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# HYPERCYCLICITY OF CONVOLUTION OPERATORS ON SPACES OF ENTIRE FUNCTIONS

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ABSTRACT. — In this paper we use Nachbin's holomorphy types to generalize some recent results concerning hypercyclic convolution operators on Fréchet spaces of entire functions of bounded type of infinitely many complex variables

RÉSUMÉ. — Dans cet article, nous utilisons les types d'holomorphic de Nachbin pour généraliser certains résultats récents concernant les opérateurs de convolutions hypercycliques sur les espaces de Fréchet de fonctions d'un nombre infini de variables complexes, entières, de type borné.

## 1. Introduction

A mapping  $f: X \rightarrow X$ , where  $X$  is a topological space, is *hypercyclic* if the set  $\{x, f(x), f^2(x), \dots\}$  is dense in  $X$  for some  $x \in X$ . In this case,  $x$  is said to be a *hypercyclic vector for  $f$* .

The study of hypercyclic translation and differentiation operators on spaces of entire functions of one complex variable can be traced back to Birkhoff [3] and MacLane [19]. Godefroy and Shapiro [14] pushed these results quite further by proving that every convolution operator on spaces of entire functions of several complex variables which is not a scalar multiple of the identity is hypercyclic. Results on the hypercyclicity of convolution operators on spaces of entire functions of infinitely many complex variables appeared later (see, e.g., [1, 17, 26, 27]). Recently, Carando, Dimant and Muro [6] proved some far-reaching results - including a solution to a problem posed in [2] - that encompass as particular cases several of the above

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mentioned results. The main tool they use are the so-called *coherent sequences of homogeneous polynomials*, introduced by themselves in [7] based on properties of polynomial ideals previously studied in [5, 4].

The aim of this paper is to generalize the results of [6]. We accomplish this task by proving results (Theorems 2.7 and 2.8) of which the main results of [6] ([6, Theorem 4.3] and [6, Corollary 4.4]) are particular cases. Furthermore we give some concrete examples (Example 3.11) that are covered by our results but not by the results of [6]. Being strictly more general than the results of [6], our results also generalize the ones first generalized by [6].

Our approach differs from the approach of [6] in our use of holomorphy types (in the sense of Nachbin [25]) instead of coherent sequences of polynomials. More precisely, we use the  $\pi_1$ - $\pi_2$ -holomorphy types introduced by the third and fourth authors in [11]. Although we already knew that  $\pi_1$ - $\pi_2$ -holomorphy types could be used in this context, it was only reading [6] that we realized that the original definitions could be refined (see Definition 2.5) to prove such general results on the hypercyclicity of convolution operators on spaces of entire functions. Holomorphy types are a somewhat old-fashioned topic in infinite-dimensional analysis, so it is quite surprising that our holomorphy type-oriented-approach turned out to be more effective than the coherent sequence-oriented-approach.

The paper is organized as follows: in Section 2 we state our main results, in Section 3 we prove that our results are more general - not only formally but also concretely - than the results of [6], and in Section 4 we prove our main results. In Section 5 we extend to our context some related results that appeared in the literature, including results on surjective hypercyclic convolution operators and connections with the existence of dense subspaces formed by hypercyclic functions for convolution operators.

Throughout the paper  $\mathbb{N}$  denotes the set of positive integers and  $\mathbb{N}_0$  denotes the set  $\mathbb{N} \cup \{0\}$ . The letters  $E$  and  $F$  will always denote complex Banach spaces and  $E'$  represents the topological dual of  $E$ . The Banach space of all continuous  $m$ -homogeneous polynomials from  $E$  into  $F$  endowed with its usual sup norm is denoted by  $\mathcal{P}(^m E; F)$ . The subspace of  $\mathcal{P}(^m E; F)$  of all polynomials of finite type is represented by  $\mathcal{P}_f(^m E; F)$ . The linear space of all entire mappings from  $E$  into  $F$  is denoted by  $\mathcal{H}(E; F)$ . When  $F = \mathbb{C}$  we write  $\mathcal{P}(^m E)$ ,  $\mathcal{P}_f(^m E)$  and  $\mathcal{H}(E)$  instead of  $\mathcal{P}(^m E; \mathbb{C})$ ,  $\mathcal{P}_f(^m E; \mathbb{C})$  and  $\mathcal{H}(E; \mathbb{C})$ , respectively. For the general theory of homogeneous polynomials and holomorphic functions we refer to Dineen [9] and Mujica [23].

## 2. Main results

In this section we state the main results of the paper and give the definitions needed to understand them.

DEFINITION 2.1. — Let  $U$  be an open subset of  $E$ . A mapping  $f: U \rightarrow F$  is said to be holomorphic on  $U$  if for every  $a \in U$  there exists a sequence  $(P^m)_{m=0}^\infty$ , where each  $P^m \in \mathcal{P}(^m E; F)$  ( $\mathcal{P}(^0 E; F) = F$ ), such that  $f(x) = \sum_{m=0}^\infty P^m(x - a)$  uniformly on some open ball with center  $a$ . The  $m$ -homogeneous polynomial  $m!P^m$  is called the  $m$ -th derivative of  $f$  at  $a$  and is denoted by  $\hat{d}^m f(a)$ . In particular, if  $P \in \mathcal{P}(^m E; F)$ ,  $a \in E$  and  $k \in \{0, 1, \dots, m\}$ , then

$$\hat{d}^k P(a)(x) = \frac{m!}{(m - k)!} \check{P}(\underbrace{x, \dots, x}_{k \text{ times}}, a, \dots, a)$$

for every  $x \in E$ , where  $\check{P}$  is the unique symmetric  $m$ -linear mapping associated to  $P$ . For any unexplained notation we refer to [9, 23, 25].

DEFINITION 2.2 (Nachbin [25]). — A holomorphy type  $\Theta$  from  $E$  to  $F$  is a sequence of Banach spaces  $(\mathcal{P}_\Theta(^m E; F))_{m=0}^\infty$ , the norm on each of them being denoted by  $\|\cdot\|_\Theta$ , such that the following conditions hold true:

- (1) Each  $\mathcal{P}_\Theta(^m E; F)$  is a linear subspace of  $\mathcal{P}(^m E; F)$ .
- (2)  $\mathcal{P}_\Theta(^0 E; F)$  coincides with  $\mathcal{P}(^0 E; F) = F$  as a normed vector space.
- (3) There is a real number  $\sigma \geq 1$  for which the following is true: given any  $k \in \mathbb{N}_0$ ,  $m \in \mathbb{N}_0$ ,  $k \leq m$ ,  $a \in E$  and  $P \in \mathcal{P}_\Theta(^m E; F)$ , we have

$$\hat{d}^k P(a) \in \mathcal{P}_\Theta(^k E; F) \text{ and}$$

$$\left\| \frac{1}{k!} \hat{d}^k P(a) \right\|_\Theta \leq \sigma^m \|P\|_\Theta \|a\|^{m-k}.$$

It is plain that each inclusion  $\mathcal{P}_\Theta(^m E; F) \subseteq \mathcal{P}(^m E; F)$  is continuous and that  $\|P\| \leq \sigma^m \|P\|_\Theta$  for every  $P \in \mathcal{P}_\Theta(^m E; F)$ .

DEFINITION 2.3 (Gupta [15, 16]). — Let  $(\mathcal{P}_\Theta(^m E; F))_{m=0}^\infty$  be a holomorphy type from  $E$  to  $F$ . A given  $f \in \mathcal{H}(E; F)$  is said to be of  $\Theta$ -holomorphy type of bounded type if

- (1)  $\frac{1}{m!} \hat{d}^m f(0) \in \mathcal{P}_\Theta(^m E; F)$  for all  $m \in \mathbb{N}_0$ ,
- (2)  $\lim_{m \rightarrow \infty} \left( \frac{1}{m!} \|\hat{d}^m f(0)\|_\Theta \right)^{\frac{1}{m}} = 0$ .

The linear subspace of  $\mathcal{H}(E; F)$  of all functions  $f$  of  $\Theta$ -holomorphy type of bounded type is denoted by  $\mathcal{H}_{\Theta_b}(E; F)$ .

*Remark 2.4.* — (a) The inequality  $\|\cdot\| \leq \sigma^m \|\cdot\|_{\Theta}$  implies that each entire mapping  $f$  of  $\Theta$ -holomorphy type of bounded type is an entire mapping of bounded type in the sense of Gupta in [16], that is,  $f$  is bounded on bounded subsets of  $E$ .

