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# ANALYSIS OF JOINT SPECTRAL MULTIPLIERS ON LIE GROUPS OF POLYNOMIAL GROWTH

**by Alessio MARTINI**

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Abstract. — We study the problem of $L^p$-boundedness ($1 < p < \infty$) of operators of the form $m(L_1, \dots, L_n)$ for a commuting system of self-adjoint left-invariant differential operators $L_1, \dots, L_n$ on a Lie group $G$ of polynomial growth, which generate an algebra containing a weighted subcoercive operator. In particular, when $G$ is a homogeneous group and $L_1, \dots, L_n$ are homogeneous, we prove analogues of the Mihlin-Hörmander and Marcinkiewicz multiplier theorems.

Résumé. — On étudie la bornitude $L^p$ ($1 < p < \infty$) des opérateurs de la forme $m(L_1, \dots, L_n)$ pour un système commutatif $L_1, \dots, L_n$ d'opérateurs différentiels autoadjoints invariants à gauche sur un groupe de Lie $G$ à croissance polynomiale, qui engendrent une algèbre contenant un opérateur sous-coercif pondéré. En particulier, quand $G$ est un groupe homogène et $L_1, \dots, L_n$ sont homogènes, on prouve des analogues des theorèmes de multiplicateurs de Mihlin-Hörmander et Marcinkiewicz.

## 1. Introduction

Let $(X, \mu)$ be a measure space, and let $L_1, \dots, L_n$ be (possibly unbounded) self-adjoint operators on $L^2(X, \mu)$ which commute strongly, *i.e.*, which admit a joint spectral resolution $E$ on $\mathbb{R}^n$. Then a joint functional calculus for $L_1, \dots, L_n$ is defined via spectral integration and, for every Borel function $m : \mathbb{R}^n \to \mathbb{C}$, the operator

$$m(L) = m(L_1, \dots, L_n) = \int_{\mathbb{R}^n} m \, dE$$

is bounded on $L^2(X, \mu)$ if and only if the function $m$ is ($E$-essentially) bounded. The characterization of the boundedness of $m(L)$ on other function spaces, such as $L^p(X, \mu)$ for $p \neq 2$, in terms of properties of the function $m$ — which will be called *(spectral) multiplier* for $L_1, \dots, L_n$ — is

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a much more difficult question, even in particular cases. Several problems and results of harmonic analysis fall into this frame, the classical examples being the Mihlin-Hörmander and Marcinkiewicz theorems for Fourier multipliers on $\mathbb{R}^n$, which give sufficient conditions for the $L^p$ boundedness ($1 < p < \infty$) of joint functions $m(-i\partial_1, \dots, -i\partial_n)$ of the partial derivatives on $\mathbb{R}^n$, in terms of smoothness properties of the multiplier $m$.

Here we are interested in the case $X = G$ is a (connected) Lie group of polynomial volume growth (with a Haar measure $\mu$) and $L_1, \dots, L_n$ belong to the algebra $\mathfrak{D}(G)$ of left-invariant differential operators on $G$, with particular reference to homogeneous operators $L_1, \dots, L_n$ on a homogeneous (nilpotent) Lie group $G$. For $n = 1$, *i.e.*, for a single operator $L = L_1$, several results of Mihlin-Hörmander type are known (see, *e.g.*, [28, 4, 17, 30, 1, 18, 20, 42, 8, 9] and references therein), dealing mainly with a sublaplacian $L$ on $G$, or more generally with a positive operator $L$ for which Gaussian-type heat kernel estimates hold. For $n > 1$, instead, most of the known results [27, 31, 32, 13, 52, 14, 15] refer to a specific class of groups $G$ (namely, the H-type groups) and to specific choices of the operators $L_1, \dots, L_n$ (*i.e.*, sublaplacians and central derivatives); a recent work of Sikora [43], which has no such restrictions (in fact it applies to more general settings than Lie groups), is however limited to the case of direct products ($G = G_1 \times \cdots \times G_n$ and each $L_j$ operates on a different factor $G_j$ of $G$), so that it does not cover the mentioned results on H-type groups.

In this paper, we propose a quite general setting where spectral multiplier theorems of Mihlin-Hörmander and Marcinkiewicz type for a system of operators can be obtained. Starting from the *weighted subcoercive operators*(1) of ter Elst and Robinson [10] — which are a large class of left-invariant differential operators on a Lie group (including positive elliptic operators, sublaplacians, and positive Rockland operators on homogeneous groups) for which Gaussian-type heat kernel estimates hold — we define a *weighted subcoercive system* to be a system $L_1, \dots, L_n$ of pairwise commuting, formally self-adjoint, left-invariant differential operators on a connected Lie group $G$ which generate a subalgebra of $\mathfrak{D}(G)$ containing a weighted subcoercive operator. An "abstract" study of weighted subcoercive systems, in relation with the algebraic structure and the representation

(1) The definition of weighted subcoercive operator given in [26], to which we refer, may appear more restrictive than the original in [10] (see [26, footnote 1] for details); however, a slight modification of the argument in [10, §11] (namely, a basis change may be needed before eliminating the "over-weight" directions) shows that the original definition is actually equivalent to the one in [26].

theory of the environment Lie group, is performed in [26], from which we get in particular that:

- the members of a weighted subcoercive system $L_1, \dots, L_n$ on $G$ are essentially self-adjoint and commute strongly (so that they admit a joint spectral resolution) in every unitary representation of $G$;
- the operators of the form $m(L) = m(L_1, \dots, L_n)$ are convolution operators:
$$m(L)\phi = \phi * \mathcal{K}_L m = \phi * \breve{m}$$
for some distribution $\breve{m} = \mathcal{K}_L m$; in fact, if $m$ is bounded and compactly supported, then $\mathcal{K}_L m \in L^2(G)$ together with all its left-invariant derivatives;
- a Plancherel formula holds:
$$\|\mathcal{K}_L m\|_{L^2(G)} = \|m\|_{L^2(\mathbb{R}^n,\sigma)}$$
for some regular Borel measure $\sigma$, which is called the *Plancherel measure* associated with $L_1, \dots, L_n$, and whose support is their joint $L^2$ spectrum;
- if $G$ is a homogeneous group (with automorphic dilations $\delta_t$), a commuting system $L_1, \dots, L_n$ of *homogeneous*, formally self-adjoint left-invariant differential operators is a weighted subcoercive system if and only if $L_1, \dots, L_n$ are *jointly injective* on the smooth vectors $v$ of every non-trivial irreducible representation $\pi$ of $G$:
$$d\pi(L_1)v = \cdots = d\pi(L_n)v = 0 \quad \Longrightarrow \quad v = 0$$
(this is a multi-variate analogue of the Rockland condition [35, 22]); in this case, we speak of a *homogeneous weighted subcoercive system* and, if $L_1, \dots, L_n$ are homogeneous of degrees $w_1, \dots, w_n$ respectively, then
$$\sigma(\epsilon_t(A)) = t^{Q_\delta}\sigma(A), \qquad \mathcal{K}_L(m \circ \epsilon_t) = t^{-Q_\delta}(\mathcal{K}_L m) \circ \delta_{t^{-1}}, \tag{1.1}$$
where $Q_\delta$ is the homogeneous dimension ($\det \delta_t = t^{Q_\delta}$) and
$$\epsilon_t(\lambda_1, \dots, \lambda_n) = (t^{w_1}\lambda_1, \dots, t^{w_n}\lambda_n)$$
are the dilations on $\mathbb{R}^n$ associated with $L_1, \dots, L_n$.

In the following, under the hypothesis that $L_1, \dots, L_n$ is a weighted subcoercive system on a Lie group $G$ of polynomial growth of degree $Q_G$, we prove weighted $L^1$ estimates for the kernels $\mathcal{K}_L m$ corresponding to multipliers $m$ with compact support, in terms of a Sobolev norm of $m$. If we suppose further that $G$ is a homogeneous group and that $L_1, \dots, L_n$ are homogeneous operators (with degrees $w_1, \dots, w_n$ and associated dilations

$\epsilon_t$), then a theorem of Mihlin-Hörmander type can be obtained: the operator $m(L)$ is of weak type $(1,1)$ and bounded on $L^p(G)$ for $1 < p < \infty$ whenever the multiplier $m$ satisfies an $L^q$ *Mihlin-Hörmander condition* of order $s$, *i.e.*,

$$\|m\|_{M_\epsilon W_q^s} \stackrel{\text{def}}{=} \sup_{t>0} \|(m \circ \epsilon_t)\, \eta\|_{W_q^s} < \infty, \tag{1.2}$$

with $q \in [2, \infty]$ and

$$s > \frac{Q_G}{2} + \frac{n-1}{q}, \tag{1.3}$$

where $W_q^s(\mathbb{R}^n)$ is the $L^q$ Sobolev space of (fractional) order $s$ and $\eta$ is a non-negative smooth cut-off function on $\mathbb{R}^n$ supported on an annulus centered at the origin. Notice that the condition (1.2) is independent of the choice of the cut-off $\eta$; moreover, an $L^\infty$ Mihlin-Hörmander condition of integral order $s$ is essentially equivalent to the pointwise conditions

$$\sup_{\lambda \neq 0} |\lambda|_\epsilon^{\|\alpha\|_\epsilon} \, |\partial^\alpha m(\lambda)| < \infty \tag{1.4}$$

for $\alpha \in \mathbb{N}^n$ with $|\alpha| = \alpha_1 + \dots + \alpha_n \leqslant s$, where $|\cdot|_\epsilon$ is a $\epsilon_t$-homogeneous norm on $\mathbb{R}^n$ and $\|\alpha\|_\epsilon = w_1 \alpha_1 + \dots + w_n \alpha_n$.

