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ON FUNCTIONS WITH A CONJUGATE

by Paul BAIRD & Michael EASTWOOD (*)

Abstract. — Harmonic functions of two variables are exactly those that admit a conjugate, namely a function whose gradient has the same length and is everywhere orthogonal to the gradient of the original function. We show that there are also partial differential equations controlling the functions of three variables that admit a conjugate.

Résumé. — Les fonctions harmoniques en deux variables sont exactement celles qui admettent une fonction conjuguée, à savoir une fonction dont le gradient a la même longueur et est partout orthogonal au gradient de la fonction d'origine. Nous montrons qu'il existe des équations aux dérivées partielles qui contrôlent également les fonctions de trois variables qui admettent une fonction conjuguée.

1. Introduction

A pair of smooth real-valued functions f and g on a Riemannian manifold M are said to be conjugate if and only if

$$\|\nabla f\| = \|\nabla g\| \quad \text{and} \quad \langle \nabla f, \nabla g \rangle = 0. \tag{1.1}$$

In this article, we shall address the following question. When does a given smooth function $f : M \to \mathbb{R}$ admit a conjugate function? When M is 2-dimensional the pair of functions $(f, g) : M \to \mathbb{R}^2$ is mutually conjugate if and only if the mapping (f, g) is conformal away from isolated points where its differential vanishes. It is well-known that, in this case, f must

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be harmonic and, conversely, a harmonic function locally always admits a conjugate, unique up to an additive constant. When M is of higher dimension, then the pair $(f, g) : M \to \mathbb{R}^2$ is said to be semiconformal. As discussed in [6], semiconformality is one of the two conditions that (f, g) be a harmonic morphism. In fact, if $M = \mathbb{R}^n$ and both f and g are polynomial, then it is the only condition [1]. In this article, we shall be concerned with f defined on an open subset in $\mathbb{R}^3$. We extend our earlier work [3] in which we derived some necessary conditions on f in order that it admit a conjugate under a non-degeneracy condition, to now obtain necessary and sufficient conditions in all cases.

An example of a pair of conjugate functions in three variables is

$$f = x_2 \frac{{x_1}^2 + {x_2}^2 + {x_3}^2}{{x_2}^2 + {x_3}^2} \qquad g = x_3 \frac{{x_1}^2 + {x_2}^2 + {x_3}^2}{{x_2}^2 + {x_3}^2}.$$

The Hopf mapping $S^3 \to S^2$ viewed in stereographic coördinates

$$f = \frac{(1 - \|x\|^2)x_2 + 2x_1x_3}{{x_2}^2 + {x_3}^2} \qquad g = \frac{(1 - \|x\|^2)x_3 - 2x_1x_2}{{x_2}^2 + {x_3}^2}$$

provides another good example. In these two cases, the pair (f, g) enjoys an evident symmetry with respect to rotation about the x_1-axis. This is not usual, as is illustrated by the following example:–

$$f = \log \sqrt{{x_1}^2 + {x_2}^2 + {x_3}^2} \qquad g = \arccos \frac{x_1}{\sqrt{{x_1}^2 + {x_2}^2 + {x_3}^3}}.$$

In all three examples, the pair (f, g) is smooth away from the x_1-axis.

We shall frequently need to manipulate tensors and for this purpose, we use Penrose's abstract index notation [13]. We shall write

$$f_i = \nabla_i f \qquad f_{ij} = \nabla_i \nabla_j f \qquad \text{et cetera},$$

where ∇_i is the flat connection on $\mathbb{R}^n$ or, more generally, the metric connection on a Riemannian manifold. Also, let us 'raise and lower' indices with the metric δ_{ij} in the usual fashion and write a repeated index to denote the invariant contraction over that index. Thus, ${f^i}_i = \Delta f$ is the Laplacian and $f^i g_i = \langle \nabla f, \nabla g \rangle$. We shall use round and square brackets to denote symmetrising and skewing over the indices they enclose. For example, $\phi_{(ij)k} = \frac{1}{2}\phi_{ijk} + \frac{1}{2}\phi_{jik}$ and $\nabla_{[i}\phi_{j]}$ is the exterior derivative of a 1-form ϕ_i.

In order to find necessary and sufficient conditions for a function f defined on an open set of $\mathbb{R}^3$ to admit a conjugate, we begin by constructing conformal invariants that reflect geometric constraints that derive from (1.1) and its derivatives.

A conformal differential invariant is a polynomial in the derivatives of f as well as the inverse (Euclidean) metric, that transforms by scaling under the action of the Möbius group on $\mathbb{R}^3 \cup \{\infty\}$ (the amount of scaling being called the *weight* of the invariant: for details see Appendix A). An elementary conformal invariant is the first order one $J := f^i f_i$ of weight -2. We shall require invariants up to third order. In Appendix A we give a more thorough treatment of conformal invariants and derive a list of those that we require; these will be labelled with uppercase Roman letters.

Higher order conformal invariants may be built from lower order ones by using simple rules. For example, if ϕ_i is a conformally invariant 1-form of weight -1, then the trace $\nabla^i \phi_i$ is conformally invariant. Applying this procedure to the 1-form $\sqrt{J} f_i$ yields $Z/\sqrt{J}$, where, up to a multiple, the operator Z is the 3-Laplacian, a well-known conformal invariant in dimension 3. The trace-free part of $\nabla_{(i}\phi_{j)}$ is invariant whenever ϕ_j has weight 2. Applying this construction to $J^{-1} f_i$ yields an invariant ψ_{ij} from which we deduce another invariant X via the formula:

$$\psi^{ij}\psi_{ij} = \tfrac{2}{3} Z^2 - JX\,.$$

The invariant X plays a fundamental role in our characterization. Its explicit expression is given in §2 below. A necessary condition that f admit a conjugate is that $X \leqslant 0$ (Theorem 2.1). In what we refer to as the *generic case* $X < 0$, there are exactly four distinct vectors (two up to sign) called *conjugate directions*, which potentially may be the gradient of a conjugate function. When $X = 0$ there are either exactly two conjugate directions, so up to sign any conjugate must be unique, or infinitely many; these two cases are distinguished by another conformal invariant derived from X and Z, which we call Y. By normalising coördinates, we explain the geometric interpretation of these conditions.

The next step is to understand when a conjugate direction ω_i is integrable and so is the gradient of a function. In §3 we show that in the generic case, integrability is equivalent to the vanishing of two polynomial expressions in ω_i and the derivatives up to third order of f (Theorem 3.1). Our objective is then to eliminate ω_i to obtain conditions involving just derivatives of f. However, a difficulty arises in that we only have explicit expressions for quadratic terms in ω_i. Thus, instead of trying to determine whether a specific conjugate direction is integrable, we ask rather that one or the other be integrable without specifying which. This leads to a set of three equations involving just quadratic terms in ω_i (Theorem 4.1). In §4, we show how to elimiate ω_i in a normalized coördinate system to give three third order differential equations in f. Each equation is a conformally invariant homogeneous expression in the derivatives of f with a certain weight and degree. To write these down in terms of conformal invariants, we explore combinations of invariants that have the same weight and degree and use ad hoc methods to equate terms. An invariant derivation without recourse to normal coördinates is given in Appendix B.

In §5 we deal with special cases, the first of which concerns functions that admit a unique conjugate direction (up to sign). In terms of conformal invariants, these are characterized by the conditions $X = 0$ and $Y \neq 0$. The analysis proceeds in a similar way to the generic case, except that now the characterization requires just two third order equations, made explicit in Corollary 5.4. The next special case concerns functions that admit infinitely many conjugates, characterized by $X = Y = 0$. Now, $J^{-1}f_j$ is a conformal Killing field, all of which can be written down explicitly, as detailed in Appendix C. This enables us to write down all conjugate pairs in this case. The final special case discusses functions of two variables that admit a conjugate (in $\mathbb{R}^3$).

Examples are discussed in §6. For the case of spherical symmetry, up to scaling and addition of a constant, $\log ||x||$ is the unique function that admits a conjugate, in fact infinitely many. If f is assumed to have cylindrical symmetry, then the corresponding examples give a nice illustration of the generic case. For a conjugate pair (f, g), fibres of the associated map into $\mathbb{R}^2$ are helices which wind around concentric cylinders; right-handed screw or left-handed screw now corresponds to the two choices of conjugate. Finally, in §7, for a function f that admits a conjugate g, we discuss how the conformal invariants $X(g)$ and $Z(g)$ of g relate to those of f. This enables us to give a characterization of 3-harmonic conjugate pairs.

2. A necessary condition

Theorem 2.1. — *Let M be an 3-dimensional Riemannian manifold and $f : M \to \mathbb{R}$ a smooth function. In order to admit a conjugate, f must satisfy the differential inequality*

$$X := 2f_i{}^j f_j f^{ik} f_k - f^i f_i f^{jk} f_{jk} + f^i f_i (f^j{}_j)^2 \leqslant 0. \tag{2.1}$$

Proof. — A proof of this theorem was given in [3]. In fact, a version was proved there valid in any dimension. Here we give a more efficient proof only valid in three dimensions. However, this proof will allow us to draw additional and useful conclusions. In addition, the method of proof (in

Lemma 2.2) will provide a good illustration of the normalisation techniques occurring throughout the rest of this article.

If f is to admit a conjugate, then there must be a closed 1-form ω_j such that

$$f^j\omega_j = 0 \quad \text{and} \quad \omega^j\omega_j = f^j f_j. \tag{2.2}$$

Indeed, (1.1) implies that we may find an ω_j that is exact. We shall show that the inequality (2.1) is necessary in order to find a closed ω_j satisfying (2.2). To proceed, let us differentiate the equations (2.2) with respect to ∇^i. We obtain

$$f^{ij}\omega_j + \omega^{ij}f_j = 0 \quad \text{and} \quad \omega^{ij}\omega_j = f^{ij}f_j. \tag{2.3}$$

Since we are supposing that $\omega_{ij} = \nabla_i\omega_j$ is symmetric we may transvect the second of these with f_i and use the first to eliminate $\omega^{ij}f_i$. This gives

$$f^{ij}\omega_i\omega_j + f^{ij}f_if_j = 0.$$

We now have the following equations

$$f^i\omega_i = 0 \qquad \omega^i\omega_i = f^if_i \qquad f^{ij}\omega_i\omega_j + f^{ij}f_if_j = 0 \tag{2.4}$$

and we claim it is a matter of algebra to show that the inequality (2.1) must hold if there is to be a solution ω_i. This is detailed in the following Lemma, which we state independently for future use. Notice that if ω_i is real then so is T_{ijk} in which case $T_{ijk}T^{ijk} \geqslant 0$. □

Lemma 2.2. — *If f_{ij} is a 3×3 symmetric matrix and f_i is a 3-vector, then*

$$(f^if_i)X + 12T_{ijk}T^{ijk} = 0 \tag{2.5}$$

where

$$T_{ijk} = f_{[i}\omega_j f_{k]\ell}\omega^\ell \tag{2.6}$$

and ω_i is any solution, real or complex, of the equations (2.4).

Proof. — If $f_i = 0$ then the conclusion is trivial. Otherwise, let us choose coördinates so that $f_1 = f_2 = 0$. We may also orthogonally diagonalise the quadratic form f_{ij} restricted to the plane orthogonal to f_i. In other words, we may further change coördinates to arrange that $f_{12} = 0$. Having made these choices, the quantity X becomes, after a short calculation,

$$X = 2(f_3)^2(f_{11} + f_{33})(f_{22} + f_{33}). \tag{2.7}$$

Another short calculation yields

$$T_{ijk}T^{ijk} = \tfrac{1}{6}(f_{22} - f_{11})^2{\omega_1}^2{\omega_2}^2 \tag{2.8}$$

whilst the equations (2.4) become

$$\omega_3 = 0 \qquad {\omega_1}^2 + {\omega_2}^2 = {f_3}^2 \qquad f_{11}{\omega_1}^2 + f_{22}{\omega_2}^2 + f_{33}{f_3}^2 = 0 \tag{2.9}$$

the second two of which may be written as

$$\begin{bmatrix} 1 & 1 \\ f_{11} + f_{33} & f_{22} + f_{33} \end{bmatrix} \begin{bmatrix} {\omega_1}^2 \\ {\omega_2}^2 \end{bmatrix} = \begin{bmatrix} {f_3}^2 \\ 0 \end{bmatrix}. \tag{2.10}$$

Now there are two cases. If $f_{11} = f_{22}$, then (2.8) implies that $T_{ijk}T^{ijk} = 0$. But (2.10) implies that $f_{11} + f_{33} = 0$ and then (2.7) shows that $X = 0$ and (2.5) reduces to $0 = 0$. On the other hand, if $f_{11} \neq f_{22}$, then we may use (2.10) to solve (2.9), obtaining

$$ {\omega_1}^2 = {f_3}^2 \frac{f_{22} + f_{33}}{f_{22} - f_{11}} \quad \text{and} \quad {\omega_2}^2 = {f_3}^2 \frac{f_{11} + f_{33}}{f_{11} - f_{22}}. \tag{2.11}$$

and compute

$$12T_{ijk}T^{ijk} = 2(f_{22} - f_{11})^2{\omega_1}^2{\omega_2}^2 = -2{f_3}^4(f_{11} + f_{33})(f_{22} + f_{33}).$$

A comparison with (2.7) immediately yields (2.5), as required. □

From now on we shall suppose that f_i is non-zero (at a particular point and hence nearby as well). In case that f admit a conjugate, it is then clear from (1.1) that the pair (f, g) is a submersion (near the point in question). The nature of the singularities of a semiconformal mapping is not known in general [2].

