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NAIVE BOUNDARY STRATA AND NILPOTENT ORBITS

by Matt KERR & Gregory PEARLSTEIN

ABSTRACT. — We give a Hodge-theoretic parametrization of certain real Lie group orbits in the compact dual of a Mumford-Tate domain, and characterize the orbits which contain a naive limit Hodge filtration. A series of examples are worked out for the groups $SU(2,1)$, Sp_4, and G_2.

RÉSUMÉ. — Nous donnons une paramétrisation de certaines orbites de groupes de Lie réels dans le dual compact d'un domaine de Mumford-Tate et une caractérisation des orbites qui contiennent une filtration limite de Hodge naïve. Une série d'exemples est élaborée pour les groupes $SU(2,1)$, Sp_4, et G_2.

1. Introduction

In a previous work [15], we introduced and studied boundary components for Mumford-Tate domains, which are homogeneous classifying spaces $D = G(\mathbb{R})^\circ/\mathcal{H}$ for Hodge structures with additional symmetries (in a Tannakian sense) [12]. Here G is a reductive, connected $\mathbb{Q}$-algebraic group, $G(\mathbb{R})^\circ$ the identity component of the (Lie) group of real points, and $\mathcal{H}$ a compact subgroup. The boundary components $B(N)$ essentially parametrize all possible LMHS (limiting mixed Hodge structures) for period maps $\Phi\colon \mathcal{S} \to \Gamma \setminus D$ into such a domain with given monodromy logarithm N, and also admit a homogeneous description. A feature of that work was the interesting representation theory that arises from considering symmetries and asymptotics of Hodge structures in tandem.

The purpose of the present study is to better understand the interaction between asymptotic Hodge theory and the $G(\mathbb{R})^\circ$-orbit structure of the compact dual $\check{D} = G(\mathbb{C})/Q$ of D. Here $Q \leqslant G(\mathbb{C})$ is a parabolic subgroup,

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and $\check{D}$ a projective variety containing D as an analytic open subset. The "naive boundary strata" of the title are the orbits in the topological boundary of D in $\check{D}$, and a natural question is which ones are Hodge-theoretically "accessible" in the sense of containing a limit point of (a lift $\widetilde{\Phi}$ of) some period map Φ. In addition to obtaining a nice answer to this question (§5.2), we shall clarify the relationship of these "boundary orbits" to the boundary components, and obtain a mixed-Hodge-theoretic parametrization of all the orbits and description of their incidence structure.

Now the traditional way to record the asymptotics of a period map Φ is via the limiting mixed Hodge structure, and not the "naive" limit point of $\widetilde{\Phi}$ in ∂D. There is a good reason for this: because of logarithmic growth of periods, the latter loses information recorded by the LMHS. One has the classic example (*cf.* [10]) of a degenerating family of genus-2 curves in two parameters, with two cycles vanishing at the origin. The degenerate fiber is a rational curve with two pairs of points identified, and the cross ratio of these 4 points is encoded in the extension data of the LMHS, while the "naive" limit and even the cohomology of the singular fiber record nothing. From an algebro-geometric point of view, then, it is unclear why one would want to study the interaction between variational Hodge theory and the orbit structure of $\check{D}$.

Our motivation for this work arose instead from a perspective heavily influenced by problems in complex geometry and representation theory, where the (finitely many) $G(\mathbb{R})^{\circ}$-orbits are objects of some importance [11, 27]. Recent work of Robles [24] has finally settled the question of maximal dimensions of integral manifolds of the infinitesimal period relation on $\check{D}$ (and hence of images of period maps in $\Gamma \setminus D$). It seems that one way of producing interesting maximal-dimensional VHS is by threading integral manifolds through "accessible" orbits in ∂D, and that this approach holds some promise for the much-studied question of smoothing Schubert varieties in $\check{D}$ in their cohomology classes. We also mention that in the forthcoming work [13], the proof of pseudo-convexity of D will be recast in terms of our Hodge-theoretic analysis of ∂D.

On the representation-theoretic side, the $G(\mathbb{R})^{\circ}$-orbits are related to the construction of infinite-dimensional unitary representations of $G(\mathbb{R})$ via parabolic induction. Moreover, one reason for writing [15] was to see for which M-T domains one might extend H. Carayol's approach [6] to putting an arithmetic structure on automorphic cohomology. Our analysis of codimension 1 boundary strata suggests (*cf.* §5.3) how to generalize his definition of Fourier coefficients to at least some cases where G is an exceptional group. We shall pursue these connections in future works.

Summary

In the remainder of the Introduction, we briefly describe the main results. Given a polarized Hodge structure $(V, {}'B, {}'\varphi_0)$ with Mumford-Tate group ${}'G$, let $\mathrm{Ad}\colon {}'G \twoheadrightarrow {}'G^{\mathrm{ad}} =: G$ be the adjoint map, and set $Z := \ker(\mathrm{Ad})(\mathbb{C}) = Z({}'G(\mathbb{C}))$, $\varphi_0 := \mathrm{Ad} \circ {}'\varphi_0$. Let Θ be the Cartan involution of G induced by conjugation by $\varphi_0(\sqrt{-1})$.

For any $g \in {}'G(\mathbb{R})^\circ$, if ${}'\varphi_1 = g^{-1} \circ {}'\varphi_0 \circ g$ satisfies $\mathrm{Ad} \circ {}'\varphi_1 = \varphi_0$, then ${}'\varphi_1/{}'\varphi_0$ is a cocharacter of Z°, whereupon it must clearly be trivial. So we have a diagram

$$\begin{tikzcd}
{}'\check{D} := {}'G(\mathbb{C})/Q_{F^\bullet_{'\varphi_0}} \arrow[d] & {}'G(\mathbb{R})^\circ/\mathcal{H}_{'\phi_0} =: {}'D \arrow[l, hook'] \arrow[d, "\cong"] \\
\check{D} := G(\mathbb{C})/Q_{F^\bullet_{\varphi_0}} & G(\mathbb{R})^\circ/\mathcal{H}_{\varphi_0} =: D \arrow[l, hook']
\end{tikzcd}$$

in which Z belongs to $Q_{F^\bullet_{'\varphi_0}}$ and the left-hand side is finite-to-one. But the parabolic subgroup $Q_{F^\bullet_{\varphi_0}}$ is necessarily connected, and so in fact ${}'\check{D}$ and $\check{D}$ are the same.

On right and left, we therefore have isomorphisms of complex manifolds (though not as homogeneous manifolds). For this reason, we may work without loss of generality in the adjoint setting. Note that we view the points of the compact dual $\check{D}$ as flags on $\mathfrak{g} := \mathrm{Lie}(G(\mathbb{C}))$. We shall make the simplifying assumptions that the polarization on $(\mathfrak{g}, \varphi_0)$ induced by ${}'B$ is a multiple of the Killing form B, and that the horizontal distribution on $\check{D}$ (equiv. ${}'\check{D}$) is bracket-generating.

In [12, (VI.B.10)], it was conjectured that one should be able to parametrize $\check{D}$ via pairs (H, χ), where $H \leqslant G_{\mathbb{R}}$ is a maximal algebraic torus and $\chi \in \mathrm{X}_*(\mathrm{H}(\mathbb{C}))$ a complex co-character of H, in $G(\mathbb{R})^\circ$-equivariant fashion. This is proved in §3 (Theorem 3.5), the take-away from which is summarized in the following

Proposition.

(a) *Given any point $F^\bullet \in \check{D}$, there exists a pair (H, χ) as above such that $F^\bullet = \mathscr{F}(H, \chi)^\bullet$ (cf. (2.1)-(2.2)).*

(b) *The pair (H, χ) determines a bigrading $\mathfrak{g}_{\mathbb{C}} = \oplus_{p,q} \mathfrak{g}_\chi^{p,q}$ with $F^\bullet = \oplus_{p \geqslant \bullet} \mathfrak{g}_\chi^{p,q}$ and $\overline{\mathfrak{g}_\chi^{p,q}} = \mathfrak{g}_\chi^{q,p}$.*

(c) *The $G(\mathbb{R})^\circ$-orbit containing $F^\bullet$ has real codim.* $\sum_{p,q>0}\dim_{\mathbb{C}}(\mathfrak{g}_\chi^{p,q})$ *in* $\check{D}$.

In §4, this is used to index the $G(\mathbb{R})^\circ$-orbits in $\check{D}$ and describe their incidence relations, beginning with the

Corollary. — *Write $F^\bullet_{\varphi_0} = \mathscr{F}(H_0,\chi_0)^\bullet$. Let $\{H_0, H_1, \ldots, H_n\}$ be representatives of the $G(\mathbb{R})^\circ$-conjugacy classes of Cartan subgroups of $G(\mathbb{R})^\circ$, obtained by Cayley transforms from H_0, with complex [resp. real] Weyl groups $W_{\mathbb{C}}(H_j)$ [resp. $W_{\mathbb{R}}^\circ(H_j)$]. Let $\chi_j \in \mathrm{X}_*(H_j(\mathbb{C}))$ be obtained from χ_0 in the same manner, with stabilizers $W_j \leqslant W_{\mathbb{C}}(H_j)$. Then $\mathscr{F}$ induces an "orbit map" from the finite set*

$$\Xi := \bigcup_j W_{\mathbb{R}}^\circ(H_j) \diagdown \big\{(H_j, w\chi_j)\,|\, j\in\{0,\ldots,n\},\, w\in W_{\mathbb{C}}(H_j)/W_j\big\}$$

onto the set of $G(\mathbb{R})^\circ$-orbits. This map is a bijection if $\check{D}$ is a complete flag variety (cf. Lemma 4.9).

Denoting analytic closure by cl, the partial order on orbits given by $\mathcal{O}_1 \geqslant \mathcal{O}_2 \iff \mathrm{cl}(\mathcal{O}_1)\supseteq \mathcal{O}_2$ is known as *Bruhat order*, and is generated (at least in the complete flag setting) by Cayley transforms and cross actions in a sense made precise in [29]. In §4.3 it is briefly explained how to understand these processes in terms of "naive" limits of Hodge flags and the framework of the Corollary.

In the remainder of the paper, we are interested in those orbits incident to D itself:

Definition. — *An orbit $\mathcal{O}\subset \mathrm{cl}(D)$ is called a* naive boundary stratum.

Given a polarized variation of Hodge structure Φ over a punctured disk with monodromy logarithm N, one can take the limit in two ways. The limiting flag $\widetilde{\mathscr{F}}(\Phi) \in \widetilde{B}(N) \subseteq \check{D}$ associated to the LMHS is obtained by first twisting by $e^{-\frac{\log(q)}{2\pi i}N}$ and then taking the $q\to 0$ limit; the *naive limit* $\widehat{\mathscr{F}}(\Phi) \in \widehat{B}(N) \subseteq \mathrm{cl}(D)$ is the limit with no twist. (See Definitions 5.1 and 5.2ff for the precise meaning of $\widetilde{B}(N)$ and $\widehat{B}(N)$.) They are related by the *naive limit map*(1)

$$\mathscr{F}^N_{\mathrm{lim}} \colon \widetilde{B}(N) \to \mathrm{cl}(D)$$

defined and studied in §5.1. For instance, if a MHS $(\widetilde{F}^\bullet, W(N)_\bullet) \in \widetilde{B}(N)$ is $\mathbb{R}$-split, then $\mathscr{F}^N_{\mathrm{lim}}(\widetilde{F}^\bullet)$ is the flag obtained by flipping the associated

(1) In forthcoming related work of Green and Griffiths, this will be called the *reduced limiting period mapping*.

bigrading about the antidiagonal; moreover, $\mathscr{F}_{\lim}^N$ factors the projection $\widetilde{B}(N) \to D(N)$ induced by passing to the $\mathbb{Q}$-splitting of the LMHS. In the classical case where D is Hermitian symmetric, the maps from open strata in a smooth toroidal compactification to those in the Baily-Borel compactification admit a natural description in terms of naive limit maps (see Theorem 5.21).

As we described above, the instinctive question is how to determine whether a given naive boundary stratum $\mathcal{O} = G(\mathbb{R})^{\circ} \cdot \mathscr{F}(H,\chi)^{\bullet}$ contains a naive flag, or equivalently some $\widehat{B}(N)$. To this end, we introduce (*cf.* §5.2) the following terminology:

- $\mathcal{O}$ is *rational* if the filtration $\widetilde{\widetilde{W}}_{\bullet} := \oplus_{-p-q \leqslant \bullet} \mathfrak{g}_{\chi}^{p,q}$ is $G(\mathbb{R})^{\circ}$-conjugate to a $\mathbb{Q}$-rational one; and
- $\mathcal{O}$ is *polarizable* if there exists a nonzero element $\widehat{N} \in \mathfrak{g}_{\mathbb{R}}^{-1,-1}$ such that $\widehat{N}^j : \mathfrak{g}_{\chi}^{p,j-p} \xrightarrow{\cong} \mathfrak{g}_{\chi}^{p-j,-p}$ $(\forall p \in \mathbb{Z},\ j \in \mathbb{N})$ and a positivity condition holds.

Two of our main general results (*cf.* Theorem 5.15*ff*) may then be stated as follows:

THEOREM. — *A stratum $\mathcal{O}$ contains a $\widehat{B}(N)$ if and only if $\mathcal{O}$ is rational and polarizable. All strata of codimension one are polarizable.*

In §6, we work out for a variety of examples (including the three G_2-domains of [12]) the complete incidence diagram and the associated bigradings for $G(\mathbb{R})^{\circ}$-orbits, and determine which of the boundary strata are polarizable.

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2. Preliminaries

Let G be a connected $\mathbb{Q}$-algebraic adjoint group, $H_{\mathbb{C}} \leqslant G_{\mathbb{C}}$ a maximal algebraic torus subgroup. The groups of complex points $G(\mathbb{C})$, $H(\mathbb{C})$ have natural Lie group structures, and we let $\mathfrak{g}$, $\mathfrak{h}$ denote the complex Lie algebras. From G, $\mathfrak{g}$ inherits an underlying $\mathbb{Q}$-Lie algebra $\mathfrak{g}_{\mathbb{Q}}$, and we let $B\colon \mathfrak{g}_{\mathbb{Q}} \times \mathfrak{g}_{\mathbb{Q}} \to \mathbb{Q}$ denote the (symmetric, nondegenerate) Killing form $B(X,Y) = \mathrm{Tr}(\mathrm{ad}\, X \circ \mathrm{ad}\, Y)$.

Consider the lattice

$$\Lambda^* := \ker\big\{\exp(2\pi i(\cdot))\colon \mathfrak{h} \to H(\mathbb{C})\big\}.$$

Sending $\phi \in \Lambda^*$ to the co-character

$$\begin{array}{rccc} \chi_\phi\colon & \mathbb{C}^* & \to & H(\mathbb{C}) \\ & z & \mapsto & e^{\log(z)\phi} \end{array}$$

yields an isomorphism

$$\Lambda^* \xrightarrow{\cong} \mathrm{X}_*(H(\mathbb{C})),$$

with inverse $\chi \mapsto \chi'(1)$. Writing $\Lambda := \mathrm{Hom}(\Lambda^*, 2\mathbb{Z}) \subset \mathfrak{h}^*$, we have

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathbb{C}\langle X_\alpha\rangle,$$

where the roots $\Delta \subset \Lambda$ generate Λ, and

$$\mathrm{ad}(h)X_\alpha = \alpha(h)X_\alpha \quad (\forall h \in \mathfrak{h}).$$

In particular, for $\chi = \chi_\phi \in \mathrm{X}_*(H(\mathbb{C}))$, we define

$$\begin{array}{rccc} \pi_\chi\colon & \Lambda & \to & \mathbb{Z} \\ & \lambda & \mapsto & \frac{1}{2}\lambda(\phi) \end{array}$$

so that $\mathrm{ad}(\phi)X_\alpha = 2\pi_\chi(\alpha)X_\alpha$, and

$$\begin{aligned} \mathrm{Ad}(\chi(z))X_\alpha &= e^{\log(z)\,\mathrm{ad}\,\phi}X_\alpha \\ &= e^{2\log(z)\pi_\chi(\alpha)}X_\alpha \\ &= z^{2\pi_\chi(\alpha)}X_\alpha. \end{aligned}$$

We shall write for $i > 0$

$$\mathscr{F}(H,\chi)^i\mathfrak{g}_{\mathbb{C}} := \bigoplus_{\left\{\substack{\alpha\in\Delta \\ \pi_\chi(\alpha)\geqslant i}\right.} \mathfrak{g}_\alpha \tag{2.1}$$

and for $i \leqslant 0$

$$\mathscr{F}(H,\chi)^i\mathfrak{g}_{\mathbb{C}} := \mathfrak{h} \oplus \bigoplus_{\left\{\substack{\alpha\in\Delta \\ \pi_\chi(\alpha)\geqslant i}\right.} \mathfrak{g}_\alpha; \tag{2.2}$$

note that the (partial) flag $\mathscr{F}(H,\chi)^\bullet$ depends only on H and π_χ.

Remark 2.1. — $G(\mathbb{R})$ and $G(\mathbb{C})$ operate on flags in $\mathfrak{g}_{\mathbb{C}}$ via Ad. This will often be tacit; that is, $\mathrm{Ad}(g)F^\bullet$ will be written $g\cdot F^\bullet$. This is especially necessary in §5 where the notation would otherwise become unwieldy.

Next, we specialize to the case where H is defined over $\mathbb{R}$. More precisely, let $\Theta = \Psi_C \in \mathrm{Aut}(G_{\mathbb{R}})$ (conjugation by C) be a Cartan involution, so that $\theta := \mathrm{Ad}\, C \in \mathrm{Aut}(\mathfrak{g}_{\mathbb{R}})$ satisfies $\theta^2 = \mathrm{id}$ and $-B(\,\cdot\,, \theta(\,\cdot\,)) > 0$. Take $H \leqslant G_{\mathbb{R}}$ to be a Θ-stable Cartan subgroup, and let $\mathfrak{g}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}}$ be the Lie algebras of $G(\mathbb{R}), H(\mathbb{R})$. We have decompositions into ± 1-eigenspaces

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{h}_{\mathbb{R}} = \mathfrak{t} \oplus \mathfrak{a} = (\mathfrak{h}_{\mathbb{R}} \cap \mathfrak{k}) \oplus (\mathfrak{h}_{\mathbb{R}} \cap \mathfrak{p})$$

of θ, and clearly $-B > 0$ [resp. < 0] on $\mathfrak{k}$ [resp. $\mathfrak{p}$]. A root $\alpha \in \Delta$ is *real* if $\alpha(\mathfrak{t}_{\mathbb{C}}) = 0$, *imaginary* if $\alpha(\mathfrak{a}_{\mathbb{C}}) = 0$, and otherwise *complex*. Indeed, we have $\Lambda^* \subset i\mathfrak{t} \oplus \mathfrak{a}$, and so the action of θ resp. $\rho :=$ complex conjugation on the root vectors $X_\alpha \in \mathfrak{g}$ sends $\alpha \mapsto -\bar{\alpha}$ resp. $\bar{\alpha}$. In particular, for α imaginary, we have

$$\begin{aligned} \theta(\alpha) = \alpha \quad \implies \quad \theta X_\alpha \in \mathbb{R}\langle X_\alpha \rangle \quad &\implies \quad \theta X_\alpha = X_\alpha \text{ [resp. } -X_\alpha] \\ &\implies \quad X_\alpha \in \mathfrak{k}_{\mathbb{C}} \text{ [resp. } \mathfrak{p}_{\mathbb{C}}] \end{aligned}$$

in which case we say α is compact [resp. noncompact] imaginary. So we have a decomposition

$$\Delta = \Delta_{\mathbb{R}} \cup \Delta_{\mathbb{C}} \cup \Delta_c \cup \Delta_n,$$

with complex roots occurring in quadruplets and other types in $\pm$ pairs. Note that every $G(\mathbb{R})^\circ$-conjugacy class of Cartans contains a Θ-stable member. We will write

$$W_{\mathbb{R}}(H) := \frac{N(G(\mathbb{R}), H(\mathbb{R}))}{H(\mathbb{R})} \leqslant \frac{N(G(\mathbb{C}), H(\mathbb{C}))}{H(\mathbb{C})} =: W_{\mathbb{C}}(H)$$

for the real and complex Weyl groups. The latter is of course generated by the reflections in the roots Δ. An algorithm for computing the real Weyl group (as implemented by the ATLAS computer software) is described in [1, sec. 6] and [26]. More germane for our purposes is the connected Weyl group

$$W^\circ_{\mathbb{R}}(H) := \frac{N\left(G(\mathbb{R})^\circ, H(\mathbb{R})^\circ\right)}{H(\mathbb{R})^\circ} \leqslant W_{\mathbb{R}}(H),$$

which contains the subgroup generated by reflections in $\Delta_{\mathbb{R}} \cup \Delta_c$ but may be larger than this unless $H(\mathbb{R})$ is compact or split.

Now assume that there exists a maximal torus $T \leqslant G_{\mathbb{R}}$ with $T(\mathbb{R})$ compact. Taking $H := T$, all roots are imaginary (in the sense that $\rho\colon \alpha \mapsto -\alpha$), and we *define* the compact [resp. noncompact] ones to be those with $-B(X_\alpha, \overline{X_\alpha}) > 0$ [resp. < 0]. (We may normalize so that $\overline{X_\alpha} = X_{-\alpha}$ resp. $-X_{-\alpha}$ for $\alpha \in \Delta_n$ resp. Δ_c.) Setting

$$\mathfrak{k} := \mathfrak{t} \oplus \Bigg(\bigoplus_{\alpha \in \Delta_c} \mathbb{C}\langle X_\alpha \rangle \Bigg) \cap \mathfrak{g}_{\mathbb{R}}, \quad \mathfrak{p} := \Bigg(\bigoplus_{\alpha \in \Delta_n} \mathbb{C}\langle X_\alpha \rangle \Bigg) \cap \mathfrak{g}_{\mathbb{R}},$$

$K := \exp(\mathfrak{k})$ is maximal compact, and the involution θ defined by linearly extending $\theta|_{\mathfrak{k}} := \mathrm{id}$, $\theta|_{\mathfrak{p}} := -\mathrm{id}$ is Cartan. In particular, T is Θ-stable, and $\Delta = \Delta_c \cup \Delta_n$.

Assume further that there exists a co-character $\chi_0 \in \mathrm{X}_*(T(\mathbb{C}))$ such that $\pi_{\chi_0}(\Delta_c) \subset 2\mathbb{Z}$ and $\pi_{\chi_0}(\Delta_n) \subset 1 + 2\mathbb{Z}$. Let φ_0 denote the restriction of

$$\mathbb{C}^* \xrightarrow{\chi_0} T(\mathbb{C}) \hookrightarrow G(\mathbb{C})$$

to $S^1 \to G(\mathbb{R})$, and $\mathrm{Ad}\colon G(\mathbb{R}) \to \mathrm{Aut}(\mathfrak{g}_{\mathbb{R}}, B)$ the adjoint homomorphism. Then $(\mathfrak{g}_{\mathbb{Q}}, \mathrm{Ad} \circ \varphi_0, -B)$ is a polarized Hodge structure of weight 0, with decomposition $\mathfrak{g}_{\mathbb{C}} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}^{j,-j}$, where

$$\mathfrak{g}^{j,-j} := \left\{\gamma \in \mathfrak{g}_{\mathbb{C}} \mid \mathrm{Ad}(\chi_0(z))\gamma = z^{2j}\gamma\right\} = \bigoplus_{\left\{\substack{\alpha\in\Delta \\ \pi_{\chi_0}(\alpha)=j}\right.} \mathbb{C}\langle X_\alpha\rangle$$

if $j \neq 0$ and $\mathfrak{g}^{0,0} := \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha\in\left(\ker \pi_{\chi_0}\right)\cap\Delta} \mathbb{C}\langle X_\alpha\rangle$. (Note that $C = \varphi(i)$.) The conjugacy class of φ_0 (or "connected Hodge domain")

$$D := G(\mathbb{R})^\circ.\varphi_0 \cong G(\mathbb{R})^\circ/\mathcal{H}_{\varphi_0}$$

parametrizes a set of $(-B)$-polarized Hodge structures on $\mathfrak{g}_{\mathbb{Q}}$ with the same Hodge numbers; a very general point in D has Mumford-Tate group G. Writing

$$F_0^k\mathfrak{g}_{\mathbb{C}} := \bigoplus_{j\geqslant k} \mathfrak{g}^{j,-j} \ \left(= \mathscr{F}(T,\chi_0)^k\mathfrak{g}_{\mathbb{C}}\right),$$

D is an analytic open subset in its compact dual

$$\check{D} := G(\mathbb{C})\cdot F_0^\bullet \cong G(\mathbb{C})/Q_{F_0^\bullet}.$$

Flags[(2)] $F^\bullet \in \check{D}$ are called *semi-Hodge*. A *Hodge* flag is one which satisfies $\mathfrak{g}_{\mathbb{C}} = \bigoplus_{p\in\mathbb{Z}} F^p \cap \overline{F^{-p}}$; equivalently, $F^\bullet$ is of the form $\mathscr{F}(T,\chi)^\bullet$ for some compact $T \leqslant G_{\mathbb{R}}$ [12, (VI.B.9)]. As above χ has an associated weight 0 (but not necessarily $(-B)$-polarized) Hodge structure $\varphi\colon S^1 \to G(\mathbb{R})$, and we write $F^\bullet = F_\varphi^\bullet$. We denote the non-Hodge locus in $\check{D}$ by $\mathfrak{Z}$, and shall (by abuse of notation) write $\varphi \in \check{D} \smallsetminus \mathfrak{Z}$.

