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GENERALIZED STAIRCASES: RECURRENCE AND SYMMETRY

by W. Patrick HOOPER & Barak WEISS (*)

Abstract. — We study infinite translation surfaces which are $\mathbb{Z}$-covers of compact translation surfaces. We obtain conditions ensuring that such surfaces have Veech groups which are Fuchsian of the first kind and give a necessary and sufficient condition for recurrence of their straight-line flows. Extending results of Hubert and Schmithüsen, we provide examples of infinite non-arithmetic lattice surfaces, as well as surfaces with infinitely generated Veech groups.

Résumé. — Nous étudions les $\mathbb{Z}$-revêtements de translation des surfaces de translation compactes. Nous donnons des conditions nécessaires pour que le groupe de Veech soit fuchsien du premier type, et une condition nécessaire et suffisante pour la récurrence du flot directionnel. En étendant des résultats de Hubert et Schmithüsen, nous donnons des exemples non-arithmétiques dont le groupe de Veech est un réseau et des exemples à groupe de Veech de type infini.

1. Introduction

The geometry of translation surfaces has been intensively studied in recent years (see [13] and [24] for definitions and a survey of recent work). While most of the work was concerned with compact surfaces, in several recent papers non-compact surfaces were also considered. For instance, in [1], the horseshoe and baker's transformations were realized by an affine transformation; [6] is a study of the geometry and dynamics of an infinite translation surface which arises as a geometric limit of compact lattice surfaces; in [9], a connection was made to $\mathbb{Z}$-valued skew products over 1-dimensional systems, and in [21], the topology of the unfolding surface for an irrational billiard was determined. Removing the restriction that

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the surface is compact gives a flexible setup and many phenomena, absent in the compact case, may be observed. For example, in the recent paper [8], Hubert and Schmithüsen made the surprising discovery that there are infinite square tiled surfaces whose Veech group is infinitely generated.

The examples studied in [9, 8] are $\mathbb{Z}$-covers of compact translation surfaces. Although this class is much smaller than the general case, it already displays many surprising features. It may be hoped that it provides a good starting point for a study of the geometry and dynamics of infinite translation surfaces. In this paper we begin the systematic study of these surfaces. Our analysis yields a bijection between $\mathbb{Z}$-covers $\widetilde{M} \to M$, ramified over a finite set $P \subset M$, and projective classes of elements $w \in H_1(M, P; \mathbb{Z})$ (Proposition 3.2). Under this bijection, *recurrent $\mathbb{Z}$-covers*, i.e. covers on which the straightline flow is recurrent in almost all directions, correspond to homology classes with vanishing holonomy (Proposition 4.3). Utilizing a theorem of Thurston which appeared in the unpublished manuscript [20], we obtain a sufficient condition ensuring that the Veech group of a cover $\widetilde{M}$ is Fuchsian of the first kind (Theorem 5.6). This result implies that any recurrent $\mathbb{Z}$-cover of a square tiled surface in genus 2 has a Veech group which is of the first kind (Corollary 5.7), extending the results of [8]. We also obtain necessary and sufficient conditions for a (finite power of a) parabolic element in the Veech group of M to lift to the Veech group of every recurrent $\mathbb{Z}$-cover $\widetilde{M}$ (Theorem 6.4). Using it one may reprove some of the results of [8] in a more general setting. We illustrate the use of our results in the last section, where we provide an example of an infinite lattice translation surface with a non-arithmetic Veech group (Proposition 7.2), and answer a question of Hubert and Schmithüsen (Section 7.2).

2. Regular covers of translation surfaces

Let M denote a compact translation surface and $P \subset M$ denote a finite (possibly empty) subset. We consider P to be a collection of punctures of the surface M and will use M° to denote $M \smallsetminus P$.

Recall that the translation surface M° comes equipped with local charts to $\mathbb{R}^2$ defined away from a discrete set of singularities, such that the transition functions are all translations [13] [24]. An *affine automorphism* of M° is a homeomorphism $f : M^\circ \to M^\circ$ which preserves the underlying affine structure of M°. The local charts identify the tangent plane $T_P M^\circ$ of every non singular point P with the plane $T_{\mathbf{0}}\mathbb{R}^2 = \mathbb{R}^2$. If f is an affine automorphism, then the induced actions on the tangent planes $T_P M^\circ \to T_{f(P)} M^\circ$, as identified with $\mathbb{R}^2$, are the same. We call this induced map *the derivative of f*, $D(f) : \mathbb{R}^2 \to \mathbb{R}^2$. Note that $D(f) \in GL(2, \mathbb{R})$, and if M is compact then $D(f)$ has determinant ± 1. The collection of all affine automorphisms of M° forms the *affine automorphism group* $\mathit{Aff}(M^\circ)$. The group $\Gamma(M^\circ) = D\big(\mathit{Aff}(M^\circ)\big) \subset GL(2, \mathbb{R})$ is called the *Veech group* of M°.

Covering space theory associates covers of a space with the subgroups of its fundamental group. A cover is called *regular* if it is associated to a normal subgroup. We consider a normal subgroup $N \subset \pi_1(M^\circ)$, and consider the associated cover $\pi : \widetilde{M} \to M^\circ$. The group $\Delta = \pi_1(M^\circ)/N$ acts on $\widetilde{M}$ as the automorphisms of the cover, with $\widetilde{M}/\Delta = M^\circ$.

We have the following from covering space theory.

Proposition 2.1. —

(1) *An element $f \in \mathit{Aff}(M^\circ)$ lifts to an $\widetilde{f} \in \mathit{Aff}(\widetilde{M})$ if and only if $f_*(N) = N$.*
(2) *An element $\widetilde{f} \in \mathit{Aff}(\widetilde{M})$ descends to an $f \in \mathit{Aff}(M^\circ)$ if and only if $\widetilde{f}$ normalizes the deck group Δ. That is, $\widetilde{f}\Delta\widetilde{f}^{-1} = \Delta$.*

Definition 2.2. — *The* affine automorphism group of a cover $\widetilde{M} \to M^\circ$ *is the group of pairs of elements $(\widetilde{f}, f) \in \mathit{Aff}(\widetilde{M}) \times \mathit{Aff}(M^\circ)$ for which $\pi \circ \widetilde{f} = f \circ \pi$. We denote this group by $\mathit{Aff}(\widetilde{M}, M^\circ)$. A necessary condition for $(\widetilde{f}, f) \in \mathit{Aff}(\widetilde{M}, M^\circ)$ is that $D(\widetilde{f}) = D(f)$. Thus we have a canonical definition of the derivative $D : \mathit{Aff}(\widetilde{M}, M^\circ) \to GL(2, \mathbb{R})$. We call the image of the group homomorphism D the* Veech group of the cover*, and denote it by $\Gamma(\widetilde{M})$.*

Let $G_N = \{f \in \mathit{Aff}(M^\circ) \ : \ f_*(N) = N\}$. For an $f \in G_N$ the action of f_* on $\pi_1(M^\circ)$ induces an action on $\Delta = \pi_1(M^\circ)/N$. The following is an immediate consequence:

Corollary 2.3. — *$\mathit{Aff}(\widetilde{M}, M^\circ) \cong \Delta \rtimes G_N$, with G_N acting on Δ as mentioned above. Indeed, we have a short exact sequence*

$$1 \to \Delta \hookrightarrow \mathit{Aff}(\widetilde{M}, M^\circ) \twoheadrightarrow G_N \to 1.$$

Note that the projection $p : \mathit{Aff}(\widetilde{M}, M) \to \mathit{Aff}(\widetilde{M})$ may not be injective. However, we do not miss much.

