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Hajime ONO, Yuji SANO & Naoto YOTSUTANI

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AN EXAMPLE OF AN ASYMPTOTICALLY CHOW UNSTABLE MANIFOLD WITH CONSTANT SCALAR CURVATURE

by Hajime ONO, Yuji SANO & Naoto YOTSUTANI (*)

Abstract. — Donaldson proved that if a polarized manifold (V, L) has constant scalar curvature Kähler metrics in $c_1(L)$ and its automorphism group $\mathrm{Aut}(V,L)$ is discrete, (V, L) is asymptotically Chow stable. In this paper, we shall show an example which implies that the above result does not hold in the case where $\mathrm{Aut}(V, L)$ is not discrete.

Résumé. — Donaldson a prouvé que, si une variété polarisée (V, L) admet une métrique kählérienne à courbure scalaire constante dans $c_1(L)$, et si son groupe d'automorphismes $\mathrm{Aut}(V, L)$ est discret, alors (V, L) est asymptotiquement stable au sens de Chow. Dans cet article, nous allons montrer un exemple qui implique que le résultat ci-dessus ne s'étend pas au cas où $\mathrm{Aut}(V, L)$ n'est pas discret.

1. Introduction

One of the main issues in Kähler geometry is the existence problem of Kähler metrics with constant scalar curvature on a given Kähler manifold. Through Yau's conjecture [20] and the works of Tian [17], Donaldson [4], this problem is formulated as follows; *The existence of Kähler metrics with constant scalar curvature in a fixed integral Kähler class would be equivalent to a suitable notion of stability of manifolds in the sense of Geometric Invariant Theory.* Though remarkable progress is made recently in this problem, we shall focus only on the related results to our purpose. Let (V, L) be an m-dimensional polarized manifold, that is to say, $L \to V$ is

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an ample line bundle over an m-dimensional compact complex manifold V. Then the first Chern class $c_1(L)$ of L can be regarded as a Kähler class of V. Let $\mathrm{Aut}(V,L)$ be the group of holomorphic automorphisms of (V,L) modulo the trivial automorphism $\mathbb{C}^\times := \mathbb{C} - \{0\}$. In [3], Donaldson proved that

THEOREM 1.1 (Donaldson). — *Let (V,L) be a polarized manifold. Assume that $\mathrm{Aut}(V,L)$ is discrete. If (V,L) has constant scalar curvature Kähler (cscK) metrics in $c_1(L)$, (V,L) is asymptotically Chow stable.*

The purpose of this paper is to show an example of asymptotically Chow unstable polarized manifolds with cscK metrics in the case where $\mathrm{Aut}(V,L)$ is not discrete. To state our result more precisely, let us recall the definition of asymptotic Chow stability and some related results. Since L is ample, V can be embedded into the projective space $\mathbb{P}(W) := \mathbb{P}(H^0(V,L^k)^*)$ for sufficiently large k as an algebraic variety $\Psi_{L^k}(V)$. For $\Psi_{L^k}(V)$, there corresponds to a point $[\mathrm{Ch}(\Psi_{L^k}(V))]$ in $\mathbb{P}[\mathrm{Sym}^d(W)^{\otimes(m+1)}]$, which is often called the Chow point (*cf.* see [13] for the full detail). Take an element $\mathrm{Ch}(\Psi_{L^k}(V))$ representing the Chow point $[\mathrm{Ch}(\Psi_{L^k}(V))]$. The action of the special linear group $\mathrm{SL}(W,\mathbb{C})$ on W is extended to the action on $\mathrm{Sym}^d(W)^{\otimes(m+1)}$. We call $\Psi_{L^k}(V)$ Chow stable if and only if the orbit $\mathrm{SL}(W,\mathbb{C})\cdot\mathrm{Ch}(\Psi_{L^k}(V))$ is closed and its stabilizer is finite. We call it Chow semistable if and only if the closure of the orbit does not contain the origin. Also we call (V,L) asymptotically Chow (semi-)stable if and only if $\Psi_{L^k}(V)$ is Chow (semi-)stable for all sufficiently large k. In this paper, we say that (V,L) is asymptotically Chow unstable if (V,L) is not asymptotically Chow semistable. Theorem 1.1 is extended by Mabuchi [11] to the case where $\mathrm{Aut}(V,L)$ is not discrete.

THEOREM 1.2 (Mabuchi). — *Let (V,L) be a polarized manifold. Assume that the obstruction introduced in* [10] *vanishes. If (V,L) has cscK metrics in $c_1(L)$, (V,L) is asymptotically Chow polystable in the sense of* [11].

The notion of polystability in the above is defined by that the orbit of $\Psi_{L^k}(V)$ with respect to the action of $\mathrm{SL}(W,\mathbb{C})$ is closed. So polystability implies semistability. The obstruction in the above is defined in [10] as a necessary condition for (V,L) to be asymptotically Chow semistable. This obstruction is reformulated by Futaki [7] in more general form by generalizing so-called Futaki invariant. The original Futaki invariant [5] is

a map $f : \mathfrak{h}(V) \to \mathbb{C}$ defined by

$$f(X) := \int_V X h_\omega \omega^m,$$

where $\mathfrak{h}(V)$ is the Lie algebra of all holomorphic vector fields on V, ω is a Kähler form and h_ω is a real-valued function defined by

$$s(\omega) - \left(\int_V s(\omega)\omega^m \Big/ \int_V \omega^m \right) = -\Delta_\omega h_\omega$$

up to addition of a constant. Here $s(\omega)$ denotes the scalar curvature of ω, $(g^{i\bar{j}})_{i\bar{j}}$ denotes the inverse of $(g_{i\bar{j}})_{i\bar{j}}$, and $\Delta_\omega := -g^{i\bar{j}}\partial_i\bar{\partial}_j$ denotes the complex Laplacian with respect to ω. It is well-known that f is independent of the choice of ω and that the vanishing of f is an obstruction for the existence of cscK metrics in the Kähler class $[\omega]$.

Now let us recall the definition of Futaki's obstruction for asymptotic Chow semistability. Let $\mathfrak{h}_0(V)$ be the Lie subalgebra of $\mathfrak{h}(V)$ consisting of holomorphic vector fields which have non-empty zero set. For any $X \in \mathfrak{h}_0(V)$, there exists a complex valued smooth function u_X such that

$$i(X)\omega = -\bar{\partial} u_X,$$

$$\int_V u_X \, \omega^m = 0. \tag{1.1}$$

Let θ be a type $(1,0)$ connection of the holomorphic tangent bundle $T'V$. Let $\Theta := \bar{\partial}\theta$, which is the curvature form with respect to θ. For $X \in \mathfrak{h}(V)$, let $L(X) := \nabla_X - \mathcal{L}_X$, where ∇_X and $\mathcal{L}_X$ are the covariant derivative by X with respect to θ and the Lie derivative respectively. Remark that $L(X)$ can be regarded as a smooth section of $\mathrm{End}(T'V)$ the endomorphism bundle of the holomorphic tangent bundle. Let ϕ be a $GL(m,\mathbb{C})$-invariant polynomial of degree p on $\mathfrak{g}l(m,\mathbb{C})$. We define $\mathcal{F}_\phi : \mathfrak{h}_0(V) \to \mathbb{C}$ by

$$\begin{aligned} \mathcal{F}_\phi(X) = (m-p+1) \int_V \phi(\Theta) \wedge u_X \, \omega^{m-p} \\ + \int_V \phi(L(X)+\Theta) \wedge \omega^{m-p+1}. \end{aligned} \tag{1.2}$$

It is proved that $\mathcal{F}_\phi(X)$ is independent of the choices of ω and θ (see [7]). Let Td^p be the p-th Todd polynomial which is a $GL(m,\mathbb{C})$-invariant polynomial of degree p on $\mathfrak{g}l(m,\mathbb{C})$. Then it is proved [7]

THEOREM 1.3 (Futaki). — *If (V,L) is asymptotically Chow semistable, then, for any $p = 1, \cdots, m$, $\mathcal{F}_{\mathrm{Td}^p}(X) = 0$ for X in a maximal reductive subalgebra $\mathfrak{h}_r(V)$ of $\mathfrak{h}_0(V)$.*

In particular $\mathcal{F}_{\mathrm{Td}^1}$ is equal to $f\,|_{\mathfrak{h}_0(V)}$ up to multiplication of a constant. The vanishing of $\mathcal{F}_{\mathrm{Td}^p}$ for all p is equivalent to the vanishing of Mabuchi's obstruction (*cf.* Proposition 4.1 in [7]).

Remark 1.4. — It might be noticed among experts that the main result in [7] derives a stronger statement than Theorem 1.3. It says that the vanishing of $\mathcal{F}_{\mathrm{Td}^p}$ for all p follows from Chow semistability of (V, L^{k_i}) for some sequence $\{k_i\}_i$ of integers (not necessarily asymptotic Chow semistability). In fact, it is proved in [7] that Chow semistability of (V, L^k) implies the equation (4.2) in [7] for a given k. The invariants $\mathcal{F}_{\mathrm{Td}^p}$ correspond to the coefficients of the polynomial in k of degree $m+1$ in the right hand of (4.2) in [7]. Hence, the vanishing of the coefficients is implied by the vanishing of the polynomial not necessarily for all k greater than some positive integer k_0 but just for finitely many k. Related to this remark, a necessary condition for Chow semistability of polarized toric manifolds is studied by the first author [16].

In [9], Futaki and the first and second authors investigated the linear dependence among $\{\mathcal{F}_{\mathrm{Td}^p}\}_p$ and proposed the following question.

Problem 1. — *Does the existence of cscK metrics induce the vanishing of $\mathcal{F}_{\mathrm{Td}^p}$ for all p?*

For $p = 1$, the existence of cscK metrics of course implies the vanishing of $\mathcal{F}_{\mathrm{Td}^1}$. If the answer were affirmative, the assumption of Theorem 1.2 could be omitted and Theorem 1.1 could be extended to the case where $\mathrm{Aut}(V, L)$ is not discrete. Note that this extension is also discussed in Conjecture 1 in [1].

Moreover, it was claimed in [9] that if a counterexample to Problem 1 exists among toric Fano manifolds with anticanonical polarization, it should be a non-symmetric toric Fano manifold with Kähler-Einstein metrics in the sense of Batyrev-Selivanova [2]. At the time when [9] was written, the existence of such toric Fano manifolds was not known. However it is discovered by Nill-Paffenholz [14] very recently. The main result of this paper is to show that one of the toric Fano manifolds in [14] is the desired example in [9]. That is to say,

Theorem 1.5. — *There exists a seven dimensional toric Fano manifold V with Kähler-Einstein metrics in $c_1(V) := c_1(K_V^{-1})$, whose $\mathcal{F}_{\mathrm{Td}^p}$ does not vanish for $2 \leqslant p \leqslant 7$. In particular, (V, K_V^{-k}) is not Chow semistable for all k greater than some positive integer k_0.*

Also Theorem 1.5 implies that the assumption about obstruction in Theorem 1.2 can not be omitted. Hence, this means that our example in Theorem 1.5 is also a counterexample to Conjecture 1 in [1].