Finally, taking Δ^+ to be a system of positive roots with $\pi_\chi(\Delta^+) \subset \mathbb{Z}_{\geqslant 0}$, assume that π_χ takes values 0 and 1 on the simple roots. Then it is known (*cf.* [24]) that the $G(\mathbb{C})$-invariant horizontal distribution $\mathcal{W} \subset T\check{D}$ given (at φ) by

$$F^{-1}\mathfrak{g}_{\mathbb{C}}/F^0\mathfrak{g}_{\mathbb{C}} \subset \mathfrak{g}_{\mathbb{C}}/F^0\mathfrak{g}_{\mathbb{C}}$$

(2) In this paper, "flag" will mean what is sometimes called a "partial flag", *i.e.* not necessarily "maximal" or "complete".

is bracket-generating; that is,

$$\mathcal{W} + [\mathcal{W}, \mathcal{W}] + [\mathcal{W}, [\mathcal{W}, \mathcal{W}]] + \cdots = T\check{D}$$

(or equivalently, $F^{-1} + [F^{-1}, F^{-1}] + \cdots = \mathfrak{g}_{\mathbb{C}}$ on the Lie algebra level).

The three assumptions delineated in the last three paragraphs will remain in effect for the rest of this paper. With T, θ as above, we can obtain θ-stable representatives of all $G(\mathbb{R})^{\circ}$-conjugacy classes of (real) Cartans by taking successive *Cayley transforms* in imaginary noncompact roots. Given $H \leqslant G_{\mathbb{R}}$ and $\alpha \in \Delta_n$, the Cayley transform in α is defined in terms of conjugation by $\mathbf{c}_\alpha := \exp\left(\frac{\pi}{4}\left(X_{-\alpha} - X_\alpha\right)\right)$, where the root vectors $X_{\pm\alpha}$ are assumed to be normalized so that $X_{-\alpha} = \overline{X_\alpha}$ and $[[X_\alpha, X_{-\alpha}], X_\alpha] = 2X_\alpha$ (in particular, $[X_\alpha, X_{-\alpha}] \in \Lambda^*$). More precisely, it replaces H by the real algebraic torus underlying $\Psi_{\mathbf{c}_\alpha}(H_{\mathbb{C}})$, which by abuse of notation shall be denoted $\Psi_{\mathbf{c}_\alpha}(H)$. This has the effect of increasing the real rank $\dim_{\mathbb{R}} \mathfrak{a}$ of H by 1, replacing $\mathfrak{h}_{\mathbb{R}}$ by $(\ker \alpha|_{\mathfrak{h}_{\mathbb{R}}}) \oplus \mathbb{R}\langle X_\alpha + X_{-\alpha}\rangle$. (Conjugation by the square $\mathbf{c}_\alpha^2$, which stabilizes H, yields the Weyl reflection in α.) Up to scaling, the new root vectors are the images of the old ones by $\mathrm{Ad}(\mathbf{c}_\alpha)$, and $\mathrm{Ad}(\mathbf{c}_\alpha)X_{\pm\alpha}$ in particular are real root vectors.

The process may be reversed by applying (inverse) Cayley transforms $\mathrm{Ad}(\mathbf{d}_\beta)$ in $\beta \in \Delta_{\mathbb{R}}$, *cf.* [16, sec. VI.7]. In more detail, if $X_{\pm\beta}$ are normalized so that $\theta(X_\beta) = -X_{-\beta}$ and $[[X_\beta, X_{-\beta}], X_\beta] = 2X_\beta$, these are given by $\mathbf{d}_\beta := \exp\left(-i\frac{\pi}{4}\left(X_{-\beta} + X_\beta\right)\right)$. Assuming additionally that $X_\beta = i\,\mathrm{Ad}(\mathbf{c}_\alpha)X_\alpha$, one has $X_{-\beta} = -i\,\mathrm{Ad}(\mathbf{c}_\alpha)X_{-\alpha}$ and

$$\mathbf{d}_\beta = e^{\mathrm{Ad}(\mathbf{c}_\alpha)\frac{\pi}{4}(X_\alpha - X_{-\alpha})} = \Psi_{\mathbf{c}_\alpha}(\mathbf{c}_\alpha^{-1}) = \mathbf{c}_\alpha^{-1}.$$

Remark 2.2. — One may pictorially represent the situation in a graph with $G(\mathbb{R})^{\circ}$-conjugacy classes of real Cartans as nodes and Cayley transforms as edges; the ATLAS software can compute these so-called Hasse diagrams (*cf.* [1], [2]).

3. From semi-Hodge flags to Cartan data

Take G, $H_0 = T_0 \leqslant G_{\mathbb{R}}$, $\chi_0 \in \mathrm{X}_*(T(\mathbb{C}))$, and $\varphi_0 \in D \subset \check{D}$ to be as in §2, with associated flag $F_0^\bullet$. The points of D are the $(-B)$-polarized Hodge structures $G(\mathbb{R})^{\circ}$-conjugate to φ_0. In this section we will give a similar characterization of the points of $\check{D}$.

3.1. The bigrading

Let $F^\bullet \in \check{D}$, with $Q_{F^\bullet} \subset G(\mathbb{C})$ the parabolic subgroup preserving $F^\bullet$. Inside $\check{D} \cong G(\mathbb{C})/Q_{F^\bullet}$ we have the connected $G(\mathbb{R})^\circ$-orbit

$$\mathcal{O}_{F^\bullet} := G(\mathbb{R})^\circ \cdot F^\bullet \cong G(\mathbb{R})^\circ / \{Q_{F^\bullet} \cap \overline{Q_{F^\bullet}} \cap G(\mathbb{R})^\circ\}.$$

Lemma 3.1. — *There exists a Cartan subgroup $H \leqslant G_{\mathbb{R}}$ with*

$$H(\mathbb{R}) \subseteq Q_{F^\bullet} \cap \overline{Q_{F^\bullet}} \cap G(\mathbb{R}).$$

Proof. — See [27, Thm. 2.6(1)] or [11, Lemma 2.1.3]. □

Fix an H (as in Lemma 3.1), and write $(\mathfrak{h})_{\mathbb{R}} := \mathrm{Lie}(H(\mathbb{R}))$, $\mathfrak{h} := \mathrm{Lie}(H(\mathbb{C}))$ for the corresponding Lie subalgebras. From $F^\bullet$ we obtain a filtration

$$\widetilde{W}_{-k}\mathfrak{g}_{\mathbb{C}} := \bigoplus_{p\in\mathbb{Z}} \left(F^p \cap \overline{F^{k-p}}\right) \mathfrak{g}_{\mathbb{C}} \tag{3.1}$$

which is stabilized by $Q_{F^\bullet} \cap \overline{Q_{F^\bullet}}$ hence by $H(\mathbb{C})$; this is of course defined over $\mathbb{R}$. Its nontriviality "measures" the failure of $F^\bullet$ to be a (pure) Hodge flag.

Lemma 3.2. — *Given $F^\bullet$ and H as above, there is a unique bigrading*

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{(p,q)\in\mathbb{Z}} \mathfrak{g}^{p,q} \tag{3.2}$$

satisfying:

(i) *each $\mathfrak{g}^{p,q}$ is a sum of root spaces of $(\mathfrak{g}, \mathfrak{h})$ (and $\mathfrak{h}$, if $(p,q) = (0,0)$);*
(ii) $F^a = \bigoplus_{\left\{\substack{(p,q)\\ p\geqslant a}\right.} \mathfrak{g}^{p,q}$;
(iii) $\overline{\mathfrak{g}^{p,q}} = \mathfrak{g}^{q,p}$;
(iv) $\widetilde{W}_{-k} = \bigoplus_{\left\{\substack{(p,q)\\ p+q\geqslant k}\right.} \mathfrak{g}^{p,q}$;
(v) $B|_{\mathfrak{g}^{p,q}\times\mathfrak{g}^{p',q'}}$ *nondegenerate for $(p',q') = (-p,-q)$, and otherwise* 0.

The $\{\dim(\mathfrak{g}^{p,q})\}$ do not depend upon the choice of H.

Proof. — We know that $F_0^\bullet := F_{\varphi_0}^\bullet = \mathscr{F}(H_0, \chi_0)^\bullet$, and that $G(\mathbb{C})$ acts transitively on $\check{D}$. Taking $g \in G(\mathbb{C})$ so that $g\cdot F_0^\bullet = F^\bullet$, we have $\mathscr{F}\left(\Psi_g(H_{0,\mathbb{C}}), \Psi_g(\chi_0)\right)^\bullet = F^\bullet$, and $\Psi_g(H_{0,\mathbb{C}}) \leqslant \Psi_g Q_{F_0^\bullet} = Q_{F^\bullet}$. Since $Q_{F^\bullet}$ is connected, any two Cartans of $Q_{F^\bullet}$ (*i.e.* Cartans of $G(\mathbb{C})$ contained in $Q_{F^\bullet}$) are conjugate by an element of $Q_{F^\bullet}$ (*cf.* [4, 11.16 and 12.1(a)]). Hence, we may arrange to have $\Psi_g(H_{0,\mathbb{C}}) = H_{\mathbb{C}}$; write $\Psi_g(H_0) = H$ and $\Psi_g(\chi_0) =: \chi_\xi =: \chi$.

Since H is real, we have $\chi, \bar{\chi} \in \mathrm{X}_*(H(\mathbb{C}))$, and so

$$\begin{aligned}\mathfrak{g}^{p,q} &:= \left\{ \gamma \in \mathfrak{g}_{\mathbb{C}} \,\middle|\, \begin{array}{l} \mathrm{Ad}(\chi(z))\gamma = z^{2p}\gamma \\ \mathrm{Ad}(\bar{\chi}(z)) = z^{2q}\gamma \end{array} \right\} \\ &= \bigoplus_{\alpha\in\Delta, \left\{\substack{\pi_\chi(\alpha)=p \\ \pi_{\bar{\chi}}(\alpha)=q}\right.} \mathbb{C}\langle X_\alpha\rangle (\oplus\mathfrak{h}, \text{ if } (p,q)=(0,0))\end{aligned}$$

gives a bigrading. Since

$$\left\{\begin{array}{l} F^a = \mathscr{F}(H,\chi) = \bigoplus_{\left\{\substack{(p,q) \\ p\geqslant a}\right.} \mathfrak{g}^{p,q} \\ \overline{F^b} = \mathscr{F}(H,\bar{\chi}) = \bigoplus_{\left\{\substack{(p,q) \\ q\geqslant b}\right.} \mathfrak{g}^{p,q}, \end{array}\right. \tag{3.3}$$

this gives (i)–(iv), and (v) follows from the fact that $X_\alpha \in \mathfrak{g}^{p,q} \iff X_{-\alpha} \in \mathfrak{g}^{-p,-q}$. Uniqueness is clear, as is the last statement since $\dim \mathfrak{g}^{p,q} = \dim\left(\{F^p \cap \overline{F^q}\}/\{F^p \cap \overline{F^{q+1}} + F^{p+1}\cap \overline{F^q}\}\right)$. □

There are several easy remarks at this point. The first is that $[\mathfrak{g}^{p,q}, \mathfrak{g}^{p',q'}] \subseteq \mathfrak{g}^{p+p',q+q'}$ and so the isotropy Lie algebra $\mathfrak{q}_{F^\bullet} := \mathrm{Lie}(Q_{F^\bullet})$ identifies with $F^0\mathfrak{g}_{\mathbb{C}}$. Next, setting

$$W_k\mathfrak{g}_{\mathbb{C}} := \bigoplus_{p+q\leqslant k} \mathfrak{g}^{p,q}$$

(which is clearly defined over $\mathbb{R}$),(3) we see that $(\mathfrak{g}_{\mathbb{R}}, F^\bullet, W_\bullet)$ give the data of a *split* $\mathbb{R}$*-mixed Hodge structure*, which is split by property (iii) in the Lemma. From the end of the proof, it satisfies the symmetry $\dim Gr^W_j\mathfrak{g} = \dim Gr^W_{-j}\mathfrak{g}$. Defining new cocharacters χ_Y, χ_ϕ by

$$Y := \frac{\xi+\bar{\xi}}{2}, \quad \phi := \frac{\xi-\bar{\xi}}{2}, \tag{3.4}$$

the mixed Hodge representation associated to $(\mathfrak{g}_{\mathbb{R}}, F^\bullet, W_\bullet)$ is

$$\begin{array}{rccccc} \tilde{\varphi}_{F^\bullet}: & \mathbb{R}^*\times S^1 & \to & H(\mathbb{R}) & \hookrightarrow & G(\mathbb{R}) \\ & (w,z) & \mapsto & \chi_Y(w)\chi_\phi(z) & =: & \tilde{\varphi}_{F^\bullet}(w,z). \end{array}$$

The composition of its complexification with Ad, mapping $\mathbb{C}^*\times\mathbb{C}^* \to \mathrm{Aut}(\mathfrak{g}, B)$, restricts on $\mathfrak{g}^{p,q}$ to multiplication by $w^{p+q}z^{p-q}$. Since we can act with $G(\mathbb{R})^\circ$ (via Ad) compatibly on $F^\bullet$,$W_\bullet$,H,ϕ,Y, and $\{\mathfrak{g}^{p,q}\}$, we have

Corollary 3.3. — *The*

$$h^{p,q}_{\mathcal{O}} := \dim_{\mathbb{C}}(\mathfrak{g}^{p,q})$$

are well-defined invariants of the $G(\mathbb{R})^\circ$-orbit $\mathcal{O}$.

(3) To a Hodge theorist, this filtration is much more familiar than $\widetilde{W}_\bullet$, but (unlike $\widetilde{W}_\bullet$) depends on the choice of H.

Finally, there is the

Proposition 3.4. — *$F^\bullet$ is Hodge $\iff \widetilde{W}_\bullet$ is trivial ($\iff W_\bullet$ is trivial).*

Proof. — Y grades $\widetilde{W}_\bullet$, so if $Gr_j^{\widetilde{W}} = \{0\}$ for $j \neq 0$ then $Y = 0$ and $\varphi := \tilde{\varphi}_{F^\bullet}|_{S^1}$ is a Hodge structure with $F^\bullet = F^\bullet_\varphi$. Conversely, if $F^\bullet$ is Hodge then the p-opposed condition $F^p \cap \overline{F^{-p+1}} = \{0\}$ holds, and so $\widetilde{W}_{-1} = \{0\}$; by symmetry, $\mathfrak{g}/\widetilde{W}_0 = \{0\}$. $\square$

3.2. From Cartan data to semi-Hodge flags

We are now prepared to parametrize the flags in $\check{D}$ by Cartan data. Let $\xi_{\mathbb{C}}$ denote the set of maximal tori $'H \leqslant G_{\mathbb{C}}$, and $\Xi_{\mathbb{C}}$ the set of pairs $('H, \chi)$ ($\chi \in \mathrm{X}_*('H(\mathbb{C}))$) $G(\mathbb{C})$-conjugate to $(H_{0,\mathbb{C}}, \chi_0)$. Define subsets $\tilde{\xi}_{\mathbb{R}} \subset \xi_{\mathbb{C}}$ resp. $\tilde{\Xi}_{\mathbb{R}} \subset \Xi_{\mathbb{C}}$ by imposing the requirement that $'H = H_{\mathbb{C}}$ for H defined over $\mathbb{R}$, and smaller subsets $\xi_{\mathbb{R}}$ resp. $\Xi_{\mathbb{R}}$ by insisting that $H(\mathbb{R})$ be compact. Finally let $\Xi^\circ_{\mathbb{R}} \subset \Xi_{\mathbb{R}}$ be the $G(\mathbb{R})^\circ$-orbit of (H_0, χ_0). Applying $\mathscr{F}$ produces a $G(\mathbb{R})^\circ$-equivariant(4) commutative diagram

$$\begin{tikzcd}
\xi_{\mathbb{R}} \arrow[r, equal] & \xi_{\mathbb{R}} \arrow[r, hook] & \tilde{\xi}_{\mathbb{R}} \arrow[r, hook] & \xi_{\mathbb{C}} \\
\Xi^\circ_{\mathbb{R}} \arrow[u, two heads, "\pi^\circ_{\mathbb{R}}"'] \arrow[r, hook] \arrow[d, "\mathscr{F}^\circ_{\mathbb{R}}"] & \Xi_{\mathbb{R}} \arrow[u, "\pi_{\mathbb{R}}"'] \arrow[r, hook] \arrow[d, "\mathscr{F}_{\mathbb{R}}"] & \tilde{\Xi}_{\mathbb{R}} \arrow[u, "\tilde{\pi}_{\mathbb{R}}"'] \arrow[r, hook] \arrow[d, "\widetilde{\mathscr{F}}_{\mathbb{R}}"] & \Xi_{\mathbb{C}} \arrow[u, two heads, "\pi_{\mathbb{C}}"'] \arrow[d, "\mathscr{F}_{\mathbb{C}}"] \\
D \arrow[r, hook] & \check{D} \setminus \mathfrak{Z} \arrow[r, hook] & \check{D} \arrow[r, equal] & \check{D}
\end{tikzcd}$$

in which surjectivity of the leftmost and rightmost upward arrows follows from the $G(\mathbb{R})^\circ$-conjugacy [resp. $G(\mathbb{R})$-conjugacy] of all compact maximal real [resp. maximal complex] tori. The desired parametrization of Hodge and semi-Hodge flags is then given by the following

Theorem 3.5. — (i) $\widetilde{\mathscr{F}}_{\mathbb{R}}$ and (ii) $\mathscr{F}_{\mathbb{R}}$ are surjective.

Proof.
(i) Given $F^\bullet \in \check{D}$, by (3.3) we have $F^\bullet = \mathscr{F}(H_{\mathbb{C}}, \chi)^\bullet$ for some $H \leqslant G_{\mathbb{R}}$. Let $g \in G(\mathbb{C})$ be such that $g \cdot F^\bullet_0 = F^\bullet$ in $\check{D}$. Then

$$F^\bullet = g \cdot \mathscr{F}\left(T_{0,\mathbb{C}}, \chi_0\right)^\bullet = \mathscr{F}\left(\Psi_g(T_{0,\mathbb{C}}), \Psi_g(\chi_0)\right)^\bullet$$

(4) $G(\mathbb{R})$-equivariant if the left-most column is removed; right-most column $G(\mathbb{C})$-equivariant.

and the $\{F^i\}$ are sums of root spaces of $\Psi_g(T_{0,\mathbb{C}})$, so that $\Psi_g(T_{0,\mathbb{C}}) \subset Q_{F^\bullet}$. We also have $H_\mathbb{C} \subset Q_{F^\bullet}$, and so (as in the proof of Lemma 3.2) $H_\mathbb{C}$ and $\Psi_g(T_{0,\mathbb{C}})$ are conjugate by $\rho \in Q_{F^\bullet}$. That is, $\Psi_{\rho g}(T_{0,\mathbb{C}}) = H_\mathbb{C}$ and $\rho g \cdot F_0^\bullet = F^\bullet$; and we conclude that $F^\bullet = \mathscr{F}\left(H_\mathbb{C}, \Psi_{\rho g}(\chi_0)\right)^\bullet \in \mathscr{F}\left(\widetilde{\Xi}_\mathbb{R}\right)$.

(ii) Given $\varphi \in \check{D} \smallsetminus \mathfrak{Z}$, there is a compact maximal torus $T \leqslant G_\mathbb{R}$ with $T(\mathbb{R}) \supset \varphi(S^1)$, and $F_\varphi^\bullet = \mathscr{F}(T_\mathbb{C}, \chi_\varphi)$. The rest of the argument is as in (i). □

Remark 3.6.

(a) (ii) is essentially part of Theorem (VI.B.9) in [12], while (i) establishes the conjecture made in Remark (VI.B.10) of [op. cit.]. [5]

(b) In the proof of (i), for any $\mu \in \mathrm{X}_*(H(\mathbb{C}))$ we have $\mu = \chi \iff \mathscr{F}(H_\mathbb{C}, \mu) = F^\bullet$. Hence $\psi_{\rho g}(\chi_0) = \chi$ and the original $(H_\mathbb{C}, \chi)$ belongs to $\widetilde{\Xi}_\mathbb{R}$. This shows that *a real-Cartan/co-character pair not in* $\widetilde{\Xi}_\mathbb{R}$ (*i.e.* not $G(\mathbb{C})$-conjugate to (T_0, χ_0)) *does not yield a flag in* $\check{D}$.

(c) Suppose $\mathscr{F}(H, \chi) = F_\varphi^\bullet$ is the flag of a $(-B)$-polarized Hodge structure φ. Then $B(\cdot, \bar{\cdot})$ is definite on each $\mathfrak{g}_\varphi^{p,-p}$. Since $G \subset \mathrm{Aut}(\mathfrak{g}, -B)$, the isotropy group $\mathcal{H}_\varphi \subset G(\mathbb{R})$ is compact. Moreover, $H(\mathbb{R})$ commutes with $\varphi(S^1)$, whereupon we have $\varphi(S^1) \subset H(\mathbb{R}) \subset \mathcal{H}_\varphi$, forcing $H(\mathbb{R})$ compact. We conclude that $\mathscr{F}\left(\widetilde{\Xi}_\mathbb{R} \smallsetminus \Xi_\mathbb{R}\right)$ *avoids the* $(-B)$*-polarized locus* (which resides between D and $\check{D} \smallsetminus \mathfrak{Z}$).

For later reference we emphasize the obvious

Proposition 3.7. — *If* $F^\bullet \in \check{D}$ *is* $\mathscr{F}_\mathbb{C}$ *of* $({}'H, \chi) \in \Xi_\mathbb{C}$, *then* ${}'H(\mathbb{C}) \subset Q_{F^\bullet}$.

Proof. — The F^i are sums of eigenspaces of $\mathrm{ad}(\chi(z))$ (which are sums of root spaces of ${}'H$), and ${}'\mathfrak{h}$ belongs to the "trivial" eigenspace hence to $F^0\mathfrak{g} (= \mathfrak{q}_{F^\bullet})$. □

3.3. Discretizing the Cartan data

Now any given $H \in \widetilde{\xi}_\mathbb{R}$ is $G(\mathbb{R})^\circ$-conjugate to some Cartan in the list of all successive Cayley transforms of H_0 in noncompact imaginary roots ([16, p. 394]). Removing all but one Cartan in each $G(\mathbb{R})^\circ$-conjugacy class, we shall fix henceforth:

- the resulting sublist $\{H_0, H_1, \ldots, H_n\} =: \tilde{\xi}_\mathbb{R}^\theta$;
- $\underline{c}(j)$:= product of Cayley transforms with $\Psi_{\underline{c}(j)}(H_0) = H_j$;
- $\chi_j(z) := e^{\log(z)\,\mathrm{ad}(\xi_j)} := \Psi_{\underline{c}(j)}(\chi_0(z)) \in \mathrm{X}_*(H_j(\mathbb{C}))$;[6]

[5] Note that the uniqueness of semi-Hodge decompositions asserted there is not correct; it depends upon the choice of Cartan.

[6] Note that $\xi_j \in \mathfrak{h}_j := \mathrm{Lie}(H_j(\mathbb{C}))$ is defined by the second equality.

- the Cartan involution $\Theta := \Psi_{\varphi_0(i)}$(=identity on H_0);
- its ± 1-eigenspaces $\mathfrak{k}, \mathfrak{p} \subset \mathfrak{g}_{\mathbb{R}}$.

For any Θ-stable H, $\alpha \in \Delta_n \implies \theta(X_{\pm\alpha}) = -X_{\pm\alpha} \implies \Theta(\mathbf{c}_\alpha) = \mathbf{c}_\alpha^{-1} \implies \Theta(\Psi_{\mathbf{c}_\alpha}(H)) = \Psi_{\mathbf{c}_\alpha^{-1}}(H) = \Psi_{\mathbf{c}_\alpha}\Psi_{\mathbf{c}_\alpha^{-2}}(H) = \Psi_{\mathbf{c}_\alpha}(H)$ since $\mathbf{c}_\alpha^{\pm 2} = w_\alpha$ is the Weyl element. Hence every $H \in \tilde{\xi}_{\mathbb{R}}^{\theta}$ is Θ-stable. If $G_{\mathbb{R}}$ is split, then we shall take H_n to be the (unique) split Cartan in $\tilde{\xi}_{\mathbb{R}}^{\theta}$.

Noting that $\xi_0 = \phi_0 \in i(\mathfrak{h}_0)_{\mathbb{R}}$, we have $\xi_j = \mathrm{Ad}(\underline{c}(j))\phi_0$. Given any $w \in W_{\mathbb{C}}(H_j)$, $\tilde{w} := \Psi_{\underline{c}(j)^{-1}}(w)$ belongs to $W_{\mathbb{C}}(H_0)$ and

$$\xi_j^{[w]} := \mathrm{Ad}(w)\xi_j = \mathrm{Ad}(\underline{c}(j))\,\mathrm{Ad}(\tilde{w})\phi_0 =: \mathrm{Ad}(\underline{c}(j))\phi_0^{[\tilde{w}]},$$

hence

$$\theta(\xi_j^{[w]}) = \mathrm{Ad}\left(\Theta(\underline{c}(j))\right)\theta(\phi_0^{[\tilde{w}]}) = \mathrm{Ad}(\overline{\underline{c}(j)})\phi_0^{[\tilde{w}]} = \overline{\mathrm{Ad}(\underline{c}(j))\overline{\phi_0^{[\tilde{w}]}}}$$

$$= -\overline{\mathrm{Ad}(\underline{c}(j))\phi_0^{[\tilde{w}]}} = -\overline{\xi_j^{[w]}}.$$

Writing $Y_j^{[w]} := \frac{\xi_j^{[w]} + \overline{\xi_j^{[w]}}}{2}$, $\phi_j^{[w]} := \frac{\xi_j^{[w]} - \overline{\xi_j^{[w]}}}{2}$,

$$\mathfrak{a}_j := (\mathfrak{h}_j)_{\mathbb{R}} \cap \mathfrak{p}, \quad \mathfrak{k}_j := (\mathfrak{h}_j)_{\mathbb{R}} \cap \mathfrak{k},$$

this yields $\phi_j^{[w]} \in i\mathfrak{t}_j$, $Y_j^{[w]} \in \mathfrak{a}_j$. This allows us to associate (nonuniquely) Hodge-compatible Cartan data to any flag in $\check{D}$:

Proposition 3.8. — *Given any $F^\bullet \in \check{D}$:*

(i) *there exist $H_j \in \tilde{\xi}_{\mathbb{R}}^{\theta}$, $w \in W_{\mathbb{C}}(H_j)$, and $g \in G(\mathbb{R})^\circ$ such that*

$$F^\bullet = \mathscr{F}\left(\Psi_g(H_j), \Psi_g(\chi_j^{[w]})\right)^\bullet \ (=: \mathscr{F}(H,\chi)^\bullet);$$

and

(ii) *referring to* (3.4)*, there is a Cartan involution $\Theta_{F^\bullet}$ and corresponding $\mathfrak{k}_{F^\bullet}, \mathfrak{p}_{F^\bullet} \subset \mathfrak{g}_{\mathbb{R}}$, $\mathfrak{t}_{F^\bullet}, \mathfrak{a}_{F^\bullet} \subset \mathfrak{h}_{\mathbb{R}}$ such that*

$$Y \in \mathfrak{a}_{F^\bullet} \text{ and } \phi \in i\mathfrak{t}_{F^\bullet}.$$

Proof.

