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Universal Taylor series, conformal mappings and boundary behaviour

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ABSTRACT. — A holomorphic function f on a simply connected domain Ω is said to possess a universal Taylor series about a point in Ω if the partial sums of that series approximate arbitrary polynomials on arbitrary compacta K outside Ω (provided only that K has connected complement). This paper shows that this property is not conformally invariant, and, in the case where Ω is the unit disc, that such functions have extreme angular boundary behaviour.

RÉSUMÉ. — On dit qu'une fonction f, qui est holomorphe sur un domaine simplement connexe Ω, possède une série universelle de Taylor autour d'un point de Ω si tout polynôme sur tout compact K en-dehors de Ω peut être approximé par des sommes partielles de cette série (pourvu que le complémentaire de K soit connexe). Cet article montre que cette propriété n'est pas invariante par transformation conforme et, dans le cas où Ω est le disque unité, que ces fonctions ont un comportement extrême dans le sens des limites angulaires.

1. Introduction

Let f be a holomorphic function on a simply connected proper subdomain Ω of the complex plane $\mathbb{C}$, let $\xi \in \Omega$ and $S_N(f, \xi)(z)$ denote the partial sum $\sum_{n=0}^{N} a_n(z - \xi)^n$ of the Taylor series of f about ξ. We call this series *universal* and write $f \in \mathcal{U}(\Omega, \xi)$ if, for every compact set $K \subset \mathbb{C} \backslash \Omega$ that has connected complement and every continuous function $g : K \to \mathbb{C}$ that is holomorphic on K°, there is a subsequence $(S_{N_k}(f, \xi))$ that converges uniformly to g on K. It is known that possession of such universal expansions is a generic property of holomorphic functions on simply connected domains (that is, $\mathcal{U}(\Omega, \xi)$ is a dense G_δ subset of the space of all

holomorphic functions on Ω endowed with the topology of local uniform convergence [17], [18]) and that the collection $\mathcal{U}(\Omega, \xi)$ is independent of the choice of the centre of expansion ξ (see [14], [16]).

However, significant questions remain open. A fundamental issue concerns conformal invariance:

PROBLEM 1. — *If $F : \Omega_0 \to \Omega$ is a conformal mapping, where Ω_0 and Ω are simply connected domains, and if $f \in \mathcal{U}(\Omega, \xi)$, does it follow that $f \circ F \in \mathcal{U}(\Omega_0, F^{-1}(\xi))$?*

Another question concerns boundary behaviour, about which there is a growing literature [18], [9], [13], [14], [7], [16], [1], [5], [11]. For example, in the case of the unit disc $\mathbb{D}$, it is known that if $f \in \mathcal{U}(\mathbb{D}, 0)$, then f does not belong to the Nevanlinna class (see [15]) and there is a residual subset Z of the unit circle $\mathbb{T}$ such that the set $\{f(r\zeta) : 0 < r < 1\}$ is unbounded for every $\zeta \in Z$ (see [4]). However, little progress has yet been made on the natural question:

PROBLEM 2. — *What can be said about the angular boundary behaviour of functions in $\mathcal{U}(\mathbb{D}, 0)$?*

I am grateful to Vassili Nestoridis for alerting me to the fact that Problem 1 had remained unresolved, and to George Costakis for drawing my attention to Problem 2. The answers are given below. Let S denote the strip $\{z \in \mathbb{C} : -1 < \operatorname{Re} z < 1\}$.

THEOREM 1. — *There is a function $f \in \mathcal{U}(S, 0)$ with the following properties:*
(i) for any conformal mapping $F : \mathbb{D} \to S$ we have $f \circ F \notin \mathcal{U}(\mathbb{D}, F^{-1}(0))$;
(ii) there exist conformal mappings $F : S \to S$ such that $f \circ F \notin \mathcal{U}(S, F^{-1}(0))$.

We define angular approach regions at a point $\zeta \in \mathbb{T}$ by

$$\Gamma_\alpha^t(\zeta) = \{z : |z - \zeta| < \alpha(1 - |z|) < \alpha t\} \quad (\alpha > 1, 0 < t \leqslant 1).$$

A boundary point ζ is called a *Fatou point* of a holomorphic function f on $\mathbb{D}$ if $\lim_{z \to \zeta, z \in \Gamma_\alpha^1(\zeta)} f(z)$ exists finitely for all α. At the opposite extreme, ζ is called a *Plessner point* of f if $f(\Gamma_\alpha^t(\zeta))$ is dense in $\mathbb{C}$ for all α and t. Plessner's theorem says that, for any holomorphic function f on $\mathbb{D}$, almost every point of $\mathbb{T}$ is either a Fatou point or a Plessner point of f (see Theorem 6.13 in [19]). Our next result shows that universal Taylor series have extreme angular boundary behaviour.