In particular cases, the previous theorem can be improved by lowering the *regularity threshold*, *i.e.,* the right-hand side of (1.3). For instance, by extending a technique due to Hebisch and Zienkiewicz [20], we prove that, if $G$ is the direct product of Euclidean and Métivier groups, then the dimension at infinity $Q_G$ can be replaced in (1.3) by the topological dimension $\dim G$; in fact, in § 3 we propose a systematic approach for exploiting such technique, by introducing the notion of *$h$-capacious groups*, for which $Q_G$ can be replaced in (1.3) by $Q_G - h$. The term $(n-1)/q$ in (1.3) can be lowered too, by determining the volume growth rate with respect to the Plancherel measure $\sigma$ of Euclidean balls with small radius: namely, if $\sigma(B(\lambda, r)) \leqslant C r^d$ for $|\lambda| = 1$ and $r \leqslant 1$, then $n - 1$ can be replaced in (1.3) by $n - d$.

Finally, a sort of product theory can be developed, by considering several homogeneous Lie groups $G_l$, each of which endowed with a homogeneous weighted subcoercive system $L_{l,1}, \dots, L_{l,n_l}$, for $l = 1, \dots, \varrho$. Let $(\epsilon_{l,t})_{t>0}$ be the dilations on $\mathbb{R}^{n_l}$ associated with the system $L_{l,1}, \dots, L_{l,n_l}$, and define the multi-parameter dilations

$$\daleth_{\vec{t}} = \epsilon_{1,t_1} \times \dots \times \epsilon_{\varrho,t_\varrho}$$

on $\mathbb{R}^{\vec{n}} = \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_\varrho}$; set moreover $X_{\vec{n}} = \{\lambda \in \mathbb{R}^{\vec{n}} \,:\, |\lambda_1| \cdots |\lambda_\varrho| = 0\}$. If $G$ is a connected Lie group, $\upsilon_l : G_l \to G$ are Lie group homomorphisms, and $L^\flat_{l,j} = \upsilon'_l(L_{l,j})$, then we have a system

$$(1.5)\qquad L^\flat_{1,1}, \dots, L^\flat_{1,n_1}, \dots, L^\flat_{\varrho,1}, \dots, L^\flat_{\varrho,n_\varrho}$$

of essentially self-adjoint, left-invariant differential operators on $G$. Under the sole hypothesis of (strong) commutativity of the operators (1.5) on $L^2(G)$, we prove a multi-variate analogue of the previous theorem: the operator $m(L^\flat)$ is bounded on $L^p(G)$ for $1 < p < \infty$ whenever the multiplier $m : \mathbb{R}^{\vec{n}} \to \mathbb{C}$ vanishes on $X_{\vec{n}}$ and satisfies an $L^q$ *Marcinkiewicz condition* of order $\vec{s} = (s_1, \dots, s_\varrho)$, *i.e.*,

$$(1.6)\qquad \|m\|_{M_\daleth S^{\vec{s}}_q W} \stackrel{\text{def}}{=} \sup_{t_1,\dots,t_\varrho>0} \|(m \circ \daleth_{\vec{t}})\, \eta_1 \otimes \cdots \otimes \eta_\varrho\|_{S^{\vec{s}}_q W} < \infty,$$

with $q \in [2, \infty]$ and

$$(1.7)\qquad s_l > \frac{Q_{G_l}}{2} + \frac{n_l - 1}{q} \qquad \text{for } l = 1, \dots, \varrho,$$

where $S^{\vec{s}}_q W(\mathbb{R}^{\vec{n}})$ is the $L^q$ Sobolev space with dominating mixed smoothness of order $\vec{s}$ (see, *e.g.*, [40, 38]) and the $\eta_l : \mathbb{R}^{n_l} \to \mathbb{R}$ are cut-off functions as before. This result can also be improved in particular cases: in fact, each of the components (1.7) of the regularity threshold can be independently lowered, by the same techniques and amounts as for the threshold (1.3) in the previous result. The hypothesis $m|_{X_{\vec{n}}} = 0$, related to the possibility that the spectral measure of $X_{\vec{n}}$ is not null, can be relaxed too, by applying iteratively our Marcinkiewicz-type result to subsystems of (1.5).

Both our theorems can be applied to the direct-product setting of [43] (in the case of homogeneous groups), and also to the systems of operators considered in the above-mentioned works on H-type groups; in fact, the results of [32, 52] are sharper than ours, since they require a weaker condition on the multiplier. On the other hand, our theorems have a much wider range of applicability, with respect both to the groups and to the systems of differential operators under consideration. In particular, the environment group $G$ in the second theorem need not be homogeneous. As a corollary, we obtain $L^p$ multiplier theorems for distinguished sublaplacians on some non-nilpotent solvable Lie groups $G$ of polynomial growth (such as the plane motion group, the oscillator groups, the diamond groups) with regularity threshold $(\dim G)/2$; to our knowledge, this threshold had been previously reached only for some homogeneous groups (*i.e.*, Heisenberg and related groups [17, 30]) and for the compact group $SU_2$ [8].

## Notation and preliminary remarks

For a topological space $X$, we denote by $C(X)$ the space of continuous (complex-valued) functions on $X$, whereas $C_0(X)$ is the subspace of continuous functions vanishing at infinity. If $X$ is a smooth manifold, then $\mathcal{D}(X)$ is the space of compactly supported smooth functions on $X$.

If $G$ is a Lie group, $f$ is a function on $G$ and $x, y \in G$, then we set

$$\mathrm{L}_x f(y) = f(x^{-1}y), \qquad \mathrm{R}_x f(y) = f(yx).$$

$\mathrm{R} : x \mapsto \mathrm{R}_x$ is the (right) regular representation of $G$. For a fixed right Haar measure $\mu$ on $G$, $\mathrm{R}_x$ is an isometry of $L^p(G)$ for $1 \leqslant p \leqslant \infty$. With respect to such measure, convolution and involution take the form

$$f * g(x) = \int_G f(xy^{-1}) g(y)\, dy, \qquad f^*(x) = \Delta(x)\overline{f(x^{-1})}$$

(where $\Delta$ is the modular function) and we set, for every representation $\pi$,

$$\pi(f) = \int_G f(x)\, \pi(x^{-1})\, dx$$

(differently from the common usage), so that in particular

$$\mathrm{R}(g) f = f * g, \qquad \pi(f * g) = \pi(g)\pi(f), \qquad \pi(Df) = d\pi(D)\pi(f)$$

for every $D \in \mathfrak{D}(G)$. We denote by $D^+$ the formal adjoint of a smooth differential operator $D$ on $G$ (with respect to the measure $\mu$).

The above conditions (1.2), (1.6) on the multiplier $m$ have been expressed in terms of Sobolev norms. In fact, there are several scales of spaces which can be used to express a differentiability condition of fractional order and with an $L^q$ flavour; since the inequalities (1.3), (1.7) involving the order are strict, it does not really matter which of the various scales is used. In the following, we will use the scale of Besov spaces $B^s_{q,r}(\mathbb{R}^n)$ and their dominating-mixed-smoothness variants $S^{\vec{s}}_{q,r}B(\mathbb{R}^{\vec{n}})$, which are particularly convenient because of their embedding and interpolation properties (see, *e.g.*, [2, 47, 49, 40, 37, 39, 38, 41] for a reference); accordingly, we will use the quantities $\|\cdot\|_{M_\epsilon B^s_{q,q}}$, $\|\cdot\|_{M_\beth S^{\vec{s}}_{q,q}B}$ in place of the ones in (1.2), (1.6).

## 2. Weighted estimates

Let $G$ be a Lie group of polynomial growth of degree $Q_G$, and set

$$|x|_G = d_G(x, e), \qquad \langle x \rangle_G = 1 + |x|_G,$$

where $d_G$ is a left-invariant connected distance on $G$ [51, § III.4] and $e \in G$ is the identity element. Let $L_1, \dots, L_n$ be a weighted subcoercive system on $G$, with associated Plancherel measure $\sigma$, and let $\mathcal{O}$ be the subalgebra of $\mathfrak{D}(G)$ generated by them. The aim of this section is to obtain inequalities of the form

$$\|\langle\cdot\rangle_G^\alpha \mathcal{K}_L m\|_{L^p(G)} \leqslant C_{K,\alpha,\beta} \|m\|_{B^\beta_{q,q}(\mathbb{R}^n)}$$

for multipliers $m$ with support contained in a fixed compact $K \subseteq \mathbb{R}^n$, and for suitable $p, q, \alpha, \beta$. In order to do this, we will subsequently perform two "changes of variables" on the spectral side: the former corresponds to choosing a system of generators of $\mathcal{O}$ made of positive weighted subcoercive operators, while the latter is the multi-variate analogue of an exponential change of variables which has often been used in the literature (see, *e.g.*, [12, § 6.B]) and which allows, together with a Fourier-series decomposition, heat kernel estimates, and Hölder's inequality, to obtain the required estimates. Properties of the Plancherel measure $\sigma$ and interpolation will then be used (as in [28] and subsequent works) to improve the obtained inequalities.

