ANNALES

DE

L'INSTITUT FOURIER

Bin ZHAO

Local Indecomposability of Hilbert Modular Galois Representations

Tome 64, nº 4 (2014), p. 1521-1560.

<http://aif.cedram.org/item?id=AIF_2014__64_4_1521_0>

Ann. Inst. Fourier, Grenoble
64, 4 (2014) 1521-1560

LOCAL INDECOMPOSABILITY OF HILBERT MODULAR GALOIS REPRESENTATIONS

by Bin ZHAO (*)

ABSTRACT. — We prove the indecomposability of the Galois representation restricted to the p-decomposition group attached to a non CM nearly p-ordinary weight two Hilbert modular form over a totally real field F under the assumption that either the degree of F over $\mathbb{Q}$ is odd or the automorphic representation attached to the Hilbert modular form is square integrable at some finite place of F.

RÉSUMÉ. — Nous prouvons l'indécomposabilité de la représentation galoisienne restreinte au groupe de p-décomposition attaché à une forme modulaire quasi-ordinaire de Hilbert sans multiplication complexe de poids 2 sous certainess hypothèses.

The main purpose of this paper is to decide the indecomposability of a Hilbert modular ordinary p-adic Galois representation restricted to the decomposition group at p, under the assumption that the representation is not of CM type. This question was originally posed by R.Greenberg. In the elliptic modular case, it was studied by Ghate and Vatsal and they gave an affirmative answer in [11] under some conditions. In a recent preprint [1], joint with Balasubramanyam, they generalized their result to the Hilbert modular case under some restrictive conditions. Their method is to study the specialization of Λ-adic forms corresponding to weight one classical forms, and they use the density of such specializations to conclude for higher weight modular forms.

In contrast to [1], our method is geometric and relies on the study of Galois representations attached to abelian varieties of GL(2)-type. More precisely, let F be a totally real field and f be a (parallel) weight two Hilbert modular form of level $\mathfrak{m}$ over F. Assume that f is a Hecke eigenform and

Keywords: Galois representation, Hilbert modular forms, complex multiplication.
Math. classification: 11F80, 11G18, 14K22.
(*) The author is partially supported by Hida's NSF grant DMS 0753991 and DMS 0854949 through UCLA graduate division.

let K_f be its Hecke field. For any prime λ of K_f over a rational prime p, let $K_{f,\lambda}$ be the completion of K_f at λ. It is well known that there is a Galois represention $\rho_f : \mathrm{Gal}(\bar{\mathbb{Q}}/F) \to GL_2(K_{f,\lambda})$ attached to f. Moreover if the eigenform f is nearly p-ordinary, then up to equivalence the restriction of ρ_f to the decomposition group $D_\mathfrak{p}$ of $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$ at $\mathfrak{p}$ is of the shape (see [35] Theorem 2 for the ordinary case and [18] Proposition 2.3 for the nearly ordinary case):

$$\rho_f|_{D_\mathfrak{p}} \sim \begin{pmatrix} \epsilon_1 & * \\ 0 & \epsilon_2 \end{pmatrix}.$$

In this paper we need to put the following technical condition on f when the degree of F over $\mathbb{Q}$ is even: there exists a finite place v of F such that π_v is square integrable (i.e. special or supercuspidal) where $\pi_f = \otimes_v \pi_v$ is the automorphic representation of $GL_2(F_\mathbb{A})$ associated to f ($F_\mathbb{A}$ is the adele ring of F). Then the main result of this paper is:

Theorem 1. — *If f does not have complex multiplication, then $\rho_f|_{D_\mathfrak{p}}$ is indecomposable.*

We will state this theorem in a little more general way as Theorem 5.6 in Section 5 and give a proof there. Here is the sketch of our argument. Under the assumption on f, there exist an abelian variety $A_{f/F}$ and a homomorphism $L \to \mathrm{End}^0(A_{f/F})$ where L/K_f is a finite extension and the degree of L over $\mathbb{Q}$ equals to the dimension of A_f, such that the Galois representation ρ_f comes from the λ-adic Tate module of A_f (at least upto a twist of a character). Hence the theorem is reduced to prove: if the abelian variety $A_{f/F}$ does not have complex multiplication, then its λ-adic Tate module $T_\lambda(A_f)$ is indecomposable as an $I_\mathfrak{p}$-module, where $I_\mathfrak{p}$ is the inertia group of $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$ at a prime $\mathfrak{p}$ of F over p. By an analysis of the endomorphism algebra of an abelian variety of $GL(2)$-type in section 1, we can always take L to be a totally real field (see Proposition 1.4). Moreover, we can assume that A_f is absolutely simple and has good reduction at $\mathfrak{p}$. Then the key argument can be divided into two steps:

First, under the assumption that $A_{f/F}$ does not have complex multiplication, we can find two distinct primes $\mathfrak{Q}$ and $\mathfrak{L}$ of F not lying over p with the following property: the abelian variety $A_{f/F}$ has good reduction at $\mathfrak{Q}$ and $\mathfrak{L}$, and if we use $A_\mathfrak{Q}$ (resp. $A_\mathfrak{L}$) to denote the reduction of A_f at $\mathfrak{Q}$ (resp. $\mathfrak{L}$), then $\mathrm{End}^0_L(A_{\mathfrak{Q}/\bar{\mathbb{F}}_q})$ and $\mathrm{End}^0_L(A_{\mathfrak{L}/\bar{\mathbb{F}}_l})$ are non-isomorphic CM quadratic extension of L (see Lemma 5.1). Here q (resp. l) is the residue characteristic of the prime $\mathfrak{Q}$ (resp. $\mathfrak{L}$). The proof is a slight modification of the argument given in [15] using Faltings's isogeny theorem, a Serre-type open image theorem due to Ribet, and some standard results on the density of primes. As is clear from the argument given in the proof of Lemma 5.1, when the prime p is ramified in the field L, we need to construct an extra auxiliary prime in our argument.

Second, we prove that if the λ-adic representation of $I_{\mathfrak{p}}$ attached to the Tate module of A_f is decomposable, it is impossible to find the primes $\mathfrak{Q}$ and $\mathfrak{L}$ with the property in the first step. The idea is that by putting polarization and level structure on $A_{f/F}$, the abelian variety $A_{f/F}$ gives rise to a point on an appropriate Hilbert modular Shimura variety. When the rational prime p is ramified in the field L, the Lie algebra $\mathrm{Lie}(A_{\mathfrak{p}/\bar{\mathbb{F}}_p})$ may not be free $\mathcal{O}_L \otimes_{\mathbb{Z}} \bar{\mathbb{F}}_p$-module of rank 1. In other words, the special fiber of the abelian variety $A_{f/F}$ at $\mathfrak{p}$ may not lie in the Hilbert modular Shimura variety considered in [15]. In our argument, we use the integral model of Hilbert modular Shimura variety considered by Deligne and Pappas in [6]. The definition of this model will be recalled in section 2 and 3. We prove that if the abelian variety $A_{f/F}$ has good reduction at the prime $\mathfrak{p}$, then it extends to an $\mathcal{O}_{F,(\mathfrak{p})}$-valued point on this Shimura variety. We also study the local properties of the above Shimura variety in section 3. In section 4 we prove that each L-linear isogeny of $A_{\mathfrak{Q}/\bar{F}_q}$ with degree prime to q induces an automorphism of the Shimura variety, and hence an automorphism of the ordinary deformation space of the mod q reduction of A_f sitting in the special fiber of x at q. Using the rigid analytic logarithms of the Serre-Tate coordinates on the ordinary deformation space (see section 4), we can prove that this automorphism must also fixes the special fiber of x at l. Then we can conclude that $\mathrm{End}^0_L(A_{\mathfrak{Q}/\bar{\mathbb{F}}_q})$ and $\mathrm{End}^0_L(A_{\mathfrak{L}/\bar{\mathbb{F}}_l})$ must be isomorphic as L-algebras. In [15], Hida proved this result under the assumption that the prime p is unramified in the base field F and the field L. What we do here is to remove the unramified assumption. Besides the different Hilbert modular Shimura variety we consider above, there are two more problems which arising from dropping the unramified conditions. If p is ramified in the base field F, the abelian variety $A_{f/F}$ may not sit in the origin of the local deformation space of its mod p fiber. We calculate its Serre-Tate coordinate in Lemma 4.3 and use eigen-coordinates to eliminate this ambiguity. If p is ramified in the field L, we have troubles on comparing the differential sheaf of $A_{f/F}$ and its Serre-Tate coordinates by the Kodaira-Spencer map. This can be overcame by a suitable base change. Both of these two technical issues will be discussed much more concretely at the beginning of section 4.

We prove the main result in Section 5, and we give an Λ-adic version of our result by applying an argument in [11],which generalizes the result of [1] unconditionally when the degree $[F:\mathbb{Q}]$ is odd and when the degree $[F:\mathbb{Q}]$ is even, we need to assume that at some finite place v of F, the representation π_v is square integrable. At the end we explain how our result can be applied to study a problem of Coleman on determining which classical elliptic modular forms lie in the image of the operator defined in [4].

Acknowledgement: I would like to express my sincere gratitude to my advisor Professor Hida Haruzo for his patient guidance as well as constant support. Without his careful explanation of his work [15] , I cannot get the approach to the work in this paper. I would like to thank Professor Don Blasius, Professor Chandrashekhar Khare, Professor Eknath Ghate and Ashay Burungale for many helpful discussions on this topic. I would also like to thank the anonymous referee for a lot of useful suggestions.

Notations: Throughout this paper, we use F to denote a totally real field with degree d over $\mathbb{Q}$ and use $\mathcal{O}_F$ to denote its integer ring. Let $\mathcal{D}=\mathcal{D}_{F/\mathbb{Q}}$ be the different of $F/\mathbb{Q}$ and $d_F = Norm_{F/\mathbb{Q}}(\mathcal{D})$ be its discriminant. For any prime $\mathfrak{p}$ of $\mathcal{O}_F$, let $\mathcal{O}_\mathfrak{p}$ (resp. $F_\mathfrak{p}$) be the completion of $\mathcal{O}_F$ (resp. F) with respect to $\mathfrak{p}$. We use $\mathbb{A}$ to denote the adele ring of $\mathbb{Q}$, and use $F_\mathbb{A}$ (resp. $F_{\mathbb{A}_f}$) to denote the adele ring (resp. finite adele ring) of F.

Fix an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$. For each rational prime p, we fix an algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and let $\mathbb{C}_p$ be the p-adic completion of $\bar{\mathbb{Q}}_p$. Fix an embedding $i_p : \bar{\mathbb{Q}} \to \mathbb{C}_p$. We also fix an algebraic closure $\bar{\mathbb{F}}_p$ of the prime field $\mathbb{F}_p$. Let $W_p = W(\bar{\mathbb{F}}_p)$ be the ring of Witt vectors of $\bar{\mathbb{F}}_p$, and L_p be its quotient field. Then L_p can be identified with the p-adic completion of the maximal unramified extension in $\bar{\mathbb{Q}}_p/\mathbb{Q}_p$.

1. Abelian Varieties of GL(2)-type

Let E be a number field with degree d over $\mathbb{Q}$, and $A_{/\overline{\mathbb{Q}}}$ be an abelian variety of dimension d. Set $\mathrm{End}^0(A_{/\overline{\mathbb{Q}}}) = \mathrm{End}(A_{/\overline{\mathbb{Q}}}) \otimes_\mathbb{Z} \mathbb{Q}$, which is a finite dimensional semisimple algebra over $\mathbb{Q}$. Suppose that we have an algebra homomorphism $E \to \mathrm{End}^0(A_{/\overline{\mathbb{Q}}})$, which identifies E with a subfield of $\mathrm{End}^0(A_{/\overline{\mathbb{Q}}})$. Recall that the abelian variety $A_{/\overline{\mathbb{Q}}}$ has complex multiplication if $\mathrm{End}^0(A_{/\overline{\mathbb{Q}}})$ contains a commutative semisimple subalgebra of dimension $2d$ over $\mathbb{Q}$. Then from [23] Section 5.3.1, we have the following two results:

Proposition 1.1. — *If $A_{/\overline{\mathbb{Q}}}$ does not have complex multiplication, then $A_{/\overline{\mathbb{Q}}}$ is isotypic (i.e. there exists a simple abelian variety $B_{/\overline{\mathbb{Q}}}$ such that $A_{/\overline{\mathbb{Q}}}$ is isogeneous to $(B_{/\overline{\mathbb{Q}}})^e$ for some $e \geqslant 1$), and $\mathrm{End}^0_E(A_{/\overline{\mathbb{Q}}}) = E$.*

Proposition 1.2. — *Under the conditions of Proposition 1.1, if we assume further that* $A_{/\overline{\mathbb{Q}}}$ *is simple, then one of the following four possibilities holds for* $D = \mathrm{End}^0(A_{/\overline{\mathbb{Q}}})$*:*

1. *E is a quadratic extension of a totally real field Z and D is a totally indefinite division quaternion algebra over Z;*
2. *E is a quadratic extension of a totally real field Z and D is a totally definite division quaternion algebra over Z;*
3. *E is a quadratic extension of a CM field Z and D is a division quaternion algebra over Z;*
4. *$E = D$ and E is totally real.*

Remark 1.3. —

1. A quaternion algebra D over a totally real field Z is called totally indefinite if for any real embedding $\tau : Z \to \mathbb{R}$, the $\mathbb{R}$-algebra $D \otimes_{Z,\tau} \mathbb{R}$ is isomorphic to the matrix algebra $M_2(\mathbb{R})$; the quaternion algebra $D_{/Z}$ is called totally definite if for any real embedding $\tau : Z \to \mathbb{R}$, the $\mathbb{R}$-algebra $D \otimes_{Z,\tau} \mathbb{R}$ is isomorphic to the Hamilton quaternion algebra $\mathbb{H}$.
2. From Proposition 1.1, we see that $\mathrm{End}^0(A_{/\overline{\mathbb{Q}}})$ is always a central simple algebra and E is a maximal commutative subfield of $\mathrm{End}^0(A_{/\overline{\mathbb{Q}}})$;
3. As remarked in [23], case 2 in Proposition 1.2 cannot happen by [32], Theorem 5(a) and Proposition 15.

Proposition 1.4. — *Under the notations and assumptions in Proposition 1.1, assume further that there exists a totally real field k such that the abelian variety $A_{/\overline{\mathbb{Q}}}$ is defined over k, and the homomorphism $E \to$* $\mathrm{End}^0(A_{/\overline{\mathbb{Q}}})$ *factors through* $\mathrm{End}^0(A_{/k})$*. Then we can find a totally real field F with degree d over $\mathbb{Q}$, which can be embedded into* $D = \mathrm{End}^0(A_{/\overline{\mathbb{Q}}})$ *as a unital subalgebra of D.*

Proof. — By Propositon 1.1, we can find a simple abelian variety $B_{/\overline{\mathbb{Q}}}$ and an integer e such that $A_{/\overline{\mathbb{Q}}}$ is isogeneous to $(B_{/\overline{\mathbb{Q}}})^e$. Hence we have an isomorphism of simple algebras $\mathrm{End}^0(A_{/\overline{\mathbb{Q}}}) \cong M_e(\mathrm{End}^0(B_{/\overline{\mathbb{Q}}}))$,and $d = e \cdot d_1$, where d_1 is the dimension of $B_{/\overline{\mathbb{Q}}}$. Since any maximal commutative subfield of $\mathrm{End}^0(A_{/\overline{\mathbb{Q}}})$ has degree d over $\mathbb{Q}$, any maximal commutative subfield of $D_1 = \mathrm{End}^0(B_{/\overline{\mathbb{Q}}})$ should have dimension $d/e = d_1$. In other words, we can find number field E_1 of degree d_1 over $\mathbb{Q}$, which can be embedded into $\mathrm{End}^0(B_{/\overline{\mathbb{Q}}})$ as a subalgebra. Since $A_{/\overline{\mathbb{Q}}}$ does not have complex multiplication, neither does $B_{/\overline{\mathbb{Q}}}$. In summary, $B_{/\overline{\mathbb{Q}}}$ satisfies all the assumptions in Proposition 1.2. Assume that $\mathrm{End}^0(B_{/\overline{\mathbb{Q}}})$ is of type 3 as in Proposition

1.2, i.e. $\mathrm{End}^0(B_{/\overline{\mathbb{Q}}})$ is a division quaternion algebra over a CM field Z and $[E_1 : Z] = 2$. Since $d_1 = [E_1 : \mathbb{Q}] = 2[Z : \mathbb{Q}]$, the degree s of Z over $\mathbb{Q}$ equals to $\frac{d_1}{2}$. Since Z is a CM field, we can find $s' = \frac{s}{2}$ different embeddings $\tau_i : Z \to \overline{\mathbb{Q}}, i = 1, ..., s'$, such that $\mathrm{Hom}_{\mathbb{Q}}(Z, \overline{\mathbb{Q}}) = \{\tau_1, ..., \tau_{s'}, \bar{\tau}_1, ..., \bar{\tau}_{s'}\}$, where $\bar{\tau}_i$ is the complex conjugation of τ_i for $i = 1, ..., s'$. Then we have an isomorphism

$$\theta : D_1 \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{\tau_i, i=1,\dots,s'} M_2(\mathbb{C}).$$

Let π_i be the composition

$$D_1 \hookrightarrow D_1 \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\theta} \prod_{\tau_i, i=1,\dots,s'} M_2(\mathbb{C}) \xrightarrow{\pi_i} M_2(\mathbb{C}),$$

where the map π_i is the i-th projection, for $i = 1, ..., s'$. Let $\bar{\pi}_i$ be the complex conjugation of π_i. Then $\{\pi_1, ..., \pi_{s'}, \bar{\pi}_1, ..., \bar{\pi}_{s'}\}$ are all the absolutely irreducible (complex) representations of D_1 (up to isomorphism).

On the other hand, we have a representation of D_1 by $\rho_1 : D_1 \to \mathrm{End}_{\mathbb{C}}(\mathrm{Lie}(B) \otimes_{\bar{\mathbb{Q}}} \mathbb{C})$. Let r_i (resp. s_i) be the multiplicity of π_i (resp $\bar{\pi}_i$) in ρ_1. Then for any $z \in Z$, the trace of $\rho_1(z)$ is given by the formula:

$$\mathrm{Tr}(\rho_1(z)) = 2\sum_{i=1}^{s'}(r_i\tau_i(z) + s_i\bar{\tau}_i(z)).$$

Since $\mathrm{Lie}(A_{/\bar{\mathbb{Q}}}) \cong (\mathrm{Lie}(B_{/\bar{\mathbb{Q}}}))^e$, we have the representation

$$\rho : D \to \mathrm{End}_{\mathbb{C}}(\mathrm{Lie}(A) \otimes_{\bar{\mathbb{Q}}} \mathbb{C}),$$

such that for any $z \in Z$,

$$\mathrm{Tr}(\rho(z)) = e\mathrm{Tr}(\rho_1(z)) = 2e\sum_{i=1}^{s'}(r_i\tau_i(z) + s_i\bar{\tau}_i(z)).$$

Since $Z \subseteq E$ and the homomorphism $E \to \mathrm{End}^0(A_{/\bar{\mathbb{Q}}})$ factors through $\mathrm{End}^0(A_{/k})$, we have $\mathrm{Tr}(\rho(z)) \in k$, for any $z \in Z$. From [32] Section 4, we have $r_i + s_i = 2$, for all $i = 1, ..., s'$. Thus for each i, either $r_i = s_i = 1$ or $r_i \cdot s_i = 0$. If $r_i \cdot s_i = 0$ for at least one i, then $\mathrm{Tr}(\rho(z))$ cannot lie in the totally real field k for all $z \in Z$ as Z is assumed to be a CM field. Hence $r_i = s_i = 1$ for all i. Then by [32] Theorem 5(e) and Proposition 19, this case cannot happen.

Combined with Remark 1.3(3), we see that $\mathrm{End}^0(B_{/\overline{\mathbb{Q}}})$ is either a totally real field or a totally indefinite division algebra over a totally real field. Then the existence of F results from:

Lemma 1.5. — *Let D be a central simple algebra over a totally real field Z with $[D : Z] = d^2$. If for all real embeddings $\tau : Z \to \mathbb{R}$, the $\mathbb{R}$-algebra*

$D \otimes_{Z,\tau} \mathbb{R}$ is isomorphic to the matrix algebra $M_d(\mathbb{R})$, then we can find a field extension F/Z with degree d such that F is totally real and can be embedded into D as an Z-subalgebra.

Proof of the lemma: We use an argument similar with the proof of Lemma 1.3.8 in [5]. It is enough to find a field extension F/Z with degree d such that F is totally real and splits D(i.e $D \otimes_Z F \cong M_d(F)$).