THEOREM 2. — *If $f \in \mathcal{U}(\mathbb{D}, 0)$, then almost every point of $\mathbb{T}$ is a Plessner point of f.*

An easy consequence of Theorem 2 is the following Baire category analogue, which strengthens the result of Bayart mentioned earlier.

COROLLARY 3. — *If $f \in \mathcal{U}(\mathbb{D}, 0)$, then there is a residual subset Z of $\mathbb{T}$ such that $\{f(r\zeta) : 0 < r < 1\}$ is dense in $\mathbb{C}$ for every $\zeta \in Z$.*

It turns out that the above solutions to Problems 1 and 2 both emerge from the same non-trivial potential theoretic insight, which we will now describe. The Poisson kernel for $\mathbb{D}$ is given by

$$P(z, \zeta) = \frac{1 - |z|^2}{|z - \zeta|^2} \quad (z \in \mathbb{D}, \zeta \in \mathbb{T}).$$

A set $E \subset \mathbb{D}$ is said to be *minimally thin at a point* $\zeta \in \mathbb{T}$ if there is a (Green) potential u on $\mathbb{D}$ such that $u \geqslant P(\cdot, \zeta)$ on E. For example, if $D \subset \mathbb{D}$ is a disc that is internally tangent to $\mathbb{T}$ at a point ζ, then $\mathbb{D} \backslash D$ is minimally thin at ζ. This follows from the facts that D is of the form $\{P(\cdot, \zeta) > c\}$ for some $c > 0$, and that $\min\{P(\cdot, \zeta), c\}$ is a potential on $\mathbb{D}$ since its greatest harmonic minorant is readily seen to be 0. More generally (see Theorem 2 in [6] and Theorem 9.5.5(iii) in [2]), if $\psi : [0, 1] \to [0, 1]$ is increasing, then the set $\{z \in \mathbb{D} : \operatorname{Re} z > 1 - \psi(|\operatorname{Im} z|)\}$ is minimally thin at 1 if and only if $\int_0^1 t^{-2} \psi(t) dt < \infty$. An introduction to the notion of minimal thinness may be found in Chapter 9 of the book [2].

The key underlying result in this paper is as follows. We abbreviate $S_N(f, 0)$ to S_N.

THEOREM 4. — *Let f be a holomorphic function on $\mathbb{D}$ and h be a positive harmonic function on $\mathbb{D}$ such that the set $\{|f| \geqslant e^h\}$ is minimally thin at $\zeta_0 \in \mathbb{T}$. If (S_{N_k}) is uniformly bounded on an open arc of $\mathbb{T}$ that contains ζ_0, then $(e^{-h} S_{N_k})$ is uniformly bounded on a set of the form $\mathbb{D} \backslash E$, where E is minimally thin at ζ_0. In the particular case where h is constant, we can thus conclude that (S_{N_k}) is uniformly bounded on $\mathbb{D} \backslash E$.*

COROLLARY 5. — *If $f \in \mathcal{U}(\mathbb{D}, 0)$ and h is a positive harmonic function on $\mathbb{D}$, then there is at most one point of $\mathbb{T}$ at which the set $\{|f| \geqslant e^h\}$ is minimally thin.*

Bayart [4] has shown that, if $f \in \mathcal{U}(\mathbb{D}, 0)$ and $a > 0$, then there is at most one point ζ of $\mathbb{T}$ such that $|f| < a$ on a disc internally tangent to $\mathbb{T}$ at ζ. Corollary 5 is a significantly stronger result, and this extra strength is crucial for our purposes. The "one point" in Corollary 5 can actually arise. This follows by choosing the set A in the following result so that $\mathbb{D} \backslash A$ is minimally thin at 1.

PROPOSITION 6. — *Let $A \subset \mathbb{D}$, where $\overline{A} \cap \mathbb{T} = \{1\}$, and let $w : \mathbb{D} \to (1, \infty)$ be a continuous function such that $w(z) \to \infty$ as $z \to 1$. Then there exists $f \in \mathcal{U}(\mathbb{D}, 0)$ such that $|f| \leqslant w$ on A. In particular, this is true for $w = e^h$, where h is a positive harmonic function on $\mathbb{D}$ that tends to ∞ at 1.*