(i) By Theorem 3.5, $F^\bullet$ is $\mathscr{F}$ of some $(H,\chi) \in \widetilde{\Xi}_{\mathbb{R}}$; clearly H is some $\Psi_g(H_j)$. So $\left(H_j, \Psi_{g^{-1}}(\chi)\right)$ is $G(\mathbb{C})$-, hence $W_{\mathbb{C}}(H_j)$-, conjugate to (H_j,χ_j).

(ii) Put $\Theta_{F^\bullet} := \Psi_{g\varphi_0(i)g^{-1}}$; then H is $\Theta_{F^\bullet}$-stable and we have $\mathfrak{a}_{F^\bullet} = \mathrm{Ad}(g)\mathfrak{a}_j$ and $\mathfrak{t}_{F^\bullet} = \mathrm{Ad}(g)\mathfrak{t}_j$, so the result follows from the above computations. □

Corollary 3.9. — *Given $F^\bullet = \mathscr{F}(H,\chi) \in \check{D}$, with the bigrading $\mathfrak{g}_{\mathbb{C}} = \oplus\mathfrak{g}^{p,q}$ associated to H, $\Theta_{F^\bullet}(\mathfrak{g}^{p,q}) = \mathfrak{g}^{-q,-p}$.*

Proof. — By Prop. 3.8(ii), $\theta_{F^\bullet}(Y) = -Y$ and $\theta_{F^\bullet}(\phi) = \phi$, whereupon the formula for $\tilde{\varphi}_{F^\bullet}$ gives

$$\begin{aligned}\mathrm{Ad}(\tilde{\varphi}_{F^\bullet}(w,z)) \circ \theta_{F^\bullet} &= \theta_{F^\bullet} \circ \mathrm{Ad}(\Theta_{F^\bullet}(\tilde{\varphi}_{F^\bullet}(w,z)) \\ &= \theta_{F^\bullet} \circ \mathrm{Ad}(\tilde{\varphi}_{F^\bullet}(w^{-1},z)).\end{aligned}$$

Restrict this to $\mathfrak{g}^{p,q}$. □

Corollary 3.10. — *If $F^\bullet \in \mathfrak{Z}$, then for some $p > 0$, $\dim_{\mathbb{C}}(\mathfrak{g}^{p,p}) \neq 0$.*

Proof. — The idea is to use the basic fact that $G(\mathbb{R})$ has a compact Cartan. Any noncompact Cartan is then an iterated Cayley transform of such and so must have a real root.

Clearly H is noncompact (*i.e.* $H \in \tilde{\xi}_{\mathbb{R}} \setminus \xi_{\mathbb{R}}$) and $Y \in \mathfrak{a}_{F^\bullet} \setminus \{0\}$ (where $\mathfrak{h}_{\mathbb{R}} = \mathfrak{a}_{F^\bullet} \oplus \mathfrak{t}_{F^\bullet}$). Since the $\dim(\mathfrak{g}^{p,q})$ depend only on $F^\bullet$, we may take H to be of minimal (positive) real rank. Suppose $\beta(Y) = 0$ $\forall \beta \in \Delta_{\mathbb{R}} (\neq \emptyset)$, and let $\widehat{H}$ be the (inverse) Cayley transform of H in some $\beta_0 \in \Delta_{\mathbb{R}}$. Then $\hat{\mathfrak{a}}_{F^\bullet} = \ker(\beta_0|_{\mathfrak{a}_{F^\bullet}}) \ni Y$ and $\hat{\mathfrak{t}}_{F^\bullet} \supset \mathfrak{t}_{F^\bullet} \ni i\phi \implies \tilde{\varphi}_{F^\bullet}$ still factors through $\widehat{H} \implies F^\bullet = \mathscr{F}(\widehat{H}, \hat{\chi})$, in contradiction to the presumed minimality of the real rank of H.

So $\beta(Y) \neq 0$ for some $\beta \in \Delta_{\mathbb{R}}$, while $\phi \in \mathfrak{t}_{F^\bullet,\mathbb{C}} \implies \beta(\phi) = 0$, whereupon

$$\mathrm{Ad}(\tilde{\varphi}_{F^\bullet}(w,z))X_\beta = \mathrm{Ad}(\chi_Y(w))X_\beta = w^{\beta(Y)}X_\beta$$

$\implies X_\beta \in \mathfrak{g}^{\frac{1}{2}\beta(Y),\frac{1}{2}\beta(Y)}$. □

Because even the real rank ($= \dim_{\mathbb{R}}(\mathfrak{a}_{F^\bullet})$) of $\mathfrak{h}_{\mathbb{R}}$ may not be unique in Proposition 3.8 (*cf.* §6), the question arises as to whether some choices are better than others. At least in one fairly general setting, we shall now see that this is so. Consider the (well-defined) real parabolic

$$\mathcal{Q} := Q_{\widetilde{W}} \leqslant G_{\mathbb{R}}$$

defined by (3.1), with Lie algebra $\mathfrak{q} = \widetilde{W}_0\mathfrak{g}_{\mathbb{R}}$. This has unipotent radical $\mathfrak{n} := \widetilde{W}_{-1}\mathfrak{g}_{\mathbb{R}}$ and, with the choice of $\mathfrak{h}$ and $\Theta_{F^\bullet}$, the natural subalgebras $\tilde{\mathfrak{m}} := \left(\bigoplus_{p+q=0} \mathfrak{g}^{p,q}\right) \cap \mathfrak{g}_{\mathbb{R}}$ and (noting $\mathfrak{a}_{F^\bullet} \subset \mathfrak{g}_{\mathbb{R}}^{0,0} \subset \tilde{\mathfrak{m}}$) $\mathfrak{a} := \ker\{\mathrm{ad}\colon \mathfrak{a}_{F^\bullet} \to \mathrm{End}(\tilde{\mathfrak{m}})\}$. Let $\mathfrak{m}$ be a direct-sum complement to $\mathfrak{a}$ in $\tilde{\mathfrak{m}}$, and $\mathcal{M}, \mathcal{A}, \mathcal{N} \leqslant G_{\mathbb{R}}$ the subgroups corresponding to $\mathfrak{m}, \mathfrak{a}, \mathfrak{n}$. This gives a *Hodge-theoretically defined* Langlands decomposition

$$\mathcal{Q} = \mathcal{MAN} \tag{3.5}$$

of the parabolic. Write $\mathcal{L} := \mathcal{Q}/\mathcal{N}$, $\mathfrak{l} := \mathrm{Lie}(\mathcal{L}) = Gr_0^{\widetilde{W}}\mathfrak{g}$.

Definition 3.11. — *$\mathcal{Q}$ (resp. $\mathfrak{q}$, $F^\bullet$) is* cuspidal $\iff$ *$\mathcal{L}/Z(\mathcal{L})$ has a compact Cartan subgroup.*

Assume that $\mathcal{Q}$ is cuspidal; then it admits a Langlands decomposition in which the reductive group has a compact Cartan. We claim that (when true) this may be demonstrated Hodge-theoretically, *i.e.* via (3.5).

Proposition 3.12. — *If $\mathcal{Q}$ is cuspidal, then we can choose $\Theta_{F^\bullet}$ and H ($\Theta_{F^\bullet}$-stable) so that $\mathfrak{a} = \mathfrak{a}_{F^\bullet}$, making $\mathfrak{t}_{F^\bullet} \subset \mathfrak{m}$ a Cartan subalgebra. The choices of H which accomplish this are of minimal real rank.*

Proof. — Suppose we have $\Theta_{F^\bullet}$, H, χ, *etc.* with $\mathfrak{a} \subsetneq \mathfrak{a}_{F^\bullet}$, and note that $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}^{0,0}$. If $\mathcal{L}/Z(\mathcal{L})$ has a compact Cartan, then the image $\widehat{H}$ of H in it may be (inverse) Cayley transformed into one. This requires the presence of a real root vector $X_\beta \in \mathfrak{l}/\mathfrak{z}(\mathfrak{l})$, which can only lie in $(\mathfrak{l}/\mathfrak{z}(\mathfrak{l}))^{0,0} = \mathfrak{g}_{\mathbb{R}}^{0,0}/\mathfrak{z}(\mathfrak{l})$. The preimage H' of $\widehat{H}' := \Psi_{\mathbf{d}_\beta}(\widehat{H})$ has $\mathfrak{h}'_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}^{0,0}$ and real rank one less than H. Evidently $\mathbf{d}_\beta$ commutes with $\tilde{\varphi}_{F^\bullet}$, and so we still have $\chi \in \mathrm{X}_*(H'_{\mathbb{C}})$. Continue until the image in $\mathcal{L}/Z(\mathcal{L})$ is compact. $\square$

4. Connected real orbits and naive boundary strata

Continuing to fix $F_0^\bullet$, (H_0, χ_0), Θ, and $\tilde{\xi}_{\mathbb{R}}$, we now turn to the enumeration and analysis of the $G(\mathbb{R})^\circ$-orbits in $\check{D}$.

4.1. Basic results on orbits

Proposition 4.1. — *Given $F^\bullet = \mathscr{F}(H,\chi) \in \check{D}$ with associated bigrading* (3.2), *the real codimension of $\mathcal{O}_{F^\bullet} := G(\mathbb{R})^\circ \cdot F^\bullet \subset \check{D}$ is*

$$c_{F^\bullet} := \sum_{(p,q)\in(\mathbb{Z}_{>0})^{\times 2}} \dim_{\mathbb{C}}(\mathfrak{g}^{p,q}).$$

Proof. — Recalling $\mathfrak{q}_{F^\bullet} = \bigoplus_{(p,q)\in\mathbb{Z}_{\geqslant 0}\times\mathbb{Z}} \mathfrak{g}^{p,q}$ and $T_{F^\bullet}\check{D} \cong \mathfrak{g}_{\mathbb{C}}/\mathfrak{q}_{F^\bullet}$, we have

$$\begin{aligned}\dim_{\mathbb{R}} T_{F^\bullet}\mathcal{O} &= \dim_{\mathbb{R}} \left(\mathfrak{g}_{\mathbb{R}}/\{\mathfrak{q}_{F^\bullet} \cap \mathfrak{g}_{\mathbb{R}}\}\right) \\ &= \dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}} - \dim_{\mathbb{C}} \left(\mathfrak{q}_{F^\bullet} \cap \mathfrak{q}_{\overline{F^\bullet}}\right)\end{aligned}$$

while

$$\begin{aligned}\dim_{\mathbb{R}} T_{F^\bullet}\check{D} &= 2\dim_{\mathbb{C}}\left(\mathfrak{g}_{\mathbb{C}}/\mathfrak{q}_{F^\bullet}\right) \\ &= 2\sum_{(p,q)\in\mathbb{Z}_{<0}\times\mathbb{Z}} \dim_{\mathbb{C}} \mathfrak{g}^{p,q} \\ &= \sum_{(p,q)\in(\mathbb{Z}\smallsetminus\{0\})\times\mathbb{Z}} \dim_{\mathbb{C}} \mathfrak{g}^{p,q} \\ &= \dim_{\mathbb{C}} \mathfrak{g}_{\mathbb{C}} - \dim_{\mathbb{C}}\left(\mathfrak{q}_{F^\bullet} \cap \mathfrak{q}_{\overline{F^\bullet}}\right) + \dim_{\mathbb{C}}\left(\bigoplus_{p,q>0} \mathfrak{g}^{p,q}\right).\end{aligned}$$

□

Corollary 4.2. — *$\mathcal{O}_{F^\bullet} \subset \check{D}$ is open $\iff$ $F^\bullet \in \check{D} \setminus \mathfrak{Z}$.*

Proof. — If $F^\bullet \in \mathfrak{Z}$, then by Corollary 3.10 $c_{F^\bullet} \neq 0$; while $F^\bullet \notin \mathfrak{Z} \implies$ only the $\{\mathfrak{g}^{p,-p}\}$ are nontrivial $\implies c_{F^\bullet} = 0$. □

In Wolf's study [27] of complex flag manifolds, it is shown that $\check{D}$ contains a unique closed orbit $\mathcal{O}_{\mathrm{c}}$ (of real codimension c_{c}); this is in the closure of all the other $G(\mathbb{R})^\circ$-orbits, and is acted upon transitively by K. One has $c_{F^\bullet} < c_{\mathrm{c}}$ for any $F^\bullet \notin \mathcal{O}_{\mathrm{c}}$; and

$$c_{\mathrm{c}} \leqslant \dim_{\mathbb{C}} \check{D}, \tag{4.1}$$

with equality if and only if $\mathcal{O}_{\mathrm{c}}$ contains a *real* flag. We shall say that a flag is of *Hodge-Tate type* if its $\dim(\mathfrak{g}^{p,q})$ are zero for $p \neq q$.

Corollary 4.3.

(i) *Equality holds in* (4.1) *if and only if $\check{D}$ contains a Hodge-Tate flag, in which case $\mathcal{O}_{\mathrm{c}}$ is the set of such flags in $\check{D}$.*

(ii) *In particular, this happens whenever G is $\mathbb{R}$-split.*

Proof.

(i) Real flags ($F^\bullet = \overline{F^\bullet}$) are obviously Hodge-Tate, and the $\{\dim(\mathfrak{g}^{p,q})\}$ are constant on orbits. Conversely, if $F^\bullet$ is Hodge-Tate, then $\dim Gr^p_{F^\bullet} = \dim Gr^p_{F_0^\bullet} = \dim Gr^{-p}_{F_0^\bullet} \implies$

$$\dim \check{D} = \sum_{p>0} \dim(\mathfrak{g}^{p,-p}_{F_0^\bullet}) = \sum_{p>0} \dim(\mathfrak{g}^{p,p}_{F^\bullet}) = c_{F^\bullet}.$$

(ii) $G_{\mathbb{R}}$ split $\implies H_n$ split $\implies \mathrm{X}_*(H_n(\mathbb{C})) = \mathrm{X}_*(H_n(\mathbb{R})) \implies \mathscr{F}(H_n, \chi_n)$ is real. □

Before proceeding to the heart of the section, we can say something about the codimension-1 orbits as well:

Corollary 4.4. — *If $c_{F^\bullet} = 1$, then $\mathfrak{g}^{1,1}$ is spanned by a single real root vector, and the other $\{\mathfrak{g}^{p,q}\}$ with $p, q > 0$ are zero.*

Proof. — By Proposition 4.1, only one $\mathfrak{g}^{p_0,q_0}$ with $p_0, q_0 > 0$ can be nonzero, and since $\dim \mathfrak{g}^{q,p} = \dim \mathfrak{g}^{p,q}$ we must have $q_0 = p_0$. Now our standing bracket-generating assumption says that

$$\mathcal{W}_{F^\bullet} + [\mathcal{W}_{F^\bullet}, \mathcal{W}_{F^\bullet}] + [\mathcal{W}_{F^\bullet}, [\mathcal{W}_{F^\bullet}, \mathcal{W}_{F^\bullet}]] + \cdots = T_{F^\bullet}\check{D},$$

whilst by Frobenius

$$T_{F^\bullet}\mathcal{O}_{F^\bullet} + [T_{F^\bullet}\mathcal{O}_{F^\bullet}, T_{F^\bullet}\mathcal{O}_{F^\bullet}] + \cdots = T_{F^\bullet}\mathcal{O}_{F^\bullet},$$

and so $\mathcal{W}_{F^\bullet}/\{\mathcal{W}_{F^\bullet}\cap T_{F^\bullet}\mathcal{O}_{F^\bullet}\}\neq\{0\}$. Taking real dimensions,

$$\begin{aligned}0 &< \dim_{\mathbb{C}}(\mathfrak{g}^{-1,-1})+2\sum_{q<-1}\dim_{\mathbb{C}}(\mathfrak{g}^{-1,q})\\ &=\dim_{\mathbb{C}}(\mathfrak{g}^{1,1})+2\sum_{q>1}\dim_{\mathbb{C}}(\mathfrak{g}^{1,q})\\ &=\dim_{\mathbb{C}}(\mathfrak{g}^{1,1})\end{aligned}$$

$\implies p_0=1.$ □

4.2. Orbit inventory

Now let $Q_j\leqslant G(\mathbb{C})$ denote the parabolic stabilizing $F_j^\bullet:=\mathscr{F}(H_j,\chi_j)^\bullet$. The Weyl subgroup

$$\mathrm{W}_j:=\frac{N(Q_j,H_j(\mathbb{C}))}{H_j(\mathbb{C})}=\operatorname{Stab}\chi_j\leqslant W_{\mathbb{C}}(H_j)$$

is generated by the reflections in the roots belonging to $\ker(\xi_j)$ [14, sec. 30.1].

Consider the finite set

$$\widetilde{\Xi}^\theta_{\mathbb{R}}:=\tilde{\pi}_{\mathbb{R}}^{-1}\left(\tilde{\xi}^\theta_{\mathbb{R}}\right)=\left\{(H_j,\chi_j^{[w]})\mid j\in\{0,\dots,n\},\,w\in W_{\mathbb{C}}(H_j)\right\}$$

of distinguished θ-stable-Cartan/co-character pairs (and its subset $\Xi^\theta_{\mathbb{R}}:=\tilde{\pi}_{\mathbb{R}}^{-1}(\{H_0\})$), where $\chi_j^{[w]}(z)=\Psi_w(\chi_j(z))$. Let $\overline{\widetilde{\Xi}^\theta_{\mathbb{R}}}$ (resp. $\overline{\Xi^\theta_{\mathbb{R}}}$) be the set of equivalence classes modulo the relation

$$(H_j,\chi_j^{[w]})\sim(H_k,\chi_k^{[w']})\iff j=k\text{ and }w'\in W^\circ_{\mathbb{R}}(H_j)\cdot w\cdot\mathrm{W}_j,$$

and $\mathcal{O}^G_{\mathbb{R}}(\check{D})$ denote the set of $G(\mathbb{R})^\circ$-orbits. Writing $W^\circ_{\mathbb{R}}(H_j)\cdot w\cdot\mathrm{W}_j=:\{w\}$ for the double cosets, we introduce the orbit map

$$\begin{array}{rccccl}\mathbf{o}\colon & \overline{\widetilde{\Xi}^\theta_{\mathbb{R}}} & \longrightarrow & \mathcal{O}^G_{\mathbb{R}}(\check{D}) & \\ & (H_j,\chi_j^{\{w\}}) & \longmapsto & \mathcal{O}_{\mathscr{F}(H_j,\chi_j^{[w]})^\bullet} & =:\mathbf{o}_j^{\{w\}}.\end{array}$$

Theorem 4.5.
- (i) *$\mathbf{o}$ is well-defined and surjective, with*[7] $\mathbf{o}^{-1}(D)=\{(H_0,\chi_0^{\{\mathbf{e}\}})\}$.
- (ii) *It restricts to a bijection between $\overline{\Xi^\theta_{\mathbb{R}}}$ and the set $\mathcal{O}^G_{\mathbb{R}}(\check{D}\smallsetminus\mathfrak{Z})$ of open orbits.*
- (iii) *The codimension-one orbits are of the form $\mathbf{o}_j^{\{w\}}$ for H_j of real rank 1.*

---

(7) $\mathbf{e}$ denotes the identity element in a Weyl group; so $\chi_0^{\{\mathbf{e}\}}=\chi_0$.

Proof.

(i)–(ii): For well-definedness, $\{w\} = \{w'\} \implies \chi_j^{[w']} = \Psi_g(\chi_j^{[w]})$ for some $g \in N(G(\mathbb{R})^\circ, H_j) \implies$

$$\mathscr{F}(H_j, \chi_j^{[w']})^\bullet = \mathscr{F}\left(\Psi_g(H_j), \Psi_g(\chi_j^{[w]})\right)^\bullet = \Psi_g\left(\mathscr{F}(H_j, \chi_j^{[w]})\right)^\bullet.$$

Surjectivity is Proposition 3.8(i).

Suppose

$$\begin{aligned} F^\bullet &:= \mathscr{F}(H_0, \chi_0^{[w]})^\bullet \\ &= \mathrm{Ad}(g)\mathscr{F}(H_0, \chi_0^{[w']})^\bullet = \mathscr{F}(\Psi_g(H_0), \Psi_g(\chi_0^{[w']}))^\bullet, \end{aligned}$$

where $g \in G(\mathbb{R})^\circ$. Using Proposition 3.7, $H_0(\mathbb{R})$ and $\Psi_g(H_0(\mathbb{R}))$ are compact maximal tori of the real reductive Lie group[8]

$$\mathcal{H}_{F^\bullet} := G(\mathbb{R})^\circ \cap Q_{F^\bullet}\ (= G(\mathbb{R})^\circ \cap Q_{F^\bullet} \cap Q_{\overline{F^\bullet}}).$$

So there exists $g' \in \mathcal{H}_{F^\bullet}$ such that $H_0 = \Psi_{g'g}(H_0)$; let $w_r \in W_{\mathbb{R}}^\circ(H_0)$ be the element represented by $g'g$. Then

$$\begin{aligned} F^\bullet = \mathrm{Ad}(g')F^\bullet = \mathscr{F}\left(\Psi_{g'g}(H_0), \Psi_{g'g}(\chi_0^{[w']})\right)^\bullet &= \mathscr{F}\left(H_0, \chi_0^{[w_r w']}\right)^\bullet \\ \implies \chi_0^{[w]} &= \chi_0^{[w_r w']} \\ \implies w\cdot \mathrm{W}_0 &= w_r w' \cdot \mathrm{W}_0 \\ \implies \{w\} &= \{w_r w'\} = \{w'\}. \end{aligned}$$

This establishes (ii), and the last statement of (i) follows from this and Remark 3.6(c).

(iii): Suppose $F^\bullet = \mathscr{F}\left(H_j, \chi_j^{[w]}\right)^\bullet$ has $c_{F^\bullet} = 1$, $\mathfrak{g}^{1,1} = \mathbb{C}\langle X_\beta\rangle$, $\mathrm{rk}_{\mathbb{R}}\, H_j$ ($= \dim_{\mathbb{R}} \mathfrak{a}_j) \geqslant 2$. Then $\mathrm{Ad}(\mathbf{d}_\beta)X_\beta \in \mathbb{C}\langle X_{\beta'}\rangle$, β' a noncompact imaginary root for $H_{j'} = \Psi_{\mathbf{d}_\beta}(H_j)$. Since $\mathrm{rk}_{\mathbb{R}}\, H_{j'} \geqslant 1$, $H_{j'}$ is not maximally compact and so has a real root β_0', necessarily orthogonal to β'. Under (conjugation by) $\mathbf{c}_{\beta'}$, β_0' goes to a real root β_0 (of H_j) orthogonal to β, and the vanishing of the $\{\mathfrak{g}^{p,p}\}_{p\neq 0,\pm 1}$ forces $X_{\pm\beta} \in \mathfrak{g}^{0,0}$. Therefore, conjugation by $\mathbf{d}_{\beta_0}$ replaces H_j by $H_{j'}$ of smaller real rank, whilst leaving Y and ϕ – hence $F^\bullet$ – alone. ☐

Remark 4.6. — Koranyi, Takeuchi and Wolf parametrized the $G(\mathbb{R})$-orbits in the case where $\check{D}$ is Hermitian symmetric (*cf.* [28, sec. 7]). Theorem 4.5 extends this to the general case.[9]

Corollary 4.7.

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[8] While $F^\bullet$ is Hodge, the corresponding Hodge structure need not be $(-B)$-polarized, and so $\mathcal{H}_{F^\bullet}$ need not be compact.

[9] We thank the referee for this remark.

(i) $|\mathcal{O}^G_{\mathbb{R}}(\check{D})| \leqslant \sum_{j=0}^n |W^\circ_{\mathbb{R}}(H_j) \setminus W_{\mathbb{C}}(H_j)/\mathrm{W}_j|$ $(< \infty)$.

(ii) $|\mathcal{O}^G_{\mathbb{R}}(\check{D} \setminus \mathfrak{Z})| = |W^\circ_{\mathbb{R}}(H_0) \setminus W_{\mathbb{C}}(H_0)/\mathrm{W}_0|$.

Remark 4.8. — $W_{\mathbb{C}}(H_j) \geqslant \mathrm{W}_j$ are the same for each j, whilst $W^\circ_{\mathbb{R}}(H_j)$ varies considerably. If $G_{\mathbb{R}}$ is split ($\implies$ H_n is), then $W_{\mathbb{C}}(H_n) = W_{\mathbb{R}}(H_n) = W^\circ_{\mathbb{R}}(H_n)$ by [5, 14.6]. Note that Corollary 4.7(ii) is due to Wolf [27, Thm. 4.9(3)].

When the isotropy group $\mathcal{H}_{\varphi_0}$ is abelian, it is a compact maximal torus in $G(\mathbb{R})^\circ$, and we say that $\check{D}$ is a *complete flag variety* (owing to how this situation most often arises).