We shall prove Theorem 1.5 by the following two ways; the direct calculation by the localization formula (Section 3), and the derivation of the Hilbert series (Section 4). In particular, our method implies the existence of Kähler-Einstein metrics on V independently of [14]. Remark that on Fano manifolds, all cscK metrics in $c_1(V)$ are equal to Kähler-Einstein metrics.

2. The Nill-Paffenholz's example

See [14] for notations and terminologies of toric geometry in this section.

First of all, let us recall toric Fano manifolds briefly. A toric variety V is an algebraic normal variety with an effective holomorphic action of $T_{\mathbb{C}} := (\mathbb{C}^{\times})^m$, where $\dim_{\mathbb{C}} V = m$. Let $T_{\mathbb{R}} := (S^1)^m$ be the real torus in $T_{\mathbb{C}}$ and $\mathfrak{t}_{\mathbb{R}}$ be the associated Lie algebra. Let $N_{\mathbb{R}} := J\mathfrak{t}_{\mathbb{R}} \simeq \mathbb{R}^m$ where J is the complex structure of $T_{\mathbb{C}}$. Let $M_{\mathbb{R}}$ be the dual space $Hom(N_{\mathbb{R}}, \mathbb{R}) \simeq \mathbb{R}^m$ of $N_{\mathbb{R}}$. Denoting the group of algebraic characters of $T_{\mathbb{C}}$ by M, then $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$. It is well-known that m-dimensional compact toric manifolds correspond to nonsingular complete fans in $\mathbb{R}^m$. Moreover when V is an m-dimensional toric Fano manifold, the corresponding fan $\Sigma_V \subset N_{\mathbb{R}} \simeq \mathbb{R}^m$ satisfies the following properties: Let $N \subset N_{\mathbb{R}}$ be the dual lattice of M,

$$G_V = \{\sigma \in N \,|\, \mathbb{R}_{>0} \cdot \sigma \in \Sigma_V \text{ and } \sigma \text{ is primitive}\}$$

and Q_V be the convex hull of G_V in $\mathbb{R}^m$. Then

(a) the set of vertices of Q_V is equal to G_V,
(b) the origin of $N_{\mathbb{R}}$ is contained in the interior of Q_V,
(c) any face of Q_V is a simplex, and
(d) the set of vertices of any facet of Q_V constitutes a basis of $N \simeq \mathbb{Z}^m \subset N_{\mathbb{R}}$.

An integral polytope satisfying the conditions (b), (c) and (d) is often called a Fano polytope. Conversely, if an m-dimensional Fano polytope $Q \subset \mathbb{R}^m$ is given, then

$$\Sigma(Q) := \{0\} \cup \{c(F)\}_{F:\text{ face of } Q}$$

is a nonsingular complete fan in $\mathbb{R}^m$. Here $c(F) = \mathbb{R}_{\geqslant 0} \cdot F \subset \mathbb{R}^m$ is the cone over F. Hence there is the m-dimensional toric Fano manifold V associated with the fan $\Sigma(Q)$. By the construction above, $Q_V = Q$.

Let V be the seven dimensional toric Fano manifold whose vertices of Fano polytope Q_V in $N_{\mathbb{R}} \simeq \mathbb{R}^7$ are given by

$$
\begin{aligned}
&(\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4\ \mathbf{v}_5\ \mathbf{v}_6\ \mathbf{v}_7\ \mathbf{v}_8\ \mathbf{v}_9\ \mathbf{v}_{10}\ \mathbf{v}_{11}\ \mathbf{v}_{12}) \\
&= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 2 & 1 & -1
\end{pmatrix}.
\end{aligned} \tag{2.1}
$$

Remark that V is isomorphic to a $\mathbb{P}^1$-bundle over $(\mathbb{P}^1)^3 \times \mathbb{P}^3$. To see this, let $\tilde{Q} \subset \mathbb{R}^6$ be the Fano polytope whose vertices are

$$
\begin{aligned}
&(\tilde{\mathbf{v}}_1\ \tilde{\mathbf{v}}_2\ \tilde{\mathbf{v}}_3\ \tilde{\mathbf{v}}_4\ \tilde{\mathbf{v}}_5\ \tilde{\mathbf{v}}_6\ \tilde{\mathbf{v}}_7\ \tilde{\mathbf{v}}_8\ \tilde{\mathbf{v}}_9\ \tilde{\mathbf{v}}_{10}) \\
&= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{pmatrix}.
\end{aligned}
$$

It is easy to see that the 6-dimensional toric Fano manifold associated to $\tilde{Q}$ is $(\mathbb{P}^1)^3 \times \mathbb{P}^3$. The projection $\pi : \mathbb{Z}^7 \to \mathbb{Z}^6,\ \pi(x_1, \ldots, x_7) = (x_1, \ldots, x_6)$ is a map of fans from $(\mathbb{Z}^7, \Sigma(Q_V))$ to $(\mathbb{Z}^6, \Sigma(\tilde{Q}))$. Hence we have an equivariant morphism $p : V \to (\mathbb{P}^1)^3 \times \mathbb{P}^3$ associated to π. We can apply Proposition 1.33 of [15] to the map of fans π. As a result, p is a $\mathbb{P}^1$- fibration on $(\mathbb{P}^1)^3 \times \mathbb{P}^3$.

THEOREM 2.1 (Nill-Paffenholz). — *The toric Fano manifold V defined by (2.1) is not symmetric, but its Futaki invariant vanishes. In particular V admits $T_{\mathbb{R}}$-invariant Kähler-Einstein metrics.*

The second statement in Theorem 2.1 follows from the fact proved by Wang-Zhu [18], which says that a toric Fano manifold admits Kähler-Einstein metrics if and only if its Futaki invariant vanishes. Here we shall explain about the symmetry of toric Fano manifolds in Theorem 2.1. Let $\mathrm{Aut}(V)$ be the group of automorphisms of V. Let $\mathcal{W}(V)$ be the Weyl group

cone	toric affine variety $\simeq \mathbb{C}^7$
$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{11}\}$	$\mathrm{Spec}(\mathbb{C}[X_1, X_2, X_3, Y_1, Y_2, Y_3, Z])$
$\{\mathbf{v}_6, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{11}\}$	$\mathrm{Spec}(\mathbb{C}[X_1^{-1}, X_2, X_3, Y_1, Y_2, Y_3, ZX_1^{-1}])$
$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_{10}, \mathbf{v}_{11}\}$	$\mathrm{Spec}(\mathbb{C}[X_1, X_2, X_3, Y_1Y_3^{-1}, Y_2Y_3^{-1}, Y_3^{-1}, ZY_3^2])$
$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{12}\}$	$\mathrm{Spec}(\mathbb{C}[X_1, X_2, X_3, Y_1, Y_2, Y_3, Z^{-1}])$
$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{11}\}$	$\mathrm{Spec}(\mathbb{C}[X_1^{-1}, X_2^{-1}, X_3, Y_1, Y_2, Y_3, ZX_1^{-1}X_2^{-1}])$
$\{\mathbf{v}_6, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{11}\}$	$\mathrm{Spec}(\mathbb{C}[X_1^{-1}, X_2, X_3, Y_1^{-1}, Y_2Y_1^{-1}, Y_3Y_1^{-1}, ZX_1^{-1}Y_1^2])$
$\{\mathbf{v}_6, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{12}\}$	$\mathrm{Spec}(\mathbb{C}[X_1^{-1}, X_2, X_3, Y_1, Y_2, Y_3, Z^{-1}X_1])$
$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{12}\}$	$\mathrm{Spec}(\mathbb{C}[X_1, X_2, X_3, Y_1^{-1}, Y_2Y_1^{-1}, Y_3Y_1^{-1}, Z^{-1}Y_1^{-2})$
$\{\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{11}\}$	$\mathrm{Spec}(\mathbb{C}[X_1^{-1}, X_2^{-1}, X_3^{-1}, Y_1, Y_2, Y_3, ZX_1^{-1}X_2^{-1}X_3^{-1}])$
$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_3, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{11}\}$	$\mathrm{Spec}(\mathbb{C}[X_1^{-1}, X_2^{-1}, X_3, Y_1^{-1}, Y_2Y_1^{-1}, Y_3Y_1^{-1}, ZX_1^{-1}X_2^{-1}Y_1^2])$
$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{12}\}$	$\mathrm{Spec}(\mathbb{C}[X_1^{-1}, X_2^{-1}, X_3, Y_1, Y_2, Y_3, Z^{-1}X_1X_2])$
$\{\mathbf{v}_6, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{12}\}$	$\mathrm{Spec}(\mathbb{C}[X_1^{-1}, X_2, X_3, Y_1^{-1}, Y_2Y_1^{-1}, Y_3Y_1^{-1}, Z^{-1}X_1Y_1^{-2}])$
$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_4, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{11}\}$	$\mathrm{Spec}(\mathbb{C}[X_1^{-1}, X_2^{-1}, X_3^{-1}, Y_1^{-1}, Y_2Y_1^{-1}, Y_3Y_1^{-1}, ZX_1^{-1}X_2^{-1}X_3^{-1}Y_1^2])$
$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_4, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{12}\}$	$\mathrm{Spec}(\mathbb{C}[X_1^{-1}, X_2^{-1}, X_3^{-1}, Y_1, Y_2, Y_3, Z^{-1}X_1X_2X_3])$
$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_3, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{12}\}$	$\mathrm{Spec}(\mathbb{C}[X_1^{-1}, X_2^{-1}, X_3, Y_1^{-1}, Y_2Y_1^{-1}, Y_3Y_1^{-1}, Z^{-1}X_1X_2Y_1^{-2}])$
$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_4, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{12}\}$	$\mathrm{Spec}(\mathbb{C}[X_1^{-1}, X_2^{-1}, X_3^{-1}, Y_1^{-1}, Y_2Y_1^{-1}, Y_3Y_1^{-1}, Z^{-1}X_1X_2X_3Y_1^{-2}])$

Table 2.1

of $\mathrm{Aut}(V)$ with respect to the maximal torus and $N_{\mathbb{R}}^{\mathcal{W}(V)}$ be the $\mathcal{W}(V)$-invariant subspace of $N_{\mathbb{R}}$. Batyrev and Selivanova [2] say that a toric Fano manifold V is symmetric if and only if $\dim N_{\mathbb{R}}^{\mathcal{W}(V)} = 0$.

Then, let us consider the symmetry of V defined by (2.1). $\mathcal{W}(V)$ contains two cyclic groups acting on $(\mathbb{P}^1)^3$ and $\mathbb{P}^3$ respectively, *i.e.,* one acts on (x_1, x_2, x_3) and the other acts on (x_4, x_5, x_6) where $(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \in N_{\mathbb{R}} \simeq \mathbb{R}^7$. Hence, we find that the dimension of $N_{\mathbb{R}}^{\mathcal{W}(V)}$ is at most one. However, since V is not symmetric, $\dim N_{\mathbb{R}}^{\mathcal{W}(V)} = 1$.