Proposition 2.4. — *If M° is not an unpunctured torus, then $p\big(\mathit{Aff}(\widetilde{M}, M^\circ)\big)$ is a finite index subgroup of $\mathit{Aff}(\widetilde{M})$.*

Proof. — Consider the group $\mathcal{T} \subset \mathit{Aff}(\widetilde{M})$ of elements ι for which $D(\iota) = I$, i.e. the group of translation automorphisms of $\widetilde{M}$. We claim that $\mathcal{T}$ acts properly discontinuously on the set of non-singular points of $\widetilde{M}$. To see this, let Q denote the union of the singularities of M with P. By assumption Q is non-empty. The surface M° has a Delaunay decomposition relative to the points in Q. See [12, §1] for background. The Delaunay decomposition of $\widetilde{M}$ relative to the lifts of Q is the lift of the decomposition of M°. A translation automorphism must permute the cells in the Delaunay decomposition, and hence is properly discontinuous.

The deck group Δ is a finite index subgroup of $\mathcal{T}$, because $\mathrm{Area}(\widetilde{M}/\Delta) = \mathrm{Area}(M) < \infty$. The group Δ is finitely generated because it is a quotient of $\pi_1(M^\circ)$, which is finitely generated. $\mathcal{T}$ is also finitely generated as it contains Δ as a finite index subgroup. An element $\widetilde{f} \in \mathit{Aff}(\widetilde{M})$ acts on $\mathcal{T}$ by conjugation, and preserves the index of subgroups. There are only finitely many subgroups of $\mathcal{T}$ with index $[\mathcal{T} : \Delta]$, because $\mathcal{T}$ is finitely generated. Thus, a finite index subgroup of $\mathit{Aff}(\widetilde{M})$ normalizes Δ. The conclusion follows by Proposition 2.1. □

Presumably, nearly every countable subgroup of $GL(2,\mathbb{R})$ arises as a Veech group of some infinite translation surface. (See [16] for an investigation of Veech groups of tame translation surfaces homeomorphic to the Loch Ness monster.) However, because Veech groups of compact translation surfaces are discrete [22], we have different answer for normal covers.

Corollary 2.5. — *If M° is not an unpunctured torus, the Veech group $\Gamma(\widetilde{M})$ is a discrete subgroup of $\widehat{SL}(2,\mathbb{R})$, the group of 2×2 real matrices of determinant ± 1.*

3. $\mathbb{Z}$-covers

We use $H_1(M,P;\mathbb{Z})$ to denote the relative homology of M with respect to the set of punctures, and $H_1(M^\circ;\mathbb{Z})$ denotes the absolute homology of the punctured surface. Intersection number is a non-degenerate bilinear form

$$i : H_1(M,P;\mathbb{Z}) \times H_1(M^\circ;\mathbb{Z}) \to \mathbb{Z}.$$

Definition 3.1. — *The* $\mathbb{Z}$-cover of M° associated to a non-zero $w \in H_1(M,P;\mathbb{Z})$ *is the cover associated to the kernel of the homomorphism*

$$\varphi_w : \pi_1(M^\circ) \to \mathbb{Z}, \quad \gamma \mapsto i(w, [\![\gamma]\!]),$$

where $[\![\gamma]\!]$ denotes the homology class of γ. We denote this cover by $\widetilde{M}_w$.

If A is a free abelian group, we use $\mathbb{P}A$ to denote $(A \smallsetminus \{\mathbf{0}\})/\sim$, where $a \sim b$ if there are non-zero $m,n \in \mathbb{Z}$ for which $ma = nb$. By non-degeneracy of the bilinear intersection form, we have:

Proposition 3.2. — *The $\mathbb{Z}$-covers $\widetilde{M}_w$ and $\widetilde{M}_{w'}$ are the same if and only if $w \sim w'$.*

Thus, the space of $\mathbb{Z}$-covers of M° is naturally identified with $\mathbb{P}H_1(M, P; \mathbb{Z})$. Statement (1) of proposition 2.1 can be restated as follows.

Proposition 3.3. — *An $f \in \mathit{Aff}(M^\circ)$ lifts to an $\widetilde{f} \in \mathit{Aff}(\widetilde{M}_w)$ if and only if $f_*(w) = \pm w$, where f_* denotes the action of f on $H_1(M, P; \mathbb{Z})$.*

We conclude this section with some remarks on the topology of $\mathbb{Z}$-covers. Since we will not be using these results in the sequel, they will be stated without proof.

Loosely speaking, we think of $\widetilde{M}_w$ as a cover of M ramified over points of P. To make this intuition precise, recall that by pulling back the Euclidean metric, we may endow a translation surface with a metric, and consider its completion. The completion of M° is M, and the map π extends to a map $\bar{\pi} : \bar{M} \to M$, where $\bar{M}$ is the completion of $\widetilde{M}_w$. It is natural to inquire whether $\bar{\pi}$ is a covering map. To this end we have:

Proposition 3.4. — *For each $p \in P$, let $U_p \subset M$ be an open disk with boundary curve γ_p such that $U_p \cap P = \{p\}$ and $\gamma_p \cap P = \varnothing$. Let $\bar{U}_p = \bar{\pi}^{-1}(U_p)$. Then $\bar{\pi}|_{\bar{U}_p}$ is a covering map if and only if $i(w, [\![\gamma_p]\!]) = 0$.*

Thus, $\bar{\pi}$ is a covering map if and only if $i(w, [\![\gamma_p]\!]) = 0$ for all $p \in P$. The following is equivalent.

Corollary 3.5. — *The map $\bar{\pi}$ is a covering map if and only if w is an element of $H_1(M; \mathbb{Z})$, viewed as a subset of $H_1(M, P; \mathbb{Z})$.*

In case $i(w, [\![\gamma_p]\!]) \neq 0$, there is no multiple of γ_p which lifts to $\widetilde{M}_w$ as a closed loop. In this case we call any $\bar{p} \in \bar{\pi}^{-1}(p)$ an *infinite singularity*, since the map $\bar{\pi} : \bar{M} \to M$ in a neighborhood of $\bar{p}$ has the structure of an 'infinite cone angle singularity' or a 'logarithmic singularity'. To compute the number of such points on $\bar{M}$, we have:

Proposition 3.6. — *Assume w is a primitive element of $H_1(M, P; \mathbb{Z})$. Suppose $p \in P$ is such that $i(w, [\![\gamma_p]\!]) \neq 0$. Then $\left|\bar{\pi}^{-1}(p)\right| = |i(w, [\![\gamma_p]\!])|$. In particular the number of infinite singularities is finite.*

If $\bar{M}$ has an infinite singularity $\bar{p}$, its metric topology is not proper. Indeed, the closure of any small ball around $\bar{p}$ is not compact. Therefore it is natural to consider $\widehat{M}$, the complement in $\bar{M}$ of the infinite singularities. That is, $\widehat{M}$ is the largest subset of $\bar{M}$ such that the restriction of $\bar{\pi}$ to $\widehat{M}$ is a covering map. Repeating the arguments of [21] and recalling terminology of [17], we may understand the topology of $\widehat{M}$. We have:

Proposition 3.7. — *If $\bar{M}$ has infinite singularities, then $\widehat{M}$ has only one end and is in the homeomorphism class of the 'Loch Ness monster', the orientable infinite genus surface with a single end. If $\bar{M}$ has no infinite singularities then it has two ends. In this case, $\bar{M}$ is either a cylinder or is homeomorphic to the orientable infinite genus surface with two non-planar ends.*

4. Recurrent $\mathbb{Z}$-covers

A translation surface has a *holonomy map* $\mathbf{hol} : H_1(M, P; \mathbb{Z}) \to \mathbb{R}^2$, obtained by developing a representative of the class into $\mathbb{R}^2$ and taking the difference of the starting and end points. For dynamical reasons, we are especially interested in $\mathbb{Z}$-covers with the following property.

