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DAVID ALBERT EDWARDS
G. F. VINCENT-SMITH

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A WEIERSTRASS-STONE THEOREM FOR CHOQUET SIMPLEXES

by D. A. EDWARDS and G. VINCENT-SMITH

1. Introduction.

One formulation, due to Kakutani [12], of the classical "Weierstrass-Stone" theorem characterizes the elements of a linear subspace L of the space C(Ω) of all real-valued continuous functions on a compact hausdorff space Ω, given that (a) L contains the constant functions, (b) L is a sublattice of C(Ω). The main theorems (2 and 6) of the present paper extend this result to the space A(X) of continuous real-valued affine functions on a compact convex set X. In the new theorems condition (b) is replaced : the conditions of theorem 2 are derived from Choquet simplex theory ; those of theorem 6 are more general. After proving theorem 2 we show, in § 4, that it can be used to prove a result of Effros about ideals in A(X) when X is a simplex. In § 5 we attempt to explain the geometrical significance of the conditions of theorem 2, and we deduce some results about pure-state-preserving affine maps between compact convex sets. § 6 shows how to reformulate our results in the terminology of Choquet boundary theory.

In the final section we leave Choquet simplex theory and discuss a general compact convex X ; we prove a density theorem which allows one to characterize, among the linear subspaces of A(X) that contain the constant functions, those that are dense.

We use many standard results of Choquet boundary theory, for which see [3, 5, 14, 15], as well as the separation theorem for Choquet simplexes, for which see [8].

A special case of theorem 2 was announced in [9, 10].

In writing this paper we have benefited from some comments of Mr. E.B. Davies.

2. Preliminaries.

We consider here a non-empty compact convex subset X of a hausdorff locally convex real topological vector space. (In the sequel we sometimes loosely refer to such an X merely as a compact convex set.) We consider too the space C(X) of real continuous functions on X, the space K(X) of convex functions in C(X), the space A(X) of all affine functions in C(X), and the space P(X) of all probability Radon measures on X. We consider a linear subspace L of A(X) that contains the constant functions, and we recall in this section some standard or near-standard elementary properties of X and its associated spaces.

For each $x \in X$ we define

$$R_x(L) = \{\mu \in P(X) : \mu(g) = g(x), \forall g \in L\}.$$

It is elementary (see [3]) that for each $\mu \in P(X)$ there is a unique point $c_\mu \in X$ such that

$$\mu(f) = f(c_\mu), \quad \forall f \in A(X).$$

Obviously $\mu \in R_x(A(X))$ if and only if $\mu \in P(X)$ with $c_\mu = x$. Now consider for each $x \in X$ the set

$$Q(x) = \{y \in X : g(y) = g(x), \forall g \in L\}.$$

This set contains x and is closed and convex. If $x \neq x'$ then $Q(x)$, $Q(x')$ are either equal or disjoint.

PROPOSITION 1. – *Given $\mu \in P(X)$ and $x \in X$, we have $\mu \in R_x(L)$ if and only if $c_\mu \in Q(x)$.*

If $c_\mu \in Q(x)$ then because each g in L is constant on the set $Q(x)$ we have

$$\mu(g) = g(c_\mu) = g(x), \quad \forall g \in L,$$

whence $\mu \in R_x(L)$. Now suppose that $\mu \in P(X)$ with $c_\mu \notin Q(x)$. Then we can find a function $g \in L$ such that $g(x) < g(c_\mu)$. But $g(c_\mu) = \mu(g)$, and so $\mu \notin R_x(L)$.

Now write, for each $f \in C(X)$ and each $x \in X$,

$$\bar{f}(x) = \inf\{g(x) : g \in L, g \geqslant f\}.$$

Then $\bar{f} : X \longrightarrow \mathbf{R}$ is concave, upper semicontinuous, and constant on each of the sets $Q(x)$; moreover $\bar{f} = f$ whenever $f \in L$. For fixed $x \in X$ the map $f \longrightarrow \bar{f}(x)$ is sublinear on C(X) ; this fact leads, via a standard Hahn-Banach argument, to

PROPOSITION 2. – *For each $x \in X$ and $f \in C(X)$,*

$$\bar{f}(x) = \max_{\mu \in R_x(L)} \mu(f) .$$

More can be said about $\bar{f}$ in two special cases.

When $L = A(X)$ we write $\hat{f}$ instead of $\bar{f}$. It is known (see [3, 4]) that x is an extreme point of X if and only if $R_x(A(X)) = \{\varepsilon_x\}$. By proposition 2 we have therefore that $\hat{f}(x) = f(x)$ for all $f \in C(X)$ whenever $x \in X_e$ (= the set of all extreme points of X) – compare proposition 6 below.

The second special case is covered by

PROPOSITION 3. – *If f is a concave function in* C(X) *then for each $x \in X$ the function $\bar{f}$ is constant on $Q(x)$ with value given by*

$$\bar{f}(x) = \max\{f(y) : y \in Q(x)\} .$$

The constancy on $Q(x)$ is obvious, as is the inequality

$$\bar{f}(x) \geqslant \max\{f(y) : y \in Q(x)\} .$$

Since f is concave we have

$$\mu(f) \leqslant f(c_\mu) , \quad \forall \mu \in P(X) .$$

By this fact and propositions 1 and 2 we now have

$$\begin{aligned} \bar{f}(x) &= \max\{\mu(f) : \mu \in P(X) , c_\mu \in Q(x)\} \\ &\leqslant \max\{f(y) : y \in Q(x)\} \leqslant \bar{f}(x) . \end{aligned}$$

In addition to what we have just proved, we shall need a number of further concepts and standard results.