Recall our standing assumption (*cf.* §2) that π_{χ_0} takes only the values 0 and 1 on the simple roots.

Lemma 4.9. — *The following are equivalent:*

(i) *$\check{D}$ is a complete flag variety;*

(ii) *$Q_{F_0^\bullet}$ is a Borel subgroup of $G(\mathbb{C})$;*

(iii) *$Q_{F^\bullet}$ is Borel for any $F^\bullet \in \check{D}$;*

(iv) *π_{χ_0} takes the value 1 on every simple root.*

Under the equivalent conditions of the Lemma, $H_0(\mathbb{R})^\circ = \mathcal{H}_{\varphi_0}$ and $\dim_{\mathbb{C}}(H_0(\mathbb{C})) = \dim_{\mathbb{C}}(Gr^0_{F^\bullet}\mathfrak{g})$, so that for any $F^\bullet = \mathscr{F}(H_j, \chi_j^{[w]})^\bullet$, from $\mathfrak{g}^{0,0} \supseteq \mathfrak{h}_j$ we have

$$\mathfrak{g}^{0,0} = \mathfrak{h}_j. \tag{4.2}$$

Moreover, since every $Q_{F_j^\bullet}$ is Borel, the $\{\mathrm{W}_j\}$ are all trivial.

Theorem 4.10. — *In the complete flag setting, $\mathbf{o}$ is a bijection and*

$$|\mathcal{O}^G_{\mathbb{R}}(\check{D})| = \sum_{j=0}^n |W^\circ_{\mathbb{R}}(H_j) \setminus W_{\mathbb{C}}(H_j)|.$$

Proof. — Suppose

$$\begin{aligned} F^\bullet &:= \mathscr{F}(H_j, \chi_j^{[w]})^\bullet \\ &= \mathrm{Ad}(g)\mathscr{F}(H_k, \chi_k^{[w']})^\bullet = \mathscr{F}(\Psi_g(H_k), \Psi_g(\chi_k^{[w']}))^\bullet, \end{aligned}$$

where $g \in G(\mathbb{R})^\circ$, and let $\mathfrak{g} = \oplus \mathfrak{g}^{p,q}$ be the bigrading induced by $F^\bullet$ and H_j. Using Proposition 3.7, $H_j(\mathbb{R})^\circ$ and $\Psi_g(H_k(\mathbb{R})^\circ)$ are maximal tori in the identity component of

$$\mathcal{H}_{F^\bullet} := G(\mathbb{R})^\circ \cap Q_{F^\bullet}(\cap Q_{\overline{F^\bullet}}),$$

and (4.2) says that $\mathcal{H}^\circ_{F^\bullet}/U(\mathcal{H}^\circ_{F^\bullet})$ is a torus of the same dimension; that is, $\mathcal{H}^\circ_{F^\bullet}$ is (connected) solvable. By [4, Prop. 19.2], there exists $\tilde{g} \in \mathcal{H}^\circ_{F^\bullet}$ such that $\Psi_{\tilde{g}}\left(\Psi_g(H_k(\mathbb{R})^\circ)\right) = H_j(\mathbb{R})^\circ$; *i.e.* $\Psi_{\tilde{g}g}(H_k) = H_j$ with $\tilde{g}g \in G(\mathbb{R})^\circ$. But then $k = j$ (*cf.* §3.3), and $\tilde{g}g$ represents an element $w_r \in W^\circ_{\mathbb{R}}(H_j)$; we have

$$\begin{aligned} F^\bullet = \mathrm{Ad}(\tilde{g})\mathscr{F}\left(\Psi_g(H_j), \Psi_g(\chi_j^{[w']})\right)^\bullet &= \mathscr{F}\left(H_j, \chi_j^{[w_r w']}\right)^\bullet \\ \implies \chi_j^{[w]} &= \chi_j^{[w_r w']} \\ \implies w\cdot \mathrm{W}_j &= w_r w'\cdot \mathrm{W}_j \\ \implies \{w\} &= \{w_r w'\} = \{w'\}, \end{aligned}$$

done. □

Continuing to assume $\check{D}$ a complete flag variety:

Corollary 4.11. — *The "real rank map" $\check{D} \to \mathbb{Z}_{>0}$ given by*

$$F^\bullet = \mathscr{F}(H, \chi)^\bullet \mapsto \dim_{\mathbb{R}} \mathfrak{a}_{F^\bullet}$$

is well-defined.

We also recover the well-known

Corollary 4.12. — *The $|W^\circ_{\mathbb{R}}(H_0) \setminus W_{\mathbb{C}}(H_0)|$ open orbits are in 1-to-1 correspondence with Weyl chambers up to reflections in the compact roots. (Explicitly, the correspondence is given by sending $\mathbf{o}_0^{\{w\}} \mapsto w\cdot C_0$, where C_0 is the chamber associated to $Q_{F_0^\bullet}$.)*

4.3. Closure order

It remains to address how the various orbits fit together. Consider the following two operations on the finite set of points $\{\mathscr{F}(H_j, \chi_j^{[w]})^\bullet\}$ in $\check{D}$:

(1) *Cayley transforms* $\mathbf{c}_\alpha$ in noncompact imaginary roots:

$$F^\bullet = \mathscr{F}\left(H_j, \chi_j^{[w]}\right)^\bullet \longmapsto \mathscr{F}\left(\Psi_{\mathbf{c}_\alpha}(H_j), \Psi_{\mathbf{c}_\alpha}(\chi_j^{[w]})\right)^\bullet =: \mathbf{c}_\alpha F^\bullet;$$

(2) *Cross actions*, *i.e.* $c_{F^\bullet}$-increasing Weyl reflections w_γ in complex roots:

$$F^\bullet = \mathscr{F}\left(H_j, \chi_j^{[w]}\right)^\bullet \longmapsto \mathscr{F}\left(H_j, \chi_j^{[w_\gamma w]}\right)^\bullet =: w_\gamma F^\bullet.$$

There is a well-developed theory of *Bruhat order* (*i.e.* closure order[10]) for the $K_{\mathbb{C}}$-orbits on complete flag varieties, where $K_{\mathbb{C}} \leqslant G(\mathbb{C})$ is the complexification of a maximal compact subgroup of $G(\mathbb{R})^\circ$.[11] (The foundational

[10] By this we mean, in general, the partial order on orbits given by $\mathcal{O}_1 \geqslant \mathcal{O}_2 \iff \mathrm{cl}(\mathcal{O}_1) \supseteq \mathcal{O}_2$.

[11] More precisely, one takes $K_{\mathbb{C}}$ to be the identity connected component of $G(\mathbb{C})^\Theta$.

article is [22]; also see the helpful recent exposition [29].) We can import these results into our setting by way of Matsuki duality, which produces a 1-to-1 correspondence between $K_{\mathbb{C}}$- and $G(\mathbb{R})^{\circ}$-orbits in a complete flag variety[12] $\check{D}$, while reversing closure order. See [17] and the Introduction to [18].

The upshot of this for us is twofold:

(a) In the general case, where $\check{D}$ is not necessarily a complete flag variety, $\mathcal{O}_{\mathbf{c}_\alpha F^\bullet}$ and $\mathcal{O}_{w_\alpha F^\bullet}$ always lie in the analytic closure $\mathrm{cl}(\mathcal{O}_{F^\bullet}) =: \mathcal{O}_{F^\bullet} \amalg \partial\mathcal{O}_{F^\bullet}$. This may also be deduced directly from the discussion below.

(b) In the complete flag case, the codimension-one inclusions obtained as in (a) generate all closure relations in the sense of [29, Theorem 3.15] (the "subexpression property" which generates more relations than mere iteration of (a)).

Remark 4.13.

(i) We should accompany these statements with the warning that our $\mathbf{c}_\alpha$ and w_γ (which operate differently from the Cayley transform and cross-action in [29]) are not well-defined operations on the level of orbits: if $w_r \in W_{\mathbb{R}}^{\circ}$, it can happen that (while $\mathcal{O}_{w_r F^\bullet} = \mathcal{O}_{F^\bullet}$) $\mathcal{O}_{\mathbf{c}_\alpha w_r F^\bullet} \neq \mathcal{O}_{\mathbf{c}_\alpha F^\bullet}$ and so forth.

(ii) On the other hand, with $\alpha \in \Delta_n(H_j)$ and $F^\bullet$ as above, we need not worry about both $\mathbf{c}_\alpha F^\bullet$ and $\mathbf{c}_{-\alpha} F^\bullet$: if $\beta \in \Delta(\Psi_{\mathbf{c}_\alpha}(H_j))$ is $\mathbf{c}_\alpha$ of α (more precisely, $X_\beta = -i(\mathrm{Ad}\,\mathbf{c}_\alpha)X_\alpha$), then $\mathbf{d}_\beta = \mathbf{c}_\alpha^{-1}$ and $\mathbf{d}_\beta^2 = w_\beta \implies w_\beta \mathbf{c}_\alpha F^\bullet = \mathbf{c}_\alpha^{-1} F^\bullet = \mathbf{c}_{-\alpha} F^\bullet \implies \mathcal{O}_{\mathbf{c}_\alpha F^\bullet} = \mathcal{O}_{\mathbf{c}_{-\alpha} F^\bullet}$.

(iii) We can further simplify computations by noticing that if $w_r \in W_{\mathbb{R}}^{\circ}(H_j)$ and $\alpha \in \Delta_n(H_j)$, then we have $w_r(\alpha) \in \Delta_n(H_j)$ and so $\mathbf{c}_\alpha w_r = w_r w_r^{-1} \mathbf{c}_\alpha w_r = w_r \mathbf{c}_{w_r(\alpha)} \implies \mathcal{O}_{\mathbf{c}_{w_r(\alpha)} F^\bullet} = \mathcal{O}_{\mathbf{c}_\alpha w_r F^\bullet}$.

Discussion. — To see what is going on in a simple case, consider an $\mathfrak{sl}_2$-triple $X_\alpha, h_\alpha, X_{-\alpha}$ with $\alpha \in \Delta_n$, $X_{-\alpha} = \overline{X_\alpha}$, $h_\alpha = [X_\alpha, X_{-\alpha}]$ and $[h_\alpha, X_{\pm\alpha}] = \pm 2X_{\pm\alpha}$. Put

$$\mathsf{F}^1 := \mathbb{C}\langle X_\alpha \rangle \subset \mathsf{F}^0 := \mathbb{C}\langle X_\alpha \rangle_B^{\perp} = \mathbb{C}\langle X_\alpha, h_\alpha \rangle \subset \mathfrak{sl}_{2,\mathbb{C}}.$$

Writing

$$\gamma_t := \exp\left\{ \frac{t}{2}(X_{-\alpha} - X_\alpha) \right\} \in SL_2(\mathbb{C}),$$

[12] In fact, this duality holds in the general case, *cf.* [23, sec. 6.6].

we have

$$\begin{aligned} h(t) &:= \mathrm{Ad}(\gamma_t)h_\alpha = (\cos t)h_\alpha + (\sin t)(X_\alpha + X_{-\alpha}), \\ X_\pm(t) &:= \mathrm{Ad}(\gamma_t)X_{\pm\alpha} \\ &= -\frac{1}{2}(\sin t)h_\alpha + \frac{1}{2}(\cos t + 1)X_{\pm\alpha} + \frac{1}{2}(\cos t - 1)X_{\mp\alpha}. \end{aligned}$$

In particular, this gives

$$\begin{aligned} h(\frac{\pi}{2}) &= \mathrm{Ad}(\mathbf{c}_\alpha)h_\alpha = X_\alpha + X_{-\alpha}, \\ X_\pm(\frac{\pi}{2}) &= \mathrm{Ad}(\mathbf{c}_\alpha)X_{\pm\alpha} = -\frac{1}{2}h_\alpha + \frac{1}{2}X_{\pm\alpha} - \frac{1}{2}X_{\mp\alpha} \end{aligned}$$

and

$$\begin{aligned} h(\pi) &= \mathrm{Ad}(w_\alpha)h_\alpha = -h_\alpha, \\ X_\pm(\pi) &= \mathrm{Ad}(w_\alpha)X_{\pm\alpha} = X_{\mp\alpha}. \end{aligned}$$

The *flag* $\mathsf{F}^1(t) := \mathbb{C}\langle X_+(t)\rangle \subset \mathsf{F}^0(t) := \mathbb{C}\langle X_+(t)\rangle_B^\perp$ is in fact in the *real* (*i.e.* $SL_2(\mathbb{R})$-)orbit of $\mathsf{F}^\bullet$ for $t \in [0, \frac{\pi}{2})$; explicitly, we have $\mathsf{F}^\bullet(t) = \mathrm{Ad}(\mu_t)\mathsf{F}^\bullet$, where $\mu_t = \mathrm{diag}\{f(t)^{\frac{1}{2}}, f(t)^{-\frac{1}{2}}\} \in SL_2(\mathbb{R})$ and $f(t) = \frac{1+\sin t}{\cos t}$. The problem at $t = \frac{\pi}{2}$ (where $\mathsf{F}^\bullet(\frac{\pi}{2}) = \mathbf{c}_\alpha\mathsf{F}^\bullet$) is that $X_+(t)$ becomes pure imaginary,[(13)] with *real span*, and the real group $SL_2(\mathbb{R})$ cannot take a non-real line to a real one. The comparable result in the general case follows from this one since the flag $F^\bullet$ on $\mathfrak{g}$ along an SL_2-orbit is determined by the restriction of $F^\bullet$ to the $\mathfrak{sl}_2$. For cross-actions, the analysis is similar except the corresponding "non-real to real" problem occurs at $t = \pi$.

The most interesting general statement (*not* assuming $\check{D}$ is a complete flag variety) we can make beyond Remark 4.13 and (a)-(b) above it, is that Cayley transforms give all the codimension-one orbits in the closure of an open orbit:

Proposition 4.14. — *Let $\mathbf{o}_0^{\{w\}}$ be any open orbit in $\check{D}$ and $\mathbf{o}_j^{\{w'\}}$ an orbit of codimension* 1*, where we may take H_j of real rank* 1 *(cf. Theorem 4.5(iii)). Then $\mathbf{o}_j^{\{w'\}} \subset \partial\mathbf{o}_0^{\{w\}} \iff (H_j, \chi_j^{[w']}) = \big(\Psi_{\mathbf{c}_\alpha}(H_0), \Psi_{\mathbf{c}_\alpha}(\chi_0^{[w_0]})\big)$ for some $\alpha \in \Delta_n$, $w_0 \in \{w\}$.*

---

(13) If one puts $X_{\pm\alpha} = \frac{1}{2}\left(\begin{smallmatrix} 1 & \mp i \\ \mp i & -1 \end{smallmatrix}\right)$, $h_\alpha = \left(\begin{smallmatrix} 0 & i \\ -i & 0 \end{smallmatrix}\right)$, then $(\mathrm{Ad}\,\mathbf{c}_\alpha)X_\alpha = -iX_\beta = -i\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. For $X_\alpha \in \mathfrak{g}^{1,-1}_{F^\bullet}$, $X_\beta \in \mathfrak{g}^{1,1}_{\mathbf{c}_\alpha F^\bullet}$. The Hodge-theoretically minded reader will no doubt think that $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ should be in $\mathfrak{g}^{-1,-1}$, since $N = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ for the corresponding VHS. This is resolved by the effect of the naive limit map in §5 below, which roughly "flips" indices $(p,q) \mapsto (-q,-p)$.

Proof. — Since "$\Longleftarrow$" is clear from the preceding discussion, we prove the converse, using the elementary observation that any codimension-1 $G(\mathbb{R})^{\circ}$-orbit can bound on at most 2 open $G(\mathbb{R})^{\circ}$-orbits.

By assumption we have $\Delta_{\mathbb{R}}(H_j) = \{\beta, -\beta\}$, so that $\mathbf{d}_\beta$ sends H_j to H_0 (the only real-rank 0 Cartan in $\tilde{\xi}^{\theta}_{\mathbb{R}}$), and β to $\alpha \in \Delta_n(H_0)$, $F^\bullet = \mathscr{F}(H_j, \chi_j^{[w']})^\bullet$ to $\mathscr{F}(H_0, \chi_0^{[w_1]})^\bullet$ (where $\chi_0^{[w_1]} = \mathbf{d}_\beta(\chi_j^{[w']})$). Since $\mathbf{c}_\alpha$ reverses the operation (see the end of §2), it gives $\partial \mathbf{o}_0^{\{w_1\}} \supset \mathbf{o}_j^{\{w'\}}$. Setting $\gamma_\alpha := \frac{\pi}{4}(X_{-\alpha} - X_\alpha)$, the Discussion above implies that $e^{t\gamma_\alpha}\mathscr{F}(H_0, \chi_0^{[w_1]}) \in \mathbf{o}_0^{\{w_1\}}$ for $t \in [0,1)$. Hence for $\epsilon \in (0,1]$, and setting $\delta_\beta := -i\frac{\pi}{4}(X_{-\beta} + X_\beta)$, we have $e^{\epsilon\delta_\beta}F^\bullet \in \mathbf{o}_0^{\{w_1\}}$.

Writing $w_\alpha \in W_{\mathbb{C}}(H_0)$ for the element induced by $\mathbf{c}_\alpha^2$, we have $\mathbf{d}_\beta^{-1} = \mathbf{c}_\alpha^2 \mathbf{d}_\beta$ hence

$$\mathbf{d}_\beta^{-1}\mathscr{F}(H_j, \chi_j^{[w']})^\bullet = \mathscr{F}(H_0, \chi_0^{[w_\alpha w_1]}).$$

Since $\mathbf{c}_{-\alpha} = \mathbf{c}_\alpha^{-1}$ is inverse to $\mathbf{d}_\beta^{-1}$, $\partial\mathbf{o}_0^{\{w_\alpha w_1\}} \supset \mathbf{o}_j^{\{w'\}}$ is given by $\mathbf{c}_{-\alpha}$. By the same argument as above, it follows that $e^{-\epsilon\delta_\beta}F^\bullet \in \mathbf{o}_0^{\{w_\alpha w_1\}}$ for $\epsilon \in (0,1]$.

As the projection of $\delta_\beta \in \mathfrak{g}^{1,1} \oplus \mathfrak{g}^{-1,-1}$ to $T_{F^\bullet}\check{D}$ is transverse to $T_{F^\bullet}\mathbf{o}_j^{\{w'\}}$, the conclusions of the previous two paragraphs establish that $\mathbf{o}_0^{\{w_1\}}$ and $\mathbf{o}_0^{\{w_\alpha w_1\}}$ are the (only) open orbits bounding on $\mathbf{o}_j^{\{w'\}}$. Now apply Theorem 4.5(ii). □

Definition 4.15. — *The* (naive) boundary strata *of the Mumford-Tate domain D are the $G(\mathbb{R})^{\circ}$-orbits in $\partial D \subset \check{D}$.*

Corollary 4.16.

(i) *To obtain (representatives of) all codimension-1 boundary strata, it suffices to consider (modulo equivalence) those $F^\bullet = \mathbf{c}_\alpha\mathscr{F}(H_0, \chi_0^{[w_r]})^\bullet$, $\alpha \in \Delta_n(H_0)$ and $w_r \in W_{\mathbb{R}}^{\circ}$, with $c_{F^\bullet} = 1$.*

(ii) *In the complete flag case, we have $c_{F^\bullet} = 1$ in (i) $\iff$ α is orthogonal to a wall of $w_r \cdot C_0$. The resulting codimension-1 stratum separates D from the open orbit corresponding to $w_\alpha w_r \cdot C_0$, the Weyl chamber "across the wall" from $w_r \cdot C_0$.*

Proof.

(i) is clear from Proposition 4.14.

For (ii), it suffices to consider the case $w_r = 1$. By Remark 4.13(ii), we may assume $\alpha \in \Delta_n^+$; $\mathbf{c}_\alpha$ transforms $H_0 \mapsto H_j$, α to $\beta \in \Delta_{\mathbb{R}}(H_j) = \{\beta, -\beta\}$, and $F_0^\bullet$ to $F^\bullet := \mathscr{F}(H_j, \chi := \Psi_{\mathbf{c}_\alpha}(\chi_0))^\bullet$ with associated Y and ϕ. Now,

$$\alpha \text{ is } \perp \text{ to a wall of } C_0 \iff$$

$$\alpha \text{ is simple for } \Delta^+ \iff$$

$$\begin{cases} \beta \text{ is simple for the system } \Delta^+(\chi) := \mathbf{c}_\alpha(\Delta^+) \\ = \pi_\chi^{-1}(\mathbb{Z}_{>0}) \cap \Delta(H_j) \text{ of positive roots.} \end{cases} \tag{4.3}$$

Since π_χ of simple roots is 1 by Lemma 4.9, and therefore exceeds 1 for any other positive root, (4.3) is equivalent to $\pi_\chi(\beta) = 1$.

If $c_{F^\bullet} = 1$, then by Corollary 4.4 $\mathfrak{g}^{1,1} = \mathbb{C}\langle X_\beta \rangle$, and so $\pi_\chi(\beta) = 1$. Conversely, assume (4.3) and consider $\gamma \in \Delta \smallsetminus \{\beta\}$ with $X_\gamma \in F^1$ ($\implies \gamma \in \Delta^+(\chi)$). Then γ is of the form $\gamma = r\beta + i\delta$, with $r \in \mathbb{R}$ and δ in some open half-plane $\mathbb{H}_j \subset \mathfrak{t}_j$, and $\bar{\gamma} = r\beta + i(-\delta) \implies \overline{X_\gamma} \notin F^1 \implies X_\gamma \notin \overline{F^1}$. So no root vectors besides X_β lie in $F^1 \cap \overline{F^1}$, and $c_{F^\bullet} = 1$.

The last statement follows from $\mathbf{c}_\alpha^2 = w_\alpha$. □

Remark 4.17.

(i) In the situation of (the proof of) Corollary 4.16, $\dim \mathfrak{a}_j = 1$, $Y \in \mathfrak{a}_j$, and $(\operatorname{ad} Y)X_\beta = 2X_\beta$ force $\{X_\beta, Y, X_{-\beta}\}$ to be an $\mathfrak{sl}_{2,\mathbb{R}}$-triple.

(ii) We can reduce the computation of $c_{F^\bullet}$ to pictures by computing the $\dim(\mathfrak{g}^{p,q})$, but the following can be faster. Let $\Delta_+(\chi_j^{\{w\}}) \subset \Delta(H_j)$ be the roots *positively graded* by $\chi_j^{\{w\}}$. (This is an actual system of positive roots if and only if $\check{D}$ is a complete flag variety.) Then

$$c_{F^\bullet} = \left| \Delta_+(\chi_j^{\{w\}}) \cap \overline{\Delta_+(\chi_j^{\{w\}})} \right| .$$

5. Boundary components and the naive limit map

The reader may have noticed the formal similarity between the $\mathbb{R}$-mixed Hodge structures associated to flags in $\check{D}$ (*cf.* §3) and limiting mixed Hodge structures. In this section, we shall elaborate on that relationship by determining precisely when a naive boundary stratum $\mathcal{O} \subset \partial D$ contains a flag $F^\bullet$ in the "naive" limit of a polarized variation of Hodge structure into $\Gamma \smallsetminus D$, $\Gamma \leqslant G(\mathbb{Q})$ a discrete group.

5.1. Limiting filtrations

For simplicity, let

$$\Phi \colon \Delta^* \to \langle T \rangle \smallsetminus D$$

be the period map associated to a PVHS over the punctured unit disk,[14] with unipotent monodromy $T \in G(\mathbb{Q})$, and $N := \log(T) \in \mathfrak{g}_{\mathbb{Q}}$. We can take the limit of Φ at the origin in two different ways:

(1) Choosing a local parameter q on Δ^* (and thus $\tau := \ell(q) := \frac{\log(q)}{2\pi i}$ on $\mathfrak{H}$)
$$\Psi := e^{-\tau N}\Phi\colon \Delta^* \to \check{D}$$
is well-defined and extends across the origin by the Nilpotent Orbit Theorem of [25].[15] Define the *limiting Hodge flag*
$$\widetilde{\mathscr{F}}_{\lim}(\Phi) := \Psi(0) \in \check{D},$$
where the tilde is a reminder of the dependence on q.

(2) Choosing a lift $\widetilde{\Phi}\colon \mathfrak{H} \to D$ of Φ, we define the *naive limiting flag* by
$$\widehat{\mathscr{F}}_{\lim}(\Phi) := \lim_{\Im(\tau)\to\infty} \widetilde{\Phi}(\tau) \in \mathrm{cl}(D),$$
where the limit is taken whilst confining $\Re(\tau)$ to an arbitrary compact interval. As we shall see, it depends only on Φ.

Remark that in (1), transversality forces $\widetilde{\mathscr{F}}_{\lim}(\Phi)^{-1} \ni N$ in the limit.

Write $\widetilde{F}^\bullet := \widetilde{\mathscr{F}}_{\lim}(\Phi)^\bullet$, and let $\widetilde{W}_\bullet := W(N)_\bullet$ denote the unique filtration on $\mathfrak{g}_{\mathbb{Q}}$ satisfying

(i) $N(\widetilde{W}_\ell \mathfrak{g}_{\mathbb{Q}}) \subset \widetilde{W}_{\ell-2}\mathfrak{g}_{\mathbb{Q}}$ $(\forall \ell)$

(ii) $N^k\colon Gr_k^{\widetilde{W}}\mathfrak{g}_{\mathbb{Q}} \to Gr_{-k}^{\widetilde{W}}\mathfrak{g}_{\mathbb{Q}}$vis an isomorphism $(\forall k \geqslant 0)$.