(b) It is clear that  $\mathcal{P}_{\Theta}(^m E; F) \subseteq \mathcal{H}_{\Theta b}(E; F)$  for each  $m \in \mathbb{N}_0$ .

For each  $\rho > 0$ , condition (2) of Definition 2.3 guarantees that the correspondence

$$f \in \mathcal{H}_{\Theta b}(E; F) \mapsto \|f\|_{\Theta, \rho} = \sum_{m=0}^{\infty} \frac{\rho^m}{m!} \|\hat{d}^m f(0)\|_{\Theta} < \infty$$

is a well defined seminorm on  $\mathcal{H}_{\Theta b}(E; F)$ . We shall henceforth consider  $\mathcal{H}_{\Theta b}(E; F)$  endowed with the locally convex topology generated by the seminorms  $\|\cdot\|_{\Theta, \rho}$ ,  $\rho > 0$ . This topology shall be denoted by  $\tau_{\Theta}$ . It is well known that  $(\mathcal{H}_{\Theta b}(E; F), \tau_{\Theta})$  is a Fréchet space (see, e.g., [11, Proposition 2.3]).

Next definitions are refinements of the concepts of  $\pi_1$ -holomorphy type and  $\pi_2$ -holomorphy type introduced in [11].

**DEFINITION 2.5.** — (a) A holomorphy type  $(\mathcal{P}_{\Theta}(^m E; F))_{m=0}^{\infty}$  from  $E$  to  $F$  is said to be a  $\pi_1$ -holomorphy type if the following conditions hold:

(a1) Polynomials of finite type belong to  $(\mathcal{P}_{\Theta}(^m E; F))_{m=0}^{\infty}$  and there exists  $K > 0$  such that

$$\|\phi^m \cdot b\|_{\Theta} \leq K^m \|\phi\|^m \cdot \|b\|$$

for all  $\phi \in E'$ ,  $b \in F$  and  $m \in \mathbb{N}$ ;

(a2) For each  $m \in \mathbb{N}_0$ ,  $\mathcal{P}_f(^m E; F)$  is dense in  $(\mathcal{P}_{\Theta}(^m E; F), \|\cdot\|_{\Theta})$ .

(b) A holomorphy type  $(\mathcal{P}_{\Theta}(^m E))_{m=0}^{\infty}$  from  $E$  to  $\mathbb{C}$  is said to be a  $\pi_2$ -holomorphy type if for each  $T \in [\mathcal{H}_{\Theta b}(E)]'$ ,  $m \in \mathbb{N}_0$  and  $k \in \mathbb{N}_0$ ,  $k \leq m$ , the following conditions hold:

(b1) If  $P \in \mathcal{P}_{\Theta}(^m E)$  and  $A: E^m \rightarrow \mathbb{C}$  is the unique continuous symmetric  $m$ -linear mapping such that  $P = \hat{A}$ , then the  $(m - k)$ -homogeneous polynomial

$$T \left( \widehat{A(\cdot)^k} \right) : E \rightarrow \mathbb{C}$$

$$y \mapsto T \left( A(\cdot)^k y^{m-k} \right)$$

belongs to  $\mathcal{P}_{\Theta}(^{m-k} E)$ ;

(b2) For constants  $C, \rho > 0$  such that

$$|T(f)| \leq C \|f\|_{\Theta, \rho} \text{ for every } f \in \mathcal{H}_{\Theta b}(E),$$

which exist because  $T \in [\mathcal{H}_{\Theta b}(E)]'$ , there is a constant  $K > 0$  such that

$$\|T(\widehat{A(\cdot)^k})\|_{\Theta} \leq C \cdot K^m \rho^k \|P\|_{\Theta} \text{ for every } P \in \mathcal{P}_{\Theta}(^m E).$$

DEFINITION 2.6. — Let  $\Theta$  be a holomorphy type from  $E$  to  $\mathbb{C}$ .

(a) For  $a \in E$  and  $f \in \mathcal{H}_{\Theta b}(E)$ , the translation of  $f$  by  $a$  is defined by

$$\tau_a f: E \longrightarrow \mathbb{C}, (\tau_a f)(x) = f(x - a).$$

By [11, Proposition 2.2] we have  $\tau_a f \in \mathcal{H}_{\Theta b}(E)$ .

(b) A continuous linear operator  $L: \mathcal{H}_{\Theta b}(E) \longrightarrow \mathcal{H}_{\Theta b}(E)$  is called a convolution operator on  $\mathcal{H}_{\Theta b}(E)$  if it is translation invariant, that is,

$$L(\tau_a f) = \tau_a(L(f))$$

for all  $a \in E$  and  $f \in \mathcal{H}_{\Theta b}(E)$ .

(c) For each functional  $T \in [\mathcal{H}_{\Theta b}(E)]'$ , the operator  $\bar{\Gamma}_{\Theta}(T)$  is defined by

$$\bar{\Gamma}_{\Theta}(T): \mathcal{H}_{\Theta b}(E) \longrightarrow \mathcal{H}_{\Theta b}(E), \bar{\Gamma}_{\Theta}(T)(f) = T * f,$$

where the convolution product  $T * f$  is defined by

$$(T * f)(x) = T(\tau_{-x} f) \text{ for every } x \in E.$$

(d)  $\delta_0 \in [\mathcal{H}_{\Theta b}(E)]'$  is the linear functional defined by

$$\delta_0: \mathcal{H}_{\Theta b}(E) \longrightarrow \mathbb{C}, \delta_0(f) = f(0).$$

The main results of this paper read as follows:

THEOREM 2.7. — Let  $E'$  be separable and  $(\mathcal{P}_{\Theta}(^m E))_{m=0}^{\infty}$  be a  $\pi_1$ -holomorphy type from  $E$  to  $\mathbb{C}$ . Then every convolution operator on  $\mathcal{H}_{\Theta b}(E)$  which is not a scalar multiple of the identity is hypercyclic.

THEOREM 2.8. — Let  $E'$  be separable,  $(\mathcal{P}_{\Theta}(^m E))_{m=0}^{\infty}$  be a  $\pi_1$ - $\pi_2$ -holomorphy type and  $T \in [\mathcal{H}_{\Theta b}(E)]'$  be a linear functional which is not a scalar multiple of  $\delta_0$ . Then  $\bar{\Gamma}_{\Theta}(T)$  is a convolution operator that is not a scalar multiple of the identity, hence hypercyclic.

Example 2.9. — To give a simple application of our main results, observe that making  $E = \mathbb{C}^n$  and  $\mathcal{P}_{\Theta}(^m \mathbb{C}^n) = \mathcal{P}(^m \mathbb{C}^n)$ , we get  $\mathcal{H}_{\Theta b}(\mathbb{C}^n) = \mathcal{H}(\mathbb{C}^n)$ , and in this case Theorem 2.7 recovers the result of Godefroy and Shapiro [14] on the hypercyclicity of convolution operators on  $\mathcal{H}(\mathbb{C}^n)$  which are not scalar multiples of the identity. More sophisticated applications will be given in Example 3.11 and in Example 5.3.

### 3. Comparison with known results

Before proceeding to the proofs we shall establish that Theorems 2.7 and 2.8 are strictly more general than [6, Theorem 4.3] and [6, Corollary 4.4], respectively. First of all we have to give the definitions needed to understand these results from [6].

DEFINITION 3.1. — For  $P \in \mathcal{P}({}^k E)$ ,  $a \in E$  and  $r \in \mathbb{N}$ ,  $P_{a^r}$  denotes the  $(k - r)$ -homogeneous polynomial on  $E$  defined by

$$P_{a^r}(x) = A(\underbrace{a, \dots, a}_{r \text{ times}}, x, \dots, x),$$

where, as before,  $A$  stands for the continuous symmetric  $k$ -linear form such that  $P(x) = A(x, \dots, x)$  for every  $x \in E$ .

DEFINITION 3.2 (Carando, Dimant, Muro [7, 6]). — For each  $k \in \mathbb{N}_0$ ,  $\mathfrak{A}_k(E)$  and  $\mathfrak{B}_k(E)$  are linear subspaces of  $\mathcal{P}({}^k E)$  containing the polynomials of finite type which are Banach spaces with norms  $\|\cdot\|_{\mathfrak{A}_k(E)}$  and  $\|\cdot\|_{\mathfrak{B}_k(E)}$ , respectively.  $\mathfrak{A}_k(E)$  and  $\mathfrak{B}_k(E)$  are also asked to be continuously contained in  $\mathcal{P}({}^k E)$ .