Definition 4.1 (Recurrent $\mathbb{Z}$-covers). — *The $\mathbb{Z}$-cover $\widetilde{M_w}$ is called* recurrent *if* $\mathbf{hol}(w) = \mathbf{0}$.

Although not explicitly stated, square-tiled covers of this type were studied before in [9] and [8]. One reason for restricting attention to recurrent $\mathbb{Z}$-covers is that non-recurrent $\mathbb{Z}$-covers have few affine symmetries. A discrete subgroup of $\widehat{SL}(2, \mathbb{R})$ is called *elementary* if it contains a finite index abelian subgroup and *non-elementary* otherwise. Conversely, a non-elementary subgroup of $\widehat{SL}(2, \mathbb{R})$ contains the free group with two generators [14, Theorem 2.9]. We have the following corollary of Proposition 3.3.

Corollary 4.2. — *If there is an $A \in \Gamma(\widetilde{M_w}) \cap SL(2, \mathbb{R})$ with $\mathrm{trace}(A) \neq \pm 2$, then $\widetilde{M_w}$ is a recurrent $\mathbb{Z}$-cover. In particular, if $\Gamma(\widetilde{M_w})$ is non-elementary, then $\widetilde{M_w}$ is a recurrent $\mathbb{Z}$-cover.*

Proof. — We prove the contrapositive. Suppose $\mathbf{hol}(w) \neq \mathbf{0}$. By Proposition 3.3, $(\tilde{f}, f) \in (\widetilde{M_w}, M^\circ)$ implies that $f_*(w) = \pm w$. Then $D(f)\big(\mathbf{hol}(w)\big) = \mathbf{hol}\big(f_*(w)\big) = \pm \mathbf{hol}(w)$. Thus,

$$\Gamma(\widetilde{M_w}) \subset \Big\{ A \in \widehat{SL}(2, \mathbb{R}) \ : \ A\big(\mathbf{hol}(w)\big) = \pm \mathbf{hol}(w) \Big\} \cong (\mathbb{R} \rtimes \mathbb{Z}/2\mathbb{Z}) \oplus \mathbb{Z}/2\mathbb{Z}.$$

We conclude $\Gamma(\widetilde{M_w})$ is abelian or contains an index two abelian subgroup. Moreover, all elements $A \in \Gamma(\widetilde{M_w}) \cap SL(2, \mathbb{R})$ have trace ± 2. □

We will now justify the term recurrent $\mathbb{Z}$-cover. Let $F_t^\theta : M \to M$ denote the straight-line flow in direction $\theta \in S^1$. Similarly, we will use $\widetilde{F}_t^\theta : \widetilde{M} \to \widetilde{M}$ to denote the straight-line flow on a $\mathbb{Z}$-cover $\widetilde{M}$ in direction θ. Recall that a measure preserving flow F_t is called *recurrent* if for any measurable set A, for a.e. $x \in A$ there is $t_n \to \infty$ such that $F_{t_n} x \in A$.

PROPOSITION 4.3 (Recurrence of the straight-line flow). — *Let $\widetilde{M}$ be a $\mathbb{Z}$-cover of M°. Then $\widetilde{M}$ is a recurrent $\mathbb{Z}$-cover if and only if for any θ for which F_t^θ is ergodic, $\widetilde{F}_t^\theta$ is recurrent.*

Proof. — We will reduce the statement to a classical result of K. Schmidt [18, Theorem 11.4] in infinite ergodic theory. Suppose (X, μ) is a finite measure space and $T : X \to X$ is a measurable transformation preserving μ which is ergodic. For a measurable $f : X \to \mathbb{Z}$, $f \in L^1(X, \mu)$, define $X_f = X \times \mathbb{Z}$ and

$$T_f : X_f \to X_f, \quad T_f(x, k) = (Tx, k + f(x))\,.$$

Then T_f is recurrent if and only if $\int f \, d\mu = 0$.

Given θ, we reduce to the above statement as follows: choose a segment α in M, which is in the direction θ' perpendicular to θ. Define $\widetilde{\alpha}$ in $\widetilde{M}$ to be the union of all lifts of α to $\widetilde{M}$. Denote by T (resp. $\widetilde{T}$) the Poincaré return map to the section α (resp. $\widetilde{\alpha}$), so that T is an interval exchange transformation. The ergodicity of T is equivalent to that of F_t^θ and the recurrence of $\widetilde{F}_t^\theta$ is equivalent to the recurrence of $\widetilde{T}$. Since continuous maps have Borel sections, we may (measurably) identify $\widetilde{M}$ with $M \times \mathbb{Z}$. In these coordinates $\widetilde{T} = T_f$ where

$$f = f^{(\theta)} : \alpha \to \mathbb{Z}, \quad f(x) = i(w, [\![\beta_x]\!]),$$

and β_x is the curve from x to Tx along the F_t^θ orbit of x, and then from Tx to x along α. Let μ be the length measure on α. Up to scaling, Lebesgue measure on M can be represented as $d\mu dt$, where dt denotes the length measure along the orbits of F_t^θ. Since f assumes finitely many values, one on each interval of continuity of T, it is in $L^1(\alpha, \mu)$.

Label by $I_1, \dots, I_\ell$ be the partition of α into intervals of continuity for T. By refining this decomposition we assume that the flow in direction θ starting from the interior of I_j does not hit a puncture in P. For each j, let β_j be a closed loop β_{x_j} as above, corresponding to some $x_j \in I_j$; the particular choice of x_j does not affect $[\![\beta_j]\!]$. Now write $\beta = \sum \mu(I_j)[\![\beta_j]\!] \in H_1(M; \mathbb{R})$. We claim that for a path γ on M representing an element of $H_1(M, P; \mathbb{Z})$,

$$i(\gamma, [\![\beta]\!]) = \mathbf{hol}_{\theta'}(\gamma),$$

i.e. the holonomy vector orthogonally projected onto the one-dimensional vector space in direction θ'. Indeed after homotoping γ off α, each positive crossing of ℓ_j means γ has crossed the rectangle above I_j, and contributes

$\mu(I_j)$ to $\mathbf{hol}_{\theta'}(\gamma)$. Therefore

$$\int f^{(\theta)}\,d\mu = \sum_j \mu(I_j) i(w, [\![\gamma_j]\!]) \\ = i(w, [\![\beta]\!]) = \mathbf{hol}_{\theta'}(w).$$

The main theorem of [11] guarantees the existence of two independent ergodic θ. We see that $\int f^{(\theta)}\,d\mu = 0$ for any ergodic direction θ on M, if and only if $\mathbf{hol}(w) = 0$. □

A similar argument was employed by Conze and Gutkin in [2] to prove recurrence of the billiard flow on some infinite billiard tables.