Let Σ be a non empty set of non-archimedean places of Z containing all the finite places where D does not split, and Σ_∞ be the set of archimedean places of Z. By the weak approximation theorem, the natural map:

$$Z \to \prod_{v \in \Sigma} Z_v \times \prod_{v \in \Sigma_\infty} Z_v$$

has dense image. Hence we can find a monic polynomial $f(X) \in Z[X]$ of degree d, such that it is sufficiently close to a monic irreducible polynomial of degree d over Z_v for all $v \in \Sigma$, and it is sufficiently close to a totally split polynomial of degree d over $\mathbb{R}$ for all $v \in \Sigma_\infty$. Set $F = Z[X]/(f(X))$. Then F/Z is a degree d field extension such that F is totally real and for any $v \in \Sigma$, there is exactly one place w of F lying over v and hence F_w/Z_v is a degree d extension of local fields.

We still need to check that F splits D. Since $D \otimes_Z F$ is a central simple algebra over F and F is a global field, it is enough to prove that for any place w of F (archimedean and non-archimedean), we have an isomorphism $D \otimes_Z F_w \cong M_d(F_w)$. Let v be the place of Z over which w lies.

If w is archimedean, then $Z_v \cong F_w \cong \mathbb{R}$, and hence

$$D \otimes_Z F_w \cong (D \otimes_Z Z_v) \otimes_{Z_v} F_w \cong M_d(\mathbb{R}), \tag{1.1}$$

by our assumption on D.

If w is non-archimedean and v is not in Σ,then $D \otimes_Z Z_v$ is already isomorphic to the matrix algebra over Z_v, so we are safe in this case.

Finally, assume that w is non-archimedean and $v \in \Sigma$. As F_w/Z_v is a degree d extension of local field, the base change from Z_v to F_w induces a homomorphism of Brauer groups $\mathrm{Br}(Z_v) \to \mathrm{Br}(F_w)$, which under the isomorphism $\mathrm{Br}(Z_v) \cong \mathrm{Br}(F_w) \cong \mathbb{Q}/\mathbb{Z}$ by local class field theory, is nothing but multiplication by d. As $[D : Z] = d^2$, the order of $D \otimes_Z Z_v$ in $\mathrm{Br}(Z_v)$ is divisible by d. This implies that $D \otimes_Z F_w$ represents the identity element in $\mathrm{Br}(F_w)$; i.e.$D \otimes_Z F_w \cong M_d(F_w)$. Hence F/Z is the desired extension. □

Hereafter we always work with the pair $(A_{/\overline{\mathbb{Q}}}, \iota : F \to \mathrm{End}^0(A_{/\overline{\mathbb{Q}}}))$, where F is a totally real field with degree d over $\mathbb{Q}$. Since the abelian variety $A_{/\overline{\mathbb{Q}}}$ is projective, we can find a number field k such that A is defined over k, and $\mathrm{End}(A_{/\overline{\mathbb{Q}}})) = \mathrm{End}(A_{/k})$. Let $\mathcal{O}_k$ be the integer ring of k, and for all prime ideals $\mathfrak{P}$ of $\mathcal{O}_k$ over some rational prime p, let $\mathcal{O}_{(\mathfrak{P})}$ be localization of $\mathcal{O}_k$ at the prime $\mathfrak{P}$ and $\mathbb{F}_{\mathfrak{P}} = \mathcal{O}_k/\mathfrak{P}$ be its residue field. As in [15], we make the following assumption:

(NLL) the abelian variety $A_{/k}$ has good reduction at $\mathfrak{P}$ and the reduction $A_0 = A \otimes_{\mathcal{O}_{\mathfrak{P}}} \mathbb{F}_{\mathfrak{P}}$ has nontrivial p-torsion $\bar{\mathbb{F}}_p$-points.

Change the abelian variety $A_{/k}$ if necessary, we can assume that ι gives a homomorphism $\iota : \mathcal{O}_F \to \mathrm{End}(A_{/k})$. Let $\bar{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_{\mathfrak{P}}$. $W_p = W(\bar{\mathbb{F}}_p)$ is the ring of Witt vectors of $\bar{\mathbb{F}}_p$. We have the decomposition of Barsotti-Tate $\mathcal{O}_F$-modules:

$$A_0[p^\infty] = \bigoplus_{\mathfrak{p}|p} A_0[\mathfrak{p}^\infty].$$

Here $\mathfrak{p}$ ranges over the primes ideals of $\mathcal{O}_F$ over p and for each $\mathfrak{p}$, let

$$A_0[\mathfrak{p}^\infty] = \lim_{\rightarrow} A_0[\mathfrak{p}^n]$$

be the $\mathfrak{p}$-divisible Barsotti-Tate group of A_0. We also define

$$T_{\mathfrak{p}}(A_0) = \lim_{\leftarrow} A_0[\mathfrak{p}^n](\bar{\mathbb{F}}_p)$$

as the $\mathfrak{p}$-divisible Tate module of A_0.

We say that a prime $\mathfrak{p}$ of $\mathcal{O}_F$ over p is ordinary if $A_0[\mathfrak{p}]$ has nontrivial $\bar{\mathbb{F}}_p$-points, otherwise we say that $\mathfrak{p}$ is local-local. When $\mathfrak{p}$ is ordinary and p is unramified in k, we have an exact sequence of Barsotti-Tate $\mathcal{O}_{\mathfrak{p}}$-modules over W_p:

$$0 \to \mu_{p^\infty} \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathfrak{p}}^* \to A[\mathfrak{p}^\infty]_{/W_p} \to F_{\mathfrak{p}}/\mathcal{O}_{\mathfrak{p}} \to 0.$$

Here $\mathcal{O}_{\mathfrak{p}}^* = \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{O}_{\mathfrak{p}}, \mathbb{Z}_p)$ is the $\mathbb{Z}_p$-dual of $\mathcal{O}_{\mathfrak{p}}$.

Let Σ_p^{ord} be the set of all ordinary primes of $\mathcal{O}_F$ over p, and Σ_p^{ll} be the set of all local-local primes. Then the condition (NLL) is equivalent to the fact that Σ_p^{ord} is not empty. Also we define:

$$A_0[p^\infty]^{ord} = \bigoplus_{\mathfrak{p}\in\Sigma_p^{ord}} A_0[\mathfrak{p}^\infty], A_0[p^\infty]^{ll} = \bigoplus_{\mathfrak{p}\in\Sigma_p^{ll}} A_0[\mathfrak{p}^\infty].$$

2. Abelian varieties with real multiplication

In this section we introduce the notion of abelian varieties with real multiplication (AVRM for short). Then we prove that by a change via an isogeny, we can make the abelian variety $A_{/k}$ considered in the previous section into an abelian variety of this type.

Fix an invertible $\mathcal{O}_F$-module L, with a notion of positivity L_+ on it: for each real embedding $\tau : F \to \mathbb{R}$, we give an orientation on the line $L \otimes_{\mathcal{O}_F,\tau} \mathbb{R}$. First we recall the following definition in [6]:

Definition 2.1. — *An L-polarized abelian scheme with real multiplication by $\mathcal{O}_F$ is the triple $(A_{/S}, \iota, \varphi)$ consisting of*

1. *$A_{/S}$ is an abelian scheme of relative dimension d;*
2. *$\iota : \mathcal{O}_F \to \mathrm{End}(A_{/S})$ is an algebra homomorphism which gives $A_{/S}$ an $\mathcal{O}_F$-module structure;*
3. *$\varphi : \underline{L} \to \mathrm{Hom}^{Sym}_{\mathcal{O}_F}(A_{/S}, A^t_{/S})$ is an $\mathcal{O}_F$-linear morphism of sheaves of $\mathcal{O}_F$-modules on the étale site $(Sch_{/S})_{ét}$ of the category of S-schemes, such that φ sends totally positive elements of L into polarizations of $A_{/S}$, and the natural morphism $\alpha : A \otimes_{\mathcal{O}_F} \underline{L} \to A^t$ is an isomorphism. Here A^t is the dual abelian scheme of A, and $\underline{L}$ is the constant sheaf valued in L, and the sheaf $\mathrm{Hom}^{Sym}_{\mathcal{O}_F}(A_{/S}, A^t_{/S})$ is defined by :*
   $(Sch_{/S})_{ét} \ni T \mapsto \mathrm{Hom}^{Sym}_{\mathcal{O}_F,T}(A_{T/T}, A^t_{T/T}) = \{\lambda : A_{T/T} \to A^t_{T/T} | \lambda$ *is $\mathcal{O}_F$-linear and symmetric*$\}$

When $L = \mathfrak{c}$ is a fractional ideal of $\mathcal{O}_F$ with the natural notion of positivity, we call the isomorphism $\alpha : A \otimes_{\mathcal{O}_F} \mathfrak{c} \to A^t$ a $\mathfrak{c}$-polarization of A (see [25]*1.0 for more discussion). We also make the convention that for $c \in \mathfrak{c}$, the morphism $\lambda(c) : A \to A^t$ is the corresponding symmetric $\mathcal{O}_F$-linear homomorphism.*

Remark 2.2. — The fppf abelian sheaf $A \otimes_{\mathcal{O}_F} \underline{L}$ is the sheafication of the functor

$$(Sch_{/S})_{\mathrm{fppf}} \ni T \mapsto A(T) \otimes_{\mathcal{O}_F} L.$$

This sheaf is represented by an abelian scheme over S, which is denoted by $A \otimes_{\mathcal{O}_F} L$. Hence the isomorphism α in (3) can be regarded as an isomorphism of abelian schemes over S.

Definition 2.3. — *Let $A_{/S}$ be an abelian scheme over a scheme S of relative dimension d, and $\iota : \mathcal{O}_F \to \mathrm{End}(A_{/S})$ be an algebra homomorphism. We say that the pair $(A_{/S}, \iota)$ satisfies the condition* (DP) *if the natural morphism $\alpha : A \otimes_{\mathcal{O}_F} \mathrm{Hom}^{Sym}_{\mathcal{O}_F}(A_{/S}, A^t_{/S}) \to A^t$ is an isomorphism. We say that the pair $(A_{/S}, \iota)$ satisfies the condition* (RA) *if Zariski locally on S, $\mathrm{Lie}(A_{/S})$ is a free $\mathcal{O}_S \otimes_{\mathbb{Z}} \mathcal{O}_F$-module of rank* 1.

We remark here that the two conditions (DP) and (RA) in Definition 2.3 can be checked at each geometric point of the base scheme S. When the pair $(A_{/S}, \iota)$ satisfies the condition (RA), we come to the notion of abelian schemes with real multiplication (by $\mathcal{O}_F$) defined in [29]. As explained in [6]2.9, when d_F is invertible on S, condition (DP) in Definition 2.3 implies (RA). For later use, we explain that condition (RA) implies (DP) under some assumption on S and by a suitable choice of the pair (L, L_+), we can make $A_{/S}$ be an L-polarized abelian scheme with real multiplication by $\mathcal{O}_F$. First we need the following:

Proposition 2.4. — ([29]*1.17,1.18) Let* $A_{/S}$ *be an abelian scheme of relative dimension d,and* $\iota : \mathcal{O}_F \to \mathrm{End}(A_{/S})$ *be an algebra homomorphism. Then the étale sheaf* $\mathrm{Hom}_{\mathcal{O}_F}^{Sym}(A_{/S}, A^t_{/S})$ *defined above is locally constant with values in a projective* $\mathcal{O}_F$*-module of rank* 1*, endowed with a notion of positivity corresponding to polarizations of* $A_{/S}$*. In particular, when* S *is normal and connected, this sheaf is constant.*

Here we remark that in [29], the abelian scheme $A_{/S}$ is assumed to satisfy condition (RA). But this condition is not necessary in the proof of the above proposition.

Now assume that S is normal and connected (e.g. S is the spectrum of the integer ring of a number field). Then from Proposition 2.4 we can find a projective $\mathcal{O}_F$-module $\mathcal{M}$ of rank 1 with a notion of positivity $\mathcal{M}_+$ and a morphism $\varphi : \underline{\mathcal{M}} \to \mathrm{Hom}_{\mathcal{O}_F}^{Sym}(A_{/S}, A^t_{/S})$. To check this φ satisfies condition (3) in Definition 2.1, we still need to check that the morphism $\alpha : A \otimes_{\mathcal{O}_F} \underline{\mathcal{M}} \to A^t$ is an isomorphism.

We can assume that $S = \mathrm{Spec}(k)$, where k is an separably closed field and we want to prove that α is an isomorphism of abelian varieties over k. Then it suffices to show that for any rational prime l, there exists $0 \neq \lambda \in \mathcal{M}$, such that $\deg(\varphi(\lambda))$ is prime to l. In fact, for any $\alpha \in \mathcal{M}$, we have a natural morphism $A \to A \otimes_{\mathcal{O}_F} \mathcal{M}$ whose effect on R-valued points is given by the formula (R is an k-algebra):

$$A(R) \ni a \mapsto a \otimes_{\mathcal{O}_F} \lambda \in A(R) \otimes_{\mathcal{O}_F} \mathcal{M}.$$

The composition of this morphism with α is $\varphi(\lambda)$. Hence $\deg(\alpha) | \deg(\varphi(\lambda))$. In particular, $\deg(\alpha)$ is prime to l. As l is arbitrary, $\deg(\alpha) = 1$ and hence α is an isomorphism.

To prove the existence of λ, we apply an argument in [12] Chapter 3 Section 5: when $char(k) > 0$, by [29]1.13, we can always lift the pair $(A_{/k}, \iota)$ to an abelian scheme with real multiplication $(\tilde{A}_{/W(k)}, \tilde{\iota})$ satisfying (RA). Here $W(k)$ is the ring of Witt vectors of k. Hence we can assume that $char(k) = 0$. By Lefschetz principle, we can assume that k is the complex filed. Then the existence of λ follows from the complex uniformization [12]Chapter 2 Section 2.2.

The following proposition tells us that when S is a scheme of characteristic 0, condition (RA) and hence (DP) is automatically satisfied.

PROPOSITION 2.5. — *Let k be a field of characteristic 0, $A_{/k}$ be an abelian variety of dimension d, and $\iota : \mathcal{O}_F \to \mathrm{End}(A_{/k})$ be an algebra homomorphism. Then $\mathrm{Lie}(A_{/k})$ is a free $\mathcal{O}_F \otimes_{\mathbb{Z}} k$-module of rank 1.*

Proof. — By Lefschetz principle we can again work over the complex field. Then the result follows from [12] Chapter 2, Corollary 2.6. □

Now we consider the object considered in Section 1. Let k be a number field, $A_{/k}$ be an abelian variety of dimension d satisfying the condition (NLL), and $\iota : F \to \mathrm{End}^0(A_{/k})$ be an algebra homomorphism. We want to prove that there is a $\mathfrak{c}$-polarized abelian variety $A'_{/\mathcal{O}_k}$ with real multiplication by $\mathcal{O}_F$ which is isogenous to $A_{/k}$.

We can find an order $\mathcal{O}$ in F which is mapped into $\mathrm{End}(A)$ under ι. By Serre's Tensor construction ([5]1.7.4.), we can find an isogeny $f : A \to A'$ over k, and the induced isomorphism $\mathrm{End}^0(A_{/k}) \to \mathrm{End}^0(A'_{/k})$ carries $\mathcal{O}_F \subseteq \mathrm{End}^0(A_{/k})$ into $\mathrm{End}(A'_{/k})$. Hence we have an algebra homomorphism $\iota' : \mathcal{O}_F \to \mathrm{End}(A'_{/k})$. By our assumption, $A_{/k}$ has good reduction at the prime $\mathfrak{P}$ of $\mathcal{O}_k$. By the criterion of Néron-Ogg-Shafarevich ([31] Section 1 Corollary 1), $A'_{/k}$ also has good reduction at $\mathfrak{P}$, and hence can be extended to an abelian scheme $A'_{/\mathcal{O}_{(\mathfrak{P})}}$ (recall that $\mathcal{O}_{(\mathfrak{P})}$ is the localization of $\mathcal{O}_k$ at the prime $\mathfrak{P}$). Since $\mathcal{O}_{(\mathfrak{P})}$ is a normal domain, by a lemma of Faltings (see [9] Lemma 1), the restriction to the generic fiber induces a bijection

$$\mathrm{End}(A'_{/\mathcal{O}_{(\mathfrak{P})}}) \to \mathrm{End}(A'_{/k}).$$

So we have an algebra homomorphism $\mathcal{O}_F \to \mathrm{End}(A'_{/\mathcal{O}_{(\mathfrak{P})}})$, which is again denoted by ι'.

From Proposition 2.4, the étale sheaf $\mathrm{Hom}^{Sym}_{\mathcal{O}_F}(A'_{/\mathcal{O}_{(\mathfrak{P})}}, A'^t_{/\mathcal{O}_{(\mathfrak{P})}})$ is a constant sheaf $\underline{\mathfrak{c}}$ for some fractional ideal $\mathfrak{c}$, with the natural notion of positivity $\mathfrak{c}_+$. Thus we have a natural isomorphism $\varphi : \underline{\mathfrak{c}} \to \mathrm{Hom}^{Sym}_{\mathcal{O}_F}(A'_{/\mathcal{O}_{(\mathfrak{P})}}, A'^t_{/\mathcal{O}_{(\mathfrak{P})}})$ which sends totally positive elements of $\mathfrak{c}$ to polarizations of $A'_{/\mathcal{O}_{(\mathfrak{P})}}$. We still need to check that the natural morphism $\alpha : A' \otimes_{\mathcal{O}_F} \underline{\mathfrak{c}} \to A'^t$ is an isomorphism over $\mathcal{O}_{(\mathfrak{P})}$. As $char(k) = 0$, by Proposition 2.5, α is an isomorphism at the generic fiber of $\mathcal{O}_{(\mathfrak{P})}$. Hence α is an isomorphism again by Faltings lemma.

In summary, we have:

PROPOSITION 2.6. — *Let $A_{/k}$ be an abelian variety of dimension d satisfying the condition* (NLL) *in Section* 1, *and $\iota : F \to \mathrm{End}^0(A_{/k})$ be an* *algebra homomorphism. Then we can find a fractional ideal $\mathfrak{c}$ and an $\mathfrak{c}$-polarized abelian scheme $(A'_{/\mathcal{O}_{(\mathfrak{P})}}, \iota', \varphi)$ with real multiplication by $\mathcal{O}_F$ such that $A_{/k}$ is k-isogenous to $A'_{/k}$.*

Remark 2.7. — Let $A_{/S}$ be an abelian scheme of relative dimension d and $\iota : \mathcal{O}_F \to \mathrm{End}(A_{/S})$ be an algebra homomorphism. By a similar argument as above, we see that if S is an integral normal scheme and the generic fiber of S is of characteristic 0, then the pair $(A_{/S}, \iota)$ must satisfy the condition (DP).

For later discussion, we need the following:

LEMMA 2.8. — *Let $A_{/S}, A'_{/S}$ be two abelian schemes of relative dimension d, and $\iota : \mathcal{O}_F \to \mathrm{End}(A_{/S}), \iota' : \mathcal{O}_F \to \mathrm{End}(A'_{/S})$ be two algebra homomorphisms. Suppose that there exists an $\mathcal{O}_F$-linear étale homomorphism of abelian schemes $f : A \to A'$. If the pair $(A_{/S}, \iota)$ satisfies the condition* (DP), *so does $(A'_{/S}, \iota')$.*

Proof. — Without loss of generality, we can assume that $S = \mathrm{Spec}(k)$ for some separably closed field k. If $char(k) = 0$, then $(A'_{/S}, \iota')$ satisfies (DP) automatically by Proposition 2.5. So we can assume that $char(k) = p > 0$. From the discussion of [12] Page $100 - 101$, the pair $(A_{/k}, \iota)$ can be lifted to characteristic 0; i.e., there exist:

(1) a normal local domain W with maximal ideal $\mathfrak{m}$ and residue field k such that the quotient field of W is of characteristic 0;
(2) an abelian scheme $\tilde{A}_{/W}$ with an $\mathcal{O}_F$-action $\tilde{\iota} : \mathcal{O}_F \to \mathrm{End}(\tilde{A}_{/W})$ such that $(A_{/k}, \iota)$ is isomorphic the the pull back of $(\tilde{A}_{/W}, \tilde{\iota})$ under the natural morphism $\mathrm{Spec}(k) \to \mathrm{Spec}(W)$.

Replacing W by its $\mathfrak{m}$-adic completion if necessary, we can assume that W is complete.

