseteq \widehat{\mathcal{C}}_{m,n}$ such that $\forall \alpha \in A,\ F_\alpha(X,Y) = X^\beta G_\alpha(X,Y)$ and $G_{\alpha_0}(0,Y) \not\equiv 0$, for some $\alpha_0 \in A$;*

b) *for every $d \in \mathbb{N}$, after admissible family, the $\mathbb{R}[\![X^*, Y]\!]$-module generated by the tuples $\{(F_{\alpha_1}, \dots, F_{\alpha_d}) : \ \alpha_1, \dots, \alpha_d \in A\}$ is finitely generated.*

The numbers m, n may change under admissible transformation.

Proof. — For the proof of a), we view $\mathcal{G}$ as a subset of $\mathbb{B}[\![X^*]\!]$, with $\mathbb{B} = \mathbb{R}[\![Y]\!]$. In [6, 4.11] the authors define the *blow-up height* of a finite set of monomials, denoted by b_X. It follows from the definition of b_X that if $b_X(\mathcal{G}_{\min}) = (0, 0)$, then there exists $\beta \in [0, \infty)^m$ such that $\mathcal{G}_{\min} = \left\{X^\beta\right\}$, which is what we want. The proof of this step is by induction on the pairs $(m, b_X(\mathcal{G}_{\min}))$, ordered lexicographically. If $m = 0$, there is nothing to prove. If $m = 1$, then $b_X(\mathcal{G}_{\min}) = (0, 0)$.

Hence we may assume that $m > 1$ and $b_X(\mathcal{G}_{\min}) \neq (0, 0)$. It follows from the proof of [6, Proposition 4.14] that there are $i, j \in \{1, \dots, m\}$ and suitable ramifications r_i^γ, r_j^δ of the variables X_i and X_j such that, after the admissible transformations $\rho_0 := r_i^\gamma \circ r_j^\delta \circ \pi_{i,j}^0$ and $\rho_\infty := r_i^\gamma \circ r_j^\delta \circ \pi_{i,j}^\infty$, the blow-up height b_X of $\mathcal{G}_{\min}$ has decreased (to see this, consider α_i, β_j in the proof of [6, Lemma 4.10] and perform the mentioned ramifications with $\gamma = \beta_j$ and $\delta = \alpha_i$). Moreover, for every $\lambda \in (0, \infty)$, after the admissible transformation $\rho_\lambda := r_i^\gamma \circ r_j^\delta \circ \pi_{i,j}^\lambda$, the series in the family $\mathcal{G}$ have one less generalised variable and one more standard variable, so m has decreased. Since admissible transformations preserve having good total support, the inductive hypothesis applies and we obtain the required conclusion.

The proof of b) is by induction on the pairs $(m + n, d)$, ordered lexicographically. Arguing by induction on d as in [9, Lemma 6.3.2], it is enough to prove the case $d = 1$. If $m + n = 1$ then, since $\mathcal{G}$ has good total support, the ideal generated by $\mathcal{G}$ is principal. Hence suppose that $m + n > 1$. Recall that, by part a) of this lemma, there are $\beta \in [0, \infty)^m$ and a collection $\{G_\alpha(X, Y) : \ \alpha \in A\} \subseteq \widehat{\mathcal{C}}_{m,n}$ such that $\forall \alpha \in A, \ F_\alpha(X, Y) = X^\beta G_\alpha(X, Y)$ and $G_{\alpha_0}(0, Y) \not\equiv 0$, for some $\alpha_0 \in A$. After a linear transformation $L_{n,c}$, we may suppose that G_{α_0} is regular of some order d in the variable Y_n.