Notice that it follows from the proof of this lemma that the equations (2.4) always have solutions if we allow ω_i to be complex and generically (in fact, precisely when $X \neq 0$) there are four solutions. Alternatively, this is geometrically clear: the first equation restricts matters to a plane wherein the second and third equations describe planar quadrics.

Perhaps our proof of Lemma 2.2 seems bizarre but, in fact, we have used a familiar technique. The Cayley-Hamilton Theorem, for example, is often proved, even for real matrices, by employing Jordan canonical form over the complex numbers. Not only that, but Lemma 2.2 can be proved without normalisation by means of the Cayley-Hamilton Theorem applied to f_{ij} restricted, as a symmetric form, to the plane orthogonal to f_i (the details of this proof being left to the reader). Another proof avoiding normalisation may be obtained by expanding the identity $0 = f_{[i}\omega_j {f_k}^k {f_{\ell]}}^\ell$. In fact, it is a consequence of Weyl's Second Fundamental Theorem of Invariant Theory [15] that dimension-dependent identities must arise by 'skewing over too many indices'. To use normalisation as we have done, however, is a simple enough method that we shall employ throughout this article.

The quantities occurring in the proof of Lemma 2.2 suggest other combinations of derivatives with geometric significance. The operator

$$f \mapsto Z \equiv f^{ij} f_i f_j + f^i f_i f^j{}_j, \tag{2.12}$$

for example is, up to a multiple, the well-known 3-Laplacian [7, 10] and in normal coördinates

$$f_1 = f_2 = f_{12} = 0 \tag{2.13}$$

at a point becomes

$$Z = {f_3}^2(f_{11} + f_{22} + 2f_{33}). \tag{2.14}$$

Also, the quantity $J \equiv f^i f_i$ is ${f_3}^2$. Therefore, from (2.7),

$$\begin{aligned} Y &\equiv Z^2 - 2JX \\ &= {f_3}^4(f_{11} + f_{22} + 2f_{33})^2 - 4(f_3)^4(f_{11} + f_{33})(f_{22} + f_{33}) \\ &= {f_3}^4(f_{11} - f_{22})^2 \end{aligned} \tag{2.15}$$

and we recognise that the vanishing of this expression when $X = 0$ is exactly the criterion discovered in the proof of Lemma 2.2 for there to be infinitely many solutions ω_i to the system (2.4). In summary, if we allow complex solutions of (2.4) then

$$\begin{aligned} X \neq 0 &\iff \exists\ 4 \text{ distinct solutions} \\ X = 0 \text{ and } Y \neq 0 &\iff \exists\ 2 \text{ distinct solutions} \\ X = 0 \text{ and } Y = 0 &\iff \exists\ \infty\text{-many solutions.} \end{aligned} \tag{2.16}$$

If we restrict attention to the case when (2.4) has real solutions, then Lemma 2.2 implies that $X \leqslant 0$ whence

$$\begin{aligned} X \neq 0 &\iff X < 0 \text{ and } \exists\ 4 \text{ distinct solutions} \\ X = 0 \text{ and } Y \neq 0 &\iff Y > 0 \text{ and } \exists\ 2 \text{ distinct solutions} \\ Y = 0 &\iff X = 0 \text{ and } \exists\ \infty\text{-many solutions,} \end{aligned} \tag{2.17}$$

the last two conclusions following from $Y = Z^2 - 2JX$ upon noting that both terms on the right hand side are non-negative.

3. Integrability of the conjugate direction: the generic case

Recall that if f is to admit a conjugate function near any particular point, then there must be a solution ω_j at that point of the algebraic equations (2.4). These three equations, specifically the third one, were derived under the assumption that ω_j extend to a closed form near the point in question but our approach from now on is to take ω_j to be defined at a particular point by the equations (2.4) and ask whether it may be extended to a smooth closed form near that point whilst maintaining (2.4). This is entirely equivalent to finding a local conjugate for f. As a matter of terminology, we shall refer to a solution ω_j of (2.4) as a conjugate direction. In case that $X < 0$ (at the point in question and hence nearby as well), we have just seen from (2.17) that there are four distinct solutions of (2.4) for ω_j. It follows that any one of these solutions uniquely and smoothly extends as a conjugate direction. Therefore, the only remaining question in case $X < 0$ is whether this extension is closed and we shall refer to this as integrability. We show that integrability is equivalent to a further two polynomial equations in ω^i and the derivatives of f.

Resolution of these further equations combined with (2.4) will lead to necessary and sufficient differential conditions on the function f in order that it admit a conjugate. All of this is under the assumption that $X < 0$ and we shall refer to this as the generic case. The case $X \equiv 0$ will be studied separately.

THEOREM 3.1. — *Let ω_j be a conjugate direction determined by* (2.4). *Then provided $X < 0$, the tensor field ω_{ij} is symmetric in its indices if and only if*

$$f^{ijk}f_if_jf_k + f^{ijk}f_i\omega_j\omega_k + 2f^{ij}f_j{}^kf_if_k - 2f^{ij}f_j{}^k\omega_i\omega_k = 0 \tag{3.1}$$

$$f^{ijk}f_if_j\omega_k + f^{ijk}\omega_i\omega_j\omega_k + 4f^{ij}f_j{}^kf_i\omega_k = 0. \tag{3.2}$$

Proof. — Since $X \neq 0$, the identity of Lemma 2.2, namely

$$f^jf_jX + 12T_{ijk}T^{ijk} = 0\,, \tag{3.3}$$

where $T_{ijk} = f_{[i}\omega_jf_{k]l}\omega^l$, shows that the vector field $f^{ij}\omega_j$ is independent of f^i and ω^i. Therefore, the tensor field ω_{ij} is symmetric in its indices if and only if

$$u^iv^j(\omega_{ij} - \omega_{ji}) = 0\,, \tag{3.4}$$

where u^i and v^j are any vector fields taken from the set $\{f^i, \omega^i, f^{ij}\omega_j\}$. Looking back at (2.3), which was obtained by differentiating (2.2), we see that

$$f^i\omega^j(\omega_{ij} - \omega_{ji}) = f^{ij}f_if_j + f^{ij}\omega_i\omega_j.$$

This already vanishes by assumption. It is our third equation from (2.4). Differentiating this third equation gives

$$\begin{aligned}0 &= f^i\nabla_i(f^{jk}f_jf_k + f^{jk}\omega_j\omega_k)\\ &= f^{ijk}f_if_jf_k + f^{ijk}f_i\omega_j\omega_k + 2f^{jk}f^i{}_jf_if_k + 2f^{jk}\omega_{ij}f^i\omega_k\,.\end{aligned}$$

We notice that the last term $f^{jk}\omega_{ij}f^i\omega_k$ occurs as the first component of the symmetry condition $f^if^{jk}\omega_k(\omega_{ij}-\omega_{ji})=0$, which therefore holds if and only if

$$f^{ijk}f_if_jf_k + f^{ijk}f_i\omega_j\omega_k + 2f^{jk}f^i{}_jf_if_k + 2f^{jk}\omega_{ji}f^i\omega_k = 0\,,$$

where we have replaced ω_{ij} by ω_{ji} in the last term. But now (2.3) shows that we can replace $\omega_{ji}f^i$ with $-f_{ji}\omega^i$. This yields (3.1). Similarly, the equation

$$0 = \omega^i\nabla_i(f^{jk}f_jf_k + f^{jk}\omega_j\omega_k) = \cdots$$

shows that the final symmetry condition $\omega^if^{jk}\omega_k(\omega_{ij}-\omega_{ji})=0$ reduces to (3.2). □

Corollary 3.2. — *Locally, a smooth function f with $X<0$ admits a smooth conjugate if and only if there is a smooth solution ω_i of the equations* (2.4), (3.1) *and* (3.2).

Proof. — Symmetry of ω_{ij} is precisely the condition that ω_i be exact and, therefore, locally of the form ∇_ig for some smooth function g. □

Of course, we know that equations (2.4) admit smooth solutions when $X<0$ so the only issue is whether we can find a solution for which (3.1) and (3.2) are also satisfied. Also, if ω_i is a solution then so is $-\omega_i$.

4. Resolution of the equations: the generic case

Throughout this section we shall suppose that $X<0$. Recall that under this hypothesis f has four conjugate directions at each point, occurring in two pairs that differ only by sign. In other words, the solutions of the equations (2.4) have the form $\{\pm\omega_i,\pm\eta_i\}$ for ω_i and η_i smooth linearly independent 1-forms. Let us consider the expressions

$$\begin{aligned}p^+ &\equiv f^{ijk}f_if_jf_k + f^{ijk}f_i\omega_j\omega_k + 2f^{ij}f_j{}^kf_if_k - 2f^{ij}f_j{}^k\omega_i\omega_k\\ p^- &\equiv f^{ijk}f_if_jf_k + f^{ijk}f_i\eta_j\eta_k + 2f^{ij}f_j{}^kf_if_k - 2f^{ij}f_j{}^k\eta_i\eta_k\\ q^+ &\equiv f^{ijk}f_if_j\omega_k + f^{ijk}\omega_i\omega_j\omega_k + 4f^{ij}f_j{}^kf_i\omega_k\\ q^- &\equiv f^{ijk}f_if_j\eta_k + f^{ijk}\eta_i\eta_j\eta_k + 4f^{ij}f_j{}^kf_i\eta_k\end{aligned}$$

According to Corollary 3.2 and the discussion that immediately follows it, we now know that f admits a conjugate if and only if

$$p^+ = q^+ = 0 \qquad \text{or} \qquad p^- = q^- = 0.$$

These two possibilities are captured by the following theorem.

Theorem 4.1. — *Locally, a smooth function f with $X < 0$ admits a smooth conjugate if and only if*

$$p^+p^- = 0 \qquad q^+q^- = 0 \qquad (p^+q^-)^2 + (p^-q^+)^2 = 0.$$

Proof. — Evidently, the vanishing of these three quantities is equivalent to $p^+ = q^+ = 0$ or $p^- = q^- = 0$. □

The condition $p^+p^- = 0$ was already resolved in [3]. We recapitulate and refine the argument as follows. Firstly, we write p^+ using normal coordinates (2.13) to discover that

$$p^+ = p_e + p_o\omega_1\omega_2 \tag{4.1}$$

where

$$\begin{aligned} p_e = {} & f_3{}^3 f_{333} + 2f_3{}^2(f_{13}{}^2 + f_{23}{}^2 + f_{33}{}^2) \\ & + (f_3 f_{113} - 2f_{11}{}^2 - 2f_{13}{}^2)\omega_1{}^2 + (f_3 f_{223} - 2f_{22}{}^2 - 2f_{23}{}^2)\omega_2{}^2 \end{aligned} \tag{4.2}$$

and

$$p_o = 2f_3 f_{123} - 4f_{13}f_{23}.$$

In normal coördinates $(\eta_1, \eta_2) = (\pm\omega_1, \mp\omega_2)$. It follows that

$$p^- = p_e - p_o\omega_1\omega_2 \tag{4.3}$$

and hence that

$$p^+p^- = p_e{}^2 - p_o{}^2\omega_1{}^2\omega_2{}^2. \tag{4.4}$$

But, since ω_i is subject to (2.4), we know that $\omega_1{}^2$ and $\omega_2{}^2$ are determined in normal coördinates by (2.11). In [3] we used this to eliminate $\omega_1{}^2$ and $\omega_2{}^2$ from p_e in (4.2) and then from p^+p^- in (4.4) to discover by trial and error that $Y^2p^+p^-$ could be written as an explicit Riemannian invariant in the derivatives of f, where Y is the invariant $Z^2 - 2JX$ from (2.15). We can argue more systematically as follows. Firstly, we may obtain η_i from ω_i without recourse to normal coördinates.