We first recall Bauer's maximum theorem [2, 3] : that every upper semicontinuous real convex function on X attains its maximum in X_e .

We also recall that a subset F of X is called a *face* if it is non-empty, closed, convex, and such that if $a, b \in X$, $0 < \lambda < 1$ and

$$\lambda a + (1 - \lambda) b \in F$$

then $a, b \in F$. (In the terminology of some writers our "face" would be a "closed convex extreme subset of X".) An extreme point is just a one-point face. If F is a face then $F_e = F \cap X_e$.

A linear subspace L of A(X) will be said to have the *Riesz separation property* if, whenever u_1, u_2, v_1, v_2 belong to L with

$$u_1 \vee u_2 \leqslant v_1 \wedge v_2 ,$$

we can find $w \in L$ such that

$$u_1 \vee u_2 \leqslant w \leqslant v_1 \wedge v_2 .$$

The same assertion, but with strict inequalities throughout, will be called the *weak Riesz separation property.*

When X is a Choquet simplex the space A(X) has these two Riesz separation properties. For this and other standard results about Choquet simplexes we refer the reader to [6, 8, 13]. By the *separation theorem* for Choquet simplexes we shall mean the statement [8] that *if* $-f, g : X \longrightarrow (-\infty, \infty]$ *are lower semicontinuous concave functions on a Choquet simplex* X *and* $f \leqslant g$ *then there is an* $h \in A(X)$ *such that* $f \leqslant h \leqslant g$.

3. First density theorem.

We continue to suppose that X is a general compact convex set. Given a subspace L of A(X) we write L_+ for the set $\{f \in L : f \geqslant 0\}$. We study in this section a linear subspace L of A(X) that satisfies the following conditions :

1) L contains the constant functions ;

2) L has the weak Riesz separation property ;

3) given $\varepsilon > 0$, $x \in X_e$, and $f \in L$ with $f(x) = 0$, we can find an element g of L_+ with $g \geqslant f$ and $g(x) < \varepsilon$.

These will be standing hypothesis for this section but we shall later sometimes require strengthened versions of 2) and 3), namely :

2_s) L has the Riesz separation property ;

3_s) for each $x \in X_e$ and each $f \in L$ with $f(x) = 0$ we can find an element g of L_+ with $g \geqslant f$ and $g(x) = 0$.

All five of the above conditions are satisfied by the space A(X) when X is a Choquet simplex (see § 2, and proposition 10 in § 5).

The first main theorem of this paper (theorem 2) characterizes the elements of $\overline{L}$ (the closure of L), given that L satisfies the conditions 1) - 3). A trivial characterization would be the statement that a function of A(X) lies in $\overline{L}$ if and only if it is constant on the set $Q(x)$ for each $x \in X$. Theorem 2 sharpens this statement significantly by using constancy only on those sets $Q(x)$ that are already *faces* (in the sense of § 2) of the compact convex set X : such faces we refer to in future as L-*faces*. All possible L-faces are provided by the following result.

PROPOSITION 4. – *If* $x \in X_e$ *then* $Q(x)$ *is a face of* X.

Choose $x \in X$ such that $Q(x)$ is not a face. Then we can find elements a, b of X and a number λ such that

$$0 < \lambda < 1, \quad c \equiv \lambda a + (1 - \lambda) b \in Q(x), \quad a \notin Q(x).$$

Then there exists an f in L such that $f(x) = 0$, $f(a) = 1$. Now take any $g \in L_+$ such that $g \geqslant f$. Then

$$g(x) = g(c) = \lambda g(a) + (1 - \lambda) g(b)$$
$$\geqslant \lambda g(a) \geqslant \lambda(a) = \lambda .$$

Since $\lambda > 0$ this implies, by condition 3), that $x \notin X_e$.

Each point of X_e lies in exactly one L-face. For the L-faces we can sharpen some of the results of § 2 concerning the sets $Q(x)$.

PROPOSITION 5. – *Let* Q *be an* L-*face of* X, *let* $x \in Q$ *and let* $\mu \in P(X)$. *Then* $\mu \in R_x(L)$ *if and only if* supp $\mu \subseteq Q$.

This is a special case of a standard result about faces. Suppose that $y \notin Q$ and that $\mu \in P(X)$ with $y \in$ supp μ. Then we can find a function $g \in L$ such that $g \mid Q = 0$ and $g(y) > 1$. Now let $G = \{z \in X : g(z) > 1\}$, so that $\mu(G) > 0$. By 3) we can now choose $f \in L_+$ so that $f \geqslant g$ and $f \mid Q < \mu(G)$. Then

$$\mu(f) \geqslant \int_G f d\mu > \mu(G) > f(x),$$

and hence $\mu \notin R_x(L)$. If, conversely, $\mu \in P(X)$ and supp $\mu \subseteq Q$ then it is trivial that $\mu \in R_x(L)$.

PROPOSITION 6. – *If* $x \in X_e$ *and* $f \in C(X)$ *then* $\bar{f}$ *is constant on* $Q(x)$ *with value given by*

$$\bar{f}(x) = \max\{f(y) : y \in Q(x)\}.$$

This follows immediately from propositions 2 and 5.

THEOREM 1. – *Let* X *be a compact convex set and let* L *be a linear subspace of* A(X) *that satisfies the conditions* 1)-3). *Then for each* $h \in K(X)$ *the family*

$$\{g \in L : g > h\}$$

is a decreasing filtering family. Consequently $\bar{h}$ *is, for each* $h \in K(X)$, *an affine upper semicontinuous function.*

It is clearly enough to prove the assertion about filtering. It will be more convenient to show that the family

$$\mathfrak{G} = \{g \in L : g < -h\}$$

is an increasing filtering family. That is, that if $g_1, g_2 \in \mathfrak{G}$ then we can find $g \in \mathfrak{G}$ such that $g \geqslant g_1 \vee g_2$.