Next, we shall consider affine toric varieties in V and the associated 7-dimensional cones. As explained above, we find that in (2.1), the first six vertices $\{\mathbf{v}_1, \ldots, \mathbf{v}_6\}$ give affine toric varieties in $(\mathbb{P}^1)^3$, the next four vertices $\{\mathbf{v}_7, \ldots, \mathbf{v}_{10}\}$ give them in $\mathbb{P}^3$, and the last two vertices $\{\mathbf{v}_{11}, \mathbf{v}_{12}\}$ give them in the $\mathbb{P}^1$-fibre. More precisely, the set of vertices of each facet of the Fano polytope defined by (2.1) consists one of $\{\mathbf{v}_1, \mathbf{v}_6\}$, one of $\{\mathbf{v}_2, \mathbf{v}_5\}$, one of $\{\mathbf{v}_3, \mathbf{v}_4\}$, three of $\{\mathbf{v}_7, \ldots, \mathbf{v}_{10}\}$ and one of $\{\mathbf{v}_{11}, \mathbf{v}_{12}\}$. Hence, the toric Fano manifold V is covered by 64 affine toric varieties, which are isomorphic to $\mathbb{C}^7$ as listed in Table 2.1. The other affine toric varieties unlisted in Table 2.1 can be obtained easily by the symmetry of V.

3. Direct computation of $\mathcal{F}_{\mathrm{Td}^p}$

First, we shall make the family $\{\mathcal{F}_{\mathrm{Td}^p}\}_p$ simpler in the case of the anticanonical polarization. For a Kähler form $\omega \in c_1(V)$, let g be the associated Kähler metric. We have the Levi-Civita connection $\theta = g^{-1}\partial g$ and its curvature form $\Theta = \bar{\partial}\theta$. Then, for the associated covariant derivative ∇ with θ, $L(X)$ can be expressed by

$$L(X) = \nabla X = \nabla_j X^i dz^j \otimes \frac{\partial}{\partial z^i}$$

where $X \in \mathfrak{h}(V)$, because ∇ is torsion free. Now assume that (V, K_V^{-1}) is a Fano manifold with anticanonical polarization. By the Calabi-Yau theorem [19], for a Kähler form $\omega \in c_1(V)$ there exists another Kähler form $\eta \in c_1(V)$ whose Ricci form ρ_η is equal to ω. For $X \in \mathfrak{h}_0(V)$ let $\tilde{u}_X$ be the Hamiltonian function with respect to ω and a different normalization from (1.1)

$$\int_V \tilde{u}_X \omega^m = -f(X).$$

Recall that $\tilde{u}_X = \Delta_\eta \tilde{u}_X$, where Δ_η is the Laplacian of η. Let

$$\begin{aligned}\mathcal{G}_{\mathrm{Td}^p}(X) := (m-p+1)\int_V \mathrm{Td}^p(\Theta_\eta) \wedge \tilde{u}_X \rho_\eta^{m-p} \\ + \int_V \mathrm{Td}^p(L_\eta(X) + \Theta_\eta) \wedge \rho_\eta^{m-p+1}.\end{aligned}$$

Here Θ_η is the curvature form of the Levi-Civita connection θ_η with respect to η and $L_\eta(X)$ is also associated with θ_η. The proof of Theorem 3.2 in [9] implies that the difference between $\mathcal{F}_{\mathrm{Td}^p}$ and $\mathcal{G}_{\mathrm{Td}^p}$ is equal to a constant multiple of $\mathcal{F}_{\mathrm{Td}^1}$ for any p.

LEMMA 3.1. — *Let V be a Fano manifold with Kähler-Einstein metrics. Then,*

$$\mathcal{F}_{\mathrm{Td}^p}(X) = \int_V (\mathrm{Td}^p \cdot c_1^{m-p+1})(L_\eta(X) + \Theta_\eta) \tag{3.1}$$

where $X \in \mathfrak{h}_0(V)$.

Proof. — Since V admits Kähler-Einstein metrics and $\mathcal{F}_{\mathrm{Td}^1}$ is proportional to the original Futaki invariant f, $\mathcal{F}_{\mathrm{Td}^1}$ vanishes. So $\mathcal{F}_{\mathrm{Td}^p}$ is equal

to $\mathcal{G}_{\mathrm{Td}^p}$. Hence, we find

$$\begin{aligned}\mathcal{F}_{\mathrm{Td}^p}(X) &= (m-p+1)\int_V \mathrm{Td}^p(\Theta_\eta)\wedge(\Delta_\eta u_X)\rho_\eta^{m-p}\\ &\quad+\int_V \mathrm{Td}^p(L_\eta(X)+\Theta_\eta)\wedge\rho_\eta^{m-p+1}\\ &= (m-p+1)\int_V \mathrm{Td}^p(\Theta_\eta)\wedge c_1(L_\eta(X))c_1(\Theta_\eta)^{m-p}\\ &\quad+\int_V \mathrm{Td}^p(L_\eta(X)+\Theta_\eta)\wedge c_1(\Theta_\eta)^{m-p+1}\\ &= \int_V \mathrm{Td}^p(L_\eta(X)+\Theta_\eta)\wedge\\ &\quad\{(m-p+1)c_1(L_\eta(X))c_1(\Theta_\eta)^{m-p}+c_1(\Theta_\eta)^{m-p+1}\}\\ &= \int_V(\mathrm{Td}^p\cdot c_1^{m-p+1})(L_\eta(X)+\Theta_\eta).\end{aligned}$$

□

Since the right hand of (3.1) is a kind of the integral invariants in [8], we can apply the localization formula in [8] for $\mathcal{F}_{\mathrm{Td}^p}$ as follows. Assume that X has only isolated zeroes $\{p_i\}$ and that $L(X)_{p_i}$ is non-degenerate at each p_i, *i.e.*,

$$\det(L(X)_{p_i}) = \det\left(\frac{\partial X^k}{\partial z^l}(p_i)\right)_{1\leqslant k,l\leqslant m}\neq 0,$$

where $(z^1,\cdots,z^m)$ are local coordinates. Then we have

$$\mathcal{F}_{\mathrm{Td}^p}(X) = \sum_{p_i}\frac{(\mathrm{Td}^p\cdot c_1^{m-p+1})(L(X)_{p_i})}{\det(L(X)_{p_i})}. \tag{3.2}$$

As for the localization formula, see also [6].

We consider the following one-parameter subgroup $\{\sigma_t\}$ in the maximal torus of $\mathrm{Aut}(V)$; it is written by

$$\begin{aligned}&\sigma_t\cdot(X_1,X_2,X_3,Y_1,Y_2,Y_3,Z)\\ &=(e^{a_1t}X_1,e^{a_2t}X_2,e^{a_3t}X_3,e^{b_1t}Y_1,e^{b_2t}Y_2,e^{b_3t}Y_3,e^{ct}Z)\end{aligned}$$

in the affine variety $\mathrm{Spec}(\mathbb{C}[X_1,X_2,X_3,Y_1,Y_2,Y_3,Z])$, which corresponds to the 7-dimensional cone generated by $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_7,\mathbf{v}_8,\mathbf{v}_9,\mathbf{v}_{11}\}$. Here, (X_1,X_2,X_3) are affine coordinates of $(\mathbb{P}^1)^3$, (Y_1,Y_2,Y_3) are affine coordinates of $\mathbb{P}^3$, and Z is an affine coordinate of the fibre. Hence, we have

$$X_1=\frac{x_0}{x_1},X_2=\frac{x_2}{x_3},X_3=\frac{x_4}{x_5},Y_1=\frac{y_0}{y_3},Y_2=\frac{y_1}{y_3},Y_3=\frac{y_2}{y_3},$$

where

$$([x_0 : x_1], [x_2 : x_3], [x_4 : x_5], [y_0 : y_1 : y_2 : y_3])$$

are homogeneous coordinates of $(\mathbb{P}^1)^3 \times \mathbb{P}^3$. Let us see σ_t in terms of another affine coordinates by using the coordinate transformations (see Table 2.1). For generic $\{a_i, b_j, c\}_{1 \leqslant i,j \leqslant 3}$, the set of fixed points of σ_t consists of the following isolated 64 points;

$$\{(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{y}, \mathbf{z}) \in V \mid \mathbf{x}_i,\ \mathbf{z} = \mathbf{p}_- \text{ or } \mathbf{p}_+,\ \ \mathbf{y} = \mathbf{p}_j\ (j = 1, 2, 3, 4)\},$$

where $\mathbf{p}_-$ denotes $[1 : 0]$, $\mathbf{p}_+$ denotes $[0 : 1]$ and $\mathbf{p}_1 = [1 : 0 : 0 : 0]$, $\mathbf{p}_2 = [0 : 1 : 0 : 0]$, $\mathbf{p}_3 = [0 : 0 : 1 : 0]$, $\mathbf{p}_4 = [0 : 0 : 0 : 1]$.

Next we shall calculate $L(X)$ at each fixed point of σ_t. For example, let us consider $L(X)$ at

$$(\mathbf{p}_+, \mathbf{p}_+, \mathbf{p}_+, \mathbf{p}_4, \mathbf{p}_+) = ([0 : 1], [0 : 1], [0 : 1], [0 : 0 : 0 : 1], [0 : 1]).$$

This point is the origin in the affine variety $\mathrm{Spec}(\mathbb{C}[X_1, X_2, X_3, Y_1, Y_2, Y_3, Z])$ associated with the 7-dimensional cone generated by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{11}\}$. The holomorphic vector field with respect to σ_t around the point is expressed by

$$\sum_{i=1}^{3} a_i X_i \frac{\partial}{\partial X_i} + \sum_{j=1}^{3} b_i Y_j \frac{\partial}{\partial Y_j} + cZ \frac{\partial}{\partial Z}.$$

Hence $L(X)$ at $(\mathbf{p}_+, \mathbf{p}_+, \mathbf{p}_+, \mathbf{p}_4, \mathbf{p}_+)$ is given by

$$L(X) = \mathrm{diag}(a_1, a_2, a_3, b_1, b_2, b_3, c).$$

For another example, let us consider $L(X)$ at

$$(\mathbf{p}_-, \mathbf{p}_+, \mathbf{p}_+, \mathbf{p}_1, \mathbf{p}_+) = ([1 : 0], [0 : 1], [0 : 1], [1 : 0 : 0 : 0], [0 : 1]).$$