Then by the SL_2-orbit theorem of [25], $\psi_q\Phi := (\widetilde{F}^\bullet, \widetilde{W}_\bullet)$ is a $\mathbb{Q}$-MHS on $\mathfrak{g}$, called the limiting mixed Hodge structure of Φ (with respect to the parameter q). Let
$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{(p,q)\in\mathbb{Z}^2} \tilde{\mathfrak{g}}_0^{p,q}$$
be the unique (Deligne) bigrading[16] such that

(a) $\widetilde{F}^a\mathfrak{g}_{\mathbb{C}} = \bigoplus_{p\geqslant a;\, q\in\mathbb{Z}} \tilde{\mathfrak{g}}_0^{p,q}$,

(b) $\widetilde{W}_b\mathfrak{g}_{\mathbb{C}} = \bigoplus_{p+q\leqslant b} \tilde{\mathfrak{g}}_0^{p,q}$,

(c) $\overline{\tilde{\mathfrak{g}}_0^{b,a}} \equiv \tilde{\mathfrak{g}}_0^{a,b} \mod \bigoplus_{p<a;\, q<b} \tilde{\mathfrak{g}}_0^{p,q}$,

---

(14) or more generally over any $\Delta_\epsilon^* := \{z \in \Delta^* \,|\, |z| < \epsilon\}$.

(15) Technically, one chooses also a lift $\widetilde{\Phi}$ to define Ψ, but here this is absorbed by the choice of q. If we start with a period map into $\Gamma \setminus D$, the lift becomes essential.

(16) More canonically, the notation is $\tilde{\mathfrak{g}}_0^{p,q} := I^{p,q}_{(\widetilde{F}^\bullet, \widetilde{W}_\bullet)}\mathfrak{g}$, *cf.* for example [20].

with *equality* in (c) if and only if $\psi_q\Phi$ is $\mathbb{R}$-split.

Now we clearly have $N \in \left(\widetilde{F}^{-1} \cap \overline{\widetilde{F}^{-1}} \cap \widetilde{W}_{-2}\right)_{\mathbb{R}} \subset \tilde{\mathfrak{g}}_{0,\mathbb{R}}^{-1,-1}$. There also exists a unique element $\delta \in \left(\bigoplus_{(p,q)\in(\mathbb{Z}_{<0})^{\times 2}} \tilde{\mathfrak{g}}_0^{p,q}\right)_{\mathbb{R}}$ (commuting with N) and a holomorphic map $\Gamma\colon \Delta \to \bigoplus_{p<0;\, q\in\mathbb{Z}} \tilde{\mathfrak{g}}_0^{p,q}$ (with $\Gamma(0) = 0$) such that, putting $\widetilde{F}^\bullet_{\mathbb{R}} := e^{-i\delta}\widetilde{F}^\bullet$, $(\widetilde{F}^\bullet_{\mathbb{R}}, \widetilde{W}_\bullet)$ is $\mathbb{R}$-split and $\widetilde{\Phi}(\tau) = e^{\tau N}e^{\Gamma(q)}\widetilde{F}^\bullet$. Writing $\mathfrak{g}_{\mathbb{C}} = \bigoplus_{(p,q)\in\mathbb{Z}^2} \tilde{\mathfrak{g}}^{p,q}$ for the bigrading associated to $\left(\widetilde{F}^\bullet_{\mathbb{R}}, \widetilde{W}_\bullet\right)$, we remark that $\widetilde{F}^\bullet_{\mathbb{R}}$ does not depend on the choice of q, while δ is still in $\bigoplus_{(p,q)\in(\mathbb{Z}_{<0})^{\times 2}} \tilde{\mathfrak{g}}^{p,q}$ and $N \in \tilde{\mathfrak{g}}^{-1,-1}$. Moreover, the element $\widetilde{Y} \in \mathrm{End}(\mathfrak{g}_{\mathbb{R}})$ defined by $\mathrm{ad}(\widetilde{Y})|_{\tilde{\mathfrak{g}}^{p,q}} = (p+q)\,\mathrm{id}_{\tilde{\mathfrak{g}}^{p,q}}$ $(\forall p,q)$ belongs to $\tilde{\mathfrak{g}}_{\mathbb{R}}^{0,0}$ (see the proof of Lemma 3.2 in [15], or below) and there is a unique $N_+ \in \tilde{\mathfrak{g}}_{\mathbb{R}}^{1,1}$ completing $(N, \widetilde{Y})$ to an $\mathfrak{sl}_2$-triple. One consequence of this is that

$$W(N_+)_{-k} = \bigoplus_{p+q\geqslant k} \tilde{\mathfrak{g}}^{p,q}. \tag{5.1}$$

Computing

$$\begin{aligned}\widetilde{\Phi}(\tau) &= e^{\tau N}e^{\Gamma(q)}e^{i\delta}\widetilde{F}^\bullet_{\mathbb{R}}\\ &= e^{\tau N}e^{\Gamma(q)}e^{-\tau N}e^{\tau N}e^{i\delta}\widetilde{F}^\bullet_{\mathbb{R}}\\ &= e^{\mathrm{Ad}(e^{\tau N})\Gamma(q)}e^{i\delta}e^{\tau N}\widetilde{F}^\bullet_{\mathbb{R}},\end{aligned}$$

we note that by [9, p. 478]

$$e^{\tau N}\widetilde{F}^\bullet_{\mathbb{R}} = e^{\frac{1}{\tau}N_+}\widehat{F}^\bullet, \tag{5.2}$$

where

$$\widehat{F}^a := \bigoplus_{p\in\mathbb{Z};\, q\leqslant -a} \tilde{\mathfrak{g}}^{p,q}. \tag{5.3}$$

So for the naive limit we have (for some $n \in \mathbb{N}$)

$$\begin{aligned}\widehat{\mathscr{F}}_{\lim}(\Phi) &= \lim_{\Im(\tau)\to\infty} e^{q\ell^n(q)\mathcal{O}(1)}e^{i\delta}e^{\frac{1}{\tau}N_+}\widehat{F}^\bullet\\ &= e^{i\delta}\widehat{F}^\bullet\\ &= \widehat{F}^\bullet,\end{aligned}$$

since $\delta \in \widehat{F}^1$. We conclude from this that the naive limit can be determined from the limiting Hodge flag, but is independent of q; in fact, it only depends on the SL_2*-orbit* $e^{\tau N}\widetilde{F}^\bullet_{\mathbb{R}}$ canonically associated to Φ, which shares its naive limit. Therefore it will suffice to restrict our investigation of which boundary strata contain a naive limit flag to limits of SL_2-orbits. The following definitions will serve to formalize these observations.

Definition 5.1. — *Given a nilpotent element $N \in \mathfrak{g}_{\mathbb{Q}}$, let $\widetilde{B}(N)$ [resp. $\widetilde{B}_{\mathbb{R}}(N)$] be (a choice of connected component*[17] *of) the subset of $\check{D}$ consisting of flags $\widetilde{F}^\bullet$ such that $e^{\tau N}\widetilde{F}^\bullet$ is a nilpotent [resp. SL_2-]orbit: that is,*

(a) $e^{\tau N}\widetilde{F}^\bullet \in D$ *for* $\Im(\tau) \gg 0$,

(b) $N\widetilde{F}^j \subset \widetilde{F}^{j-1}$ $(\forall j)$,

(c) $\left(\widetilde{F}^\bullet, W(N)_\bullet\right)$ *is $\mathbb{R}$-split.*

The (Hodge-theoretic, rational) boundary component *associated to N is*

$$B(N) := \mathrm{Ad}(e^{\mathbb{C}\langle N\rangle}) \diagdown \widetilde{B}(N),$$

with $\mathbb{R}$-split locus $B_{\mathbb{R}}(N) := \mathrm{Ad}(e^{\mathbb{R}\langle N\rangle}) \diagdown \widetilde{B}_{\mathbb{R}}(N)$.

Let N be such that $\widetilde{B}(N) \neq \emptyset$. Given $\widetilde{F}^\bullet \in \widetilde{B}(N)$, $e^{\tau N}\widetilde{F}^\bullet$ may be regarded as a period map $\Phi_{(\widetilde{F}^\bullet,N)} \colon \Delta^*_\epsilon \to \langle e^N\rangle \diagdown D$ with LMHS $\psi_q\Phi_{(\widetilde{F}^\bullet,N)} = \left(\widetilde{F}^\bullet, W(N)_\bullet\right)$. Clearly, we may regard $\widetilde{B}(N)$ as the set of possible LMHS for period maps with local monodromy e^N.

Definition 5.2. — *The* naive limit map

$$\begin{array}{cccc} \mathscr{F}^N_{\lim} \colon & \widetilde{B}(N) & \longrightarrow & \mathrm{cl}(D) \\ & \widetilde{F}^\bullet & \longmapsto & \widehat{\mathscr{F}}_{\lim}\left(\Phi_{(\widetilde{F}^\bullet,N)}\right) \end{array}$$

sends nilpotent orbits to their naive limit flags.

Now, it is clear that $\mathscr{F}^N_{\lim}$ factors through $\widetilde{B}_{\mathbb{R}}(N)$ (and $B(N)$, hence $B_{\mathbb{R}}(N)$): we have a diagram

$$\begin{tikzcd}
\widetilde{B}(N) \arrow[dr, two heads, "\sigma_{\mathbb{R}}"'] \arrow[drr, two heads, "\mathscr{F}^N_{\lim}"] & & & \\
\widetilde{B}_{\mathbb{R}}(N) \arrow[u, hook] \arrow[r, equal] & \widetilde{B}_{\mathbb{R}}(N) \arrow[r, two heads, "\mathscr{F}^N_{\lim}"'] & \widehat{B}(N) \arrow[r, hook] & \mathrm{cl}(D),
\end{tikzcd}$$

where $\sigma_{\mathbb{R}}$ is the canonical splitting described above, and

$$\widehat{B}(N) := \mathscr{F}^N_{\lim}\left(\widetilde{B}(N)\right) = \mathscr{F}^N_{\lim}\left(\widetilde{B}_{\mathbb{R}}(N)\right).$$

Proposition 5.3. — *A naive boundary stratum $\mathcal{O}$ contains a naive limit of a VHS if and only if $\mathcal{O}$ contains a $\widehat{B}(N)$.*

There are several important remarks concerning the naive limit map $\mathscr{F}^N_{\lim}$. First, we have the perhaps surprising

[17] The excellent reasons for making such a choice are described in the Introduction to [15].

THEOREM 5.4. — *Viewed as a mapping from $\widetilde{B}(N)$ to $\check{D}$, $\mathscr{F}^N_{\lim}$ is holomorphic.*

Proof. — Referring to (5.3) and the discussion preceding it, we have

$$\begin{aligned}\widehat{F}^a &= \bigoplus_{q\leqslant -a} I^{p,q}_{\left(e^{-i\delta}\widetilde{F}^\bullet,\widetilde{W}_\bullet\right)} = e^{-i\delta}\bigoplus_{q\leqslant -a} I^{p,q}_{\left(\widetilde{F}^\bullet,\widetilde{W}_\bullet\right)}\\ &= \bigoplus_{q\leqslant -a} I^{p,q}_{\left(\widetilde{F}^\bullet,\widetilde{W}_\bullet\right)} = \sum_{\ell\in\mathbb{Z}} \widetilde{W}_\ell \cap \widetilde{F}^{\ell+a}.\end{aligned}$$

(See the appendix of [20] regarding the last step.) So $\widehat{F}^\bullet$ depends holomorphically on $\widetilde{F}^\bullet$. □

Next, we may regard $\mathscr{F}^N_{\lim}$ as sending a ($\mathbb{Q}$-)LMHS $\left(\widetilde{F}^\bullet, W(N)_\bullet\right)$ to the $\mathbb{R}$-MHS $\left(\widehat{F}^\bullet, W(N_+)_\bullet\right)$, where $\widehat{F}^\bullet = \mathscr{F}^N_{\lim}(\widetilde{F}^\bullet)$ and N_+ is as in the argument above. By (5.1) and (5.3), this has Deligne bigrading

$$\hat{\mathfrak{g}}^{p,q} := \tilde{\mathfrak{g}}^{-q,-p}.$$

In other words, viewing MHS in terms of their bigradings, on the $\mathbb{R}$-split locus $\widetilde{B}_{\mathbb{R}}(N)$ the naive limit map *is nothing but the reflection about the antidiagonal.* As an easy consequence, the upside-down "weight" filtration (3.1) attached to $\widehat{F}^\bullet \in \check{D}$ is completely determined by N:

$$\sum_{p\in\mathbb{Z}} \widehat{F}^p \cap \overline{\widehat{F}^{j-p}} = W(N)_{-j}\ (= \widetilde{W}_{-j}).$$

Furthermore, since Hodge tensors remain Hodge in the limit, the mixed Hodge representation $\tilde{\varphi}_{\widetilde{F}^\bullet}(w,z)$ attached to the bigrading factors through $G(\mathbb{R})$, forcing the associated $i\tilde{\phi}$ and $\widetilde{Y}$ into $\tilde{\mathfrak{g}}^{0,0}_{\mathbb{R}}$. For $\widetilde{F}^\bullet \in \widetilde{B}_{\mathbb{R}}(N)$, the "flip" merely sends these to $i\hat{\phi} := i\tilde{\phi}$ and $\widehat{Y} := -\widetilde{Y}$.[18] Taking a Cartan subalgebra $\mathfrak{h} \ni \widetilde{Y}, \tilde{\phi}$, $\mathfrak{h}$ lives in $\tilde{\mathfrak{g}}^{0,0}$, whereupon the entirety of Lemma 3.2 holds with $\mathfrak{g}^{p,q} := \hat{\mathfrak{g}}^{p,q}$, $F^\bullet := \widehat{F}^\bullet$, and $\widetilde{W}_\bullet := W(N)_\bullet$. So the LMHS provides Cartan/co-character data for the naive limit flag; in particular:

PROPOSITION 5.5. —*In the complete flag case, and more generally whenever* $\dim_{\mathbb{C}} \mathfrak{g}^{0,0} = \operatorname{rank}(G_{\mathbb{C}})$, $\mathscr{F}^N_{\lim}|_{\widetilde{B}_{\mathbb{R}}(N)}$ *factors unambiguously through* $\widetilde{\Xi}_{\mathbb{R}}$.

Finally, we observe that it is possible to make $\mathscr{F}_{\lim}$ even more "symmetric", by extending Definition 5.1 to the setting of $\mathbb{R}$-nilpotent orbits, *i.e.* where $N \in \mathfrak{g}_{\mathbb{R}}$. The resulting *real boundary components* $\widetilde{B}(N) \supset$

(18) From this perspective, the "loss of extension-class information" we shall describe later seems rather surprising, but has the heuristic explanation that "more Hodge tensors reside at the bottom of $\mathfrak{sl}_2$-chains than at the top": flipping them to the top annihilates some extensions.

$\widetilde{B}_{\mathbb{R}}(N)$ can now only be regarded as parametrizing $\mathbb{R}$-LMHS $\left(\widetilde{F}^{\bullet}, W(N)_{\bullet}\right)$ (with the attendant *much* coarser equivalence relation[19]), an apparent weakness. On the other hand, a short computation with formula (5.2) shows that if we let $G(\mathbb{R})^{\circ}$ act on everything in sight $((N, Y, N_+), \widetilde{F}^{\bullet}_{\mathbb{R}}, \widehat{F}^{\bullet}, \tilde{\mathfrak{g}}^{p,q}$, *etc.*) then the naive limit becomes a $G(\mathbb{R})^{\circ}$-equivariant map

$$\mathscr{F}^{\mathcal{N}}_{\lim}\colon \bigcup_{N\in\mathcal{N}} \widetilde{B}_{\mathbb{R}}(N) \longrightarrow \mathrm{cl}(D), \tag{5.4}$$

where $\mathcal{N}$ is any *nilpotent orbit* (that is, the $G(\mathbb{R})^{\circ}$-orbit of a nilpotent element[20]) in $\mathfrak{g}_{\mathbb{R}}$. The image of (5.4) is obviously a boundary stratum, which we shall denote by $\widehat{\mathcal{B}}(\mathcal{N})$. In this sense, if one flag in a stratum is a naive limit, they all are. On the other hand, there can exist strata of the form $\widehat{\mathcal{B}}(\mathcal{N})$ that do not contain a $\widehat{B}(N)$ for $N \in \mathfrak{g}_{\mathbb{Q}}$ (*cf.* §6.2.3), essentially because there can exist nilpotent orbits with no rational points.

Remark 5.6. — We have chosen for simplicity to suppress rational nilpotent (simplicial) cones $\sigma = \mathbb{R}_{\geqslant 0}\langle N_1, \dots, N_r\rangle \subset \mathfrak{g}_{\mathbb{R}}$ of rank $r > 1$ and their corresponding boundary components $\widetilde{B}(\sigma)$ (for which we refer to [15, sec. 5]). In fact, given $\widetilde{F}^{\bullet} \in \widetilde{B}(\sigma)$, $\lim_{\Im(\tau)\to\infty} e^{\tau N}\widetilde{F}^{\bullet}$ is independent of the choice of $N \in \sigma^{\circ}$ (interior of σ), and this produces a holomorphic map $\mathscr{F}^{\sigma}_{\lim}\colon B(\sigma) \to \check{D}$, which image $\widehat{B}(\sigma)$. (Here $B(\sigma) := e^{\langle\sigma\rangle_{\mathbb{C}}} \backslash \widetilde{B}(\sigma)$ is the set of *σ-nilpotent orbits*, where $\langle\sigma\rangle_{\mathbb{C}}$ denotes the complex vector space spanned by σ.) In general we have $\widetilde{B}(\sigma) = \cap_{N\in\sigma^{\circ}} \widetilde{B}(N)$ hence $\widehat{B}(\sigma) = \cap_{N\in\sigma^{\circ}} \widehat{B}(N)$.

There is however a special case which is important for Theorem 5.21 below: that of D Hermitian, with $\mathcal{W} = TD$. Given σ (with nonempty $\widetilde{B}(\sigma)$) and $\widetilde{F}^{\bullet} \in \widetilde{B}(\sigma)$, let $\mathcal{Q} \leqslant G(\mathbb{R})$ be the parabolic subgroup with $\mathrm{Lie}(\mathcal{Q}) = W(\sigma)_0\mathfrak{g}_{\mathbb{R}}$, and $\{\tilde{\mathfrak{g}}^{p,q}\}_{-1\leqslant p,q\leqslant 1}$ the bigrading of $\mathfrak{g}_{\mathbb{C}}$ attached to $(\widetilde{F}^{\bullet}, W(\sigma)_{\bullet})$. Then the description of $\widetilde{B}(\sigma)$ in [15, sec. 7] leads at once to $\widetilde{B}(\sigma) = e^{\tilde{\mathfrak{g}}^{-1,-1}}\mathcal{Q}\cdot\widetilde{F}^{\bullet} = \widetilde{B}(N)$, and hence also $\widehat{B}(\sigma) = \widehat{B}(N)$, for any $N \in \sigma^{\circ}$.

5.2. Main results

Returning to the question motivating this section, we wish to determine when a naive boundary stratum contains a $\widehat{B}(N)$ (or more generally, is a $\widehat{\mathcal{B}}(\mathcal{N})$). The key is given by the following two definitions, which concern the situation

$$F^{\bullet} = \mathscr{F}(H,\chi)^{\bullet} \in \mathcal{O} \subset \mathrm{cl}(D)$$

[19] Obviously, we are not going modulo this relation, or $\widetilde{B}_{\mathbb{R}}(N)$ would reduce to a point.
[20] This is the usual meaning of the term in Lie theory (as opposed to Hodge theory).

together with its associated

- bigrading (*cf.* Lemma 3.2)

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{p,q} \mathfrak{g}^{p,q}, \quad \overline{\mathfrak{g}^{p,q}} = \mathfrak{g}^{q,p}, \tag{5.5}$$

- filtration

$$\widetilde{W}_{-j}\mathfrak{g}_{\mathbb{C}} := \sum_{p\in\mathbb{Z}} F^p \cap \overline{F^{j-p}} = \bigoplus_{p+q\geqslant j} \mathfrak{g}^{p,q}$$

  defined over $\mathbb{R}$, and
- $\mathbb{R}$-parabolic subalgebra $\mathfrak{q} := \widetilde{W}_0\mathfrak{g}$.

Definition 5.7. — *$\mathcal{O}$ is* rational $\iff$ *$\widetilde{W}_\bullet$ is $G(\mathbb{R})^\circ$-conjugate to a filtration defined over $\mathbb{Q}$.*

Remark 5.8. — In Definition 5.7, it suffices to assume $\mathfrak{q}$ is $G(\mathbb{R})^\circ$-conjugate to a $\mathbb{Q}$-parabolic, provided $\mathfrak{g}_1 := \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}^{p,1-p}$ bracket-generates $\widetilde{W}_{-1}$. (This does *not* follow from our bracket-generating assumption on the horizontal distribution.) This is because $\mathfrak{q} = \mathrm{Lie}(Q) = \widetilde{W}_0$ defined over $\mathbb{Q} \implies \widetilde{W}_{-1} = \mathrm{Lie}(U(Q))$ and $\widetilde{W}_1 = \widetilde{W}_{-1}^{\perp}$ are defined over $\mathbb{Q}$, and bracket-generation then implies $\widetilde{W}_{-2} = [\widetilde{W}_{-1}, \widetilde{W}_{-1}]$ and so forth, so that all filtrands are defined over $\mathbb{Q}$.

Definition 5.9.

(a) *$\mathcal{O}$ is* polarizable *if and only if there exists $\widehat{N} \in \mathfrak{g}_{\mathbb{R}}^{-1,-1}$ such that:*
  (i) *$\widehat{N}^j$ gives isomorphisms $\mathfrak{g}^{p,j-p} \xrightarrow{\cong} \mathfrak{g}^{p-j,-p}$ for each p, j; and*
  (ii) *$i^{-j}(-1)^{p+1}B(v, \widehat{N}^j\bar{v}) > 0$ for each p, j, and non-zero $v \in \widehat{P}^{p,j-p} := \mathfrak{g}^{p,j-p} \cap \ker(\widehat{N}^{j+1})$.*

(b) *$\mathcal{O}$ belongs to the* nilpotent closure $\mathrm{ncl}(D)$ $\iff$ *there exists a nilpotent $\widehat{N} \in F^{-1} \cap \mathfrak{g}_{\mathbb{R}}$ such that $e^{iy\widehat{N}}F^\bullet \in D$ for $y > 0$. (Clearly $\mathrm{ncl}(D) \subseteq \mathrm{cl}(D)$.)*

The criteria (a) and (b) are useful in different situations, and will turn out to be equivalent (*cf.* Theorem 5.15 below). Evidently (b) is independent of the choice of $F^\bullet \in \mathcal{O}$ and H, and hence well-defined. (That the same is true for (a) follows from the proof of Theorem 5.15.)

Remark 5.10. — Unlike the $\{\mathfrak{g}^{p,q}\}$, the $\widehat{P}^{p,q}$ need not be sums of root spaces, precisely because $\widehat{N}$ need not be a multiple of a root vector, *cf.* §6.2.1.

An additional criterion, which will make an appearance in the examples in §6, is given by

Definition 5.11. — *A boundary stratum $\mathcal{O}$ [resp. boundary component $B(N)$] is* cuspidal $\iff$ $\mathfrak{q}$ *[resp. $W(N)_0\mathfrak{g}$] is a cuspidal parabolic subalgebra.*

Since the anti-diagonal flip sends $W(N)_0\mathfrak{g}$ exactly to $\mathfrak{q}$, the naive limit map sends cuspidal boundary components to cuspidal strata.

Proposition 5.12. — *Codimension-one boundary strata are cuspidal.*

Proof. — This is an immediate consequence of Corollary 4.16(i), as the Cartan $\Psi_{\mathbf{c}_\alpha}(H_0)$ will have real rank 1, with "noncompact part" $\mathcal{A} = e^{\mathbb{R}\langle Y\rangle}$ centralizing the Levi. □

To put definition 5.9(a) in context, recall the notion of a polarized $\mathbb{R}$-MHS (on $(\mathfrak{g}, -B)$), which for us shall mean a triple $(W_\bullet, F^\bullet, N)$ such that:

(I) $W_\bullet$ is an increasing filtration of $\mathfrak{g}_\mathbb{R}$, and $(F^\bullet, W_\bullet)$ is an $\mathbb{R}$-MHS (not necessarily split), with associated Deligne bigrading $\{\mathfrak{g}^{p,q}\}$ of $\mathfrak{g}_\mathbb{C}$;

(II) N is a nilpotent element of $F^{-1}\cap\mathfrak{g}_\mathbb{R}$, with $W(N)_\bullet = W_\bullet$ (which implies $N\in\mathfrak{g}^{-1,-1}$); and

(III) the Hodge structure induced by $F^\bullet$ on
$$\ker\left\{N^{j+1}\colon Gr^W_j \to Gr^W_{-j-2}\right\} =: P_j$$
is polarized by $-B(\cdot, N^j(\cdot))$, for each $j \geqslant 0$.

Conditions 5.9(a)(i,ii) are nothing but a translation of (II,III) for the specific (split) setting considered there. We shall require a couple of lemmas, from the work of Cattani, Kaplan and Schmid (*cf.* [25, Thm. 6.16], [9, Cor. 3.13], [8, (2.18)]):

Lemma 5.13. — *If $e^{zN}F^\bullet$ is an $\mathbb{R}$-nilpotent orbit, then $(W(N)_\bullet, F^\bullet, N)$ is a polarized $\mathbb{R}$-MHS.*

Lemma 5.14. — *If $(W(N)_\bullet, F^\bullet, N)$ is a polarized $\mathbb{R}$-MHS, then $e^{zN}F^\bullet$ is an $\mathbb{R}$-nilpotent orbit; if it is $\mathbb{R}$-split, then $e^{zN}F^\bullet\in D$ for $\Im(z)>0$.*

We are now ready to prove the first main theorem of this section:

Theorem 5.15. — *For $\mathcal{O}\subset \mathrm{cl}(D)$, the following are equivalent:*

(A) *$\mathcal{O}\subset \mathrm{ncl}(D)$;*

(B) *$\mathcal{O}$ is of the form $\widehat{\mathcal{B}}(\mathcal{N})$;*

(C) *$\mathcal{O}$ is polarizable.*

Proof.