Let D be a disc contained in $\mathbb{D}$ that is internally tangent to $\mathbb{T}$ at the point 1. As noted in the proof of Proposition 5.6 in [14], no member of $\mathcal{U}(\mathbb{D}, 0)$, when restricted to D, can have a limit at 1. However, by the above proposition, there exists f in $\mathcal{U}(\mathbb{D}, 0)$ satisfying $|f(z)| \leqslant |z - 1|^{-1/2}$ on D, whence $(z - 1)f(z) \to 0$ as $z \to 1$ in D. Thus the function $z \mapsto (z-1)f(z)$ does not belong $\mathcal{U}(\mathbb{D}, 0)$. This answers a question of Costakis [8], who had asked whether the property of having a universal Taylor series is preserved under multiplication by non-constant polynomials. Similarly, no antiderivative of this function f can belong to $\mathcal{U}(\mathbb{D}, 0)$. This gives a negative answer to another question of Costakis (private communication), about whether antiderivatives of universal Taylor series are necessarily universal. (The corresponding question for derivatives remains open.) Costakis has also observed that Theorem 2 above and Theorem 1.2 of [3] together show that each member of $\mathcal{U}(\mathbb{D}, 0)$ must tend to ∞ along some path to the boundary.

We will prove Theorem 4 in the next section and subsequently proceed to the remaining proofs.

2. Proof of Theorem 4

Let $D(z, r)$ denote the open disc of centre z and radius r, let $C(K)$ denote the space of real-valued continuous functions on a compact set K, and let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denote the extended complex plane. If $U \subset \widehat{\mathbb{C}}$ is open, we denote by $G_U(\cdot, \zeta)$ the Green function for U with pole at $\zeta \in U$, and assign this function the value 0 outside U.

Now let f be a holomorphic function on $\mathbb{D}$ and h be a positive harmonic function on $\mathbb{D}$ such that the set $\{|f| \geqslant e^h\}$ is minimally thin at $\zeta_0 \in \mathbb{T}$. We define

$$U = \left\{ z \in D(3\zeta_0/4, 1/4) : |f(z)| < e^h \right\}.$$

Then U is open and $\overline{U} \cap \mathbb{T} = \{\zeta_0\}$. Also, $\mathbb{D} \backslash U$ is minimally thin at ζ_0, since

$$\mathbb{D} \backslash U = [\mathbb{D} \backslash D(3\zeta_0/4, 1/4)] \cup \{|f| \geqslant e^h\}$$

and the union of two sets that are minimally thin at ζ_0 is also minimally thin at ζ_0. Let μ_z denote harmonic measure for U and $z \in U$. For each $z \in U$ we define a modified measure μ_z^* on ∂U by writing

$$d\mu_z^*(\zeta) = \frac{\log(1/\,|\zeta|)}{\log(1/\,|z|)} d\mu_z(\zeta) \quad (\zeta \in \partial U).$$

These are probability measures since the function $\zeta \mapsto \log(1/\,|\zeta|)$ is harmonic on $\mathbb{C}\backslash\{0\}$.

We will make use of some key facts about minimal thinness from [2]. The first of these, Theorem 9.6.2, describes how the minimal thinness of $\mathbb{D}\backslash U$ at ζ_0 affects the behaviour of positive superharmonic functions v on U near ζ_0. Specifically, it tells us that, for each such v, there is a set $E(v) \subset \mathbb{D}$, minimally thin at ζ_0, and a number $l(v) \in (0, \infty]$, such that

$$\frac{v(z)}{\log(1/\,|z|)} = \frac{v(z)}{G_{\mathbb{D}}(z,0)} \to l(v) \quad (z \to \zeta_0, z \in \mathbb{D}\backslash E(v)).$$

If $\phi \in C(\partial U)$, then $\left|\int \phi d\mu_z^*\right| \leqslant \max_{\partial U} |\phi|$ for all $z \in U$. By considering separately the cases where v is given by

$$z \mapsto \int \phi^{\pm}(\zeta) \log(1/\,|\zeta|) d\mu_z(\zeta),$$

we now see that there is a set $E_\phi \subset \mathbb{D}$, minimally thin at ζ_0, and a number $l_\phi \in \mathbb{R}$, such that

$$\int \phi d\mu_z^* = \frac{1}{\log(1/\,|z|)} \int \phi(\zeta) \log(1/\,|\zeta|) d\mu_z(\zeta)$$
$$\to l_\phi \quad (z \to \zeta_0, z \in \mathbb{D}\backslash E_\phi).$$