Since $f : A \to A'$ is étale and $\mathcal{O}_F$-linear, $C = ker(f)$ is a finite étale $\mathcal{O}_F$-submodule of $A_{/k}$. Then we can lift C to an étale $\mathcal{O}_F$-submodule $\tilde{C}_{/W}$ of $\tilde{A}_{/W}$. Let $\tilde{A}'_{/W}$ be the quotient of $\tilde{A}_{/W}$ by $\tilde{C}_{/W}$, with the natural homomorphism $\tilde{\iota}' : \mathcal{O}_F \to \mathrm{End}(\tilde{A}'_{/W})$ induced from $\tilde{A}_{/W}$. By the above construction it is easy to see that $(\tilde{A}'_{/W}, \tilde{\iota}')$ lifts $(A'_{/k}, \iota')$. Then from Remark 2.7, $(A'_{/k}, \iota')$ satisfies (DP). □

3. Hilbert modular Shimura variety

Fix a finite set of primes Ξ. Set

$$\mathbb{Z}_{(\Xi)} = \{\frac{m}{n} \in \mathbb{Q} | m, n \in \mathbb{Z}, (n, p) = 1, \forall p \in \Xi\}.$$

Then define $\mathcal{O}_{(\Xi)} = \mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_{(\Xi)}$, and $\mathcal{O}^{\times}_{(\Xi),+}$ as the set of totally positive units in $\mathcal{O}_{(\Xi)}$. Also we define:

$$\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z},\ \widehat{\mathbb{Z}}^{(\Xi)} = \varprojlim \mathbb{Z}/n\mathbb{Z},\ \mathbb{Z}_{\Xi} = \prod_{l \in \Xi} \mathbb{Z}_l,$$

where in the first inverse limit, n ranges over all positive numbers, and in the second inverse limit, n ranges over all positive integers prime to Ξ. Let $\mathbb{A}$ be the adele ring of $\mathbb{Q}$. Then set

$$\mathbb{A}^{(\Xi\infty)} = \{x \in \mathbb{A} | x_l = x_\infty = 0, \forall l \in \Xi\},$$

and $F_{\mathbb{A}^{(\Xi\infty)}} = F \otimes_{\mathbb{Q}} \mathbb{A}^{(\Xi\infty)}$.

Define the algebraic group $G = \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}(GL(2))$ and let Z be its center. K is an open compact subgroup of $G(\widehat{\mathbb{Z}})$ which is maximal at Ξ, in the sense that $K = G(\mathbb{Z}_\Xi) \times K^{(\Xi)}$, where

$$K^{(\Xi)} = \{x \in K | x_p = 1 \text{ for all } p \in \Xi\}.$$

Definition 3.1. — *Define the functor* $\mathcal{E}'^{(\Xi)}_K : Sch_{/\mathbb{Z}_{(\Xi)}} \to Set$, *such that for each* $\mathbb{Z}_{(\Xi)}$*-scheme* S, $\mathcal{E}'^{(\Xi)}_K(S) = [(A_{/S}, \iota, \bar{\lambda}, \bar{\eta}^{(\Xi)})]$. *Here* $[(A_{/S}, \iota, \bar{\lambda}, \bar{\eta}^{(\Xi)})]$ *is the set of isomorphism classes of quadruples* $(A_{/S}, \iota, \bar{\lambda}, \bar{\eta}^{(\Xi)})$ *consisting of:*

(1) *an abelian scheme* $A_{/S}$ *of relative dimension* d*;*
(2) *an algebra homomorphism* $\iota : \mathcal{O}_F \to \mathrm{End}(A_{/S})$ *such that the pair* $(A_{/S}, \iota)$ *satisfies the condition* (DP) *(see Definition 2.3);*
(3) *a subset* $\{\lambda \circ \iota(b) : b \in \mathcal{O}^{\times}_{(\Xi),+}\}$ *of* $\mathrm{Hom}(A_{/S}, A^t_{/S}) \otimes_{\mathbb{Z}} \mathbb{Q}$*, where* $\lambda : A_{/S} \to A^t_{/S}$ *is an* $\mathcal{O}_F$*-linear polarization of* A*, whose degree is prime to* Ξ*;*
(4) $\bar{\eta}^{(\Xi)}$ *is a rational* K*-level structure of the abelian scheme* $A_{/S}$ *(see Remark 3.3 below).*

An isomorphism from one quadruple $(A_{/S}, \iota, \bar{\lambda}, \bar{\eta}^{(\Xi)})$ *to another* $(A'_{/S}, \iota', \bar{\lambda}', \bar{\eta}'^{(\Xi)})$ *is an element* $f \in \mathrm{Hom}(A_{/S}, A'_{/S}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(\Xi)}$ *whose degree is prime to* Ξ *such that:*

(1) $f \circ \iota(b) = \iota'(b) \circ f$ *for all* $b \in \mathcal{O}_F$*;*
(2) $f^t \circ \bar{\lambda}' \circ f = \bar{\lambda}$ *as subsets of* $\mathrm{Hom}(A_{/S}, A^t_{/S}) \otimes_{\mathbb{Z}} \mathbb{Q}$*;*
(3) *we have the equality of level stuctures:* $V^{(\Xi)}(f)(\bar{\eta}^{(\Xi)}) = \bar{\eta}'^{(\Xi)}$.

Now we choose a representative $I = \{\mathfrak{c}\}$ of fractional ideals in the finite class group

$$Cl(K) = (F_{\mathbb{A}^{(\Xi\infty)}})^{\times} / \mathcal{O}^{\times}_{(\Xi),+} \det(K).$$

For each $\mathfrak{c}$, fix an $\mathcal{O}_F$-lattice $L_\mathfrak{c} \subseteq V = F^2$ such that $\wedge(L_\mathfrak{c} \wedge L_\mathfrak{c}) = \mathfrak{c}^*$. Here $\wedge : V \wedge V \to F$ is the alternating form given by $((a_1, a_2), (b_1, b_2)) \mapsto a_1 b_2 - a_2 b_1$.

Definition 3.2. — *Define the functor* $\mathcal{E}^{(\Xi)}_{K,\mathfrak{c}} : Sch_{/\mathbb{Z}_{(\Xi)}} \to Set$,*such that for each* $\mathbb{Z}_{(\Xi)}$*-scheme* S, $\mathcal{E}^{(\Xi)}_{K,\mathfrak{c}}(S) = \{(A_{/S}, \iota, \phi, \bar{\alpha}^{(\Xi)})\}_{/\cong}$, *where* $\{(A_{/S}, \iota, \phi, \bar{\alpha}^{(\Xi)})\}_{/\cong}$ *is the set of isomorphic classes of quadruples* $(A_{/S}, \iota, \phi, \bar{\alpha}^{(\Xi)})$ *consisting of*

1. *an abelian scheme* $A_{/S}$ *of relative dimension* d*;*
2. *an algebra homomorphism* $\iota : \mathcal{O}_F \to \mathrm{End}(A_{/S})$ *such that the pair* $(A_{/S}, \iota)$ *satisfies the condition* (DP) *(see Definition 2.3);*
3. *a* $\mathfrak{c}$*-polarization* $\phi : A \otimes_{\mathcal{O}_F} \mathfrak{c} \to A^t$ *of* $A_{/S}$ *(see Definition 2.1);*
4. $\bar{\alpha}^{(\Xi)}$ *is an integral* K*-level structure of the abelian scheme* $A_{/S}$ *(see Remark 3.3 below).*

An isomorphism from one quadruple $(A_{/S}, \iota, \phi, \bar{\alpha}^{(\Xi)})$ *to another* $(A'_{/S}, \iota', \phi', \bar{\alpha}'^{(\Xi)})$ *is an isomorphism* $f : A \to A'$ *of abelian schemes over* S *such that*

1. $f \circ \iota(b) = \iota'(b) \circ f$ *for all* $b \in \mathcal{O}_F$*;*
2. $f^t \circ \phi' \circ (f \otimes_{\mathcal{O}_F} Id_\mathfrak{c}) = \phi : A \otimes_{\mathcal{O}_F} \mathfrak{c} \to A^t$*;*
3. *we have an equality of integral level structures:* $T^{(\Xi)}(f)(\bar{\alpha}^{(\Xi)}) = \bar{\alpha}'^{(\Xi)}$.

Remark 3.3. — Here we briefly recall the notion of level structures on an abelian scheme with real multiplication. As in Definition 3.1 and 3.2, we fix an abelian scheme $A_{/S}$ and a homomorphism $\iota : \mathcal{O}_F \to \mathrm{End}(A_{/S})$. Take a point $s \in S$ and let $\bar{s} : \mathrm{Spec}(k(\bar{s})) \to S$ be a geometric point of S over s, where $k(\bar{s})$ is a separably closed field extension of the residue field $k(s)$ of S at the point s. Consider the prime-to-Ξ Tate module

$$T^\Xi(A_{\bar{s}}) = \varprojlim_N A[N](k(\bar{s})),$$

where N runs through all positive integers prime to Ξ, and set $V^\Xi(A_{\bar{s}}) = T^\Xi(A_{\bar{s}}) \otimes_\mathbb{Z} \mathbb{Z}_\Xi$, which is a free $F_{\mathbb{A}^{(\Xi\infty)}}$-module of rank 2. When N is invertible on S, the finite scheme $A[N]$ is étale over S. The algebraic fundamental group $\pi(S, \bar{s})$ acts on $A[N](k(\bar{s}))$, and hence on $T^\Xi(A_{\bar{s}})$ and $V^\Xi(A_{\bar{s}})$. This action is compatible with the action of $G(\widehat{\mathbb{Z}}^{(\Xi)})$ (resp. $G(F_{\mathbb{A}^{(\Xi\infty)}})$) on $T^\Xi(A_{\bar{s}})$ (resp. $V^\Xi(A_{\bar{s}})$).

We define a sheaf of sets $ILV^{(\Xi)} : (Sch_{/S})_{ét} \to Set$ on the étale site of the category of S-schemes such that for any connected S-scheme S', we have:

$$ILV^{(\Xi)}(S') = H^0(\pi(S',\bar{s}'),\ \mathrm{Isom}_{\mathcal{O}_F}(L_{\mathfrak{c}} \otimes_{\mathcal{O}_F} \widehat{\mathbb{Z}}^{(\Xi)}, T^{\Xi}(A_{\bar{s}'}))),$$

where $\bar{s}'$ is a geometric point of S' over a point s' of S'. The étale sheaf $ILV^{(\Xi)}$ is independent of the choice of s' (see [20] Section 6.4.1). The group $G(\widehat{\mathbb{Z}}^{(\Xi)})$ acts on the sheaf $ILV^{(\Xi)}$ through its action on the Tate module $T^{\Xi}(A_{\bar{s}'})$, and we denote by $ILV^{(\Xi)}/K$ the quotient sheaf of $ILV^{(\Xi)}$ under the group action of $K^{(\Xi)}$. An integral K-level structure of $A_{/S}$ is a section $\bar{\alpha}^{(\Xi)} \in ILV^{(\Xi)}/K(S)$. Similarly we define another sheaf $RLV^{(\Xi)}$: $(Sch_{/S})_{ét} \to Set$ such that for any connected S-scheme S', we have:

$$RLV^{(\Xi)}(S') = H^0(\pi(S',\bar{s}'), \mathrm{Isom}_{\mathcal{O}_F}(V \otimes_{\mathbb{Z}} \mathbb{A}^{(\Xi\infty)}, V^{\Xi}(A_{\bar{s}'}))),$$

and define the quotient sheaf $RLV^{(\Xi)}/K$ in the same way. Then a rational K-level structure of $A_{/S}$ is a section $\bar{\eta}^{(\Xi)} \in RLV^{(\Xi)}/K(S)$.

Suppose that we have another abelian scheme $A'_{/S}$ and a homomorphism $\iota' : \mathcal{O}_F \to \mathrm{End}(A'_{/S})$. We can similarly define two étale sheaves $ILV'^{(\Xi)}$ and $RLV'^{(\Xi)}$ replacing $A_{/S}$ by $A'_{/S}$ in the above construction. If $f : A \to A'$ is an $\mathcal{O}_F$-linear isomorphism of abelian schemes, the isomorphism f induces an isomorphism of Tate modules $T^{(\Xi)}(A_{\bar{s}}) \cong T^{(\Xi)}(A'_{\bar{s}})$ for any geometric point $\bar{s}$ of S. Hence f induces an isomorphism of étale sheaves $T^{(\Xi)}(f) : ILV^{(\Xi)} \to ILV'^{(\Xi)}$ which is compatible with the $G(\widehat{\mathbb{Z}}^{(\Xi)})$-action. Thus f also induces an isomorphism $T^{(\Xi)}(f) : ILV^{(\Xi)}/K \to ILV'^{(\Xi)}/K$ for all subgroup K of $G(\widehat{\mathbb{Z}})$. For any integral K-level structure $\bar{\alpha}^{(\Xi)} \in ILV^{(\Xi)}/K(S)$, we use $T^{(\Xi)}(f)(\bar{\alpha}^{(\Xi)})$ to denote its image under the isomorphism $T^{(\Xi)}(f)$. Similarly if $f : A \to A'$ is an $\mathcal{O}_F$-linear prime-to-Ξ isogeny of abelian schemes, then f induces an isomorphism $V^{(\Xi)}(A_{\bar{s}}) \cong V^{(\Xi)}(A'_{\bar{s}})$ and hence isomorphisms of étale sheaves $V^{(\Xi)}(f) : RLV^{(\Xi)} \to RLV'^{(\Xi)}$ and $V^{(\Xi)}(f) : RLV^{(\Xi)}/K \to RLV'^{(\Xi)}/K$. For any rational K-level structure $\bar{\eta}^{(\Xi)} \in RLV^{(\Xi)}/K(S)$, we use $V^{(\Xi)}(f)(\bar{\eta}^{(\Xi)})$ to denote its image under the isomorphism $V^{(\Xi)}(f)$. We refer to [21] Section 4.3.1 for more discussion on this topic.

Theorem 3.4. — *When K is small enough (e.g. $\det(K^{(\Xi)}) \cap \mathcal{O}_+^\times \subseteq (K^{(\Xi)} \cap Z(\mathbb{Z}))^2$), then we have a natural isomorphism of functors:*

$$i : \coprod_{\mathfrak{c}\in I} \mathcal{E}^{(\Xi)}_{K,\mathfrak{c}} \to \mathcal{E}'^{(\Xi)}_{K}.$$

The proof is essentially given in [20] Section 4.2.1 so we omit the proof here. The only thing we want to remark here is that for any quadruple $(A_{/S}, \iota, \bar{\lambda}, \bar{\eta}^{(\Xi)})$ considered in Definition 3.1, we can find an abelian scheme $A'_{/S}$ with real multiplication ι',and an $\mathcal{O}_F$-linear prime-to-Ξ isogeny f :

$A \to A'$ of abelian schemes over S such that $A'_{/S}$ admits an integral level structure. Since S is a $\mathbb{Z}_{(\Xi)}$-scheme, the isogeny f is étale. From Lemma 2.8, the pair $(A'_{/S}, \iota')$ also satisfies the condition (DP). Then we can follow the argument in [20] Section 4.2.1 to conclude this theorem.

From [6], the functor $\mathcal{E}^{(\Xi)}_{K,\mathfrak{c}}$ is representable. By Theorem 3.4, when K is small enough, we can assume that the functor $\mathcal{E}'^{(\Xi)}_K$ is represented by a $\mathbb{Z}_{(\Xi)}$-scheme $Sh^{(\Xi)}_K$. From [6] Theorem 2.2, the scheme $Sh^{(\Xi)}_K$ is flat of complete intersection over $\mathbb{Z}_{(\Xi)}$, and smooth over $\mathbb{Z}_{(\Xi)}[\frac{1}{d_F}]$.

Now we take the projective limit of $Sh^{(\Xi)}_K$ for various K, and get a $\mathbb{Z}_{(\Xi)}$-scheme $Sh^{(\Xi)}$. It is clear that $Sh^{(\Xi)}_{/\mathbb{Z}_{(\Xi)}}$ represents the moduli problem $\mathcal{E}'^{(\Xi)} : Sch_{/\mathbb{Z}_{(\Xi)}} \to Set$, such that for each $\mathbb{Z}_{(\Xi)}$-scheme S, $\mathcal{E}'^{(\Xi)}_K(S) = [(A_{/S}, \iota, \bar{\lambda}, \eta^{(\Xi)})]$. where $[(A_{/S}, \iota, \bar{\lambda}, \bar{\eta}^{(\Xi)})]$ is the set of isomorphism classes of quadruples $(A_{/S}, \iota, \bar{\lambda}, \eta^{(\Xi)})$ considered in Definition 3.1, except that $\eta^{(\Xi)} \in RLA^{(\Xi)}(S)$ is a rational level structure instead of a rational K-level structure for some open compact subgroup K. An isomorphism from one quadruple $(A_{/S}, \iota, \bar{\lambda}, \eta^{(\Xi)})$ to another $(A'_{/S}, \iota', \bar{\lambda}', \eta'^{(\Xi)})$ is an element $f \in \mathrm{Hom}(A_{/S}, A'_{/S}) \otimes_{\mathbb{Z}} \mathbb{Z}_{(\Xi)}$ whose degree is prime to Ξ such that it satisfies the first two conditions in Definition 3.1, and also $V^{(\Xi)}(f)(\eta^{(\Xi)}) = \eta'^{(\Xi)}$ instead of that last condition there.

For any $g \in G(F_{\mathbb{A}^{(\Xi\infty)}})$, the map sending each quadruple $(A_{/S}, \iota, \bar{\lambda}, \eta^{(\Xi)})$ to another quadruple $(A_{/S}, \iota, \bar{\lambda}, g(\eta^{(\Xi)}))$ induces an automorphism of the functor $\mathcal{E}'^{(\Xi)}$, and hence an automorphism of the Shimura variety $Sh^{(\Xi)}_{/\mathbb{Z}_{(\Xi)}}$ by universality. We still denote this action by g.

For simplicity we denote the Shimua variety $Sh^{(\Xi)}_{/\mathbb{Z}_{(\Xi)}}$ by $X_{/\mathbb{Z}_{(\Xi)}}$ in the following discussion. Pick a closed point $x_p \in X(\bar{\mathbb{F}}_p)$. Let K be a neat subgroup of $G(F_{\mathbb{A}^{(\Xi\infty)}})$. Then the natural morphism $X \to X_K = X/K$ is étale. Let $\mathcal{O}_{X,x_p}$ and $\mathcal{O}_{X_K,x_p}$ be the stalk of X and X_K at x_p, respectively. The completion of $\mathcal{O}_{X,x_p}$ is canonically isomorphic to the completion of $\mathcal{O}_{X_K,x_p}$, and we denote this completion by $\widehat{\mathcal{O}}_{x_p}$. Suppose that x_p is represented by a quadruple $(A_{0/\bar{\mathbb{F}}_p}, \iota_0, \phi_0, \bar{\alpha_0}^{(\Xi)}) \in \mathcal{E}^{(\Xi)}_{K,\mathfrak{c}}(\bar{\mathbb{F}}_p)$.

Let $CL_{/W_p}$ be the category of complete local W_p-algebras with residue field $\bar{\mathbb{F}}_p$. Consider the local deformation functor $\widehat{D}_p : CL_{/W_p} \to Set$, given by

$$\widehat{D}_p(R) = \{(A_{/R}, \iota_R, \phi_R) | (A_{/R}, \iota_R, \phi_R) \times_R \bar{\mathbb{F}}_p \cong (A_{0/\bar{\mathbb{F}}_p}, \iota_0, \phi_0)\}_{/\cong},$$

here the triple $(A_{/R}, \iota_R, \phi_R)$ consists of an abelian A schemes over R, an algebra homomorphism $\iota_R : \mathcal{O}_F \to \mathrm{End}(A_{/R})$ and a $\mathfrak{c}$-polarization ϕ_R of

$A_{/R}$. An isomorphism from a triple $(A_{/R}, \iota_R, \phi_R)$ to another $(A'_{/R}, \iota'_R, \phi'_R)$ is an isomorphism $f : A \to A'$ of abelian schemes over R such that

(1) for all $a \in \mathcal{O}_F$, we have $f \circ \iota_R(a) = \iota'_R(a) \circ f : A \to A'$;
(2) $f^t \circ \phi'_R \circ (f \otimes Id_{\mathfrak{c}}) = \phi_R : A \otimes_{\mathcal{O}_F} \mathfrak{c} \to A^t$.

Define a functor $DEF_p : CL_{/W_p} \to Set$ by the formula:

$$DEF_p(R) = \{(D_{/R}, \Lambda_R, \varepsilon_R)\}_{/\cong},$$

where $D_{/R}$ is a Barsotti-Tate $\mathcal{O}_F$-module over R, $\Lambda_R : D \otimes_{\mathcal{O}_F} \mathfrak{c} \to D^t$ is an $\mathcal{O}_F$-linear isomorphism of Barsotti-Tate $\mathcal{O}_F$-modules over R (D^t is the Cartier dual of D), and $\varepsilon_R : D_0 = D \otimes_R \bar{\mathbb{F}}_p \to A_0[p^\infty]$ is an isomorphism of Barsotti-Tate $\mathcal{O}_F$-modules over the special fiber $\mathrm{Spec}(\bar{\mathbb{F}}_p)$ of $\mathrm{Spec}(R)$.

For any triple $(A_{/R}, \iota_R, \phi_R)$ in $\widehat{D}_p(R)$, let $A[p^\infty]_{/R}$ be its p-divisible Barsotti-Tate $\mathcal{O}_F$-module over R. The $\mathfrak{c}$-polarization ϕ_R of $A_{/R}$ gives an isomorphism $\Lambda_R : A[p^\infty] \otimes_{\mathcal{O}_F} \mathfrak{c} \to A^t[p^\infty] \cong (A[p^\infty])^t$. The isomorphism $(A_{/R}, \iota_R, \phi_R) \times_R \bar{\mathbb{F}}_p \cong (A_{0/\bar{\mathbb{F}}_p}, \iota_0, \phi_0)$ gives an isomorphism $\varepsilon_R : A[p^\infty] \otimes_R \bar{\mathbb{F}}_p \to A_0[p^\infty]$. By the Serre-Tate deformation theory ([26] Theorem 1.2.1), we have:

Proposition 3.5. — *The above association*

$$(A_{/R}, \iota_R, \phi_R) \mapsto (A[p^\infty]_{/R}, \Lambda_R, \varepsilon_R)$$

induces an equivalence of functors $\widehat{D}_p \to DEF_p$.