(C) $\Longrightarrow$ (A): Let $F^\bullet\in\mathcal{O}$, $\{\mathfrak{g}^{p,q}\}$, $\widehat{N}\in\mathfrak{g}_\mathbb{R}^{-1\cdot -1}$ be as in Definition 5.9(a) and $W_\bullet = \bigoplus_{p+q\leqslant\bullet}\mathfrak{g}^{p,q}$. (Note that $\widehat{N}$ must be nilpotent.) By 5.9(a)(i), we have $W_\bullet = W(\widehat{N})_\bullet$, whereupon 5.9(a)(i–ii) and (5.5) $\implies$ $(W_\bullet, F^\bullet, \widehat{N})$ is a polarized split $\mathbb{R}$-MHS. By Lemma 5.14, $e^{iy\widehat{N}}F^\bullet \in D$ for $y > 0$.

$\underline{\text{(B)} \implies \text{(C)}}$: (B) says that there exists a polarized $\mathbb{R}$-mixed Hodge structure $(W(N)_\bullet, \widetilde{F}^\bullet, N)$, without loss of generality $\mathbb{R}$-split, such that $\lim_{y\to\infty} e^{iyN}\widetilde{F}^\bullet =: F^\bullet \in \mathcal{O}$. Let $\{\mathfrak{g}^{p,q}\}$ be the associated bigrading, $N_+ \in \tilde{\mathfrak{g}}^{-1,-1}$ be as in the discussion preceding (5.1), and $\widehat{N} := -N_+$. Then $\mathfrak{g}^{p,q} := \tilde{\mathfrak{g}}^{-q,-p}$ is the bigrading associated to $(F^\bullet, W(\widehat{N})_\bullet)$ and it is an easy exercise to check that $(W(\widehat{N})_\bullet, F^\bullet, \widehat{N})$ is a polarized $\mathbb{R}$-MHS. It is obviously split, and (C) follows at once.

$\underline{\text{(A)} \implies \text{(B)}}$: Let $F^\bullet \in \mathcal{O}$, $\{\mathfrak{g}^{p,q}\}$ be as in the discussion preceding Definitions 5.7 and 5.9, and put $W_\bullet := \bigoplus_{p+q\leqslant\bullet} \mathfrak{g}^{p,q}$, $\mathfrak{g}_j := \bigoplus_{p+q=j} \mathfrak{g}^{p,q}$. By assumption, $\widehat{N} \in F^{-1}\cap\mathfrak{g}_\mathbb{R}$ is nilpotent with $e^{iy\widehat{N}}F^\bullet \in D$ for $y > 0$. The projection of $\widehat{N}$ to $\mathfrak{g}_\mathbb{R}^{-1,-1}$ still satisfies this hypothesis, since $i\widehat{N} \in F^{-1} \cap \overline{F^{-1}}$ while $F^{-1} \cap \overline{F^{-1}} \cap (F^0 + \overline{F^0})$ belongs to $\mathfrak{g}_\mathbb{R} + F^0$. Hence we may assume $\widehat{N} \in \mathfrak{g}_\mathbb{R}^{-1,-1}$.

By Lemma 5.13, $(W(\widehat{N})_\bullet, F^\bullet, \widehat{N})$ is a polarized $\mathbb{R}$-MHS. To deduce that it is split, we will show that $W(\widehat{N})_\bullet = W_\bullet$. If this is not the case, then for some $j \geqslant 0$ the map

$$\nu_j \colon \mathfrak{g}_j \to \mathfrak{g}_{-j}$$

induced by $\widehat{N}^j$ is not an isomorphism, and there exists

$$\alpha \in F^p \cap \overline{F^{j-p}} \cap \ker \widehat{N}^j.$$

But then $\alpha \in \overline{F^{j-p} \cap \ker \widehat{N}^j} \implies$

$$\begin{aligned} e^{iy\widehat{N}}\alpha \in e^{iy\widehat{N}}\left(\overline{F^{j-p} \cap \ker \widehat{N}^j}\right) &= \overline{e^{-iy\widehat{N}}}\left(\overline{F^{j-p} \cap \ker \widehat{N}^j}\right) \\ &= \overline{e^{iy\widehat{N}}\left\{e^{-2iy\widehat{N}}(F^{j-p} \cap \ker \widehat{N}^j)\right\}} \\ &\subseteq \overline{e^{iy\widehat{N}}F^{-p+1}}, \end{aligned}$$

where the last inclusion is argued as follows: given $\beta \in F^{j-p} \cap \ker \widehat{N}^j$,

$$\begin{aligned} e^{-2iy\widehat{N}}\beta = \beta - 2iy\widehat{N}\beta - \frac{4y^2}{2}\widehat{N}^2\beta + \cdots + \frac{(-2i)^{j-1}}{(j-1)!}y^{j-1}\widehat{N}^{j-1}\beta + 0 \\ \in F^{(j-p)-(j-1)} = F^{-p+1}. \end{aligned}$$

So

$$(0 \neq)\, e^{iy\widehat{N}}\alpha \in e^{iy\widehat{N}}F^p \cap \overline{e^{iy\widehat{N}}F^{-p+1}} =: F_y^p \cap \overline{F_y^{-p+1}},$$

where $F_y^\bullet \in D$ is a Hodge flag (for $y > 0$). This violates the p-opposed condition on a Hodge flag.

We conclude that $(W(\widehat{N})_\bullet = W_\bullet, F^\bullet, \widehat{N})$ is a split polarized $\mathbb{R}$-MHS. Let $\widehat{Y} \in \mathfrak{g}_{\mathbb{R}}^{0,0}$ be the element inducing the grading $\{\mathfrak{g}_j\}$, and $\widehat{N}_+ \in \mathfrak{g}_{\mathbb{R}}^{1,1}$ complete $\widehat{N}$, $\widehat{Y}$ to an $\mathfrak{sl}_2$-triple. Then setting $\widetilde{F}^{-a} := \bigoplus_{p\in\mathbb{Z};\, q\leqslant a} \mathfrak{g}^{p,q}$, $\widetilde{W}_{-b} := \bigoplus_{p+q\geqslant b} \mathfrak{g}^{p,q}$, $(\widetilde{N}, \widetilde{Y}, \widetilde{N}_+) := (-\widehat{N}_+, -\widehat{Y}, -\widehat{N})$, we have $\widetilde{W}_\bullet = W(\widetilde{N})_\bullet$, and $(W(\widetilde{N})_\bullet, \widetilde{F}^\bullet, \widetilde{N})$ is a split polarized $\mathbb{R}$-MHS. At this point, formula (5.2) applies to give $\lim_{y\to\infty} e^{iy\widetilde{N}} \widetilde{F}^\bullet = \lim_{y\to\infty} e^{-\frac{i}{y}\widehat{N}} F^\bullet = F^\bullet$, completing the proof. □

Proposition 5.16.

(i) *Any codimension-one boundary stratum is polarizable.*
(ii) *Suppose D is* strongly classical, *in the sense that π_{χ_0} takes values in $\{-1, 0, 1\}$. (This implies in particular that D is Hermitian symmetric.) Then all boundary strata are polarizable.*
(iii) *If $\mathfrak{g}_{\mathcal{O}}^{-1,-1} = \{0\}$, then $\mathcal{O}$ is not polarizable.*

Proof.

(i) By Corollary 4.4, $\mathfrak{g}^{-1,-1}$ (for some $F^\bullet \in \mathcal{O}$) is spanned by a single real root vector X. Clearly iX spans the normal tangent space $(\mathcal{N}_{\mathcal{O}/\check{D}})_{F^\bullet}$, and so X or $-X$ satisfies Definition 5.9(b). Now use the implication (A) $\Longrightarrow$ (C).

(ii) The normal space $(\mathcal{N}_{\mathcal{O}/\check{D}})_{F^\bullet}$ identifies naturally with $i\mathfrak{g}_{\mathbb{R}}^{-1,-1}$, yielding a diffeomorphism between a ball $B \ni F^\bullet$ and a ball in $i\mathfrak{g}_{\mathbb{R}}^{-1,-1}$. We must have $B \cap D \neq \emptyset$, and so there exists $N \in \mathfrak{g}_{\mathbb{R}}^{-1,-1} \smallsetminus \{0\}$ such that $e^{iN} F^\bullet \in D$. The same argument as in the proof of (A) $\Longrightarrow$ (B) (with $y\widehat{N}$ replaced by N) in Theorem 5.15 shows that $W_\bullet = W(N)_\bullet$, whereupon direct calculation establishes (ii) in Definition *5.9(a)*. (For example, given $v \in \ker(N^2) \subset \mathfrak{g}^{1,0}$, we have $e^{iN} v \in F_0^1 = \mathfrak{g}_0^{1,-1}$ where $F_0^\bullet \in D$, so that $0 < B(e^{iN} v, \overline{e^{iN} v}) = -2iB(v, N\bar{v})$.)

(iii) is obvious. □

Our second result completely characterizes when $\mathcal{O}$ has a flag occurring as the naive limit of a $\mathbb{Q}$-VHS into a discrete quotient $\Gamma \smallsetminus D$.

Theorem 5.17. — *Let $\mathcal{O} \subset \mathrm{cl}(D)$ be a boundary stratum. Then $\mathcal{O}$ contains a $\widehat{B}(N)$ ($N \in \mathfrak{g}_{\mathbb{Q}}$) $\Longleftrightarrow$ $\mathcal{O}$ is rational and polarizable.*

Proof. — Only "$\Longleftarrow$" requires proof. By polarizablility, $\mathcal{O}$ contains the naive limit of an element $(\widetilde{F}^\bullet, \widetilde{W}_\bullet = W(N)_\bullet) \in \widetilde{B}(N)$, $N \in \mathfrak{g}_{\mathbb{R}}$. Bearing in mind the antidiagonal flip, $\widetilde{W}_\bullet$ is the $\widetilde{W}_\bullet$ in Definition 5.7, and by rationality we may assume it is defined over $\mathbb{Q}$. The issue is whether we can orbit by

$Q(\mathbb{R})$ to get N into $\mathfrak{g}_{\mathbb{Q}}$. But $W_{-2}\mathfrak{g}_{\mathbb{R}}$ is exactly $N(W_0\mathfrak{g}_{\mathbb{R}})$, and for $\gamma \in W_0\mathfrak{g}_{\mathbb{R}}$,

$$\frac{d}{dt}\left(\mathrm{Ad}(e^{t\gamma})N\right)|_{t=0} = \mathrm{ad}(\gamma)N = -N(\gamma).$$

Hence $T_N\left(\mathrm{Ad}\, Q(\mathbb{R})\cdot N\right) = W_{-2}\mathfrak{g}_{\mathbb{R}}$, and $\mathrm{Ad}\, Q(\mathbb{R}).N$ contains an open subset of $W_{-2}\mathfrak{g}_{\mathbb{R}}$ centered about N. Since $\mathbb{Q}$-points are dense in $\widetilde{W}_{-2}\mathfrak{g}_{\mathbb{R}}$, we are done. $\square$

Remark 5.18. — It suffices to assume $\mathfrak{q}$ is $G(\mathbb{R})^{\circ}$-conjugate to a $\mathbb{Q}$-parabolic if $\mathfrak{g}_1$ bracket-generates $\widetilde{W}_{-1}$ (*cf.* Remark 5.8) or if $\widetilde{W}_{-1} = \widetilde{W}_{-2}$.

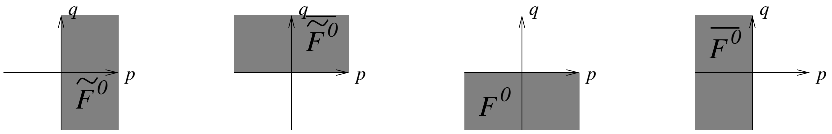
5.3. Dimension formulas, and a classical digression

In the remainder of this section, we turn our attention to the naive limit map

$$\mathscr{F}^N_{\lim}\colon \widetilde{B}(N) \twoheadrightarrow \widehat{B}(N)$$

and its image, where $N \in \mathfrak{g}_{\mathbb{Q}} \setminus \{0\}$ is nilpotent and $\widetilde{B}(N) \neq \emptyset$.

Let $\widetilde{F}^{\bullet} \in \widetilde{B}_{\mathbb{R}}(N)$ resp. $F^{\bullet} := \mathscr{F}^N_{\lim}(\widetilde{F}^{\bullet}) \in \widehat{B}(N)$ be given, with associated MHS $(\widetilde{F}^{\bullet}, \widetilde{W}_{\bullet} := W(N)_{\bullet})$ resp. $(F^{\bullet}, W_{\bullet} := W(N_+)_{\bullet})$[21] and bigradings $\tilde{\mathfrak{g}}^{p,q}$ resp. $\mathfrak{g}^{p,q} = \tilde{\mathfrak{g}}^{-q,-p}$ with dimension $h^{p,q} := \dim_{\mathbb{C}} \tilde{\mathfrak{g}}^{p,q} = \dim_{\mathbb{C}} \mathfrak{g}^{p,q}$. In $\tilde{\mathfrak{g}}^{\bullet,\bullet}$ indexing we have the pictures



For $p+q \geqslant 0$, the primitive subspaces are $\widetilde{P}^{p,q} := \{\ker(N^{p+q+1}) \subseteq \tilde{\mathfrak{g}}^{p,q}\}$ and

$$P^{p,q} := N^{p+q}(\widetilde{P}^{p,q}) = \{\ker(N) \subseteq \mathfrak{g}^{p,q}\},$$

of dimension $h^{p,q}_{\mathrm{prim}}$; for $p+q \leqslant 0$ we set $z^{p,q} := h^{-q,-p}_{\mathrm{prim}}$. Write $\tilde{\mathfrak{g}}_j := \bigoplus_{p+q=j} \tilde{\mathfrak{g}}^{p,q} = \mathfrak{g}_{-j}$, and $\mathfrak{q} = \widetilde{W}_0\mathfrak{g} = \mathrm{Lie}(Q)$. Of course, $c_{F^{\bullet}} > 0$ and $\widehat{B}(N) \subset \mathcal{O}_{F^{\bullet}} \subset \partial D$.

We recall the following basic material on boundary components from [15, sec. 7].[22] Let $Z(N)$ denote the centralizer of N in G, with unipotent radical U_N and Levi subgroup $G_N \cong Z(N)/U_N$; write

$$\mathscr{G}(N) := U_N(\mathbb{C}) \rtimes G_N(\mathbb{R}) \geqslant U_N(\mathbb{R}) \rtimes G_N(\mathbb{R}) = Z(N)(\mathbb{R}).$$

[21] As above, N_+ is determined by (N, Y) where Y arises from the Deligne bigrading associated to $(\widetilde{F}^{\bullet}, \widetilde{W}_{\bullet})$.

[22] In this paragraph, all groups are tacitly identity components of the group written.

On the Lie algebra level, we have $\mathfrak{z}(N) := \mathrm{Lie}(Z(N)(\mathbb{C})) = \ker(\mathrm{ad}\, N) \subset W_0\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{u}_N := \mathrm{Lie}(U_N(\mathbb{C})) = \mathfrak{z}(N) \cap W_{-1}\mathfrak{g}_{\mathbb{C}}$, and $\mathfrak{g}_N := \mathrm{Lie}(G_N(\mathbb{C})) = \mathfrak{g}_0 \cap \mathfrak{z}(N)$. Letting $Z(N)(\mathbb{R})$ resp. $\mathscr{G}(N)$ act on $(\widetilde{F}^\bullet, \widetilde{W}_\bullet)$ gives isomorphisms[23]

$$\begin{aligned} \widetilde{B}_{\mathbb{R}}(N) &\cong Z(N)(\mathbb{R})/\big\{Z(N)(\mathbb{R}) \cap Q_{\widetilde{F}^\bullet}\big\}, \\ \widetilde{B}(N) &\cong \mathscr{G}(N)/\big\{\mathscr{G}(N) \cap Q_{\widetilde{F}^\bullet}\big\}. \end{aligned}$$

Passing to the associated graded (or quotienting by U_N) gives projections from both to the Mumford-Tate domain

$$D(N) \cong G_N(\mathbb{R})/\mathcal{H}_N$$

(where $\mathrm{Lie}(\mathcal{H}_N) = \mathfrak{g}_{\mathbb{R}}^{0,0} \cap \mathfrak{z}(N)$), which is Hermitian symmetric if $\mathfrak{g}_N \subset \mathfrak{g}^{-1,1} \oplus \mathfrak{g}^{0,0} \oplus \mathfrak{g}^{1,-1}$. The boundary components $B_{\mathbb{R}}(N)$ and $B(N)$ are obtained by quotienting out by $e^{\mathbb{R}\langle N\rangle}$ resp. $e^{\mathbb{C}\langle N\rangle}$ on the left. Finally, from the equivariant nature of $\mathscr{F}^N_{\lim}$, it is evident that

$$\widehat{B}(N) = Z(N)(\mathbb{R})\cdot F^\bullet \cong Z(N)(\mathbb{R})/\{Z(N)(\mathbb{R}) \cap Q_{F^\bullet}\}. \tag{5.6}$$

Like $Q_{\widetilde{F}^\bullet} \cap Z(N)(\mathbb{R})$, $Q_{F^\bullet} \cap Z(N)(\mathbb{R})$ projects to $\mathcal{H}_N$ in $G_N(\mathbb{R})$; and so $\widehat{B}(N)$, too, maps (holomorphically) to $D(N)$. In a diagram, we have

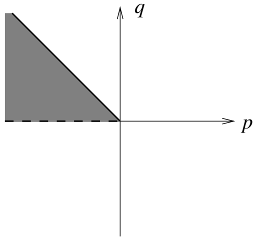
$$\begin{tikzcd}
 & B(N) \arrow[ddl, two heads] \arrow[dr, two heads] & & \\
 & B_{\mathbb{R}}(N) \arrow[u, hook, "(\beta)"'] \arrow[r, two heads, "(\gamma)"] \arrow[dl, two heads] & \widehat{B}(N) \arrow[r, hook, "(\delta)"] \arrow[dll, two heads, "(\alpha)"] & \mathcal{O}_{F^\bullet} \subset \partial D, \\
D(N) & & &
\end{tikzcd} \tag{5.7}$$

and one might wonder (for example) when (α)-(δ) are isomorphisms. To state the next result, consider the regions

I : $p < 0$, $q \geqslant 0$, and $p + q \leqslant 0$
I′: $q > 0$ and $p + q \leqslant 0$
I″: $q > 0$ and $p + q < 0$
II: $p < 0$ and $q < 0$

(23) [15, sec. 7] describes $\widetilde{B}_{\mathbb{R}}(N)$, $\widetilde{B}(N)$, and $D(N)$ as orbits of possibly smaller groups. This is a refinement of the presentation here, which follows from the coarser [15, Lemma 3.3].

in $\mathbb{Z}^2$; for example, $\mathbf{I}'$ is



and $\mathbf{I} \amalg \mathbf{II} \amalg \bar{\mathbf{I}} = \mathbf{I}' \amalg \mathbf{II} \amalg \overline{\mathbf{I}'}$ consist of pairs $(p,q) \neq (0,0)$ with $p+q \leqslant 0$. Write $d := \dim_{\mathbb{C}} \check{D}$.

PROPOSITION 5.19 (Dimension formulas).

(i) $\dim_{\mathbb{R}} B(N) = 2\sum_{(p,q)\in\mathbf{I}} z^{p,q} + 2\left(\sum_{(p,q)\in\mathbf{II}'} z^{p,q} - 1\right)$.

(ii) $\dim_{\mathbb{R}} B_{\mathbb{R}}(N) = 2\sum_{(p,q)\in\mathbf{I}} z^{p,q} + \left(\sum_{(p,q)\in\mathbf{II}'} z^{p,q} - 1\right)$.

(iii) $\dim_{\mathbb{R}} \widehat{B}(N) = 2\sum_{(p,q)\in\mathbf{I}'} z^{p,q}$.

(iv) $\dim_{\mathbb{R}} D(N) = 2\sum_{p<0} z^{p,-p}$.

(v) $\dim_{\mathbb{R}} \mathcal{O}_{F^\bullet} = \sum_{p<0 \text{ or } q>0} h^{p,q} = 2d - c_{F^\bullet}$,

where we recall $c_{F^\bullet} = \sum_{(p,q)\in\mathbf{II}'} h^{p,q}$.

Proof. — Only (i)–(iii) require justification. These results follow from computing tangent spaces:

$$\begin{aligned} T_{\widetilde{F}^\bullet}\widetilde{B}_{\mathbb{R}}(N) &\cong \mathfrak{z}(N)_{\mathbb{R}}/\left\{\mathfrak{z}(N)_{\mathbb{R}} \cap \widetilde{F}^0(\cap\overline{\widetilde{F}^0})\right\} \\ &= \mathfrak{z}(N)_{\mathbb{R}}/\mathfrak{z}(N)^{0,0}_{\mathbb{R}} \cong \left(\bigoplus_{(p,q)\neq(0,0)} P^{p,q}\right)_{\mathbb{R}}; \end{aligned}$$

similarly,

$$T_{\widetilde{F}^\bullet}\widetilde{B}(N) \cong \mathfrak{z}(N)/\left\{\mathfrak{z}(N)\cap\widetilde{F}^0\right\} = \bigoplus_{p\in\mathbb{Z};\, q>0} P^{p,q}.$$

To get tangent spaces to $B(N)$ and $B_{\mathbb{R}}(N)$, quotient out the span of N in $P^{1,1}_{\mathbb{R}}$ resp. $P^{1,1}$. Noting that $\widetilde{W}_0 \cap F^0 \supset \widetilde{W}_0 \cap \widetilde{F}^0$ and $\mathfrak{z}(N) \subset \widetilde{W}_0 \implies \mathfrak{z}(N)\cap F^0 \supset \mathfrak{z}(N)\cap\widetilde{F}^0$,

$$\begin{aligned} T_{F^\bullet}\widehat{B}(N) &\cong \mathfrak{z}(N)_{\mathbb{R}}/\{\mathfrak{z}(N)\cap F^0(\cap\overline{F^0})\} \\ &\cong \mathfrak{z}(N)\cap\left(\bigoplus_{p<0 \text{ or } q>0} \tilde{\mathfrak{g}}^{p,q}\right) = \left(\bigoplus_{p<0 \text{ or } q>0} P^{p,q}\right)_{\mathbb{R}}. \end{aligned}$$

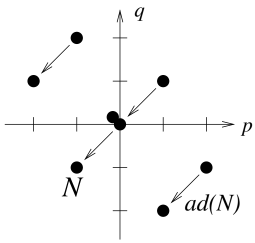
□

One immediate consequence is that if $F^\bullet$ is Hodge-Tate, then $\widehat{B}(N)$ is a point (namely, $F^\bullet$). We also have:

Corollary 5.20. — *The maps in* (5.7) *are isomorphisms under the following conditions:*

(α): $z^{p,q} = 0$ *for* $(p,q) \in \mathbf{I}''$*;*
(β): $c_{F^\bullet} = 1$*;*
(γ): $c_{F^\bullet} = 1$*, and* $z^{p,0} = 0$ *for* $p < 0$*; or equivalently,* $z^{p,q} = 0$ *for* $(p,q) \in \mathbf{II} \smallsetminus \{(1,1)\}$ *and* $z^{-1,-1} = 1$*;*
(δ): *never.*

On an infinitesimal level, the $P^{p,q}$ with $p+q=0$ and $p \neq 0$ parametrize the Hodge structure given by $\widetilde{F}^\bullet$ on the associated graded $\bigoplus_i Gr_i^{\widetilde{W}}$; the $P^{p,q}$ with $p+q>0$ parametrize extension classes. The information lost by the naive limit map (γ) is precisely that which is parametrized by the $P^{p,q}$ with $p \geqslant 0$ and $q \geqslant 0$ (except for $\langle N \rangle \subset P^{1,1}$). Carayol's nonclassical $SU(2,1)$ example ([6, 15]), with mixed-Hodge diagram



is one instance where (γ) is an isomorphism.

At another extreme is the *strongly classical* case, where D is Hermitian symmetric with $F^{-1}\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}$, so that (for $\Gamma \leqslant G(\mathbb{Q})$ neat arithmetic) $X := \Gamma \smallsetminus D$ is a connected Shimura variety. It is known when D is a Siegel space $\mathfrak{H}_g$ (*cf.* [10, 7]) that smooth toroidal compactifications (as in [3]) are obtained by adding in quotients of our $B(\sigma)$'s (parametrizing σ-nilpotent orbits), and as we shall see this holds more generally. The Baily-Borel compactification, on the other hand, is obtained by using Γ-invariant sections of $K_D^{\otimes M}$ (for some $M \gg 0$) to embed X in a projective space, and then taking (Zariski or analytic) closure. Heuristically, since $K_D \cong \bigwedge^d(F^{-1}/F^0)$ measures exactly the changes in the Hodge flag (in any direction), the limits of sections of any $K_D^{\otimes M}$ keep track of limits of flags. This suggests that the subvarieties being glued in are quotients of $\widehat{B}(\sigma)$'s.

More precisely, a *Baily-Borel boundary component* $\mathbf{F}$ of D is a holomorphic path component in $\mathrm{cl}(D) \setminus D$, *i.e.* an equivalence class under the relation: $\mathbf{F} \sim \mathbf{F}' \iff \exists$ holomorphic $\mu\colon \Delta \to \check{D}$ with $\mathbf{F}, \mathbf{F}' \in \mu(\Delta) \subset \mathrm{cl}(D)$ [19, sec. V.2]. The (singular) Baily-Borel compactification X^* is a disjoint union of X and finitely many *B-B boundary strata* $\Gamma_{\mathbf{F}} \setminus \mathbf{F}$, where $\mathbf{F}$ runs over Γ-equivalence classes of *rational* boundary components (see the proof below). Let $\bar{X}$ denote a smooth toroidal compactification with $\bar{X} \setminus X = \cup Y_i$ a strict normal crossing divisor; writing $Y_I = \cap_{i\in I} Y_i$, the *AMRT boundary strata* are the $Y_I \setminus \cup_{J \supsetneq I} Y_J$. By [3, Prop. III.5.3], there is a natural holomorphic map $\pi\colon \bar{X} \twoheadrightarrow X^*$ with $\pi|_X = id_X$, sending ARMT strata to B-B strata.