To prove this, write $u = g_1 \vee g_2$ and consider the family

$$\mathfrak{F} = \{v \in L : v > u\}.$$

By condition 2) on L this is a decreasing filtering family. It follows that $\bar{u}$ is upper semicontinuous finite and affine. Moreover, by proposition 6 we have for all $x \in X_e$ that $\bar{u}(x) = u(x)$, and hence $\bar{u}(x) < -h(x)$, for all $x \in X_e$. But $h + \bar{u}$ is upper semicontinuous and convex, and so attains its maximum in X_e. Therefore

$$u(x) \leqslant \bar{u}(x) < -h(x), \quad \forall x \in X.$$

Consequently, for each $x \in X$ there is a function $v_x \in \mathfrak{F}$ such that $v_x(x) < -h(x)$. Then for all y in a neighbourhood G_x of x in X we have

$$u(y) < v_x(y) < -h(y).$$

But we can choose finitely many points $x_1, x_2, \ldots, x_n$ in X so that

$$X = G_{x_1} \cup G_{x_2} \cup \ldots \cup G_{x_n}.$$

Then

$$u < v_{x_1} \wedge v_{x_2} \wedge \ldots \wedge v_{x_n} \equiv w < -h.$$

By condition 2) we can find a function g in L such that $u < g < w$. Then

$$g_1 \vee g_2 < g < -h,$$

and the proof is complete.

By proposition 3 we now have immediately

COROLLARY 1. – *Under the conditions of theorem* 1, *if* $h \in A(X)$ *then the function* $\bar{h}$ *is upper semicontinuous and affine. Moreover, for each* $x \in X$, $\bar{h}$ *is constant on* $Q(x)$ *with value given by*

$$\bar{h}(x) = \max\{h(y) : y \in Q(x)\}.$$

The following result will be needed later.

COROLLARY 2. – *Under the conditions of theorem* 1, *if* h *in* K(X) *is constant on each* L-*face of* X *then* $\bar{h} = \hat{h}$.

It is obvious that $\bar{h} \geqslant \hat{h}$. If however g in A(X) is such that $g \geqslant h$ then, by proposition 6, $g(x) \geqslant \bar{h}(x)$ for all $x \in X_e$. But, by theorem 1, $\bar{h} - g$ is affine and upper semicontinuous, so that, by the maximum theorem, $g \geqslant \bar{h}$. Consequently $\hat{h} \geqslant \bar{h}$ and the proof is complete.

The "Weierstrass-Stone" theorem of the title can now be stated.

THEOREM 2. – *Let* X *be a compact convex set and let* L *be a linear subspace of* A(X) *that satisfies conditions* 1)-3). *Then*

$$\bar{L} = \{f \in A(X) : f \text{ is constant on each L-face of } X\}.$$

Suppose that f in A(X) is constant on each L-face. Then, by corollary 2 to theorem 1, $\bar{f} = f$. By theorem 1 and Dini's theorem we now have $f \in \bar{L}$. Conversely, it is trivial that every function in $\bar{L}$ is constant on each L-face of X.

COROLLARY. – *Let* L *be a linear subspace of* A(X) *that satisfies conditions* 1)-3) *together with*

4) L *separates the points of* X_e .

Then X *is a Choquet simplex and* L *is dense in* A(X).

For an L-face Q we have $Q_e = Q \cap X_e$. Consequently condition 4) implies, by the Krein-Milman theorem, that $Q(x) = \{x\}$ for each $x \in X_e$, and theorem 2 now supplies the desired density statement. Moreover, by corollary 2 to theorem 1, the function $\hat{h}$ is affine for each $h \in K(X)$. X is therefore a Choquet simplex.

We note here that theorem 2 implies one of the classical formulations of the Weierstrass-Stone theorem : *Suppose that* Ω *is a compact hausdorff space and that* M *is a linear sublattice of* $C(\Omega)$ *that contains the constant functions. Then* $f \in C(\Omega)$ *belongs to* $\overline{M}$ *if and only if, for each* $\omega \in \Omega$, f *is constant on the set* $\Omega(\omega)$, *where*

$$\Omega(\omega) = \{\zeta \in \Omega : g(\zeta) = g(\omega) \ \forall g \in M\}.$$

In order to deduce this result from theorem 2 one considers the set $X = P(\Omega)$ of all Radon probability measures on Ω, with the weak topology induced by the natural pairing with $C(\Omega)$. Then X is a Choquet simplex and the pairing with $C(\Omega)$ induces an isometric positive linear isomorphism between $C(\Omega)$ and A(X). This isomorphism maps M onto a linear subspace L of A(X). It is immediate that L satisfies conditions 1)-3) and easily shown that a function in A(X) is constant on the L-faces of X if and only if it arises from a function in $C(\Omega)$ that is constant on each of the sets $\Omega(\omega)$. An application of theorem 2 now supplies the above characterization of $\overline{M}$. The details may be left as an exercise.

In fact theorem 2 is strictly more general than the result about $C(\Omega)$ that we have just mentioned, since it can occur that neither of the two spaces L, $\overline{L}$ is a vector lattice. In fact we know, for instance, that when X is a Choquet simplex the space A(X) is a vector lattice if and only if X_e is closed, and that X_e may fail to be closed [2, 13].

4. Approximation in ideals.

We suppose throughout this section that X is a Choquet simplex.