Lemma 4.2. — *The conjugate direction η_i is determined by the conjugate direction ω_i via the formula*

$$\sqrt{Y}\eta_i = 2f^{jk}f_j\omega_k f_i + (Z - 2f^{jk}f_jf_k)\omega_i - 2Jf_i{}^j\omega_j. \tag{4.5}$$

Proof. — Since it is evidently coördinate-free, we may verify this formula in normal coördinates (2.13). Substituting from (2.14) we see that the right hand side of (4.5) becomes

$$2(f_{13}f_3\omega_1 + f_{23}f_3\omega_2)f_i + {f_3}^2(f_{11} + f_{22})\omega_i - 2{f_3}^2{f_i}^j\omega_j$$

In more detail,

i	right hand side of (4.5)
1	${f_3}^2(f_{11} + f_{22})\omega_1 - 2{f_3}^2(f_{11}\omega_1) = {f_3}^2(f_{22} - f_{11})\omega_1$
2	${f_3}^2(f_{11} + f_{22})\omega_2 - 2{f_3}^2(f_{22}\omega_2) = {f_3}^2(f_{11} - f_{22})\omega_2$
3	$2(f_{13}f_3\omega_1 + f_{23}f_3\omega_2)f_3 - 2{f_3}^2(f_{13}\omega_1 + f_{23}\omega_2) = 0$

On the other hand, from (2.15) the left hand side of (4.5) becomes

$$\sqrt{{f_3}^4(f_{11} - f_{22})^2}\eta_i$$

and the whole of (4.5) reduces to $(\eta_1, \eta_2) = \pm(\omega_1, -\omega_2)$ depending on the sign chosen for the square root of Y. □

Note that since $Y > 0$ when $X < 0$ we could always insist of taking the positive square root of Y in (4.5) to obtain a consistent smooth choice of conjugate direction η_i once ω_i is chosen. In any case, now let us consider p_e in more detail. From (4.1) and (4.3) we see that

$$p_e = \tfrac{1}{2}(p^+ + p^-). \tag{4.6}$$

Note that p^+ does not see the sign of ω_i and p^- does not see the sign of η_i. Moreover, interchanging ω_i and η_i interchanges p^+ and p^-. Hence, from (4.6) we see that p_e depends only on the derivatives of f. In principle, we could now use (4.5) to substitute for η_i in p^-. We conclude that Yp_e is a polynomial in f_i, f_{ij}, f_{ijk}, and ω_i, which is actually independent of ω_i when (2.4) holds. Equation (2.4) may now be used to eliminate ω_i from Yp_e leaving a polynomial in f_i, f_{ij}, f_{ijk}. In practice, this is quite an intricate matter, which we consign to §B. The result is:

$$Yp_e = \tfrac{1}{2}Y(p^+ + p^-) = \tfrac{1}{2}(ZS - 2XR + 2XY),$$

where R and S are two further conformal invariants derived in §A.

Let us apply similar reasoning to some of the other quantities occurring above. From (4.1) and (4.3) we see that

$$p_0\omega_1\omega_2 = \tfrac{1}{2}(p^+ - p^-).$$

As we have already observed, interchanging ω_i and η_i interchanges p^+ and p^-, hence changing the sign of $p^+ - p^-$. As is readily verified in normal coördinates, another quantity with this property is

$$E \equiv \epsilon^{ijk} f_i\omega_j {f_k}^\ell\omega_\ell$$

where ϵ^{ijk} is a choice of volume form, uniquely normalised up to sign by $\epsilon^{ijk}\epsilon_{ijk} = 6$. Specifically, if we further constrain our normal coördinates (2.13) by requiring that $\epsilon^{123} = 1$, then

$$E = f_3(f_{22} - f_{11})\omega_1\omega_2.$$

As above, it follows that we may use (4.5) to eliminate η_i from

$$YEp_o\omega_1\omega_2 = \tfrac{1}{2}E(Yp^+ - Yp^-).$$

Moreover, this quantity is stable under interchange of ω_i and η_i. It must be a polynomial in f_i, f_{ij}, f_{ijk} alone, which is given by:

$$YEp_o\omega_1\omega_2 = \tfrac{1}{2}E(Yp^+ - Yp^-) = -\tfrac{1}{4}JXV,$$

where this calculation is once more detailed in §B and V is one of our list of conformal invariants derived in §A. But from Lemma 2.2, we have the identity

$$E^2 = -\tfrac{1}{2}J^2X. \tag{4.7}$$

We conclude that

$$\begin{aligned} P &\equiv 8Y^2p^+p^- = 2Y^2(p^+ + p^-)^2 - 2Y^2(p^+ - p^-)^2 \\ &= 2(ZS - 2XR + 2XY)^2 + XV^2. \end{aligned}$$

The vanishing of P is then our fourth conformally invariant condition (in addition to the first three (2.4)), obtained in [3], for the existence of a conjugate in the generic case $X < 0$. We now proceed similarly to obtain the two other conditions to provide a necessary and sufficient set of conditions.

First we observe that $Q \equiv Y\sqrt{Y}q^+q^-$ is conformally invariant, where we use Lemma 4.2 to define η_i by a choice of square root for Y. Certainly it is a Riemannian invariant and we shall compute it in normal coördinates (2.13). According to the proof of Lemma 4.2, we may take

$$\sqrt{Y} = {f_3}^2(f_{22} - f_{11}) \qquad \eta_1 = \omega_1 \qquad \eta_2 = -\omega_2,$$

in which case

$$q^+ = q_1\omega_1 + q_2\omega_2 \quad \text{and} \quad q^- = q_1\omega_1 - q_2\omega_2,$$

where

$$\begin{aligned} q_1 &= {f_3}^2 f_{133} + f_{111}{\omega_1}^2 + 3f_{122}{\omega_2}^2 + 4f_3f_{13}(f_{11} + f_{33}) \\ q_2 &= {f_3}^2 f_{233} + f_{222}{\omega_2}^2 + 3f_{112}{\omega_1}^2 + 4f_3f_{23}(f_{22} + f_{33}) \end{aligned}$$

so that

$$Q = Y\sqrt{Y}q^+q^- = {f_3}^6(f_{22} - f_{11})^3({q_1}^2{\omega_1}^2 - {q_2}^2{\omega_2}^2)$$

from which $\omega_1{}^2$ and $\omega_2{}^2$ may be eliminated with (2.11). The result is a polynomial expression in f and its derivatives. In terms of the various conformal invariants developed in §A it turns out that

$$\begin{aligned} Q \;=\; & \tfrac{1}{6}JZB - \tfrac{1}{4}JU - \tfrac{1}{4}ZS^2 \\ & + X(XZ^3 - JX^2Z + 6W + \tfrac{1}{4}JM - \tfrac{2}{7}ZXR + \tfrac{5}{7}RS \\ & \qquad - \tfrac{15}{7}N + \tfrac{2}{9}ZA - \tfrac{9}{10}F - \tfrac{2}{21}ZK + \tfrac{10}{21}T + \tfrac{6}{25}G - \tfrac{17}{42}JD), \end{aligned}$$

as may be verified in normal form (2.13).

The final condition $(p^+q^-)^2 + (p^-q^+)^2 = 0$ can similarly be expressed in terms of conformal invariants; although we do not attempt to write down the expression, we discuss how this can be done in §B.

5. Special cases

5.1. Functions with a unique conjugate direction

Suppose now that f is a function that admits a unique conjugate direction up to sign. By (2.17), this occurs when $X = 0$ and $Y > 0$. We first prove an analogue of Theorem 3.1.

Theorem 5.1. — *Let ω_j be a conjugate direction determined by* (2.4), *with $X = 0$ and $Y > 0$. Then the tensor field ω_{ij} is symmetric in its indices if and only if*

$$\epsilon^{ijk} f_i \omega_j \left(J f_k{}^{lm} f_l \omega_m - 2 f_{kl} f^l (f^{mn} f_m \omega_n) \right) = 0 \tag{5.1}$$

$$\epsilon^{ijk} f_i \omega_j \left(J f_k{}^{lm} \omega_l \omega_m + f_k{}^l f_l (f^{mn} f_m f_n + Z) \right) = 0 \tag{5.2}$$

Proof. — As in the proof of Theorem 3.1, ω_{ij} is symmetric in its indices if and only if $u^i v^j(\omega_{ij} - \omega_{ji}) = 0$, where u^i and v^j are linearly independent vector fields. However, since $X = 0$, by Lemma 2.2, the vector field $f^{ij}\omega_j$ is a linear combination of f^i and ω^i and we have to use an alternative. A judicious choice turns out to be the vector field

$$\nu^i = \epsilon^{ijk} f_j f_k{}^l \omega_l - \epsilon^{jkl} f_j \omega_k f_l{}^i \,.$$

A short calculation using the identity

$$\epsilon^{ijk}\epsilon_{lmn} = 6\delta^i_{[l}\delta^j_m\delta^k_{n]} \,,$$

shows that $\epsilon_{ijk}\nu^i f^j \omega^k = -Z/J$, which is non-zero by hypothesis (since $Y = Z^2$). In particular, ν^i has a non-zero component orthogonal to f^i and

ω^i. In order to bring this vector field into play, rather than differentiate the third equation from (2.4), we differentiate the equation:

$$\epsilon^{ijk} f_i \omega_j f_k{}^l \omega_l = 0\,. \tag{5.3}$$

This gives

$$\epsilon^{ijk} f_i \omega_j f_{klm} \omega^l + \epsilon^{ijk} f_{mi} \omega_j f_k{}^l \omega_l - \omega_{mi} \nu^i = 0\,. \tag{5.4}$$

First transvect this with f^m. Then the resulting symmetry condition $f^m \nu^i(\omega_{mi} - \omega_{im}) = 0$ holds if and only if

$$\epsilon^{ijk} f_i \omega_j f_{klm} f^l \omega^m + \epsilon^{ijk} f_{im} f^m \omega_j f_k{}^l \omega_l - f^m \omega_{im} \nu^i = 0\,.$$

But from (2.3), the last term can be replaced by $\omega^m f_{im} \nu^i$ which is equal to $[\epsilon^{lmn} f_l f_m{}^r f_r \omega_n / (f^s f_s)] f^{ij} f_i \omega_j$ (since $\epsilon^{ijk} f_j \omega_k f_{im} \omega^m = 0$ by (5.3)). On multiplying through by J, we obtain the equation

(5.5)
$$J \epsilon^{ijk} f_i \omega_j f_k{}^{lm} f_l \omega_n - J \epsilon^{ijk} f_i{}^l f_l f_j{}^m \omega_m \omega_k - (\epsilon^{ijk} f_i \omega_j f_k{}^l f_l) f^{mn} f_m \omega_n = 0\,.$$

However, from (5.3) we deduce the identity

$$J f_{jm} \omega^m + (f^{kl} f_k f_l) \omega_j - (f^{kl} f_k \omega_l) f_j = 0\,.$$

Indeed, the left-hand side is both orthogonal and colinear to the span of f_j and ω_j. On replacing $J f_{jm} \omega_m$ by $(f^{kl} f_k \omega_l) f_j - (f^{kl} f_k f_l) \omega_j$ in the middle term of (5.5), we obtain (5.1). Similarly, on transvecting (5.4) with ω^m, we conclude that the symmetry condition $\omega^m \nu^i (\omega_{mi} - \omega_{im}) = 0$ is equivalent to (5.2). □

As for the generic case, we can summarise the conditions that f admits a conjugate as follows.

Corollary 5.2. — *Locally, a smooth function f with $X = 0$ admits a smooth conjugate if and only if there is a smooth solution ω_i of the equations* (2.4), (5.1) *and* (5.2).

We can express these conditions in terms of the derivatives of f either by using invariant arguments, or by expressing them in normal coördinates. To do this invariantly, the following lemma can be employed to eliminate quadratic terms in ω^i.

Lemma 5.3. — *Suppose $X = 0$ and $Y \neq 0$. Let Q^{ij} be any symmetric form. Then*

$$Z Q^{ij} \omega_i \omega_j = -Z Q^{ij} f_i f_j + 2 J Q^{ij} f_i f_j{}^k f_k + J^2 (f_k{}^k Q_l{}^l - Q^{kl} f_{kl})\,. \tag{5.6}$$

Proof. — Recall that $E \equiv \epsilon^{ijk} f_i \omega_j f_k{}^\ell \omega_\ell$ satisfies $E^2 = -J^2 X/2$, so that

$$X = 0 \quad \Leftrightarrow \quad E = 0 \quad \Leftrightarrow \quad J f_{jk}\omega^k + (f^{kl} f_k f_l)\omega_j - (f^{kl} f_k \omega_\ell) f_j = 0\,, \tag{5.7}$$

where the latter equality occurs since the LHS is both orthogonal and colinear to the span of f_j and ω_j. We then apply this to the identity given by transvecting $f_{[i}\omega_j f_k{}^k Q_{l]}{}^l = 0$ with $f^i \omega^j$. An alternative proof is simply to check that the formula holds in the Riemannian normalisation. □

Equation (5.1) can now be written in the form $Q^{ij}\omega_i\omega_j = 0$, where

$$\begin{aligned} Q^{ij} \quad = \quad & -\epsilon^{ikl} f_k (J f_l{}^{mj} f_m - 2 f_{lm} f^m (f^{nj} f_n)) \\ & -\epsilon^{jkl} f_k (J f_l{}^{mi} f_m - 2 f_{lm} f^m (f^{ni} f_n)), \end{aligned}$$

which, by Lemma 5.3 can be written as an invariant expression in the derivatives of f. However, it is more direct and somewhat simpler to just write out (5.1) in the Riemannian normalisation.