A sequence  $\mathfrak{A}(E) = \{\mathfrak{A}_k(E)\}_{k \in \mathbb{N}_0}$  is said to be a coherent sequence of homogeneous polynomials if there exist positive constants  $C$  and  $D$  such that the following conditions hold for all  $k$ :

- (a) For each  $P \in \mathfrak{A}_{k+1}(E)$  and  $a \in E$ ,  $P_a \in \mathfrak{A}_k(E)$  and

$$\|P_a\|_{\mathfrak{A}_k(E)} \leq C \|P\|_{\mathfrak{A}_{k+1}(E)} \|a\|.$$

- (b) For each  $P \in \mathfrak{A}_k(E)$  and  $\gamma \in E'$ ,  $\gamma P \in \mathfrak{A}_{k+1}(E)$  and

$$\|\gamma P\|_{\mathfrak{A}_{k+1}(E)} \leq D \|\gamma\| \|P\|_{\mathfrak{A}_k(E)}.$$

As usual, for  $k = 0$ ,  $\mathfrak{A}_0(E)$  is the 1-dimensional space of constant functions on  $E$ , that is  $\mathfrak{A}_0(E) = \mathbb{C}$ .

The coherent sequence  $\mathfrak{A}(E) = \{\mathfrak{A}_k(E)\}_k$  is said to be multiplicative if there exists  $M > 0$  such that  $PQ \in \mathfrak{A}_{k+l}(E)$  and

$$\|PQ\|_{\mathfrak{A}_{k+l}(E)} \leq M^{k+l} \|P\|_{\mathfrak{A}_k(E)} \|Q\|_{\mathfrak{A}_l(E)},$$

whenever  $P \in \mathfrak{A}_k(E)$  and  $Q \in \mathfrak{A}_l(E)$ .

Remark 3.3. — Note that the case  $k = 0$  implies that the constant  $C$  of condition 3.2(a) is greater than or equal to 1. From [4, Theorem 3.2] it follows that every coherent sequence  $\{\mathfrak{A}_k(E)\}_{k \in \mathbb{N}_0}$  is a holomorphy type with constant  $\sigma = C$ . So,

$$\|P\| \leq C^k \|P\|_{\mathfrak{A}_k(E)}$$

for all  $P \in \mathfrak{A}_k(E)$  and  $k \in \mathbb{N}_0$ .

Let  $\mathfrak{A}(E) = \{\mathfrak{A}_k(E)\}_k$  be a coherent sequence of homogeneous polynomials on  $E$ . Since  $\mathfrak{A}(E)$  is a holomorphy type by Remark 3.3, we can consider the space  $\mathcal{H}_{\mathfrak{A}(E)b}(E)$  of holomorphic functions of  $\mathfrak{A}(E)$ -holomorphy type of bounded type according to Definition 2.3. Following the notation of [6] we shall henceforth represent this space by the symbol  $\mathcal{H}_{b\mathfrak{A}}(E)$ . So  $\mathcal{H}_{b\mathfrak{A}}(E)$  becomes a Fréchet space with the topology generated by the family of seminorms  $\{p_\rho\}_{\rho>0}$ , where

$$p_\rho(f) = \sum_{k=0}^{\infty} \frac{\rho^k}{k!} \|\hat{d}^k f(0)\|_{\mathfrak{A}_k(E)},$$

for  $f \in \mathcal{H}_{b\mathfrak{A}}(E)$ .

Next we define the polynomial Borel transform in the context of coherent sequences:

DEFINITION 3.4. — Let  $\mathfrak{A}(E) = \{\mathfrak{A}_k(E)\}_k$  be a coherent sequence. For each  $k$  the polynomial Borel transform is defined by

$$B_k: \mathfrak{A}_k(E)' \longrightarrow \mathcal{P}({}^k E') \quad , \quad B_k(\varphi)(\gamma) = \varphi(\gamma^k).$$

From now on, the expression  $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$  will always mean that the polynomial Borel transform  $B_k: \mathfrak{A}_k(E)' \longrightarrow \mathfrak{B}_k(E')$  is an isometric isomorphism.

The main hypercyclicity results of [6] are the following:

THEOREM 3.5 ([6] Theorem 4.3). — Suppose that  $E'$  is separable. Let  $\{\mathfrak{B}_k(E')\}_k$  be a coherent sequence and  $\{\mathfrak{A}_k(E)\}_k$  be such that  $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$  for every  $k$ . Then, every convolution operator on  $\mathcal{H}_{b\mathfrak{A}}(E)$  which is not a scalar multiple of the identity is hypercyclic.

COROLLARY 3.6 ([6] Corollary 4.4). — Suppose that  $E'$  is separable. Let  $\{\mathfrak{B}_k(E')\}_k$  be a coherent multiplicative sequence and  $\{\mathfrak{A}_k(E)\}_k$  be such that  $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$  for every  $k$ . For every  $\varphi \in [\mathcal{H}_{b\mathfrak{A}}(E)]'$  which is not a scalar multiple of  $\delta_0$ , the operator

$$T_\varphi: \mathcal{H}_{b\mathfrak{A}}(E) \longrightarrow \mathcal{H}_{b\mathfrak{A}}(E) \quad , \quad T_\varphi(f) = \varphi * f,$$

is hypercyclic.

The next result proves that Theorem 3.5 is a particular case of Theorem 2.7:



PROPOSITION 3.7. — *Let  $\{\mathfrak{B}_k(E')\}_k$  be a coherent sequence and  $\{\mathfrak{A}_k(E)\}_k$  be such that  $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$  for all  $k$ . Then  $\{\mathfrak{A}_k(E)\}_k$  is a  $\pi_1$ -holomorphy type from  $E$  to  $\mathbb{C}$ .*

*Proof.* — By [6, Proposition 2.5] we know that  $\{\mathfrak{A}_k(E)\}_k$  is a coherent sequence, hence it is a holomorphy type by Remark 3.3. As to condition 2.5(a2), [6, Lemma 2.1] shows that

$$\overline{\mathcal{P}_f(kE)}^{\mathfrak{A}_k} = \mathfrak{A}_k(E) \text{ for every } k.$$

So all that is left to check is the inequality in condition (a1) of Definition 2.5. By assumption we know that  $\|T\|_{\mathfrak{A}_k(E)'} = \|B_k(T)\|_{\mathfrak{B}_k(E')}$  for every  $T \in \mathfrak{A}_k(E)'$ . Let  $C$  be the constant of condition 3.2(a) for the coherent sequence  $\{\mathfrak{B}_k(E')\}_k$ . By the inequality in Remark 3.3,

$$\|B_k(T)\| \leq C^k \|B_k(T)\|_{\mathfrak{B}_k(E')},$$

for all  $T \in \mathfrak{A}_k(E)'$  and  $k \in \mathbb{N}_0$ . Thus,

$$\begin{aligned} \|\phi^k\|_{\mathfrak{A}_k(E)} &= \sup_{\|T\|_{\mathfrak{A}_k(E)'}=1} |T(\phi^k)| = \sup_{\|T\|_{\mathfrak{A}_k(E)'}=1} |B_k(T)(\phi)| \\ &\leq \|\phi\|^k \cdot \sup_{\|T\|_{\mathfrak{A}_k(E)'}=1} \|B_k(T)\| \\ &\leq \|\phi\|^k \cdot C^k \cdot \sup_{\|T\|_{\mathfrak{A}_k(E)'}=1} \|B_k(T)\|_{\mathfrak{B}_k(E')} \\ &= \|\phi\|^k \cdot C^k \cdot \sup_{\|T\|_{\mathfrak{A}_k(E)'}=1} \|T\|_{\mathfrak{A}_k(E)'} = C^k \cdot \|\phi\|^k, \end{aligned}$$

for all  $\phi \in E'$  and  $k \in \mathbb{N}_0$ . □

To continue we need the following result:

PROPOSITION 3.8 ([6] Lemma 3.3). — *Let  $\{\mathfrak{B}_k(E')\}_k$  be a coherent multiplicative sequence and  $\{\mathfrak{A}_k(E)\}_k$  be such that  $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$  for every  $k$ . Let  $k \geq l$ ,  $P \in \mathfrak{A}_k(E)$  and  $\varphi \in \mathfrak{A}_{k-l}(E)'$  be given. Then the  $l$ -homogeneous polynomial  $x \in E \mapsto \varphi(P_{x^l}) \in \mathbb{C}$  belongs to  $\mathfrak{A}_l(E)$  and*

$$\|x \mapsto \varphi(P_{x^l})\|_{\mathfrak{A}_l(E)} \leq M^k \|\varphi\|_{\mathfrak{A}_{k-l}(E)'} \|P\|_{\mathfrak{A}_k(E)}.$$