Let $\hat{Y} = (Y_1, \dots, Y_{n-1})$. By the formal Weierstrass Division for generalised power series (see [6, 4.17]), for every $\alpha \in A$ there are $Q_\alpha \in \mathbb{R}[\![X^*, Y]\!]$ and $B_{\alpha,0}, \dots, B_{\alpha,d-1} \in \mathbb{R}[\![X^*, \hat{Y}]\!]$ such that $G_\alpha = G_{\alpha_0} Q_\alpha + R_\alpha$, where $R_\alpha(X, Y) = \sum_{i=0}^{d-1} B_{\alpha,i}\left(X, \hat{Y}\right) Y_n^i$. It is proven in [6, p. 4390] that the total support of the collection $\{B_{\alpha,j} : \ \alpha \in A, \ j = 0, \dots, d-1\}$ is contained in the good set $\Sigma\mathrm{Supp}\,(\mathcal{G})$ of all finite sums (done component-wise) of elements of $\mathrm{Supp}\,(\mathcal{G})$. Hence, by the inductive hypothesis on the total number of variables, after admissible family acting on $\left(X, \widehat{Y}\right)$, the $\mathbb{R}[\![X^*, \hat{Y}]\!]$-module generated by $\mathcal{B} = \{B_\alpha = (B_{\alpha,0}, \dots, B_{\alpha,d-1}) :\ \alpha \in A\}$ is finitely generated. Therefore, there are $\alpha_1, \dots, \alpha_q \in A$ and for all $\alpha \in A$ there are $C_{\alpha,1}, \dots, C_{\alpha,q} \in \mathbb{R}[\![X^*, \hat{Y}]\!]$ such that $B_\alpha = \sum_{j=1}^{q} C_{\alpha,j} B_{\alpha_j}$. Putting everything together, we obtain that, for every $\alpha \in A$,

$$F_\alpha = \left(Q_\alpha - \sum_{j=1}^{q} C_{\alpha,j} Q_{\alpha_j} \right) F_{\alpha_0} + \sum_{j=1}^{q} C_{\alpha,j} F_{\alpha_j}.$$

□

Proof of Proposition 4.3. — Write $F(X,Y,Z) = \sum_{\alpha \in A} F_\alpha (X,Y) Z^\alpha$ and consider the family $\mathcal{G} = \{F_\alpha (X,Y) :\ \alpha \in A\}$, which is contained in $\widehat{\mathcal{C}}_{m,n}$ by Conditions 2 and 5 in 2.4. Note that $A \subseteq [0, \infty)$ is a well ordered set and $\mathcal{G}$ has good total support.

By Lemma 4.4, after admissible family acting on (X,Y), the $\mathbb{R}[\![X^*, Y]\!]$-ideal generated by $\mathcal{G}$ is finitely generated. Hence we can apply the Inductive Hypothesis 4.1 simultaneously to the generators and obtain that, after admissible family acting on (X,Y), the generators are normal and linearly ordered by division. Hence, there is $\alpha_1 \in A$ and for all $\alpha \in A$ there is $Q_\alpha \in \mathbb{R}[\![X^*, Y]\!]$ such that $F_\alpha = F_{\alpha_1} \cdot Q_\alpha$. Notice that, since F_{α_1} is normal, by monomial division $Q_\alpha \in \widehat{\mathcal{C}}_{m,n}$ (by Remark 2.5, the inverse of a unit belonging to $\hat{\mathcal{C}}$ also belongs to $\hat{\mathcal{C}}$). This allows us to write

$$F(X,Y,Z) = \sum_{\alpha < \alpha_1} F_\alpha (X,Y) Z^\alpha + F_{\alpha_1} (X,Y) Z^{\alpha_1} U(X,Y,Z),$$

where $U(X,Y,Z) = 1 + \sum_{\alpha > \alpha_1} Q_\alpha (X,Y) Z^{\alpha - \alpha_1}$. The series $G(X,Y,Z) = \sum_{\alpha < \alpha_1} F_\alpha (X,Y) Z^\alpha$ belongs to $\widehat{\mathcal{C}}_{m,n,1}$ by Condition 5 in 2.4, hence $U \in \left(\widehat{\mathcal{C}}_{m,n,1}\right)^\times$. We repeat the above argument for G. This procedure will provide, after admissible family acting on (X,Y), a decreasing sequence $\alpha_1 > \alpha_2 > \dots$ which is necessarily finite (say, of length d), since A is well-ordered. Now it is enough to rename $H_i := Q_{\alpha_i}$ for $i = 1, \dots, d$ and factor out H_1 to obtain the required finite presentation. □

We can now finish the proof of Theorem 3.7 by showing how to reduce the order of a finite presentation for F.

Proof of Theorem 3.7. — In what follows, up to suitable reflections, there is no harm in considering the variables (X,Y) as generalised, hence, to simplify the notation, we will suppose $Y = \emptyset$.

Suppose first that $F \in \widehat{\mathcal{C}}_{m,1}$, i.e. Z is a standard variable. By Proposition 4.3, we may suppose that F admits a finite presentation as in Definition 4.2. Since the exponents α_i are in $\mathbb{N}$, we have that G is regular of order α_1 in the variable Z.

If $\alpha_1 = 1$, then we perform the Tschirnhausen transformation translating Z by the solution to the implicit function problem $G = 0$, and obtain that F is normal.