Corollary 4.4. — *If $\mathbf{hol}(w) = 0$, the straightline flow $\widetilde{F}_t^\theta$ on $\widetilde{M}_w$ is recurrent for a.e. θ.*

Proof. — Combine Proposition 4.3 with the famous result of Kerckhoff, Masur and Smillie [11]. □

5. Veech groups of recurrent $\mathbb{Z}$-covers

Let $H \subset \mathbb{R}^2$. We define $K(H)$ to be the smallest extension field of $\mathbb{Q}$ for which there is an $A \in GL(2, \mathbb{R})$ such that $A(H) \subset K(H)^2 \subset \mathbb{R}^2$. The *holonomy field* of a translation surface M is the field $k = K\big(\mathbf{hol}(H_1(M;\mathbb{Z}))\big)$. The holonomy field was first introduced and studied by Gutkin and Judge [4]. We will follow the treatment of the holonomy field given in the appendix of [10]. It is known (see [10], Theorem 28) that if M is compact and there is a pseudo-Anosov homeomorphism in *Aff*(M), then k is a field extension of $\mathbb{Q}$ of degree at most the genus g of M. Moreover, the image $\mathbf{hol}\big(H_1(M;\mathbb{Z})\big)$ is a $\mathbb{Z}$-module of rank $2[k:\mathbb{Q}]$.

It follows from the work of Kenyon and Smillie that if *Aff*(M°) contains a pseudo-Anosov homeomorphism then

$$K\big(\mathbf{hol}(H_1(M;\mathbb{Z}))\big) = K\big(\mathbf{hol}(H_1(M,P;\mathbb{Z}))\big).$$

We unambiguously declare this the holonomy field in this case, and we use k to denote this field.

Definition 5.1 (Holonomy-free subspaces). — *The* holonomy-free subspaces of homology *are $W = \ker \mathbf{hol} \subset H_1(M,P;\mathbb{Z})$ of relative homology, and $W_0 = W \cap H_1(M;\mathbb{Z})$ of absolute homology.*

The $\mathbb{Z}$-modules W_0 and W have ranks given by the following equations.

$$\text{rk } W_0 = \text{rk } H_1(M;\mathbb{Z}) - 2[k:\mathbb{Q}] = 2(g - [k:\mathbb{Q}]).$$

$$\text{rk } W = \text{rk } H_1(M,P;\mathbb{Z}) - 2[k:\mathbb{Q}] = \begin{cases} 2(g-[k:\mathbb{Q}]) + \#P - 1 & \text{if } P \neq \emptyset \\ 2(g-[k:\mathbb{Q}]) & \text{otherwise.} \end{cases}$$

The affine automorphism group $\mathit{Aff}(M^\circ)$ acts on homology and preserves the subspaces W_0 and W. Thus, we have the following group homomorphisms.

$$\psi_0 : \mathit{Aff}(M^\circ) \to \mathit{Aut}(W_0), \quad f \mapsto f_*|_{W_0}.$$
$$\psi : \mathit{Aff}(M^\circ) \to \mathit{Aut}(W), \quad f \mapsto f_*|_{W}.$$

The following statement follows immediately from Proposition 3.3. It explains our interest in these homomorphisms.

Proposition 5.2. — *Let $f \in \mathit{ker}\ \psi$. For each $w \in W$, there is an $\widetilde{f} \in \mathit{Aff}(\widetilde{M_w})$ such that $(\widetilde{f}, f) \in \mathit{Aff}(\widetilde{M_w}, M^\circ)$. The subgroup*

$$\{(\widetilde{f}, f) \in \mathit{Aff}(\widetilde{M_w}, M^\circ) \ : \ f \in \mathit{ker}\ \psi\}$$

is normal inside $\mathit{Aff}(\widetilde{M_w}, M^\circ)$.

The elements of $\mathit{Aff}(M^\circ)$ permute the punctures. Let $\rho : \mathit{Aff}(M^\circ) \to \mathit{Sym}(P)$ be the map which assigns to an $f \in \mathit{Aff}(M^\circ)$ the permutation induced on P. We have the following.

Proposition 5.3. — *$\psi(\mathit{ker}\ \psi_0 \cap \mathit{ker}\ \rho)$ is abelian of rank at most $(\text{rk } W_0)(\text{rk } W - \text{rk } W_0)$. Thus, there is an exact sequence*

$$1 \to \mathit{ker}\ \psi \hookrightarrow \mathit{ker}\ \psi_0 \twoheadrightarrow A \to 1$$

where $A \subset \mathbb{Z}^{(\text{rk } W_0)(\text{rk } W - \text{rk } W_0)} \rtimes \mathit{Sym}(P)$ has a finite index free abelian subgroup.

Proof. — Enumerate $P = \{p_1, \ldots, p_n\}$, and let $\gamma_i \in H_1(M^\circ;\mathbb{Z})$ be the homology class of a loop which travels clockwise around p_i for $i = 1, \ldots, n$. Let $J : W \to \mathbb{Z}^n$ denote the function

$$J(w) = \big(i(w,\gamma_1), \ldots, i(w,\gamma_n)\big) \in \mathbb{Z}^n.$$

Note that for all $f \in \mathit{Aff}(M^\circ)$ we have $J \circ f_*(w) = \rho(f) \circ J(w)$, where the permutation $\rho(f)$ is acting as a permutation matrix. In addition, $J(w)$ determines the coset of W/W_0 which contains w. The following statements follow from this discussion.

(1) $\mathit{ker}\ J = W_0$.
(2) If $f \in \mathit{ker}\ \rho$, then $f_*(w) - w \in W_0$ for all $w \in W$.

By definition, if $f \in \mathit{ker}\ \psi_0$, then $f_*(w_0) = w_0$ for all $w_0 \in W_0$. For $f \in \mathit{ker}\ \psi_0 \cap \mathit{ker}\ \rho$, let $h_f : W/W_0 \to W_0$ denote the map $w + W_0 \mapsto f_*(w) - w$. This is well defined by the above discussion. Moreover, we can recover $\psi(f) = f_*|_W$ via the formula $\psi(f)(w) = w + h_f(w + W_0)$. If $f, g \in \mathit{ker}\ \psi_0 \cap \mathit{ker}\ \rho$,

$$\begin{aligned}\psi(g \circ f)(w) &= \psi(g)\big(w + h_f(w + W_0)\big)\\ &= w + h_f(w + W_0) + h_g\big(w + h_f(w + W_0) + W_0\big)\\ &= w + h_f(w + W_0) + h_g(w + W_0).\end{aligned}$$

So $\psi(\mathit{ker}\ \psi_0 \cap \mathit{ker}\ \rho)$ is abelian group. Moreover, an element $\psi(f)$ of this group is uniquely determined by the linear map $h_f : W/W_0 \to W_0$. It can be observed that $W/W_0 \cong \mathbb{Z}^{\mathrm{rk}\ W - \mathrm{rk}\ W_0}$ and $W_0 \cong \mathbb{Z}^{\mathrm{rk}\ W_0}$. Hence, the space of all possible h_f is isomorphic to $\mathbb{Z}^{(\mathrm{rk}\ W_0)(\mathrm{rk}\ W - \mathrm{rk}\ W_0)}$. □

If G is a discrete subgroup of $GL(2,\mathbb{R})$, we will use $\Lambda G \subset \mathbb{RP}^1$ to denote the limit set of the projection of G to $PGL(2,\mathbb{R}) = \mathrm{Isom}(\mathbb{H}^2)$. A subgroup G of $GL(2,\mathbb{R})$ or $PGL(2,\mathbb{R})$ is elementary if and only if ΛG contains two or fewer points. See [14] for background on the limit set and for the following.