PROPOSITION 3.9. — *Let  $\{\mathfrak{B}_k(E')\}_k$  be a coherent multiplicative sequence and  $\{\mathfrak{A}_k(E)\}_k$  be such that  $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$  for every  $k$ . Then  $\mathfrak{A}(E) = \{\mathfrak{A}_k(E)\}_k$  is a  $\pi_2$ -holomorphy type from  $E$  to  $\mathbb{C}$ .*

*Proof.* — Again by [6, Proposition 2.5] we get that  $\{\mathfrak{A}_k(E)\}_k$  is a coherent sequence, so the space  $\mathcal{H}_{b\mathfrak{A}}(E)$  is well defined. Let  $T \in [\mathcal{H}_{b\mathfrak{A}}(E)]'$  and  $m, k \in \mathbb{N}_0$  with  $k \leq m$  be given. Note that

$$T\left(\widehat{A(\cdot)^k}\right) = (x \mapsto T(P_{x^{m-k}}))$$

for every  $P \in \mathfrak{A}_m(E)$ , where  $A$  is the  $m$ -linear symmetric mapping on  $E^m$  such that  $P = \hat{A}$ . Therefore from Proposition 3.8 it follows that  $T(\widehat{A(\cdot)^k})$  belongs to  $\mathfrak{A}_{m-k}(E)$  and

$$\begin{aligned} \|T(\widehat{A(\cdot)^k})\|_{\mathfrak{A}_{m-k}(E)} &= \|x \mapsto T(P_{x^{m-k}})\|_{\mathfrak{A}_{m-k}(E)} \\ &\leq M^m \cdot \|T|_{\mathfrak{A}_k(E)}\|_{\mathfrak{A}_k(E)'} \cdot \|P\|_{\mathfrak{A}_m(E)}, \end{aligned}$$

where  $T|_{\mathfrak{A}_k(E)}$  obviously means the restriction of  $T$  to  $\mathfrak{A}_k(E)$ . Since  $T \in [\mathcal{H}_{b\mathfrak{A}}(E)]'$ , there are  $C > 0$  and  $\rho > 0$  such that

$$|T(f)| \leq C \cdot p_\rho(f)$$

for every  $f \in \mathcal{H}_{b\mathfrak{A}}(E)$ . In particular,

$$|T(Q)| \leq C \cdot p_\rho(Q) = C \cdot \rho^k \cdot \|Q\|_{\mathfrak{A}_k(E)}$$

for every  $Q \in \mathfrak{A}_k(E)$ , so

$$\|T|_{\mathfrak{A}_k(E)}\|_{\mathfrak{A}_k(E)'} \leq C \cdot \rho^k.$$

Therefore,

$$\begin{aligned} \|T(\widehat{A(\cdot)^k})\|_{\mathfrak{A}_{m-k}(E)} &\leq M^m \cdot \|T|_{\mathfrak{A}_k(E)}\|_{\mathfrak{A}_k(E)'} \cdot \|P\|_{\mathfrak{A}_m(E)} \\ &\leq C \cdot M^m \cdot \rho^k \cdot \|P\|_{\mathfrak{A}_m(E)}, \end{aligned}$$

which completes the proof. □

A combination of Proposition 3.7 with Proposition 3.9 makes clear that Corollary 3.6 is a particular case of Theorem 2.8:

**COROLLARY 3.10.** — *Let  $\{\mathfrak{B}_k(E')\}_k$  be a coherent multiplicative sequence and  $\{\mathfrak{A}_k(E)\}_k$  be such that  $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$  for every  $k$ . Then  $\mathfrak{A}(E) = \{\mathfrak{A}_k(E)\}_k$  is a  $\pi_1$ - $\pi_2$ -holomorphy type.*

Now we prove that our results are more than formal generalizations of the known results, in the sense that there are concrete cases covered by our results and not covered by the results of [6]. Of course it is enough to give examples of  $\{\mathfrak{A}_k(E)\}_k$  such that:

- (i)  $\{\mathfrak{A}_k(E)\}_k$  is a  $\pi_1$ - $\pi_2$ -holomorphy type,
- (ii)  $\mathfrak{A}_k(E)' = \mathfrak{B}_k(E')$  for every  $k$ ,
- (iii)  $\{\mathfrak{B}_k(E')\}_k$  fails to be a coherent sequence.

**Example 3.11.** — (a) Consider the space  $\mathcal{P}_{(p,m(s;q))}({}^m E)$  of all absolutely  $(p, m(s; q))$ -summing  $m$ -homogeneous polynomials on  $E$  introduced by Matos [21, Section 3], where  $0 < q \leq s \leq +\infty$  and  $p \geq q$ . In general  $(\mathcal{P}_{(p,m(s;q))}({}^m E))_{m \in \mathbb{N}_0}$  is not a holomorphy type, hence fails to be a coherent sequence. For example, making  $s = q = p > 1$ , the space

$\mathcal{P}_{(p,m(p;p))}({}^m E)$  coincides with the space of absolutely  $p$ -summing  $m$ -homogeneous polynomials (see [21, p. 843]), which is not a holomorphy type by [8, Example 3.2].

On the other hand, Matos proved in [22, Section 8.2] that if  $E'$  has the bounded approximation property, then the Borel transform  $\mathcal{B}_{\tilde{N},(s;(r,q))}$  establishes an isometric isomorphism between  $[\mathcal{P}_{\tilde{N},(s;(r,q))}({}^m E)]'$  and  $\mathcal{P}_{(s',m(r';q'))}({}^m E')$ , where  $\mathcal{P}_{\tilde{N},(s;(r,q))}({}^m E)$  denotes the space of all  $(s;(r,q))$ -quasi-nuclear  $m$ -homogeneous polynomials on  $E$  (as usual  $s', r', q'$  denote the conjugates of  $s, r, q$ , respectively). So

$$(3.1) \quad [\mathcal{P}_{\tilde{N},(s;(r,q))}({}^m E)]' = \mathcal{P}_{(s',m(r';q'))}({}^m E') \text{ for every } m.$$

The proof that  $\mathcal{P}_{\tilde{N},(s;(r,q))}({}^m E)$  is a  $\pi_1$ -holomorphy type can be found in [22, Sections 8.2 and 8.3] and that it is a  $\pi_2$ -holomorphy type in [22, Proposition 9.1.5].

(b) X. Mujica proved in [24, Teorema 2.5.1] that if  $E'$  has the bounded approximation property,  $p \geq 1$  and  $F$  is reflexive, then the Borel transform  $\mathcal{B}_{\sigma(p)}$  establishes an isometric isomorphism between  $[\mathcal{P}_{\sigma(p)}({}^m E; F)]'$  and  $\mathcal{P}_{\tau(p)}({}^m E'; F')$ , where  $\mathcal{P}_{\sigma(p)}({}^m E; F)$  denotes the space of all  $\sigma(p)$ -nuclear  $m$ -homogeneous polynomials from  $E$  into  $F$ , and  $\mathcal{P}_{\tau(p)}({}^m E'; F')$  denotes the space of all  $\tau(p)$ -summing  $m$ -homogeneous polynomials from  $E'$  into  $F'$ . Making  $F = \mathbb{C}$  we get

$$[\mathcal{P}_{\sigma(p)}({}^m E)]' = \mathcal{P}_{\tau(p)}({}^m E') \text{ for every } m.$$

Again, and for the same reason,  $(\mathcal{P}_{\tau(p)}({}^m E'))_{m=0}^\infty$  is not a holomorphy type in general, consequently it fails to be a coherent sequence. Condition (a1) of Definition 2.5 follows easily because  $\mathcal{P}_{\sigma(p)}$  is a polynomial ideal. Condition (a2) is proved in [24, Proposição 2.4.4], so  $\mathcal{P}_{\sigma(p)}({}^m E)$  is a  $\pi_1$ -holomorphy type. The fact that  $\mathcal{P}_{\sigma(p)}({}^m E)$  is a  $\pi_2$ -holomorphy type is proved in [24, Lema 3.2.6] with  $K = 1$ .