Now let (ϕ_n) be a dense sequence in $C(\partial U)$. Lemma 9.3.1 in [2] allows us to construct a set $E^* \subset \mathbb{D}$, minimally thin at ζ_0, and a sequence of positive numbers (ρ_n), decreasing to 0, such that

$$E_{\phi_n} \cap D(\zeta_0, \rho_n) \subset E^* \quad (n \in \mathbb{N}).$$

Thus, for each $n \in \mathbb{N}$, the function $z \mapsto \left(\int \phi_n d\mu_z^*\right)$ converges to a finite limit as $z \to \zeta_0$ in $\mathbb{D}\backslash E^*$. It follows that the limit measure

$$\nu_0 = \lim_{z \to \zeta_0, z \in \mathbb{D}\backslash E^*} \mu_z^* \tag{1}$$

exists in the sense of w^*-convergence of measures. (The argument we have used in this paragraph can be regarded as a minimal fine topology analogue of that used in Doob [10] to construct fine harmonic measure at an irregular boundary point of a domain.)

Clearly ν_0 is a probability measure on ∂U. We will now show that $\nu_0(\{\zeta_0\}) = 0$. Since $\mathbb{D}\backslash U$ is minimally thin at ζ_0, we can combine Theorems 9.2.7, 9.3.3(ii) and equation (9.2.4) in [2] to see that there is a Green potential v_0 on $\mathbb{D}$ and a set $E_0 \subset \mathbb{D}$, minimally thin at ζ_0, such that

$$\frac{v_0(z)}{\log(1/|z|)} \to \infty \quad (z \to \zeta_0, z \in \mathbb{D}\backslash U)$$

and

$$\frac{v_0(z)}{\log(1/|z|)} \to 1 \quad (z \to \zeta_0, z \in U\backslash E_0).$$

Let $\varepsilon > 0$. Then there exists $r > 0$ such that

$$v_0(z) > \varepsilon^{-1} \log\left(1/|z|\right) \quad (z \in (\mathbb{D}\backslash U) \cap D(\zeta_0, r))$$

and

$$v_0(z) < 2 \log\left(1/|z|\right) \quad (z \in (U\backslash E_0) \cap D(\zeta_0, r)).$$

Hence

$$\mu_z^*(D(\zeta_0, r) \cap \partial U) = \frac{1}{\log(1/|z|)} \int_{D(\zeta_0,r)\cap\partial U} \log(1/|\zeta|) d\mu_z(\zeta)$$

$$\leqslant \frac{1}{\log(1/|z|)} \int_{D(\zeta_0,r)\cap\partial U} \varepsilon v_0 d\mu_z$$

$$\leqslant \varepsilon \frac{v_0(z)}{\log(1/|z|)}$$

$$< 2\varepsilon \quad (z \in (U\backslash E_0) \cap D(\zeta_0, r)),$$

by the superharmonicity of v_0, and so $\nu_0(D(\zeta_0, r) \cap \partial U) \leqslant 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\nu_0(\{\zeta_0\}) = 0$ as claimed.

Let I be an open arc of $\mathbb{T}$ containing ζ_0 on which (S_{N_k}) is uniformly bounded, and let $\psi_j : \partial U\backslash\{\zeta_0\} \to \mathbb{R}$ be the function given by

$$\psi_j(z) = \begin{cases} -\frac{1}{2} & (|z| < 1 - \frac{1}{j}) \\ G_{\widehat{\mathbb{C}}\backslash\overline{I}}(z, \infty)/\log(1/|z|) & (1 > |z| \geqslant 1 - \frac{1}{j}) \end{cases}. \quad (2)$$

It is easy to check that $G_{\widehat{\mathbb{C}}\backslash\overline{I}}(z, \infty)/\log(1/|z|)$ has a finite (positive) limit as $z \to \zeta_0$ in $\mathbb{D}$. Thus ψ_j, extended by this limiting value, is upper semicontinuous and bounded above on ∂U. Further, $\psi_j \downarrow -1/2$ on $\partial U\backslash\{\zeta_0\}$ as $j \to \infty$. Hence we can find $j_0 \in \mathbb{N}$ such that

$$\int \psi_{j_0} d\nu_0 < 0. \quad (3)$$

For each $k \in \mathbb{N}$ we define the subharmonic function

$$u_k = \frac{1}{N_k} \log |S_{N_k} - f| \quad \text{on } \mathbb{D}. \quad (4)$$

Since $S_{N_k} - f$ has a zero of order (at least) N_k at 0, the function $u_k(z) - \log|z|$ is also subharmonic on $\mathbb{D}$. Further, $\limsup_{k\to\infty} u_k \leqslant 0$. Thus it follows from the maximum principle that

$$\limsup_{k\to\infty} u_k(z) \leqslant \log|z| \quad \text{on } \mathbb{D}.$$

Hence (see Corollary 5.7.2 in [2]) we can choose $k_0 \in \mathbb{N}$ such that

$$u_k(z) \leqslant \frac{\log|z|}{2} \quad (|z| \leqslant 1 - \frac{1}{j_0}, k \geqslant k_0). \tag{5}$$