We recall that an *order ideal* of A(X) is a linear subspace J of A(X) such that whenever $u, v \in J$ and $f \in A(X)$ and $u \leqslant f \leqslant v$ it follows that $f \in J$. Following Effros [11], we define an *ideal* of A(X) to be a positively generated order ideal : that is, an order ideal J such that $J = J_+ - J_+$.

For example, one can take a face F of X and write

$$J_F = \{f \in A(X) : f|F = 0\}.$$

Then J_F is obviously an order ideal, and by the separation theorem for simplexes it is easy to see that it is actually an ideal.

We note that if J is an ideal and $u, v \in J$ then there is a function $w \in J$ such that $w \geqslant u \vee v$. For suppose $u_1, v_1 \in J_+$ with $u_1 \geqslant u$, $v_1 \geqslant v$. Then it is enough to take $w = u_1 + v_1$. Another simple observation is that J satisfies condition 2_s) of § 3, because A(X) does and J is an order ideal.

PROPOSITION 7. – *If* J *is a proper ideal of* A(X) *and*

$$Q = \{x \in X : f(x) = 0, \forall f \in J\}$$

then Q *is a face of* X.

First suppose, if possible, that $Q = \emptyset$. Then for each $x \in X$ we can find $f_x \in J_+$ such that $f_x(x) > 0$. We then have $f_x(y) > 0$ for all y in a neighbourhood of x. By an obvious covering argument we can now select $x_1, x_2, \ldots, x_n$ so that $f \equiv \sum_{r=1}^{n} f_{x_r}$ is strictly positive on X. Since $f \in J$ this implies, J being an order ideal, that $J = A(X)$.

Next, since J is positively generated we have

$$Q = \{x \in X : f(x) = 0, \forall f \in J_+\} .$$

But this, by the argument used in § 3, implies that Q is a face.

THEOREM 3 (*Effros*). – *Let* X *be a Choquet simplex and let* J *be a proper ideal of* A(X). *Then the closure of* J *is the set of all functions in* A(X) *that vanish identically on the face*

$$Q = \{x \in X : f(x) = 0, \forall f \in J\} .$$

(*Consequently* $\overline{J}$ *is also an ideal*).

For Effros' own proof see [11]. Our main object here is to give a new proof of this result by use of theorem 2. We in fact prove slightly more than we need for this purpose.

Form the direct sum $J \oplus \mathbf{R}$, consisting of all functions in A(X) of the form $f + \alpha$, where $f \in J$ and α is a real constant. The sum is direct by proposition 7.

PROPOSITION 8. – *The linear subspace* $J \oplus \mathbf{R}$ *of* A(X) *satisfies the conditions* 1), 2_s), 3_s) *of* § 3.

We write L for $J \oplus \mathbf{R}$. We have to prove 2_s) and 3_s).

To prove 2_s) suppose that $u_1, u_2, v_1, v_2 \in L$ with

$$u_1 \vee u_2 \leqslant v_1 \wedge v_2 .$$

The difference $v_2 - v_1$ is constant on Q and we can take this constant to be $\alpha \geqslant 0$. Now let $v_3 = v_2 - \alpha$, so that v_1 and v_3 agree on Q with common value β, say. By a remark preceding proposition 7 we can find u_3 in L such that

$$u_3 \leqslant v_1 \wedge v_3 , \quad u_3 | Q = \beta .$$

But now

$$u_1 \vee u_2 \vee u_3 \leqslant v_1 \wedge v_2$$

and hence, by the Riesz property for A(X), we can find w in A(X) such that

$$u_1 \vee u_2 \vee u_3 \leqslant w \leqslant v_1 \wedge v_2 .$$

Since u_3 and v_1 agree on Q we have $w|Q = \beta$ and

$$u_3 - \beta \leqslant w - \beta \leqslant v_1 - \beta .$$

The two end terms are in J, so $w \in L$, and so L does have property 2_s).

To prove property 3_s) note first that for an $x \in Q \cap X_e$ the statement is immediate, from $J = J_+ - J_+$. Now take $x \in X_e \backslash Q$ and let $f \in J$ with $f(x) = \alpha$, and write $f_1 = f - \alpha$. Choose $g \in J_+$ with $g \geqslant f$ and distinguish the two cases (i) $\alpha \geqslant 0$, (ii) $\alpha < 0$.

In case (i) we have $f_1 \leqslant f$ and hence $f_1 \vee 0 \leqslant g$. By the separation theorem we can choose $h \in A(X)$ so that

$$f_1 \vee 0 \leqslant h \leqslant g, \quad h(x) = 0 .$$

But now $h \in J_+ \subseteq L_+$, $h \geqslant f_1$ and $h(x) = 0$.

In case (ii) we have $f \vee \alpha \leqslant g$. By the separation theorem we can find $h \in A(X)$ such that

$$f \vee \alpha \leqslant h \leqslant g, \quad h(x) = \alpha .$$

Then $h \in J$. Writing $h_1 = h - \alpha$ we have $h_1 \in L_+$, $h_1 \geq f_1$, $h_1(x) = 0$. This completes the proof of property 3_s) for L.

To apply theorem 2 usefully we now need to describe the L-faces of X. Obviously Q is one of these. To find the others consider $Q(x)$ when $x \in X_e \backslash Q$. This is an L-face disjoint from Q and is the closed convex hull of

$$Q(x)_e = Q(x) \cap X_e .$$

Suppose $y \in X_e \backslash Q$, $y \neq x$. We show that $y \notin Q(x)$. For we can find $f \in J_+$ such that $f(x) > 0$. If $f(y) \neq f(x)$ then it is obvious that $y \notin Q(x)$. If $f(y) = f(x)$ we can find, by the separation theorem, a function $g \in A(X)$ such that

$$-f \leq g \leq f, \quad g(x) = -f(x), \quad g(y) = f(y) .$$

Then $g \in J$ with $g(x) \neq g(y)$, so again $y \notin Q(x)$. Thus $Q(x)_e = \{x\}$, and hence $Q(x) = \{x\}$, whenever $x \in X_e \backslash Q$. Thus the L-faces comprise the face Q together with all sets of the form $\{x\}$, where $x \in X_e \backslash Q$.