As in [26, § 3.2], we can find a polynomial $p_* \geqslant 0$ on $\mathbb{R}^n$ such that, if

$$p_0(\lambda) = p_*(\lambda) + \sum_{j=1}^n \lambda_j^2 + 1, \qquad p_\nu(\lambda) = p_0(\lambda) + \lambda_\nu \quad \text{for } \nu = 1, \dots, n,$$

then $p_*(L), p_0(L), p_1(L), \dots, p_n(L)$ are all positive and weighted subcoercive, and moreover $p_0(L), \dots, p_n(L)$ generate $\mathcal{O}$. Let $p : \mathbb{R}^n \to \mathbb{R}^{1+n}$ be the map whose components are the polynomials $p_0, \dots, p_n$. For $l \in \mathbb{Z}^{1+n}$, set

$$E_l(\lambda) = e^{il \cdot e^{-p(\lambda)}} - 1 = e^{i(l_0 e^{-p_0(\lambda)} + l_1 e^{-p_1(\lambda)} + \dots + l_n e^{-p_n(\lambda)})} - 1.$$

Then $E_l \in C_0(\mathbb{R}^n)$, and in fact

$$E_l = \sum_{0 \neq k \in \mathbb{N}^{1+n}} \frac{(il_0)^{k_0} \cdots (il_n)^{k_n}}{k_0! \cdots k_n!} e^{-k_0 p_0} \cdots e^{-k_n p_n},$$

with uniform convergence on $\mathbb{R}^n$. This means that, if $h_{\nu,t}$ is the heat kernel of $p_\nu(L)$ for $\nu = 0, \dots, n$ (with $h_{\nu,0}$ denoting the Dirac delta at the identity of $G$), then

$$\breve{E}_l = \sum_{0 \neq k \in \mathbb{N}^{1+n}} \frac{(il_0)^{k_0} \cdots (il_n)^{k_n}}{k_0! \cdots k_n!} h_{0,k_0} * \dots * h_{n,k_n} \tag{2.1}$$

with convergence in the norm of the corresponding convolution operators on $L^2(G)$.

Lemma 2.1. — *There exists $C > 0$ such that*

$$\|\breve{E}_l\|_2 \leqslant C|l| \qquad \text{for all } l \in \mathbb{Z}^{1+n}.$$

*Proof.* — We have

$$|E_l(\lambda)| \leqslant |l \cdot e^{-p(\lambda)}| \leqslant \sum_{\nu=0}^{n} |l_\nu| e^{-p_\nu(\lambda)} \leqslant (1+n)|l| e^{-p_*(\lambda)},$$

so that in particular, if $f = e^{p_*} E_l$, then

$$\|\breve{E}_l\|_2 = \|f(L)\mathcal{K}_L(e^{-p_*})\|_2 \leqslant \|f\|_\infty \|\mathcal{K}_L(e^{-p_*})\|_2 \leqslant (1+n)\|\mathcal{K}_L(e^{-p_*})\|_2 |l|,$$

which is the conclusion. □

Lemma 2.2. — *There exist $c, \omega > 0$ such that*

$$\|\breve{E}_l\|_{L^2(G, e^{2|x|_G}\, dx)} \leqslant c e^{\omega |l|} \qquad \textit{for all } l \in \mathbb{Z}^{1+n}.$$

*Proof.* — Since all the connected left-invariant distances on $G$ are equivalent in the large [51, Proposition III.4.2], by interpolating the inequalities (e) and (f) of [26, Theorem 2.3], we have that there exist $c \geqslant 1$ and $\omega > 0$ such that

$$\|h_{\nu,t}\, e^{|\cdot|_G}\|_q \leqslant c e^{\omega t} \qquad \text{for } t \geqslant 1,\ \nu = 0, \dots, n \text{ and } q \in [1, \infty].$$

By Young's inequality and submultiplicativity of $e^{|\cdot|_G}$, we then get

$$\|(h_{0,k_0} * \dots * h_{n,k_n})\, e^{|\cdot|_G}\|_q \leqslant c^{1+n} e^{\omega(k_0 + \dots + k_n)}$$

for $k \in \mathbb{N}^{1+n} \smallsetminus \{0\}$ and $q \in [1, \infty]$. This means in particular that the series in (2.1) converges absolutely in $L^2(G, e^{2|x|_G}\, dx)$, with

$$\sum_{0 \neq k \in \mathbb{N}^{1+n}} \left\| \frac{(il_0)^{k_0} \cdots (il_n)^{k_n}}{k_0! \cdots k_n!} h_{0,k_0} * \dots * h_{n,k_n} \right\|_{L^2(G, e^{2|x|_G}\, dx)} \leqslant c^{1+n} e^{e^{\omega}|l|},$$

and we are done. □

Lemma 2.3. — *For all $\alpha \geqslant 0$, we have*

$$\|\breve{E}_l\|_{L^2(G, \langle x \rangle_G^{2\alpha}\, dx)} \leqslant C_\alpha |l|^{\alpha+1} \qquad \textit{for } l \in \mathbb{Z}^{1+n}.$$

*Proof.* — By Lemma 2.2, it is sufficient to check the estimate for $|l|$ large, but then

$$\int_G |\breve{E}_l(x)|^2 \langle x \rangle_G^{2\alpha}\, dx \leqslant \int_{|x|_G \leqslant \omega|l|} + \int_{|x|_G > \omega|l|}$$
$$\leqslant (1+\omega|l|)^{2\alpha} \|\breve{E}_l\|_2^2 + \sup_{r > \omega|l|} \frac{(1+r)^{2\alpha}}{e^{2r}} \|\breve{E}_l\|^2_{L^2(G, e^{2|x|_G}\, dx)} \leqslant C_\alpha |l|^{2(\alpha+1)}$$

by Lemmata 2.1 and 2.2. □

LEMMA 2.4. — *Let $K \subseteq \mathbb{R}^n$ be compact. For every $f \in \mathcal{D}(\mathbb{R}^n)$ supported in $K$, there exists $g \in \mathcal{D}(\mathbb{T}^{1+n})$, depending linearly on $f$, such that*

$$f(\lambda) = g\left(e^{ie^{-p(\lambda)}}\right) = g\left(e^{ie^{-p_0(\lambda)}}, \dots, e^{ie^{-p_n(\lambda)}}\right), \tag{2.2}$$

$$g(1, \dots, 1) = 0, \tag{2.3}$$

$$\|g\|_{B^s_{2,2}(\mathbb{T}^{1+n})} \leqslant C_{K,s} \|f\|_{B^s_{2,2}(\mathbb{R}^n)} \qquad \text{for all } s \geqslant 0. \tag{2.4}$$

*In particular, if $g(e^{it}) = \sum_{l \in \mathbb{Z}^{1+n}} \hat{g}(l) e^{il\cdot t}$ is the Fourier-series development of $g$, then we have*

$$f = \sum_{0 \neq l \in \mathbb{Z}^{1+n}} \hat{g}(l) E_l, \tag{2.5}$$

*with uniform convergence on $\mathbb{R}^n$.*

*Proof.* — Since $K \subseteq \mathbb{R}^n$ is compact and the polynomials $p_0, \dots, p_n$ are strictly positive, $p(K)$ is a compact subset of $\Omega = ]0, +\infty[^{1+n}$. Therefore we can choose $\psi_K \in \mathcal{D}(\Omega)$ such that $\psi_K|_{p(K)} \equiv 1$. If we put

$$\tilde{f}(y) = f(y_1 - y_0, \dots, y_n - y_0)\psi_K(y) \qquad \text{for } y \in \mathbb{R}^{1+n},$$

we then have that $\tilde{f} \in \mathcal{D}(\Omega)$, $f = \tilde{f} \circ p$ and

$$\|\tilde{f}\|_{B^s_{2,2}(\mathbb{R}^{1+n})} \leqslant C_{K,s} \|f\|_{B^s_{2,2}(\mathbb{R}^n)} \qquad \text{for all } s \geqslant 0,$$

since the change of variables has maximal rank.