This point is the origin in $\mathrm{Spec}(\mathbb{C}[X_1^{-1}, X_2, X_3, Y_1^{-1}, Y_2 Y_1^{-1}, Y_3 Y_1^{-1}, Z X_1^{-1} Y_1^2])$ associated with the 7-dimensional cone generated by $\{\mathbf{v}_6, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{11}\}$. The holomorphic vector field with respect to σ_t around the point is expressed by

$$\begin{aligned} -a_1 U_1 \frac{\partial}{\partial U_1} + \sum_{i=2}^{3} a_i U_i \frac{\partial}{\partial U_i} - b_1 U_4 \frac{\partial}{\partial U_4} + \sum_{j=2}^{3} (b_j - b_1) U_{3+j} \frac{\partial}{\partial U_{3+j}} \\ + (c - a_1 + 2b_1) U_7 \frac{\partial}{\partial U_7}, \end{aligned}$$

where

$$\begin{aligned} U_1 := X_1^{-1}, U_2 := X_2, U_3 := X_3, U_4 := Y_1^{-1}, U_5 := Y_2 Y_1^{-1}, U_6 := Y_3 Y_1^{-1}, \\ U_7 := Z X_1^{-1} Y_1^2. \end{aligned}$$

no.	fixed pt	$L(X)$
1-1	$(+++, \mathbf{p}_1, \pm)$	$(a_1, a_2, a_3, b_2 - b_1, b_3 - b_1, -b_1, \pm(c + 2b_1))$
1-2	$(-++, \mathbf{p}_1, \pm)$	$(-a_1, a_2, a_3, b_2 - b_1, b_3 - b_1, -b_1, \pm(c - a_1 + 2b_1))$
1-3	$(+-+, \mathbf{p}_1, \pm)$	$(a_1, -a_2, a_3, b_2 - b_1, b_3 - b_1, -b_1, \pm(c - a_2 + 2b_1))$
1-4	$(++-, \mathbf{p}_1, \pm)$	$(a_1, a_2, -a_3, b_2 - b_1, b_3 - b_1, -b_1, \pm(c - a_3 + 2b_1))$
1-5	$(+--, \mathbf{p}_1, \pm)$	$(a_1, -a_2, -a_3, b_2 - b_1, b_3 - b_1, -b_1, \pm(c - a_2 - a_3 + 2b_1))$
1-6	$(-+-, \mathbf{p}_1, \pm)$	$(-a_1, a_2, -a_3, b_2 - b_1, b_3 - b_1, -b_1, \pm(c - a_1 - a_3 + 2b_1))$
1-7	$(--+, \mathbf{p}_1, \pm)$	$(-a_1, -a_2, a_3, b_2 - b_1, b_3 - b_1, -b_1, \pm(c - a_1 - a_2 + 2b_1))$
1-8	$(---, \mathbf{p}_1, \pm)$	$(-a_1, -a_2, -a_3, b_2 - b_1, b_3 - b_1, -b_1, \pm\big(c - \sum a_i + 2b_1\big))$
2-1	$(+++, \mathbf{p}_2, \pm)$	$(a_1, a_2, a_3, b_1 - b_2, b_3 - b_2, -b_2, \pm(c + 2b_2))$
2-2	$(-++, \mathbf{p}_2, \pm)$	$(-a_1, a_2, a_3, b_1 - b_2, b_3 - b_2, -b_2, \pm(c - a_1 + 2b_2))$
2-3	$(+-+, \mathbf{p}_2, \pm)$	$(a_1, -a_2, a_3, b_1 - b_2, b_3 - b_2, -b_2, \pm(c - a_2 + 2b_2))$
2-4	$(++-, \mathbf{p}_2, \pm)$	$(a_1, a_2, -a_3, b_1 - b_2, b_3 - b_2, -b_2, \pm(c - a_3 + 2b_2))$
2-5	$(+--, \mathbf{p}_2, \pm)$	$(a_1, -a_2, -a_3, b_1 - b_2, b_3 - b_2, -b_2, \pm(c - a_2 - a_3 + 2b_2))$
2-6	$(-+-, \mathbf{p}_2, \pm)$	$(-a_1, a_2, -a_3, b_1 - b_2, b_3 - b_2, -b_2, \pm(c - a_1 - a_3 + 2b_2))$
2-7	$(--+, \mathbf{p}_2, \pm)$	$(-a_1, -a_2, a_3, b_1 - b_2, b_3 - b_2, -b_2, \pm(c - a_1 - a_2 + 2b_2))$
2-8	$(---, \mathbf{p}_2, \pm)$	$(-a_1, -a_2, -a_3, b_1 - b_2, b_3 - b_2, -b_2, \pm\big(c - \sum a_i + 2b_2\big))$
3-1	$(+++, \mathbf{p}_3, \pm)$	$(a_1, a_2, a_3, b_1 - b_3, b_2 - b_3, -b_3, \pm(c + 2b_3))$
3-2	$(-++, \mathbf{p}_3, \pm)$	$(-a_1, a_2, a_3, b_1 - b_3, b_2 - b_3, -b_3, \pm(c - a_1 + 2b_3))$
3-3	$(+-+, \mathbf{p}_3, \pm)$	$(a_1, -a_2, a_3, b_1 - b_3, b_2 - b_3, -b_3, \pm(c - a_2 + 2b_3))$
3-4	$(++-, \mathbf{p}_3, \pm)$	$(a_1, a_2, -a_3, b_1 - b_3, b_2 - b_3, -b_3, \pm(c - a_3 + 2b_3))$
3-5	$(+--, \mathbf{p}_3, \pm)$	$(a_1, -a_2, -a_3, b_1 - b_3, b_2 - b_3, -b_3, \pm(c - a_2 - a_3 + 2b_3))$
3-6	$(-+-, \mathbf{p}_3, \pm)$	$(-a_1, a_2, -a_3, b_1 - b_3, b_2 - b_3, -b_3, \pm(c - a_1 - a_3 + 2b_3))$
3-7	$(--+, \mathbf{p}_3, \pm)$	$(-a_1, -a_2, a_3, b_1 - b_3, b_2 - b_3, -b_3, \pm(c - a_1 - a_2 + 2b_3))$
3-8	$(---, \mathbf{p}_3, \pm)$	$(-a_1, -a_2, -a_3, b_1 - b_3, b_2 - b_3, -b_3, \pm\big(c - \sum a_i + 2b_3\big))$
4-1	$(+++, \mathbf{p}_4, \pm)$	$(a_1, a_2, a_3, b_1, b_2, b_3, \pm c)$
4-2	$(-++, \mathbf{p}_4, \pm)$	$(-a_1, a_2, a_3, b_1, b_2, b_3, \pm(c - a_1))$
4-3	$(+-+, \mathbf{p}_4, \pm)$	$(a_1, -a_2, a_3, b_1, b_2, b_3, \pm(c - a_2))$
4-4	$(++-, \mathbf{p}_4, \pm)$	$(a_1, a_2, -a_3, b_1, b_2, b_3, \pm(c - a_3)))$
4-5	$(+--, \mathbf{p}_4, \pm)$	$(a_1, -a_2, -a_3, b_1, b_2, b_3, \pm(c - a_2 - a_3))$
4-6	$(-+-, \mathbf{p}_4, \pm)$	$(-a_1, a_2, -a_3, b_1, b_2, b_3, \pm(c - a_1 - a_3))$
4-7	$(--+, \mathbf{p}_4, \pm)$	$(-a_1, -a_2, a_3, b_1, b_2, b_3, \pm(c - a_1 - a_2))$
4-8	$(---, \mathbf{p}_4, \pm)$	$(-a_1, -a_2, -a_3, b_1, b_2, b_3, \pm\big(c - \sum a_i\big))$

Table 3.1

Hence $L(X)$ at $(\mathbf{p}_-, \mathbf{p}_+, \mathbf{p}_+, \mathbf{p}_1, \mathbf{p}_+)$ is given by

$$L(X) = \operatorname{diag}(-a_1, a_2, a_3, -b_1, b_2 - b_1, b_3 - b_1, c - a_1 + 2b_1).$$

As for the other fixed points, the computations of $L(X)$ are given by the Table 3.1; As for the notation of the column of fixed points, for example, $(++-, \mathbf{p}_1, -)$ means a fixed point $(\mathbf{p}_+, \mathbf{p}_+, \mathbf{p}_+, \mathbf{p}_1, \mathbf{p}_-) = ([0:1], [0:1], [0:$

$1], [1 : 0 : 0 : 0], [1 : 0])$. Remark that $L(X)$ is non-degenerate at any fixed point for generic $\{a_i, b_j, c\}_{i,j=1,2,3}$.

Finally, we shall list below the results of calculations of $\mathcal{F}_{\mathrm{Td}^p}$ $(2 \leqslant p \leqslant 7)$ with respect to the holomorphic vector field induced by $\{\sigma_t\}$ for generic $\{a_i, b_j, c\}_{i,j=1,2,3}$. As for $p = 1$, it suffices to consider f instead of $\mathcal{F}_{\mathrm{Td}^1}$. We have the localization formula for f independently of Lemma 3.1 (*cf.* [6]). The formula for f is same as (3.2), but it holds without assuming the existence of Kähler-Einstein metrics. By using it (not Lemma 3.1) we can prove that f vanishes on $\mathfrak{h}_0(V)$. See Appendix for the calucation. Combining this and [18], we can prove that V admits Kähler-Einstein metrics independently of [14]. Then, we apply Lemma 3.1 to $\mathcal{F}_{\mathrm{Td}^p}$ $(p \geqslant 2)$. Since the computations are quite enormous, we use the computer algebra system "Maxima".[(1)] However, in order to see that V is a counterexample to Problem 1, it is sufficient to check that $\mathcal{F}_{\mathrm{Td}^p}$ does not vanish for some $\{a_i, b_j, c\}$ and some $2 \leqslant p \leqslant 7$. It is still tough, but it would be able to check without computer. For the readers convenience, we put all the data needed to compute in the case where $(a_1, a_2, a_3, b_1, b_2, b_3, c) = (1, 1, 1, 1, 2, 3, 4)$ and $p = 2$ in Appendix.

$p = 2$**:**

$$\begin{aligned} 12\mathcal{F}_{\mathrm{Td}^2}(X) &= \sum_{\mathbf{q}:\ \text{fixed pt}} \frac{(c_1^2 + c_2)c_1^6(L(X)_{\mathbf{q}})}{\det(L(X)_{\mathbf{q}})} \\ &= \sum_{\mathbf{q}:\ \text{fixed pt}} \frac{(c_2 c_1^6)(L(X)_{\mathbf{q}})}{\det(L(X)_{\mathbf{q}})} \\ &= 13056\Big(\sum a_i - \sum b_i - 2c\Big). \end{aligned} \tag{3.3}$$

$p = 3$**:**

$$\begin{aligned} 24\mathcal{F}_{\mathrm{Td}^3}(X) &= \sum_{\mathbf{q}:\ \text{fixed pt}} \frac{c_2 c_1^6(L(X)_{\mathbf{q}})}{\det(L(X)_{\mathbf{q}})} \\ &= 12\mathcal{F}_{\mathrm{Td}^2}(X) \\ &= 13056\Big(\sum a_i - \sum b_i - 2c\Big). \end{aligned}$$

$p = 4$**:**

$$\begin{aligned} 720\mathcal{F}_{\mathrm{Td}^4}(X) &= \sum_{\mathbf{q}:\ \text{fixed pt}} \frac{(-c_1^4 + 4c_1^2c_2 + 3c_2^2 + c_1c_3 - c_4)c_1^4(L(X)_{\mathbf{q}})}{\det(L(X)_{\mathbf{q}})} \\ &= 94080\Big(\sum a_i - \sum b_i - 2c\Big). \end{aligned}$$

(1) Maxima is available from http://maxima.sourceforge.net/.