By theorem 2, $\overline{L}$ therefore consists of all the functions in A(X) that are constant on the face Q. Consequently $\overline{J}$ consists of all the functions in A(X) that vanish identically on Q. This concludes the proof of theorem 3.

It may be useful to note here that once the closed ideals of A(X) are known it is easy to study the quotient of A(X) by such an ideal. This has been done by Effros [11], but a more substantial use of the separation theorem simplifies the argument a good deal.

In fact we know [7] that if Q is a face of the simplex X then Q is a simplex. Given a proper face Q we form the ideal J_Q and also the restriction map

$$f \longrightarrow f|Q \equiv \rho_Q f$$

from A(X) into A(Q). This map has J_Q as kernel and, by the separation theorem, it maps $A(X)_+$ *onto* $A(Q)_+$. Again by the separation theorem one shows easily that

$$\| \rho_Q f \| = \inf_{g \in J_Q} \| f - g \| .$$

Thus there is a natural isomorphism between $A(X)/J_Q$ and A(Q) which is positive in both directions, and isometric.

5. The dual map.

Given a compact convex set X and a linear subspace L of A(X), one can form the Banach spaces L^* and $A(X)^*$ and the map $A(X)^* \longrightarrow L^*$ dual to the natural injection $L \longrightarrow A(X)$. We shall suppose that L satisfies condition 1), and seek to elucidate conditions 2) and 3) by study of L^* and the dual map.

Note first that if $\varphi \in L^*$ and $\varphi \geqslant 0$ then $\varphi(1) = \|\varphi\|$. For when $-1 \leqslant f \leqslant 1$ we have $-\varphi(1) \leqslant \varphi(f) \leqslant \varphi(1)$ and so $|\varphi(f)| \leqslant \varphi(1)$, with equality when $f = 1$. Now consider the *state-space* of L :

$$Y = \{\varphi \in L^* : \varphi \geqslant 0, \varphi(1) = 1\}.$$

Then Y is a convex subset of L^* that is compact for the topology $\sigma(L^*, L)$. The pairing between L and L^* induces by restriction a linear map $L \longrightarrow A(Y)$ that is onto a dense subspace, isometric and positive. The only perhaps delicate point here is the isometry, and for this one may proceed as in [7]. The density statement is proved by use of the Hahn-Banach theorem. We can accordingly identify the partially ordered Banach spaces $\bar{L}$ and A(Y). It is therefore clear, from the statements of § 2, that conditions 2) and 2_s) are equivalent for $\bar{L}$, and hold good if and only if Y is a Choquet simplex.

PROPOSITION 9. – *The space* L *has the property* 2) *if and only if* Y *is a Choquet simplex.*

Since L^* coïncides with the order dual of L, L^* is a vector lattice whenever L has the weak Riesz separation property : this requires an obvious adaptation of an argument in [16]. Conversely, if Y is a simplex then, by a result of Andô [1] and Lindenstrauss [13] (see [8] for a simple proof), $\bar{L}$ has the Riesz property and hence L has the weak Riesz property.

We next seek to elucidate condition 3).

PROPOSITION 10. – *Let* Y *be a Choquet simplex, let* M *be a dense linear subspace of* A(Y) *that contains the constant functions, and let* $y \in Y$. *Then the following statements are equivalent :*

(i) $y \in Y_e$;

(ii) *for each* $f \in A(Y)$ *with* $f(y) = 0$ *we can find a function* $g \in A(Y)_+$ *with* $g \geqslant f$ *and* $g(y) = 0$;

(iii) *for each* $\varepsilon > 0$ *and each* $f \in M$ *with* $f(y) = 0$ *we can find a function* $g \in M_+$ *with* $g \geqslant f$ *and* $g(y) < \varepsilon$.

If $y \in Y_e$ then $\{y\}$ is a face of Y and the remarks at the beginning of § 4 supply a proof of (ii). Thus (i) $\Longrightarrow$ (ii).

Next (ii) implies (iii). For let f satisfy the conditions of (iii). Then by (ii) we can find $h \in A(Y)_+$ such that $h \geqslant f$, $h(y) = 0$. Since M is dense in A(Y) we can now find $g \in M$ such that $h < g < h + \varepsilon$. Clearly $g \in M_+$, $g \geqslant f$ and $g(y) < \varepsilon$.

Finally to show that (iii) implies (i) note that M must separate the points of Y. Take y in Y to be not an extreme point. Then we can find $a, b \in Y$ with $a \neq b$ and $y = (a + b)/2$, and then $f \in M$ such that $-f(a) = f(b) = 1$. Then $f(y) = 0$, but for each $g \in M_+$ with $g \geqslant f$ we have $g(y) \geqslant 1/2$. This concludes the proof.

Further light is now thrown on condition 3) by the following considerations. Given that L satisfies the condition 1) we can construct Y and the natural map $A(X)^* \longrightarrow L^*$, as already explained. Since $L^* = A(Y)^*$, it is clear from the construction of Y that the map $A(X)^* \longrightarrow A(Y)^*$, restricted to those functionals that are evaluations at the points of X, induces an onto continuous map $\pi : X \longrightarrow Y$. We shall denote by $\tilde{\pi}$ the map $P(X) \longrightarrow P(Y)$ induced by π.