From the proof of Lemma 2.2, we see that $X = 0$ implies that the product $\omega_1\omega_2 = 0$. Thus (5.1) becomes:

$$f_3{}^3(\omega_1{}^2 - \omega_2{}^2)(f_3 f_{123} - 2 f_{13} f_{23}) = 0\,.$$

But since $Y \neq 0$ ($f_{22} - f_{11} \neq 0$), $\omega_1{}^2 - \omega_2{}^2 = J$ and this is equivalent to

$$f_3{}^5(f_3 f_{123} - 2 f_{13} f_{23}) = 0\,,$$

which we recognize to be a multiple of V (which is given in normal coördinates by $4J^2 f_3(f_{22} - f_{11})(f_3 f_{123} - 2 f_{13} f_{23})$). Thus (2.4) and (5.1) correspond to the conformally invariant condition $V = 0$.

We give an invariant treatment of (5.2) as follows. Differentiate the right-hand identity of (5.7):

$$\begin{aligned} 0 \;&=\; \nabla_i (J f_{jk}\omega^k + (f^{kl} f_k f_l)\omega_j - (f^{kl} f_k \omega_l) f_j) \\ &=\; 2(f_{il} f^l) f_{jk}\omega^k + J f_{ijk}\omega^k + J f_j{}^k \omega_{ik} + f_{ikl} f^k f^l \omega_j + 2 f^{kl} f_{ik} f_l \omega_j \\ & \quad + f^{kl} f_k f_l \omega_{ij} - f_{ikl} f^k \omega^l f_j - f^{kl} f_{ik}\omega_l f_j - f^{kl} f_k \omega_{il} f_j - f^{kl} f_k \omega_l f_{ij}\,. \end{aligned} \tag{5.8}$$

Note that for the moment we do not assume symmetry of ω_{ij}.

Recall the fundamental identities: $\omega^{ij}\omega_j = f^{ij} f_j$ and $\omega^{ij} f_j = -f^{ij}\omega_j$. Transvect (5.8) with ω^j to obtain:

$$\begin{aligned} 0 \quad = \quad & -2 f_{il} f^l f^{jk} f_j f_k + J f_{ijk}\omega^j\omega^k + J(f_j{}^k \omega^j)\omega_{ik} \\ & + J f_{ikl} f^k f^l + 2 J f^{kl} f_{ik} f_l + (f^{kl} f_k f_l)(f_{ij} f^j) - f^{kl} f_k \omega_l f_{ij}\omega^j\,. \end{aligned}$$

From (5.7), $J f_j{}^k \omega^j = (f^{lm} f_l \omega_m) f^k - (f^{lm} f_l f_m)\omega^k$, so that

$$\begin{aligned} J(f_j{}^k\omega^j)\omega_{ik} \quad &= \quad (f^{lm} f_l \omega_m)\omega_{ik} f^k - (f^{lm} f_l f_m)\omega_{ik}\omega^k \\ &= \quad -(f^{lm} f_l \omega_m) f_{ik}\omega^k - (f^{lm} f_l f_m) f_{ik} f^k\,, \end{aligned}$$

which gives the identity:

$$Jf_{ijk}\omega^j\omega^k + Jf_{ijk}f^jf^k - 2(f^{kl}f_kf_l)f_{ij}f^j - 2(f^{kl}f_k\omega_l)f_{ij}\omega^j + 2Jf^{kl}f_{ik}f_l = 0.$$

From this, we deduce that (5.2) has the equivalent form:

$$\epsilon^{ijk}f_i\omega_j\left(-Jf_k{}^{lm}f_lf_m - 2Jf_{kl}f^{lm}f_m + f_k{}^lf_l(3f^{mn}f_mf_n + Z)\right) = 0 \Leftrightarrow$$

$$\epsilon^{ijk}f_i\omega_j\left(-\sigma_k + J(J\nabla_k(\Delta f) - \tfrac{1}{2}\Delta f\nabla_kJ)\right) = 0\,, \tag{5.9}$$

where σ_k is the conformally invariant 1-form given by Theorem A.3 of Appendix A. Even though $J\nabla_k(\Delta f) - \frac{1}{2}\Delta f\nabla_kJ$ is not itself conformally invariant, its component orthogonal to the span of f_i and ω_i is, so the left-hand side of (5.9) is conformally invariant. Now square this and use Lemma 5.3 to eliminate quadratic terms in ω_i. We obtain an identity involving only the derivatives of f, which we identify in terms of conformal invariants as:

$$\tfrac{25}{14}N + \tfrac{3}{5}G + \tfrac{3}{4}F + \tfrac{1}{21}T - \tfrac{17}{21}ZK - \tfrac{7}{9}ZA = 0\,. \tag{5.10}$$

Corollary 5.4. — *Locally, a smooth function f with $X = 0$ admits a smooth conjugate if and only if $V \equiv 0$ and* (5.10) *are satisfied.*

5.2. Functions that admit infinitely many conjugates

When X and Y both vanish, the function f admits infinitely many conjugate directions. The following gives a complete description.

Theorem 5.5. — *Suppose f is a smooth real-valued non-constant function such that its invariants X and Y both vanish. Then, up to scale and conformal transformation, f is one of the following*

$$x_1 \qquad \log({x_1}^2 + {x_2}^2 + {x_3}^2) \qquad \arctan\left(\frac{x_3}{x_2}\right) \qquad \frac{x_1}{{x_1}^2 + {x_2}^2 + {x_3}^2}. \tag{5.11}$$

Proof. — From (A.5) we deduce immediately that $\phi_{ij} = 0$. But

$$\phi_{ij} = \text{the symmetric trace-free part of } J^2\nabla_i[J^{-1}f_j]$$

whose vanishing is precisely saying that $J^{-1}f_j$ is a conformal Killing field V_j all of which can be written down explicitly. Following [8],

$$V_j = -s_j - m_{jk}x^k + \lambda x_j + x_jr_kx^k - \tfrac{1}{2}r_jx_kx^k$$

where s_j and r_j are arbitrary vectors, λ is a arbitrary constant, and m_{ij} is an arbitrary skew matrix. We may invert

$$V_j = (f^kf_k)^{-1}f_j \iff f_j = (V^kV_k)^{-1}V_j$$

and inquire whether f_j is closed. As a condition on V_j, this reads

$$V^k V_k \nabla_{[i} V_{j]} + 2V^k V_{[i} \nabla_{j]} V_k = 0, \tag{5.12}$$

the consequences of which are best viewed using a normal form for V_j such as those provided by Theorem C.5 in §C. Specifically, matrices of the form (C.6) provide conformal Killing fields of the form

$$\lambda\Big(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_3}\Big) + \mu\Big(x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}\Big)$$

in accordance with the conventions of [8]. However, only when $\mu = 0$ or $\lambda = 0$ is (5.12) satisfied. When both vanish, we obtain the linear functions which are equivalent under scaling and conformal transformation to the first of (5.11). Otherwise we obtain the second two, respectively. Matrices from the next group provide nothing new but matrices of the form (C.7) correspond to the conformal Killing fields

$$\mu\Big(x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}\Big) - ({x_1}^2 - {x_2}^2 - {x_3}^2)\frac{\partial}{\partial x_1} - 2x_1 x_2 \frac{\partial}{\partial x_2} - 2x_1 x_3 \frac{\partial}{\partial x_3}$$

and (5.12) is satisfied precisely when $\mu = 0$. This gives rise to the final possibility for f in the list (5.11). □

In fact, all of the functions with $X = Y = 0$ admit, not only infinitely many conjugate directions, but infinitely many conjugates. According to Theorem 5.5, it suffices to check this for the four cases (5.11). The first three of these are discussed in detail elsewhere in this article, specifically in §6.1, §6.3, and §6.2 respectively. Finally, the functions

$$f = \frac{x_1}{{x_1}^2 + {x_2}^2 + {x_3}^2} \qquad g = \frac{x_2 \cos\theta + x_3 \sin\theta}{{x_1}^2 + {x_2}^2 + {x_3}^2}$$

form a conjugate pair for any θ.

5.3. Functions of two variables that admit a conjugate in $\mathbb{R}^3$

Let $f = f(x_2, x_3)$ be a function of two variables only. Then many conformal invariants simplify and in the case of a unique conjugate direction, the equations have a simple interpretation. As a first observation, it is easily checked that X factors as a product:

$$X = (\Delta f)(f^i \nabla_i J)\,,$$

so that we also have

$$Z = \tfrac{1}{2} f^i \nabla_i J + J\Delta f\,, \quad Y = \left(\tfrac{1}{2} f^i \nabla_i J - J\Delta f\right)^2\,.$$

Furthermore, by its expression in the Riemannian normalisation, one sees that $V \equiv 0$. In particular, the fourth condition for a conjugate: $P \equiv 0$ simplifies to

$$ZS - 2XR + 2XY = 0\,.$$

Now suppose $X = 0$ and $Y > 0$. Then either $\Delta f = 0$, in which case $\omega = (0, -f_3, f_2)$ is, up to sign, the unique integrable conjugate direction and we are in the case of a planar function with planar conjugate, or $f^i\nabla_i J = 0$ and $\Delta f \neq 0$. We can now exploit Theorem 5.1. Since (5.1) is equivalent to $V \equiv 0$, this is vacuous. However, (5.2) now comes into play. By going into the Riemannian normalisation, one sees that the third order terms of this equation vanish, and it becomes:

$$(\epsilon^{ijk} f_i \omega_j f_k{}^l f_l)(f^{mn} f_m f_n + Z) = 0\,.$$

However, since $\Delta f \neq 0$, it is also the case that $f^{mn} f_m f_n + Z \neq 0$ and the equation becomes

$$\epsilon^{ijk} f_i \omega_j f_k{}^l f_l = 0\,.$$

Let us write this out explicitly in coördinates:

$$-\omega_1 f_2 f_3{}^l f_l + \omega_1 f_3 f_2{}^l f_l = 0\,.$$

But ω_1 must be non-zero otherwise we are once more in the situation of a planar function with a planar conjugate whence $\Delta f = 0$, contrary to our hypothesis. On combining this with the condition $f^i\nabla_i J = 0$, we obtain the simultaneous equations in $f_2{}^k f_k$ and $f_3{}^k f_k$:

$$\left\{\begin{array}{rcl} f_3 f_2{}^k f_k - f_2 f_3{}^k f_k & = & 0 \\ f_2 f_2{}^k f_k + f_3 f_3{}^k f_k & = & 0 \end{array}\right.$$

Since $f_2{}^2 + f_3{}^2 \neq 0$, these only admit the solution $f_2{}^k f_k = f_3{}^k f_k = 0$. But this implies that

$$\nabla_l (f^k f_k) = 0 \quad \Leftrightarrow \quad ||\nabla f|| = \text{constant}\,.$$

The unique conjugate direction is thus given up to sign by

$$\omega = (\sqrt{f_2{}^2 + f_3{}^2}, 0, 0).$$

Furthermore this case occurs precisely when f satisfies the eikonal equation $||\nabla f||^2 = \text{constant}$. This should be compared with the example of a function having spherical symmetry as discussed in §6.3 below, where now the conjugate must satisfy an eikonal equation, even though there is no conformal transformation which sends concentric spheres to parallel planes.

6. Some examples

In general, it is not the case that a function will admit a conjugate, even locally. For example, the function $f = x_1x_2x_3$ has the property that $X = 6f^2$. In particular X cannot be $\leqslant 0$ on any open set, so that f does not admit a conjugate on any open set.

Recall from the Introduction that the pair (f, g) of a function and its conjugate define a semi-conformal mapping into $\mathbb{R}^2$. In the analytic category, such mappings arise (i) as the extension to the boundary at infinity of a harmonic morphism on the associated heaven space of the domain, see [5]; (ii) from local CR hypersurfaces in the standard Levi-indefinite hyperquadric in $\mathbb{C}P_3$, see [4]. The latter perspective leads to an explicit construction of semiconformal mappings from a holomorphic function of two complex variables, which, in a first form was given in [12] then refined in [4]. In what follows, we highlight some particular cases of interest when a function f admits a conjugate function.