$p = 5$**:**

$$1440\mathcal{F}_{\mathrm{Td}^5}(X) = \sum_{\mathbf{q}:\ \text{fixed pt}} \frac{(-c_1^3c_2 + 3c_1c_2^2 + c_1^2c_3 - c_1c_4)c_1^3(L(X)_{\mathbf{q}})}{\det(L(X)_{\mathbf{q}})}$$
$$= 28800\Big(\sum a_i - \sum b_i - 2c\Big).$$

$p = 6$**:**

$$60480\mathcal{F}_{\mathrm{Td}^6}(X) = \sum_{\mathbf{q}:\ \text{fixed pt}} \Bigg(\frac{(2c_1^6 - 12c_1^4c_2 + 11c_1^2c_2^2 + 10c_3^2 + 5c_1^3c_3)c_1^2(L(X)_{\mathbf{q}})}{\det(L(X)_{\mathbf{q}})}$$
$$+ \frac{(11c_1c_2c_3 - c_3^2 - 5c_1^2c_4 - 9c_2c_4 - 2c_1c_5 + 2c_6)c_1^2(L(X)_{\mathbf{q}})}{\det(L(X)_{\mathbf{q}})}\Bigg)$$
$$= 82176\Big(\sum a_i - \sum b_i - 2c\Big).$$

$p = 7$**:**

$$120960\mathcal{F}_{\mathrm{Td}^7}(X) = \sum_{\mathbf{q}:\ \text{fixed pt}} \Bigg(\frac{(11c_1^2c_2c_3 - 9c_1c_2c_4 + 2c_1c_6 - 2c_1^2c_5)c_1(L(X)_{\mathbf{q}})}{\det(L(X)_{\mathbf{q}})}$$
$$+ \frac{(2c_1^3c_4 - c_1c_3^2 - 2c_1^4c_3 + 10c_1c_2^3 - 10c_1^3c_2^2 + 2c_1^5c_2)c_1(L(X)_{\mathbf{q}})}{\det(L(X)_{\mathbf{q}})}\Bigg)$$
$$= 16128\Big(\sum a_i - \sum b_i - 2c\Big).$$

Remark that all $\mathcal{F}_{\mathrm{Td}^p}$ $(2 \leqslant p \leqslant 7)$ are proportional to each other. This result is consistent with the fact that $\dim N_{\mathbb{R}}^{\mathcal{W}(V)} = 1$. Therefore we can conclude that even if a Fano manifold admits Kähler-Einstein metrics (*i.e.*, cscK metrics), $\{\mathcal{F}_{\mathrm{Td}^p}\}_{p=1,\dots,m}$ may not vanish. The proof of the main theorem is completed.

4. The derivatives of the Hilbert series

In [9], Futaki and the first two authors showed a relation between the obstructions to asymptotic Chow semistability and the derivatives of the Hilbert series. In the present section, we will see that we can also show Theorem 1.5 using such relation.

We first review the definition and some properties of the Hilbert series. See [9] for more details. Let V be a toric Fano m-fold and Q be the corresponding Fano polytope. The polar dual P of Q, which is the Delzant polytope of (V, K_V^{-1}) in $M_{\mathbb{R}} \simeq \mathbb{R}^m$, is defined as

$$P := \{w \in \mathbb{R}^m \mid \langle \mathbf{v}_j, w\rangle \geqslant -1\}$$

where $\mathbf{v}_j \in \mathbb{Z}^m$ is a vertex of Q for each j.

We call the convex rational polyhedral cone

$$\mathcal{C}^* := \{y \in \mathbb{R}^{m+1} \mid \langle \lambda_j, y\rangle \geqslant 0\}$$

the toric diagram of V, where $\lambda_j = (\mathbf{v}_j, 1) \in \mathbb{Z}^{m+1}$. Note here that this cone is a pointed cone in $\mathbb{R}^{m+1}$, that is to say, $\mathcal{C}^* \cap (-\mathcal{C}^*) = \{0\}$. We can also represent $\mathcal{C}^*$ by

$$\mathcal{C}^* = \{c_1\boldsymbol{\mu}_1 + \cdots + c_k\boldsymbol{\mu}_k \,|\, c_i \geqslant 0,\ i = 1, \cdots, k\}$$

where $\boldsymbol{\mu}_j = (\mathbf{w}_j, 1) \in \mathbb{Z}^{m+1}$ and $\mathbf{w}_1, \cdots, \mathbf{w}_k$ are the vertices of the Delzant polytope P. Then we can define the (multi-graded) Hilbert series $C(\mathbf{x}, \mathcal{C}^*)$ of the rational cone $\mathcal{C}^*$ by

$$C(\mathbf{x}, \mathcal{C}^*) = \sum_{\mathbf{a} \in \mathcal{C}^* \cap \mathbb{Z}^{m+1}} \mathbf{x}^{\mathbf{a}} \qquad (\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_{m+1}^{a_{m+1}}).$$

As proved in [12], the Hilbert series $C(\mathbf{x}, \mathcal{C}^*)$ can be written as a rational generating function of the form

$$C(\mathbf{x}, \mathcal{C}^*) = \frac{K_{\mathcal{C}^*}(\mathbf{x})}{(1 - \mathbf{x}^{\boldsymbol{\mu}_1}) \cdots (1 - \mathbf{x}^{\boldsymbol{\mu}_k})}$$

where $K_{\mathcal{C}^*}(\mathbf{x})$ is a Laurent polynomial. Using Brion's formula, we are able to calculate the right hand side of the above equation as follows, see [9];

$$C(\mathbf{x}, \mathcal{C}^*) = \sum_{j=1}^{k} \frac{1}{1 - \mathbf{x}^{\boldsymbol{\mu}_j}} \prod_{b=1}^{m} \frac{1}{(1 - \tilde{\mathbf{x}}^{\mathbf{e}_{j,b}})},$$

where $\mathbf{e}_{j,1}, \cdots, \mathbf{e}_{j,m} \in \mathbb{Z}^m$ denote the generators of the edges emanating from a vertex $\mathbf{w}_j$ and $\tilde{\mathbf{x}} = (x_1, \cdots, x_m)$.

Let $\mathcal{C}_R$ be the convex polytope defined as

$$\mathcal{C}_R = \{\xi \in \mathcal{C} \,|\, \xi = (\mathbf{b}, m+1)\},$$

where $\mathcal{C}$ is the interior of the dual cone of $\mathcal{C}^*$. For $\xi = (\mathbf{b}, m+1) \in \mathcal{C}_R$ we write

$$\mathbf{e}^{-t\xi} = (e^{-b_1 t}, \cdots, e^{-b_m t}, e^{-(m+1)t})$$

and consider

$$C(\mathbf{e}^{-t\xi}, \mathcal{C}^*) = \frac{K_{\mathcal{C}^*}(\mathbf{e}^{-t\xi})}{(1 - e^{-t\langle \boldsymbol{\mu}_1, \xi\rangle}) \cdots (1 - e^{-t\langle \boldsymbol{\mu}_k, \xi\rangle})}.$$

For each fixed $\xi \in \mathcal{C}_R$, the Laurent expansion of $C(\mathbf{e}^{-t\xi}, \mathcal{C}^*)$ at $t = 0$ is written as

$$C(\mathbf{e}^{-t\xi}, \mathcal{C}^*) = \frac{C_{-m-1}(\mathbf{b})}{t^{m+1}} + \frac{C_{-m}(\mathbf{b})}{t^m} + \frac{C_{-m+1}(\mathbf{b})}{t^{m-1}} + \cdots.$$

In [9], it was shown that the following relation between the invariants $\mathcal{F}_{\mathrm{Td}^p}$ and the derivatives of $C_i(\mathbf{b})$ at $\mathbf{b} = \mathbf{0}$.

THEOREM 4.1 ([9]). — *The linear span of the derivatives* $d_0C_i(\mathbf{b})$, $i = -m-1, -m, \ldots$, *coincides with the linear span of* $\mathcal{F}_{\mathrm{Td}^1}, \ldots \mathcal{F}_{\mathrm{Td}^m}$ *restricted to* $\mathfrak{t} \otimes \mathbb{C} \simeq \mathbb{C}^m$.

Let V be the toric Fano manifold defined by (2.1) and P be the Delzant polytope of (V, K_V^{-1}). Then we can calculate $\{\mathbf{w}_j, \mathbf{e}_{j,b}\}_{(1\leqslant j\leqslant 64,\ 1\leqslant b\leqslant 7)}$ explicitly from the argument in Section 2, and so we see whether all $\mathcal{F}_{Td^p}$ vanish or not. Note that these 64 verticies of P correspond to the facets of the Fano polytope defined by (2.1). However it is difficult to check it directly, because in our case the Hilbert series has too many terms.

To solve this problem, we use the following proposition.

PROPOSITION 4.2. — *If* $\mathcal{F}_{\mathrm{Td}^p} = 0$ *for each* $p = 1, \ldots, m$ *then*

$$\frac{\partial}{\partial x} C(x^{n_1}, \ldots, x^{n_m}, e^{-(m+1)t})_{|x=1} = 0 \tag{4.1}$$

for any $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{N}^m$.

Proof. — If $\mathcal{F}_{\mathrm{Td}^p} = 0$ for each $p = 1, \ldots, m$ then

$$\frac{\partial}{\partial b_i} C(e^{-b_1 t}, \ldots, e^{-b_m t}, e^{-(m+1)t})_{|\mathbf{b}=0} = 0$$

holds for each $i = 1, \ldots, m$ by Theorem 4.1. Hence we easily see the proposition by the chain rule. □

The left hand side of (4.1) for the toric Fano 7-fold associated with (2.1) is computable with computer. The combinatorial data we need is in Appendix. For example, in consequence of the Maple calculation, we can see

$$\frac{\partial}{\partial x} C(x^{n_1}, \cdots, x^{n_7}, e^{-8t})_{|x=1} = -\frac{184e^{-8t}(2e^{-32t} + 31e^{-24t} + 70e^{-16t} + 31e^{-8t} + 2)}{(-1 + e^{-8t})^7} \neq 0$$

for $(n_1, n_2, n_3, n_4, n_5, n_6, n_7) = (1, 2, 3, 4, 5, 6, 7)$.

Thus, by Proposition 4.2, Theorem 1.5 has been proved.