Referring to Remark 5.6 for $B(\sigma)$, $\widehat{B}(\sigma)$, and $\mathscr{F}^{\sigma}_{\lim}$, we have

Theorem 5.21. — *In the strongly classical case:*

(a) *The map $\mathscr{F}^{\sigma}_{\lim}\colon B(\sigma) \twoheadrightarrow \widehat{B}(\sigma)$ sends LMHS (up to $e^{\langle\sigma\rangle_{\mathbb{C}}}$) to their associated graded.*

(b) *The restriction of $\pi\colon \bar{X} \twoheadrightarrow X^*$ to each AMRT stratum identifies naturally, for some rational nilpotent cone $\sigma \subset \mathfrak{g}_{\mathbb{R}}$, with the map*

$$\Gamma_\sigma \setminus B(\sigma) \xrightarrow[\mathscr{F}^{\sigma,\Gamma}_{\lim}]{} \!\!\!\!\!\rightarrow \overline{\Gamma_\sigma} \setminus \widehat{B}(\sigma)$$

induced by $\mathscr{F}^{\sigma}_{\lim}$, where $\Gamma_\sigma = \mathrm{Stab}(\sigma) \leqslant \Gamma$ and $\overline{\Gamma_\sigma} = \Gamma_\sigma / \{\Gamma_\sigma \cap U(Z(\sigma))\}$.

Proof.

(a) We must show $\mathscr{F}_{\lim}$ loses all extension information, or equivalently that (α) is an isomorphism. But this is clear since the nonzero $\tilde{\mathfrak{g}}^{p,q}$ are in the square $[-1,1]^{\times 2}$.

(b) We will begin by verifying the following:

Claim. — *The rational B-B boundary components are precisely the $\{\widehat{B}(\sigma)\}$ with σ rational, or equivalently (in view of Remark 5.6) the $\widehat{B}(N)$ with $N \in \mathfrak{g}_{\mathbb{Q}}$.*

Let $\mathbf{F} \subset \mathrm{cl}(D) \setminus D$ be any B-B boundary component. There exists a homomorphism $\psi\colon U(1) \times SL_2(\mathbb{R}) \to G(\mathbb{R})$ such that

$$f(g\langle i\rangle)(e^{i\theta}) := (\mathrm{Ad} \circ \psi)\left(z, g \cdot \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right) \cdot g^{-1}\right)$$

defines a symmetric holomorphic map $f\colon \mathfrak{H} \to D$ with

$$\widehat{F}^{\bullet} := \lim_{\Im(\tau)\to\infty} f(\tau) \in \mathbf{F}$$

[3, Thm. III.3.3]. That is, f is an SL_2-orbit with naive limit $\widehat{F}^\bullet$. Writing $N := d\psi\left(0, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$, $\mathfrak{q} := W(N)_0\mathfrak{g}_{\mathbb{R}}$ is the Lie algebra of the stabilizer of $\mathbf{F}$, a maximal parabolic subgroup $\mathcal{Q}$ acting transitively on $\mathbf{F}$ [3, Thm. III.3.7]. Since f is an SL_2-orbit, we have $f(\tau) = e^{\tau N}\widetilde{F}^\bullet$ for some $\widetilde{F}^\bullet \in \check{D}$ with $(\widetilde{F}^\bullet, W(N)_\bullet)$ $\mathbb{R}$-split; write $\mathfrak{g}_{\mathbb{C}} = \bigoplus_{-1\leqslant p,q\leqslant 1} \tilde{\mathfrak{g}}^{p,q}$ for the associated bigrading. As $\tilde{\mathfrak{g}}^{0,0}(= \hat{\mathfrak{g}}^{0,0}) \twoheadrightarrow \mathfrak{q}/\mathfrak{z}(N)_{\mathbb{R}}$, (5.6) gives $\widehat{B}(N) = Z(N)(\mathbb{R})\cdot \widehat{F}^\bullet = \mathcal{Q}\cdot \widehat{F}^\bullet = \mathbf{F}$.

Now $\mathbf{F}$ is by definition *rational* iff $\mathcal{Q} = Q(\mathbb{R})$ for some rational parabolic subgroup $Q \leqslant G$. By [3, p. 141 (3-4)], this is the case exactly when $\psi|_{SL_2}$ can be taken to be defined over $\mathbb{Q}$, which is equivalent to $N \in \mathfrak{g}_{\mathbb{Q}}$. This proves the Claim.

Continuing to fix $\widetilde{F}^\bullet$ and $\widehat{F}^\bullet$ as above, set $U_{\mathbb{C}} = e^{\tilde{\mathfrak{g}}_{\mathbb{C}}^{-1,-1}}$ and $D(\mathbf{F}) := U_{\mathbb{C}} \cdot \mathcal{Q}\cdot \widetilde{F}^\bullet \subset \check{D}$. Taking $\widetilde{G}_\ell \leqslant G(\mathbb{R})$ to be the subgroup with Lie algebra $\tilde{\mathfrak{g}}_\ell := [\tilde{\mathfrak{g}}_{\mathbb{R}}^{-1,-1}, \tilde{\mathfrak{g}}_{\mathbb{R}}^{1,1}] \subset \tilde{\mathfrak{g}}_{\mathbb{R}}^{0,0}(= \hat{\mathfrak{g}}_{\mathbb{R}}^{0,0})$, we consider the open cone $C(\mathbf{F}) := \mathrm{Ad}(\widetilde{G}_\ell)N \subset \tilde{\mathfrak{g}}_{\mathbb{R}}^{-1,-1}$ [3, Thm. III.4.1]. For any $N' \in \mathrm{Ad}(g_\ell)N \in C(\mathbf{F})$, $W(N')_\bullet = \mathrm{Ad}(g_\ell)W(N)_\bullet = W(N)_\bullet$ and $e^{\tau N'}\widetilde{F}^\bullet = g_\ell e^{\tau N} g_\ell^{-1}\widetilde{F}^\bullet = g_\ell e^{\tau N}\widetilde{F}^\bullet$ is a nilpotent orbit with naive limit $g_\ell\widehat{F}^\bullet = \widehat{F}^\bullet$. As in Remark 5.6 we therefore have $\widetilde{B}(N') = D(\mathbf{F}) = \widetilde{B}(N)$; conclude that for any simplicial cone $\sigma \subset \mathrm{cl}(C(\mathbf{F}))$ with $\sigma^\circ \subset C(\mathbf{F})$, $\widetilde{B}(\sigma) = \widetilde{B}(N) = D(\mathbf{F})$ and $\widehat{B}(\sigma) = \widehat{B}(N) = \mathbf{F}$.

Define a map $\pi_{\mathbf{F}} \colon D(\mathbf{F}) \to \mathbf{F}$ by $\pi_{\mathbf{F}}(uq\widetilde{F}^\bullet) = uq\widehat{F}^\bullet$ [3, III(4.2)ff]. Writing $q \in \mathcal{Q}$ as zg_ℓ ($z \in Z(N)(\mathbb{R})$, $q \in \widetilde{G}_\ell$), we have

$$\begin{aligned} uq\widehat{F}^\bullet = uzg_\ell\widehat{F}^\bullet = uz\widehat{F}^\bullet &= \lim_{\Im(\tau)\to\infty} uze^{\tau N}\widetilde{F}^\bullet \\ &= \lim_{\Im(\tau)\to\infty} e^{\tau N}uz\widetilde{F}^\bullet = \lim_{\Im(\tau)\to\infty} e^{\tau N}uq\widetilde{F}^\bullet, \end{aligned}$$

so $\pi_{\mathbf{F}}$ is just the naive limit map $\mathscr{F}^N_{\mathrm{lim}} = \mathscr{F}^\sigma_{\mathrm{lim}}$.

Let $\{\sigma^{\mathbf{F}}_\alpha\}$ be the Γ-admissible decomposition of $\mathrm{cl}(C(\mathbf{F}))$ into rational polyhedral cones (*cf.* [3, Defn. III.5.1]) involved in the construction of $\bar{X}$. (Since $\bar{X}$ is smooth, they will be simplicial.) Amongst these, let σ be one the cones with interior contained in $C(\mathbf{F})$.[24] From the proof of [3, Thm. III.5.2], one finds that the associated AMRT stratum is given by $\Gamma_\sigma e^{\langle\sigma\rangle_{\mathbb{C}}} \backslash D(\mathbf{F}) = \Gamma_\sigma \backslash B(\sigma)$, and its map to $\Gamma_{\mathbf{F}} \backslash \mathbf{F} = \overline{\Gamma_\sigma} \backslash \widehat{B}(\sigma)$ is induced by $\pi_{\mathbf{F}}$, completing the proof. □

Remark 5.22. — $\mathscr{F}^{\sigma,\Gamma}_{\mathrm{lim}}$ may be understood as a morphism, defined over a number field, from a (connected) mixed Shimura variety to a (connected) pure Shimura variety [21, sec. 12.6].

(24) Here $\dim(\sigma) \leqslant \dim C(\mathbf{F})$; strict inequality leads to $\mathbb{C}^*$ factors in $\Gamma_\sigma \backslash B(\sigma)$.

5.4. Parabolic induction and parabolic orbits

For applications of this material to representation theory, which we plan to pursue in subsequent work, the following Hodge-theoretic approach to an Iwasawa decomposition will be of use. Writing $F^\bullet = \mathscr{F}(H,\chi)^\bullet$, let Θ be as in Proposition 3.8 so that (by Corollary 3.9) $\Theta(\mathfrak{g}^{p,q}) = \mathfrak{g}^{-q,-p}$. For each (p,q) with $p+q \neq 0$, the $(+1)$- and (-1)-eigenspaces of Θ on $(\mathfrak{g}^{p,q} \oplus \mathfrak{g}^{q,p} \oplus \mathfrak{g}^{-q,-p} \oplus \mathfrak{g}^{-p,-q})_{\mathbb{R}}$ are clearly both of (real) dimension $2h^{p,q}$. On the anti-diagonal line, we can use the fact that $(\widetilde{W}_\bullet, \widetilde{F}^\bullet, N)$ is a polarized MHS on $(\mathfrak{g}, -B)$ to deduce that the sign of Θ on $N^j \widetilde{P}^{p+j,j-p} \subset \tilde{g}^{p,-p}$ is $(-1)^{j+p}$, determining its eigenspaces in $\mathfrak{g}^{0,0}_{\mathbb{R}}$ and $(\mathfrak{g}^{p,-p} \oplus \mathfrak{g}^{-p,p})_{\mathbb{R}}$. The sum $\mathfrak{k}$ of all the $(+1)$-eigenspaces is the Lie algebra of a maximal compact subgroup $K \leqslant G(\mathbb{R})$.

Now assume $\mathcal{Q} := Q(\mathbb{R})$ is cuspidal, with Langlands decomposition $\mathcal{Q} = \mathcal{MAN}$ as in Proposition 3.12; in particular, $\mathcal{MA} = G_0(\mathbb{R})$ and $\mathcal{N} = U(\mathcal{Q})$. (Note that $\mathrm{Lie}(\mathcal{Q}) = \mathfrak{q}_{\mathbb{R}} = \widetilde{W}_0 \mathfrak{g}_{\mathbb{R}}$, $\mathrm{Lie}(\mathcal{N}) = \widetilde{W}_{-1}\mathfrak{g}_{\mathbb{R}}$, and $\mathcal{M}$ is reductive with compact Cartan, but possibly not connected.) Then $\mathfrak{k} \oplus \mathfrak{q}_{\mathbb{R}}$ evidently gives all of $\mathfrak{g}_{\mathbb{R}}$, and so

$$G(\mathbb{R})^\circ = K\mathcal{Q} = K\mathcal{MAN}.$$

Since $\mathcal{Q}$ contains the stabilizer $Q_{F^\bullet}(\cap Q_{F^\bullet}) \cap G(\mathbb{R})^\circ$ of $F^\bullet$ in $G(\mathbb{R})^\circ$, we have a natural fibration

$$\mathcal{O}_{F^\bullet} \overset{\pi}{\twoheadrightarrow} G(\mathbb{R})^\circ/\mathcal{Q} \cong K/K \cap \mathcal{M} =: \mathcal{K}_{F^\bullet}$$

over a compact base. Its fibers, one of which contains $\widehat{B}(N)$, are of real dimension $2\sum_{(p,q)\in \mathbf{I}'} h^{p,q}$ $(\geqslant \dim \widehat{B}(N))$.

Remark 5.23. — By Proposition 5.19(iii), if $h^{p,q} = z^{p,q}$ for all $(p,q) \in \mathbf{I}'$, then the fibers are $G(\mathbb{R})^\circ$-translates of $\widehat{B}(N)$. For instance, this holds in the strongly classical case. More generally, it is related to forthcoming work of Robles on the CR-structure of the $G(\mathbb{R})^\circ$-orbits.

Associated to any complex representation $\rho\colon \mathcal{Q} \to \mathrm{Aut}(V)$ is a vector bundle

$$\mathcal{V}_\rho := \frac{G(\mathbb{R})^\circ \times V}{\mathcal{Q}} \twoheadrightarrow \mathcal{K}_{F^\bullet},$$

and we may define a representation $\mathrm{Ind}_{\mathcal{Q}}^{G(\mathbb{R})^\circ}(\rho)$ of $G(\mathbb{R})^\circ$ by letting the latter act by left translation on the space of C^∞ $\mathbb{C}$-valued sections of $\mathcal{V}_\rho$. In order for boundary components to provide a useful framework for studying these representations, we should have at least $\mathcal{M}/Z(\mathcal{M}) \subseteq G_N(\mathbb{R})$, or equivalently

$$\begin{cases} z^{p,-p} = h^{p,-p} \text{ for } p \neq 0 \quad \text{and} \\ \mathfrak{g}^{0,0}/\mathfrak{z}(\mathfrak{g}_0) \cong \{\ker(N) \subset \mathfrak{g}^{0,0}\}. \end{cases}$$

In this situation, one can begin with a representation μ of $G_N(\mathbb{R})$,[25] for instance on a coherent cohomology group $H^i(D(N), \mathcal{O}(\mathcal{V}'))$ (with $\mathcal{V}'$ a holomorphic homogeneous vector bundle over $D(N)$), together with a character σ of $\mathcal{A}$, and pull $\sigma\mu$ back to $\mathcal{Q}$ (via the projection $\mathcal{Q} \twoheadrightarrow \mathcal{Q}/Z(\mathcal{M})\mathcal{N}$).

Two special cases of interest are:

(A) when $\mathcal{O}_{F^\bullet}$ has $c_{F^\bullet} = 1$, $\dim(Gr^W_{\pm 2}\mathfrak{g}) = 1$, and $\dim(Gr^W_{\pm k}\mathfrak{g}) = 0$ for $k > 2$; and

(B) when $\mathcal{O}_{F^\bullet}$ is the closed orbit, $(F^\bullet, W_\bullet)$ Hodge-Tate, and G_N trivial.

In case (B), $\mathcal{M}$ is finite, and for $\sigma = \Delta_{\mathcal{Q}}^{\frac{1}{2}}$ ($\Delta_{\mathcal{Q}} :=$ modular character[26]) together with an appropriate choice of μ, $\mathrm{Ind}_{\mathcal{Q}}^{G(\mathbb{R})}(\sigma\mu)$ is the direct sum of the TDLDS (totally degenerate limits of discrete series) for $G(\mathbb{R})^\circ$.

In another (but related) direction, we expect in some cases (including (A) above), the $Q(\mathbb{C})$-orbit of $\widetilde{F}^\bullet$ to play a role in generalizing H. Carayol's results on Fourier coefficients [6] for nonclassical automorphic cohomology classes (in some $H^i(\Gamma \backslash D, \mathcal{O}(\mathcal{V}))$). More precisely, $(Q(\mathbb{C})\cdot \widetilde{F}^\bullet) \cap D$ will project to a sort of punctured tubular neighborhood of $\Gamma_N \backslash B(N)$ in $\Gamma \backslash D$, whenever

$$\dim_{\mathbb{C}}(Q(\mathbb{C})\cdot \widetilde{F}^\bullet) = \sum_{(p,q)\in \mathbf{I}\amalg\mathbf{II}'} h^{p,q}$$

equals $\dim_{\mathbb{C}}(B(N)) + 1$, which is to say when $z^{p,q} = h^{p,q}$ for $(p,q) \neq (0,0)$. (Basically, this gives a homogeneous structure to a union of nilpotent orbits.) The pullback of a cohomology class to this neighborhood then is expected to have a Laurent expansion "about" $\Gamma_N \backslash B(N)$, with coefficients lying in groups of the form $\left\{H^i(\Gamma_N \backslash B(N), \mathcal{O}(\mathcal{L}^{\otimes k} \otimes W))\right\}_{k\in\mathbb{Z}}$. This will be taken up in a future work.

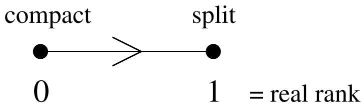
6. Examples

The simplest nontrivial case is, of course, the upper half-plane, Let $G = PGL_2$, $D = \mathfrak{H}$, $\check{D} = \mathbb{P}^1$, where we think of $\mathfrak{H}$ as parametrizing polarized

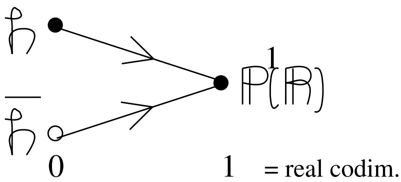
[25] optionally twisted by a character of the component group of $\mathcal{M}$

[26] The modular (or modulus) character $\Delta_{\mathcal{Q}} \colon \mathcal{Q} \to \mathbb{C}^*$ of a parabolic subgroup $\mathcal{Q}$ with unipotent radical $\mathcal{N}$ is given by $|\det(\mathrm{Ad}_{\mathrm{Lie}(\mathcal{N})}(\cdot))|$. The reason for including the factor σ is so that $\mathrm{Ind}_{\mathcal{Q}}^{G(\mathbb{R})}$ takes unitary representations to unitary representations.

Hodge structures on $\mathfrak{g} = \mathfrak{sl}_2$ with $h^{-1,1} = h^{0,0} = h^{1,-1} = 1$. The group $G(\mathbb{R})$ has two components, with $G(\mathbb{R})^\circ \cong SL_2(\mathbb{R})/\{\pm\,\mathrm{id}\}$ and $[(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})]$ in the non-identity component, so that $W_{\mathbb{R}}^\circ$ is trivial and $W_{\mathbb{R}} = W_{\mathbb{C}} \cong \mathbb{Z}/2\mathbb{Z}$. The two nontrivial $G(\mathbb{R})^\circ$-conjugacy classes of Cartan subgroups in $G(\mathbb{R})^\circ$ are depicted in a Hasse diagram



where the arrow denotes a Cayley transform. The $G(\mathbb{R})^\circ$-orbits in $\check{D}$ are of course

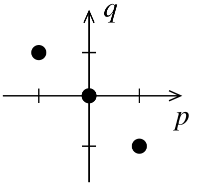
(6.1)


where the segments denote incidence: that is, the orbit on the right endpoint is contained in the closure of the left-endpoint orbit.

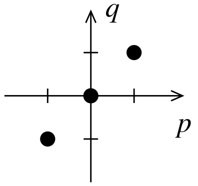
We call (6.1) an *enhanced Hasse diagram*, and produce them for a number of other examples in §6.1. The "enhancements" are as follows:

- a solid vertex corresponds to an orbit in $\mathrm{ncl}(D)$ (*i.e.* polarizable);
- a vertex with an "$\times$" denotes an orbit in $\mathrm{cl}(D) \smallsetminus \mathrm{ncl}(D)$ (*i.e.* non-polarizable);
- an open vertex signifies an orbit not in $\mathrm{cl}(D)$;
- a solid edge with [resp. without] an arrow is an incidence obtained via a Cayley transform [resp. cross-action]; and
- a dotted edge is an incidence deduced from the subexpression property (*cf.* §4.3 and [29, Thm. 3.15]).

Orbits will be labeled as in §4.2 (viz., $\mathbf{o}_j^{\{w\}}$), and we shall indicate as well the $\{\dim(\mathfrak{g}^{p,q})\}$ attached to each orbit: for PGL_2 this is simply



(open orbits)



(closed orbit)

where a dot stands for a single complex dimension. We call these *mixed Hodge diagrams*.

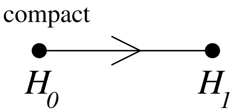
In §6.2 we will discuss a few simple “negative examples” which motivated the definitions in §5.2.

6.1. Enhanced Hasse diagrams

This subsection treats the cases where G is (a $\mathbb{Q}$-form of) $SU(2,1)^{\mathrm{ad}}$, PSp_4, and ($\mathbb{R}$-split) G_2, briefly illustrating the method for $SU(2,1)^{\mathrm{ad}}$ and merely describing results for the other two groups.

6.1.1. $G = SU(2,1)^{\mathrm{ad}}$

$G(\mathbb{R})\,(= G(\mathbb{R})^{\circ})$ has two conjugacy classes of Cartan subgroups



and no Cartan of real rank 2. In the root diagram associated to a choice of (real) Cartan, we denote by

a noncompact imaginary [resp. compact imaginary, real, complex] root.[27] A character $\chi \in \mathrm{X}_*(H(\mathbb{C}))$ is depicted by shading half of the root diagram, which is meant to heuristically indicate $\chi^{-1}(\mathbb{R}_{\geqslant 0}) \subset \Lambda \otimes \mathbb{R}$. We begin with (H_0, χ_0), apply a Cayley transform to get (H_1, χ_1), then apply $W_{\mathbb{C}}$ to both, and finally, group the results in $W_{\mathbb{R}}^{\circ}(H_0)$- resp. $W_{\mathbb{R}}^{\circ}(H_1)$-orbits. These latter are labeled by their images under the orbit map (§4.2), with elements of $W_{\mathbb{C}}$ written in terms of reflections in simple roots.

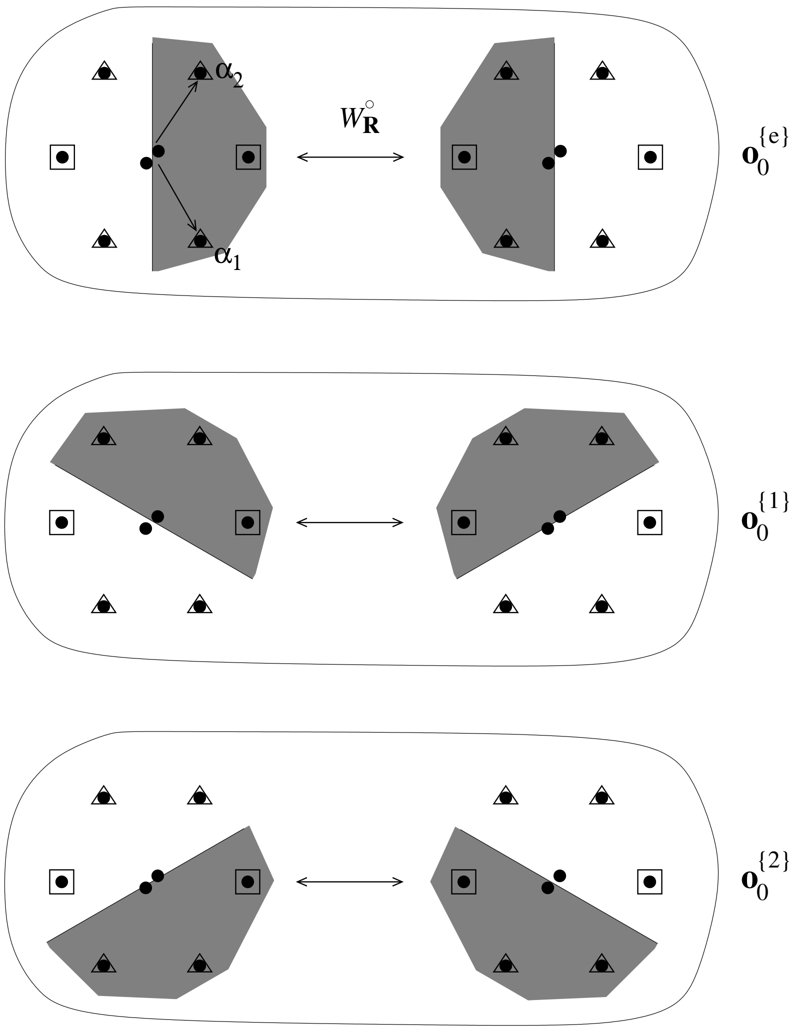
Now, the results will depend upon D, for which there are essentially two choices compatible with the assumptions of §2. For both, we let V be a 6-dimensional $\mathbb{Q}$-vector space, with a symplectic form Q and a decomposition $V_{\mathbb{Q}(i)} = V_+ \oplus \overline{V_+}$ with $Q(V_+, V_+) = 0$. We assume that the Hermitian form $H(v,w) := -2iQ(v,\bar{w})$ on V_+ has signature $(2,1)$, so that the projectivization of those $v \in V_+$ with $H(v,v) < 0$ yields a (Picard) 2-ball $\mathbb{B} \subset \mathbb{P}(V_+)$. This parametrizes Hodge structures on V with Hodge numbers $(3,3)$, and a nontrivial involution (with eigenspaces $V_+, \overline{V_+}$). Alternatively, one can consider Carayol's nonclassical domain $\mathbb{D}$ parametrizing point-line

[27] We represent the Cartan subalgebra in our root diagram by two bullets at the origin.

pairs (p, L) in $\mathbb{P}(V_+)$ with $L \cap \mathbb{B} \neq \emptyset$ and $p \in L \smallsetminus (L \cap \mathrm{cl}(\mathbb{B}))$, or equivalently HS-with-involution on V with numbers $(1, 2, 2, 1)$. The corresponding Hodge structures on $\mathfrak{g}$ have Hodge numbers $(2, 4, 2)$ for $\mathbb{B}$ and $(1, 2, 2, 2, 1)$ for $\mathbb{D}$.