6.1. Linear and quadratic functions

Any linear function f admits infinitely many conjugate functions, also linear; indeed the two invariants X and Y both vanish identically. The only quadratic function that admits a conjugate is, up to isometries and scaling, $f = {x_1}^2 - {x_2}^2 - {x_3}^2$. Note that f has an isolated critical point at the origin, however its conjugate $g = x_1\sqrt{{x_2}^2 + {x_3}^2}$, although of class C^1 at the origin, is not smooth there. It is unknown if a pair of smooth conjugate functions (f, g) can have an isolated critical point. When they are harmonic and so determine a harmonic morphism, this is impossible [6].

6.2. Cylindrical symmetry

Let $r^2 = {x_2}^2 + {x_3}^2$ and suppose that $f = f(r)$ so that its level sets are concentric cylinders. Then by solving the equations (2.4), we obtain the conjugate direction:–

$$(\omega_1, \omega_2, \omega_3) = \left(\sqrt{f'^2 + rf'f''}, x_3\sqrt{\frac{-f'f''}{r}}, -x_2\sqrt{\frac{-f'f''}{r}}\right)$$

whose four-valuedness corresponds to taking different signs for the square roots. Then for any branch, $d\omega = 0$ if and only if

$$f'^2 + rf'f'' = C,$$

where C is a constant which is $\geqslant 0$. This has as first integral:–

$$f'^2 = \frac{A}{r^2} + C, \tag{6.1}$$

where $A \geqslant 0$ is a constant, and ω is now given by

$$(\omega_1, \omega_2, \omega_3) = \left(\sqrt{C}, \frac{x_3\sqrt{A}}{r^2}, -\frac{x_2\sqrt{A}}{r^2}\right)$$

Then $X = 2Cf'f''/r = -2AC/r^4$ is $\leqslant 0$ with the inequality strict provided neither of A nor C vanish.

In fact we can integrate (6.1) explicitly to obtain

$$f = \begin{cases} \sqrt{A}\ln\left\{\frac{\sqrt{A+Cr^2}-\sqrt{A}}{\sqrt{C}r}\right\} + \sqrt{A+Cr^2} & (C > 0) \\ \sqrt{A}\ln r \quad (C = 0) \end{cases}$$

The conjugate function is given by $g = \sqrt{C}x_1 - \sqrt{A}\arctan(x_3/x_2)$, interpolating between the two special case given by $A = 0$ ($f = \sqrt{C}r$) and $C = 0$ ($f = \sqrt{A}\ln r$). In fact the mapping (f, g) has fibres which are helices lying on the cylinders $r =$ constant. When $C = 0$ these helices become circles lying in planes orthogonal to the x_1-axis and when $A = 0$ they become lines parallel to the x_1-axis. Geometrically, we can interpret the four-valuedness of ω as corresponding to the choice of a right-hand screw or a left-hand screw for the helices, together with a choice of orientation. In the special cases we obtain just two equal and opposite directions.

6.3. Spherical symmetry

Let $r^2 = {x_1}^2 + {x_2}^2 + {x_3}^2$ and suppose that $f = f(r)$ depends on the radial coordinate only. Then

$$X = 2f'(r)^2\left(f''(r) + \frac{f'(r)}{r}\right)^2$$

so that if f is to admit a conjugate, the necessary condition $X \leqslant 0$ forces f to be either constant or to satisfy the differential equation

$$f''(r) + \frac{f'(r)}{r} = 0\,.$$

This has general solution $f = A\log r + B$, where A and B are arbitrary constants. For convenience, we take $f = \log r$. Note that spherical symmetry implies that $Y \equiv 0$ and so there are infinitely many conjugate directions. In fact any conjugate function g must satisfy $\partial g/\partial r = 0$ and $||\nabla g|| = 1/r$. Thus g is determined by its values on say the sphere $r = 1$, where it must satisfy the equation $||\nabla g|| = 1$. Such an equation is know as an *eikonal equation* and solutions are determined by initial data on a hypersurface (i.e. a curve) in the sphere S^2. It should be noted that the sphere S^2 does not admit a nowhere vanishing vector field and since we require $||\nabla g|| = 1$, then g cannot be globally defined on S^2. Thus even though the function f defined on $\mathbb{R}^3 \setminus \{0\}$ admits infinitely many different conjugate functions in a neighbourhood of any point of its domain, the domain of any of these conjugate functions cannot coincide with that of f.

6.4. An Ansatz

The following Ansatz provides a method of obtaining many pairs of conjugate functions. Let $h(x, y)$ satisfy the partial differential equation:

$$\left(\frac{\partial h}{\partial x}\right)^2 + 4y\left(\frac{\partial h}{\partial y}\right)^2 + 4h\frac{\partial h}{\partial y} = 0\,. \tag{6.2}$$

Then the functions

$$\begin{cases} f &= x_2 h(x_1, {x_2}^2 + {x_3}^2) \\ g &= x_3 h(x_1, {x_2}^2 + {x_3}^2) \end{cases}$$

are conjugate. For example, by taking $h = (x^2/y) + 1$, we obtain the pair of conjugate functions of the Introduction. A straightforward calculation shows that the only product solutions $h(x, y) = u(x)v(y)$ to (6.2), have the form

$$h = \frac{be^{cx}e^{\sqrt{1-c^2y}}}{1+\sqrt{1-c^2y}},$$

where b and c are constants. In fact, with reference to §5.1, every solution obtained by this Ansatz satisfies $X \equiv 0$.

7. Invariants of the conjugate

For a function f which admits a conjugate g, we can ask which of its properties are shared by its conjugate. More specifically, can we express the conformal invariants of g in terms of those of f? For the invariant X, this turns out to be simply done. In order to be clear on which invariants are being considered, in this section we shall write $X(f)$ and $X(g)$ and so on, for the invariants of the respective functions.

Theorem 7.1. — *If f admits a conjugate function g, then $X(f) = X(g)$.*

Proof. — In addition to (2.4), we have the identities:

$$g^{ij}f_j + f^{ij}g_j = 0 \quad \text{and} \quad g^{ij}g_j = f^{ij}f_j\,.$$

Set $v_i = \epsilon_{ijk}f^jg^k$. Then we can decompose g_{ij} in terms of a symmetric basis:

$$\begin{aligned} g_{ij} &= \frac{1}{J^2}(g^{kl}f_kf_l)f_if_j + \frac{1}{J^2}(g^{kl}g_kg_l)g_ig_j + \frac{1}{J^4}(g^{kl}v_kv_l)v_iv_j \\ &\quad + \frac{2}{J^2}(g^{kl}f_kg_l)f_{(i}g_{j)} + \frac{2}{J^3}(g^{kl}f_kv_l)f_{(i}v_{j)} + \frac{2}{J^3}(g^{kl}g_kv_l)g_{(i}v_{j)} \\ &= \frac{1}{J}(g^k{}_k)(J\delta_{ij} - f_if_j - g_ig_j) - \frac{2}{J}f_{k(i}g^kf_{j)} + \frac{2}{J}f_{k(i}f^kg_{j)} \\ &\quad + \frac{1}{J^2}(f^{kl}f_kg_l)(f_if_j - g_ig_j) - \frac{2}{J^2}(f^{kl}f_kf_l)f_{(i}g_{j)}\,. \end{aligned}$$

As a first application of this formula, we deduce the identity:

$$f^{ij}g_{ij} - (f^i{}_i)(g^j{}_j) = 0\,. \tag{7.1}$$

Furthermore

$$g^{ij}g_{ij} = (g^k{}_k)^2 + \frac{2}{J}f_{kj}g^kf^{lj}g_l + \frac{2}{J}f_{kj}f^kf^{lj}f_l - \frac{2}{J^2}(f^{kl}f_kg_l)^2 - \frac{2}{J^2}(f^{kl}f_kf_l)^2\,,$$

which implies that

$$X(g) = -2f_{kj}g^kf^{lj}g_l + \frac{2}{J}(f^{kl}f_kg_l)^2 + \frac{2}{J}(f^{kl}f_kf_l)^2\,.$$

In normal coördinates, on applying (2.11), the RHS equals

$$-2{g_1}^2f_{11}^2 - 2{g_2}^2{f_{22}}^2 + 2{f_3}^2{f_{33}}^2 = 2{f_3}^2(f_{11}+f_{33})(f_{22}+f_{33})\,,$$

which is precisely $X(f)$. □

Corollary 7.2. — *If f admits a conjugate function g, then for any $\epsilon \in \mathbb{R}$, the function $f + \epsilon g$ admits $g - \epsilon f$ as a conjugate and $X(f + \epsilon g) = (1+\epsilon^2)^2X(f)$.*

Proof. — That $f + \epsilon g$ and $g - \epsilon f$ are conjugates, is easily checked. Then

$$\begin{aligned} X(f+\epsilon g) &= X(f) \\ &\quad + \epsilon\{4({f_i}^jg_j + {g_i}^jf_j)f^{ik}f_k - 2J(g^{ij}f_{ij} - (f^i{}_i)(g^j{}_j))\} \\ &\quad + \epsilon^2\{4{f_i}^jf_jg^{ik}g_k + 4{f_i}^jg_jg^{ik}f_k + 2{g_i}^jf_jg^{ik}f_k + 2{f_i}^jg_jf^{ik}g_k \\ &\qquad - J[f^{ij}f_{ij} + g^{ij}g_{ij} - (f^i{}_i)^2 - (g^j{}_j)^2]\} \\ &\quad + \epsilon^3\{4({f_i}^jg_j + {g_i}^jf_j)g^{ik}g_k - 2J(g^{ij}f_{ij} - (f^i{}_i)(g^j{}_j))\} \\ &\qquad + \epsilon^4X(g)\,. \end{aligned}$$

But the coefficients of the odd powers of ϵ vanish on account of (2.4) and (7.1), so from Theorem 7.1, we obtain

$$X(f+\epsilon g) = X(f) + \epsilon^2(X(f)+X(g)) + \epsilon^4 X(g) = (1+\epsilon^2)^2 X(f)\,.$$

□

Note that if we view the pair (f,g) of a function and its conjugate as defining a semiconformal map into $\mathbb{R}^2$, then the replacement of (f,g) by $(f+\epsilon g, g-\epsilon f)$ amounts to multiplication of $f+ig$ by $1-i\epsilon$ when we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. Indeed, semiconformality is preserved under conformal transformations of both the domain and codomain.

To calculate the invariant $Z(g)$ in terms of invariants of f turns out to be more challenging. In fact $Z(g)$ depends on the choice of conjugate direction, so that, in the generic case, the appropriate quantity to consider is the product $\sqrt{Y}Z(\omega)Z(\eta)$. This can be calculated by the methods of §B to produce an expression involving third order derivative of f which we don't attempt to write down. On the other hand, information about $Z(g)$ can be obtained as in the above Corollary.

Lemma 7.3. — *If f admits a conjugate g, then we have*

$$Z(f+\epsilon g) = (1+\epsilon^2)(Z(f)+\epsilon Z(g))\,.$$

Furthermore,

$$Z(g) = \frac{d}{d\epsilon} Z(f+\epsilon g)|_{\epsilon=0}\,;$$

that is, $Z(g) = \mathcal{Z}_f(g)$ where $\mathcal{Z}_f$ is the linearisation of the operator Z at f.

In fact the latter part of the lemma is easily deduced directly from (2.4):

$$Z(g) = g^{ij}g_ig_j + (g^i g_i)(g^j{}_j) = f^{ij}f_ig_j + (f^i f_i)(g^j{}_j)\,,$$

where, for a given f with ∇f non-zero, the RHS is now a linear operator on g, which, since the principal term is the Laplacian, is elliptic.

Proof. — We have:

$$\begin{aligned} Z(f+\epsilon g) &= Z(f) + \epsilon(g^{ij}f_if_j + 2f^{ij}g_if_j + J\Delta g) \\ &\qquad \epsilon^2(2g^{ij}g_if_j + f^{ij}g_ig_j + J\Delta f) + \epsilon^3 Z(g) \\ &= Z(f) + \epsilon(f^{ij}f_ig_j + J\Delta g) + \epsilon^2(f^{ij}f_if_j + J\Delta f) + \epsilon^3 Z(g) \\ &= (1+\epsilon^2)(Z(f)+\epsilon Z(g))\,. \end{aligned}$$

The last part of the lemma now follows from this formula, or as indicated above, directly from (2.4). □

An interesting problem is to characterize those conjugate pairs that are 3-harmonic, i.e. conjugate pairs (f, g) satisfying $Z(f) = Z(g) = 0$, for then the mapping (f, g) determines a 3-harmonic morphism [11]. If both X and Y vanish, then so does Z and we have a complete description in this case given by Theorem 5.5. Up to conformal transformation, the different conjugate 3-harmonic pairs are given by

$$(x_1, x_2), \quad \left(\tfrac{1}{2}\log({x_1}^2 + {x_2}^2 + {x_3}^2), \arctan(x_3/x_2)\right),$$
$$\left(\frac{x_1}{{x_1}^2 + {x_2}^2 + {x_3}^2}, \frac{x_2}{{x_1}^2 + {x_2}^2 + {x_3}^2}\right).$$

More generally, by the homogeneity of $Z(f)$ in f, the function f is 3-harmonic if and only if it satisfies the linearisation of Z at f: $\mathcal{Z}_f(f) = 0$, so that by Lemma 7.3, $\mathcal{Z}_f(f) = \mathcal{Z}_f(g) = 0$ is a necessary and sufficient condition for a conjugate pair (f, g) to be 3-harmonic.