Lemma 5.4 (Limit sets of normal subgroups). — *Suppose G is a non-elementary discrete subgroup of $GL(2,\mathbb{R})$ or $PGL(2,\mathbb{R})$. If N is a non-trivial normal subgroup of G, then $\Lambda N = \Lambda G$.*

Theorem 5.5. — *If $D\big(\mathit{Aff}(M^\circ)\big)$ is non-elementary and $D(\mathit{ker}\ \psi_0)$ is non-trivial, then*

$$\Lambda D\big(\mathit{Aff}(M^\circ)\big) = \Lambda D(\mathit{ker}\ \psi_0) = \Lambda D(\mathit{ker}\ \psi).$$

In this case, $\Lambda\Gamma(\widetilde{M}_w) = \Lambda D\big(\mathit{Aff}(M^\circ)\big)$ for all recurrent $\mathbb{Z}$-covers $\widetilde{M}_w$ of M°.

Proof. — If $D(\mathit{ker}\ \psi_0)$ is non-trivial, then by a direct application of Lemma 5.4, $\Lambda D\big(\mathit{Aff}(M^\circ)\big) = \Lambda D(\mathit{ker}\ \psi_0)$. In particular, $D(\mathit{ker}\ \psi_0)$ is non-elementary and thus contains a free group with two generators [14, Theorem 2.9]. By Proposition 5.3, $D(\mathit{ker}\ \psi)$ is a finite index subgroup of the kernel of a map from $D(\mathit{ker}\ \psi_0)$ to an abelian group. Hence, $D(\mathit{ker}\ \psi)$ is non-empty. By another application of Lemma 5.4, we see $\Lambda D(\mathit{ker}\ \psi) = \Lambda D(\mathit{ker}\ \psi_0)$. □

A *Fuchsian group of the first kind* is a discrete subgroup Γ of $\mathrm{Isom}(\mathbb{H}^2)$ (or some other linear group which acts isometrically on $\mathbb{H}^2$) for which $\Lambda\Gamma = \mathbb{RP}^1$.

Theorem 5.6. — *Suppose $D\big(\mathit{Aff}(M^\circ)\big)$ is a lattice and that $\mathrm{rk}\ W_0 \leqslant 2$. Then $D(\mathit{ker}\ \psi)$ is a Fuchsian group of the first kind. In particular, for any $w \in W$, $\Gamma(\widetilde{M}_w)$ is Fuchsian of the first kind.*

Proof. — By Theorem 5.5, it is sufficient to show that $D(\mathit{ker}\,\psi_0)$ is nontrivial. Note that rk W_0 is even. If rk $W_0 = 0$, then $\mathit{ker}\,\psi_0 = \mathit{Aff}(M^\circ)$. The more difficult case is when rk $W_0 = 2$. We will assume that $\mathit{ker}\,\psi_0$ is empty and derive a contradiction.

By the Selberg lemma, the group $D\big(\mathit{Aff}(M^\circ)\big)$ contains a finite index subgroup Γ which is torsion free [14, Theorem 2.29]. As observed by Veech [22], $\mathbb{H}^2/D\big(\mathit{Aff}(M^\circ)\big)$ is not co-compact. Therefore, Γ is isomorphic to the fundamental group of the punctured surface $\mathbb{H}^2/\Gamma$, which is a free group. This free group Γ pulls back to a free group $F \subset \mathit{Aff}(M^\circ)$ such that $D|_F$ is injective.

Since rk $W_0 = 2$, $\psi_0 : F \to \widehat{SL}(2,\mathbb{Z})$, where $\widehat{SL}(2,\mathbb{Z})$ denotes the set of 2×2 matrices of determinant ± 1. By our assumption from the first paragraph, $\psi_0|_F$ is injective. Without loss of generality, we may assume that $\psi_0(F) \subset SL(2,\mathbb{Z})$. (If not, replace F by the index two subgroup for which this is true.)

Summarizing the previous two paragraphs, we have two faithful representations, $D|_F$ and $\psi_0|_F$, of F into $SL(2,\mathbb{R})$. We will derive a contradiction from properties of these representations. These representations satisfy the following statements for all $f \in F$.

(1) If $D(f)$ is parabolic, then $\psi_0(f)$ is also parabolic.
(2) If $D(f)$ is hyperbolic, then $2 \leqslant |\mathrm{tr}\ \psi_0(f)| < |\mathrm{tr}\ D(f)|$.

Statement 1 is true because if $f \in \mathit{Aff}(M^\circ)$ is a parabolic, then some power of f is a multi-twist of M°. All eigenvalues of the action of a multi-twist on homology are 1. In particular, the eigenvalues for the action of f on homology are all of modulus 1. Thus, $\psi_0(f)$ is either elliptic or parabolic. But, if $\psi_0(f)$ is elliptic, then ψ_0 is not faithful. If $D(f)$ is hyperbolic, then $f \in \mathit{Aff}(M^\circ)$ is a pseudo-Anosov homeomorphism. Let λ be the eigenvalue of $D(f)$ with largest magnitude. A theorem of Fried implies that λ is also the eigenvalue with largest magnitude of the action of f_* on $H_1(M^\circ;\mathbb{Z})$, and also that λ occurs with multiplicity one [3]. In particular, the eigenvalues of $\psi_0(f) = f_*|_{W_0}$ have modulus strictly less than $|\lambda|$. Again, $\psi_0(f)$ is not elliptic since ψ_0 is assumed to be faithful.

Now consider the quotient surfaces $S_1 = \mathbb{H}^2/D(F)$ and $S_2 = \mathbb{H}^2/\psi_0(F)$. For $i = 1, 2$, let g_i denote the genus of S_i and let $n_i \geqslant 1$ denote the number of ends. We have $F = \pi_1(S_1) = \pi_1(S_2)$, so this induces a homotopy equivalence $\phi : S_1 \to S_2$. Thus, we have that rk $F = 2g_i + n_i - 1$ for each i. By statement 1 above, we have $n_1 \leqslant n_2$. We will show that $g_1 = g_2$ and $n_1 = n_2$.

An element of the fundamental group of a surface is called *peripheral* if it is homotopic to a puncture. Assume that $n_1 < n_2$. Let $\gamma_1, \ldots, \gamma_{n_1} \in \pi_1(S_1)$ denote disjoint peripheral curves. Note that the homology classes of these curves are linearly dependent. Let $\gamma'_j = \phi_*(\gamma_j) \in \pi_1(S_2)$. Note that since S_2 has $n_2 > n_1$ punctures, the homology classes of the curves $\gamma'_1, \ldots, \gamma'_{n_1}$ are linearly independent. This contradicts either the fact that ϕ is a homotopy equivalence, or that $n_1 < n_2$. Thus, $n_1 = n_2$.