5. Appendix

5.1. Combinatorial data of the Nill-Paffenholz's example

In this subsection we shall list up the necessary combinatorial data of Nill-Paffenholz's example for the calculation in Section 4.

- The vertices of the Fano polytope Q are given by (2.1).
- The 64 vertices of the polar polytope P are given by

$$(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \cdots, \mathbf{w}_{64}) = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & 2 & -1 & -1 & -1 & 2 & -1 & -1 & -1 & 2 & -1 \\ -1 & -1 & -1 & -1 & -1 & 2 & -1 & -1 & -1 & 2 & -1 & -1 & -1 & 2 & -1 & -1 \\ -1 & -1 & -1 & -1 & 2 & -1 & -1 & -1 & 2 & -1 & -1 & -1 & 2 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 5 & -1 & -1 & -1 & -1 \\ -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 5 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & -1 & 2 & -1 & -1 & 2 & -1 & 2 & -1 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & -1 & 0 \\ -1 & 2 & -1 & -1 & 2 & -1 & -1 & 2 & 0 & -1 & 2 & 2 & 2 & 2 & 2 & -1 & 2 & 0 \\ 2 & -1 & -1 & -1 & 2 & 2 & 0 & -1 & -1 & -1 & 2 & 2 & 2 & 2 & -1 & 2 & 2 & 0 \\ 1 & 1 & 1 & 5 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 5 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 & 2 & -1 & 0 & 2 & 2 & -1 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & -1 \\ 0 & 2 & -1 & 2 & 0 & 2 & -1 & 2 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 \\ 0 & -1 & 2 & 2 & 0 & -1 & 2 & 2 & 0 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 5 & -1 & -1 & -1 & -1 & -1 & -1 & 5 & 5 \\ -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 5 & -1 & 5 & 5 & 5 & -1 & -1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 & 5 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 5 & 5 & 5 & -1 & -1 \\ 1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\left.\begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 5 & -1 & -1 & -1 & 5 & -1 & -1 & -1 & -1 & -1 & -1 & 5 \\ -1 & -1 & -1 & -1 & -1 & 5 & -1 & -1 & 5 & 5 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 5 & 5 & -1 & -1 & 5 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}\right).$$

- The neighbors of each vertex of P are listed in Table 5.1 (Here, vertices v and u of P are called neighbors if the interval $[u, v]$ is an edge of P). The other sets of neighbors unlisted in Table 5.1 can be obtained by the symmetry of V.

5.2. Computation data in Section 3

In this subsection, we list all of the data, which are needed to compute f and $\mathcal{F}_{\mathrm{Td}^2}$. First, we compute that $f \equiv 0$ by using its original localization formula. Since f is a linear function in the variables a_i,b_j, c $(1 \leqslant i, j \leqslant 3)$ and is symmetric among $\{a_i\}_{1 \leqslant i \leqslant 3}$ and among $\{b_j\}_{1 \leqslant j \leqslant 3}$ due to the symmetry of V, we can write

$$f(X) = A \sum_{i=1}^{3} a_i + B \sum_{j=1}^{3} b_j + Cc$$

vertex	associated cone	neighbors
$\mathbf{w}_1$	$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{11}\}$	$\mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5, \mathbf{w}_6, \mathbf{w}_7, \mathbf{w}_8$
$\mathbf{w}_3$	$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_{10}, \mathbf{v}_{11}\}$	$\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4, \mathbf{w}_{13}, \mathbf{w}_{14}, \mathbf{w}_{15}, \mathbf{w}_{16}$
$\mathbf{w}_7$	$\{\mathbf{v}_6, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{11}\}$	$\mathbf{w}_1, \mathbf{w}_{11}, \mathbf{w}_{15}, \mathbf{w}_{19}, \mathbf{w}_{22}, \mathbf{w}_{24}, \mathbf{w}_{26}$
$\mathbf{w}_8$	$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{12}\}$	$\mathbf{w}_1, \mathbf{w}_{12}, \mathbf{w}_{16}, \mathbf{w}_{20}, \mathbf{w}_{23}, \mathbf{w}_{25}, \mathbf{w}_{26}$
$\mathbf{w}_{19}$	$\{\mathbf{v}_6, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{11}\}$	$\mathbf{w}_4, \mathbf{w}_7, \mathbf{w}_{11}, \mathbf{w}_{15}, \mathbf{w}_{31}, \mathbf{w}_{32}, \mathbf{w}_{53}$
$\mathbf{w}_{20}$	$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{12}\}$	$\mathbf{w}_4, \mathbf{w}_8, \mathbf{w}_{12}, \mathbf{w}_{16}, \mathbf{w}_{51}, \mathbf{w}_{52}, \mathbf{w}_{53}$
$\mathbf{w}_{24}$	$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{11}\}$	$\mathbf{w}_6, \mathbf{w}_7, \mathbf{w}_{28}, \mathbf{w}_{31}, \mathbf{w}_{36}, \mathbf{w}_{40}, \mathbf{w}_{56}$
$\mathbf{w}_{26}$	$\{\mathbf{v}_6, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{12}\}$	$\mathbf{w}_7, \mathbf{w}_8, \mathbf{w}_{47}, \mathbf{w}_{50}, \mathbf{w}_{53}, \mathbf{w}_{55}, \mathbf{w}_{56}$
$\mathbf{w}_{27}$	$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_4, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{11}\}$	$\mathbf{w}_{28}, \mathbf{w}_{29}, \mathbf{w}_{30}, \mathbf{w}_{31}, \mathbf{w}_{32}, \mathbf{w}_{33}, \mathbf{w}_{34}$
$\mathbf{w}_{28}$	$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_4, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{11}\}$	$\mathbf{w}_{21}, \mathbf{w}_{22}, \mathbf{w}_{24}, \mathbf{w}_{27}, \mathbf{w}_{29}, \mathbf{w}_{30}, \mathbf{w}_{35}$
$\mathbf{w}_{31}$	$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_3, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{11}\}$	$\mathbf{w}_{18}, \mathbf{w}_{19}, \mathbf{w}_{24}, \mathbf{w}_{27}, \mathbf{w}_{36}, \mathbf{w}_{40}, \mathbf{w}_{44}$
$\mathbf{w}_{34}$	$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_4, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{12}\}$	$\mathbf{w}_{27}, \mathbf{w}_{35}, \mathbf{w}_{39}, \mathbf{w}_{43}, \mathbf{w}_{44}, \mathbf{w}_{57}, \mathbf{w}_{64}$
$\mathbf{w}_{35}$	$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_4, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{12}\}$	$\mathbf{w}_{28}, \mathbf{w}_{34}, \mathbf{w}_{39}, \mathbf{w}_{43}, \mathbf{w}_{54}, \mathbf{w}_{55}, \mathbf{w}_{56}$
$\mathbf{w}_{44}$	$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_3, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{12}\}$	$\mathbf{w}_{31}, \mathbf{w}_{34}, \mathbf{w}_{52}, \mathbf{w}_{53}, \mathbf{w}_{56}, \mathbf{w}_{60}, \mathbf{w}_{61}$
$\mathbf{w}_{53}$	$\{\mathbf{v}_6, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{10}, \mathbf{v}_{12}\}$	$\mathbf{w}_{19}, \mathbf{w}_{20}, \mathbf{w}_{26}, \mathbf{w}_{44}, \mathbf{w}_{47}, \mathbf{w}_{50}, \mathbf{w}_{57}$
$\mathbf{w}_{56}$	$\{\mathbf{v}_6, \mathbf{v}_5, \mathbf{v}_3, \mathbf{v}_7, \mathbf{v}_8, \mathbf{v}_9, \mathbf{v}_{12}\}$	$\mathbf{w}_{24}, \mathbf{w}_{25}, \mathbf{w}_{26}, \mathbf{w}_{35}, \mathbf{w}_{44}, \mathbf{w}_{60}, \mathbf{w}_{61}$

Table 5.1

for some real numbers A, B and C. To show the vanishing of f, it suffices to show that f vanishes with respect to at least three cases, for example,

$$(a_1, a_2, a_3, b_1, b_2, b_3, c) = \begin{cases} (-1, -1, -1, 1, 2, 3, 1) \\ (1, 1, 1, 1, 2, 3, 4) \\ (-2, -2, -2, 1, 2, 3, 1). \end{cases}$$

The data of the first case is given in Table 5.2. In the columns of $c_1(L(X))$ in Table 5.2, the first element corresponds to (+)-case and the other to (−)-case. Our computation is divided into the four parts of Table 5.2, which are labeled $\{(1\text{-}i)\}_{1\leqslant i\leqslant 8}$, $\{(2\text{-}i)\}_{1\leqslant i\leqslant 8}$, $\{(3\text{-}i)\}_{1\leqslant i\leqslant 8}$ and $\{(4\text{-}i)\}_{1\leqslant i\leqslant 8}$. The sum among $\{(1\text{-}i)\}_{1\leqslant i\leqslant 8}$ is given by

$$\frac{2^8}{6} - \frac{(-4)^8}{6} + 3\left(-\frac{5^8}{8} + \frac{(-3)^8}{8}\right) + 3\left(\frac{8^8}{10} - \frac{(-2)^8}{10}\right) - \frac{11^8}{12} + \frac{(-1)^8}{12} = -12985056.$$

The sum among $\{(2\text{-}i)\}_{1\leqslant i\leqslant 8}$ is given by

$$-\frac{0^8}{10} + \frac{(-10)^8}{10} + 3\left(\frac{3^8}{12} - \frac{(-9)^8}{12}\right) + 3\left(-\frac{6^8}{14} + \frac{(-8)^8}{14}\right) + \frac{9^8}{16} - \frac{(-7)^8}{16} = 4805280.$$