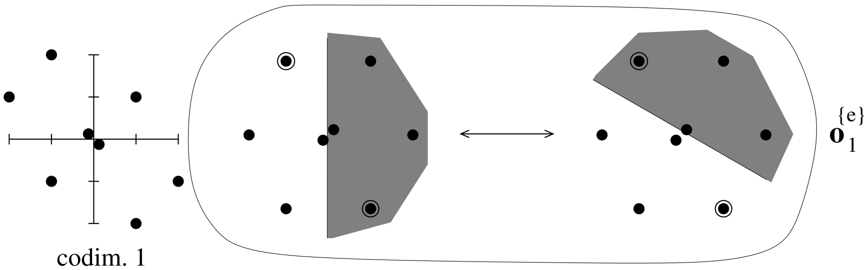
We carry out the procedure for $\mathbb{D}$ first: since it is a complete flag domain, Theorem 4.10 applies.

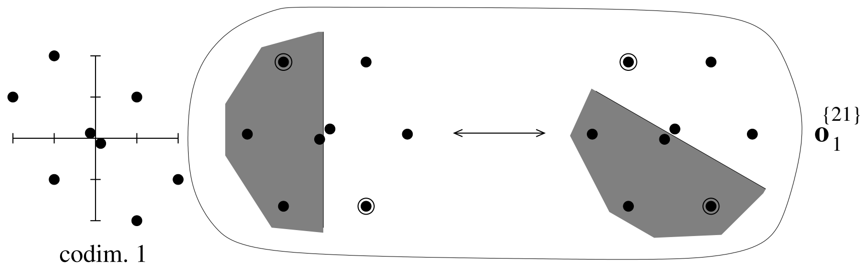
From H_0, we obtain the (three) open orbits:

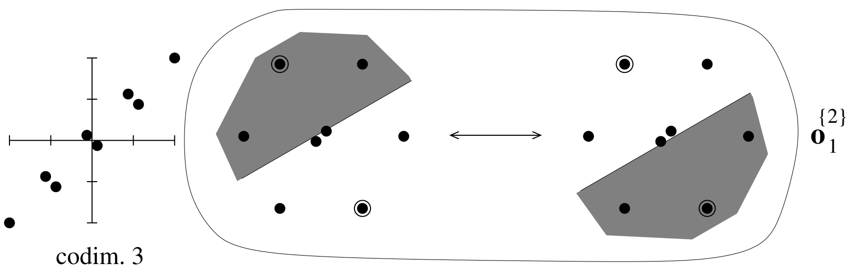


(Note that $W_{\mathbb{R}}(H_0) \cong \mathbb{Z}/2\mathbb{Z}$ is generated by reflection in $\boxed{\bullet}$.)

From H_1, we get three more orbits whose codimensions can be read off from the accompanying mixed Hodge diagrams:





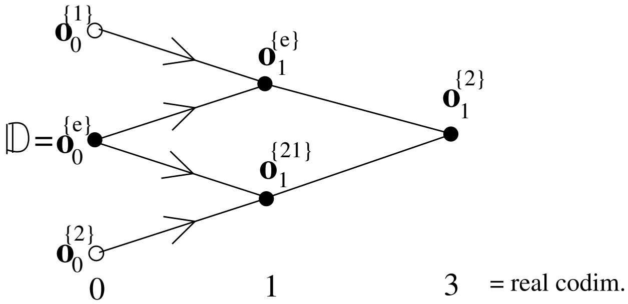


(Here $W_{\mathbb{R}}(H_1) \cong \mathbb{Z}/2\mathbb{Z}$ is generated by reflection in a real root.)

For the enhanced Hasse diagram, we read off inclusions

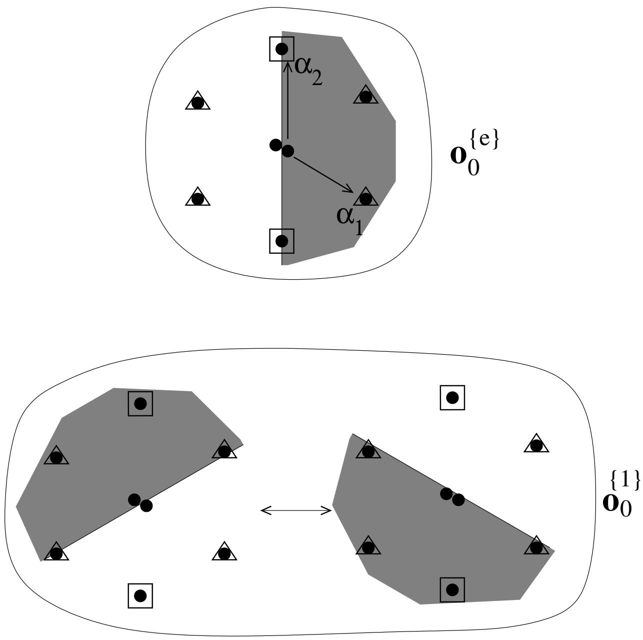
$$\mathrm{cl}(\mathbf{o}_0^{\{1\}}) \supset \mathbf{o}_1^{\{e\}} \subset \mathrm{cl}(\mathbb{D}) \supset \mathbf{o}_1^{\{21\}} \subset \mathrm{cl}(\mathbf{o}_0^{\{2\}})$$

from visually "obvious" Cayley transforms,(28) obtaining the left half of



The right-hand inclusions are, as the diagram indicates, given by cross-actions.

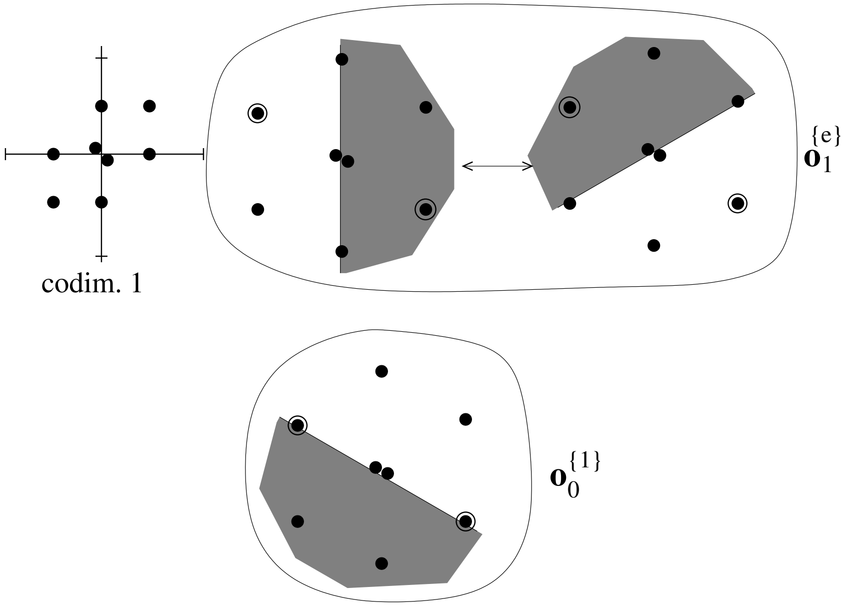
Turning to the classical domain $\mathbb{B}$, we have (from H_0) only 2 open orbits:(29)



(28) What seems clear from the pictures may be justified more carefully using §4.3; this is left to the reader.

(29) The "photographic negatives" of these pictures do not appear because they cannot be obtained from the first picture via $W_{\mathbb{C}}$.

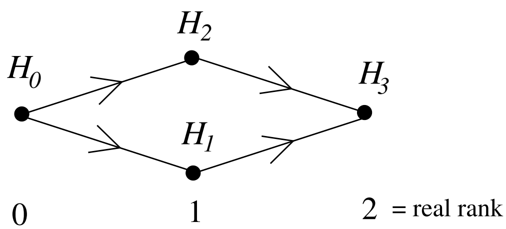
From H_1, we get



where the last $W_{\mathbb{R}}$-orbit demonstrates that the orbit map $\mathbf{o}$ need not be one-to-one in the non-complete-flag case. The incidence diagram is the same as for PGL_2, since $\mathbf{o}_0^{\{e\}} = \mathbb{B}$, $\mathbf{o}_0^{\{1\}} = \mathbb{P}(V_+) \smallsetminus \mathrm{cl}(\mathbb{B})$, and $\mathbf{o}_1^{\{e\}} = \partial\mathbb{B} \cong S^3$.

6.1.2. $G = PSp_4$

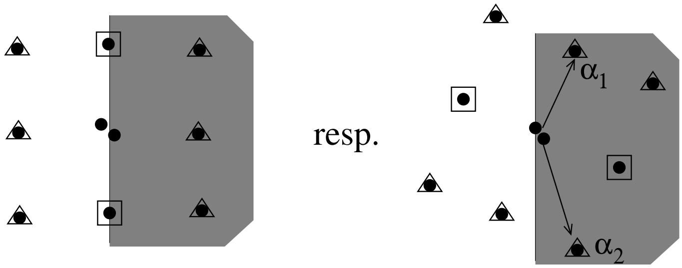
The Hasse diagram for the Cartan subgroups of $G(\mathbb{R})^\circ$ is



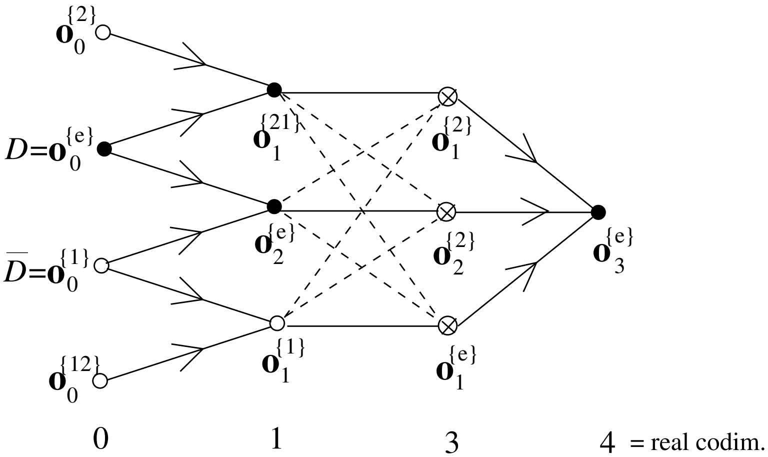
where H_1 [resp. H_2] is obtained from H_0 by the Cayley transform in a long [resp. short] root. We shall only consider the two cases[30] where D is the period domain for rank 4 Hodge structures of weight 1 resp. 3 with Hodge

[30] There is also a weight 2 case with numbers $(1, 2, 1)$, left to the reader.

numbers $(2,2)$ resp. $(1,1,1,1)$ (inducing weight zero HS of type $(3,4,3)$ resp. $(1,1,2,2,2,1,1)$ on $\mathfrak{g}$). The pictures corresponding to (H_0, χ_0) are[31]

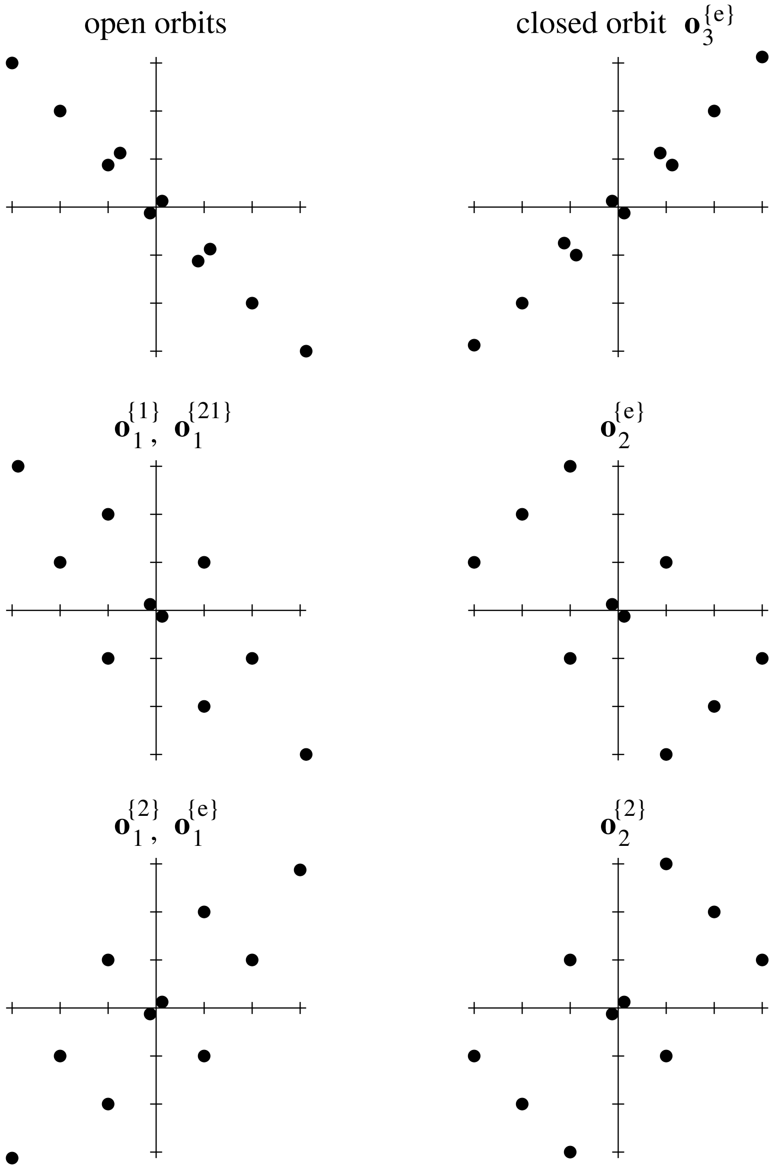


and the enhanced Hasse diagram in the first (Siegel $\mathfrak{H}_2$) case is, up to labeling, the same as for the Carayol domain. The second (complete flag) case $D = D_{(1,1,1,1)}$ has enhanced Hasse diagram



(31) with apologies to the reader for taking α_1 to be the short root (meaning that the Cayley transform in α_1 gives H_2 and vice versa).

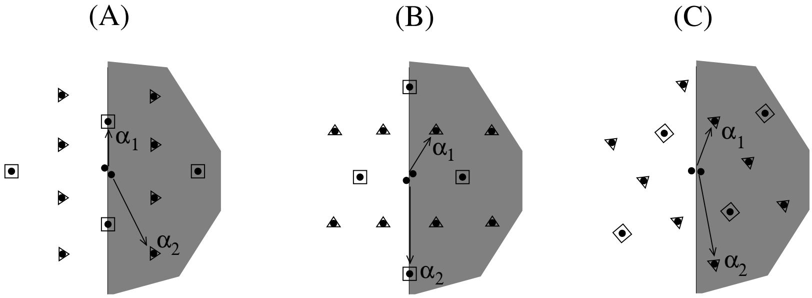
in which we obtain the first examples of non-polarizable boundary strata. Of the mixed Hodge diagrams



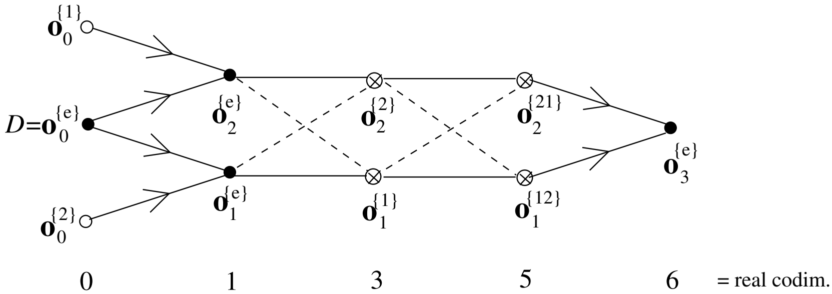
the first four correspond to boundary components in [15], whereas the bottom two have $\dim(\mathfrak{g}^{-1,-1}) = 0$ making the codimension-three substrata obviously non-polarizable.

6.1.3. $G = \text{split } G_2$

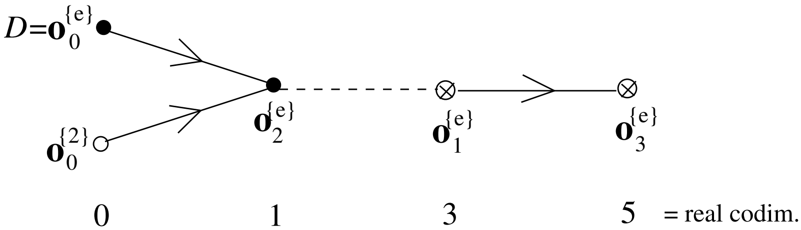
Note that $G(\mathbb{R}) = G(\mathbb{R})^\circ$. There are three cases, corresponding to polarized HS's with Hodge numbers (A) $(2,3,2)$, (B) $(1,2,1,2,1)$, resp. (C) $(1,1,1,1,1,1,1)$ on the standard (7-dimensional) representation and (A) $(1,4,4,4,1)$, (B) $(2,1,2,4,2,1,2)$, resp. (C) $(1,1,1,1,2,2,2,1,1,1,1)$ on $\mathfrak{g}$ (see [12, Chap. 4]). The Cartan diagram is as for PSp_4,[32] and the (H_0, χ_0) pictures are



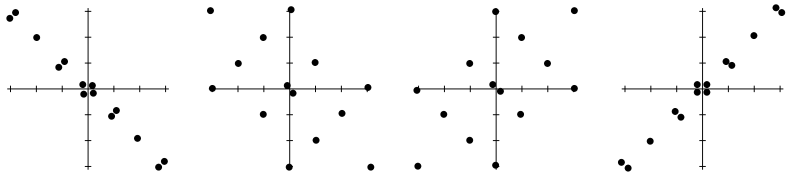
(Note that $W^\circ_{\mathbb{R}}(H_j)$ is generated by reflections in Δ_c resp. $\Delta_{\mathbb{R}}$ for $j = 0$ resp. 3, and by the reflection in $\Delta_{\mathbb{R}}$ and $(-\,\mathrm{id})$ for $j = 1$ and 2.) The complete flag case is (C), with enhanced Hasse diagram



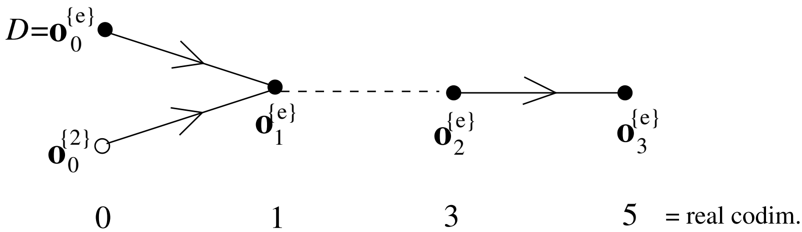
We omit the mixed Hodge diagrams, which are unwieldy, but include them for (B)

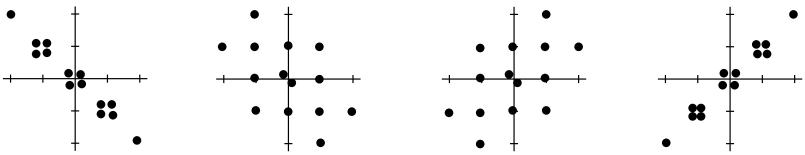


[32] same meaning for H_1 vs. H_2, and the same absurd convention on α_1 vs. α_2



and for (A)





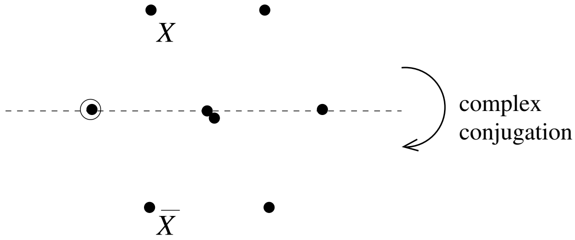
One can check that (A) $\mathbf{o}_2^{\{e\}}$ and $\mathbf{o}_3^{\{e\}}$ are both in the image of a $B(N)$ (see [15, sec. 8]), while polarizability of (B) $\mathbf{o}_2^{\{e\}}$ and (A) $\mathbf{o}_1^{\{e\}}$ is of course covered by Proposition 5.16(i). Note that (B) $\mathbf{o}_3^{\{e\}}$ gives an example of a non-polarizable closed orbit, since the displayed bigrading cannot satisfy Definition 5.9(a)(i) (take $p = 3$, $j = 6$).

6.2. Counterexamples

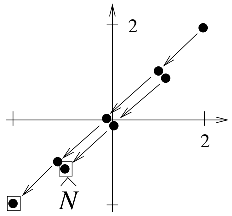
The 3 vignettes with which we conclude this paper illustrate, for polarizable boundary strata, the potential failure of cuspidality, of rationality, and of a stronger notion of polarizability.

6.2.1. $\mathfrak{g}^{-1,-1}$ need not contain a real root

To see this, we have to consider strata of codimension strictly larger than 1. Let D be Carayol's nonclassical domain (*cf.* §6.1), and $\mathcal{O}_{\mathrm{c}} \subset \check{D}$ the (polarizable) closed orbit. The Cartan H determined by $F^\bullet \in \mathcal{O}_{\mathrm{c}}$ has real rank 1, which gives a root diagram



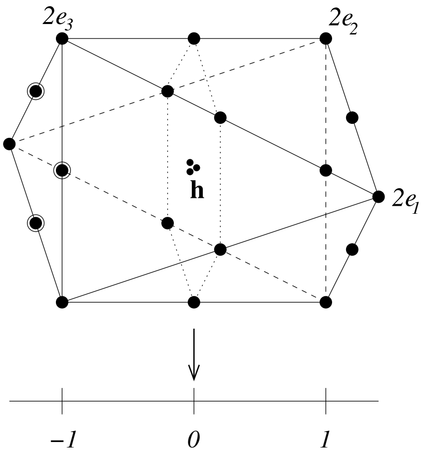
The associated bigrading is



where $\widehat{N} \in \mathfrak{g}_{\mathbb{R}}^{-1\cdot -1}$ is as in Definition 5.9 and arrows denote the action of $\operatorname{ad} N$. Clearly $\mathfrak{g}^{-1,-1} = \mathbb{C}\langle X, \bar{X}\rangle$ and $\widehat{N}$ is a multiple of $X + \bar{X}$.

6.2.2. A noncuspidal boundary component

This time, in addition to codimension > 1, we have to start with a Mumford-Tate group of rank at least 3. Taking $G = Sp_6$, we consider the Siegel domain $D = \mathfrak{H}_3$ parametrizing Hodge structures φ of type $(3, 3)$. The associated projection π_χ on roots takes the form



sending $ae_1 + be_2 + ce_3 \mapsto \frac{1}{2}(a + b - c)$.

On the other hand, with H $\mathbb{R}$-split, the same picture describes $\pi_{\chi_{\widetilde{Y}}}$ for the (Hodge-Tate) limiting mixed Hodge structures parametrized by $B_{\mathbb{R}}(N)$, where N is the sum of root vectors for the 3 circled roots ($e_3 - e_1$, $e_3 - e_2$, $-e_1 - e_2$). The subalgebra $\ker(\mathrm{ad}\,\widetilde{Y}) \cong Gr_0^{\widetilde{W}}\mathfrak{g}$, whose root system maps to 0 in the picture, is $\mathfrak{sl}_3$. Since SL_3 has no compact Cartan, $\mathfrak{q} = \widetilde{W}_0\mathfrak{g}$ is not a cuspidal parabolic subgroup. The boundary component $B(N)$ maps to the closed orbit $\mathcal{O}_{\mathrm{c}} \subset \check{D}$, and so $\mathcal{O}_{\mathrm{c}}$ is not cuspidal.

6.2.3. A non-rational $G(\mathbb{R})^{\circ}$-orbit in ∂D

Take D once more to be Carayol's $(1,2,2,1)$-domain, but this time with G a $\mathbb{Q}$-anisotropic form of $SU(2,1)$. More precisely, let V_+ be a 3-dimensional vector space over $\mathbb{Q}(i)$, H a $\mathbb{Q}(i)$-Hermitian form of signature $(2,1)$ on V_+ hat does not represent zero, and Q the alternating form on $V := \mathrm{Res}_{\mathbb{Q}(i)/\mathbb{Q}}(V_+)$ given by minus the imaginary part of H. So

$$G = \mathrm{Res}_{\mathbb{Q}(i)/\mathbb{Q}}\left(\mathrm{Aut}(V_+, H)\right) = \mathrm{Res}_{\mathbb{Q}(i)/\mathbb{Q}}\left(GL(V_+)\right) \cap Sp(V, Q),$$

and extending Q to $V_{\mathbb{Q}(i)} = V_+ \oplus \overline{V_+}$, we have $H(v,v) = -2iQ(v,\bar{v})$.

Now, we know that the resulting domain D has polarizable boundary strata, which then come from $\mathbb{R}$-limit mixed Hodge structures. But if the stratum is rational in our sense, then $\widetilde{W}_{\bullet} = W(N)_{\bullet}$ can be defined over $\mathbb{Q}$ (with weights on V centered about 3, not 0). Taking any nonzero rational vector $w \in W_2V$, we have $w = v + \bar{v}$ for $(0 \neq)\, v \in W_2V_+$, and $0 = Q\left(W_2W_{\mathbb{Q}(i)}, W_2V_{\mathbb{Q}(i)}\right)$ forces

$$0 = -2iQ(v,\bar{v}) = H(v,v),$$

in contradiction to anisotropy.

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