By the previous two paragraphs, we may take the homotopy equivalence $\phi : S_1 \to S_2$ to be a homeomorphism. In addition, these surfaces have the same number of parabolic cusps. Thus $\psi_0(F)$ is a lattice in $SL(2,\mathbb{Z})$. For non-peripheral $\beta \in \pi_1(S_1)$ let $\ell_1(\beta)$ denote the length of the geodesic representative on S_1, and let $\ell_2(\beta)$ denote the length of the geodesic representative of $\phi_*(\beta)$. Theorem 3.1 of [20] states that

$$\sup_{\beta \in \pi_1(S_1)} \frac{\ell_2(\beta)}{\ell_1(\beta)} \geqslant 1,$$

with equality only if $S_1 = S_2$. (This holds for any pair of complete, finite area, hyperbolic structures on the same surface.) This contradicts statement (2). $\square$

The following immediate consequence illustrates the use of Theorem 5.6.

Corollary 5.7. — *If M is any translation surface of genus 1 or 2 with non-elementary Veech group, then the Veech group of any recurrent $\mathbb{Z}$-cover has the same limit set. In particular, if M is a square tiled surface of genus 1 or 2 then the Veech group of any recurrent $\mathbb{Z}$-cover is Fuchsian of the first kind.*

6. Multi-twists

A *multi-twist* is an $f \in \mathit{Aff}(M^\circ)$ which preserves the cylinders in a cylinder decomposition and for which $D(f)$ is parabolic with eigenvalue 1. It is well known that if M is compact, and $D(f)$ is parabolic then some power of f is a multi-twist. The action of a multi-twist f on $H_1(M,P;\mathbb{Z})$ is given by the formula

$$f_* : x \mapsto x + \sum_j i(x, \gamma_j^\circ) t_j \gamma_j, \tag{6.1}$$

where j varies over the cylinders in the preserved decomposition. Here $\gamma_j \in H_1(M,P;\mathbb{Z})$ and $\gamma_j^\circ \in H_1(M^\circ;\mathbb{Z})$ denote the homology classes of the core curve in cylinder j (although the curves are the same they represent elements in different homology spaces, and we will use different notation to distinguish them). We denote by $\langle\gamma_j\rangle$ and $\langle\gamma_j^\circ\rangle$ the $\mathbb{Z}$-module spanned by these curves in their respective homology groups. The restriction of the action of f on cylinder j is a Dehn twist. The number $t_j \in \mathbb{Z}$ is the twist number of this Dehn twist. Each t_j is non-zero and they all have the same sign. If this sign is positive f is performing left Dehn twists and if it is negative f is performing right Dehn twists.

Let $\phi = f_* - I$. That is,

$$\phi : H_1(M,P;\mathbb{Z}) \to H_1(M,P;\mathbb{Z}), \quad x \mapsto \sum_j i(x,\gamma_j^\circ)t_j\gamma_j. \tag{6.2}$$

A direct application of Proposition 3.3 yields the following.

Proposition 6.1. — *The multi-twist $f \in Aff(M^\circ)$ lifts to an $\widetilde{f} \in Aff(\widetilde{M}_w, M^\circ)$ if and only if $\phi(w) = 0$.*

A linear map g on a vector space V is called *unipotent of index n* if $(g-I)^n(V) = \mathbf{0}$.

Lemma 6.2. —

(1) *$f_* : H_1(M,P;\mathbb{Z}) \to H_1(M,P;\mathbb{Z})$ is unipotent of index 2. In particular, $ker\ \phi = Fix(f_*^k)$ for all non-zero $k \in \mathbb{Z}$.*
(2) *$\phi\big(H_1(M,P;\mathbb{Z})\big)$ is a submodule of $\langle\gamma_j\rangle$ of full rank. Moreover, this rank is bounded from above by the genus of M.*
(3) *If $D\big(Aff(M^\circ)\big)$ is non-elementary, then both $\mathbf{hol} \circ \phi\big(H_1(M,P;\mathbb{Z})\big)$ and $\mathbf{hol}(ker\ \phi)$ are $\mathbb{Z}$-modules of rank $[k:\mathbb{Q}]$, where k is the holonomy field.*

Proof. — We prove these statements in order. For all $x \in H_1(M,P;\mathbb{Z})$, $\phi(x)$ is a linear combination of the $\{\gamma_j\}$. But, $i(\gamma_i,\gamma_j^\circ) = 0$ for all i and j. This implies statement (1).

From equation (6.2), we infer that $\phi\big(H_1(M,P;\mathbb{Z})\big) \subset \langle\gamma_j\rangle$. Consider the map $\pi : H_1(M^\circ;\mathbb{Z}) \to H_1(M,P;\mathbb{Z})$ induced by the inclusion of $M^\circ \hookrightarrow M$. Define the map

$$\eta : H_1(M,P;\mathbb{Z}) \to \langle\gamma_j^\circ\rangle, \quad x \mapsto \sum_j i(x,\gamma_j^\circ)t_j\gamma_j^\circ.$$

Note that $\pi\circ\eta = \phi$. We claim that the image of η is a $\mathbb{Z}$-module of rank equal to rk $\langle\gamma_j^\circ\rangle$. If this is true, then the conclusion follows as $\pi\left(\langle\gamma_j^\circ\rangle\right) = \langle\gamma_j\rangle$. We now prove this claim. By non-degeneracy of $i : H_1(M,P;\mathbb{Z}) \times H_1(M^\circ;\mathbb{Z}) \to \mathbb{Z}$, it is equivalent to show that if $x \in ker(\eta)$ then $i(x,\gamma^\circ) = 0$ for all $\gamma^\circ \in \langle\gamma_j^\circ\rangle$. We will prove the contrapositive of this statement. Suppose

$i(x,\gamma^\circ) \neq 0$ for some $\gamma^\circ \in \langle \gamma_j^\circ \rangle$. Then $i(x,\gamma_k^\circ) \neq 0$ for some k. We compute

$$i\big(x,\eta(x)\big) = i\big(x, \sum_j i(x,\gamma_j^\circ) t_j \gamma_j^\circ\big) = \sum_j t_j i(x,\gamma_j^\circ)^2.$$

Recall that each t_j is non-zero and has the same sign. In addition, $i(x,\gamma_k^\circ) \neq 0$, so $i\big(x,\phi(x)\big) \neq 0$. Therefore, $\eta(x) \neq 0$.

The inequality rk $\langle \gamma_j \rangle \leqslant$ *genus*(M) follows from topology. Note that the core curves of cylinders are disjoint. Cutting along $g+1$ closed curves on a surface of genus g necessarily disconnects the surface. Hence, the maximal rank of the span of the $\{\gamma_j\}$ is *genus*(M), because the γ_j have disjoint representatives.