Notice now that the map

$$\Phi : \Omega \ni y \mapsto e^{ie^{-y}} = (e^{ie^{-y_0}}, \dots, e^{ie^{-y_n}}) \in \mathbb{T}^{1+n}$$

is a smooth diffeomorphism with its image, which is an open subset of $\mathbb{T}^{1+n}$ not containing $(1, \dots, 1)$. The function $g = \tilde{f} \circ \Phi^{-1} \in \mathcal{D}(\Phi(\Omega))$ can be then extended by zero to a smooth function on $\mathbb{T}^{1+n}$, and we have clearly

$$\|g\|_{B^s_{2,2}(\mathbb{T}^{1+n})} \leqslant C_{K,s} \|\tilde{f}\|_{B^s_{2,2}(\mathbb{R}^{1+n})} \qquad \text{for all } s \geqslant 0.$$

The construction shows that $g$ depends linearly on $f$ and satisfies (2.2)-(2.4). In particular, we have $\sum_{l \in \mathbb{Z}^{1+n}} \hat{g}(l) = 0$, so that the Fourier decomposition of $g$ can be rewritten as

$$g(e^{it}) = \sum_{0 \neq l \in \mathbb{Z}^{1+n}} \hat{g}(l)(e^{il\cdot t} - 1)$$

(with uniform convergence since $g$ is smooth), which gives (2.5). $\square$

PROPOSITION 2.5. — *Let $K \subseteq \mathbb{R}^n$ be compact, $\alpha \geqslant 0$, $\beta > \alpha + (n+3)/2$. For all $f \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp} f \subseteq K$, we have*

$$\|\breve{f}\|_{L^2(G, \langle x \rangle_G^{2\alpha}\, dx)} \leqslant C_{K,\alpha,\beta} \|f\|_{B^\beta_{2,2}(\mathbb{R}^n)}.$$

*Proof.* — Let $g \in \mathcal{D}(\mathbb{T}^{1+n})$ be given by Lemma 2.4. Then

$$\breve{f} = \sum_{0 \neq l \in \mathbb{Z}^{1+n}} \hat{g}(l) \breve{E}_l$$

in the norm of convolution operators on $L^2(G)$. However, the series in the right-hand side converges absolutely in $L^2(G, \langle x \rangle_G^{2\alpha}\, dx)$, since

$$\begin{aligned}\sum_{0 \neq l \in \mathbb{Z}^{1+n}} |\hat{g}(l)|\, \|\breve{E}_l\|_{L^2(G, \langle x \rangle_G^{2\alpha}\, dx)} &\leqslant C_\alpha \sum_{0 \neq l \in \mathbb{Z}^{1+n}} |\hat{g}(l)|\, |l|^{\alpha+1} \\ &\leqslant C_{\alpha,\beta} \|g\|_{B^\beta_{2,2}(\mathbb{T}^{1+n})} \leqslant C_{K,\alpha,\beta} \|f\|_{B^\beta_{2,2}(\mathbb{R}^n)}\end{aligned}$$

by Lemma 2.3 and Hölder's inequality, and the conclusion follows. □

The previous proposition contains a "preliminary version" of the required inequalities, which we are now going to sharpen by interpolation with the Plancherel formula. In order to control the $L^2(\sigma)$ norm with a Besov norm, we will use a refined trace theorem due to Triebel.

Let $\tau$ be a (positive) regular Borel measure on $\mathbb{R}^n$, and let $0 \leqslant d \leqslant n$; we say that $\tau$ is *locally $d$-bounded* on an open $\Omega \subseteq \mathbb{R}^n$ if, for every compact $K \subseteq \Omega$ and for $0 \leqslant \gamma < d$, there exist $C, \bar{r} > 0$ such that

$$\tau(B(\lambda, r)) \leqslant C r^\gamma \qquad \text{for } \lambda \in K \text{ and } r \leqslant \bar{r}.$$

Notice that every regular Borel measure $\tau$ is locally 0-bounded on the whole $\mathbb{R}^n$; moreover, if $\tau$ is homogeneous with respect to some system of dilations $\epsilon_t$ on $\mathbb{R}^n$ (*i.e.*, if $\tau(\epsilon_t(A)) = t^a \tau(A)$ for some $a \geqslant 0$ and every Borel $A \subseteq \mathbb{R}^n$), then $\tau$ is locally 1-bounded on $\mathbb{R}^n \setminus \{0\}$.

Lemma 2.6. — *Let $\tau$ be a regular Borel measure which is locally $d$-bounded on an open $\Omega \subseteq \mathbb{R}^n$. If $s > (n-d)/2$ and $K \subseteq \Omega$ is compact, then*

$$\|f\|_{L^2(\tau)} \leqslant C_{K,s} \|f\|_{B^s_{2,2}(\mathbb{R}^n)}$$

*for every $f \in \mathcal{D}(\mathbb{R}^n)$ with* $\operatorname{supp} f \subseteq K$.

*Proof.* — If $d = 0$, then $\|f\|_\infty \leqslant C_s \|f\|_{B^s_{2,2}(\mathbb{R}^n)}$ and the result is trivial. Suppose instead that $0 < d \leqslant n$, and let $K \subseteq \Omega$ be compact, $\varepsilon > 0$. Choose a compact neighborhood $K' \subseteq \Omega$ of $K$, and let $C, \bar{r} > 0$ such that

$$\tau(B(\lambda, r)) \leqslant C r^{d-\varepsilon} \qquad \text{for } \lambda \in K' \text{ and } 0 < r \leqslant \bar{r}.$$

Let moreover $\bar{r}' = \min\{\bar{r}, \operatorname{dist}(\Omega \setminus \mathring{K}', K)\}$, $C' = \max\{C, \tau(K)/(\bar{r}')^{d-\varepsilon}\}$. The identity $\tau_K(E) = \tau(E \cap K)$ defines a positive regular Borel measure $\tau_K$ on $\mathbb{R}^n$, which coincides with $\tau$ on $K$, and with $\operatorname{supp} \tau_K \subseteq K$. Moreover

$$\tau_K(B(\lambda, r)) \leqslant C' r^{d-\varepsilon} \qquad \text{for every } r > 0 \text{ and } \lambda \in \mathbb{R}^n,$$

by construction. Therefore, since the Besov space $B^s_{2,2}(\mathbb{R}^n)$ coincides with the Triebel-Lizorkin space $F^s_{2,2}(\mathbb{R}^n)$, by [50, Corollary 9.8(ii)] we have

$$\|f\|_{L^2(\tau)} = \|f\|_{L^2(\tau_K)} \leqslant C_{K,s} \|f\|_{B^s_{2,2}(\mathbb{R}^n)}$$

for $s > (n-d)/2 + \varepsilon/2$ and $f \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp} f \subseteq K$. The conclusion follows from the arbitrariness of $\varepsilon > 0$. □

Theorem 2.7. — *Suppose that, for some open $\Omega \subseteq \mathbb{R}^n$, the Plancherel measure $\sigma$ is locally $d$-bounded on $\Omega$. Let $K \subseteq \Omega$ be compact, $D \in \mathfrak{D}(G)$, $p, q \in [1, \infty]$, $\alpha \geqslant 0$,*

$$\beta > \alpha + Q_G \left( \frac{1}{\min\{2,p\}} - \frac{1}{2} \right) + \frac{n}{q} - \frac{d}{\max\{2,q\}}.$$

*For all $m \in B^\beta_{q,q}(\mathbb{R}^n)$ with* $\operatorname{supp} m \subseteq K$, *we have*

$$\| \langle \cdot \rangle_G^\alpha \, D \breve{m} \|_{L^p(G)} \leqslant C_{K,D,\alpha,\beta,p,q} \|m\|_{B^\beta_{q,q}(\mathbb{R}^n)}.$$

*Proof.* — Consider first the case $p = 2$, $D = 1$. Let $\xi \in \mathcal{D}(\mathbb{R}^n)$ be such that $\operatorname{supp} \xi \subseteq \Omega$, $\xi|_K \equiv 1$, and let $K' \subseteq \Omega$ be a compact neighborhood of $\operatorname{supp} \xi$. Proposition 2.5, together with the continuous inclusion $B^\beta_{q,2}(\mathbb{R}^n) \subseteq B^{\beta - n/2}_{2,2}(\mathbb{R}^n)$, then yields, for $m \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp} m \subseteq K'$, that

$$\|\breve{m}\|_{L^2(G, \langle x \rangle_G^{2\alpha} \, dx)} \leqslant C_{K,\alpha,\beta,q} \|m\|_{B^\beta_{q,2}(\mathbb{R}^n)} \tag{2.6}$$

for $\beta > \alpha + (2n+3)/2$. By the use of a suitable approximate identity, (2.6) can be easily extended to all $m \in B^\beta_{q,2}(\mathbb{R}^n)$ with $\operatorname{supp} m \subseteq \operatorname{supp} \xi$. Hence, if we consider the linear map $M : m \mapsto \mathcal{K}_L(m\xi)$, then we have that

(2.7) $M$ is bounded $B^\beta_{q,2}(\mathbb{R}^n) \to L^2(G, \langle x \rangle_G^{2\alpha} \, dx)$ for $\beta > \alpha + (2n+3)/2$.

On the other hand, for $\alpha = 0$, the Plancherel formula and Lemma 2.6 give

$$M \text{ is bounded } B^\beta_{q,2}(\mathbb{R}^n) \to L^2(G) \text{ for } \beta > \frac{n}{q} - \frac{d}{\max\{2,q\}} \tag{2.8}$$

(this is clear for $q = 2$ and $q = \infty$; for $1 \leqslant q < 2$, we exploit the continuous inclusion $B^\beta_{q,2}(\mathbb{R}^n) \subseteq B^{\beta - n/q + n/2}_{2,2}(\mathbb{R}^n)$; for $2 < q < \infty$, we interpolate). Therefore, by interpolating (2.7) and (2.8), we get

$$M \text{ is bounded } B^\beta_{q,2}(\mathbb{R}^n) \to L^2(G, \langle x \rangle_G^{2\alpha} \, dx) \text{ for } \beta > \alpha + \frac{n}{q} - \frac{d}{\max\{2,q\}}.$$

In order to conclude, it is sufficient to notice that, if $\beta > \alpha + n/q - d/\max\{2,q\}$, then for any $\beta' \in \left]\alpha + n/q - d/\max\{2,q\}, \beta\right[$ we have the continuous inclusion $B^\beta_{q,q}(\mathbb{R}^n) \subseteq B^{\beta'}_{q,2}(\mathbb{R}^n)$ , and moreover $Mm = \breve{m}$ for every $m$ with $\operatorname{supp} m \subseteq K$.