We define two more functors $DEF_p^? : CL_{/W_p} \to Set, ? = ord, ll$, by:

$$DEF_p^?(R) = \{(D^?, \phi^?, \varepsilon^?)\}_{/\cong},$$

here in the triple $(D^?, \phi^?, \varepsilon^?)$, $D^?$ is a Barsotti-Tate $\mathcal{O}_F$-module over R,$\phi^? : D^? \otimes_{\mathcal{O}_F} \mathfrak{c} \to (D^?)^t$ is an isomorphism of Barsotti-Tate $\mathcal{O}_F$-modules over R, and $\varepsilon^? : D^? \otimes_R \bar{\mathbb{F}}_p \to A_0[p^\infty]^?$ is an isomorphism over $\bar{\mathbb{F}}_p$.

Similar with [15] Proposition 1.2, we have the following facts:

(1) the functor DEF_p is represented by the formal scheme $\widehat{S}_{p/W_p}$ associated to $\widehat{\mathcal{O}}_{x_p}$;
(2) there is a natural equivalence of functors: $DEF_p \cong DEF_p^{ord} \times DEF_p^{ll}$,and hence the formal $\widehat{S}_{p/W_p}$ is a product of two formal schemes $\widehat{S}^{ord}_{p/W_p}$ and $\widehat{S}^{ll}_{p/W_p}$ such that $DEF_p^?$ is represented by $\widehat{S}^?_{p/W_p}$ for $? = ord, ll$;
(3) For each $\mathfrak{p} \in \Sigma_p^{ord}$, fix an isomorphism $\mathcal{O}_{\mathfrak{p}} \cong T_{\mathfrak{p}}(A_0)$ (recall that $T_{\mathfrak{p}}(A_0)$ is the $\mathfrak{p}$-adic Tate module of A_0 defined in Section 1). Since $\mathfrak{c}$ is prime to p, by the $\mathfrak{c}$-polarization ϕ_0, we also have an isomorphism

$\mathcal{O}_{\mathfrak{p}} \cong T_{\mathfrak{p}}(A_0^t)$. Then $\widehat{S}^{ord}_{p/W_p}$ is a smooth formal scheme over W_p which is isomorphic to

$$\prod_{\mathfrak{p}\in\Sigma^{ord}} \mathrm{Hom}(T_{\mathfrak{p}}(A_0)\otimes_{\mathcal{O}_{\mathfrak{p}}} T_{\mathfrak{p}}(A_0^t), \widehat{\mathbb{G}}_m) \cong \prod_{\mathfrak{p}\in\Sigma^{ord}} \mathrm{Hom}(\mathcal{O}_{\mathfrak{p}}, \widehat{\mathbb{G}}_m)$$
$$= \prod_{\mathfrak{p}\in\Sigma^{ord}} \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathfrak{p}}^*,$$

here $\mathcal{O}_{\mathfrak{p}}^* = \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{O}_{\mathfrak{p}}, \mathbb{Z}_p)$.

In fact, for any triple $(A_{/R}, \iota_R, \phi_R)$ in $\widehat{D}_p(R)$, the level structure $\bar{\alpha_0}^{(\Xi)}$ on A_0 can be extended uniquely to a level structure on $A_{/R}$. Then the functor $\widehat{D}_p$, and hence the functor DEF_p by Proposition 3.5, is represented by the the formal scheme $\widehat{S}_{p/W_p} = Spf(\widehat{\mathcal{O}}_{x_p})$.

For a triple $(D_{/R}, \Lambda_R, \varepsilon_R) \in DEF_p(R)$, we have a canonical decomposition the Barsotti-Tate $\mathcal{O}_F$-module $D = D^{ord} \times D^{ll}$, where D^{ord} is the maximal ordinary Barsotti-Tate $\mathcal{O}_F$-submodule of D, and D^{ll} is its local-local complement. From this we have a morphism

$$DEF_p(R) \ni (D_{/R}, \Lambda_R, \varepsilon_R) \mapsto$$
$$\{(D^{ord}_{/R}, \Lambda_R|_{D^{ord}}, \varepsilon_R|_{D^{ord}}), (D^{ll}_{/R}, \Lambda_R|_{D^{ll}}, \varepsilon_R|_{D^{ll}})\} \in DEF_p^{ord}(R)\times DEF_p^{ll}(R),$$

from which we get a equivalence of functors between DEF_p and $DEF_p^{ord} \times DEF_p^{ll}$. Hence the formal scheme $\widehat{S}_{p/W_p}$ is a product of two formal schemes $\widehat{S}^{ord}_{p/W_p} \times \widehat{S}^{ll}_{p/W_p}$.

In contrast with [15] Proposition 1.2, the formal scheme $\widehat{S}_{p/W_p}$ may not be smooth when p divides the discriminant d_F of F since the Shimura variety $Sh^{(p)}_{/\mathbb{Z}_{(p)}}$ we consider here is not smooth. But from the Serre-Tate deformation theory, the formal scheme $\widehat{S}^{ord}_{p/W_p}$ is always smooth, and this is the part we are interested in.

4. Eigen coordinates

At the beginning of this section we set up some notations. Let $k \subseteq \bar{\mathbb{Q}}$ be a number field and Ξ be a finite set of primes. For each $p \in \Xi$, choose a finite extension $\widetilde{L}_p$ of L_p in $\mathbb{C}_p$ such that:

(1) $k \subseteq i_p^{-1}(\widetilde{L}_p)$;
(2) $i_p^{-1}(\widetilde{L}_p)$ contains the Galois closure of F in $\bar{\mathbb{Q}}$.

Denote by $\widetilde{W}_p$ the valuation ring of $\widetilde{L}_p$. Then define:

$$\widetilde{\mathcal{W}}_\Xi = \bigcap_{p\in\Xi} i_p^{-1}(\widetilde{W}_p) \subseteq \bar{\mathbb{Q}}, \widetilde{\mathcal{W}}_k = \widetilde{\mathcal{W}}_\Xi \cap k.$$

The ring $\widetilde{\mathcal{W}}_\Xi$ is a semilocal ring, and for each $l \in \Xi$, there is a unique maximal ideal $\mathfrak{m}_l$ with residue characteristic l. Let $\widetilde{\mathcal{L}}_\Xi$ be the quotient field of $\widetilde{\mathcal{W}}_\Xi$.

Suppose that the quadruple $(A_{/\widetilde{\mathcal{W}}_\Xi}, \iota, \bar{\lambda}, \eta^{(\Xi)})$ represents a point $x \in Sh^{(\Xi)}(\widetilde{\mathcal{W}}_\Xi)$ such that the image of x lies in $Sh^{(\Xi)}(\widetilde{\mathcal{W}}_k)$. For each $p \in \Xi$, x induces an $\bar{\mathbb{F}}_p$-valued point $x_p \in Sh^{(\Xi)}(\bar{\mathbb{F}}_p)$. Then the quadruple $(A_{\mathfrak{P}/\bar{\mathbb{F}}_p}, \iota_{\mathfrak{P}}, \bar{\lambda}_{\mathfrak{P}}, \eta_{\mathfrak{P}}^{(\Xi)})$ obtained by mod p reduction represents the point x_p.

This section is the most important part of this paper. We give a sketch of what we want to do in this section before we start the down to earth arguments.

First we construct a torus $R^\times_{(\Xi)}$ acting on the Hilbert modular Shimura variety which fixes the closed point x_p. Hence this action induces an automorphism on the formal completion $\widehat{S}_p$ of the Shimura variety $Sh^{(\Xi)}$ at the closed point x_p. From the previous section, we have a decomposition $\widehat{S}_{p/W_p} = \widehat{S}^{ord}_{p/W_p} \times \widehat{S}^{ll}_{p/W_p}$. Then we recall the construction of $\hat{\rho}$-eigen σ-coordinates in [15] and give the explicit expression of the action of $R^\times_{(\Xi)}$ on these coordinates. When the ind-étale exact sequence of the Barsotti-Tate $\mathcal{O}_\mathfrak{p}$-module $A[\mathfrak{p}^\infty]$ splits over $\widetilde{W}_p$, we calculate its Serre-Tate coordinates in Lemma 4.3. It turns out that when p is ramified in the base field (so $W_p \neq \widetilde{W}_p$) this Serre-Tate coordinate is a p-th power root of unity and the abelian variety $A_{/\widetilde{W}_p}$ is isogenous to an abelian variety whose Serre-Tate coordinate at $\mathfrak{p}$ is 1. From the construction of the eigencoordinates, the $\hat{\rho}$-eigen σ-coordinates of these abelian varieties are all 0 for any embedding $\sigma : F \to \bar{\mathbb{Q}}_p$ which induces the prime $\mathfrak{p}$ in F. Since we can change our abelian variety by an isogenous abelian variety, the eigen coordinates should be the right object to study.

The above calculation is local at p. We want to transit the action of $R^\times_{(\Xi)}$ on $\widehat{S}_{p/W_p}$ to the deformation space $\widehat{S}_{l/W_l}$ for some other prime l with the property that there exists a prime $\mathfrak{L}$ of k over l and $A_{/\widetilde{\mathcal{W}}_\Xi}$ has partially ordinary reduction at $\mathfrak{L}$. Let $\pi : A \to \mathrm{Spec}(\widetilde{\mathcal{W}}_\Xi)$ be the structure morphism and set $\omega = \pi_*(\Omega_{A/\widetilde{\mathcal{W}}_\Xi})$ which is an $\mathcal{O}_F \otimes_\mathbb{Z} \widetilde{\mathcal{W}}_\Xi$-module and define $\omega^{\otimes 2} = \omega \otimes_{\mathcal{O}_F\otimes_\mathbb{Z}\widetilde{\mathcal{W}}_\Xi} \omega$. This is the global object which allows us to compare the action of $R^\times_{(\Xi)}$ at different local deformation space. The sheaf $\omega^{\otimes 2}$ is related with the Serre-Tate coordinates (or the eigen coordinates) through the

Kodaira-Spencer map. The Kodaira-Spencer map is not an isomorphism in general if the reduction of $A_{/\widetilde{\mathcal{W}}_\Xi}$ at $\mathfrak{P}$ is not ordinary. We want to have decomposition of $\omega^{\otimes 2}$ by its $\mathcal{O}_F \otimes_{\mathbb{Z}} \widetilde{\mathcal{W}}_\Xi$-module structure as in [25]. Recall $I = \mathrm{Hom}(F, \bar{\mathbb{Q}})$. The natural homomorphism

$$\mathcal{O}_F \otimes_{\mathbb{Z}} \widetilde{\mathcal{W}}_\Xi \to \widetilde{\mathcal{W}}_\Xi^I, a \otimes b \mapsto (\sigma(a) \cdot b)_{\sigma \in I}$$

is not an isomorphism when the prime $p \in \Xi$ is ramified in $\mathcal{O}_F$. It becomes an isomorphism when base change to the quotient field $\widetilde{\mathcal{L}}_\Xi$ of $\widetilde{\mathcal{W}}_\Xi$. On the other hand, the formation of the sheaf $\omega_{/\widetilde{\mathcal{W}}_\Xi}$ is compatible with arbitrary base change. So we can decompose the sheaf $\omega^{\otimes 2} \otimes_{\widetilde{\mathcal{W}}_\Xi} \widetilde{\mathcal{L}}_\Xi$ as a direct sum $\oplus_{\sigma \in I} \tilde{\omega}^{\otimes 2\sigma}$ such that on $\tilde{\omega}^{\otimes 2\sigma}$, the ring $\mathcal{O}_F$ acts by the embedding $\sigma : F \to \bar{\mathbb{Q}}$. Under this decomposition and the Kodaira-Spencer map, we can compare the endomorphism algebras of the reductions of $A_{/\widetilde{\mathcal{W}}_\Xi}$ at different primes and get our main result Theorem 4.4 at the end of this section.

4.1. Construction and properties of eigen coordinates

By [15] Lemma 2.2, we have

Lemma 4.1. — *If $A_{\mathfrak{P}/\bar{\mathbb{F}}_p}$ is not supersingular (i.e. $\Sigma_p^{ord} \neq \emptyset$), then there exists a CM quadratic extension M of F, and an isomorphism of F-algebras $\theta_{\mathfrak{P}} : M \cong \mathrm{End}_F^0(A_{\mathfrak{P}/\bar{\mathbb{F}}_p})$. Set $R = M \cap \theta_{\mathfrak{P}}^{-1}(\mathrm{End}_{\mathcal{O}_F}(A_{\mathfrak{P}/\bar{\mathbb{F}}_p}))$, which is an order in M. If a prime ideal $\mathfrak{p}$ in $\mathcal{O}_F$ belongs to Σ_p^{ord}; i.e. $A_{\mathfrak{P}}[\mathfrak{p}]$ has nontrivial $\bar{\mathbb{F}}_p$-rational points, then $\mathfrak{p}$ splits into two primes $\mathcal{P}\bar{\mathcal{P}}$ in R with $\mathcal{P} \neq \bar{\mathcal{P}}$.*

As in [15], we make the convention that we choose $\mathcal{P}$ such that $A_{\mathfrak{P}}[\mathcal{P}]$ is connected and $A_{\mathfrak{P}}[\bar{\mathcal{P}}]$ is étale.

By the above lemma, we have an isomorphism $M \otimes_F F_{\mathfrak{p}} \cong F_{\mathfrak{p}} \times F_{\mathfrak{p}}$, such that the first factor corresponds to $\mathcal{P}$ and the second factor corresponds to $\bar{\mathcal{P}}$. As M can be naturally embedded into $M \otimes_F F_{\mathfrak{p}}$, we have two embeddings from M to $F_{\mathfrak{p}}$, which correspond to the two factors of $F_{\mathfrak{p}} \times F_{\mathfrak{p}}$. We always regard M as a subfield of $F_{\mathfrak{p}}$ by the first embedding, while the second embedding is denoted by $c : M \hookrightarrow F_{\mathfrak{p}}$.

Let $R_{(\Xi)} = R \otimes_{\mathbb{Z}} \mathbb{Z}_{(\Xi)}$. For $\alpha \in R_{(\Xi)}^{\times}$, $\theta_{\mathfrak{P}}(\alpha)$ is a prime-to-Ξ isogeny of $A_{\mathfrak{P}/\bar{\mathbb{F}}_p}$, and hence induces an endomorphism of $V^{(\Xi)}(A_{\mathfrak{P}})$. We still denote this endomorphism by $\theta_{\mathfrak{P}}(\alpha)$. Define a map $\hat{\rho} : R_{(\Xi)}^{\times} \to G(F_{\mathbb{A}^{(\Xi\infty)}})$ such that for each $\alpha \in R_{(\Xi)}^{\times}$, $\hat{\rho}(\alpha)$ is given by the formula: $\eta_{\mathfrak{P}}^{(\Xi)} \circ \hat{\rho}(\alpha) = \theta_{\mathfrak{P}}(\alpha) \circ \eta_{\mathfrak{P}}^{(\Xi)}$.

Fix a prime-to-Ξ polarization $\lambda_{\mathfrak{P}}$ of $A_{\mathfrak{P}}$ as a representative of $\bar{\lambda}_{\mathfrak{P}}$. Under the isomorphism $\theta_{\mathfrak{P}}$, the Rosati involution associated to $\lambda_{\mathfrak{P}}$ on $\mathrm{End}^0_F(A_{\mathfrak{P}/\bar{\mathbb{F}}_p})$ induces a positive involution on field M. As M is CM, this involution must be the complex conjugation on M. Hence for any $\alpha \in R^{\times}_{(\Xi)}$, $\lambda_{\mathfrak{P}}^{-1} \circ \theta_{\mathfrak{P}}(\alpha)^t \circ \lambda_{\mathfrak{P}} = \theta_{\mathfrak{P}}(\bar{\alpha})$. Then $\theta_{\mathfrak{P}}(\alpha)^t \circ \lambda_{\mathfrak{P}} \circ \theta_{\mathfrak{P}}(\alpha) = \lambda_{\mathfrak{P}} \circ \theta_{\mathfrak{P}}(\bar{\alpha}) \circ \theta_{\mathfrak{P}}(\alpha) = \lambda_{\mathfrak{P}} \circ \theta_{\mathfrak{P}}(\alpha\bar{\alpha})$. Since $\alpha\bar{\alpha} \in \mathcal{O}^{\times}_{(\Xi),+}$, we have $\theta_{\mathfrak{P}}(\alpha)^t \circ \bar{\lambda}_{\mathfrak{P}} \circ \theta_{\mathfrak{P}}(\alpha) = \bar{\lambda}_{\mathfrak{P}}$. So $\theta_{\mathfrak{P}}(\alpha)$ is an isogeny from the quadruple $(A_{\mathfrak{P}/\bar{\mathbb{F}}_p}, \iota_{\mathfrak{P}}, \bar{\lambda}_{\mathfrak{P}}, \eta^{(\Xi)}_{\mathfrak{P}})$ to $(A_{\mathfrak{P}/\bar{\mathbb{F}}_p}, \iota_{\mathfrak{P}}, \bar{\lambda}_{\mathfrak{P}}, \theta_{\mathfrak{P}}(\alpha)(\eta^{(\Xi)}_{\mathfrak{P}})) = (A_{\mathfrak{P}/\bar{\mathbb{F}}_p}, \iota_{\mathfrak{P}}, \bar{\lambda}_{\mathfrak{P}}, \hat{\rho}(\alpha)(\eta^{(\Xi)}_{\mathfrak{P}}))$ in the sense of Definition 3.1; in other words, the automorphism $g = \hat{\rho}(\alpha)$ of the Shimura variety $Sh^{(\Xi)}_{/\mathcal{W}_\Xi} = Sh^{(\Xi)}_{/\mathbb{Z}_{(\Xi)}} \times_{\mathbb{Z}_{(\Xi)}} \mathcal{W}_\Xi$ fixes the closed point x_p.

Denote the formal scheme $\widehat{S}_{p/W_p}$ as the completion of the Shimura variety $Sh^{(\Xi)}_{/\mathcal{W}_\Xi}$ along the closed point x_p, and $\nu_p : \widehat{S}_{p/W_p} \to Sh^{(\Xi)}_{/W_p}$ is the natural morphism. As explained in Section 3, $\widehat{S}_{p/W_p}$ is the product of two formal schemes $\widehat{S}^{ord}_{p/W_p}$ and $\widehat{S}^{ll}_{p/W_p}$, and if we fix an isomorphism $\mathcal{O}_{\mathfrak{p}} \cong T_{\mathfrak{p}}(A_{\mathfrak{P}})$ for each $\mathfrak{p} \in \Sigma^{ord}$, then $\widehat{S}^{ord}_{p/W_p}$ is isomorphic to $\Pi_{\mathfrak{p}\in\Sigma^{ord}}\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathcal{O}^*_{\mathfrak{p}}$. By deformation theory, we have a Serre-Tate coordinate $t_{\mathfrak{p}} \in \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathcal{O}^*_{\mathfrak{p}}$ for each $\mathfrak{p} \in \Sigma^{ord}$. Then for each object $\mathcal{R}$ in the category $CL_{/W_p}$, and an $\mathcal{R}$-valued point $x \in \widehat{S}_p(\mathcal{R})$, the Serre-Tate coordinate gives us an element $t_{\mathfrak{p}}(x) \in \widehat{\mathbb{G}}_m(\mathcal{R}) \otimes_{\mathbb{Z}_p} \mathcal{O}^*_{\mathfrak{p}} = (1+\mathfrak{m}_{\mathcal{R}}) \otimes_{\mathbb{Z}_p} \mathcal{O}^*_{\mathfrak{p}}$, here $\mathfrak{m}_{\mathcal{R}}$ is the maximal ideal of $\mathcal{R}$. In particular, when $\mathcal{R}$ is a subring of $\mathbb{C}_p$, we can consider the p-adic logarithm $log_p : \mathcal{R} \to \mathbb{C}_p$. Consider the following map:

$$log_p \otimes Id : (1+\mathfrak{m}_{\mathcal{R}}) \otimes_{\mathbb{Z}_p} \mathcal{O}^*_{\mathfrak{p}} \to \mathbb{C}_p \otimes_{\mathbb{Z}_p} \mathcal{O}^*_{\mathfrak{p}} \cong \mathrm{Hom}(\mathcal{O}_{\mathfrak{p}}, \mathbb{C}_p) \cong \prod_{\sigma: F\to\bar{\mathbb{Q}}, \sigma\sim\mathfrak{p}} \mathbb{C}_p.$$

Here the notation $\sigma \sim \mathfrak{p}$ means that the composite map $i_p \circ \sigma : F \to \bar{\mathbb{Q}}_p$ induces the prime $\mathfrak{p}$ of F. For such σ, let π_σ be the projection of $\Pi_{\sigma:F\to\bar{\mathbb{Q}},\sigma\sim\mathfrak{p}}\mathbb{C}_p$ to its σ-factor. Then we get an element $\tau_\sigma(x) = \pi_\sigma \circ (log_p \otimes Id)(t_{\mathfrak{p}}(x)) \in \mathbb{C}_p$. The association $x \in \widehat{S}_p(\mathcal{R}) \mapsto \tau_\sigma(x) \in \mathbb{C}_p$ gives p-adic rigid analytic functions on the rigid analytic space $(\widehat{S}^{ord}_p)^{p-an}$ associated to $\widehat{S}^{ord}_p$.