Now we will consider statement (3). Since $D\big(\mathit{Aff}(M^\circ)\big)$ is non-elementary we can conjugate $f \in \mathit{Aff}(M^\circ)$ to obtain a new $f' \in \mathit{Aff}(M^\circ)$ so that $D(f')$ has an eigenvector distinct from the eigenvector of $D(f)$. By applying an element of $SL(2,\mathbb{R})$ to M°, we may assume without loss of generality that the derivatives are of the form

$$D(f) = \begin{bmatrix} 1 & \sqrt{\mu} \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad D(f') = \begin{bmatrix} 1 & 0 \\ \pm\sqrt{\mu} & 1 \end{bmatrix},$$

for some $\mu > 0$. Then the surface M can be obtained from Thurston's construction of flat surfaces admitting pseudo-Anosov automorphisms [19, §6]. In particular, the widths of all horizontal cylinders (resp. vertical cylinders) appear as the entries of an eigenvector of a Perron-Frobenius matrix with eigenvalue μ. So from the theory of such matrices, we know $\mu \in k$ and

$$\text{rk } \mathbf{hol}_x\big(H_1(M,P;\mathbb{Z})\big) = \text{rk } \mathbf{hol}_y\big(H_1(M,P;\mathbb{Z})\big) = \text{rk } \mathbb{Z}[\mu] = [k:\mathbb{Q}],$$

where $\mathbf{hol}_x(\gamma)$ and rk $\mathbf{hol}_y(\gamma)$ denote the x- and y-coordinates of the holonomy map respectively, and $\mathbb{Z}[\mu]$ is the $\mathbb{Z}$-module generated by μ. For all $\gamma \in H_1(M,P;\mathbb{Z})$, we have

$$\mathbf{hol} \circ \phi(\gamma) = D(f)\mathbf{hol}(\gamma) - \mathbf{hol}(\gamma) = \big(\mu \mathbf{hol}_y(\gamma), 0\big).$$

We conclude

$$\text{rk } \mathbf{hol} \circ \phi\big(H_1(M,P;\mathbb{Z})\big) = \text{rk } \mathbf{hol}_y\big(H_1(M,P;\mathbb{Z})\big) = [k:\mathbb{Q}].$$

On the other hand, $\gamma \in \mathit{ker}\ \phi$ if and only if $\mathbf{hol}_y(\gamma) = 0$. Thus

$$\text{rk } \mathbf{hol}(\mathit{ker}\ \phi) = \text{rk } \mathbf{hol}_x\big(H_1(M,P;\mathbb{Z})\big) = [k:\mathbb{Q}].$$

□

We first establish a corollary of statement (1) of the lemma.

Corollary 6.3. — *Let $f \in \mathit{Aff}(M^\circ)$ be a multi-twist, and let $w \in W$. If $f_*(w) \neq w$, then $D\big(\mathit{Aff}(\widetilde{M}_w, M^\circ)\big)$ is infinite index in $D\big(\mathit{Aff}(M^\circ)\big)$.*

Recall the definition of the holonomy-free subspace W of $H_1(M,P;\mathbb{Z})$. Proposition 3.3 stated that an element $f \in \mathit{Aff}(M^\circ)$ lifted to an affine automorphism $\widetilde{f} \in \mathit{Aff}(\widetilde{M}_w, M)$ if and only if $f_*(w) = \pm w$.

Theorem 6.4 (Lifting multi-twists). — *Assume $f \in \mathit{Aff}(M^\circ)$ is a multi-twist and that $D\big(\mathit{Aff}(M^\circ)\big)$ is non-elementary. Let the notation be as above.*

$$\text{rk } W - \text{rk}(W \cap \mathit{ker}\ \phi) = \text{rk } \langle\gamma_j\rangle - [k:\mathbb{Q}] \leqslant g - [k:\mathbb{Q}]. \tag{6.3}$$

In particular, f_* acts trivially on W if and only if rk $\langle\gamma_j\rangle = [k:\mathbb{Q}]$.

Proof. — By linearity of ϕ and statement (2) of Lemma 6.2,

$$\text{rk}(\mathit{ker}\,\phi) = \text{rk } H_1(M,P;\mathbb{Z}) - \text{rk } \phi(H_1(M,P;\mathbb{Z})) = \text{rk } W + 2[k:\mathbb{Q}] - \text{rk } \langle\gamma_j\rangle.$$

Now, note that $W \cap \mathit{ker}\ \phi = \mathit{ker}\ \mathbf{hol}|_{\mathit{ker}\ \phi}$. By linearity of **hol**, we have

$$\text{rk}(\mathit{ker}\ \phi) = \text{rk}(W \cap \mathit{ker}\ \phi) + \text{rk } \mathbf{hol}(\mathit{ker}\ \phi) = \text{rk}(W \cap \mathit{ker}\ \phi) + [k:\mathbb{Q}],$$

with the last equality following from statement (3) of the lemma. Subtracting these two equations gives (6.3). The inequality follows from statement (2) of the lemma. □

As an illustration of the use of Theorem 6.4, we deduce:

Corollary 6.5. — *Suppose M is square-tiled and has a cylinder decomposition in which all cylinders are homologous in $H_1(M,P;\mathbb{Z})$. Then the Veech group of any recurrent $\mathbb{Z}$-cover is Fuchsian of the first kind.*

Proof. — In this case rk$\langle\gamma_j\rangle = 1$ and $k = \mathbb{Q}$, so $f_* \in \mathit{ker}\ \psi_0$. Since Df_* is nontrivial, the result follows from Theorem 5.5. □

Remark 6.6. — In [5, Theorem 2], Hubert and Schmithüsen define a class of $\mathbb{Z}$-covers of square tiled surfaces $\mathcal{O}^\infty \to \mathcal{O}$. They show that if $\mathcal{O}$ has a one-cylinder decomposition, then the Veech group of $\mathcal{O}^\infty$ is Fuchsian of the first kind. Thus Corollary 6.5 is an extension of the results of [8].

7. Examples

7.1. Square tiled surfaces with homologous cylinders

We give a construction of a square tiled surface with a horizontal cylinder decomposition all of whose cylinders are homologous. (In fact the reader may verify that all such surfaces arise via this construction.)

Let $C_0, \ldots, C_{k-1}$ be cylinders all with the same rational circumference c, and each with rational width. For each $i = 0, \ldots, k-1$ pick a rational interval exchange of $T_i : [0, c) \to [0, c)$. Use T_i to identify the bottom edge of C_i to the top edge of $C_{i+1 (\mathrm{mod}\ k)}$. Call the resulting surface M, and let $P \subset M$ be a finite set of points with rational coordinates. Then there is a horizontal cylinder decomposition of M°, all of whose cylinders are homologous. So, by Corollary 6.5, any recurrent $\mathbb{Z}$-cover of M° has a Veech group which is Fuchsian of the first kind.

The term *eierlegende Wollmilchsau* refers to the square tiled surface, W, whose properties were first studied by Herrlich and Schmithüsen [5]. It can be obtained by the above construction. See figure 7.1. This is a surface of genus three with four cone singularities, each with cone angle 4π. Let P denote the set of these singularities. The Veech group of W° is $\widehat{SL}(2, \mathbb{Z})$, the group of integer matrices of determinant ± 1.

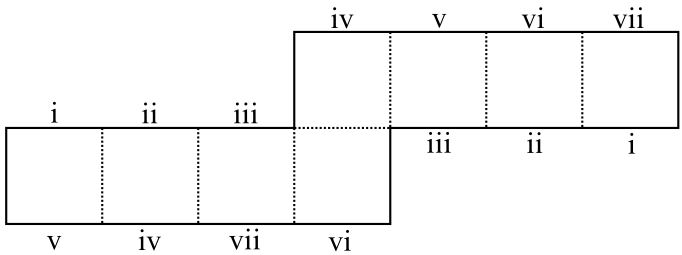

Figure 7.1. The eierlegende Wollmilchsau surface. Horizontal edges are glued together as indicated by the roman numerals. Vertical edges are glued to their opposite (by horizontal translations).