Also, by Bernstein's lemma (see Theorem 5.5.7 in [20]),

$$\log|S_{N_k}| \leqslant N_k G_{\widehat{\mathbb{C}\setminus \overline{I}}}(\cdot, \infty) + \log\left(\sup_{\overline{I}}|S_{N_k}|\right).$$

We know that there exists $a \geqslant 1$ such that $|S_{N_k}| \leqslant a$ on I for all k. On $\overline{U} \cap \mathbb{D}$ we thus have

$$u_k \leqslant \frac{1}{N_k} \log\left(2\max\left\{|S_{N_k}|, |f|\right\}\right)$$

$$\leqslant \frac{1}{N_k}\left(\log 2 + \max\left\{N_k G_{\widehat{\mathbb{C}\setminus\overline{I}}}(\cdot, \infty) + \log a, h\right\}\right)$$

$$\leqslant G_{\widehat{\mathbb{C}\setminus\overline{I}}}(\cdot, \infty) + \frac{\log 2a + h}{N_k}. \tag{6}$$

Using the subharmonicity of $u_k - (\log 2a + h)/N_k$ and its upper boundedness on U, and then (2), (5) and (6), we see that

$$u_k(z) - \frac{\log 2a + h(z)}{N_k} \leqslant \int_{\mathbb{D}\cap\partial U}\left(u_k - \frac{\log 2a + h}{N_k}\right)d\mu_z$$

$$\leqslant \int_{\partial U}\psi_{j_0}(\zeta)\log(1/|\zeta|)d\mu_z(\zeta)$$

$$= \log(1/|z|)\int_{\partial U}\psi_{j_0}d\mu_z^* \quad (z \in U, k \geqslant k_0). \tag{7}$$

By (1), the upper semicontinuity of ψ_{j_0}, and (3), there exists $r_1 \in (0,1)$ such that

$$\int_{\partial U}\psi_{j_0}d\mu_z^* < 0 \quad (z \in U \cap D(\zeta_0, r_1)\setminus E^*). \tag{8}$$

Combining (7) and (8) with (4), we see that

$$e^{-h}|S_{N_k} - f| \leqslant 2a \quad \text{on } U \cap D(\zeta_0, r_1)\setminus E^* \text{ when } k \geqslant k_0,$$

and the conclusion of Theorem 4 follows on defining

$$E = (\mathbb{D}\setminus U) \cup (\mathbb{D}\setminus D(\zeta_0, r_1)) \cup E^*,$$

which is minimally thin at ζ_0.

3. The remaining proofs

Proof of Corollary 5. — Let $f \in \mathcal{U}(\mathbb{D}, 0)$ and suppose that, for some positive harmonic function h on $\mathbb{D}$, the set $\{|f| \geqslant e^h\}$ is minimally thin at two distinct points $\zeta_1, \zeta_2 \in \mathbb{T}$. Further, let I be an open arc of $\mathbb{T}$ containing ζ_1 and ζ_2 such that $\bar{I} \neq \mathbb{T}$. In view of the Poisson integral representation of positive harmonic functions on $\mathbb{D}$ we can easily modify h to obtain another such function h_1 that vanishes continuously on a closed subarc I_1 of I lying between ζ_1 and ζ_2 and such that the set $\{|f| \geqslant e^{h_1}\}$ remains minimally thin at ζ_1, ζ_2. By universality we can find a subsequence (S_{N_k}) that is uniformly convergent to 0 on the set $\{r\zeta : \zeta \in \bar{I}, 1 \leqslant r \leqslant 2\}$. Theorem 4 then tells us that there is a set $E \subset \mathbb{D}$, which is minimally thin at both ζ_1 and ζ_2, such that $(e^{-h_1} S_{N_k})$ is uniformly bounded on $\mathbb{D} \backslash E$. By Theorem 8 of [12] we can choose line segments $L_1, L_2 \subset \mathbb{D} \backslash E$ with endpoints at ζ_1, ζ_2, respectively. Since (S_{N_k}) is locally uniformly convergent on $\mathbb{D}$, it follows from the maximum principle that $(\log |S_{N_k}| - h_1)$ is uniformly bounded on a domain Ω whose boundary is contained in the union of L_1, L_2, I and a suitable closed line segment in $\mathbb{D}$ joining L_1 to L_2. Hence (S_{N_k}) is uniformly bounded on the set $\omega = \{r\zeta : \zeta \in I_1, 0 < r < 2\}$. This leads to the conclusion that $S_{N_k} \to 0$ on ω and thus $f \equiv 0$, which is a contradiction. $\quad \square$