Suppose that $\alpha_1 > 1$. We follow, up to suitable reflections and ramifications, the algorithm for decreasing the order of regularity in the proof of [18, Theorem 2.5], which we briefly summarise (the details can be found in [21, Section 4.2.2]). By the Taylor formula, there are series $A_1, \ldots, A_d \in \widehat{\mathcal{C}}_m$, with $A_i(0) = 0$, and a unit $U \in \left(\widehat{\mathcal{C}}_{m,1}\right)^{\times}$ such that

$$G(X, Z) = A_d(X) + \ldots + A_1(X) Z^{\alpha_1 - 1} + U(X, Z) Z^{\alpha_1}.$$

After a Tschirnhausen translation, we may assume that $A_1 = 0$. We apply the Inductive Hypothesis 4.1 simultaneously to the A_i in such a way that, after admissible family acting on X, the A_i are normal, i.e. $A_i(X) = X^{\beta_i} U_i(X)$ for some $\beta_i \in \mathbb{K}^m$, $U_i \in \left(\widehat{\mathcal{C}}_m\right)^{\times}$, and for some $l \in \{2, \ldots, d\}$ the series $A_l^{1/l}$ divides all the series $A_i^{1/i}$ (notice that if $F \in \widehat{\mathcal{C}}_{0,m+1}$, i.e. all the variables are standard, then we can start the proof by first ramifying the variables X with exponent $d!$, in order to ensure that only natural exponents appear in the series $A_l^{1/l}$).

Let $j \in \{1, \ldots m\}$ be such that the variable X_j appears with a nonzero exponent in the monomial X^{β_l} and consider the family of blow-up transformations $\left\{\pi_{m+1,j}^{\lambda} : \ \lambda \in \mathbb{R} \cup \{\pm\infty\}\right\}$.

After the transformations $\pi_{m+1,j}^{\pm\infty}$, the series G has the form $Z^{\alpha_1} V(X, Z)$, where $V \in \left(\widehat{\mathcal{C}}_{m,1}\right)^{\times}$, so in this case F is normal, and we are done.

After the transformation $\pi_{m+1,j}^{0}$, the exponent of X_j in the monomial X^{β_l} has decreased by the quantity l. By repeating the procedure and applying it to the other variables appearing with a nonzero exponent in the monomial X^{β_l} , we can reduce the order of regularity of G to $\alpha_1 - l$.

For $\lambda \in \mathbb{R} \setminus \{0\}$, after the transformation $\pi_{m+1,j}^{\lambda}$, thanks to the fact that $A_1 = 0$, the order of G is at most $\alpha_1 - 1$.

This shows that, in the case when Z is a standard variable, after admissible family almost respecting Z, the series F is normal.

Now suppose that $F \in \widehat{\mathcal{C}}_{m+1,0}$, i.e. Z is a generalised variable. By Proposition 4.3, we may suppose that F admits a finite presentation as in Definition 4.2. We can apply the Inductive Hypothesis 4.1 simultaneously to

$H_1,\dots,H_d$ in such a way that, after admissible family, we have

$$G(X,Z)=Z^{\alpha_1}\tilde{U}_1(X,Z)+X^{\Gamma_2}Z^{\alpha_2}\tilde{U}_2(X,Z)+\dots+X^{\Gamma_d}Z^{\alpha_d}\tilde{U}_d(X,Z),$$

for some units $\tilde{U}_i\in\left(\widehat{\mathcal{C}}_{m+1,0}\right)^{\times}$, and the exponents $\Gamma_i=\left(\gamma_i^{(1)},\dots,\gamma_i^{(m)}\right)$ are such that the monomials $\left\{X^{\frac{\Gamma_i}{\alpha_1-\alpha_i}}\ :\ i=2,\dots,d\right\}$ are linearly ordered by division. Let $i_0\in\{2,\dots,d\}$ be smallest with the property that

$$(\#)\qquad \forall i\in\{2,\dots,d\},\ \forall j\in\{1,\dots,m\},\quad \frac{\gamma_{i_0}^{(j)}}{\alpha_1-\alpha_{i_0}}\leqslant\frac{\gamma_i^{(j)}}{\alpha_1-\alpha_i}.$$

Suppose $\gamma_{i_0}^{(1)}\neq 0$ and perform a ramification of the variable X_1 with exponent $\gamma:=\frac{\gamma_{i_0}^{(1)}}{\alpha_1-\alpha_{i_0}}$. We consider the family of blow-up transformations $\left\{\pi_{m+1,1}^{\lambda}:\ \lambda\in[0,\infty]\right\}$.