Take now an arbitrary $D \in \mathfrak{D}(G)$. For $m \in B^{\beta}_{\infty,\infty}(\mathbb{R}^n)$ with $\operatorname{supp} m \subseteq K$, set $m_0 = m e^{p_*}$, $\xi = e^{-p_*}$; then $\breve{m} = \breve{m}_0 * \breve{\xi}$, so that, by Young's inequality,

$$\|\langle\cdot\rangle_G^\alpha \, D\breve{m}\|_2 \leqslant \|\langle\cdot\rangle_G^\alpha \, \breve{m}_0\|_2 \|\langle\cdot\rangle_G^\alpha \, D\breve{\xi}\|_1 \lesssim \|m_0\|_{B^{\beta}_{q,q}(\mathbb{R}^n)} \lesssim \|m\|_{B^{\beta}_{q,q}(\mathbb{R}^n)}$$

for $\beta > \alpha + n/q - d/\max\{2,q\}$. This concludes the proof for the case $p = 2$.

The case $1 \leqslant p < 2$ follows by Hölder's inequality, since $\int_G \langle x\rangle_G^{-\alpha} \, dx < \infty$ for $\alpha > Q_G$.

Let now $p = \infty$. If $\zeta \in \mathcal{D}(\mathbb{R}^n)$ is nonnegative and $\zeta(e) > 0$, and if we set $w_\alpha = \langle\cdot\rangle_G^\alpha * \zeta$ for $\alpha \geqslant 0$, then $w_\alpha$ is smooth and nonnegative,

$$\langle x\rangle_G^\alpha \leqslant C_\alpha w_\alpha(x), \qquad D w_\alpha(x) \leqslant C_{D,\alpha} \langle x\rangle_G^\alpha$$

for all $D \in \mathfrak{D}(G)$. If $\mathfrak{D}_*$ is a basis of the $A \in \mathfrak{D}(G)$ of order up to $\lfloor(\dim G)/2\rfloor + 1$, then we have, for $\beta > \alpha + n/q - d/\max\{2,q\}$ and $D \in \mathfrak{D}(G)$,

$$\begin{aligned} \|\langle\cdot\rangle_G^\alpha \, D\breve{m}\|_\infty &\lesssim \|w_\alpha \, D\breve{m}\|_\infty \lesssim \sum_{A \in \mathfrak{D}_*} \|A(w_\alpha D\breve{f})\|_2 \\ &\lesssim \sum_{A_1, A_2 \in \mathfrak{D}_*} \|(A_1 w_\alpha)(A_2 D\breve{m})\|_2 \lesssim \sum_{A \in \mathfrak{D}_*} \|\langle\cdot\rangle_G^\alpha \, AD\breve{m}\|_2 \lesssim \|m\|_{B^{\beta}_{q,q}} \end{aligned}$$

by Sobolev's embedding, Leibniz's rule and the case $p = 2$.

The remaining case $2 < p < \infty$ follows by interpolation. □

## 3. Improved weighted estimates

The weighted estimates given by Theorem 2.7 for $p = 1$ yield a "weak multiplier theorem" for a weighted subcoercive system $L_1, \dots, L_n$ on a Lie group $G$ of polynomial growth: the operator $m(L)$ is bounded on $L^p(G)$ for $1 \leqslant p \leqslant \infty$ if the multiplier $m$ is compactly supported and sufficiently smooth; more precisely, by taking $q = \infty$, we require an order of smoothness $\gamma > Q_G/2$, where $Q_G$ is the dimension at infinity of $G$. If $G = \mathbb{R}^n$, then $Q_G$ coincides with the topological dimension $\dim G = n$; for non-abelian (simply connected) nilpotent groups, however, $Q_G > \dim G$. Nevertheless, for a particular class of 2-step nilpotent groups (namely, Heisenberg and related groups) multiplier theorems have been proved with $(\dim G)/2$ as the regularity threshold [17, 30, 20, 32, 52]. In this section, we extend to our context of weighted subcoercive systems the technique of Hebisch and Zienkiewicz [20], which allows in some cases to lower the threshold in the weighted $L^1$ estimates.

Let $G$ be a nilpotent Lie group, with Lie algebra $\mathfrak{g}$. Let $\mathfrak{z}$ be the center of $\mathfrak{g}$ and set

$$\mathfrak{y} = \{v \in \mathfrak{g} : [v, \mathfrak{g}] \subseteq \mathfrak{z}\}; \tag{3.1}$$

$\mathfrak{y}$ is a characteristic ideal of $\mathfrak{g}$ containing $\mathfrak{z}$ (in fact, it is the term following $\mathfrak{z}$ in the ascending central series of $\mathfrak{g}$). Let moreover $P : \mathfrak{g} \to \mathfrak{g}/\mathfrak{z}$ be the canonical projection. The bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ induces, by restriction, passage to the quotient and transposition, another bilinear map

$$J : \mathfrak{g}/\mathfrak{z} \times \mathfrak{z}^* \to \mathfrak{y}^*,$$

which we will call the *capacity map* of $\mathfrak{g}$, and is uniquely determined by

$$J(P(x), \tau)(y) = \tau([x, y])$$

for $x \in \mathfrak{g}$, $y \in \mathfrak{y}$, $\tau \in \mathfrak{z}^*$. The group $G$ is said to be an *H-type group* if there exists an inner product on $\mathfrak{g}$ such that, for every $\tau \in \mathfrak{z}^*$ of norm 1, the map $J(\cdot, \tau) : \mathfrak{g}/\mathfrak{z} \to \mathfrak{y}^*$ is an isometric embedding (this condition implies that $\mathfrak{g} = \mathfrak{y}$, so that $G$ is 2-step). If $G$ is an H-type group, then in particular

$$|J(\bar{x}, \tau)| \geqslant |\bar{x}||\tau| \tag{3.2}$$

for suitable norms on $\mathfrak{g}/\mathfrak{z}$, $\mathfrak{z}^*$ and $\mathfrak{y}^*$; the validity of such an inequality defines the class of *Métivier groups*, which has been introduced in the study of analytic hypoellipticity of Rockland operators [29, 21]; this class is strictly larger than that of H-type groups (see [33] for an example), but is still contained in the class of 2-step groups.

In the following, we consider a more general inequality of the form

$$|J(\bar{x}, \tau)| \geqslant w(\bar{x})\zeta(\tau)$$

for some non-negative functions $w : \mathfrak{g}/\mathfrak{z} \to \mathbb{R}$, $\zeta : \mathfrak{z}^* \to \mathbb{R}$, which may hold also on higher-step groups. Rewritten as

$$w(\bar{x})^\gamma \leqslant |J(\bar{x}, \tau)|^\gamma \zeta(\tau)^{-\gamma}$$

for some $\gamma > 0$, this inequality will be interpreted via the spectral theorem, in order to control a multiplication operator (corresponding to $w(\bar{x})^\gamma$) with a function of the central derivatives (corresponding to $\zeta(\tau)^{-\gamma}$); in this interpretation, it turns out that $|J(\bar{x}, \tau)|^2$ corresponds to a sum of products of left- and right-invariant differential operators on $G$, therefore the term $|J(\bar{x}, \tau)|^\gamma$ can be dominated by an *a priori* estimate for a weighted subcoercive operator on the direct product $G \times G$.

In order to fill in the details, it is convenient to introduce some notation. For every smooth differential operator $D$ on $G$, the identity

$$(Df)^* = D^\circ f^* \tag{3.3}$$

defines another differential operator $D^\circ$ on $G$; the map $D \mapsto D^\circ$ is a conjugate-linear involutive automorphism of the unital algebra of all smooth differential operators on $G$, which maps left-invariant operators to right-invariant ones and *vice versa*.

The Lie algebra $\tilde{\mathfrak{g}}$ of the direct product $\tilde{G} = G \times G$ is canonically isomorphic to $\mathfrak{g} \oplus \mathfrak{g}$; we define the correspondence $D \mapsto D^\bullet$ on $\mathfrak{D}(\tilde{G})$ as the unique conjugate-linear automorphism of the unital algebra $\mathfrak{D}(\tilde{G}) \cong \mathrm{U}((\mathfrak{g} \oplus \mathfrak{g})_{\mathbb{C}})$ extending the Lie algebra automorphism $(X,Y) \mapsto (Y,X)$ of $\mathfrak{g} \oplus \mathfrak{g}$.

Let $\xi$ be the unitary representation of $\tilde{G}$ on $L^2(G)$ given by $\xi(x,y)f = \mathrm{R}_x \, \mathrm{L}_y \, f$. Then, for every $D \in \mathfrak{D}(\tilde{G})$, $d\xi(D)$ is a smooth differential operator on $G$, and

$$d\xi(D^\bullet) = d\xi(D)^\circ.$$

Finally, for $D \in \mathfrak{D}(G)$, let $\tilde{D} \in \mathfrak{D}(\tilde{G})$ be defined by $\tilde{D}(f \otimes g) = (Df) \otimes g$, so that in particular $d\xi(\tilde{D}) = D$.