no.	fixed pt	$L(X)$	$\det L(X)$	$c_1(L(X))$
1-1	$(+++, \mathbf{p}_1, \pm)$	$(-1,-1,-1,1,2,-1,\pm 3)$	± 6	$(2,-4)$
1-2	$(-++, \mathbf{p}_1, \pm)$	$(1,-1,-1,1,2,-1,\pm 4)$	∓ 8	$(5,-3)$
1-3	$(+-+, \mathbf{p}_1, \pm)$	$(-1,1,-1,1,2,-1,\pm 4)$	∓ 8	$(5,-3)$
1-4	$(++-, \mathbf{p}_1, \pm)$	$(-1,-1,1,1,2,-1,\pm 4)$	∓ 8	$(5,-3)$
1-5	$(+--, \mathbf{p}_1, \pm)$	$(-1,1,1,1,2,-1,\pm 5)$	± 10	$(8,-2)$
1-6	$(-+-, \mathbf{p}_1, \pm)$	$(1,-1,1,1,2,-1,\pm 5)$	± 10	$(8,-2)$
1-7	$(--+, \mathbf{p}_1, \pm)$	$(1,1,-1,1,2,-1,\pm 5)$	± 10	$(8,-2)$
1-8	$(---, \mathbf{p}_1, \pm)$	$(1,1,1,1,2,-1,\pm 6)$	∓ 12	$(11,-1)$
2-1	$(+++, \mathbf{p}_2, \pm)$	$(-1,-1,-1,-1,1,-2,\pm 5)$	∓ 10	$(0,-10)$
2-2	$(-++, \mathbf{p}_2, \pm)$	$(1,-1,-1,-1,1,-2,\pm 6)$	± 12	$(3,-9)$
2-3	$(+-+, \mathbf{p}_2, \pm)$	$(-1,1,-1,-1,1,-2,\pm 6)$	± 12	$(3,-9)$
2-4	$(++-, \mathbf{p}_2, \pm)$	$(-1,-1,1,-1,1,-2,\pm 6)$	± 12	$(3,-9)$
2-5	$(+--, \mathbf{p}_2, \pm)$	$(-1,1,1,-1,1,-2,\pm 7)$	∓ 14	$(6,-8)$
2-6	$(-+-, \mathbf{p}_2, \pm)$	$(1,-1,1,-1,1,-2,\pm 7)$	∓ 14	$(6,-8)$
2-7	$(--+, \mathbf{p}_2, \pm)$	$(1,1,-1,-1,1,-2,\pm 7)$	∓ 14	$(6,-8)$
2-8	$(---, \mathbf{p}_2, \pm)$	$(1,1,1,-1,1,-2,\pm 8)$	± 16	$(9,-7)$
3-1	$(+++, \mathbf{p}_3, \pm)$	$(-1,-1,-1,-2,-1,-3,\pm 7)$	± 42	$(-2,-16)$
3-2	$(-++, \mathbf{p}_3, \pm)$	$(1,-1,-1,-2,-1,-3,\pm 8)$	∓ 48	$(1,-15)$
3-3	$(+-+, \mathbf{p}_3, \pm)$	$(-1,1,-1,-2,-1,-3,\pm 8)$	∓ 48	$(1,-15)$
3-4	$(++-, \mathbf{p}_3, \pm)$	$(-1,-1,1,-2,-1,-3,\pm 9)$	∓ 48	$(1,-15)$
3-5	$(+--, \mathbf{p}_3, \pm)$	$(-1,1,1,-2,-1,-3,\pm 9)$	± 54	$(4,-14)$
3-6	$(-+-, \mathbf{p}_3, \pm)$	$(1,-1,1,-2,-1,-3,\pm 9)$	± 54	$(4,-14)$
3-7	$(--+, \mathbf{p}_3, \pm)$	$(1,1,-1,-2,-1,-3,\pm 9)$	∓ 54	$(4,-14)$
3-8	$(---, \mathbf{p}_3, \pm)$	$(1,1,1,-2,-1,-3,\pm 10)$	∓ 60	$(7,-13)$
4-1	$(+++, \mathbf{p}_4, \pm)$	$(-1,-1,-1,1,2,3,\pm 1)$	∓ 6	$(4,2)$
4-2	$(-++, \mathbf{p}_4, \pm)$	$(1,-1,-1,1,2,3,\pm 2)$	± 12	$(7,3)$
4-3	$(+-+, \mathbf{p}_4, \pm)$	$(-1,1,-1,1,2,3,\pm 2)$	± 12	$(7,3)$
4-4	$(++-, \mathbf{p}_4, \pm)$	$(-1,-1,1,1,2,3,\pm 3)$	± 12	$(7,3)$
4-5	$(+--, \mathbf{p}_4, \pm)$	$(-1,1,1,1,2,3,\pm 3)$	∓ 18	$(10,4)$
4-6	$(-+-, \mathbf{p}_4, \pm)$	$(1,-1,1,1,2,3,\pm 3)$	∓ 18	$(10,4)$
4-7	$(--+, \mathbf{p}_4, \pm)$	$(1,1,-1,1,2,3,\pm 3)$	∓ 18	$(10,4)$
4-8	$(---, \mathbf{p}_4, \pm)$	$(1,1,1,1,2,3,\pm 4)$	± 24	$(13,5)$

Table 5.2

The sum among $\{(3\text{-}i)\}_{1\leqslant i\leqslant 8}$ is given by

$$\frac{(-2)^8}{42}-\frac{(-16)^8}{42}+3\left(-\frac{1^8}{48}+\frac{(-15)^8}{48}\right)+3\left(\frac{4^8}{54}-\frac{(-14)^8}{54}\right)-\frac{7^8}{60}+\frac{(-13)^8}{60}$$
$$=-10565664.$$

The sum among $\{(4\text{-}i)\}_{1\leqslant i\leqslant 8}$ is given by

$$-\frac{4^8}{6}+\frac{2^8}{6}+3\left(\frac{7^8}{12}-\frac{3^8}{12}\right)+3\left(-\frac{10^8}{18}+\frac{4^8}{18}\right)+\frac{13^8}{24}-\frac{5^8}{24}=18745440.$$

Then, the total sum is equal to zero.

The data of the second case is given in Table 5.4. In this case, Table 5.4 coincides with Table 5.2 up to order. For example, the row (1-1) and (1-2) in Table 5.4 coincide with the row (1-8) and (1-7) in Table 5.2 respectively. Hence, f in this case also vanishes.

The data of the third case is given in Table 5.3.

Then, the sum among $\{(1\text{-}i)\}_{1\leqslant i\leqslant 8}$ is given by

$$\frac{(-1)^8}{48}-\frac{(-7)^8}{48}+3\left(-\frac{5^8}{80}+\frac{(-5)^8}{80}\right)+3\left(\frac{11^8}{112}-\frac{(-3)^8}{112}\right)-\frac{17^8}{114}+\frac{(-1)^8}{114}$$
$$=-42821280.$$

The sum among $\{(2\text{-}i)\}_{1\leqslant i\leqslant 8}$ is given by

$$-\frac{(-3)^8}{80}+\frac{(-13)^8}{80}+3\left(\frac{3^8}{112}-\frac{(-11)^8}{112}\right)+3\left(-\frac{9^8}{144}+\frac{(-9)^8}{144}\right)+\frac{15^8}{176}-\frac{(-7)^8}{176}$$
$$=18984096.$$

The sum among $\{(3\text{-}i)\}_{1\leqslant i\leqslant 8}$ is given by

$$\frac{(-5)^8}{336}-\frac{(-19)^8}{336}+3\left(-\frac{1^8}{432}+\frac{(-17)^8}{432}\right)+3\left(\frac{7^8}{528}-\frac{(-15)^8}{528}\right)-\frac{13^8}{624}+\frac{(-13)^8}{624}$$
$$=-16631520.$$

The sum among $\{(4\text{-}i)\}_{1\leqslant i\leqslant 8}$ is given by

$$-\frac{1^8}{48}+\frac{(-1)^8}{48}+3\left(\frac{7^8}{144}-\frac{1^8}{144}\right)+3\left(-\frac{13^8}{240}+\frac{3^8}{20}\right)+\frac{19^8}{336}-\frac{5^8}{336}=40468704.$$

Then, the total sum is equal to zero.

Next, we compute $\mathcal{F}_{\mathrm{Td}^2}(X)$, where X is the holomorphic vector field associated with σ_t for when $(a_1,a_2,a_3,b_1,b_2,b_3,c)=(1,1,1,1,2,3,4)$.

The data is given in Table 5.4. Since $\mathcal{F}_{\mathrm{Td}^1}$ vanishes, it is sufficient to check that

$$\sum_{\mathbf{q}:\,\text{fixed pt}}\frac{(c_2c_1^6)(L(X)_{\mathbf{q}})}{\det(L(X)_{\mathbf{q}})} \tag{5.1}$$

no.	fixed pt	$L(X)$	$\det L(X)$	$c_1(L(X))$
1-1	$(+++, \mathbf{p}_1, \pm)$	$(-2,-2,-2,1,2,-1,\pm 3)$	± 48	$(-1,-7)$
1-2	$(-++, \mathbf{p}_1, \pm)$	$(2,-2,-2,1,2,-1,\pm 5)$	∓ 80	$(5,-5)$
1-3	$(+-+, \mathbf{p}_1, \pm)$	$(-2,2,-2,1,2,-1,\pm 5)$	∓ 80	$(5,-5)$
1-4	$(++-, \mathbf{p}_1, \pm)$	$(-2,-2,2,1,2,-1,\pm 5)$	∓ 80	$(5,-5)$
1-5	$(+--, \mathbf{p}_1, \pm)$	$(-2,2,2,1,2,-1,\pm 7)$	± 112	$(11,-3)$
1-6	$(-+-, \mathbf{p}_1, \pm)$	$(2,-2,2,1,2,-1,\pm 7)$	± 112	$(11,-3)$
1-7	$(--+, \mathbf{p}_1, \pm)$	$(2,2,-2,1,2,-1,\pm 7)$	± 112	$(11,-3)$
1-8	$(---, \mathbf{p}_1, \pm)$	$(2,2,2,1,2,-1,\pm 9)$	∓ 144	$(17,-1)$
2-1	$(+++, \mathbf{p}_2, \pm)$	$(-2,-2,-2,-1,1,-2,\pm 5)$	∓ 80	$(-3,-13)$
2-2	$(-++, \mathbf{p}_2, \pm)$	$(2,-2,-2,-1,1,-2,\pm 7)$	± 112	$(3,-11)$
2-3	$(+-+, \mathbf{p}_2, \pm)$	$(-2,2,-2,-1,1,-2,\pm 7)$	± 112	$(3,-11)$
2-4	$(++-, \mathbf{p}_2, \pm)$	$(-2,-2,2,-1,1,-2,\pm 7)$	± 112	$(3,-11)$
2-5	$(+--, \mathbf{p}_2, \pm)$	$(-2,2,2,-1,1,-2,\pm 9)$	∓ 144	$(9,-9)$
2-6	$(-+-, \mathbf{p}_2, \pm)$	$(2,-2,2,-1,1,-2,\pm 9)$	∓ 144	$(9,-9)$
2-7	$(--+, \mathbf{p}_2, \pm)$	$(2,2,-2,-1,1,-2,\pm 9)$	∓ 144	$(9,-9)$
2-8	$(---, \mathbf{p}_2, \pm)$	$(2,2,2,-1,1,-2,\pm 11)$	± 176	$(15,-7)$
3-1	$(+++, \mathbf{p}_3, \pm)$	$(-2,-2,-2,-2,-1,-3,\pm 7)$	± 336	$(-5,-19)$
3-2	$(-++, \mathbf{p}_3, \pm)$	$(2,-2,-2,-2,-1,-3,\pm 9)$	∓ 432	$(1,-17)$
3-3	$(+-+, \mathbf{p}_3, \pm)$	$(-2,2,-2,-2,-1,-3,\pm 9)$	∓ 432	$(1,-17)$
3-4	$(++-, \mathbf{p}_3, \pm)$	$(-2,-2,2,-2,-1,-3,\pm 9)$	∓ 432	$(1,-17)$
3-5	$(+--, \mathbf{p}_3, \pm)$	$(-2,2,2,-2,-1,-3,\pm 11)$	± 528	$(7,-15)$
3-6	$(-+-, \mathbf{p}_3, \pm)$	$(2,-2,2,-2,-1,-3,\pm 11)$	± 528	$(7,-15)$
3-7	$(--+, \mathbf{p}_3, \pm)$	$(2,2,-2,-2,-1,-3,\pm 11)$	± 528	$(7,-15)$
3-8	$(---, \mathbf{p}_3, \pm)$	$(2,2,2,-2,-1,-3,\pm 13)$	∓ 624	$(13,-13)$
4-1	$(+++, \mathbf{p}_4, \pm)$	$(-2,-2,-2,1,2,3,\pm 1)$	∓ 48	$(1,-1)$
4-2	$(-++, \mathbf{p}_4, \pm)$	$(2,-2,-2,1,2,3,\pm 3)$	± 144	$(7,1)$
4-3	$(+-+, \mathbf{p}_4, \pm)$	$(-2,2,-2,1,2,3,\pm 3)$	± 144	$(7,1)$
4-4	$(++-, \mathbf{p}_4, \pm)$	$(-2,-2,2,1,2,3,\pm 3)$	± 144	$(7,1)$
4-5	$(+--, \mathbf{p}_4, \pm)$	$(-2,2,2,1,2,3,\pm 5)$	∓ 240	$(13,3)$
4-6	$(-+-, \mathbf{p}_4, \pm)$	$(2,-2,2,1,2,3,\pm 5)$	∓ 240	$(13,3)$
4-7	$(--+, \mathbf{p}_4, \pm)$	$(2,2,-2,1,2,3,\pm 5)$	∓ 240	$(13,3)$
4-8	$(---, \mathbf{p}_4, \pm)$	$(2,2,2,1,2,3,\pm 7)$	± 336	$(19,5)$