Remark 4.2. — From the above construction, we can see that actually the eigen coordinates take values in the valuation ring of the field $\mathbb{C}_p$. But in later argument, we need to invert the prime p when comparing the eigen coordinates and the invariant differential sheaf of $A_{\widetilde{\mathcal{W}}_\Xi}$ by the Kodaira-Spencer map. Hence we always regard the coordinates τ_σ as $\mathbb{C}_p$-valued functions on the formal scheme $\widehat{S}^{ord}_p$ or $\mathbb{C}_p$-valued rigid analytic functions on $(\widehat{S}^{ord}_p)^{p-an}$.

Since the action of $g = \hat{\rho}(\alpha)$ on the Shimura variety $Sh^{(\Xi)}_{/\mathcal{W}_\Xi}$ fixes the closed point x_p, this action also preserves the formal schemes $\widehat{S}^{ord}_p$ and $\widehat{S}^{ll}_p$, and hence $g = \hat{\rho}(\alpha)$ acts on the function τ_σ for each $\sigma \sim \mathfrak{p}$, $\mathfrak{p} \in \Sigma^{ord}$. By [22] Lemma 3.3, the action of $g = \hat{\rho}(\alpha)$ on the Serre-Tate coordinate $t_\mathfrak{p}$ is given by the formula $g(t_\mathfrak{p}) = t_\mathfrak{p}^{\alpha^{1-c}}$. (See the explanation after Lemma 4.1 for the two embeddings of M to $F_\mathfrak{p}$). Then by the construction of τ_σ, we see that the action of $g = \hat{\rho}(\alpha)$ on the function τ_σ is given by the formula: $g(\tau_\sigma) = \tau_\sigma \circ \hat{\rho}(\alpha) = i_p \circ \sigma(\alpha^{1-c}) \cdot \tau_\sigma$. We remark here that $i_p \circ \sigma : F \to \bar{\mathbb{Q}}_p$ naturally extends to an embedding $i_p \circ \sigma : F_\mathfrak{p} \to \bar{\mathbb{Q}}_p$, and hence the expression $i_p \circ \sigma(\alpha^{1-c})$ is well defined. As in [15], the function τ_σ is called a $\hat{\rho}$-eigen σ-coordinate.

Now consider the original point $x \in Sh^{(\Xi)}(\widetilde{\mathcal{W}_\Xi})$, which is represented by the quadruple $(A_{/\widetilde{\mathcal{W}}_\Xi}, \iota, \bar{\lambda}, \eta^{(\Xi)})$.

Lemma 4.3. — *Assume that we have a prime $\mathfrak{p} \in \Sigma^{ord}$, such that the exact sequence of Barsotti-Tate $\mathcal{O}_\mathfrak{p}$-modules :*

$$0 \to \mu_{p^\infty} \otimes_{\mathbb{Z}_p} \mathcal{O}^*_\mathfrak{p} \to A[\mathfrak{p}^\infty] \to F_\mathfrak{p}/\mathcal{O}_\mathfrak{p} \to 0$$

splits over over $\widetilde{W}_p$. In this case, the Serre-Tate coordinate $t_\mathfrak{p}(x)$ for the prime $\mathfrak{p}$ at the point x must be a p-th power root of unity. In particular, for the $\hat{\rho}$-eigen coordinate we have $\tau_\sigma(x) = 1$ for all $\sigma \sim \mathfrak{p}$.

This fact is proved in [2] Section 7 or [14] Section 6.3.4 in the case of elliptic curves. The higher dimensional case is considered in [7] when the abelian variety has ordinary reduction at $\mathfrak{P}$. Since the discussion in the partially ordinary case may not exist in the references, for the sake of completeness we give a proof here.

Proof. — First we assume that the ring $R = M \cap \theta^{-1}_\mathfrak{P}(\mathrm{End}_{\mathcal{O}_F}(A_{\mathfrak{P}/\bar{\mathbb{F}}_p}))$ in Lemma 4.1 is the integer ring $\mathcal{O}_M$ of M. From Lemma 4.1 , the prime $\mathfrak{p}$ in $\mathcal{O}_F$ splits into two primes $\mathcal{P}$ and $\bar{\mathcal{P}}$ in $\mathcal{O}_M$ such that the finite group scheme $A_\mathfrak{P}[\mathcal{P}]_{/\bar{\mathbb{F}}_p}$ (resp. $A_\mathfrak{P}[\bar{\mathcal{P}}]_{/\bar{\mathbb{F}}_p}$) is connected (resp. étale).

From the splitting of the exact sequence

$$0 \to \mu_{p^\infty} \otimes_{\mathbb{Z}_p} \mathcal{O}^*_\mathfrak{p} \to A[\mathfrak{p}^\infty] \to F_\mathfrak{p}/\mathcal{O}_\mathfrak{p} \to 0$$

over $\widetilde{W}_p$, for each integer n, there exists a finite subgroup scheme $A[\bar{\mathcal{P}}^n]_{/\widetilde{W}_p}$ of $A[\mathfrak{p}^n]_{/\widetilde{W}_p}$ which projects isomorphically to $A_\mathfrak{P}[\bar{\mathcal{P}}]_{/\bar{\mathbb{F}}_p}$ under the reduction map. Denote the quotient abelian scheme $(A/A[\bar{\mathcal{P}}^n])_{/\widetilde{W}_p}$ by $A'_{n/\widetilde{W}_p}$ and let $\pi_n : A \to A'_n$ be the natural projection defined over $\widetilde{W}_p$.

As M is a number field, there exists a positive integer N and an element $a \in \mathcal{O}_M$ such that $\bar{\mathcal{P}}^N = (a)$ in $\mathcal{O}_M$. Under the isomorphism $\theta_{\mathfrak{P}} : M \cong \mathrm{End}^0_F(A_{\mathfrak{P}/\bar{\mathbb{F}}_p})$, the element $a \in \mathcal{O}_M$ gives an isogeny of $A_{\mathfrak{P}/\bar{\mathbb{F}}_p}$ whose kernel is $A_{\mathfrak{P}}[\bar{\mathcal{P}}^N]_{/\bar{\mathbb{F}}_p}$, which is still denoted by a.

From the above construction, the projection $\pi_N : A \to A'_N$ is a lifting of the isogeny $a : A_{\mathfrak{P}} \to A_{\mathfrak{P}}$ to $\widetilde{W}_p$. From [26] Theorem 2.1(4) or [7] Formula 3.7.2, we have the following equation:

$$t_{\mathfrak{p}}(A'_{N/\widetilde{W}_p}; a(\alpha), \alpha') = t_{\mathfrak{p}}(A_{/\widetilde{W}_p}; \alpha, \bar{a}(\alpha')),$$

for $\alpha, \alpha' \in T_{\mathfrak{p}}A_{\mathfrak{P}}(\bar{\mathbb{F}}_p)$. Here $\bar{a}$ is the complex conjugate of a in M. From our choice of the element $a \in \mathcal{O}_M$, the action of a (resp. $\bar{a}$) on $T_{\mathfrak{p}}A_{\mathfrak{P}}(\bar{\mathbb{F}}_p)$ is divisible by p (resp. invertible). Hence the above equation tells us that the Serre-Tate coordinate $t_{\mathfrak{p}}(A_{/\widetilde{W}_p}; \alpha, \alpha')$ is a p-th power. Now we replace a by arbitrary power of a, and repeat the above argument. It follows that $t_{\mathfrak{p}}(A_{/\widetilde{W}_p}; \alpha, \alpha')$ is a p^n-th power for all $n \geqslant 1$. As $t_{\mathfrak{p}}(A_{/\widetilde{W}_p}; \alpha, \alpha') \in \widehat{\mathbb{G}}_m(\widetilde{W}_p)$, we have $t_{\mathfrak{p}}(A_{/\widetilde{W}_p}; \alpha, \alpha') = 1$ for all $\alpha, \alpha' \in T_{\mathfrak{p}}A_{\mathfrak{P}}(\bar{\mathbb{F}}_p)$. So we have $t_{\mathfrak{p}}(x) = 1$.

In the general case, as the ring R is an order in M, we can find a positive integer m such that $ma \in R$. We replace a by ma in the above argument, and it is easy to see that $t_{\mathfrak{p}}(x)^m = 1$ in this setting. As the Serre-Tate coordinate $t_{\mathfrak{p}}(x)$ belongs to $\widehat{\mathbb{G}}_m(\widetilde{W}_p)$, we can take m as a power of p, as desired. □

4.2. Comparison of endomorphism algebras at different special fibers

In this section we want to compare the endomorphism algebras of the special fibers of the abelian scheme $A_{/\widetilde{\mathcal{W}}_\Xi}$. The key ingredient is the Kodaira-Spencer map, which we will recall below.

As we can regard $x \in Sh^{(\Xi)}(\widetilde{\mathcal{W}}_\Xi)$ as a $\widetilde{W}_p$-rational point the point x actually sits in the formal scheme $\widehat{S}_{p/W_p}$, in other words, if we regard x as a morphism $\mathrm{Spec}(\widetilde{\mathcal{W}}_\Xi) \to Sh^{(\Xi)}$, then this morphism factors through $\nu_p : \widehat{S}_p \to Sh^{(\Xi)}$.

Let $(A_p^{univ}, \iota_p^{univ}, \phi_p^{univ})$ be the universal object over $\widehat{S}_p$. Let $\pi_p : A_p^{univ} \to \widehat{S}_p$ be the structure morphism and $e_p : \widehat{S}_p \to A_p^{univ}$ be the morphism corresponding to the identity element. Consider the sheaf $\omega_p^{univ} = (\pi_p)_*(\Omega_{A_p^{univ}/\widehat{S}_p}) = e_p^*(\Omega_{A_p^{univ}/\widehat{S}_p})$ over $\widehat{S}_{p/W_p}$, which has a natural $\mathcal{O}_{\widehat{S}_p} \otimes_{\mathbb{Z}} \mathcal{O}_F$-module structure, and compatible with arbitrary base change. Set

$(\omega_p^{univ})^{\otimes 2} = \omega_p^{univ} \otimes_{(\mathcal{O}_{\widehat{S}_p} \otimes_{\mathbb{Z}} \mathcal{O}_F)} \omega_p^{univ}$. Then we have the Kodaira-Spencer map:

$$KS : (\omega_p^{univ})^{\otimes 2} \to \Omega_{\widehat{S}_p/W_p}.$$

We remark here that the Kodaira-Spencer map is $\mathcal{O}_{\widehat{S}_p} \otimes_{\mathbb{Z}} \mathcal{O}_F$-linear and compatible with the $g = \hat{\rho}(\alpha)$-action on both sides.

By the isomorphism $\widehat{S}_p \cong \widehat{S}_p^{ord} \times \widehat{S}_p^{ll}$ over W_p, we have the decomposition: $\Omega_{\widehat{S}_p/W_p} = (\pi^{ord})^* \Omega_{\widehat{S}_p^{ord}/W_p} \oplus (\pi^{ll})^* \Omega_{\widehat{S}_p^{ll}/W_p}$, where $\pi^{ord} : \widehat{S}_p \to \widehat{S}_p^{ord}$ and $\pi^{ll} : \widehat{S}_p \to \widehat{S}_p^{ll}$ are the natural projection. Since $\widehat{S}_p^{ord} \cong \Pi_{\mathfrak{p} \in \Sigma^{ord}} \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathfrak{p}}^*$, if we set $\widehat{S}_{\mathfrak{p}} = \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}_p} \mathcal{O}_{\mathfrak{p}}^*$, then we have $\Omega_{\widehat{S}_p^{ord}/W_p} = \oplus_{\mathfrak{p} \in \Sigma^{ord}} (\pi_{\mathfrak{p}})^* \Omega_{\widehat{S}_{\mathfrak{p}}/W_p}$, where $\pi_{\mathfrak{p}} : \widehat{S}_p^{ord} \to \widehat{S}_{\mathfrak{p}}$ is the natural projection. To express the g-action on $\Omega_{\widehat{S}_p^{ord}/W_p}$ in a simple way, we base change this module to $\widetilde{L}_p$, i.e. we consider $\Omega_{\widehat{S}_p^{ord}/W_p} \otimes_{W_p} \widetilde{L}_p = \Omega_{\widehat{S}_p^{ord}/\widetilde{L}_p} = \oplus_{\mathfrak{p} \in \Sigma^{ord}} \Omega_{\widehat{S}_{\mathfrak{p}}/\widetilde{L}_p}$, which is free of finite rank over $(\widehat{S}_p^{ord})_{/\widetilde{L}_p}$. Moreover, for each $\mathfrak{p} \in \Sigma^{ord}$, the set $\{d\tau_\sigma | \tau \sim \mathfrak{p}\}$ forms a basis of the module $\Omega_{\widehat{S}_{\mathfrak{p}}/\widetilde{L}_p}$ over $\widehat{S}_{\mathfrak{p}}$, here τ_σ's are the $\hat{\rho}$-eigen coordinates constructed above.

On the other hand, we consider the cotangent bundle $(\omega_p^{univ})^{\otimes 2} \otimes_{W_p} \widetilde{L}_p = (\widetilde{\omega}_p^{univ})^{\otimes 2}$, which has a natural $\mathcal{O}_F \otimes_{\mathbb{Z}} \widetilde{L}_p$-module structure. By our construction of $\widetilde{L}_p$, for any embedding $\sigma : F \to \bar{\mathbb{Q}}$, $\sigma(\mathcal{O}_F)$ is contained in $i_p^{-1}(\widetilde{L}_p)$. Hence we have the isomorphism $\mathcal{O}_F \otimes_{\mathbb{Z}} \widetilde{L}_p \cong \Pi_{\sigma : F \to \bar{\mathbb{Q}}} \widetilde{L}_p$. By this isomorphism we can decompose the $\mathcal{O}_F \otimes_{\mathbb{Z}} \widetilde{L}_p$-module $(\widetilde{\omega}_p^{univ})^{\otimes 2}$ as $(\widetilde{\omega}_p^{univ})^{\otimes 2} = \oplus_{\sigma : F \to \bar{\mathbb{Q}}} (\widetilde{\omega}_p^{univ})^{\otimes 2\sigma}$ such that on the bundle $(\widetilde{\omega}_p^{univ})^{\otimes 2\sigma}$, $\mathcal{O}_F$ acts through the embedding σ.

Then by [25] section 1.0, for each $\mathfrak{p} \in \Sigma^{ord}$, the Kodaira-Spencer map induces an isomorphism

$$\bigoplus_{\sigma : F \to \bar{\mathbb{Q}}, \sigma \sim \mathfrak{p}} (\widetilde{\omega}_p^{univ})^{\otimes 2\sigma} \to (\pi_{\mathfrak{p}} \circ \pi^{ord})^* \Omega_{\widehat{S}_{\mathfrak{p}}/\widetilde{L}_p},$$

under which the bundle $(\widetilde{\omega}_p^{univ})^{\otimes 2\sigma}$ corresponds to the sub-bundle generated by $d\tau_\sigma$. Hence the action of $g = \hat{\rho}(\alpha)$ preserves each $(\widetilde{\omega}_p^{univ})^{\otimes 2\sigma}$ and acts it by multiplying the scalar $i_p \circ \sigma(\alpha^{1-c})$. Moreover, as we assume that $\tau_\sigma(x) = 0$ for all $\sigma \sim \mathfrak{p}$, g also preserves $(\widetilde{\omega}_p^{univ})^{\otimes 2\sigma}(x)$.

Now we can state the main result in this section:

Theorem 4.4. — *Fix an embedding $\sigma_1 : F \to \bar{\mathbb{Q}}$, such that $i_p \circ \sigma_1$ induces $\mathfrak{p}$. If there exists some prime $l \neq p$ in Ξ, such that the prime $\mathfrak{l}$*

induced from $i_l \circ \sigma_1$ *belongs to* Σ_l^{ord}*, then we have an isomorphism of* F*-algebras:* $\mathrm{End}_F^0(A_{\mathfrak{P}/\bar{F}_p}) \cong \mathrm{End}_F^0(A_{\mathfrak{L}/\bar{F}_l})$*. Here* $A_{\mathfrak{L}/\bar{F}_l}$ *sits in the quadruple* $(A_{\mathfrak{L}/\bar{\mathbb{F}}_l}, \iota_{\mathfrak{L}}, \bar{\lambda}_{\mathfrak{L}}, \eta_{\mathfrak{L}}^{(\Xi)})$ *obtained by mod* l *reduction of the point* $x \in Sh^{(\Xi)}(\widetilde{\mathcal{W}}_\Xi)$*.*

Proof. — Set $\omega = \pi_*(\Omega^1_{A/\widetilde{\mathcal{W}}_\Xi})$, which is naturally an $\mathcal{O}_F \otimes_{\mathbb{Z}} \widetilde{\mathcal{W}}_\Xi$-module. Again we set $\omega^{\otimes 2} = \omega \otimes_{(\mathcal{O}_F \otimes_{\mathbb{Z}} \widetilde{\mathcal{W}}_\Xi)} \omega$. The base change $\omega^{\otimes 2} \otimes_{\widetilde{\mathcal{W}}_\Xi} \widetilde{\mathcal{L}}_\Xi$ is an $\mathcal{O}_F \otimes_{\mathbb{Z}} \widetilde{\mathcal{L}}_\Xi$-module. By our construction of $\widetilde{\mathcal{L}}_\Xi$, we have an isomorphism:

$$\mathcal{O}_F \otimes_{\mathbb{Z}} \widetilde{\mathcal{L}}_\Xi \cong \bigoplus_{\sigma : F \to \bar{\mathbb{Q}}} \widetilde{\mathcal{L}}_\Xi.$$

From this we have the decomposition: $\omega^{\otimes 2} \otimes_{\widetilde{\mathcal{W}}_\Xi} \widetilde{\mathcal{L}}_\Xi = \oplus_{\sigma : F \to \bar{\mathbb{Q}}} \tilde{\omega}^{\otimes 2\sigma}$.

Since the formation of the cotangent sheaf ω_p^{univ} over $\widehat{S}_p$ is compatible with arbitrary base change, by the Cartesian diagram:

$$\begin{array}{ccc} A & \longrightarrow & A_p^{univ} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\widetilde{L}_p) & \xrightarrow{\ x\ } & \widehat{S}_p, \end{array}$$

we see that $\tilde{\omega}^{\otimes 2\sigma} \otimes_{\widetilde{\mathcal{L}}_\Xi} \widetilde{L}_p = (\omega_p^{univ})^{\otimes 2\sigma}(x)$. As $g = \hat{\rho}(\alpha)$ acts on the Shimura variety $Sh^{(\Xi)}_{/\widetilde{\mathcal{W}}_\Xi}$, g sends the bundle $\omega^{\otimes 2} \otimes_{\widetilde{\mathcal{W}}_\Xi} \widetilde{\mathcal{L}}_\Xi$ and hence each factor $\tilde{\omega}^{\otimes 2\sigma}$ to the corresponding bundles over $g(x)$. As g preserves $(\omega_p^{univ})^{\otimes 2\sigma}(x)$ for all $\sigma \sim \mathfrak{p}$, it also preserves $\tilde{\omega}^{\otimes 2\sigma}$. In particular, g preserves $\tilde{\omega}^{\otimes 2\sigma_1}$.