Lemma 3.1. — *Let $B = B^+ \in \mathfrak{D}(G)$ be weighted subcoercive, and set $\Delta = B^2$. Then $\tilde{\Delta} + \tilde{\Delta}^\bullet$ is positive weighted subcoercive on $\tilde{G}$.*

*Proof.* — For $D \in \mathfrak{D}(G)$, let $\overline{D} \in \mathfrak{D}(G)$ be the differential operator uniquely determined by the identity $\overline{Df} = \overline{D}\,\overline{f}$. The map $D \mapsto \overline{D}$ defines a conjugate-linear involutive automorphism of the unital algebra $\mathfrak{D}(G)$, and it is easily proved that $\tilde{D}^\bullet(f \otimes g) = f \otimes (\overline{D}g)$. In particular, we have

$$(\tilde{\Delta} + \tilde{\Delta}^\bullet)(f \otimes g) = (B^2 f) \otimes g + f \otimes (\overline{B}^2 g).$$

In view of [26, Theorem 5.4], since $B$ is self-adjoint and weighted subcoercive, in order to conclude it will be sufficient to show that $\overline{B}$ is weighted subcoercive too.

As in [26, §2], fix a weighted structure of $\mathfrak{g}$ and a weighted subcoercive form $C$ such that $d\,\mathrm{R}_G(C) = B$. If $\overline{C}$ is the form defined by $\overline{C}(\alpha) = \overline{C(\alpha)}$, then it is easy to see that $\overline{B} = d\,\mathrm{R}_G(\overline{C})$, and, on the other hand,

$$\Re\langle \phi, d\,\mathrm{R}_G(\overline{C})\phi\rangle = \Re\langle \overline{\phi}, d\,\mathrm{R}_G(C)\overline{\phi}\rangle,$$

thus $\overline{C}$ is also weighted subcoercive by definition. □

Let $L_1, \dots, L_n \in \mathfrak{D}(G)$ be a weighted subcoercive system on the nilpotent Lie group $G$, and let $\Delta = p(L)^2$, where $p$ is a real polynomial such that $p(L)$ is weighted subcoercive. We define

$$\tilde{A} = (\tilde{\Delta} + \tilde{\Delta}^\bullet)/2, \qquad A = d\xi(\tilde{A}) = (\Delta + \Delta^\circ)/2.$$

By Lemma 3.1, $\tilde{A}$ is a (left-invariant) positive weighted subcoercive operator on $\tilde{G}$, whereas $A$ is a differential operator on $G$ which in general is neither left- nor right-invariant; since $\tilde{A}, \tilde{\Delta}, \tilde{\Delta}^\bullet$ form a weighted subcoercive system, the corresponding operators $A, \Delta, \Delta^\circ$ in the representation $\xi$ admit a joint spectral resolution.

Let $h_t$ ($t > 0$) be the convolution kernel of $e^{-t\Delta}$.

LEMMA 3.2. — *Suppose that $u \in L^2(G)$ commutes with all the $h_t$ ($t > 0$). For all Borel $m : \mathbb{R} \to \mathbb{C}$, $u$ is in the domain of $m(\Delta)$ if and only if it is in the domain of $m(A)$, and in this case*

$$m(A)u = m(\Delta)u. \tag{3.4}$$

*Proof.* — From (3.3) we easily deduce

$$e^{-t\Delta^\circ} f = (f^* * h_t)^* = h_t * f$$

and

$$e^{-tA} f = e^{-t\Delta/2} e^{-t\Delta^\circ/2} f = h_{t/2} * f * h_{t/2},$$

so that $e^{-tA}u = e^{-t\Delta}u$. If $\xi_t(\lambda) = e^{-t\lambda}$ and $\mathcal{J}_0 = \operatorname{span}\{\xi_t : t > 0\}$, then we obtain (3.4) for $m \in \mathcal{J}_0$. It is not difficult to extend (3.4) to $m \in C_0(\mathbb{R})$ by the Stone-Weierstrass theorem, and then to all Borel $m : \mathbb{R} \to \mathbb{C}$ by the spectral theorem and dominated convergence. □

LEMMA 3.3. — *Let $X \in \mathfrak{g}$. Then, for all $v \in \mathfrak{g}$,*

$$(X + X^\circ)|_{\exp(v)} = d\exp_v([v, X]).$$

*Proof.* — The semigroup associated to $\tilde{X} + \tilde{X}^\bullet$ is $t \mapsto (\exp(tX), \exp(tX))$, so that, for all $f \in \mathcal{D}(G)$, $v \in \mathfrak{g}$,

$$(X + X^\circ)|_{\exp(v)} f = \left.\frac{d}{dt}\right|_{t=0} f(\exp(-tX)\exp(v)\exp(tX)).$$

Since $\exp(-tX)\exp(v)\exp(tX) = \exp(\operatorname{Ad}(\exp(-tX))(v))$, we have

$$\left.\frac{d}{dt}\right|_{t=0} (\exp(-tX)\exp(v)\exp(tX)) = d\exp_v(\operatorname{ad}(-X)(v)) = d\exp_v([v, X]),$$

which is the conclusion. □

In the following, we will identify $G$ with $\mathfrak{g}$ via the exponential map. Choose a basis $\nu_1, \dots, \nu_r$ of $(\mathfrak{g}/\mathfrak{z})^*$ and a basis $T_1, \dots, T_d$ of $\mathfrak{z}$, and set $P_j = \nu_j \circ P$. The functions $P_j : G \to \mathbb{R}$ can be thought of as multiplication operators on $L^2(G)$, and it is not difficult to show that the operators

$$P_1, \dots, P_r, -iT_1, \dots, -iT_d$$

are (essentially) self-adjoint on $L^2(G)$ and commute strongly pairwise, so that they admit a joint spectral resolution.

Through the chosen bases, $J$ can be identified with a bilinear map $\mathbb{R}^r \times \mathbb{R}^d \to \mathfrak{y}^*$. Therefore, for every $Y \in \mathfrak{y}$, we have a bilinear form $J(\cdot,\cdot)(Y) : \mathbb{R}^r \times \mathbb{R}^d \to \mathbb{R}$, which in fact is a polynomial; we can then evaluate this particular polynomial in the operators $P_1,\dots,P_r,-iT_1,\dots,-iT_d$, and denote by $J(P,-iT)(Y)$ the resulting operator on $L^2(G)$. Finally, choose an inner product on $\mathfrak{y}$ (which induces an inner product on $\mathfrak{y}^*$) and an orthonormal basis $\{Y_l\}_l$ of $\mathfrak{y}$; then also the map $|J(\cdot,\cdot)|^2$ is a polynomial, thus as before we can consider the operator $|J(P,-iT)|^2$ on $L^2(G)$, and clearly $|J(P,-iT)|^2 = \sum_l (J(P,-iT)(Y_l))^2$.

Lemma 3.4. — *For all $Y \in \mathfrak{y}$, $J(P,-iT)(Y)$ is a differential operator on $G$; more precisely, $J(P,-iT)(Y) = -i(Y + Y^\circ)$. In particular*

$$|J(P,-iT)|^2 = -\sum_l (Y_l + Y_l^\circ)^2 = d\xi\left(-\sum_l (\tilde{Y}_l + \tilde{Y}_l^\bullet)^2\right).$$

*Proof.* — Let $\hat{T}_1,\dots,\hat{T}_d \in \mathfrak{z}^*$ and $\hat{\nu}_1,\dots,\hat{\nu}_r \in \mathfrak{g}/\mathfrak{z}$ be the dual bases of $T_1,\dots,T_d$ and $\nu_1,\dots,\nu_r$ respectively. Then, by bilinearity, for every $Y \in \mathfrak{y}$,

$$J(P,-iT)(Y) = -i\sum_{j,k} J(\hat{\nu}_j,\hat{T}_k)(Y) P_j T_k.$$

This shows that $J(P,-iT)(Y)$ is a differential operator on $G$. In fact, for all $x \in G = \mathfrak{g}$, we have $\sum_j P_j(x)\hat{\nu}_j = P(x)$, therefore

$$\begin{aligned} J(P,-iT)(Y)|_x &= -i\sum_k J(P(x),\hat{T}_k)(Y) T_k \\ &= -i\sum_k \hat{T}_k([x,Y]) T_k = -i[x,Y] = -i(Y+Y^\circ)|_x \end{aligned}$$

by Lemma 3.3 (notice that, since $T_1,\dots,T_d$ are central, they are constant vector fields in exponential coordinates). □

Since $T_1,\dots,T_d$ are central, the left-invariant differential operators

$$L_1,\dots,L_n,-iT_1,\dots,-iT_d \tag{3.5}$$

on $G$ are a weighted subcoercive system. We can thus consider the Plancherel measure $\sigma'$ on $\mathbb{R}^n \times \mathfrak{z}^*$ associated to this system, which can be shown not to depend on the choice of the basis of $\mathfrak{z}$.