Proposition 7.1. — *Any recurrent $\mathbb{Z}$-cover of W° has a Veech group that contains the congruence 4 subgroup of $SL(2, \mathbb{Z})$.*

Proof. — The horizontal direction has a multi-twist ϕ in a pair of homologous cylinders with derivative $D(\phi) = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$. For any $B \in \widehat{SL}(2, \mathbb{Z}) = \Gamma(W^\circ)$, there is a multi-twist ϕ_B in a pair of homologous cylinders with derivative $D(\phi_B) = B D(\phi) B^{-1}$. By Corollary 6.5, each ϕ_B lifts to any recurrent $\mathbb{Z}$-cover. The derivatives of these elements generate the congruence 4 subgroup of $SL(2, \mathbb{Z})$. $\square$

7.2. A question of Hubert and Schmithüsen

We consider a surface defined in [8]. Let $Z_{3,1}$ be as in figure 7.2, let w be the cycle marked on figure 7.2 and let $Z_{3,1}^{\infty}$ be the corresponding $\mathbb{Z}$-cover. Since $\mathbf{hol}(w) = 0$ this is a recurrent $\mathbb{Z}$-cover. Hubert and Schmithüsen proved that the Veech group of $Z_{3,1}$ is not a lattice, but, since the genus of $Z_{3,1}$ is 2, rk $W_0 = 2$ so Theorem 5.6 implies that the Veech group of $Z_{3,1}^{\infty}$ is Fuchsian of the first kind. This answers a question raised in [8].

Since the Veech group of $Z_{3,1}$ is of the first kind but is not a lattice, it is infinitely generated. Note that a similar argument was employed in [7] and [15] to produce compact translation surfaces with infinitely generated Veech groups, and again in [8] to proved the existence of non-compact square-tiled surfaces with infinitely generated Veech group.

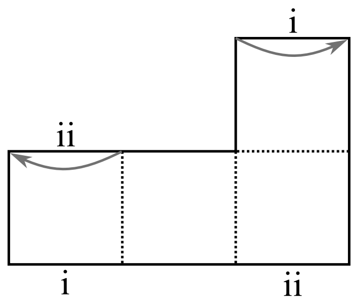

Figure 7.2. The surface $Z_{3,1}$ and the cycle w.

7.3. A double cover of the octagon

Let X denote the polygon shown on the left side of figure 7.3. The translation surface O is obtained by applying the Zemlyakov-Katok unfolding construction to X [23]. The surface O is a double cover of the regular octagon with opposite sides identified, as depicted on the right side of figure 7.3. The surface O is of genus 3 with two cone singularities, each with cone angle 6π.

Let P consist of the two singularities of O. The orientation preserving part of the Veech group is generated by the derivatives of the following affine automorphisms.

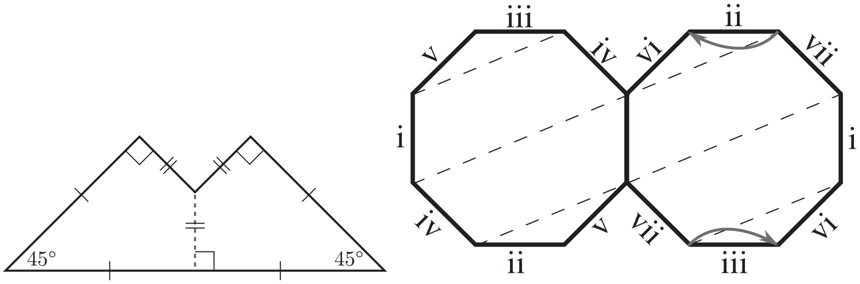

Figure 7.3. The polygon X and the surface O.

- $h \in \mathit{Aff}(O^\circ)$ is the right multi-twist in the horizontal cylinder decomposition. We have $D(h) = \begin{bmatrix} 1 & 2+\sqrt{2} \\ 0 & 1 \end{bmatrix}$.
- $g \in \mathit{Aff}(O^\circ)$ is the right multi-twist in the cylinder decomposition in the direction of angle $\pi/4$. We have $D(g) = \begin{bmatrix} -\sqrt{2} & 1+\sqrt{2} \\ -1-\sqrt{2} & 2+\sqrt{2} \end{bmatrix}$.
- $f \in \mathit{Aff}(O^\circ)$ is the right multi-twist in the cylinder decomposition in the direction of angle $\pi/8$. $D(f) = \begin{bmatrix} -1-\sqrt{2} & 4+3\sqrt{2} \\ -\sqrt{2} & 3+\sqrt{2} \end{bmatrix}$.
- The two elements in $\mathit{Aff}(O^\circ)$ with derivative $-I$.

The orientation preserving part of the Veech group $D\big(\mathit{Aff}(O^\circ)\big)$ is an index two subgroup of a $(4,\infty,\infty)$-triangle group.

Proposition 7.2. — *For any $w \in W \subset H_1(O,P;\mathbb{Z})$, there is a lift of $f \in \mathit{Aff}(O^\circ)$ to $D\big(\mathit{Aff}(\widetilde{O}_w, O^\circ)\big)$. In particular, $D\big(\mathit{Aff}(\widetilde{O}_w, O^\circ)\big)$ is always a Fuchsian group of the first kind.*

Proof. — The affine automorphism f is a multi-twist which preserves a cylinder decomposition consisting of two cylinders. By the multi-twist theorem, it fixes all of W. By Theorem 5.5, $D\big(\mathit{Aff}(\widetilde{O}_w)\big)$ is a Fuchsian group of the first kind. □

The following gives an example of an infinite translation surface with non-arithmetic Veech group which is a lattice.

Proposition 7.3. — *There exists a $w_1 \in W$ for which $D\big(\mathit{Aff}(\widetilde{O}_{w_1}, O^\circ)\big)$ is an infinitely generated Fuchsian group of the first kind, and a $w_2 \in W$ for which $D\big(\mathit{Aff}(\widetilde{O}_{w_2}, O^\circ)\big)$ contains the lattice $\langle D(f), D(g), D(h)\rangle \subset D\big(\mathit{Aff}(O^\circ)\big)$.*

Proof. — We saw in the previous proposition that f always lifts. As O is genus 3, the multi-twist theorem implies that $\mathit{Fix}_W(g_*)$ and $\mathit{Fix}_W(h_*)$

are at worst codimension 1 inside W. Note that $dim\ W = 3$. Thus, we can find a non-zero $w_2 \in Fix_W(g_*) \cap Fix_W(h_*)$. As $D\big(Aff(O^\circ)\big)$ is generated by $\langle D(f), D(g), D(h)\rangle$, we see $D\big(Aff(\widetilde{O}_{w_2}, O^\circ)\big) = D\big(Aff(O^\circ)\big)$.

To see that there is a $w_1 \in W$ for which $D\big(Aff(\widetilde{O}_{w_1}, O^\circ)\big)$ is infinitely generated, it is sufficient to show that $D\big(Aff(\widetilde{O}_{w_2}, O^\circ)\big)$ is infinite index in $D\big(Aff(O^\circ)\big)$. By Corollary 6.3 and the multi-twist theorem, it is sufficient to check that the span of the core curves of a cylinder decomposition span a rank three submodule of $H_1(O, P; \mathbb{Z})$. This is true for both the horizontal direction and the direction of angle $\pi/4$. □

It turns out that there is only one non-zero $w \in W$ up to scaling which is fixed by f_*, g_* and h_*. This w is the homology class shown in grey in figure 7.3.

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