Proof of Theorem 2. — Let $f \in \mathcal{U}(\mathbb{D}, 0)$ and suppose that the set of Plessner points of f does not have full arclength measure. We fix $\alpha > 1$ and $0 < t \leqslant 1$, and define

$$J_a = \{\zeta \in \mathbb{T} : |f| \leqslant a \text{ on } \Gamma_\alpha^t(\zeta)\} \qquad (a > 0).$$

By Plessner's theorem the set J_a will then have positive arclength measure provided we choose a large enough. Let $F = \cup_{\zeta \in J_a} \Gamma_\alpha^t(\zeta)$. The set $\mathbb{D} \backslash F$ is then minimally thin at almost every point of J_a, by Lemma 9.7.5 of [2] and the conformal invariance of minimal thinness. This leads to a contradiction, in view of the Corollary 5 and the fact that $|f| \leqslant a$ on F. $\quad \square$

Proof of Corollary 3. — Let $f \in \mathcal{U}(\mathbb{D}, 0)$, and let Z denote the set of all $\zeta \in \mathbb{T}$ such that $\{f(r\zeta) : 0 < r < 1\}$ is dense in $\mathbb{C}$. If $\zeta \in \mathbb{T} \backslash Z$, then we can choose $p \in \mathbb{Q} + i\mathbb{Q}$ and a positive rational number q such that

$$\{f(r\zeta) : 0 < r < 1\} \subset \mathbb{C} \backslash D(p, q). \qquad (9)$$

We write $E_{p,q}$ for the collection of all points $\zeta \in \mathbb{T}$ satisfying (9). Thus $E_{p,q}$ is closed and $\mathbb{T} \backslash Z = \cup_{p,q} E_{p,q}$. If Z were not residual, then there would exist p, q as above such that $E_{p,q}$ has non-empty interior J relative to $\mathbb{T}$. It follows

that f does not take values in $D(p, q)$ on the sector $\{r\zeta : 0 < r < 1, \zeta \in J\}$. This contradicts the conclusion of Theorem 2, so Z must be residual. □

Proof of Proposition 6. — Without loss of generality we may assume that A is closed relative to $\mathbb{D}$ and that $\overline{A} \cup \overline{D}(0, n/(n+1))$ has connected complement for each $n \in \mathbb{N}$. By Lemma 2.1 of [17] there is a countable collection $\mathcal{K}$, of compact sets $K \subset \mathbb{C}\backslash\mathbb{D}$ with connected complement, having the following property: if $L \subset \mathbb{C}\backslash\mathbb{D}$ is compact and $\mathbb{C}\backslash L$ is connected, then $L \subseteq K$ for some $K \in \mathcal{K}$. It is easy to see that $\overline{A} \cup \overline{D}(0, n/(n+1)) \cup K$ has connected complement for every $K \in \mathcal{K}$. Now let $\mathcal{P}$ be the collection of all complex polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$, let $((K_n, p_n))$ be an enumeration of $\mathcal{K} \times \mathcal{P}$, and let $d_n = \max_{z \in K_n} |z|$. We inductively define a sequence of polynomials (q_n) as follows.

Since $w(z) \to \infty$ as $z \to 1$ we can choose $n_1 \in \mathbb{N}$ large enough so that $|z^{n_1} p_1(1)| \leqslant w(z)/2^2$ on $A \cup \overline{D}(0, 1/2)$. We then define

$$p_1^*(z) = \begin{cases} z^{-n_1} p_1(z) & \text{if } |z| \geqslant 1 \\ p_1(1) & \text{if } |z| < 1 \end{cases},$$

use Mergelyan's theorem to choose a polynomial q_1^* such that

$$|q_1^* - p_1^*| < (2^2 d_1^{n_1})^{-1} \quad \text{on } \overline{A} \cup \overline{D}(0, 1/2) \cup K_1,$$

and define $q_1(z) = z^{n_1} q_1^*(z)$. Since $w \geqslant 1$ and $d_1 \geqslant 1$, we have

$$|q_1(z)| \leqslant |z^{n_1}| |q_1^*(z) - p_1^*(z)| + |z^{n_1} p_1(1)|$$
$$\leqslant 2^{-2} + 2^{-2} w(z)$$
$$\leqslant 2^{-1} w(z) \qquad (z \in A \cup \overline{D}(0, 1/2))$$

and

$$|p_1(z) - q_1(z)| = |z^{n_1}| |q_1^*(z) - p_1^*(z)| \leqslant 2^{-2} \quad (z \in K_1).$$