The core of the technique under discussion is contained in the following

Proposition 3.5. — *Suppose that, for some nonnegative Borel functions $w : \mathfrak{g}/\mathfrak{z} \to \mathbb{R}$ and $\zeta : \mathfrak{z}^* \to \mathbb{R}$, we have*

$$|J(\bar{x},\tau)| \geqslant w(\bar{x})\,\zeta(\tau) \qquad \text{for all } \bar{x} \in \mathfrak{g}/\mathfrak{z},\ \tau \in \mathfrak{z}^*.$$

*If $K \subseteq \mathbb{R}^n$ is compact and $\gamma \geqslant 0$, then, for all $m \in \mathcal{D}(\mathbb{R}^n)$ with* $\operatorname{supp} m \subseteq K$,

$$\| |w \circ P|^\gamma \breve{m} \|_2^2 \leqslant C_{K,\gamma} \int_{\mathbb{R}^n \times \mathfrak{z}^*} |m(\lambda)|^2 \, \zeta(\tau)^{-2\gamma} \, d\sigma'(\lambda, \tau).$$

*Proof.* — From the hypothesis we deduce, by the spectral theorem,

$$\| |w \circ P|^\gamma f \|_2 \leqslant C_\gamma \| |J(P, -iT)|^\gamma \zeta(-iT)^{-\gamma} f \|_2$$

for $f \in L^2(G)$. By Lemma 3.4, $|J(P, -iT)|^2 = d\xi(D)$ for some $D \in \mathfrak{D}(\tilde{G})$; since $\tilde{A}$ is weighted subcoercive on $\tilde{G}$, by [26, Theorem 2.3(iii)], for some polynomial $q_\gamma$ we have, in the representation $\xi$,

$$\| |J(P, -iT)|^\gamma \psi \|_2 \leqslant C_\gamma \| q_\gamma(A) \psi \|_2,$$

therefore, by putting the two inequalities together, we get

$$\| |w \circ P|^\gamma f \|_2 \leqslant C_\gamma \| \zeta(-iT)^{-\gamma} q_\gamma(A) f \|_2$$

(since the $T_j$ commute strongly with $A$). In particular, if we take $f = \breve{m}$,

$$\| |w \circ P|^\gamma \breve{m} \|_2 \leqslant C_\gamma \| \zeta(-iT)^{-\gamma} q_\gamma(A) \breve{m} \|_2 = C_\gamma \| \zeta(-iT)^{-\gamma} q_\gamma(\Delta) \breve{m} \|_2$$

by Lemma 3.2, since $\breve{m}$ commutes with all the $h_t$. On the other hand, by the Plancherel formula for the system (3.5),

$$\| \zeta(-iT)^{-\gamma} q_\gamma(\Delta) \breve{m} \|_2^2 \leqslant C_{K,\gamma} \int_{\mathbb{R}^n \times \mathfrak{z}^*} |\breve{m}(\lambda)|^2 \, \zeta(\tau)^{-2\gamma} \, d\sigma'(\lambda, \tau),$$

where $C_{K,\gamma} = \sup_{\lambda \in K} q_\gamma(p(\lambda)^2)^2$, and we are done. □

Simple manipulations give a slightly more general form of the previous estimate:

Corollary 3.6. — *Suppose that, for some nonnegative Borel functions $w_j : \mathfrak{g}/\mathfrak{z} \to \mathbb{R}$ and $\zeta_j : \mathfrak{z}^* \to \mathbb{R}$ ($j = 1, \dots, h$), we have*

$$|J(\bar{x}, \tau)| \geqslant w_j(\bar{x}) \, \zeta_j(\tau) \qquad \textit{for all } \bar{x} \in \mathfrak{g}/\mathfrak{z},\ \tau \in \mathfrak{z}^*,$$

*and set $\tilde{w}_j(x) = 1 + w_j(P(x))$. If $K \subseteq \mathbb{R}^n$ is compact, then for all $m \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp} m \subseteq K$ and for all $\vec{\gamma} = (\gamma_1, \dots, \gamma_h) \geqslant 0$ we have*

$$\| \tilde{w}_1^{\gamma_1} \cdots \tilde{w}_h^{\gamma_h} \, \breve{m} \|_{L^2(G)}^2 \leqslant C_{K,\vec{\gamma}} \int_{\mathbb{R}^n \times \mathfrak{z}^*} |m(\lambda)|^2 \prod_{j=1}^h (1 + \zeta_j(\tau)^{-2\gamma_j}) \, d\sigma'(\lambda, \tau).$$

*Proof.* — If we set, for $I \subseteq \{1, \dots, h\}$,

$$\gamma_I = \sum_{j \in I} \gamma_j, \qquad w_{\vec{\gamma},I}(\bar{x}) = \prod_{j \in I} w_j(\bar{x})^{\gamma_j/\gamma_I}, \qquad \zeta_{\vec{\gamma},I}(\tau) = \prod_{j \in I} \zeta_j(\tau)^{\gamma_j/\gamma_I},$$

then clearly

$$|J(\bar{x}, \tau)| \geqslant w_{\vec{\gamma},I}(\bar{x}) \, \zeta_{\vec{\gamma},I}(\tau) \qquad \text{for all } \bar{x} \in \mathfrak{g}/\mathfrak{z},\ \tau \in \mathfrak{z}^*,$$

and moreover

$$\prod_{j=1}^{h} \tilde{w}_j^{2\gamma_j} \leqslant C_{\vec{\gamma}} \sum_{I \subseteq \{1,\dots,h\}} (w_{\vec{\gamma},I} \circ P)^{2\gamma_I}, \quad \prod_{j=1}^{h} (1 + \zeta_j^{-2\gamma_j}) = \sum_{I \subseteq \{1,\dots,h\}} \zeta_{\vec{\gamma},I}^{-2\gamma_I},$$

therefore the conclusion follows by repeated application of Proposition 3.5. □

Under some particular hypotheses, we may therefore control a weighted $L^2$ norm of $\breve{m}$ in terms of an $L^2(\sigma_{\tilde{\zeta}})$ norm of $m$, where $\sigma_{\tilde{\zeta}}$ is the push-forward of

$$\tilde{\zeta}(\tau)\, d\sigma'(\lambda, \tau) \tag{3.6}$$

on the first factor of $\mathbb{R}^n \times \mathfrak{z}^*$, for some nonnegative function $\tilde{\zeta} : \mathfrak{z}^* \to \mathbb{R}$.

Lemma 3.7. — (i) *Suppose that $\tilde{\zeta} \in L^1_{\mathrm{loc}}(\mathfrak{z}^*)$ is nonnegative. Then $\sigma_{\tilde{\zeta}}$ is a regular Borel measure on $\mathbb{R}^n$.*

(ii) *Suppose moreover that $G$ is a homogeneous group, with dilations $\delta_t$ and homogeneous dimension $Q_\delta$, and that $L_1, \dots, L_n$ is a homogeneous system, with associated dilations $\epsilon_t$. If $\tilde{\zeta}$ is homogeneous of degree $a$, i.e., $\tilde{\zeta}(\tau \circ \delta_t) = t^a \tilde{\zeta}(\tau)$, then $\sigma_{\tilde{\zeta}}$ is homogeneous of degree $Q_\delta + a$, i.e., $\sigma_{\tilde{\zeta}}(\epsilon_t(A)) = t^{Q_\delta + a} \sigma_{\tilde{\zeta}}(A)$.*

*Proof.* — (i) Let $K \subseteq \mathbb{R}^n$ be compact. By [26, Lemma 3.16], the canonical projection $\mathbb{R}^n \times \mathfrak{z}^* \to \mathbb{R}^n$ is a proper continuous map when restricted to $\operatorname{supp} \sigma'$, hence there is a compact $K' \subseteq \mathfrak{z}^*$ such that $(K \times \mathfrak{z}^*) \cap \operatorname{supp} \sigma' \subseteq K \times K'$, and consequently

$$\sigma_{\tilde{\zeta}}(K) \leqslant C_K \int_{K \times K'} e^{-2p(\lambda)^2}\, \tilde{\zeta}(\tau)\, d\sigma'(\lambda, \tau) = C_K \|(\tilde{\zeta} \chi_{K'})^{1/2}(-iT) h_1\|_2^2,$$

by the Plancherel formula. On the other hand, since $h_1$ is in the Schwartz class, the last quantity is easily seen to be finite by using the Euclidean Fourier transform and the fact that $(\tilde{\zeta} \chi_{K'})^{1/2} \in L^2(\mathfrak{z}^*)$. We have thus proved that $\sigma_{\tilde{\zeta}}$ is finite on compacta; by [36, Theorem 2.18], this means that $\sigma_{\tilde{\zeta}}$ is a regular Borel measure on $\mathbb{R}^n$.

(ii) Without loss of generality, we may take the basis $T_1, \dots, T_d$ of $\mathfrak{z}$ as composed by $\delta_t$-homogeneous elements; thus (3.5) is a homogeneous system, and the associated dilations $\epsilon'_t$ on $\mathbb{R}^n \times \mathfrak{z}^*$ are given by $\epsilon'_t(\lambda, \tau) = (\epsilon_t(\lambda), \tau \circ \delta_t)$. By (1.1), $\sigma'$ is $\epsilon'_t$-homogeneous of degree $Q_\delta$. Therefore, if $\tilde{\zeta}$ is homogeneous of degree $a$, then clearly the measure (3.6) is homogeneous of degree $Q_\delta + a$; since the canonical projection $\mathbb{R}^n \times \mathfrak{z}^* \to \mathbb{R}^n$ intertwines the two system of dilations, we infer that also $\sigma_{\tilde{\zeta}}$ is homogeneous of degree $Q_\delta + a$. □

Via interpolation, we then obtain an improvement of Theorem 2.7, where the role of the Plancherel measure $\sigma$ is now played by some $\sigma_{\tilde{\zeta}}$.