THEOREM 4. – *Let* X *be a compact convex set and let* L *be a linear subspace of* A(X) *that satisfies the conditions* 1) *and* 2). *Then the following assertions are equivalent :*

(i) L *satisfies condition* 3) ;

(ii) $\pi(X_e) \subseteq Y_e$;

(iii) *for each* $x \in X_e$ *the set* $Q(x)$ *is a face of* X ;

(iv) $\tilde{\pi} : P(X) \longrightarrow P(Y)$ *maps maximal measures onto maximal measures.*

Condition (ii) is often expressed by saying that π is pure-state-preserving.

For the proof note first that the implication (i) $\Longrightarrow$ (iii) is the content of proposition 4 and that (iv) $\Longrightarrow$ (ii) is trivial. Moreover we have, by propositions 9 and 10, that $A(Y) \circ \pi$ satisfies 1)-3) whenever (ii) is true. But L is dense in $A(Y) \circ \pi$ and L contains the constants. Consequently (ii) implies (i).

Now denote by M the image of L in A(Y) under the natural map $L \longrightarrow A(Y)$, so that

$$L = \{g \circ \pi : g \in M\}.$$

We shall prove the implication (i) $\longrightarrow$ (iv). Choose a maximal $\mu \in P(X)$. To show that $\tilde{\pi}\mu$ is maximal in P(Y) we prove that

$$(\tilde{\pi}\mu)(f) = (\tilde{\pi}\mu)(\hat{f})$$

for all $f \in K(Y)$. Now because M is dense in A(Y) we have, for all $y \in Y$,

$$\hat{f}(y) = \inf\{g(y) : g \in M, g > f\}.$$

Note now that $f \circ \pi \in K(X)$ and that $g \in M$ satisfies $g > f$ if and only if $g \circ \pi > f \circ \pi$. This remark, with the fact that L satisfies conditions 1)-3) shows, by theorem 1, that

$$\{g \in M : g > f\}$$

is a decreasing filtering family.

Therefore

$$\begin{aligned}(\tilde{\pi}\mu)(\hat{f}) &= \inf\{(\tilde{\pi}\mu)(g) : g \in M, g > f\} \\ &= \inf\{\mu(g \circ \pi) : g \in M, g > f\} \\ &= \inf\{\mu(\nu) : \nu \in L, \nu > f \circ \pi\} \\ &= \mu(\overline{f \circ \pi}).\end{aligned}$$

Thus it is enough to show that

$$\mu(f \circ \pi) = \mu(\overline{f \circ \pi}).$$

But μ is maximal in P(X) and by corollary 2 to theorem 1 we have $\overline{f \circ \pi} = \widehat{f \circ \pi}$, so the result is clear. I.e. (i) implies (iv).

Finally we show that (iii) implies (ii). For let $x \in X_e$ and suppose

$$\pi(x) = \lambda u + (1 - \lambda)\nu$$

with $0 < \lambda < 1$, and $u, \nu \in Y$. Since π is onto Y we can find $a, b \in X$ such that $\pi(a) = u$, $\pi(b) = \nu$. Then

$$\pi(x) = \pi(\lambda a + (1 - \lambda) b)$$

and hence

$$\lambda a + (1 - \lambda) b \in \pi^{-1} \pi(x) = Q(x) .$$

By (iii) this implies that $a, b \in Q(x)$, whence $\pi(a) = \pi(b) = \pi(x)$, which shows that $\pi(x) \in Y_e$.

We have now obtained the implications

$$\text{(i)} \Longrightarrow \text{(iii)} \Longrightarrow \text{(ii)} \Longrightarrow \text{(i)} \Longrightarrow \text{(iv)} \Longrightarrow \text{(ii)} ,$$

so the proof is complete.

COROLLARY 1. – *If the linear subspace* L *of* A(X) *satisfies the conditions* 1)-3) *then its closure satisfies* 1), 2_s), 3_s).

This follows immediately from the preceding discussion.

COROLLARY 2. – *Let* X *be a compact convex set, let* Y *be a Choquet simplex, and let* $\pi : X \longrightarrow Y$ *be a continuous affine map. Then the following statements are equivalent :*

(i) $\pi(X_e) \subseteq Y_e$;

(ii) π *maps faces of* X *onto faces of* Y ;

(iii) $\tilde{\pi} : P(X) \longrightarrow P(Y)$ *maps maximal measures onto maximal measures.*

Here (iii) implies (i) trivially, and (ii) implies (iii) by the argument of theorem 4, with $\pi(X)$ playing the role of Y and $A(\pi(X)) \circ \pi$ the role of L in that theorem.

The only new point is that (i) $\longrightarrow$ (ii). This is a result of E.B. Davies and the second author, and will appear in a forthcoming paper. We omit the proof, remarking only that it can be made to rest on the following statement : *if* K *is a non-empty compact convex subset of the Choquet simplex* Y *and* $K_e \subseteq Y_e$ *then* K *is a face of* Y.

COROLLARY 3. – *Let* X, Y, π *meet the conditions of the preceding corollary and let* $L = A(Y) \circ \pi$. *Then* L *consists of those functions in* A(X) *that are constant on each set of the form* $\pi^{-1} \pi(x)$, *where* $x \in X_e$. *Consequently, the following statements are equivalent :*

(i) π *is one-one ;*

(ii) π *is one-one on* X_e ;

(iii) $A(X) = A(Y) \circ \pi$.