Appendix A. Conformal invariants

Suppose f is a smooth function defined on an open subset $U \subseteq \mathbb{R}^3$. As usual, we denote the partial derivatives of f by subscripts

$$f_i = \frac{\partial f}{\partial x^i}, \quad f_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}, \quad f_{ijk} = \frac{\partial^3 f}{\partial x^i \partial x^j \partial x^k}, \quad \ldots.$$

Equivalently, we may regard these quantities as tensors obtained by repeated application of the flat connection ∇_i corresponding to the flat metric δ_{ij}. Suppose Ω is a smooth non-vanishing function defined on U such that $\hat{\delta}_{ij} \equiv \Omega^2 \delta_{ij}$ is also flat. If we let $\Upsilon_i = \nabla_i \log \Omega$, then it is well-known [6] that these functions are precisely the solutions of

$$\nabla_i \Upsilon_j = \Upsilon_i \Upsilon_j - \tfrac{1}{2}\delta_{ij}\Upsilon^k \Upsilon_k$$

and that all solutions are obtained by the conformal transformations of the round sphere S^3 viewed as flat-to-flat conformal rescalings via stereographic projection. Let $\hat{\nabla}_i$ denote the metric connection for $\hat{\delta}_{ij}$ and write

$$\hat{f} = f, \quad \hat{f}_i = \hat{\nabla}_i f, \quad \hat{f}_{ij} = \hat{\nabla}_i \hat{\nabla}_j f, \quad \hat{f}_{ijk} = \hat{\nabla}_i \hat{\nabla}_j \hat{\nabla}_k f, \quad \ldots.$$

A conformal differential invariant of f of weight w is a polynomial

$$I = I(\delta^{ij}, f, f_i, f_{ij}, f_{ijk}, \ldots)$$

in the derivatives of f and the inverse metric δ^{ij} with the property that it is invariant under arbitrary coördinate transformation and

$$I(\hat{\delta}^{ij}, \hat{f}, \hat{f}_i, \hat{f}_{ij}, \hat{f}_{ijk}, \ldots) = \Omega^w I(\delta^{ij}, f, f_i, f_{ij}, f_{ijk}, \ldots) \tag{A.1}$$

for all flat-to-flat conformal rescalings Ω. As detailed in [9], this notion of invariance is the same as requiring equivariance under the action of $\mathrm{SO}(4,1)$ on the 3-sphere, with $\mathbb{R}^3 \hookrightarrow S^3$ by stereographic projection. It is straightforward to write down explicit formulae for the effect of flat-to-flat rescalings on derivatives

$$\text{(A.2)}\qquad \begin{array}{rcl} \hat{f}_i & = & f_i \\ \hat{f}_{ij} & = & f_{ij} - 2\Upsilon_{(i}f_{j)} + \delta_{ij}\Upsilon^k f_k \\ \hat{f}_{ijk} & = & f_{ijk} - 6\Upsilon_{(i}f_{jk)} + 3\delta_{(ij}\Upsilon^p f_{k)p} \\ & \vdots & \qquad + 6\Upsilon_{(i}\Upsilon_j f_{k)} - 3\delta_{(ij}\Upsilon_{k)}\Upsilon^p f_p - \frac{3}{2}\Upsilon^p\Upsilon_p\delta_{(ij}f_{k)} \end{array}$$

with a view to verifying (A.1) by direct calculation. It is difficult to find conformal invariants from this direct point of view. Certainly $J \equiv \delta^{ij}f_i f_j = f^i f_i$ is an invariant of weight -2. Perhaps the simplest second order invariant is

$$Z \equiv f^{ij}f_i f_j + Jf^j{}_j.$$

It has weight -4 but it is usual to omit the powers of Ω in verifying invariance (this is easily made precise by regarding the invariant as taking its values in an appropriate line-bundle). Specifically, as a linear combination of complete contractions it is manifestly invariant under coördinate transformation and

$$\begin{array}{rcl} \hat{f}^{ij}\hat{f}_i\hat{f}_j & = & f^{ij}f_i f_j - \Upsilon^i f_i f^j f_j = f^{ij}f_i f_j - J\Upsilon^k f_k \\ \hat{J}\hat{f}^j{}_j & = & Jf^j{}_j + J\Upsilon^k f_k \end{array}$$

whence

$$\hat{f}^{ij}\hat{f}_i\hat{f}_j + \hat{J}\hat{f}^j{}_j = f^{ij}f_i f_j + Jf^j{}_j,$$

as required. The familiar quantity

$$\text{(A.3)}\qquad X = 2f_i{}^j f_j f^{ik} f_k - f^i f_i f^{jk} f_{jk} + f^i f_i (f^j{}_j)^2$$

is a conformal invariant of weight -6. That it is a polynomial in the derivatives of f invariant under coördinate change is already manifest. Its conformal invariance, however, is most easily seen from the identity of Lemma 2.2:–

$$JX + 12T_{ijk}T^{ijk} = 0, \quad \text{where } T_{ijk} = f_{[i}\omega_j f_{k]l}\omega^l.$$

Here, recall that ω_j is any solution of the equations (2.4). We make take $\hat{\omega}_i = \omega_i$ to obtain a solution of the conformally transformed equations resulting from (A.2). Then

$$\hat{f}_{kl}\hat{\omega}^l = f_{kl}\omega^l - \Upsilon^l\omega_l f_k + \Upsilon^l f_l \omega_k$$

and so T_{ijk} is a conformally invariant tensor (of weight -2). Notice, however, that T_{ijk} is not an expression solely in f and its derivatives but also involves ω_j. It may also be imaginary-valued. It is only in the combination $T_{ijk}T^{ijk}$ that ω_j can be eliminated using the relations (2.4). Of course, it is also possible to check the conformal invariance of X directly from the expression (A.3).

In the remainder of this section we construct an extensive menagerie of conformal differential invariants of f. It is possible, in principle [9], to list all such invariants. In practise, however, it is easier to construct invariants by a number of tricks (see [14]). Apart from the particular invariant V constructed below, these will turn out to be sufficient for our purposes. The new connection $\hat{\nabla}_i$ is related to ∇_i by

$$\hat{\nabla}_i\phi_j = \nabla_i\phi_j - \Upsilon_i\phi_j - \Upsilon_j\phi_i + \delta_{ij}\Upsilon^k\phi_k$$

when acting on an arbitrary 1-form ϕ_j. It follows that

$$\nabla^i[\Omega^{-1}\phi_i] = \Omega^{-1}\nabla^i\phi_i,$$

which we will more conveniently express by saying if ϕ_i has conformal weight -1, then $\phi_i \mapsto \nabla^i\phi_i$ is conformally invariant. Similarly,

$$\phi_j \mapsto \nabla_{(i}\phi_{j)} - \tfrac{1}{3}\nabla^k\phi_k\delta_{ij}$$

is conformally invariant when ϕ_j has weight 2. Where J does not vanish we may consider the smooth 1-form $J^{1/2}f_i$. It has weight -1 whence

$$J^{1/2}\nabla^j[J^{1/2}f_j]$$

is conformally invariant (of weight -4). As written here, this is not a polynomial but if we expand it we obtain

$$\tfrac{1}{2}[\nabla^j J]f_j + J\nabla^j f_j = f^{ij}f_if_j + f^if_if^j{}_j\,,$$

which is a perfectly good polynomial. It follows that this is an invariant whether or not J vanishes. It is our previous invariant Z. Another viewpoint on this construction is that $f^j\nabla_jJ + 2J\nabla_jf^j$ is a conformally invariant bilinear differential pairing between f_i and J. There are many such pairings on $\mathbb{R}^3$ as follows.

LEMMA A.1. — *The following pairings are conformally invariant.*

$$\underbrace{\psi}_{\text{weight } v} \times \underbrace{\phi}_{\text{weight } w} \mapsto \underbrace{v\psi\nabla_i\phi - w\phi\nabla_i\psi}_{\text{weight } v+w}$$

$$\underbrace{\psi_i}_{\text{weight } v} \times \underbrace{\phi}_{\text{weight } w} \mapsto \underbrace{(v+1)\psi^i\nabla_i\phi - w\phi\nabla_i\psi^i}_{\text{scalar of weight } v+w-2}$$

$$\textit{ditto} \mapsto \underbrace{v\psi_{[i}\nabla_{j]}\phi + w\phi\nabla_{[i}\psi_{j]}}_{\text{skew of weight } v+w}$$

$$\textit{ditto} \mapsto \underbrace{\begin{aligned}(v-2)[\psi_{(i}\nabla_{j)}\phi - \tfrac{1}{3}\delta_{ij}\psi^k\nabla_k\phi]\\ - w\phi[\nabla_{(i}\psi_{j)} - \tfrac{1}{3}\delta_{ij}\nabla^k\psi_k]\end{aligned}}_{\text{symmetric trace-free of weight } v+w}$$

Proof. — These are all easily verified by direct calculation. Alternatively, we may employ evident variations on the trick used so far. For example, for non-vanishing ψ and ϕ we may write the first pairing as

$$\phi^{-v+1}\psi^{w+1}\nabla_i[\phi^v\psi^{-w}],$$

which is clearly invariant since $\phi^v\psi^{-w}$ has weight zero. All of these pairings are similarly based on well-known conformally invariant linear differential operators. □

Notice that the bundles occurring in these pairings are irreducible in the sense that they are associated to irreducible representations of the orthogonal group. These are the bundles between which it is relatively straightforward to find invariant pairings. Here are two more examples that we shall need.

LEMMA A.2. — *The following pairings are conformally invariant for ψ of weight v and ϕ_{ij} being symmetric trace-free and of weight w.*

$$\psi \times \phi_{ij} \mapsto \underbrace{v\psi\nabla^i\phi_{ij} - (w+1)\phi_{ij}\nabla^i\psi}_{\text{weight } v+w-2}$$

$$\psi \times \phi_{ij} \mapsto \underbrace{\begin{aligned}v\psi[\nabla_{(i}\phi_{jk)} - \tfrac{2}{5}\delta_{(ij}\nabla^l\phi_{k)l}]\\ - (w-4)[\phi_{(ij}\nabla_{k)}\psi - \tfrac{2}{5}\delta_{(ij}\phi_{k)l}\nabla^l\psi]\end{aligned}}_{\text{symmetric trace-free of weight } v+w}$$

Proof. — Easily verified by direct calculation. □

In fact, all the invariant pairings that we shall need may be constructed from invariant linear differential operators. (There are, however, many invariant pairings that do not arise in this way.) We are now able to list almost all the conformal invariants that we shall use.

Theorem A.3. — *The following are conformal differential invariants of a smooth function f locally defined on $\mathbb{R}$.*

$$J \equiv f^i f_i \qquad Z \equiv f^{ij} f_i f_j + J f^j{}_j$$
$$X \equiv 2 f_i{}^j f_j f^{ik} f_k - J f^{jk} f_{jk} + J (f^j{}_j)^2$$

If we now define

$$\sigma_i \equiv J \nabla_i Z - 2 Z \nabla_i J \qquad \tau_i \equiv J \nabla_i X - 3 X \nabla_i J$$
$$\phi_{ij} \equiv J f_{ij} - 2 f_{(i} f_{j)}{}^k f_k - \tfrac{1}{3} J f^k{}_k \delta_{ij} + \tfrac{2}{3} f^{kl} f_k f_l \delta_{ij},$$

then the following are also conformal invariants.

$$R \equiv f^i \sigma_i \qquad S \equiv f^i \tau_i \qquad A \equiv \sigma^i \sigma_i \qquad B \equiv \tau^i \tau_i$$
$$D \equiv \sigma^i \tau_i \qquad T \equiv \phi^{ij} \sigma_i \sigma_j \qquad U \equiv \phi^{ij} \tau_i \tau_j$$

If we now define

$$\rho_{ijk} \equiv J \nabla_{(i} \phi_{jk)} - 3 \phi_{(ij} \nabla_{k)} J - \tfrac{2}{5} \delta_{(ij} \nabla^l \phi_{k)l} + \tfrac{6}{5} \delta_{(ij} \phi_{k)l} \nabla^l J$$
$$\lambda_j \equiv 2 J \nabla^i \phi_{ij} - \phi_{ij} \nabla^i J,$$

then the following are also conformal invariants.

$$F \equiv \rho^{ijk} \phi_{ij} \lambda_k \qquad G \equiv \phi^{ij} \lambda_i \lambda_j \qquad K \equiv \sigma^i \lambda_i$$
$$M \equiv \tau^i \lambda_i \qquad N \equiv \sigma_i \rho^{ijk} \phi_{jk} \qquad W \equiv \rho^{ijk} \rho_{ij}{}^l \phi_{kl}.$$

Proof. — We have already observed that J, Z, and X are conformally invariant. The remaining invariants in this theorem are manufactured from these basic ones by using Lemmata A.1 and A.2 as appropriate. □

There is one more invariant that we shall need and its construction is slightly different. Let Q^{ij} be any symmetric form and set $\upsilon = \epsilon^{jkl}(J f_k{}^i Q_{ij} - f^i Q_{ij} f_{km} f^m) f_l$. Then the following identity holds:

$$Y(Q^{ij}\omega_i\omega_j - Q^{ij}\eta_i\eta_j) = 4E\upsilon \tag{A.4}$$

(recall that $E \equiv \epsilon^{ijk} f_i \omega_j f_k{}^\ell \omega_\ell$). In the case when $Q^{ij} = f^{ijk} f_k - 2 f^{ik} f_k{}^j$, one may check that υ is conformally invariant. It is convenient and consistent with [3] to define the related conformal invariant $V = 4J\upsilon$. It has a different character to our previous invariants in that it changes sign under change of orientation. It is said to be an odd invariant.

