ANNALES DE L'INSTITUT FOURIER

GIOVANNI VIDOSSICH

Characterization of separability for *LF*-spaces

Annales de l'institut Fourier, tome 18, nº 2 (1968), p. 87-90

<http://www.numdam.org/item?id=AIF_1968__18_2_87_0>

Ann. Inst. Fourier, Grenoble
18, 2 (1968), 87-90.

CHARACTERIZATION OF SEPARABILITY FOR *LF*-SPACES

by Giovanni VIDOSSICH

This note characterizes the separability of *LF*-spaces by five equivalent conditions, one being that all members of a given defining sequence (=*suite de définition* according to [2]) are separable. These conditions imply that the dual space must be hereditarily separable and Lindelöf for the weak* topology (= topology $\sigma(X', X)$ of [2]).

Concerning uniform spaces, we shall employ the terminology (and results) of the first two chapters of [4]. We shall denote by

$$\mathbf{F}$$

the scalar field, which is **R** or **C**; and we shall say

weak topology induced by $H \subseteq F(X, Y)$

the less fine topology on X making continuous all members of H (caution that this is a purely topological definition). Finally, an $\aleph_0$-space is — according to [5] — a regular space X where there exists a countable *pseudobase* $\mathfrak{P}$, i.e. a countable $\mathfrak{P} \subseteq \mathfrak{P}(X)$ such that for every compact $K \subseteq X$ and every open $U \subseteq X$ which contains K it follows $K \subseteq P \subseteq U$ for a suitable $P \in \mathfrak{P}$.

THEOREM. — *Let X be an LF-space and $(E_n)_{n=1}^{\infty}$ a defining sequence of X. The following statements are pairwise equivalent:*

(1) *X is separable.*

(2) *X is weakly separable.*

(3) *Every weakly* compact subset of X' is weakly* metrizable.*

(4) *Every equicontinuous subset of* X' *has a countable base for the weak* topology.*

(5) *Every* E_n *is separable.*

(6) X *is an* $\aleph_0$*-space.*

Proof. — (1) $\to$ (2) : Clear.

(2) $\to$ (3) : Let $e: X \to X''$ be the canonical map $x \longmapsto (f(x))_{f \in X'}$ and K a weakly* compact subset of X'. Then $e' : x \longmapsto e(x)|_K$ is a continuous map from X equipped with the weak topology into the topological subspace $e'(X)$ of $F_p(K, \mathbf{F})$ (= product space of Card(K) copies of $\mathbf{F}$). By (2), $e'(X)$ is separable: let H be a countable dense subset of it. The weak* topology of K is exactly the weak topology induced by $e'(X) \subseteq F(K, \mathbf{F})$: by [3, p. 175, Footnote], this topology equals the weak topology induced by $H \subseteq F(K, \mathbf{F})$ and therefore it is metrizable.

(3) $\to$ (4) : Because the weak* closure of an equicontinuous set is weakly* compact by [2, Th. 3].

(4) $\to$ (5) : By [2, Cor. to Th. 3], there is a linear homeomorphism e from X onto a subspace $e(X)$ of $L_{\mathfrak{S}}(X', \mathbf{F})$, $\mathfrak{S}$ being a suitable cover of X' consisting of weakly* compact subsets of X' and $L_{\mathfrak{S}}(X', \mathbf{F})$ the space of weakly* continuous linear functionals on X' with the topology of uniform convergence on members of $\mathfrak{S}$. By [2, Th. 3], the members of $\mathfrak{S}$ are equicontinuous and hence weakly* metrizable by (4). By a well known theorem contained in [5, (J) and (D)], $C_u(K, \mathbf{F})$ (= uniform space made of all weakly* continuous maps $K \to \mathbf{F}$ and the uniformity of uniform convergence) is separable for all $K \in \mathfrak{S}$ and therefore the uniformity of $C_u(K, \mathbf{F})$ has a basis of *countable* uniform coverings (as follows easily, if you want, from [4, ii. 33 and ii. 9]). Consequently the uniformity π of $\prod_{K \in \mathfrak{S}} C_u(K, \mathbf{F})$ — and hence the trace of π on every subset — has a basis of *countable* uniform covers as it follows directly from the definition of product uniformity [4, Exercise ii. 2] (alternatively, this result may be deduced from [1, Prop. 3] and [4, ii. 33 and ii. 9]). It is well known that there is a uniform embedding $e^* : L_{\mathfrak{S}}(X', \mathbf{F}) \to \prod_{K \in \mathfrak{S}} C_u(K, \mathbf{F})$, the last space being equipped with the product uniformity π.

By what has been proved, the uniformity induced by π on $e^*(e(X))$ has a basis consisting of countable uniform coverings: consequently — because a linear homeomorphism is a uniform isomorphism for the (canonical) uniformities of linear topological spaces — the (canonical) uniformity of the linear topological space X has a basis of countable uniform covers, as well as its trace on every E_n. But this trace coincides with the (canonical) uniformity of the linear topological space $E_n (n \in \mathbf{Z}^+)$, hence it is metrizable and consequently separable (if $\{(U_{m,n})_{m=1}^{\infty} | n \in \mathbf{Z}^+\}$ is a countable base of countable uniform covers for the uniformity of E_n and if $x_{m,n}$ is an element of $U_{m,n}$ whenever this set is not empty, then $\{x_{m,n} | m, n \in \mathbf{Z}^+\}$ is dense in E_n).

(5) → (6): By [2, Prop. 4], every compact subset of X is contained in some E_n. This, together with the fact that X induces the original topology on each E_n, implies that $\bigcup_{n=1}^{\infty} \mathfrak{P}_n$ is a countable pseudobase for X whenever $\mathfrak{P}_n$ is for $E_n (n \in \mathbf{Z}^+)$.

(6) → (1): By [5, (D) and (E)]. ▌

We remark that the idea of countable uniform covers may be used to show directly that every metrizable subgroup of a separable topological group must be separable.

The above theorem points out some important examples of non-metrizable $\aleph_0$-spaces. [5, (J) and 10,4] imply some results on spaces of mappings between separable *LF*-spaces, of which we note only the following one.

Corollary. — *If an LF-space X is separable, then X' is weakly* hereditarily Lindelöf and separable.*

Proof. — By « (1) ←→ (6) » of the above theorem and [5, (J) and (D), (E)]. ▌

BIBLIOGRAPHIE

[1] H. H. Corson, Normality in subsets of product spaces, *Amer. J. Math.* **81** (1959), 785-796.

[2] J. Dieudonné and L. Schwartz, La dualité dans les espaces ($\mathfrak{F}$) and ($\mathfrak{LF}$), *Ann. Inst. Fourier* **1** (1949-50), 61-101.
[3] A. Grothendieck, Critères de compacité dans les espaces fonctionnels généraux, *Amer. J. Math.* **74** (1952), 168-186.
[4] J. R. Isbell, *Uniform Spaces*, American Math. Society, Providence, 1964
[5] E. Michael, $\aleph_0$-spaces, *J. Math. Mec.* 15 (1966), 983-1022.

Manuscrit reçu le 16 février 1968.

Giovanni Vidossich,
Vle XX Sett. 225,
54031 Avenza (Italie).