By corollary 2 and the separation theorem it is enough to prove this for the special case when π is onto Y. In that case the map $f \longrightarrow f \circ \pi$ from A(Y) into A(X) is a positive isometry, and the result follows from theorems 4 and 2. The conditions (i)-(iii) of course here imply that X is a simplex.

6. A Choquet boundary formulation.

In the classical Weierstrass-Stone theorem one characterizes the elements of $\bar{N}$, given that N is a certain type of subspace of $C(\Omega)$, Ω being a compact hausdorff space. That characterization is in terms of behaviour on Ω, and makes no explicit use of the state-space (see § 5) of N. The purpose of this section is to find an analogous formulation of theorem 2.

Consider a linear subspace H of $C(\Omega)$ that contains the constant functions. For each $\omega \in \Omega$ we write

$$S_\omega = \{\mu \in P(\Omega) : \mu(g) = g(\omega), \forall g \in H\},$$

$$\Omega(\omega) = \{\zeta \in \Omega : g(\zeta) = g(\omega), \forall g \in H\}.$$

We recall that $\omega \in \Omega$ is a *Choquet boundary point* for H if

$$\mu \in S_\omega \Longrightarrow \text{supp}\, \mu \subseteq \Omega(\omega).$$

The Choquet boundary $\partial_H \Omega$ is the set of all such ω. We call a subset F of Ω a *quasi-face* for H if it is a closed non-empty subset such that whenever $\omega \in \Omega$ and $\mu \in S_\omega$ we have $\omega \in F$ if and only if $\text{supp}\, \mu \subseteq F$. If F is a quasi-face and $\omega \in F$ then $\Omega(\omega) \subseteq F$. Also $\omega \in \Omega$ is in $\partial_H \Omega$ if and only if $\Omega(\omega)$ is a quasi-face : thus the Choquet boundary points are those that lie in minimal quasi-faces.

Now consider the state-space

$$X = \{\varphi \in H^* : \varphi \geqslant 0, \varphi(1) = 1\}$$

of H, again with the relative topology from $\sigma(H^*, H)$. By a straightforward adaptation of an argument of Bishop and de Leeuw [4] one can show that, for the special case in which $\Omega = X$, the quasi-faces just defined coïncide with the faces of X, as defined in § 2. In general the connection between the quasi-faces of Ω and the faces of X is given by the following considerations.

By the Hahn-Banach theorem and the Riesz representation theorem each state (i.e. point of the state-space) of H has a representation $f \longrightarrow \mu(f)$ for some $\mu \in P(\Omega)$; and, conversely, every such map is a state. It follows that there is a natural map $e : P(\Omega) \longrightarrow X$ (onto), and this is easily shown to be continuous and affine. Writing $\tilde{e}(\omega) \equiv e(\varepsilon_\omega)$ we obtain a continuous map $\tilde{e} : \Omega \longrightarrow X$. The connection between quasi-faces in Ω and faces in the compact convex sets $P(\Omega)$ and X is given by

PROPOSITION 11. – *Let* Q *be a face in* X. *Then* $e^{-1}(Q)$ *is a face in* $P(\Omega)$, $\tilde{e}^{-1}(Q)$ *is a quasi-face in* Ω, *and*

$$e^{-1}(Q) = \{\mu \in P(\Omega) : \tilde{e}(\mathrm{supp}\,\mu) \subseteq Q\}.$$

Here it is immediate that $e^{-1}(Q)$ is a face of P(Q). On the other hand we know that for each non-empty compact set $K \subseteq \Omega$ the set

$$E = \{\mu \in P(\Omega) : \mathrm{supp}\,\mu \subseteq K\}$$

is a face of $P(\Omega)$, that all faces of $P(\Omega)$ are of this form, and that

$$E_e = \{\varepsilon_\omega \ : \omega \in K\}.$$

It now follows that, for all $\mu \in P(\Omega)$,

$$\mathrm{supp}\,\mu \subseteq \tilde{e}^{-1}(Q) \Longleftrightarrow \mu \in e^{-1}(Q),$$

which makes the rest of proposition 11 obvious.

COROLLARY 1. – $\tilde{e}$ *maps* $\partial_H \Omega$ *onto* X_e.

Now recall from § 5 that there is a natural positive isometry $j : \overline{H} \longrightarrow A(X)$ (onto).

COROLLARY 2. – *If* $f \in \overline{H}$ *and* Q *is a face of* X *then* $j(f)$ *is constant on* Q *if and only if* f *is constant on* $\tilde{e}^{-1}(Q)$.

A computation to show that $e : P(\Omega) \longrightarrow X$ is the map between state-spaces dual to $j^{-1} : A(X) \longrightarrow \overline{H}$ makes this clear.

We now introduce a second subspace N of $C(\Omega)$. But H will remain in play as a fixed subspace of $C(\Omega)$, and everything will be relative to it.

We take N to be a linear subspace of H and consider the following conditions :

A) N contains the constant functions ;

B) N has the weak Riesz separation property ;

C) given $\varepsilon > 0$, $\omega \in \partial_H \Omega$ and $f \in N$ such that $f(\omega) = 0$, we can find $g \in N_+$ with $g \geqslant f$ and $g(\omega) < \varepsilon$.

The map $j : \overline{H} \longrightarrow A(X)$ induces an identification of N as a partially ordered normed vector space with a linear subspace of A(X). It is clear that the conditions A), B), C) correspond respectively to the conditions 1), 2), 3) on the subspace $j(N)$. By proposition 11 we can reformulate part of theorem 4 as follows.

PROPOSITION 12. – *If* N *has the properties* A), B) *then the following statements are equivalent*

(i) N *satisfies condition* C) ;

(ii) $\partial_H \Omega \subseteq \partial_N \Omega$.