### 4. Proofs of the main results

The first step is the definition of the Borel transform. A holomorphy type from  $E$  to  $F$  shall be denoted by either  $\Theta$  or  $(\mathcal{P}_\Theta({}^m E; F))_{m=0}^\infty$ .

DEFINITION 4.1. — (a) Let  $\Theta$  be a  $\pi_1$ -holomorphy type from  $E$  to  $F$ . It is clear that the Borel transform

$$\mathcal{B}_\Theta : [\mathcal{P}_\Theta({}^m E; F)]' \longrightarrow \mathcal{P}({}^m E'; F') , \mathcal{B}_\Theta T(\phi)(y) = T(\phi^m y),$$

for  $T \in [\mathcal{P}_\Theta({}^m E; F)]'$ ,  $\phi \in E'$  and  $y \in F$ , is well defined and linear. Moreover,  $\mathcal{B}_\Theta$  is continuous and injective by conditions (a1) and (a2) of Definition 2.5. So, denoting the range of  $\mathcal{B}_\Theta$  in  $\mathcal{P}({}^m E'; F')$  by  $\mathcal{P}_{\Theta'}({}^m E'; F')$ , the correspondence

$$\mathcal{B}_\Theta T \in \mathcal{P}_{\Theta'}({}^m E'; F') \mapsto \|\mathcal{B}_\Theta T\|_{\Theta'} := \|T\|,$$

defines a norm on  $\mathcal{P}_{\Theta'}({}^m E'; F')$ . In this fashion the spaces  $([\mathcal{P}_\Theta({}^m E; F)]', \|\cdot\|)$  and  $(\mathcal{P}_{\Theta'}({}^m E'; F'), \|\cdot\|_{\Theta'})$  are isometrically isomorphic.

(b) Let  $(\mathcal{P}_\Theta({}^m E))_{m=0}^\infty$  be a  $\pi_1$ -holomorphy type from  $E$  to  $\mathbb{C}$ . A holomorphic function  $f \in \mathcal{H}(E')$  is said to be of  $\Theta'$ -exponential type if:

- (b1)  $\hat{d}^m f(0) \in \mathcal{P}_{\Theta'}({}^m E')$  for every  $m \in \mathbb{N}_0$ ;
- (b2) There are constants  $C \geq 0$  and  $c > 0$  such that

$$\|\hat{d}^m f(0)\|_{\Theta'} \leq Cc^m,$$

for all  $m \in \mathbb{N}_0$ .

The vector space of all such functions is denoted by  $\text{Exp}_{\Theta'}(E')$ .

The change we made in the definition of  $\pi_1$ -holomorphy types does not affect the validity of [11, Corollary 2.1]. So if  $\Theta$  is a  $\pi_1$ -holomorphy type from  $E$  to  $\mathbb{C}$ , then all nuclear entire functions of bounded type belong to  $\mathcal{H}_{\Theta b}(E)$ . In particular, the functions of the form  $e^\phi$ , for  $\phi \in E'$ , belong to  $\mathcal{H}_{\Theta b}(E)$ . The proof of [11, Theorem 2.1] is not affected either:

PROPOSITION 4.2 ([11] Theorem 2.1). — *If  $\Theta$  is a  $\pi_1$ -holomorphy type from  $E$  to  $\mathbb{C}$ , then the Borel transform*

$$\mathcal{B}: [\mathcal{H}_{\Theta b}(E)]' \longrightarrow \text{Exp}_{\Theta'}(E'), \quad \mathcal{B}T(\phi) = T(e^\phi),$$

for all  $T \in [\mathcal{H}_{\Theta b}(E)]'$  and  $\phi \in E'$ , is an algebraic isomorphism.

PROPOSITION 4.3. — *Let  $\Theta$  be a  $\pi_1$ -holomorphy type from  $E$  to  $\mathbb{C}$  and  $U$  be a non-empty open subset of  $E'$ . Then the set*

$$S = \text{span}\{e^\phi : \phi \in U\}$$

is dense in  $\mathcal{H}_{\Theta b}(E)$ .

*Proof.* — Assume that  $S$  is not dense in  $\mathcal{H}_{\Theta b}(E)$ . In this case, the geometric Hahn-Banach Theorem gives a nonzero functional  $T \in [\mathcal{H}_{\Theta b}(E), \tau_\Theta]'$  that vanishes on  $\overline{S}$ . In particular  $T(e^\phi) = 0$  for each  $\phi \in U$ . So  $\mathcal{B}T(\phi) = T(e^\phi) = 0$  for every  $\phi \in U$ . Thus  $\mathcal{B}T$  is a holomorphic function that vanishes on the open non-void set  $U$ . It follows that  $\mathcal{B}T \equiv 0$  on  $E'$ . Since  $\mathcal{B}$  is injective by Proposition 4.2,  $T \equiv 0$ . This contradiction proves that  $S$  is dense in  $\mathcal{H}_{\Theta b}(E)$ . □

Let  $\Theta$  be a holomorphy type from  $E$  to  $\mathbb{C}$ . The linear space of all convolution operators on  $\mathcal{H}_{\Theta b}(E)$  is denoted by  $\mathcal{O}(\mathcal{H}_{\Theta b}(E))$ . We define the map  $\Gamma_{\Theta}$  by

$$\begin{aligned} \Gamma_{\Theta} : \mathcal{O}(\mathcal{H}_{\Theta b}(E)) &\longrightarrow [\mathcal{H}_{\Theta b}(E)]' \\ L &\mapsto \Gamma_{\Theta}(L) : \mathcal{H}_{\Theta b}(E) \longrightarrow \mathbb{K} \\ f &\mapsto \Gamma_{\Theta}(L)(f) := (L(f))(0). \end{aligned}$$

Remember the definition of  $\delta_0$  to see that  $\Gamma_{\Theta}(L) = \delta_0 \circ L$ . It is clear that  $\Gamma_{\Theta}$  is a well defined linear map.

LEMMA 4.4. — *Let  $\Theta$  be a  $\pi_1$ -holomorphy type from  $E$  to  $\mathbb{C}$  and  $L \in \mathcal{O}(\mathcal{H}_{\Theta b}(E))$  be given. Then:*

- (a)  $L(e^{\phi}) = \mathcal{B}(\Gamma_{\Theta}(L))(\phi) \cdot e^{\phi}$  for every  $\phi \in E'$ .
- (b)  $L$  is a scalar multiple of the identity if and only if  $\mathcal{B}(\Gamma_{\Theta}(L))$  is constant.

*Proof.* — (a) Since  $\Gamma_{\Theta}(L) \in [\mathcal{H}_{\Theta b}(E)]'$ , from Theorem 4.2 we know that

$$\mathcal{B}(\Gamma_{\Theta}(L))(\phi) = \Gamma_{\Theta}(L)(e^{\phi}) = L(e^{\phi})(0)$$

for each  $\phi \in E'$ . Therefore

$$\begin{aligned} L(e^{\phi})(y) &= [\tau_{-y}(L(e^{\phi}))](0) \\ &= [L(\tau_{-y}(e^{\phi}))](0) \\ &= [L(e^{\phi(y)} \cdot e^{\phi})](0) \\ &= e^{\phi(y)} \cdot L(e^{\phi})(0) \\ &= e^{\phi(y)} \cdot \mathcal{B}(\Gamma_{\Theta}(L))(\phi) \\ &= (\mathcal{B}(\Gamma_{\Theta}(L))(\phi) \cdot e^{\phi})(y), \end{aligned}$$

for all  $y \in E$ .

(b) Let  $\lambda \in \mathbb{C}$  be such that  $\mathcal{B}(\Gamma_{\Theta}(L))(\phi) = \lambda$  for every  $\phi \in E'$ . By (a) it follows that

$$L(e^{\phi}) = \mathcal{B}(\Gamma_{\Theta}(L))(\phi) \cdot e^{\phi} = \lambda e^{\phi}$$

for every  $\phi \in E'$ . The continuity of  $L$  and the denseness of  $\{e^{\phi} : \phi \in E'\}$  in  $\mathcal{H}_{\Theta b}(E)$  (Proposition 4.3) yield that  $L(f) = \lambda f$  for every  $f \in \mathcal{H}_{\Theta b}(E)$ .