As $\tilde{\omega}^{\otimes 2\sigma_1} \otimes_{\widetilde{\mathcal{L}}_\Xi} \widetilde{L}_l = (\widetilde{\omega}_l^{univ})^{\otimes 2\sigma_1}(x)$, g also preserves the fiber $(\widetilde{\omega}_l^{univ})^{\otimes 2\sigma_1}(x)$ of the bundle $(\widetilde{\omega}_l^{univ})^{\otimes 2\sigma_1}$ at the point x_l and acts on it by multiplication by $i_l \circ \sigma_1(\alpha)$. Hence g must act on the eigen coordinate $\tau_{\sigma_1,l}(x)$ by multiplying $i_l \circ \sigma_1(\alpha)$, and g preserves the sub-bundle of $\Omega_{\widehat{S}_l/W_l}(x)$ generated by $d\tau_{\sigma_1,l}(x)$. If g sends $x_l \in Sh^{(\Xi)}(\bar{\mathbb{F}}_l)$ to another point $x_l' \neq x_l$, the action of g has to move the deformation space $\widehat{S}_l$ over x_l to the deformation space $\widehat{S}_l'$ over x_l', where $\widehat{S}_l'$ is the completion of $Sh^{(\Xi)}_{/\widetilde{\mathcal{W}}_\Xi}$ along the closed point x_l'. Then g induces an isomorphism of cotangent bundles $g : \Omega_{\widehat{S}_l/W_l}(x) \to \Omega_{\widehat{S}_l'/W_l}(g(x))$ and hence g cannot preserve any sub-bundle of $\widehat{S}^{ord}_{l/W_l}(x)$, which is a contradiction. So g fixes the point x_l, i.e. there exists a prime-to-Ξ isogney $\tilde{\theta}_{\mathfrak{L}}(\alpha)$ of $A_{\mathfrak{L}}$,such that $\tilde{\theta}_{\mathfrak{L}}(\alpha) \circ \eta_{\mathfrak{L}}^{(\Xi)} = \eta_{\mathfrak{L}}^{(\Xi)} \circ \hat{\rho}(\alpha)$, and hence establishes an isomorphism from the quadruple $(A_{\mathfrak{L}/\bar{\mathbb{F}}_l}, \iota_{\mathfrak{L}}, \bar{\lambda}_{\mathfrak{L}}, \eta_{\mathfrak{L}}^{(\Xi)})$ to the quadruple $(A_{\mathfrak{L}/\bar{\mathbb{F}}_l}, \iota_{\mathfrak{L}}, \bar{\lambda}_{\mathfrak{L}}, \eta_{\mathfrak{L}}^{(\Xi)} \circ \hat{\rho}(\alpha))$. The association $\alpha \mapsto \tilde{\theta}_{\mathfrak{L}}(\alpha)$

gives us an embedding $M \hookrightarrow \mathrm{End}^0_F(A_{\mathfrak{L}/\bar{F}_l})$. Since $\mathrm{End}^0_F(A_{\mathfrak{L}/\bar{F}_l})$ is also a CM quadratic extension of F by Lemma 4.1, this embedding must be an isomorphism. Hence we get the desired isomorphism of F-algebras. □

5. Main result on local indecomposability and applications

Let k be a number field. Suppose that we are given an abelian variety $A_{/k}$ and an algebra homomorphism $\iota : \mathcal{O}_F \to \mathrm{End}(A_{/k})$ (recall that F is a totally real field of degree d over $\mathbb{Q}$ and $\mathcal{O}_F$ is its integer ring). Assume that there is a prime ideal $\mathfrak{P}$ of k over a rational prime p, such that $A_{/k}$ satisfies the condition (NLL) in section 1. From Proposition 1.1, the abelian variety $A_{/\bar{\mathbb{Q}}} = A_{/k} \times_k \bar{\mathbb{Q}}$ is isotypic. Without loss of generality, we can assume that $A_{/k}$ is absolutely simple. Let $I_{\mathfrak{P}}$ be the inertia group of $\mathrm{Gal}(\bar{\mathbb{Q}}/k)$ at the prime $\mathfrak{P}$.

Before we give the main result, we need the following:

Lemma 5.1. — *If the abelian variety $A_{/\bar{\mathbb{Q}}} = A_{/k} \times_k \bar{\mathbb{Q}}$ does not have complex multiplication, we can find two primes $\mathfrak{L}$ and $\mathfrak{Q}$ of k lying over l and q respectively (p, l, q are distinct primes), such that $A_{/k}$ has good reduction at $\mathfrak{L}$ and $\mathfrak{Q}$, and F-algebras $\mathrm{End}^0_F(A_{\mathfrak{L}/\bar{\mathbb{F}}_l})$ and $\mathrm{End}^0_F(A_{\mathfrak{Q}/\bar{\mathbb{F}}_q})$ are non-isomorphic CM quadratic extension of F , here $A_{\mathfrak{L}/\bar{\mathbb{F}}_l}$ (resp. $A_{\mathfrak{Q}/\bar{\mathbb{F}}_q}$) is the reduction of $A_{/k}$ at $\mathfrak{L}$ (resp. $\mathfrak{Q}$).*

Proof. — Fix an embedding $\sigma : F \to \bar{\mathbb{Q}}$ such that the composition $i_l \circ \sigma$ induces the prime $\mathfrak{p}$. From [15] Proposition 7.1, the set

$$\{\mathfrak{L} | \mathfrak{L} \text{ is a prime of } k \text{ over a rational prime } l \neq p \\ \text{such that } A_{/k} \text{ has good reduction at } \mathfrak{L}, \text{ and } \Sigma_l^{ord} \neq \emptyset\}$$

has Dirichlet density 1. On the other hand, the primes $\mathfrak{l}$ in F which splits completely over $\mathbb{Q}$ also has Dirichlet density 1, we can find a prime $\mathfrak{L}$ of k over a rational prime l such that:

(1) l is unramified in F;
(2) $A_{/k}$ has good reduction at $\mathfrak{L}$ and Σ_l^{ord} contains the prime $\mathfrak{l}$ induced by $i_l \circ \sigma$ and $\mathfrak{l}$ splits over $\mathbb{Q}$.

Let $A_{\mathfrak{L}/\bar{\mathbb{F}}_l}$ be the reduction of $A_{/k}$ at $\mathfrak{L}$, and set $M_{\mathfrak{L}} = \mathrm{End}^0_F(A_{\mathfrak{L}/\bar{\mathbb{F}}_l})$. By Lemma 4.1, $M_{\mathfrak{L}}$ is a quadratic CM extension of the field F.

Now by an argument in [15] Proposition 5.1 we can find a prime $\mathfrak{Q}$ of k over a rational prime $q \neq p, l$, such that

(1) $A_{/k}$ has good reduction at $\mathfrak{Q}$;

(2) Σ_q^{ord} contains the prime induced by $i_q \circ \sigma$;
(3) $M_{\mathfrak{Q}} = \mathrm{End}_F^0(A_{\mathfrak{Q}/\bar{\mathbb{F}}_q})$ is a CM quadratic extension of F which is non-isomorphic to $M_{\mathfrak{L}}$.

For completeness, we give a sketch of the construction of $\mathfrak{Q}$ and refer to [15] Proposition 5.1 for more details. We use D to denote the division algebra $\mathrm{End}^0(A_{/k})$ and let Z be the center of D. From Proposition 1.2, Z is totally real and either $Z = F = D$ or D is a quaternion division algebra over Z and $[F : Z] = 2$.

For any prime $\mathfrak{q}$ of F, we fix an isomorphism $T_{\mathfrak{q}}(A) \cong (\mathcal{O}_{F,\mathfrak{q}})^2$, and denote by $r_{\mathfrak{q}} : \mathrm{Gal}(\bar{\mathbb{Q}}/k) \to GL_2(\mathcal{O}_{F,\mathfrak{q}})$ as the induced Galois representation on $T_{\mathfrak{q}}(A)$. Define the algebra $C_{\mathfrak{q}} = Z_{\mathfrak{q}}[r_{\mathfrak{q}}(\mathrm{Gal}(\bar{\mathbb{Q}}/k))]$ as the subalgebra of $\mathrm{End}^0_{Z_{\mathfrak{q}}}(T_{\mathfrak{q}}(A)) = \mathrm{End}_{\mathcal{O}_{Z,\mathfrak{q}}}(T_{\mathfrak{q}}(A)) \otimes_{\mathcal{O}_{Z,\mathfrak{q}}} Z_{\mathfrak{q}}$, of $r_{\mathfrak{q}}$ generated over $Z_{\mathfrak{q}}$ by the image of $r_{\mathfrak{q}}$. Then by Faltings' isogeny theorem, $C_{\mathfrak{q}}$ is either isomorphic to a quaternion division algebra over $Z_{\mathfrak{q}}$ or isomorphic to $M_2(Z_{\mathfrak{q}})$. In the case $\mathfrak{q} = \mathfrak{l}$, $C_{\mathfrak{l}}$ is isomorphic to $M_2(F_{\mathfrak{l}}) = M_2(Z_{\mathfrak{l}})$. Under this assumption,we can apply an argument in [30] Chapter 4 to prove that the image $\mathrm{Im}(r_{\mathfrak{l}})$ contains an open subgroup of $SL_2(\mathbb{Z}_l) \subseteq C_{\mathfrak{l}}^{\times}$.

Choose a quadratic ramified extension $K/\mathbb{Q}_l$. Since $F_{\mathfrak{l}}/\mathbb{Q}_l$ is unramified, K and $F_{\mathfrak{l}}$ are linearly disjoint over $\mathbb{Q}_l$. Let L be the compositum field of K and $F_{\mathfrak{l}}$. Define the torus $T_{/\mathcal{O}_{F,\mathfrak{l}}}$ of $GL_{2/\mathcal{O}_{F,\mathfrak{l}}}$ as the norm 1 subgroup of $Res_{\mathcal{O}_L/\mathcal{O}_{F,\mathfrak{l}}}(\mathbb{G}_m)$; i.e.

$$T(\mathcal{O}_{F,\mathfrak{l}}) = \{x \in \mathcal{O}_L^{\times} | Norm_{L/F_{\mathfrak{l}}}(x) = 1\}.$$

Hence $T_{/\mathcal{O}_{F,\mathfrak{l}}}$ is a maximal anisotropic torus of $GL_{2/\mathcal{O}_{F,\mathfrak{l}}}$, and $T(\mathcal{O}_{F,\mathfrak{l}}) \cap SL_2(\mathbb{Z}_l)$ is a maximal anisotropic torus of $GL_{2/\mathbb{Z}_l}$.

Choose $\alpha \in T(\mathcal{O}_{F,\mathfrak{l}}) \cap \mathrm{Im}(r) \cap SL_2(\mathbb{Z}_l)$, such that α has two different eigenvalues in $\bar{\mathbb{Q}}_l$. Then $T(\mathcal{O}_{F,\mathfrak{l}})$ is the centralizer T_α of α in $GL_2(\mathcal{O}_{F,\mathfrak{l}})$. Since the isomorphism classes of maximal torus in $GL_{2/\mathcal{O}_{F,\mathfrak{l}}}$ is finite, the isomorphism class of the centralizer of α is determined by α mod p^j, for some integer j large enough. In other word, if $\beta \in SL_2(\mathbb{Z}_l)$, such that $\alpha \equiv \beta$ mod p^j, then the centralizer T_β of β is isomorphic to $T_\alpha = T$. By Chebotarev density, we can find a prime $\mathfrak{Q}$ of k over a rational prime $q \neq p, l$, such that $A_{/k}$ has good reduction at $\mathfrak{Q}$ and $r(Frob_{\mathfrak{Q}}) \equiv \alpha \bmod p^j$. Hence the commutator $T_{r(Frob_{\mathfrak{Q}})}$ of $r(Frob_{\mathfrak{Q}})$ is isomrphic to T. Let $M_{\mathfrak{Q}}$ be the field generated over F by the eigenvalues of $r(Frob_{\mathfrak{Q}})$. By the above construction, $\mathfrak{l}$ does not split in $M_{\mathfrak{Q}}$, and hence $M_{\mathfrak{Q}}$ is not isomorphic to $M_{\mathfrak{L}}$. Further by [15] Proposition 7.1, we can assume that Σ_q^{ord} contains the prime induce from $i_q \circ \sigma$.

Then it is clear from the above construction that the primes $\mathfrak{Q}$ and $\mathfrak{L}$ satisfy the desired property. □

Now we can state and prove the main theorem in this paper:

Theorem 5.2. — *Under the above notations and assumptions, suppose further that $A_{/\bar{\mathbb{Q}}} = A_{/k} \times_k \bar{\mathbb{Q}}$ does not have complex multiplication, then for any $\mathfrak{p} \in \Sigma_p^{ord}$, the $\mathfrak{p}$-adic Tate module $T_\mathfrak{p}(A)$ of A is indecomposable as an $I_\mathfrak{P}$-module.*

Proof. — Let the prime $\mathfrak{Q}$ and $\mathfrak{L}$ be the primes of k in the previous lemma. Define a finite set of primes $\Xi = \{p, q, l\}$. For this set Ξ, we define the semilocal ring $\widetilde{\mathcal{W}}_k$ as in section 4. Hence the abelian variety $A_{/k}$ can be extended to an abelian scheme $A_{/\widetilde{\mathcal{W}}_k}$. From Proposition 2.6, replacing $A_{/k}$ by an isogenous abelian variety if necessary, we can assume that the abelian scheme $A_{/\widetilde{\mathcal{W}}_k}$ admits an $\mathcal{O}_F$-action $\iota : \mathcal{O}_F \to \mathrm{End}(A_{/\widetilde{\mathcal{W}}_k})$ and a $\mathfrak{c}$-polarization ϕ for some fractional ideal $\mathfrak{c}$ of F. Then by choosing a integral level structure α^Ξ of A, we get a quadruple $(A_{/\widetilde{\mathcal{W}}_k}, \iota, \phi, \alpha^\Xi)$, which represents a point in the Shimura variety $x \in Sh^{(\Xi)}(\widetilde{\mathcal{W}}_k)$.

Now assume that the Tate module $T_\mathfrak{p}(A)$ is decomposable as an $I_\mathfrak{P}$-module. Then the exact sequence of Barsotti-Tate $\mathcal{O}_\mathfrak{p}$-modules over $\widetilde{W}_p$:

$$0 \to \mu_{p^\infty} \otimes_{\mathbb{Z}_p} \mathcal{O}_\mathfrak{p} \to A[\mathfrak{p}^\infty] \to F_\mathfrak{p}/\mathcal{O}_\mathfrak{p} \to 0$$

splits. Then by Theorem 4.4, we must have isomorphisms of F-algebras: $M_\mathfrak{Q} \cong \mathrm{End}^0_F(A_{\mathfrak{Q}/\bar{\mathbb{F}}_p})$ and $M_\mathfrak{L} \cong \mathrm{End}^0_F(A_{\mathfrak{L}/\bar{\mathbb{F}}_p})$. But this contradicts with our construction $M_\mathfrak{Q} \not\cong M_\mathfrak{L}$. Hence $T_\mathfrak{p}(A)$ must be indecomposable as an $I_\mathfrak{P}$-module. □

5.1. Application to Hilbert modular Galois represenations

As the first application of Theorem 5.2, we study the Galois representation attached to certain Hilbert modular forms. First we recall the notions of Hilbert modular forms and Hecke operators.

Let $\mathrm{I} = \mathrm{Hom}_\mathbb{Q}(F, \bar{\mathbb{Q}})$, and let $\mathbb{Z}[\mathrm{I}]$ be the set of formal $\mathbb{Z}$-linear combinations of elements in I. Then $\mathbb{Z}[\mathrm{I}]$ can be identified with the character group $X(T)$ of the torus T. Take $k = (k_\sigma)_{\sigma \in \mathrm{I}}$ such that $k_\sigma \geqslant 2$ for all $\sigma \in \mathrm{I}$ and all the k_σ's have the same parity. Set $t = (1, \dots, 1) \in \mathbb{Z}[\mathrm{I}]$ and $n = k - 2t$. Choose $v = (v_\sigma)_{\sigma \in \mathrm{I}}$ such that $v_\sigma \geqslant 0$, for all σ, $v_\sigma = 0$ for at least one σ, and there exists $\mu \in \mathbb{Z}$ such that $n + 2v = \mu t \in \mathbb{Z}[\mathrm{I}]$. Then define $w = v + k - t$.

Recall that in Section 3 we define the algebraic group $G = Res_{\mathcal{O}_F/\mathbb{Z}}(GL_2)$ and $T = Res_{\mathcal{O}_F/\mathbb{Z}}(\mathbb{G}_m)$. Denote by $\nu : G \to T$ the reduced norm morphism. Fix an open subgroup U of $G(\widehat{\mathbb{Z}}) = GL_2(\widehat{\mathcal{O}}_F)$ where $\widehat{\mathcal{O}}_F = \mathcal{O}_F \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} = \Pi_{\mathfrak{p}} \mathcal{O}_{F,\mathfrak{p}}$. In the last product, $\mathfrak{p}$ ranges over all the prime ideals of $\mathcal{O}_F$ and $\mathcal{O}_{F,\mathfrak{p}}$ is the completion of $\mathcal{O}_F$ at $\mathfrak{p}$. Let $F_{\mathbb{A}} = F \otimes_{\mathbb{Z}} \mathbb{A}$ be the adele ring of F. We can decompose the group $G(F_{\mathbb{A}})$ as the product $G_\infty \times G_f$, where G_∞ (resp. G_f) is the infinite (resp. finite) part of $G(F_{\mathbb{A}})$, and for each $u \in G(F_{\mathbb{A}})$, we have the corresponding decomposition $u = u_\infty u_f$.

Let $\mathfrak{h}$ be the complex upper half plane and $i = \sqrt{-1} \in \mathfrak{h}$. Let $\mathfrak{h}^{\mathrm{I}}$ be the product of d copies of $\mathfrak{h}$ indexed by elements in I and $z_0 = (i, \ldots, i) \in \mathfrak{h}^{\mathrm{I}}$. Define a function $j : G_\infty \times \mathfrak{h}^{\mathrm{I}} \to \mathbb{C}^{\mathrm{I}}$ by the formula:

$$\left(\begin{pmatrix} a_\tau & b_\tau \\ c_\tau & d_\tau \end{pmatrix}, z_\tau \right)_{\tau \in \mathrm{I}} \mapsto (c_\tau z_\tau + d_\tau)_{\tau \in \mathrm{I}}.$$

Definition 5.3. — *Define the space of Hilbert modular cusp forms $S_{k,w}(U;\mathbb{C})$ as the set of functions $f : G(F_{\mathbb{A}}) \to \mathbb{C}$ satisfying the following conditions:*

(1) *$f|_{k,w}u = f$, for all $u \in UC_{\infty+}$ where $C_{\infty+} = (\mathbb{R}^\times \cdot SO_2(\mathbb{R}))^{\mathrm{I}} \subseteq G_\infty$, and*
$$f|_{k,w}u(x) = j(u_\infty, z_0)^{-k} v(u_\infty)^w f(xu^{-1});$$
(2) *$f(ax) = f(x)$ for all $a \in G(\mathbb{Q}) = GL_2(F)$;*
(3) *For any $x \in G_f$, the function $f_x : \mathfrak{h}^{\mathrm{I}} \to \mathbb{C}$ defined by $u_\infty(z_0) \mapsto j(u_\infty, z_0)^k v(u_\infty)^{-w} f(xu_\infty)$ for $u_\infty \in G_\infty$ is holomorphic;*
(4) *$\int_{F_{\mathbb{A}}/F} f\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} x\right) da = 0$ for all $x \in G(F_{\mathbb{A}})$ and additive Haar measure da on $F_{\mathbb{A}}/F$.*

When $F = \mathbb{Q}$, we also add the following condition: the function $|\mathrm{Im}(z)^{k/2} f_x(z)|$ is uniformly bounded on $\mathfrak{h}$ for all $x \in G_f = GL_2(\mathbb{A}_f)$.

Fix an integral ideal $\mathfrak{m}$ of F, we define three open subgroups of $GL_2(\widehat{\mathcal{O}}_F)$:

$$U_0(\mathfrak{m}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\widehat{\mathcal{O}}_F) | c \in \mathfrak{m}\widehat{\mathcal{O}}_F \right\},$$

$$U_1(\mathfrak{m}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\widehat{\mathcal{O}}_F) | c \in \mathfrak{m}\widehat{\mathcal{O}}_F, a \equiv 1 \; mod \; \mathfrak{m}\widehat{\mathcal{O}}_F \right\},$$

$$U(\mathfrak{m}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\widehat{\mathcal{O}}_F) | c \in \mathfrak{m}\widehat{\mathcal{O}}_F, a \equiv d \equiv 1 \; mod \; \mathfrak{m}\widehat{\mathcal{O}}_F \right\},$$

and set $S_{k,w}(\mathfrak{m}, \mathbb{C}) = S_{k,w}(U_1(\mathfrak{m}), \mathbb{C})$.

Let U, U' be two open compact subgroups of G_f and fix $x \in G_f$. Define a Hecke operator

$$[UxU'] : S_{k,w}(U;\mathbb{C}) \to S_{k,w}(U';\mathbb{C}), f \mapsto \sum_i f|_{k,w}x_i,$$

where $\{x_i\}$ is a set of representatives of the left cosets $U\backslash UxU'$; i.e., we have $UxU' = \coprod Ux_i$ and when we consider the action $f|_{k,w}x_i$, we regard $x_i \in G_f$ as an element in $G(F_{\mathbb{A}})$ such that its infinite part consists of d copies of identity matrices. For all prime ideal $\mathfrak{q}$ of F, fix a uniformizer $\pi_\mathfrak{q}$ of $F_\mathfrak{q}$, and define the Hecke operator

$$T(\mathfrak{q}) = \left[U \begin{pmatrix} 1 & 0 \\ 0 & \beta_\mathfrak{q} \end{pmatrix} U\right] : S_{k,w}(U;\mathbb{C}) \to S_{k,w}(U;\mathbb{C}),$$

where $\beta_\mathfrak{q} \in F_{\mathbb{A}_f}^\times$ is the finite idele whose $\mathfrak{q}$-component is $\pi_\mathfrak{q}$ and all the other components are 1. For each fractional ideal $\mathfrak{n}$ of F, set $\alpha = \Pi_\mathfrak{q} \pi_\mathfrak{q}^{v_\mathfrak{q}(\mathfrak{n})} \in F_{\mathbb{A}_f}^\times$, and define the Hecke operator

$$\langle \mathfrak{n} \rangle = \left[U \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} U\right] : S_{k,w}(U;\mathbb{C}) \to S_{k,w}(U;\mathbb{C}).$$

Let $f \in S_{k,w}(\mathfrak{m}, \mathbb{C})$ be a normalized Hilbert modular eigenform in the sense that for any prime ideal $\mathfrak{q}$ of F, there exists $c(\mathfrak{q}, f) \in \bar{\mathbb{Q}}$ and $d(\mathfrak{q}, f) \in \bar{\mathbb{Q}}$ such that $T(\mathfrak{q})(f) = c(\mathfrak{q}, f) \cdot f$ and $\langle \mathfrak{q} \rangle(f) = d(\mathfrak{q}, f) \cdot f$. Let K_f be the field generated over $\mathbb{Q}$ by all the $c(\mathfrak{a}, f)$'s and $d(\mathfrak{a}, f)$'s. Shimura proved that K_f is a number field which is either totally real or CM. Denote by $\mathcal{O}_f$ the integer ring of K_f.