Table 5.3

does not vanish. Then, we calculate (5.1) separately as follows. The sum among $\{(1\text{-}i)\}_{1\leqslant i\leqslant 8}$ is given by

$$\begin{aligned}&-\frac{38\cdot 11^6}{12}-\frac{22}{12}+3\left(\frac{15\cdot 8^6}{10}+\frac{15\cdot 2^6}{10}\right)-3\frac{8\cdot 3^6}{8}-\frac{7\cdot 2^6}{6}+\frac{4^6}{6}\\&=-4431588,\end{aligned}$$

no.	fixed pt	$L(X)$	$\det L(X)$	$c_1(L(X))$	$c_2(L(X))$
1-1	$(++, \mathbf{p}_1, \pm)$	$(1,1,1,1,2,-1,\pm 6)$	∓ 12	$(11,-1)$	$(38,-22)$
1-2	$(-+, \mathbf{p}_1, \pm)$	$(-1,1,1,1,2,-1,\pm 5)$	± 10	$(8,-2)$	$(15,-15)$
1-3	$(+-, \mathbf{p}_1, \pm)$	$(1,-1,1,1,2,-1,\pm 5)$	± 10	$(8,-2)$	$(15,-15)$
1-4	$(++, \mathbf{p}_1, \pm)$	$(1,1,-1,1,2,-1,\pm 5)$	± 10	$(8,-2)$	$(15,-15)$
1-5	$(+--, \mathbf{p}_1, \pm)$	$(1,-1,-1,1,2,-1,\pm 4)$	∓ 8	$(5,-3)$	$(0,-8)$
1-6	$(-+-, \mathbf{p}_1, \pm)$	$(-1,1,-1,1,2,-1,\pm 4)$	∓ 8	$(5,-3)$	$(0,-8)$
1-7	$(--+, \mathbf{p}_1, \pm)$	$(-1,-1,1,1,2,-1,\pm 4)$	∓ 8	$(5,-3)$	$(0,-8)$
1-8	$(---, \mathbf{p}_1, \pm)$	$(-1,-1,-1,1,2,-1,\pm 3)$	± 6	$(2,-4)$	$(-7,-1)$
2-1	$(++, \mathbf{p}_2, \pm)$	$(1,1,1,-1,1,-2,\pm 8)$	± 16	$(9,-7)$	$(4,-12)$
2-2	$(-+, \mathbf{p}_2, \pm)$	$(-1,1,1,-1,1,-2,\pm 7)$	∓ 14	$(6,-8)$	$(-11,3)$
2-3	$(+-, \mathbf{p}_2, \pm)$	$(1,-1,1,-1,1,-2,\pm 7)$	∓ 14	$(6,-8)$	$(-11,3)$
2-4	$(++, \mathbf{p}_2, \pm)$	$(1,1,-1,-1,1,-2,\pm 7)$	∓ 14	$(6,-8)$	$(-11,3)$
2-5	$(+--, \mathbf{p}_2, \pm)$	$(1,-1,-1,-1,1,-2,\pm 6)$	± 12	$(3,-9)$	$(-18,18)$
2-6	$(-+-, \mathbf{p}_2, \pm)$	$(-1,1,-1,-1,1,-2,\pm 6)$	± 12	$(3,-9)$	$(-18,18)$
2-7	$(--+, \mathbf{p}_2, \pm)$	$(-1,-1,1,-1,1,-2,\pm 6)$	± 12	$(3,-9)$	$(-18,18)$
2-8	$(---, \mathbf{p}_2, \pm)$	$(-1,-1,-1,-1,1,-2,\pm 5)$	∓ 10	$(0,-10)$	$(-17,33)$
3-1	$(++, \mathbf{p}_3, \pm)$	$(1,1,1,-2,-1,-3,\pm 10)$	∓ 60	$(7,-13)$	$(-34,26)$
3-2	$(-+, \mathbf{p}_3, \pm)$	$(-1,1,1,-2,-1,-3,\pm 9)$	± 54	$(4,-14)$	$(-41,49)$
3-3	$(+-, \mathbf{p}_3, \pm)$	$(1,-1,1,-2,-1,-3,\pm 9)$	± 54	$(4,-14)$	$(-41,49)$
3-4	$(++, \mathbf{p}_3, \pm)$	$(1,1,-1,-2,-1,-3,\pm 9)$	± 54	$(4,-14)$	$(-41,49)$
3-5	$(+--, \mathbf{p}_3, \pm)$	$(1,-1,-1,-2,-1,-3,\pm 8)$	∓ 48	$(1,-15)$	$(-40,72)$
3-6	$(-+-, \mathbf{p}_3, \pm)$	$(-1,1,-1,-2,-1,-3,\pm 8)$	∓ 48	$(1,-15)$	$(-40,72)$
3-7	$(--+, \mathbf{p}_3, \pm)$	$(-1,-1,1,-2,-1,-3,\pm 8)$	∓ 48	$(1,-15)$	$(-40,72)$
3-8	$(---, \mathbf{p}_3, \pm)$	$(-1,-1,-1,-2,-1,-3,\pm 7)$	± 42	$(-2,-16)$	$(-31,95)$
4-1	$(++, \mathbf{p}_4, \pm)$	$(1,1,1,1,2,3,\pm 4)$	± 24	$(13,5)$	$(68,-4)$
4-2	$(-+, \mathbf{p}_4, \pm)$	$(-1,1,1,1,2,3,\pm 3)$	∓ 18	$(10,4)$	$(37,-5)$
4-3	$(+-, \mathbf{p}_4, \pm)$	$(1,-1,1,1,2,3,\pm 3)$	∓ 18	$(10,4)$	$(37,-5)$
4-4	$(++, \mathbf{p}_4, \pm)$	$(1,1,-1,1,2,3,\pm 3)$	∓ 18	$(10,4)$	$(37,-5)$
4-5	$(+--, \mathbf{p}_4, \pm)$	$(1,-1,-1,1,2,3,\pm 2)$	± 12	$(7,3)$	$(14,-6)$
4-6	$(-+-, \mathbf{p}_4, \pm)$	$(-1,1,-1,1,2,3,\pm 2)$	± 12	$(7,3)$	$(14,-6)$
4-7	$(--+, \mathbf{p}_4, \pm)$	$(-1,-1,1,1,2,3,\pm 2)$	± 12	$(7,3)$	$(14,-6)$
4-8	$(---, \mathbf{p}_4, \pm)$	$(-1,-1,-1,1,2,3,\pm 1)$	∓ 6	$(4,2)$	$(-1,-7)$

Table 5.4

the sum among $\{(2\text{-}i)\}_{1\leqslant i\leqslant 8}$ is given by

$$\frac{4\cdot 9^6}{16}+\frac{12\cdot 7^6}{16}+3\left(\frac{11\cdot 6^6}{14}+\frac{3\cdot 8^6}{14}\right)-3\left(\frac{18\cdot 3^6}{12}+\frac{18\cdot 9^6}{12}\right)+\frac{33\cdot 10^6}{10}$$
$$=1404828,$$

the sum among $\{(3\text{-}i)\}_{1\leqslant i\leqslant 8}$ is given by

$$\frac{34\cdot 7^6}{60}+\frac{26\cdot 13^6}{60}-3\left(\frac{41\cdot 4^6}{54}+\frac{49\cdot 14^6}{54}\right)+3\left(\frac{40}{48}+\frac{72\cdot 15^6}{48}\right)$$
$$-\frac{31\cdot 2^6}{42}-\frac{95\cdot 16^6}{42}=-5038812,$$

and the sum among $\{(4\text{-}i)\}_{1\leqslant i\leqslant 8}$ is given by

$$\frac{68\cdot 13^6}{24}+\frac{4\cdot 5^6}{24}-3\left(\frac{37\cdot 10^6}{18}+\frac{5\cdot 4^6}{18}\right)+3\left(\frac{14\cdot 7^6}{12}+\frac{6\cdot 3^6}{12}\right)$$
$$+\frac{4^6}{6}-\frac{7\cdot 2^6}{6}=7921956.$$

Therefore, we get the total sum

$$\sum_{\mathbf{q}:\,\text{fixed pt}}\frac{(c_2c_1^6)(L(X)_{\mathbf{q}})}{\det(L(X)_{\mathbf{q}})}=-4431588+1404828-5038812+7921956$$
$$=-143616$$
$$=13056\times(-11)\neq 0. \tag{5.2}$$

The last equality (5.2) confirms the result (3.3) in Section 3.

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Hajime ONO
Tokyo University of Science
Faculty of Science and Technology
Department of Mathematics
2641 Yamazaki, Noda
Chiba 278-8510 (Japan)
ono_hajime@ma.noda.tus.ac.jp

Yuji SANO
Kumamoto University
Graduate School of Science and Technology
2-39-1, Kurokami
Kumamoto, 860-8555 (Japan)
sano@sci.kumamoto-u.ac.jp

Naoto YOTSUTANI
University of Science and Technology of China
School of Mathematics
Hefei, Anhui
230026 P.R. (China)
naotoy@ustc.edu.cn