Given that N satisfies A)-C) we can by theorem 2 characterize the elements of $j(\overline{N})$. We want now to characterize the elements of N in terms of behaviour on Ω. By propositions 4 and 11 we see that

$$\{\zeta \in \Omega : g(\zeta) = g(\omega), \forall g \in N\}$$

is a quasi-face (for H) of Ω whenever $\omega \in \partial_H \Omega$. Such quasi-faces we shall call N-quasi-faces. By corollary 2 to proposition 11 we can now reformulate theorem 2 as follows.

THEOREM 5. – *Let* N *be a linear subspace of* H *that satisfies the conditions* A)-C). *Then*

$$\overline{N} = \{f \in \overline{H} : f \text{ is constant on each N-quasi-face of } \Omega\}$$

COROLLARY. – *If, under the conditions of theorem* 5, N *separates every two points of* $\partial_H \Omega$ *that are separated by* H *then* N *is dense in* H.

Many of the other statements of § § 2-5 have reformulations of this type, but it is now merely an exercise to find such.

Examples of subspaces of $C(\Omega)$ that possess the Riesz separation property have been given for instance by Lindenstrauss [13].

7. A general density theorem.

The conditions 1)-3) of § 3 are intimately related, as we saw in § § 3-5, to Choquet simplex theory. We show now that a recent unpublished result of Mr F. Jellett makes it possible to replace conditions 2) and 3) in theorem 2 by a single more general condition not directly related to simplex theory or to vector lattices. The proof of the resulting density theorem (theorem 6, below) is similar to that of theorem 2, with some short-circuiting.

For the rest of this paper X will be a general compact convex set. Jellett's result is

PROPOSITION 13. – *Let* L *be a linear subspace of* A(X) *that contains the constant functions and suppose that, for each* $f \in A(X)$, *the set*

$$\{g \in L : g > f\}$$

is a decreasing filtering family. Then L *satisfies condition* 3), *and hence* $Q(x)$ *is a face of* X *whenever* $x \in X_e$.

Since proposition 4 depends only on conditions 1) and 3) it is enough here to prove 3).

Choose $x \in X_e$ and let $f \in L$ with $f(x) = 0$. Let $u = f \vee 0$. Then $u \in C(X)$ and $u(x) = 0$, and so, by the remarks following proposition 2, $\hat{u}(x) = 0$. Therefore, given $\varepsilon > 0$, we can find a function $h \in A(X)$ such that $u < h$ and $h(x) < \varepsilon$. By the filtering condition we can now choose $g \in L$ so that $u < g < h$. Then $f \vee 0 < g$ and $g(x) < \varepsilon$, so the proof is complete.

Given the conditions of proposition 13 we can introduce, as in § 3, the L-faces of X. Propositions 5 and 6 remain good, since they depend only on 1) and 3).

THEOREM 6. – *Let* L *be a linear subspace of* A(X) *that contains the constant functions, and suppose that, for each* $f \in$ A(X), *the set*

$$\{g \in \mathrm{L} : g > f\}$$

is a decreasing filtering family. Then

$$\bar{\mathrm{L}} = \{f \in \mathrm{A(X)} : f \textit{ is constant on each } \mathrm{L}\text{-face of X}\}.$$

By theorem 1 this result is more general than theorem 2 (on this point see corollary 1 below). It is also closer to the classical Weierstrass-Stone theorem. Theorem 6 however states conditions directly involving all the elements of A(X), whereas the conditions of theorem 2 involved only the functions in L ; theorem 2 may accordingly have some advantages. Theorem 6 makes it possible to give a shorter proof of theorem 3 than that in § 4.

To prove theorem 6 consider a function $f \in$ A(X) that is constant on each L-face. Obviously $\bar{f} \geqslant \hat{f} = f$. By proposition 6 we have $\bar{f}(x) = f(x)$ for all $x \in \mathrm{X}_e$. By the filtering condition $\bar{f} - f$ is upper semicontinuous and affine. Therefore, by the maximum theorem, $\bar{f} = f$. Invoking the filtering condition once more we have, by Dini's theorem, $f \in \bar{\mathrm{L}}$. The converse half of theorem 6 is trivial.

That the filtering condition is natural in this connection is made clear by

COROLLARY 1. – *Let* L *be a linear subspace of* A(X) *that contains the constant functions. Then* L *is dense in* A(X) *if and only if the following two conditions hold :*

(i) L *separates the points of* X_e ;

(ii) L *satisfies the filtering condition of theorem* 6.

This result is now in fact quite obvious.

The foregoing discussion can be translated into Choquet boundary language, by use of the argument of § 6. We state one result in this form.

COROLLARY 2. – *Let* Ω *be a compact hausdorff space, let* M, N *be linear subspaces of* C(Ω) *that contain the constant functions and let* M $\subseteq$ N. *Then* M *is dense in* N *if and only if*

(i) M *separates each pair of points of* $\partial_N \Omega$ *that are separated by* N ; *and*

(ii) *for each* $f \in$ N *the set*

$$\{g \in M : g > f\}$$

is a decreasing filtering family.

This is hardly more than a translation of corollary 1, and the proof may be left to the reader. Note the consequence *that* M *is dense in* C(Ω) *if and only if* (a) M *separates the points of* Ω *and* (b) *for each* $f \in C(\Omega)$ *the set* $\{g \in M : g > f\}$ *is a decreasing filtering family.* This last statement can also be proved by adapting the usual proof of the Weierstrass-Stone theorem.

Further work, by Mr Jellett, on the filtering condition of theorem 6 will appear elsewhere.

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D.A. EDWARDS,
Mathematical Institute,
24-29 St Giles
Oxford (England)