For such an f, let $\pi_f = \otimes \pi_v$ be the automorphic representation of $GL_2(F_{\mathbb{A}})$ on the linear span of all the right translations of f by elements of $GL_2(F_{\mathbb{A}})$, here $F_{\mathbb{A}}$ is the adele ring of F,and π_v is a representation of $GL_2(F_v)$ for each finite place v of F. We assume that one of the following two statements holds:

(1) $[F : \mathbb{Q}]$ is odd;
(2) there exists some finite place v of F such that π_v is square integrable.

For such an eigenform f, the following result is known (see [21] Theorem 2.43 for details and historical remarks). For each prime λ of $\mathcal{O}_f$ over a rational prime p, there is a continuous representation $\rho_{f,\lambda} : \mathrm{Gal}(\bar{\mathbb{Q}}/F) \to GL_2(\mathcal{O}_{f,\lambda})$, which is unramified outside primes dividing $\mathfrak{m}p$ such that for any primes $\mathfrak{q} \nmid \mathfrak{m}p$, we have:

$$\mathrm{trace}(\rho_{f,\lambda}(Frob_\mathfrak{q})) = c(\mathfrak{q}, f), \text{ and } \det(\rho_{f,\lambda}(Frob_\mathfrak{q})) = d(\mathfrak{q}, f)N\mathfrak{q}.$$

Here $\mathcal{O}_{f,\lambda}$ is the completion of $\mathcal{O}_f$ at λ, $Frob_{\mathfrak{q}}$ is the Frobenius of $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$ at $\mathfrak{q}$,and for any ideal $\mathfrak{b}$ of $\mathcal{O}_F$, $N\mathfrak{b}$ is the cardinality number of the ring $\mathcal{O}_F/\mathfrak{b}$.

Fix a prime $\mathfrak{p}$ of $\mathcal{O}_F$ over a rational prime p, let $D_{\mathfrak{p}}$(resp. $I_{\mathfrak{p}}$) be the decomposition group (resp. inertia group) of $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$ at $\mathfrak{p}$. Let λ be a prime of $\mathcal{O}_f$ over p. From [35] Lemma 2.1.5, if $c(\mathfrak{p}, f)$ is a unit mod λ, then the restriction of $\rho_{f,\lambda}$ to $D_{\mathfrak{p}}$ is upper triangular, i.e. there exist two characters ϵ_1, ϵ_2 of $D_{\mathfrak{p}}$, such that

$$\rho_{f,\lambda}|_{D_{\mathfrak{p}}} \sim \begin{pmatrix} \epsilon_1 & * \\ 0 & \epsilon_2 \end{pmatrix}.$$

Lemma 5.4. — *Suppose that $k = 2t$ and f is nearly $\mathfrak{p}$-ordinary in the sense that $c(\mathfrak{p}, f)$ is a unit mod λ. Then there exists an abelian variety $A_{f/F}$, a finite extension L/K_f and an homomorphism $L \to \mathrm{End}^0(A_{f/F})$ such that degree of L over $\mathbb{Q}$ equals to the dimension of A_f and up to a character the λ-adic representation $\rho_{f,\lambda}$ comes from the Tate module of A_f.*

Proof. — As the Hecke operator $T(\mathfrak{p})$ acts nontrivially on f, from [18] Corollary 2.2, the local representation $\pi_{\mathfrak{p}}$ of $GL_2(F_{\mathfrak{p}})$ is either a principal representation $\pi(\xi_{\mathfrak{p}}, \eta_{\mathfrak{p}})$ or a special representation $\sigma(\xi_{\mathfrak{p}}, \eta_{\mathfrak{p}})$. From the argument in [18] Section 2, we can find a finite character $\chi : F_{\mathbb{A}}^{\times}/F^{\times} \to \bar{\mathbb{Q}}^{\times}$ ($F_{\mathbb{A}}$ is the adele ring of F) such that the $\mathfrak{p}$-component of χ satisfies $\chi_{\mathfrak{p}} = \xi_{\mathfrak{p}}$ on $\mathcal{O}_{F,\mathfrak{p}}^{\times}$ and unramified at every infinite place of F. Then the argument in [18] Section 2 implies that the automorphic representation $\chi \otimes \pi$ corresponds to a primitive $\mathfrak{p}$-ordinary newform f_0. If we regard the representations $\rho_{f,\lambda}$ and $\rho_{f_0,\lambda}$ as representations in $GL_2(\bar{\mathbb{Q}}_p)$, then they are related by the formula $\rho_{f,\lambda} \otimes \chi^{-1} = \rho_{f_0,\lambda}$. It is enough to prove the statement for the newform f_0 and henceforth we assume that the Hilbert modular form f is a primitive $\mathfrak{p}$-ordinary newform with character ψ for some idele class character ψ of F with finite order.

From [16] Theorem 4.4 or [34] Theorem 2.1, there exists an abelian variety A_f defined over F , a finite extension L/K_f whose degree equals to the dimension of A_f and an embedding $\theta : L \to \mathrm{End}(A_{f/F})$ such that the λ-adic representation associated to the Tate module of A_f is isomorphic to $\rho_{f,\lambda}$. Moreover the number field L is either totally real or CM. To be more precise, there exists an integer e such that $dim(A_{f/F}) = e[K_f : \mathbb{Q}]$. When $[F : \mathbb{Q}]$ is odd, $e = 1$ and there is nothing to explain in this situation. When $[F : \mathbb{Q}]$ is even, e can be bigger than 1, and a priori the p-adic Tate module of $A_{f/F}$ gives us a representation of $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$ in $GL_2(L_{\lambda})$,where L_{λ} is a finite extension of $K_{f,\lambda}$. Since this representation is odd, by choosing suitable eigenvectors of a complex conjugation $c \in \mathrm{Gal}(\bar{\mathbb{Q}}/F)$ as basis for $T_p(A_f)$, we can realize this representation in $GL_2(K_{f,\lambda})$. (See [35] Section 2.1 for details.) □

Remark 5.5. — As $c(\mathfrak{p}, \lambda)$ is a unit mod λ, the abelian variety A_f has potentially semistable reduction at $\mathfrak{p}$ by the lemma in [34] Section 2. More precisely, if we denote by F_ψ the number field corresponding to the character ψ by class field theory, then A_f has semistable reduction over F_ψ. In fact, choose a prime λ' of $\mathcal{O}_f$ over a rational prime $l \neq p$ and consider the λ'-adic representation $\rho_{f,\lambda'}$. When $\mathfrak{p}$ does not divide the level $\mathfrak{m}$, the abelian variety A_f has good reduction at $\mathfrak{p}$ because the representation $\rho_{f,\lambda'}$ is unramified at $\mathfrak{p}$. If $\mathfrak{p}$ divides $\mathfrak{m}$, one can consider the complex representation $\sigma_\mathfrak{p}$ of the local Weil-Deligne group $W'_{F_\mathfrak{p}}$ of F at $\mathfrak{p}$ associated to $\rho_{f,\lambda'}$ (see [33]). Then by a result of Carayol [3], we have an isomorphism $\pi(\sigma_\mathfrak{p}) \cong \pi_\mathfrak{p}$, where $\pi(\sigma_\mathfrak{p})$ is the representation of $GL_2(F_\mathfrak{p})$ associated to $\sigma_\mathfrak{p}$ under the local Langlands correspondence. In particular, the Euler factor $L(\pi_\mathfrak{p}, s)$ of the L-series at $\mathfrak{p}$ is given by $(1 - c(\mathfrak{p}, f)N\mathfrak{p}^{-s})^{-1}$. As $c(\mathfrak{p}, f) \neq 0$ by assumption, $L(\pi_\mathfrak{p}, s)$ is nontrivial. Hence $\pi_\mathfrak{p}$ is either a special representation $\sigma(\alpha_\mathfrak{p}, \beta_\mathfrak{p})$ or a principal series representation $\pi(\alpha_\mathfrak{p}, \beta_\mathfrak{p})$, where $\alpha_\mathfrak{p}, \beta_\mathfrak{p}$ are two quasi-characters of $F_\mathfrak{p}^\times$. In the first case, from [34] Theorem 2.2, the reduction of A_f at $\mathfrak{p}$ is purely multiplicative. From the uniformization result in [28], $\rho_{f,\lambda}|_{I_\mathfrak{p} \cap \mathrm{Gal}(\bar{\mathbb{Q}}/F_\psi)}$ is indecomposable. As $I_\mathfrak{p} \cap \mathrm{Gal}(\bar{\mathbb{Q}}/F_\psi)$ is a subgroup of $I_\mathfrak{p}$ with finite index, and $char(K_f) = 0$, the representation $\rho_{f,\lambda}|_{I_\mathfrak{p}}$ is also indecomposable. In the second case, as the Euler factor $L(\pi_\mathfrak{p}, s) \neq 1$, one of the quasi-characters $\alpha_\mathfrak{p}, \beta_\mathfrak{p}$ is unramified. By comparing the determinant of the two representations $\pi_\mathfrak{p}$ and $\sigma_\mathfrak{p}$, we see that the product $\psi_\mathfrak{p}^{-1}\alpha_\mathfrak{p}\beta_\mathfrak{p}$ is unramified, where $\psi_\mathfrak{p}$ is the $\mathfrak{p}$-component of the idele class character ψ. Hence over F_ψ, both quasi-characters $\alpha_\mathfrak{p}$ and $\beta_\mathfrak{p}$ are unramified. Then from the criterion of Néron-Ogg-Shafarevich, the abelian variety A_f has good reduction over F_ψ at $\mathfrak{p}$.

Now we would like to prove the following:

THEOREM 5.6. — *Under the above notations and assumptions in lemma 5.4, if f does not have complex multiplication, then the representation $\rho_{f,\lambda}|_{I_\mathfrak{p}}$ is indecomposable.*

Proof. — From Lemma 5.4 and Remark 5.5 we can assume that A_f has good reduction over F_ψ. From Proposition 1.2, we see that $A_{f/\bar{\mathbb{Q}}}$ is isotypic; i.e. there exists a simple abelian variety $B_{/\bar{\mathbb{Q}}}$ such that there exists an isogeny $\varphi : A_f \to B^e$ for some integer $e \geqslant 1$. This isogeny induces an isomorphism of simple algebras $i : \mathrm{End}^0(A_{f/\bar{\mathbb{Q}}}) \to \mathrm{End}^0((B_{/\bar{\mathbb{Q}}})^e)$. Hence we have an embedding $\theta_B = i \circ \theta : L \to \mathrm{End}^0((B_{/\bar{\mathbb{Q}}})^e)$.

From Proposition 1.4, we can find a totally real field F_B and a homomorphism $\iota_B : F_B \to \mathrm{End}^0(B_{/\bar{\mathbb{Q}}})$, such that $[F_B : \mathbb{Q}] = dim(B_{/\bar{\mathbb{Q}}})$. Let Z be the center of the division algebra $\mathrm{End}^0(B_{/\bar{\mathbb{Q}}})$. From the proof of Proposition 1.4, if we identify F_B as a subalgebra of $\mathrm{End}^0(B_{/\bar{\mathbb{Q}}})$ by ι_B, then $Z \subseteq F_B$ and $[F_B : Z] \leqslant 2$.

If $[F_B : Z] = 1$, we have $F_B = Z$ and hence $F_B \subseteq \theta_B(L)$. Since both A_f and B are projective varieties, we can find a finite extension M of F_ψ such that

(1) the abelian variety B is defined over M;
(2) we have the equalities of endomorphism algebras: $\mathrm{End}(A_{f/\bar{\mathbb{Q}}}) = \mathrm{End}(A_{f/M})$ and $\mathrm{End}(B_{/\bar{\mathbb{Q}}}) = \mathrm{End}(B_{/M})$.
(3) the isogeny φ is defined over M.

Under the above notations, the isogeny φ gives an isomorphism of p-adic Tate modules $T_p(B) \otimes_{F_B} L \cong T_p(A)$, which is equivariant under the action of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/M)$.

If $[F_B : Z] = 2$, F_B may not contained in the image $\theta_B(L)$. In this case, we can find a quadratic extension K/L such that F_B can be embedded into K. As the homomorphism $\theta : L \to \mathrm{End}^0(A_{f/\bar{\mathbb{Q}}})$ identifies L with a maximal commutative subfield of the simple algebra $\mathrm{End}^0(A_{f/\bar{\mathbb{Q}}})$, we can extend this homomorphism to a homomorphism $\theta' : K \to \mathrm{End}^0(A^2_{f/\bar{\mathbb{Q}}})$, which identifies K with a maximal commutative subfield of $\mathrm{End}^0(A^2_{f/\bar{\mathbb{Q}}})$. Similarly we can extend the homomorphism θ_B to a homomorphism $\theta'_B : K \to \mathrm{End}^0(B^{2e}_{/\bar{\mathbb{Q}}})$. Since $A^2_{f/\bar{\mathbb{Q}}}$ is isogeneous to $B^{2e}_{/\bar{\mathbb{Q}}}$, the simple algebras $\mathrm{End}^0(A^2_{f/\bar{\mathbb{Q}}})$ and $\mathrm{End}^0(B^{2e}_{/\bar{\mathbb{Q}}})$ are isomorphic. Since all automorphisms of a simple algebra are inner, by choosing a suitable isogeny from $A^2_{f/\bar{\mathbb{Q}}}$ to $B^{2e}_{/\bar{\mathbb{Q}}}$, we have an isomorphism $i' : \mathrm{End}^0(A^2_{f/\bar{\mathbb{Q}}}) \cong \mathrm{End}^0(B^{2e}_{/\bar{\mathbb{Q}}})$, such that $i' \circ \theta' = \theta'_B : K \cong \mathrm{End}^0(B^{2e}_{/\bar{\mathbb{Q}}})$. By the same argument as above, we can find a finite extension M/F_ψ such that we have an isomorphism of p-adic Tate modules: $T_p(B) \otimes_{F_B} K \cong T_p(A_f) \otimes_L K$, which is equivariant under the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/M)$.

As $B^e_{/M}$ is isogenous to $A_{f/M}$, $B_{/M}$ has good reduction at a prime $\mathfrak{p}'$ of M over the prime $\mathfrak{p}$ of F. By Theorem 5.2, for any place λ_B of F_B such that the λ_B-divisible Barsotti-Tate module of $B_{/M}$ is ordinary, the corresponding λ_B-adic Tate module is indecomposable as a $\mathrm{Gal}(\bar{\mathbb{Q}}/M) \cap I_{\mathfrak{p}}$-module. By the above isomorphism of Tate modules, we see that $\rho_{f,\lambda}|_{Gal(\bar{\mathbb{Q}}/M)\cap I_{\mathfrak{p}}}$

is indecomposable. Since $\mathrm{Gal}(\bar{\mathbb{Q}}/M) \cap I_{\mathfrak{p}}$ is a subgroup of $I_{\mathfrak{p}}$ with finite index, and $char(K_f) = 0$, the representation $\rho_{f,\lambda}|_{I_{\mathfrak{p}}}$ must be also indecomposable. □

From Theorem 5.6, we can prove a result on local indecomposability of Λ-adic Galois representations. First we briefly recall the definition of ordinary Hecke algebras defined in [17] Section 3.

Let Φ be the Galois closure of F in $\bar{\mathbb{Q}}$. The embedding $i_p : \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$ induces a p-adic valuation on Φ and we denote by $\mathcal{O}_\Phi$ the valuation ring. Let K be a finite extension of the p-adic closure of Φ in $\bar{\mathbb{Q}}_p$, and $\mathcal{O}_K$ be the valuation ring of K. Let F_∞/F be the maximal abelian extension of F unramified outside p and ∞, and Z be its Galois group. Let Z_1 be the torsion free part of Z. Let $\Lambda = \mathcal{O}_K[[Z_1]]$ be the continuous group algebra of Z_1 over $\mathcal{O}_K$. Then Λ is (noncanonically) isomorphic to the formal power series ring of $1+\delta$ variables over $\mathcal{O}_K$, where δ is the defect in Leopoldt's conjecture. Let $\chi : \mathrm{Gal}(\bar{\mathbb{Q}}/F) \to \mathbb{Z}_p^\times$ be the cyclotomic character. The restriction of χ to Z_1 gives a character of Z_1, which is still denoted by χ. For any integer $k \geqslant 2$ and a finite order character $\epsilon : Z_1 \to \bar{\mathbb{Q}}_p$. The character $\epsilon\chi^{k-1} : Z_1 \to \bar{\mathbb{Q}}_p$ gives a homomorphism $\kappa_{k,\epsilon} : \Lambda \to \bar{\mathbb{Q}}_p$.

For any two open compact subgroups U, U' of G_f and $x \in G_f$, we have the modified Hecke operator defined in [17] Section 3:

$$(UxU') : S_{k,w}(U;\mathbb{C}) \to S_{k,w}(U';\mathbb{C}).$$

For each prime ideal $\mathfrak{q}$ of F, set

$$T_0(\mathfrak{q}) = \left(U \begin{pmatrix} 1 & 0 \\ 0 & \beta_{\mathfrak{q}} \end{pmatrix} U\right) : S_{k,w}(U;\mathbb{C}) \to S_{k,w}(U;\mathbb{C}),$$

where $\beta_{\mathfrak{q}}$ is the same as in the definition of $T(\mathfrak{q})$.

Fix an integral ideal $\mathfrak{n}$ of F which is prime to p, and for each integer $\alpha \geqslant 1$, set $S_{k,w}(\mathfrak{n}p^\alpha;\mathbb{C}) = S_{k,w}(U_1(\mathfrak{n} \cap U(p^\alpha));\mathbb{C})$. Define the Hecke algebra $h_{k,w}(\mathfrak{n}p^\alpha;\mathcal{O}_\Phi)$ as the $\mathcal{O}_\Phi$-subalgebra of $\mathrm{End}_{\mathbb{C}}(S_{k,w}(\mathfrak{n}p^\alpha;\mathbb{C}))$ generated by all the $T_0(\mathfrak{q})$'s over $\mathcal{O}_\Phi$ and define $h_{k,w}(\mathfrak{n}p^\alpha;\mathcal{O}_K) = h_{k,w}(\mathfrak{n}p^\alpha;\mathcal{O}_\Phi) \otimes_{\mathcal{O}_\Phi} \mathcal{O}_K$. Inside $h_{k,w}(\mathfrak{n}p^\alpha;\mathcal{O}_K)$ we have the p-adic ordinary projector $e_\alpha = \lim_{n\to\infty} T_0(p)^{n!}$ and we have the ordinary Hecke algebra $h_{k,w}^{ord}(\mathfrak{n}p^\alpha;\mathcal{O}_K) = e_\alpha h_{k,w}(\mathfrak{n}p^\alpha;\mathcal{O}_K)$. For $\beta \geqslant \alpha \geqslant 0$, we have a natural surjective $\mathcal{O}_K$-algebra homomorphism $h_{k,w}^{ord}(\mathfrak{n}p^\beta;\mathcal{O}_K) \to h_{k,w}^{ord}(\mathfrak{n}p^\alpha;\mathcal{O}_K)$, and we define

$$h_{k,w}^{ord}(\mathfrak{n}p^\infty;\mathcal{O}_K) = \varprojlim_\alpha h_{k,w}^{ord}(\mathfrak{n}p^\alpha;\mathcal{O}_K).$$

From [17] Theorem 3.3, the ordinary Hecke algebra $h_{k,w}^{ord}(\mathfrak{n}p^\infty;\mathcal{O}_K)$ is a torsion free Λ-module of finite type, and the isomorphism class of

$h^{ord}_{k,w}(\mathfrak{n}p^\infty; \mathcal{O}_K)$ as an $\mathcal{O}_K$-algebra only depends on the class of v in $\mathbb{Z}[\mathrm{I}]/\mathbb{Z}t$, and hence we denote this algebra by $h^{ord}_v(\mathfrak{n}p^\infty; \mathcal{O}_K)$.