Next, given $q_1, ..., q_{k-1}$, where $k \geqslant 2$, we choose $n_k > \deg q_{k-1}$ large enough such that

$$\left| z^{n_k} \left(p_k - \sum_1^{k-1} q_j \right)(1) \right| \leqslant 2^{-k-1} w(z) \quad \text{on } A \cup \overline{D}(0, k/(k+1)),$$

define

$$p_k^*(z) = \begin{cases} z^{-n_k}(p_k - \sum_1^{k-1} q_j)(z) & \text{if } |z| \geqslant 1 \\ (p_k - \sum_1^{k-1} q_j)(1) & \text{if } |z| < 1 \end{cases},$$

use Mergelyan's theorem to choose a polynomial q_k^* such that

$$|q_k^* - p_k^*| < (2^{k+1} d_k^{n_k})^{-1} \quad \text{on } \overline{A} \cup \overline{D}(0, k/(k+1)) \cup K_k,$$

and define $q_k(z) = z^{n_k} q_k^*(z)$. Thus

$$|q_k(z)| \leqslant |z^{n_k}| |q_k^*(z) - p_k^*(z)| + |z^{n_k} p_k^*(z)|$$

$$\leqslant 2^{-k-1} + \left| z^{n_k} \left(p_k - \sum_1^{k-1} q_j \right) (1) \right|$$

$$\leqslant 2^{-k-1} + 2^{-k-1} w(z)$$

$$\leqslant 2^{-k} w(z) \qquad (z \in \overline{A} \cup \overline{D}(0, k/(k+1)))$$

and

$$\left| p_k(z) - \sum_1^k q_j(z) \right| = |z^{n_k}| |q_k^*(z) - p_k^*(z)| \leqslant 2^{-k-1} \quad (z \in K_k).$$

It is now easy to see that the series $\sum q_n$ converges locally uniformly on $\mathbb{D}$ to a holomorphic function f such that $|f| \leqslant w$ on A, and that

$$\left| p_k - S_{n_{k+1}-1} \right| = \left| p_k - \sum_1^k q_j \right| \leqslant 2^{-k-1} \quad \text{on} \quad K_k \qquad (k \in \mathbb{N}).$$

Thus $f \in \mathcal{U}(\mathbb{D}, 0)$, as claimed. $\qquad\qquad \square$

A related result for the strip S, given below, will be used in the proof of Theorem 1.

PROPOSITION 7. — *Let A be a bounded subset of S such that $\overline{A} \cap \partial S = \{\pm 1\}$, and let $w : S \to (1, \infty)$ be a continuous function such that $w(z) \to \infty$ as $z \to \pm 1$. Then there exists $f \in \mathcal{U}(S, 0)$ such that $|f| \leqslant w$ on A. In particular, this is true for $w = e^h$, where h is any positive harmonic function on S that tends to ∞ at ± 1.*

Proof. — Let $F_+ : \{\operatorname{Re} z < 1\} \to \mathbb{D}$ be a conformal map such that $F_+(0) = 0$ and with boundary limit $F_+(1) = 1$, and let $F_-(z) = F_+(-z)$. Thus F_- is a conformal map from $\{\operatorname{Re} z > -1\}$ to $\mathbb{D}$ and $F_-(-1) = 1$. We exhaust S by the rectangles

$$R_n = \{|\operatorname{Re} z| \leqslant n/(n+1), \ |\operatorname{Im} z| \leqslant n\} \qquad (n \in \mathbb{N}).$$

We may assume that A is closed relative to S and that $\overline{A} \cup R_n$ has connected complement for each n. Let $w_n = \max_{R_n} w$. Also, let K_n and p_n be as in the proof of Proposition 6, except that the sets K_n now lie outside S rather than $\mathbb{D}$.