Conversely, let  $\lambda \in \mathbb{C}$  be such that  $L(f) = \lambda f$  for every  $f \in \mathcal{H}_{\Theta b}(E)$ . Calling on (a) again we get

$$\lambda e^{\phi} = L(e^{\phi}) = \mathcal{B}(\Gamma_{\Theta}(L))(\phi) \cdot e^{\phi},$$

hence  $\mathcal{B}(\Gamma_{\Theta}(L))(\phi) = \lambda$  for every  $\phi \in E'$ . □

In the proof of our main result we shall use the following criterion, which was obtained, independently, by Kitai [18] and Gethner and Shapiro [13]:

**THEOREM 4.5** (Hypercyclicity Criterion). — *Let  $X$  be a separable Fréchet space. A continuous linear operator  $T: X \rightarrow X$  is hypercyclic if there are dense subsets  $Z, Y \subseteq X$  and a map  $S: Y \rightarrow Y$  satisfying the following three conditions:*

- (a) For each  $z \in Z$ ,  $T^n(z) \rightarrow 0$  when  $n \rightarrow \infty$ ;
- (b) For each  $y \in Y$ ,  $S^n(y) \rightarrow 0$  when  $n \rightarrow \infty$ ;
- (c)  $T \circ S = I_Y$ .

The last ingredient we need to give the proof of Theorem 2.7 is the next result.

**PROPOSITION 4.6.** — *Let  $(\mathcal{P}_\Theta(mE))_{m=0}^\infty$  be a  $\pi_1$ -holomorphy type from  $E$  to  $\mathbb{C}$ . Then the set*

$$B = \{e^\phi : \phi \in E'\}$$

*is a linearly independent subset of  $\mathcal{H}_{\Theta b}(E)$ .*

*Proof.* — We have already remarked that  $\{e^\phi : \phi \in E'\} \subseteq \mathcal{H}_{\Theta b}(E)$  whenever  $\Theta$  is a  $\pi_1$ -holomorphy type. Given  $a \in E$ , from [11, Proposition 3.1(i)] we know that the differentiation operator

$$D_a : \mathcal{H}_{\Theta b}(E) \rightarrow \mathcal{H}_{\Theta b}(E), \quad D_a(f) = df(\cdot)(a)$$

is well defined. Now one just has to follow the lines of the proof of [1, Lemma 2.3] to get the result.  $\square$

*Proof of Theorem 2.7.* — Let  $L: \mathcal{H}_{\Theta b}(E) \rightarrow \mathcal{H}_{\Theta b}(E)$  be a convolution operator which is not a scalar multiple of the identity. We shall show that  $L$  satisfies the Hypercyclicity Criterion of Theorem 4.5. First of all, since  $E'$  is separable and  $\Theta$  is a  $\pi_1$ -holomorphy type, we have that  $\mathcal{H}_{\Theta b}(E)$  is separable as well. We have already remarked that  $\mathcal{H}_{\Theta b}(E)$  is a Fréchet space. By  $\Delta$  we mean the open unit disk in  $\mathbb{C}$ . Consider the sets

$$V = \{\phi \in E' : |\mathcal{B}(\Gamma_\Theta(L))(\phi)| < 1\} = \mathcal{B}(\Gamma_\Theta(L))^{-1}(\Delta)$$

and

$$W = \{\phi \in E' : |\mathcal{B}(\Gamma_\Theta(L))(\phi)| > 1\} = \mathcal{B}(\Gamma_\Theta(L))^{-1}(\mathbb{C} - \bar{\Delta}).$$

Since  $L$  is not a scalar multiple of the identity, Lemma 4.4(b) yields that  $\mathcal{B}(\Gamma_\Theta(L))$  is non constant. Therefore, it follows from Liouville's Theorem that  $V$  and  $W$  are non-empty open subsets of  $E'$ . Consider now the following subspaces of  $\mathcal{H}_{\Theta b}(E)$ :

$$H_V = \text{span}\{e^\phi : \phi \in V\} \text{ and } H_W = \text{span}\{e^\phi : \phi \in W\}.$$

By Proposition 4.3 we know that both  $H_V$  and  $H_W$  are dense in  $\mathcal{H}_{\Theta b}(E)$ .

Let us deal with  $H_V$  first. Given  $\phi \in V$ , from Lemma 4.4(a) we have

$$L(e^\phi) = \mathcal{B}(\Gamma_\Theta(L))(\phi) \cdot e^\phi \in H_V.$$

So  $L(H_V) \subseteq H_V$  because  $L$  is linear. Applying Lemma 4.4(a) and the linearity of  $L$  once again we get

$$L^n(e^\phi) = [\mathcal{B}(\Gamma_\Theta(L))(\phi)]^n \cdot e^\phi$$

for all  $n \in \mathbb{N}$  and  $\phi \in V$ . Consequently,

$$L^n(f) = [\mathcal{B}(\Gamma_\Theta(L))(\phi)]^n \cdot f$$

for all  $n \in \mathbb{N}$  and  $f \in H_V$ . Since  $|\mathcal{B}(\Gamma_\Theta(L))(\phi)| < 1$  whenever  $\phi \in V$ , it follows that  $L^n(f) \rightarrow 0$  when  $n \rightarrow \infty$  for each  $f \in H_V$ .

Now we handle  $H_W$ . For each  $\phi \in W$ ,  $\mathcal{B}(\Gamma_\Theta(L))(\phi) \neq 0$ , so we can define

$$S(e^\phi) := \frac{e^\phi}{\mathcal{B}(\Gamma_\Theta(L))(\phi)} \in \mathcal{H}_{\Theta b}(E).$$

By Proposition 4.6,  $\{e^\phi : \phi \in W\}$  is a linearly independent set, so we can extend  $S$  to  $H_W$  by linearity. Therefore  $S(H_W) \subseteq H_W$  and

$$S^n(f) = \frac{f}{[\mathcal{B}(\Gamma_\Theta(L))(\phi)]^n}$$

for all  $f \in H_W$  and  $n \in \mathbb{N}$ . Since  $|\mathcal{B}(\Gamma_\Theta(L))(\phi)| > 1$  whenever  $\phi \in W$ , it follows that  $S^n(f) \rightarrow 0$  when  $n \rightarrow \infty$  for each  $f \in H_W$ .

Finally,  $L \circ S(f) = f$  for every  $f \in H_W$ , so  $L$  is hypercyclic. □

Let us proceed to the proof of Theorem 2.8. The next result is needed. It is an adaptation of [11, Theorem 3.1] to the new definition of  $\pi_2$ -holomorphy type. In this case it is worth giving the details.

**PROPOSITION 4.7.** — *If  $(\mathcal{P}_\Theta(mE))_{m=0}^\infty$  is a  $\pi_2$ -holomorphy type from  $E$  to  $\mathbb{C}$ ,  $T \in [\mathcal{H}_{\Theta b}(E)]'$  and  $f \in \mathcal{H}_{\Theta b}(E)$ , then  $T * f \in \mathcal{H}_{\Theta b}(E)$  and the mapping  $T*$  defines a convolution operator on  $\mathcal{H}_{\Theta b}(E)$ .*

*Proof.* — Since  $T \in [\mathcal{H}_{\Theta b}(E)]'$ , there are constants  $C \geq 0$  and  $\rho > 0$  such that

$$|T(f)| \leq C \|f\|_{\Theta, \rho}$$

for all  $f \in \mathcal{H}_{\Theta b}(E)$ . By [11, Proposition 3.1],

$$\begin{aligned} (T * f)(x) &= T(\tau_{-x}f) = T\left(\sum_{m=0}^\infty \frac{1}{m!} \hat{d}^m f(x)\right) \\ (4.1) \quad &= \sum_{m=0}^\infty \frac{1}{m!} \sum_{k=0}^\infty \frac{1}{k!} T(\widehat{d^{k+m} f(0)(\cdot)^k})(x) \end{aligned}$$

for every  $x \in E$ . By Definition 2.5(b) there is a constant  $K$  such that

$$T \left( \widehat{d^{k+m} f(0)(\cdot)^k} \right) \in \mathcal{P}_\Theta({}^m E) \text{ and}$$

$$\left\| T \left( \widehat{d^{k+m} f(0)(\cdot)^k} \right) \right\|_\Theta \leq CK^{m+k} \rho^k \left\| \hat{d}^{m+k} f(0) \right\|_\Theta$$