It is useful to record the conformal weight and homogeneity in f for each of the invariants of Theorem A.3 together with V:–

	J	Z	X	R	S	V	A	B	D
weight	-2	-4	-6	-8	-10	-11	-14	-18	-16
degree	2	3	4	6	7	8	10	12	11

	T	U	F	G	K	M	N	W
weight	-18	-22	-18	-18	-14	-16	-18	-18
degree	13	15	13	13	10	11	13	13

Any polynomial combination with consistent total weight will also be invariant. For example, the quantity $Y = Z^2 - 2JX$ introduced in (2.15) is a conformal invariant of weight -8 (and homogeneity 6). Other evident invariants are not necessarily new. For example, it is easily verified by direct computation that

$$\phi^{ij}\phi_{ij} = \tfrac{2}{3}Z^2 - JX. \tag{A.5}$$

This gives yet another verification that X is conformally invariant.

Appendix B. Invariant derivation of certain equations

Our aim is to eliminate ω_i from polynomial expressions of the form $F(f_i, f_{ij}, f_{ijk}, ..., \omega_i)$, given that the equations (2.4) hold. We suppose that $X < 0$, so that in particular $Y > 0$. Recall that

$$\eta_i = \frac{1}{\sqrt{Y}}\left\{2(f^{kl}f_k\omega_l)f_i + (Jf_k{}^k - f^{kl}f_kf_l)\omega_i - 2Jf_i{}^k\omega_k\right\},$$

gives the other conjugate direction, where an ambiguity of sign arises with the choice of square root.

Lemma B.1. — *Let Q^{ij} be any symmetric form. Then*

$$\begin{aligned} Y(Q^{ij}\omega_i\omega_j + Q^{ij}\eta_i\eta_j) &= 2Q^{ij}f_if_j(JX - Z^2) \\ &\quad +2J^2Q_j{}^j(Zf_l{}^l - X) - 2J^2ZQ^{ij}f_{ij} + 4JZQ^{ij}f_i{}^kf_kf_j \end{aligned} \tag{B.1}$$

$$\sqrt{Y}Q^{ij}\omega_i\eta_j = -ZQ^{ij}f_if_j + 2JQ^{ij}f_if_j{}^kf_k + J^2(f_k{}^kQ_l{}^l - Q^{kl}f_{kl}) \tag{B.2}$$

Proof. — Both formulae can be deduced by skew-symmetrising over the indices of an appropriate 4-tensor. For example, to derive the second, consider the four tensor: $T_{ijkl} = f_i\omega_jf_k{}^kQ_l{}^l$ and apply the identity: $T_{[ijkl]} = 0$. On transvecting first with f^i, then with ω^j and applying (2.4), the result follows. □

Now let us find invariant proofs of some of the identities of §4. Recall

$$\begin{array}{lcl} p^+ & \equiv & f^{ijk}f_i\omega_j\omega_k + f^{ijk}f_if_jf_k - 2f^{ij}f_j{}^k\omega_i\omega_k + 2f^{ij}f_j{}^kf_if_k \\ p^- & \equiv & f^{ijk}f_i\eta_j\eta_k + f^{ijk}f_if_jf_k - 2f^{ij}f_j{}^k\eta_i\eta_k + 2f^{ij}f_j{}^kf_if_k \\ q^+ & \equiv & f^{ijk}\omega_i\omega_j\omega_k + f^{ijk}f_if_j\omega_k + 4f^{ij}f_j{}^kf_i\omega_k \\ q^- & \equiv & f^{ijk}\eta_i\eta_j\eta_k + f^{ijk}f_if_j\eta_k + 4f^{ij}f_j{}^kf_i\eta_k\,. \end{array}$$

Theorem B.2. — *The following identities hold:*

$$Y(p^+ + p^-) = ZS - 2XR + 2XY \tag{B.3}$$

$$Y(p^+ - p^-) = EV/J\,. \tag{B.4}$$

where X, Y, Z, R, S, V are the standard conformal invariants and where $E \equiv \epsilon^{ijk}f_i\omega_jf_k{}^\ell\omega_\ell$.

Proof. — The first identity is an application of (B.1), where we have set $Q^{ij} = f^{ijk}f_k - 2f^{ik}f_k{}^j$. For the second, we apply (A.4) with symmetric form $Q^{ij} = f^{ijk}f_k - 2f^{ik}f_k{}^j$. □

Note that both the LHS and the RHS of equation (B.4) change sign under the interchange of the conjugate directions, the equation itself being well-defined and independent of this operation.

The condition $p^+p^- \equiv 0$ of Theorem 4.1 now follows since

$$4Y^2p^+p^- = Y^2(p^+ + p^-)^2 - Y^2(p^+ - p^-)^2.$$

On applying (4.7), this gives the necessary condition $P \equiv 0$ of §4:

$$2(ZS - 2XR + 2XY)^2 + XV^2 = 0\,.$$

Now consider the remaining conditions. We claim that we can use (B.1) and (B.2) to write $q^{ijk}\omega_i\omega_j\omega_k$ as a linear form in ω_i, where q^{ijk} is any symmetric tensor.

For this, first set $Q^{ij} = q^{ijk}\omega_k$. Then from (B.1),

$$\begin{aligned} Yq^{ijk}\omega_i\omega_j\omega_k &= -Yq^{ijk}\eta_i\eta_j\omega_k + 2q^{ijk}f_if_j\omega_k(JX - Z^2) \\ &\quad + 2J^2q_j{}^{jk}\omega_k(Zf_l{}^l - X) - 2J^2Zq^{ijk}f_{ij}\omega_k + 4JZq^{ijk}f_i{}^lf_lf_j\omega_k\,. \end{aligned} \tag{B.5}$$

We now have to calculate $Yq^{ijk}\eta_i\eta_j\omega_k$. For this we set $Q^{ij} = \sqrt{Y}q^{ijk}\eta_k$ and apply (B.2):

$$\begin{aligned} &Yq^{ijk}\eta_i\eta_j\omega_k = \sqrt{Y}Q^{ij}\omega_i\eta_j \\ &= -Zq^{ijk}f_if_jv_k + 2Jq^{ijk}f_if_j{}^lf_lv_k + J^2(f_j{}^jq_l{}^{lk}v_k - q^{ijk}f_{ij}v_k) \end{aligned}$$

where $v_i = \sqrt{Y}\eta_i = 2(f^{kl}f_k\omega_l)f_i + (Jf_k{}^k - f^{kl}f_kf_l)\omega_i - 2Jf_i{}^k\omega_k$. On expanding the right-hand side and substituting into (B.5), we obtain:

$$\begin{aligned}&Yq^{ijk}\omega_i\omega_j\omega_k = \omega_i\Big\{q^{ijk}f_jf_k(-Y - 2Zf^{lm}f_lf_m)\\&+Jq^{ij}{}_j\Big[Y + Z(f^{mn}f_mf_n) - 2(f^{mn}f_mf_n)^2\Big]\\&+J(Z + 2f^{lm}f_lf_m)(2q^{ijk}f_j{}^lf_lf_k - Jq^{ijk}f_{jk}) - 2JZq^{jkl}f_jf_kf_l{}^i\\&-2f^{in}f_n\Big[2Jq^{jkl}f_jf_kf_l{}^mf_m + J^2f_j{}^jq_k{}^{kl}f_l - J^2q^{jkl}f_{jk}f_l - Zq^{jkl}f_jf_kf_l\Big]\\&+4J^2q^{jkl}f_jf_k{}^mf_mf_l{}^i + 2J^3(f_j{}^jq_k{}^{kl}f_l{}^i - q^{jkl}f_{jk}f_l{}^i)\Big\},\end{aligned}$$

as claimed.

We can now express Yq^+ by setting $q^{ijk} = f^{ijk}$ and then adding $Y(f^{ijk}f_jf_k\omega_i + 4f^{jk}f_k{}^if_j\omega_i)$:

$$\begin{aligned}&Yq^+ = \omega_i\Big\{ - 2Zf^{ijk}f_jf_k(f^{lm}f_lf_m)\\&+Jf^{ij}{}_j\Big[Y + Z(f^{mn}f_mf_n) - 2(f^{mn}f_mf_n)^2\Big]\\&+J(Z + 2f^{lm}f_lf_m)(2f^{ijk}f_j{}^lf_lf_k - Jf^{ijk}f_{jk}) - 2JZq^{jkl}f_jf_kf_l{}^i\\&-2f^{in}f_n\Big[2Jf^{jkl}f_jf_kf_l{}^mf_m + J^2f_j{}^jf_k{}^{kl}f_l - J^2f^{jkl}f_{jk}f_l - Zf^{jkl}f_jf_kf_l\Big]\\&+4J^2f^{jkl}f_jf_k{}^mf_mf_l{}^i + 2J^3(f_j{}^jf_k{}^{kl}f_l{}^i - f^{jkl}f_{jk}f_l{}^i) + 4f^{jk}f_k{}^if_j\Big\}\end{aligned}$$

This has the form $Yq^+ \equiv \alpha^i\omega_i$, where each α^i is an explicit Riemannian invariant polynomial expression in f_i, f_{ij}, f_{ijk}, which at each point is defined up to addition of an arbitrary linear combination:

$$af^i + b[(f^{kl}f_kf_l)\omega^i + Jf^{ik}\omega_k]\,.$$

By symmetry, we must also have $Yq^- = \alpha^i\eta_i$. Then the fifth condition $Y\sqrt{Y}q^+q^- \equiv 0$ has the form $r^{ij}\omega_i\eta_j = 0$, where r^{ij} is the symmetric form $r^{ij} = (1/\sqrt{Y})\alpha^i\alpha^j$. We can now apply (B.2) to give an invariant derivation of the quantity $Q \equiv Y\sqrt{Y}q^+q^-$ of §4.

The final equation of Theorem 4.1 is $(p^+q^-)^2 + (p^-q^+)^2 = 0$. But

$$\begin{aligned}&4\{(p^+q^-)^2 + (p^-q^+)^2\} = \{(p^+ + p^-)^2 + (p^+ - p^-)^2\}\{(q^+)^2 + (q^-)^2\}\\&-2(p^+ + p^-)(p^+ - p^-)\{(q^+)^2 - (q^-)^2\}\,,\end{aligned}$$

which we can see as a product of conformally invariant terms that we can deal with. First, multiply the whole expression by $Y^3\sqrt{Y}$. Then $Y(p^+ + p^-)$ is given by (B.3), whilst $Y(p^+ - p^-)$ is given by (B.4). On the other hand,

$$Y\sqrt{Y}((q^+)^2 + (q^-)^2) = r^{ij}\omega_i\omega_j + r^{ij}\eta_i\eta_j,$$

which can be expressed using (B.1) above, whereas

$$Y\sqrt{Y}((q^+)^2 - (q^-)^2) = r^{ij}\omega_i\omega_j - r^{ij}\eta_i\eta_j$$

can be expressed using (A.4). Note that the result involves E^2, which by (4.7) can be written in terms of conformal invariants of §A.