Now set $\mathbf{h} = h^{ord}_0(\mathfrak{n}p^\infty; \mathcal{O}_K)$. Fix $\mathrm{Spec}(\Lambda_L) \to \mathrm{Spec}(\mathbf{h})$ a (reduced) irreducible component of $\mathbf{h}$ and let $\mathcal{F} : \mathbf{h} \to \Lambda_L$ be the corresponding homomorphism. Then Λ_L is finite free over Λ, and the quotient field L of Λ_L is a finite extension of the quotient field of Λ. Let P be a $\bar{\mathbb{Q}}_p$-valued point of Λ_L, and let $\varphi_P : \Lambda_L \to \bar{\mathbb{Q}}_p$ be the corresponding homomorphism. The point P is called an arithmetic point if φ_P is an extension of $\kappa_{k,\epsilon}$ for some k and ϵ. If P is an arithmetic point, then the composition $\varphi_P \circ \mathcal{F} : \mathbf{h} \to \bar{\mathbb{Q}}_p$ gives the Hecke eigenvalues of a classical Hilbert modular form f of weight k and tame level $\mathfrak{n}$. We also say that the Hilbert modular form f corresponds to P, and f belongs to the family $\mathcal{F}$. We say that $\mathcal{F}$ has complex multiplication if there exists an arithmetic point P in $\mathcal{F}$, such that the corresponding Hilbert modular form has complex multiplication. Once this is the case, then for all arithmetic point in $\mathcal{F}$, the corresponding Hilbert modular form also has complex multiplication.

It's well known that there is a 2-dimensional Galois representation $\rho_{\mathcal{F}} : \mathrm{Gal}(\bar{\mathbb{Q}}/F) \to GL_2(L)$ attached to $\mathcal{F}$ such that for each prime $\mathfrak{p}$ of F over p, the restriction of $\rho_{\mathcal{F}}$ to the decomposition $D_\mathfrak{p}$ is upper triangula;i.e. $\rho_{\mathcal{F}}|_{D_\mathfrak{p}}$ is of the shape:

$$\rho_{\mathcal{F}}|_{D_\mathfrak{p}} \sim \begin{pmatrix} \delta_\mathfrak{p} & u_\mathfrak{p} \\ 0 & \varepsilon_\mathfrak{p} \end{pmatrix},$$

here $\delta_\mathfrak{p}, \varepsilon_\mathfrak{p} : D_\mathfrak{p} \to \Lambda_L$ are two characters of $D_\mathfrak{p}$.

Theorem 5.7. — *Suppose that $\mathcal{F}$ does not have complex multiplication, and $\mathcal{F}$ has an arithmetic point P which corresponds to a weight 2 Hilbert modular form satisfying the condition required in Theorem 5.6. Then there exists a proper closed subscheme S of $\mathrm{Spec}(\Lambda_L)$ such that for all arithmetic points P of $\mathrm{Spec}(\Lambda_L)$ outside S which corresponds to a classical form f, the representation $\rho_f|_{D_\mathfrak{p}}$ is indecomposable, where ρ_f is the Galois representation attached to f. In particular, when Leopoldt conjecture holds for F and p, then for all but finitely many classical forms f belonging to $\mathcal{F}$, the representation $\rho_f|_{D_\mathfrak{p}}$ is indecomposable.*

The proof follows essentially from the argument in [11] Theorem 18. For the sake of completeness, we give a proof here.

Proof. — By the assumption and Theorem 5.6, the representation $\varphi_P \circ \rho_{\mathcal{F}}|_{D_\mathfrak{p}}$ is indecomposable. Hence $\rho_{\mathcal{F}}|_{D_\mathfrak{p}}$ is indecomposable either. Define $c_\mathfrak{p} = \varepsilon_\mathfrak{p}^{-1} \cdot u_\mathfrak{p} : D_\mathfrak{p} \to \Lambda_L$. Then it's easy to check that $c_{\mathcal{F}}$ satisfies the cocycle condition and $\rho_{\mathcal{F}}|_{D_\mathfrak{p}}$ is indecomposable if and only if the class

$[c_\mathfrak{p}]$ of $c_\mathfrak{p}$ in $H^1(D_\mathfrak{p}, \Lambda_L(\delta_\mathfrak{p}\varepsilon_\mathfrak{p}^{-1}))$ is nontrivial. Since Λ_L is finite over Λ, the residue field of Λ_L is finite and let q be its order. Let E_1 be the compositum of the finitely many tamely ramified abelian extension of $F_\mathfrak{p}$ whose order divides $q-1$, and E_2 be the maximal abelian pro-p-extension of $F_\mathfrak{p}$. Denote by E the compositum field of E_1 and E_2 and set $H = \mathrm{Gal}(\bar{\mathbb{Q}}_p/E) \subseteq D_\mathfrak{p}$. Then the characters $\delta_\mathfrak{p}$ and $\varepsilon_\mathfrak{p}$ are trivial when restricted to H. Hence the restriction of $\rho_\mathcal{F}$ to H is of the shape:

$$\rho_\mathcal{F}|_H \sim \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix},$$

for some (additive) homomorphism $\lambda : H \to \Lambda_L$. From [11] Lemma 19, the restriction

$$H^1(D_\mathfrak{p}, \Lambda_L(\delta_\mathfrak{p}\varepsilon_\mathfrak{p}^{-1})) \to H^1(H, \Lambda_L(\delta_\mathfrak{p}\varepsilon_\mathfrak{p}^{-1}))$$

is injective. Since $[c_\mathfrak{p}]$ is nontrivial in $H^1(D_\mathfrak{p}, \Lambda_L(\delta_\mathfrak{p}\varepsilon_\mathfrak{p}^{-1}))$, the homomorphism $\lambda : H \to \Lambda_L$ is nontrivial. Let I be the ideal of Λ_L generated by $\lambda(H)$. Then I is nonzero and I defines a proper closed subscheme S of $\mathrm{Spec}(\Lambda_L)$. If f is a classical Hilbert modular form in $\mathcal{F}$, then $\rho_f|_H$ is decomposable if and only if f corresponds an arithmetic point in S. Hence for any arithmetic point P of $\mathcal{F}$ outside S, which corresponds to the modular form f, the representation $\rho_f|_H$, and hence $\rho_f|_{D_\mathfrak{p}}$ is indecomposable. $\square$

Now we consider the nearly ordinary case. Let

$$\mathcal{O}_{F,p} = \varprojlim \mathcal{O}_F/p^n\mathcal{O}_F$$

be the p-adic completion of $\mathcal{O}_F$ at p, and U_F be the torsion free part of $\mathcal{O}_{F,p}^\times$. Then set $\Gamma = Z_1 \times U_F$ and let $\Lambda' = \mathcal{O}_K[[\Gamma]]$ be the continuous group algebra. For any finite character $\varepsilon : \Gamma \to \bar{\mathbb{Q}}_p^\times$, we have another character

$$\Gamma = Z_1 \times U_F \to \bar{\mathbb{Q}}_p^\times, (a,d) \mapsto \chi(a)^\mu d^v \varepsilon((a,d)),$$

which induces a homomorphism $\kappa_{n,v,\varepsilon} : \Lambda' \to \bar{\mathbb{Q}}_p$.

We briefly recall the definition of nearly ordinary Hecke algebras defined in [19] Section 1. For any $\alpha \geqslant 1$, set $U_\alpha = U_1(\mathfrak{n}) \cap U(p^\alpha)$, and let $\mathbb{h}_{k,w}(\mathfrak{n}p^\alpha; \mathcal{O}_\Phi)$ be the $\mathcal{O}_\Phi$-subalgebra of $\mathrm{End}_\mathbb{C}(S_{k,w}(\mathfrak{n}p^\alpha; \mathbb{C}))$ generated by all the Hecke operators $(U_\alpha x U_\alpha)$ for $x \in U_0(\mathfrak{n}p^\alpha)$ over $\mathcal{O}_\Phi$. Set $\mathbb{h}_{k,w}(\mathfrak{n}p^\alpha; \mathcal{O}_K) = \mathbb{h}_{k,w}(\mathfrak{n}p^\alpha; \mathcal{O}_\Phi) \otimes_{\mathcal{O}_\Phi} \mathcal{O}_K$. Applying the ordinary projector e_α we get the nearly ordinary Hecke algebra $\mathbb{h}_{k,w}^{n.ord}(\mathfrak{n}p^\alpha; \mathcal{O}_K)$, and by taking limit, we have the Hecke algebra $\mathbb{h}_{k,w}^{n.ord}(\mathfrak{n}p^\infty; \mathcal{O}_K)$. From [19] Theorem 2.3, the Hecke algebra $\mathbb{h}_{k,w}^{n.ord}(\mathfrak{n}p^\infty; \mathcal{O}_K)$ are all isomorphic to each other for all pair (k,w) as $\mathcal{O}_K$-algebras and denote this algebra by $\mathbb{h}^{n.ord}(\mathfrak{n}p^\infty; \mathcal{O}_K)$, which is a torsion free Λ'-module of finite type. Let $\mathrm{Spec}(\Lambda'_L)$ be an irreducible component of $\mathrm{Spec}(\mathbb{h}^{n.ord}(\mathfrak{n}p^\infty; \mathcal{O}_K))$ and let

$\mathcal{F} : \mathbb{h}^{n.ord}(\mathfrak{n}p^\infty; \mathcal{O}_K) \to \Lambda'_L$ be the corresponding homomorphism. We know that Λ'_L is free of finite rank over Λ'. A $\bar{\mathbb{Q}}_p$-rational point $P \in \mathrm{Spec}(\Lambda'_L)(\bar{\mathbb{Q}}_p)$ is called an arithmetic point if the corresponding homomorphism φ_P extends $\kappa_{n,v,\varepsilon}$ for some n, v. For such an arithmetic point, the composition $\varphi_P \circ (\mathcal{F})$ gives the eigenvalues of a Hilbert modular form of weight (k, w) and tame level $\mathfrak{m}$.

For such an $\mathcal{F}$, we have a two dimensional Galois representation $\rho_{\mathcal{F}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\Lambda'_L)$ such that for any prime $\mathfrak{p}$ of F over p, the restriction $\rho_{\mathcal{F}}|_{D_\mathfrak{p}}$ is upper triangular. Similarly with Theorem 5.7, we have the following result:

Theorem 5.8. — *Suppose that $\mathcal{F}$ does not have complex multiplication, and $\mathcal{F}$ has an arithmetic point P which corresponds to a (parallel) weight 2 Hilbert modular form satisfying the condition required in Theorem 5.6. Then there exists a proper closed subscheme S of $\mathrm{Spec}(\Lambda'_L)$ such that for all arithmetic points P of $\mathrm{Spec}(\Lambda'_L)$ outside S which corresponds to a classical form f, the representation $\rho_f|_{D_\mathfrak{p}}$ is indecomposable, where ρ_f is the Galois representation associated to f.*

5.2. Application to a question of Coleman

In the rest of this paper, we work with elliptic modular forms. Let $p > 3$ be a prime number and N be a positive number prime to p. For each integer k we use $M_k^\dagger(\Gamma_1(N))$ (resp. $S_k^\dagger(\Gamma_1(N))$) to denote the space of overconvergent p-adic modular forms (resp. cuspforms) of level N over $\mathbb{C}_p$ (see [24] for the definitions). In [4] Proposition 6.3, Coleman proved that there is a linear map $\theta^{k-1} : M_{2-k}^\dagger(\Gamma_1(N)) \to M_k^\dagger(\Gamma_1(N))$ such that the effect of θ^{k-1} on the q-expansions is given by the differential operator $(q\frac{d}{dq})^{k-1}$. Also there is an operator U on $M_k^\dagger(\Gamma_1(N))$ such that if $F(q) = \Sigma_{n\geqslant 0} a_n q^n$ is an overconvergent modular form, then $U(F)(q) = \Sigma_{n\geqslant 0} a_{pn} q^n$. Recall that if F is a generalized eigenvector for U with eigenvalue λ in the sense that there exists some $n \geqslant 1$ such that $(U - \lambda)^n(F) = 0$, then the p-adic valuation of λ is called the slope of F. From [4] Lemma 6.3, if $f \in S_k^\dagger(\Gamma_1(N))$ is a normalized classical eigenform of slope strictly smaller than $k - 1$, then f cannot be in the image of θ^{k-1}. On the other hand, a classical eigenform cannot have slope larger than $k - 1$. Then it remains to consider the remaining boundary case; i.e. overconvergent modular forms of slope one less than the weight. In [4] Proposition 7.1, Coleman proved that for $k \geqslant 2$, every classical CM cuspidal eigenform of weight k and slope $k - 1$ is in the image of θ^{k-1}. Then he asked whether there is non-CM classical cusp forms in the image of θ^{k-1}. Since the only possible slope for new forms of weight k is $\frac{k}{2} - 1$ (see [13] Section 4), it's enough to consider old forms.

Let $g = \Sigma_{n \geqslant 1} a_n q^n$ be a classical normalized eigenform of level N and weight $k \geqslant 2$. Denote by $K_g = \mathbb{Q}(a_n | n = 1, 2, \ldots)$ the Hecke field of g, which is known to be a number field. For each prime $\mathfrak{p}$ of K_g over the rational prime p, it induces an embedding $i_{\mathfrak{p}} : K_g \to \bar{\mathbb{Q}}_p$ and let $v_{\mathfrak{p}}$ be the corresponding valuation on K_g. Then we can regard g as a modular form over $\bar{\mathbb{Q}}_p$ by $i_{\mathfrak{p}}$. As explained in [13] Section 4, one can attach to g two oldforms on $\Gamma_1(N) \cap \Gamma_0(p)$ whose slopes add up to $k - 1$. When the eigenform g is $\mathfrak{p}$-ordinary; i.e. $v_{\mathfrak{p}}(a_p) = 1$, one of the associated oldforms has slope 0 and the other has slope $k - 1$. We denote the latter oldform by f. What we can prove is the following:

Proposition 5.9. — *Let g be a weight two normalized classical cusp eigenform on $\Gamma_1(N)$ with the Hecke field K_g. Suppose that there exists a prime $\mathfrak{p}$ of K_g over the rational prime p such that g is $\mathfrak{p}$-ordinary, and the associated slope one oldform f is in the image of the operator θ. Then g is a CM eigenform.*

Proof. — Let $\rho_{g,\mathfrak{p}} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_2(K_{g,\mathfrak{p}})$ be the $\mathfrak{p}$-adic Galois representation attached to g. As explained in [8] Proposition 1.2 or [10] Proposition 11, when f is in the image of θ, the restriction of $\rho_{g,\mathfrak{p}}$ to an inertia group I_p of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ at p splits as the direct sum of the trivial character and the character χ_p, where χ_p is the p-adic cyclotomic character. Then from Theorem 5.6, the eigenform g must have complex multiplication. □

Remark 5.10. — In [8] Theorem 1.3, Emerton proved that if the assumption in the above proposition is true for all primes $\mathfrak{p}$ of K_g over p, then g is a CM eigenform. Hence the above proposition can be regarded as an improvement of his theorem. Also in [10] Section 6, Ghate discussed the case when p divides the level N. In this case he explained that one can also attach to the eigenform g a primitive form f with the same weight and level as g. Then he proved that f is in the image of θ if and only if the restriction of $\rho_{g,\mathfrak{p}}$ to the inertia group I_p splits (we need to emphasize here that Ghate's argument works for all weights, but we restrict ourselves to the weight two case where Theorem 5.6 is applicable). Hence the result in Theorem 5.6 also applies and the above proposition still holds in this case.

BIBLIOGRAPHY

[1] B. Balasubramanyam, E. Ghate & V. Vatsal, "On local Galois representations associated to ordinary Hilbert modular forms", Preprint.

[2] M. Brakočević, "Anticyclotomic p-adic L-function of central critical Rankin-Selberg L-value", IMRN. doi:10.1093/imrn/rnq275.

[3] H. Carayol, "Sur les représentations l-adiques associées aux formes modulaires de Hilbert", *Ann. Sci. École Norm. Sup. (4)* **19** (1986), p. 409-468.

[4] R. Coleman, "Classical and overconvergent modular forms", *Invent. Math.* **124** (1996), no. 1-3, p. 215-241.

[5] B. Conrad, C.-L. Chai & F. Oort, "CM Liftings", 2011.

[6] A. de Jong & R. Noot, "Jacobians with complex multiplication", in *Arithmetic Algebraic Geometry* (G. van der Geer, F. Oort and J.Steenbrink, ed.), Progress in Math., vol. 89, Brikhäuser, Boston, 1991, p. 177-192.

[7] P. Deligne & G. Pappas, "Singularités des espaces de modules de Hilbert, en les caractéristiques divisant le discriminant", *Compositio Math.* **90** (1994), p. 59-79.

[8] M. Emerton, "A p-adic variational Hodge conjecture and modular forms with complex multiplication", preprint available at Emerton's homepage: http://www.math.uchicago.edu/~emerton/pdffiles/cm.pdf.

[9] G. Faltings, "Finiteness theorems for abelian varieties over number fields", in *Arithmetic geometry* (G. Cornell and J. Silverman, ed.), Springer-Verlag, New York, 1986, p. 9-27.

[10] E. Ghate, "Ordinary forms and their local Galois representations", in *Algebra and number theory*, Hindustan Book Agency, Delhi, 2005, p. 226-242.

[11] E. Ghate & V. Vatsal, "On the local behaviour of ordinary Λ-adic representations", *Ann. Inst. Fourier (Grenoble)* **54** (2004), p. 2143-2162.

[12] E. Goren, *Lectures on Hilbert Modular Varieties and Modular Forms*, CRM monograph series, no. 14, American Mathematical Soc., 2001.

[13] F. Gouvêa & B. Mazur, "Families of modular eigenforms", *Math. Comp.* **58** (1992), no. 198, p. 793-805.

[14] H. Hida, "Elliptic Curves and Arithmetic Invariant", book manuscript to be published by Springer.

[15] ———, "Local indecomposability of Tate modules of non CM abelian varieties with real multiplication", to appear in J. Amer. Math. Soc., preprint available at Hida's homepage: http://www.math.ucla.edu/~hida/AbNSS.pdf.

[16] ———, "On abelian varieties with complex multiplication as factors of the Jacobians of Shimura curves", *Amer. J. Math.* **103** (1981), p. 727-776.

[17] ———, "On p-adic Hecke algebras for GL_2 over totally real fields", *Ann. of Math.* **128** (1988), p. 295-384.

[18] ———, "Nearly ordinary Hecke algebras and Galois representations of several variables", in *Algebraic analysis, geometry, and number theory* (Baltimore, MD, 1988), Johns Hopkins Univ. Press, Baltimore, MD, 1989, p. 115-134.

[19] ———, "On nearly ordinary Hecke algebras for $GL(2)$ over totally real fields", in *Algebraic number theory*, Adv. Stud. Pure Math., 17, Academic Press, Boston, MA, 1989, p. 139-169.

[20] ———, *p-adic automouphism forms on Shimura varieties*, Springer Monographs in Mathematics, Springer-Verlag, 2004.

[21] ———, *Hilbert modular forms and Iwasawa theory*, Oxford Mathematical Monographs, Oxford University Press, 2006.

[22] ———, "The Iwasawa μ-invariant of p-adic Hecke L-functions", *Ann. of Math.* **172** (2010), p. 41-137.

[23] ———, *Geometric Modular Forms and Elliptic Curves*, second ed., World Scientific, Singapore, 2012.

[24] N. Katz, "p-adic properties of modular schemes and modular forms", in *Modular functions of one variable* III, Lecture Notes in Math., vol. 350, Springer, Berlin, 1973, p. 69-190.

[25] N. M. Katz, "p-adic L-functions for CM fields", *Invent. Math.* **49** (1978), p. 199-297.

[26] ———, "Serre-Tate local moduli", in *Algebraic surfaces (Orsay, 1976-78)*, Lecture Notes in Math., vol. 868, Springer, Berlin-New York, 1981, p. 138-202.

[27] D. Mumford, *Geometric Invariant Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, vol. 34, Springer-Verlag, Berlin-New York, 1965.

[28] ———, "An analytic construction of degenerating abelian varieties over complete rings", *Compositio Math.* **24** (1972), p. 239-272.

[29] M. Rapoport, "Compactifications de l'espace de modules de Hilbert-Blumenthal", *Compositio Math.* **36** (1978), no. 3, p. 255-335.

[30] K. A. Ribet, "Galois action on division points of abelian varieties with real multiplications", *Amer. J. Math.* **98** (1976), p. 751-804.

[31] J.-P. Serre & J. Tate, "Good reduction of abelian varieties", *Ann. of Math.* **88** (1965), p. 492-517.

[32] G. Shimura, "On analytic families of polarized abelian varieties and automorphic functions", *Ann. of Math.* **78** (1963), p. 149-192.

[33] J. Tate, "Number theoretic background", in *Automorphic forms, representations and L-functions*, Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977, part 2, p. 3-26.

[34] A. Wiles, "On p-adic representations for totally real fields", *Ann. of Math.* **123** (1986), p. 407-456.

[35] ———, "On ordinary λ-adic representations associated to modular forms", *Invent. math.* **94** (1988), p. 529-573.

Manuscrit reçu le 6 novembre 2012,
accepté le 17 mai 2013.

Bin ZHAO
Department of Mathematics,
UCLA,
Los Angeles, CA 90095-1555,
USA
zhaobin@math.ucla.edu