We inductively define a sequence of polynomials (q_n) as follows. Given $k \in \mathbb{N}$ and $q_1, ..., q_{k-1}$, let m_{k-1} denote the degree of $\sum_1^{k-1} q_j$. (We define

$m_0 = 0$.) By Cauchy's estimates we can choose $\delta_k \in (0,1)$ small enough such that, if g is holomorphic on $\mathbb{D}$ and $|g| < \delta_k$ on $\overline{D}(0, 1/2)$, then

$$|S_N(g, 0)| \leqslant 2^{-k} \quad \text{on} \quad K_1 \cup ... \cup K_k \quad (N = 0, 1, ..., m_{k-1}). \quad (10)$$

Since $|F_\pm| < 1$ and $w(z) \to \infty$ as $z \to \pm 1$, we can choose $n_k \in \mathbb{N}$ large enough so that

$$|F_\pm(z)|^{n_k} \left| \left(p_k - \sum_1^{k-1} q_j \right) (\pm 1) \right| \leqslant 2^{-k-2} \delta_k \frac{w(z)}{w_k} \quad (z \in A \cup R_k) \quad (11)$$

and also that $b_k, c_k \in D(0, 2^{-k-1} \delta_k)$, where

$$b_k = \{F_-(1)\}^{n_k} \left(p_k - \sum_1^{k-1} q_j \right) (-1), \quad c_k = \{F_+(-1)\}^{n_k} \left(p_k - \sum_1^{k-1} q_j \right) (1).$$

The function p_k^* defined by

$$p_k^*(z) = \begin{cases} p_k(z) - \sum_1^{k-1} q_j(z) + b_k & (\operatorname{Re} z \geqslant 1) \\ \{F_+(z)\}^{n_k} \left(p_k - \sum_1^{k-1} q_j \right)(1) \\ \quad + \{F_-(z)\}^{n_k} \left(p_k - \sum_1^{k-1} q_j \right)(-1) & (|\operatorname{Re} z| < 1) \\ p_k(z) - \sum_1^{k-1} q_j(z) + c_k & (\operatorname{Re} z \leqslant -1) \end{cases}$$

is continuous at ± 1 and holomorphic outside $\{|\operatorname{Re} z| = 1\}$, so by Mergelyan's theorem we can choose a polynomial q_k such that

$$|q_k - p_k^*| < 2^{-k-1} \delta_k \quad \text{on} \quad \overline{A} \cup R_k \cup K_k.$$

In view of (11),

$$|q_k| \leqslant |q_k - p_k^*| + |p_k^*| \leqslant 2^{-k-1} \delta_k + 2^{-k-1} \delta_k \frac{w(z)}{w_k} \quad \text{on } A \cup R_k,$$

so

$$|q_k| \leqslant 2^{-k} w \quad \text{on} \quad A, \qquad |q_k| \leqslant 2^{-k} \delta_k \quad \text{on} \quad R_k.$$

Also,

$$\left| p_k - \sum_1^k q_j \right| \leqslant |q_k - p_k^*| + \left| p_k^* - \left(p_k - \sum_1^{k-1} q_j \right) \right|$$

$$\leqslant 2^{-k-1} \delta_k + \max\{|b_k|, |c_k|\} \leqslant 2^{-k} \quad \text{on} \quad K_k. \quad (12)$$

We can clearly also arrange that the sequence (δ_k) is decreasing.

It follows that the series $\sum q_k$ converges locally uniformly on S to a holomorphic function f satisfying $|f| \leqslant w$ on A. Further, $|\sum_k^\infty q_j| < \delta_k$ on

R_k, which contains $\overline{D}(0, 1/2)$, so

$$\left| p_k - S_{m_{k-1}}(f, 0) \right| \leqslant \left| p_k - \sum_1^{k-1} q_j \right| + \left| S_{m_{k-1}}(\sum_k^{\infty} q_j, 0) \right| \leqslant 2^{1-k} \quad \text{on} \ \ K_k,$$

by (12) and (10). Thus $f \in \mathcal{U}(S, 0)$, as required. □

Proof of Theorem 1. — The notion of minimal thinness at a boundary point of S is defined in the same way as for $\mathbb{D}$, except that we now use the Poisson kernel for S. Let h be a positive harmonic function on S such that $h(z) \to \infty$ as $z \to \pm 1$, let $w = e^h$, and let A be a bounded, relatively closed subset of S such that $\overline{A} \cap \partial S = \{\pm 1\}$ and $S \backslash A$ is minimally thin at ± 1. Next, let $f \in \mathcal{U}(S, 0)$ be as in Proposition 7.

Part (i) of Theorem 1 now follows from Corollary 5, and the conformal invariance of harmonicity and minimal thinness.

To prove part (ii) we choose a conformal mapping $F : S \to S$ which sends two (distinct) points of $\{\operatorname{Re} z = 1\}$ to the boundary points ± 1. The argument used to prove Corollary 5 is readily adapted to show that, if $f_1 \in \mathcal{U}(S, 0)$ and h is a positive harmonic function on S, then the set $\{|f_1| \geqslant e^h\}$ cannot be minimally thin at more than one point of $\{\operatorname{Re} z = 1\}$. Part (ii) now follows again from the conformal invariance of harmonicity and minimal thinness. □

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