Appendix C. Normalising conformal Killing fields

The conformal Killing fields on $\mathbb{R}^3$ form a finite-dimensional vector space on which $\mathrm{O}(4,1)$ acts via the conformal automorphisms of S^3. It is the adjoint representation $\mathfrak{o}(4,1)$ and so the question of normalising a conformal Killing field up to conformal transformations comes down to finding canonical representatives for the orbits of this action. This is a question of linear algebra, which may be stated more generally as follows. Suppose we are given a real symmetric $n \times n$ matrix H of Lorentzian signature meaning that there is a real invertible $n \times n$ matrix such that

$$A^tHA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{C.1}$$

Suppose N is a real skew $n \times n$ matrix. We would like to find a real invertible $n \times n$ matrix A such that A^tHA and A^tNA are placed in some canonical form. For example, we may insist on (C.1) for A^tHA but following [8, 9] we normally prefer (written in block form)

$$A^tHA = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \mathrm{Id} & 0 \\ 1 & 0 & 0 \end{bmatrix}, \tag{C.2}$$

where Id is the $(n-2) \times (n-2)$ identity matrix.

Lemma C.1. — *Suppose H is a real symmetric $n \times n$ matrix of Lorentzian signature and N is a real skew $n \times n$ matrix. Suppose that, regarded as a complex matrix, $H^{-1}N$ has only one eigenvector up to scale. Then, the eigenvalue is zero, it must be that $n = 3$, and we can find an invertible real 3×3 matrix A such that*

$$A^tHA = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \textit{and} \quad A^tNA = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix}.$$

Proof. — Notice that

$$H \mapsto A^tHA \quad \text{and} \quad N \mapsto A^tNA \qquad \Longrightarrow \; H^{-1}N \mapsto A^{-1}H^{-1}NA.$$

Therefore, without loss of generality, we may suppose that $H^{-1}N$ is in Jordan canonical form. Our hypothesis says that there is just one Jordan block with the eigenvalue λ down the diagonal. But

$$\operatorname{tr}(H^{-1}N) = \operatorname{tr}(N^t(H^t)^{-1}) = -\operatorname{tr}(NH^{-1}) = -\operatorname{tr}(H^{-1}N)$$

so $\lambda = 0$. In particular, the eigenspace is the same as the kernel of N. Suppose u is a non-zero vector in this kernel and consider

$$u^\perp \equiv \{v \text{ s.t. } u^tHv = 0\}.$$

Since

$$u^tHH^{-1}Nv = u^tNv = v^tN^tu = -v^tNu = 0,$$

it follows that $H^{-1}N$ preserves $u^\perp$. The hypothesis that $H^{-1}N$ has only one eigenvector up to scale now forces $u \in u^\perp$. In other words u is null, i.e. $u^tHu = 0$. It is well-known that $\mathrm{O}(n-1,1)$ acts transitively on the null vectors. Therefore we may suppose that

$$H = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \mathrm{Id} & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

It follows that

$$N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & M & -r \\ 0 & r^t & 0 \end{bmatrix}$$

where M is a skew $(n-2) \times (n-2)$ matrix. Therefore,

$$H^{-1}N = \begin{bmatrix} 0 & r^t & 0 \\ 0 & M & -r \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad (H^{-1}N)^2 = \begin{bmatrix} 0 & r^tM & -r^tr \\ 0 & M^2 & -Mr \\ 0 & 0 & 0 \end{bmatrix}$$

From the Jordan canonical form of $H^{-1}N$ we see that, not only does its trace vanish, but also the traces of its higher powers. In particular,

$$0 = \operatorname{tr}((H^{-1}N)^2) = \operatorname{tr}(M^2)$$

and since M is skew it follows that $M = 0$ and hence that $\operatorname{rank} N = 2$. Since the kernel of N is supposedly 1-dimensional, $n = 3$ is forced and

$$N = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -r \\ 0 & r & 0 \end{bmatrix}.$$

Finally, if we take

$$A = \begin{bmatrix} \mu^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu \end{bmatrix},$$

then

$$A^tHA = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A^tNA = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mu r \\ 0 & \mu r & 0 \end{bmatrix}$$

and so we can insist that $\mu r = -2$ if we so wish. $\square$

LEMMA C.2. — *Suppose H is a real symmetric $n\times n$ matrix of Lorentzian signature and N is a real skew $n \times n$ matrix. Then the eigenvalues of $H^{-1}N$ lie on the real or imaginary axes.*

Proof. — Suppose that $x + iy$ is an eigenvalue, i.e.

$$H^{-1}N(u + iv) = (x + iy)(u + iv) \quad \text{for some } u + iv \neq 0. \tag{C.3}$$

Writing out the real and imaginary parts separately gives

$$H^{-1}Nu = xu - yv \quad \text{and} \quad H^{-1}Nv = yu + xv. \tag{C.4}$$

We argue by contradiction, supposing that neither x nor y vanishes. In this case we see from (C.4) that neither u nor v vanishes. Because N is skew, we see from (C.4) that

$$\begin{aligned}0 = u^tNu = u^tHH^{-1}Nu = xu^tHu - yu^tHv,\\ 0 = v^tNv = v^tHH^{-1}Nv = yv^tHu + xv^tHv.\end{aligned}$$

Therefore

$$xu^tHu = yu^tHv = yv^tHu = -xv^tHv.$$

Since we are supposing that $x \neq 0$, we conclude that $u^tHu = -v^tHv$. Again using (C.4), we now find that

$$0 = u^tNv + v^tNu = u^tH^{-1}HNv + v^tH^{-1}HNu = 2yu^tHu + 2xu^tHv,$$

whence

$$0 = xu^tHu - yu^tHv \quad \text{and} \quad 0 = yu^tHu + xu^tHv.$$

Therefore $(x^2 + y^2)u^tHu = 0$ and so $u^tHu = 0$. Bearing in mind our assumption that $y \neq 0$, we have found two real vectors u and v with

$$u \neq 0, \quad v \neq 0, \quad u^tHu = 0, \quad v^tHv = 0, \quad u^tHv = 0.$$

For H of Lorentzian signature this forces $v = tu$ for some $t \in \mathbb{R}$. Substituting back into (C.3) and taking out a factor of $(1 + it)$ gives

$$H^{-1}Nu = (x + iy)u$$

and hence that $y = 0$, our required contradiction. $\square$

LEMMA C.3. — *Suppose H is a real symmetric $n\times n$ matrix of Lorentzian signature and N is a real skew $n \times n$ matrix. Suppose that $H^{-1}N$ has a non-zero real eigenvalue λ. Then $-\lambda$ is also an eigenvalue and we can find an invertible real $n \times n$ matrix A such that (in block form)*

$$A^tHA = \begin{bmatrix} 0 & 0 & 1\\ 0 & \mathrm{Id} & 0\\ 1 & 0 & 0\end{bmatrix} \quad \textit{and} \quad A^tNA = \begin{bmatrix} 0 & 0 & \lambda\\ 0 & M & 0\\ -\lambda & 0 & 0\end{bmatrix},$$

where M is a skew $(n-2)\times(n-2)$ matrix.

Proof. — Certainly, we may arrange that A^tHA is of the required form and we shall suppose, without loss of generality, that H is already normalised like this. Write $H^{-1}Nu=\lambda u$ for $u\neq 0$. Then

$$0=u^tNu=u^tHH^{-1}Nu=\lambda u^tHu$$

so $u^tHu=0$. Therefore, by a suitable A we may arrange

$$u=\begin{bmatrix}0\\0\\1\end{bmatrix},\quad\text{and this forces}\quad H^{-1}N=\begin{bmatrix}\cdot&\cdot&0\\\cdot&\cdot&0\\\cdot&\cdot&\lambda\end{bmatrix}.$$

Bearing in the mind that N is skew, this implies

$$N=\begin{bmatrix}\cdot&\cdot&\lambda\\\cdot&\cdot&0\\-\lambda&0&0\end{bmatrix},\quad\text{and then } H^{-1}N=\begin{bmatrix}-\lambda&0&0\\\cdot&\cdot&0\\\cdot&\cdot&\lambda\end{bmatrix}.$$

It follows that $-\lambda$ is an eigenvalue, say $H^{-1}Nv=-\lambda v$ for some $v\neq 0$ and, reasoning as above, $v^tHv=0$. Since u and v are not proportional, we may scale them so that $u^tHv=1$. Finally, if we arrange that

$$v=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad\text{then}\ \ H^{-1}N=\begin{bmatrix}-\lambda&0&0\\0&\cdot&0\\0&\cdot&\lambda\end{bmatrix}.$$

This immediately implies that N has the desired form. □

With these Lemmata on hand we are now in a position to establish a general canonical form. As already mentioned, we shall prefer (C.2) for A^tHA. When $n=2$ there is almost nothing more to do:–

$$A^tHA=\begin{bmatrix}0&1\\1&0\end{bmatrix}\quad\text{and}\quad A^tNA=\begin{bmatrix}0&\lambda\\-\lambda&0\end{bmatrix}$$

simply because N is skew. It remains to observe that we can change the sign of λ using

$$A=\begin{bmatrix}0&1\\1&0\end{bmatrix}$$

but that λ^2 is well-defined because the characteristic polynomial

$$\det(H^{-1}N-t\,\mathrm{id})=t^2-\lambda^2$$

is invariant. The first interesting case is $n=3$.

Theorem C.4. — *Suppose H is a real symmetric 3×3 matrix of Lorentzian signature and N is a real skew 3×3 matrix. Then we can find an invertible real 3×3 matrix A such that*

$$A^tHA = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and A^tNA is

$$\begin{bmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ -\lambda & 0 & 0 \end{bmatrix} \quad \text{or} \quad \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & -\lambda \\ 0 & \lambda & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{bmatrix}. \tag{C.5}$$

Furthermore, these three possible canonical forms are distinct apart from changing the sign of λ in the first two cases and the coincidence of the first two cases when $\lambda = 0$.

Proof. — If $H^{-1}N$ has only one eigenvector up to scale, then Lemma C.1 applies and we obtain the third case of (C.5). Else, Lemma C.2 implies that either all eigenvalues are real or they are $i\lambda, -i\lambda, 0$ for some $\lambda \neq 0$.

Firstly, let us suppose they are all real. They could still all be zero in which case the kernel N is at least 2-dimensional. But the rank of a skew matrix is always even so then $N = 0$. Otherwise, if $\lambda \neq 0$ is a real eigenvalue, then Lemma C.3 gives the first of (C.5).

When $i\lambda$ is an eigenvalue, then

$$H^{-1}N(u + iv) = i\lambda(u + iv)$$

implies that

$$H^{-1}Nu = -\lambda v \quad \text{and} \quad H^{-1}Nv = \lambda u.$$

It follows that

$$\begin{aligned} &0 = u^tNu = u^tHH^{-1}Nu = -\lambda u^tHv \\ &0 = u^tNv + v^tNu = u^tHH^{-1}Nv + v^tHH^{-1}Nu = \lambda(u^tHu - v^tHv) \end{aligned}$$

and so if $\lambda \neq 0$, we conclude that

$$u^tHu = v^tHv \quad \text{and} \quad u^tHv = 0.$$

In this case, by a suitable A we may arrange

$$u = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

from which the second of (C.5) is forced. Interchanging u and v changes the sign of λ. Otherwise, the distinctions between these canonical forms is clear from the Jordan canonical form of $H^{-1}N$ and its characteristic polynomial. $\square$

It is easy to generalise these canonical forms to $n \times n$ matrices. The only one we shall need is the 5×5 case and we state it here.

Theorem C.5. — *Suppose H is a real symmetric 5×5 matrix of Lorentzian signature and N is a real skew 5×5 matrix. Then we can find an invertible real 5×5 matrix A such that*

$$A^t H A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $A^t N A$ is

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & -\mu & 0 & 0 \\ -\lambda & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} \textit{well-defined up to} \\ (\lambda,\mu) \mapsto (-\lambda,\mu) \textit{ or } (\lambda,-\mu) \\ \textit{or } (-\lambda,-\mu), \end{array} \tag{C.6}$$

or

$$\begin{bmatrix} 0 & \lambda/\sqrt{2} & 0 & 0 & 0 \\ -\lambda/\sqrt{2} & 0 & 0 & 0 & -\lambda/\sqrt{2} \\ 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & -\mu & 0 & 0 \\ 0 & \lambda/\sqrt{2} & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} \textit{well-defined up to} \\ (\lambda,\mu) \mapsto (-\lambda,\mu) \textit{ or } (\lambda,-\mu) \\ \textit{or } (-\lambda,-\mu) \textit{ or } (\mu,\lambda) \\ \textit{or } (-\mu,\lambda) \textit{ or } (\mu,-\lambda) \\ \textit{or } (-\mu,-\lambda), \end{array}$$

or

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & -\mu & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \end{bmatrix} \quad \textit{well-defined up to } \mu \mapsto -\mu. \tag{C.7}$$

Furthermore, these canonical forms are distinct except for the evident coincidence of the first two cases when $\lambda = 0$.

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