for all  $k, m \in \mathbb{N}_0$ . For  $\rho_0 > \rho$  we can write

$$\begin{aligned} & \left\| \sum_{k=0}^\infty \frac{1}{k!} T \left( \widehat{d^{k+m} f(0)(\cdot)^k} \right) \right\|_\Theta \leq \sum_{k=0}^\infty \frac{1}{k!} \left\| T \left( \widehat{d^{k+m} f(0)(\cdot)^k} \right) \right\|_\Theta \\ & \leq \sum_{k=0}^\infty \frac{1}{k!} CK^{m+k} \rho^k \left\| \hat{d}^{m+k} f(0) \right\|_\Theta \\ & \leq \sum_{k=0}^\infty \frac{1}{k!} CK^{m+k} \rho_0^k \left\| \hat{d}^{m+k} f(0) \right\|_\Theta \\ & = C \frac{m!}{\rho_0^m} \sum_{k=0}^\infty \frac{(m+k)!}{m!k!} \cdot \frac{K^{m+k}}{(m+k)!} \rho_0^{m+k} \left\| \hat{d}^{m+k} f(0) \right\|_\Theta \\ & \leq C \frac{m!}{\rho_0^m} \sum_{k=0}^\infty \frac{(2K)^{m+k}}{(m+k)!} \rho_0^{m+k} \left\| \hat{d}^{m+k} f(0) \right\|_\Theta \\ & = C \frac{m!}{\rho_0^m} \left\| \sum_{k=m}^\infty \frac{1}{k!} \hat{d}^k f(0) \right\|_{\Theta, 2K\rho_0} \leq C \frac{m!}{\rho_0^m} \|f\|_{\Theta, 2K\rho_0} < \infty. \end{aligned}$$

This means that

$$P_m = \sum_{k=0}^\infty \frac{1}{k!} T \left( \widehat{d^{k+m} f(0)(\cdot)^k} \right)$$

belongs to  $\mathcal{P}_\Theta({}^m E)$  and

$$(4.2) \quad \|P_m\|_\Theta \leq C \frac{m!}{\rho_0^m} \|f\|_{\Theta, 2K\rho_0}.$$

Hence

$$\lim_{m \rightarrow \infty} \left( \frac{1}{m!} \|P_m\|_\Theta \right)^{\frac{1}{m}} \leq \frac{1}{\rho_0}$$

for every  $\rho_0 > \rho$ . This implies that

$$\lim_{m \rightarrow \infty} \left( \frac{1}{m!} \|P_m\|_\Theta \right)^{\frac{1}{m}} = 0.$$



Therefore, it follows from (4.1) that  $(T * f) = \sum_{m=0}^{\infty} \frac{1}{m!} P_m \in \mathcal{H}_{\Theta b}(E)$ . It is clear that  $T*$  is linear. For  $\rho_1 > 0$ , from (4.2) we get

$$\begin{aligned} \|T * f\|_{\Theta, \rho_1} &= \sum_{m=0}^{\infty} \frac{\rho_1^m}{m!} \|P_m\|_{\Theta} \\ &\leq \sum_{m=0}^{\infty} \frac{C \rho_1^m}{m!} \frac{m!}{(\rho_1 + \rho)^m} \|f\|_{\Theta, 2K(\rho_1 + \rho)} \\ &= \left( \sum_{m=0}^{\infty} \frac{C \rho_1^m}{(\rho_1 + \rho)^m} \right) \|f\|_{\Theta, 2K(\rho_1 + \rho)}, \end{aligned}$$

proving that  $T*$  is continuous. Now we have

$$\begin{aligned} (T * \tau_a f)(x) &= T(\tau_{-x} \circ \tau_a f) = T(\tau_{-x+a} f) \\ &= (T * f)(-(-x + a)) \\ &= (T * f)(x - a) = \tau_a(T * f)(x), \end{aligned}$$

for all  $x, a \in E$ . This completes the proof that  $T*$  is a convolution operator. □

*Proof of Theorem 2.8.* — The operator  $\bar{\Gamma}_{\Theta}(T)$  is a convolution operator for each  $T \in [\mathcal{H}_{\Theta b}(E)]'$  by Proposition 4.7. Suppose that there is  $\lambda \in \mathbb{C}$  such that  $\bar{\Gamma}_{\Theta}(T)(f) = \lambda \cdot f$  for all  $f \in \mathcal{H}_{\Theta b}(E)$ . Then

$$\lambda \cdot f(x) = \bar{\Gamma}_{\Theta}(T)(f)(x) = (T * f)(x) = T(\tau_{-x} f)$$

for every  $x \in E$ . In particular,

$$\lambda \cdot \delta_0(f) = \lambda \cdot f(0) = T(\tau_0 f) = T(f)$$

for every  $f \in \mathcal{H}_{\Theta b}(E)$ . Hence  $T = \lambda \cdot \delta_0$ . This contradiction shows that  $\bar{\Gamma}_{\Theta}(T)$  is not a scalar multiple of the identity, hence hypercyclic by Theorem 2.7. □

### 5. Further results

In this section we show that several related results that appear in the literature have analogues in the context of  $\pi_j$ -holomorphy types,  $j = 1, 2$ .

We start with an analogue of [2, Corollary 8]:

**PROPOSITION 5.1.** — *If  $E'$  is separable and  $(\mathcal{P}_{\Theta}(^n E))_{n=0}^{\infty}$  is a  $\pi_1$ -holomorphy type from  $E$  to  $\mathbb{C}$ , then every nonzero convolution operator on  $\mathcal{H}_{\Theta b}(E)$  has dense range.*

*Proof.* — Let  $L \neq 0$  be a convolution operator. If  $L$  is a scalar multiple of the identity, then clearly  $L$  is surjective. Suppose now that  $L$  is not a scalar multiple of the identity. By Proposition 4.3,  $\text{span}\{e^\phi : \phi \in E'\}$  is dense in  $\mathcal{H}_{\Theta b}(E)$ . By Lemma 4.4,  $L(e^\phi) = \mathcal{B}(\Gamma_\Theta(L))(\phi) \cdot e^\phi$  for every  $\phi \in E'$ , and this implies that each  $e^\phi$  belongs to the range of  $L$ . Therefore,

$$\mathcal{H}_{\Theta b}(E) = \overline{\text{span}\{e^\phi : \phi \in E'\}}^{\tau_\theta} = \overline{L(\mathcal{H}_{\Theta b}(E))}^{\tau_\theta}.$$

□

We can go farther with  $\pi_1$ - $\pi_2$ -holomorphy types. The following result is closely related to a result of Malgrange [20] on the existence of solutions of convolution equations. Its proof follows the same steps of the proof of [11, Theorem 4.4]:

**THEOREM 5.2.** — *Let  $(\mathcal{P}_\Theta({}^n E))_{n=0}^\infty$  be a  $\pi_1$ - $\pi_2$ -holomorphy type from  $E$  to  $\mathbb{C}$  such that  $\text{Exp}_{\Theta'}(E')$  is closed under division, that is: if  $f, g \in \text{Exp}_{\Theta'}(E')$  are such that  $g \neq 0$  and  $f/g$  is holomorphic, then  $f/g \in \text{Exp}_{\Theta'}(E')$ . Then every nonzero convolution operator on  $\mathcal{H}_{\Theta b}(E)$  is surjective.*

*Example 5.3.* — (a) Let  $E'$  have the bounded approximation property and  $(\mathcal{P}_N({}^m E))_{m=0}^\infty$  be the holomorphy type of nuclear homogeneous polynomials on  $E$ . To see that this is a  $\pi_1$ - $\pi_2$ -holomorphy type, regard it as a particular case of the quasi-nuclear holomorphy types considered in Example 3.11(a) or see it directly in [15, page 15 and Lemma 7.2]. By [15, Proposition 7.2], in this case the role of  $\text{Exp}_{\Theta'}(E')$  is played by the space  $\text{Exp}(E')$  of all entire mappings of exponential-type on  $E'$  [15, Definition 7.5]. Also,  $\text{Exp}(E')$  is closed under division [15, Proposition 8.1]. Hence, it follows from Theorem 5.2 that every nonzero convolution operator on  $\mathcal{H}_{Nb}(E)$  is surjective.

(b) As we saw in Example 3.11(a), if  $E'$  has the bounded approximation property, then  $(\mathcal{P}_{\tilde{N},(s;(r,q))}({}^m E))_{m=0}^\infty$  is a  $\pi_1$ - $\pi_2$ -holomorphy type, and, according to the duality (3.1), in this case the role of  $\text{Exp}_{\Theta'}(E')$  is played by  $\text{Exp}_{(s',m(r';q'))}(E')$ . Making  $A = B = 0$  in [10, Theorem 3.8] one gets that  $\text{Exp}_{(s',m(r';q'))}(E')$  is closed under division (alternatively, see [22, Theorem 5.4.8]). Hence, it follows from Theorem 5.2 that every nonzero convolution operator on  $\mathcal{H}_{\tilde{N},(s;(r,q))b}^\sim(E)$  is surjective.