After the transformation $\pi_{m+1,1}^{\infty}$, we can write

$$G(X,Z)=Z^{\alpha_1}\left[\tilde{U}_1(X,Z)+X^{\Gamma_2}Z^{\beta_2}\tilde{U}_2(X,Z)+\dots+X^{\Gamma_d}Z^{\beta_d}\tilde{U}_d(X,Z)\right],$$

where $\beta_i:=\frac{\gamma_i^{(1)}}{\gamma_{i_0}^{(1)}}(\alpha_1-\alpha_{i_0})+\alpha_i-\alpha_1$ is nonnegative, thanks to (#). Notice that, since by (#) every $\gamma_i^{(1)}$ is nonzero, the expression between square brackets is a unit. Hence in this case F has a finite presentation of order 1, i.e. F is normal, and we are done.

After the transformation $\pi_{m+1,1}^{0}$, we can write

$$\begin{aligned}G(X,Z)=&X_1^{\gamma\alpha_1}\left[Z^{\alpha_1}\tilde{U}_1(X,Z)+X^{\Delta_2}Z^{\alpha_2}\tilde{U}_2(X,Z)+\dots\right.\\&\left.+X^{\Delta_d}Z^{\alpha_d}\tilde{U}_d(X,Z)\right],\end{aligned}$$

where $\Delta_i=\left(\delta_i^{(1)},\dots,\delta_i^{(m)}\right):=\left(\gamma_i^{(1)}-\gamma_{i_0}^{(1)}\frac{\alpha_1-\alpha_i}{\alpha_1-\alpha_{i_0}},\gamma_i^{(2)},\dots,\gamma_i^{(m)}\right)$. Remark that, by (#), the exponents $\delta_i^{(1)}$ are nonnegative and $\delta_{i_0}^{(1)}=0$. Hence, up to factoring out by a power of X_1, the variable X_1 does not appear any more in the i_0^{th} term of the above finite presentation. By repeating this step with the other variables X_j such that $\gamma_{i_0}^{(j)}\neq 0$, we obtain

$$\begin{aligned}G(X,Z)=&X^{\Delta}\left[Z^{\alpha_{i_0}}V(X,Z)+X^{\Delta'_{i_0+1}}Z^{\alpha_{i_0+1}}\tilde{U}_{i_0+1}(X,Z)+\dots\right.\\&\left.+X^{\Delta'_d}Z^{\alpha_d}\tilde{U}_d(X,Z)\right],\end{aligned}$$

where $V \in \left(\widehat{\mathcal{C}}_{m+1,0}\right)^{\times}$, the components of Δ are $\frac{\alpha_1\gamma_{i_0}^{(j)}}{\alpha_1-\alpha_{i_0}}$ and the components of Δ'_i are $\gamma_i^{(j)} - \gamma_{i_0}^{(j)}\frac{\alpha_1-\alpha_i}{\alpha_1-\alpha_{i_0}}$. Hence F has a finite presentation of order $d - i_0 + 1$.

If $\lambda \in (0,\infty)$, then after the transformation $\pi^{\lambda}_{m+1,1}$, the variable Z is standard and we have reduced to the case $F \in \widehat{\mathcal{C}}_{m,1}$. $\square$

Remark 4.5. — In the case when the set of admissible exponents is $\mathbb{N}$ the proof of Theorem 3.7 can be simplified. In fact, by Noetherianity of $\mathbb{R}[\![X,Y]\!]$, the $\mathbb{R}[\![X,Y]\!]$-ideal generated by the family $\mathcal{G}$ is finitely generated and one obtains immediately a "formal" finite presentation for F, where the units are formal power series, not necessarily belonging to $\widehat{\mathcal{C}}$. After monomialising the generators and factoring out an appropriate monomial, this automatically implies that F is regular of some order in the variable Z. Hence we can dispense with Proposition 4.3 and implement directly the last part of the proof of Theorem 3.7.

This argument also implies that in the real analytic setting, in order to obtain regularity in a chosen variable Z, there is no need to prove a convergent version of the finite presentation in Definition 4.2. In their proof of quantifier elimination for the real field with restricted analytic functions and the function $x \mapsto 1/x$, Denef and van den Dries prove such a convergent version (see [3, Lemma 4.12]), by invoking a consequence of faithful flatness in [13, (4C)(ii)]. Our remark implies that this is not necessary.

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