Now we establish a connection with the fashionable subject of lineability (for detailed information see, e.g., [12] and references therein).

DEFINITION 5.4. — A subset  $A$  of an infinite-dimensional topological vector space  $E$  is said to be dense-lineable in  $E$  if  $A \cup \{0\}$  contains a dense subspace of  $E$ .

Next result is closely related to (actually is a generalization of) [2, Corollary 12]:

PROPOSITION 5.5. — Let  $E'$  be separable,  $(\mathcal{P}_\Theta(nE))_{n=0}^\infty$  be a  $\pi_1$ -holomorphy type from  $E$  to  $\mathbb{C}$  and  $L$  be a convolution operator on  $\mathcal{H}_{\Theta b}(E)$  that is not a scalar multiple of the identity. Then the set of hypercyclic functions for  $L$  is dense-lineable in  $\mathcal{H}_{\Theta b}(E)$ .

Proof. — The convolution operator  $L$  is hypercyclic by Theorem 2.7, so we can take a hypercyclic function  $f$  for  $L$ . Define

$$M = \left\{ \sum_{i=0}^m \lambda_i L^i(f) : m \in \mathbb{N}_0, \lambda_0, \lambda_1, \dots, \lambda_m \in \mathbb{C} \right\},$$

where  $L^0$  denotes the identity on  $\mathcal{H}_{\Theta b}(E)$ . Clearly  $M$  is a vector subspace of  $\mathcal{H}_{\Theta b}(E)$  and, since  $\{L^n(f) : n \in \mathbb{N}_0\}$  is contained in  $M$ ,  $M$  is a dense subset of  $\mathcal{H}_{\Theta b}(E)$ . Now we only have to prove that every nonzero element of  $M$  is hypercyclic for  $L$ , that is, for every  $g \in M, g \neq 0$ , the set  $\{g, L(g), \dots, L^n(g), \dots\}$  is dense in  $\mathcal{H}_{\Theta b}(E)$ . If  $g \in M, g \neq 0$ , then  $g = \sum_{i=0}^m \lambda_i L^i(f)$ , with  $\lambda_0, \lambda_1, \dots, \lambda_m \in \mathbb{C}$ . Note that  $\sum_{i=0}^m \lambda_i L^i \neq 0$  because  $g \neq 0$ . Since  $\sum_{i=0}^m \lambda_i L^i$  is a convolution operator, it follows from Proposition 5.1 that  $\sum_{i=0}^m \lambda_i L^i$  has dense range. Using that  $\{L^n(f) : n \in \mathbb{N}_0\}$  is dense in  $\mathcal{H}_{\Theta b}(E)$  and that  $\sum_{i=0}^m \lambda_i L^i$  is continuous and has dense range, we conclude that the set

$$\sum_{i=0}^m \lambda_i L^i(\{L^n(f) : n \in \mathbb{N}_0\})$$

is dense in  $\mathcal{H}_{\Theta b}(E)$ . So,

$$\begin{aligned} \{g, L(g), \dots, L^n(g), \dots\} &= \{L^n(g) : n \in \mathbb{N}_0\} \\ &= \left\{ L^n \left( \sum_{i=0}^m \lambda_i L^i(f) \right) : n \in \mathbb{N}_0 \right\} \\ &= \sum_{i=0}^m \lambda_i L^i(\{L^n(f) : n \in \mathbb{N}_0\}) \end{aligned}$$

is dense in  $\mathcal{H}_{\Theta b}(E)$ , proving that  $g$  is hypercyclic for  $L$ . □

Combining Theorem 2.8 with Proposition 5.5 we get:

**COROLLARY 5.6.** — *Let  $E'$  be separable,  $(\mathcal{P}_\Theta(mE))_{m=0}^\infty$  be a  $\pi_1$ - $\pi_2$ -holomorphy type and  $T \in [\mathcal{H}_{\Theta b}(E)]'$  be a linear functional which is not a scalar multiple of  $\delta_0$ . Then the set of hypercyclic functions for  $\bar{\Gamma}_\Theta(T)$  is dense-lineable in  $\mathcal{H}_{\Theta b}(E)$ .*

A result similar to [6, Proposition 4.1] is the following:

**PROPOSITION 5.7.** — *Let  $(\mathcal{P}_\Theta(mE))_{m=0}^\infty$  be a  $\pi_1$ -holomorphy type from  $E$  to  $\mathbb{C}$ . Then for every convolution operator  $L: \mathcal{H}_{\Theta b}(E) \rightarrow \mathcal{H}_{\Theta b}(E)$ , the functional  $\Gamma_\Theta(L)$  is the unique functional in  $[\mathcal{H}_{\Theta b}(E)]'$  such that  $L(f) = \Gamma_\Theta(L) * f$  for every  $f \in \mathcal{H}_{\Theta b}(E)$ .*

*Proof.* — Let  $L: \mathcal{H}_{\Theta b}(E) \rightarrow \mathcal{H}_{\Theta b}(E)$  be a convolution operator. By the definition of  $\Gamma_\Theta$  we have that  $\Gamma_\Theta \in [\mathcal{H}_{\Theta b}(E)]'$  and

$$\begin{aligned} L(f)(x) &= L(f)(0 - (-x)) = [\tau_{-x}L(f)](0) \\ &= L(\tau_{-x}f)(0) = \Gamma_\Theta(L)(\tau_{-x}f) \\ &= \Gamma_\Theta(L) * f(x) \end{aligned}$$

for all  $f \in \mathcal{H}_{\Theta b}(E)$  and  $x \in E$ . Thus,  $L(f) = \Gamma_\Theta(L) * f$  for every  $f \in \mathcal{H}_{\Theta b}(E)$ . Let us prove the uniqueness: if  $S \in [\mathcal{H}_{\Theta b}(E)]'$  is such that  $L(f) = S * f$ , then

$$L(e^\phi)(x) = S * e^\phi(x) = S(\tau_{-x}e^\phi) = S(e^\phi) \cdot e^{\phi(x)} = \mathcal{BS}(S) \cdot e^{\phi(x)}$$

for all  $\phi \in E'$  and  $x \in E$ . Hence  $L(e^\phi) = \mathcal{BS}(S) \cdot e^\phi$  for every  $\phi \in E'$ . It follows from Lemma 4.4(a) that  $\mathcal{B}(\Gamma_\Theta(L))(\phi) = \mathcal{BS}(S)$  for every  $\phi \in E'$ . So  $S = \Gamma_\Theta(L)$  by the injectivity of the Borel transform (Proposition 4.2).  $\square$

We finish the paper exploring the multiplicative structure of  $[\mathcal{H}_{\Theta b}(E), \tau_\Theta]'$ :

**DEFINITION 5.8.** — *Let  $(\mathcal{P}_\Theta(mE))_{m=0}^\infty$  be a  $\pi_2$ -holomorphy type from  $E$  to  $\mathbb{C}$ . For  $T_1, T_2 \in [\mathcal{H}_{\Theta b}(E)]'$  we define the convolution product of  $T_1$  and  $T_2$  in  $[\mathcal{H}_{\Theta b}(E)]'$  by*

$$T_1 * T_2 := \Gamma_\Theta(O_1 \circ O_2) \in [\mathcal{H}_{\Theta b}(E)]',$$

where  $O_1 = T_1*$  and  $O_2 = T_2*$ .

It is easy to see that  $[\mathcal{H}_{\Theta b}(E)]'$  is an algebra under this convolution product with unity  $\delta_0$ . Furthermore, the convolution product satisfies the following property:

$$(T_1 * T_2) * f = T_1 * (T_2 * f),$$

for all  $T_1, T_2 \in [\mathcal{H}_{\Theta b}(E)]'$  and  $f \in \mathcal{H}_{\Theta b}(E)$ .

The same proof of [11, Theorem 3.3] provides the following analogue of [6, Corollary 4.2]:

**THEOREM 5.9.** — *If  $(\mathcal{P}_\Theta({}^m E))_{m=0}^\infty$  is a  $\pi_1$ - $\pi_2$ -holomorphy type, then the Borel transform is an algebra isomorphism between  $[\mathcal{H}_{\Theta b}(E), \tau_\Theta]'$  and  $\text{Exp}_{\Theta'}(E')$ .*

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