PROPOSITION 3.8. — (i) *Suppose that, for some nonnegative Borel functions* $w_j : \mathfrak{g}/\mathfrak{z} \to \mathbb{R}$ *and* $\zeta_j : \mathfrak{z}^* \to \mathbb{R}$ $(j = 1, \dots, h)$, *we have*

$$|J(\bar{x}, \tau)| \geqslant w_j(\bar{x})\, \zeta_j(\tau) \qquad \text{for all } \bar{x} \in \mathfrak{g}/\mathfrak{z},\ \tau \in \mathfrak{z}^*, \tag{3.7}$$

*and set* $\tilde{w}_j(x) = 1 + w_j(P(x))$. *Suppose moreover that, for some* $\gamma_1, \dots, \gamma_h > 0$, *if* $\tilde{\zeta}_{\vec{\gamma}} = \prod_{j=1}^h (1 + \zeta_j^{-2\gamma_j})$, *then the measure* $\sigma_{\tilde{\zeta}_{\vec{\gamma}}}$ *is locally* $d$*-bounded on some open* $\Omega \subseteq \mathbb{R}^n$. *If* $K \subseteq \Omega$ *is compact,* $q \in [1, \infty]$, $\alpha \geqslant 0$,

$$\beta > \alpha + \frac{n}{q} - \frac{d}{\max\{2, q\}},$$

*then, for all* $m \in \mathcal{D}(\mathbb{R}^n)$ *with* $\operatorname{supp} m \subseteq K$,

$$\|\langle \cdot \rangle_G^\alpha\, \tilde{w}_1^{\gamma_1} \cdots \tilde{w}_h^{\gamma_h}\, \breve{m}\|_{L^2(G)} \leqslant C_{K,\alpha,\vec{\gamma},\beta} \|m\|_{B^\beta_{q,q}(\mathbb{R}^n)}.$$

(ii) *Suppose in addition that* $\int_G \langle x \rangle_G^{-2\alpha} \prod_{j=1}^h \tilde{w}_j(x)^{-2\gamma_j}\, dx < \infty$ *for* $\alpha > \bar{\alpha}_{\vec{\gamma}}$. *If* $K \subseteq \Omega$ *is compact,* $q \in [1, \infty]$, $\alpha \geqslant 0$,

$$\beta > \alpha + \bar{\alpha}_{\vec{\gamma}} + \frac{n}{q} - \frac{d}{\max\{2, q\}},$$

*then, for all* $m \in \mathcal{D}(\mathbb{R}^n)$ *with* $\operatorname{supp} m \subseteq K$,

$$\|\langle \cdot \rangle_G^\alpha\, \breve{m}\|_{L^1(G)} \leqslant C_{K,\alpha,\vec{\gamma},\beta} \|m\|_{B^\beta_{q,q}(\mathbb{R}^n)}.$$

*Proof.* — (i) Since $\sigma_{\tilde{\zeta}_{\vec{\gamma}}}$ is locally $d$-bounded, the function $\zeta_j$ cannot be everywhere null, therefore (3.7) and the bilinearity of $J$ imply $w_j(P(x)) \leqslant C\langle x \rangle_G^\theta$ for some $C, \theta \geqslant 0$, thus also $\prod_{j=1}^h \tilde{w}_j(x)^{2\gamma_j} \leqslant C_{\vec{\gamma}} \langle x \rangle_G^{2\theta(\gamma_1 + \cdots + \gamma_h)}$ for some $C_{\vec{\gamma}} \geqslant 0$.

Let $\psi \in \mathcal{D}(\mathbb{R}^n)$ such that $\psi|_K = 1$ and $K' = \operatorname{supp} \psi \subseteq \Omega$. The operator $m \mapsto \mathcal{K}_L(m\psi)$ is then continuous

$$B^\beta_{q,q}(\mathbb{R}^n) \to L^2\Big(G, \langle x \rangle_G^{2\alpha} \prod_{j=1}^h \tilde{w}_j(x)^{2\gamma_j}\, dx\Big)$$

for $\alpha \geqslant 0$, $\beta > \alpha + \theta(\gamma_1 + \cdots + \gamma_h) + n$ by Theorem 2.7, whereas it is continuous

$$B^\beta_{q,q}(\mathbb{R}^n) \to L^2\Big(G, \prod_{j=1}^h \tilde{w}_j(x)^{2\gamma_j}\, dx\Big)$$

for $\beta > n/q - d/\max\{2, q\}$ by Corollary 3.6 and Lemma 2.6 (*cf.* the proof of Theorem 2.7). The conclusion then follows by interpolation.

(ii) It follows from (i) by Hölder's inequality. □

The hypotheses of the previous proposition are quite involved, and it is not particularly clear which classes of groups and systems of operators satisfy them. Hebisch and Zienkiewicz [20] treat explicitly the case of direct products of H-type groups; however, as it is mentioned in a remark at the end of [20], there are further cases of homogeneous groups for which this technique gives an improvement of the weighted $L^1$ estimates. In order to attempt a systematic treatment of these various cases, we introduce the following definition: for $h \in \mathbb{N}$, we say that a homogeneous Lie group $G$ is *$h$-capacious* if there exist linearly independent homogeneous elements $\omega_1, \dots, \omega_h \in (\mathfrak{g}/\mathfrak{z})^*$ and linearly independent homogeneous elements $z_1, \dots, z_h \in \mathfrak{z}$ such that, for $j = 1, \dots, h$,

$$(3.8) \qquad |J(\bar{x}, \tau)| \geqslant |\omega_j(x)||\tau(z_j)| \qquad \text{for all } \bar{x} \in \mathfrak{g}/\mathfrak{z},\ \tau \in \mathfrak{z}^*.$$

Clearly, every homogeneous group is 0-capacious. In the following proposition, we give some criteria which may be of some use in showing that a certain homogeneous group is $h$-capacious. Let us denote by

$$(3.9) \qquad \mathfrak{g}_{[1]} = \mathfrak{g}, \qquad \mathfrak{g}_{[r+1]} = [\mathfrak{g}, \mathfrak{g}_{[r]}]$$

the descending central series of a Lie algebra $\mathfrak{g}$.

Proposition 3.9. — *Let $G$ be a homogeneous group, with dilations $\delta_t$.*

(i) *If $G$ is a Métivier group (with any family of automorphic dilations), then $\mathfrak{z} = [\mathfrak{g}, \mathfrak{g}]$ and $G$ is $(\dim \mathfrak{z})$-capacious.*
(ii) *Suppose that, for some $r \geqslant 2$, $\dim \mathfrak{g}_{[r]} = 1$. Then $G$ is 1-capacious.*
(iii) *If $\mathfrak{g}$ admits a $\mathbb{C}$-linear structure which is compatible with its homogeneous Lie algebra structure, and if moreover $\dim_{\mathbb{C}} \mathfrak{g}_{[r]} = 1$ for some $r \geqslant 2$, then $\mathfrak{g}$ is 2-capacious.*
(iv) *Suppose that $G = G_1 \times G_2$, where $G_1$ and $G_2$ are homogeneous Lie groups with dilations $\delta_{1,t}$ and $\delta_{2,t}$ respectively, so that $\delta_t = \delta_{1,t} \times \delta_{2,t}$. If $G_1$ is $h_1$-capacious and $G_2$ is $h_2$-capacious, then $G$ is $(h_1 + h_2)$-capacious.*

*Proof.* — (i) Notice that the capacity map $J$ takes its values in the subspace of $\mathfrak{y}^*$ corresponding to $(\mathfrak{y}/\mathfrak{z})^*$. The condition (3.2) implies that $J(\cdot, \tau) : \mathfrak{g}/\mathfrak{z} \to (\mathfrak{y}/\mathfrak{z})^*$ is injective for $\tau \neq 0$, and that $J(\bar{x}, \cdot) : \mathfrak{z}^* \to (\mathfrak{y}/\mathfrak{z})^*$ is injective for $\bar{x} \neq 0$. Therefore $\dim \mathfrak{g} \leqslant \dim \mathfrak{y}$, so that $\mathfrak{g} = \mathfrak{y}$ and $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{z}$; on the other hand, $\dim(\mathfrak{g}/\mathfrak{z}) \geqslant \dim \mathfrak{z}$.

The $\delta_t$ are automorphisms, hence $\mathfrak{z}$ is a homogeneous ideal. Thus, if $h = \dim \mathfrak{z}$, we can choose linearly independent homogeneous elements $z_1, \dots, z_h$ of $\mathfrak{z}$, and also linearly independent homogeneous $\omega_1, \dots, \omega_h \in (\mathfrak{g}/\mathfrak{z})^*$, since

$h \leqslant \dim(\mathfrak{g}/\mathfrak{z})$. Modulo a suitable renormalization, from (3.2) we infer (3.8) for $j = 1, \ldots, h$.

If $[\mathfrak{g}, \mathfrak{g}]$ were strictly contained in $\mathfrak{z}$, then we would find $\tau \in \mathfrak{z}^* \setminus \{0\}$ such that $\tau|_{[\mathfrak{g},\mathfrak{g}]} = 0$, but then also $J(\cdot, \tau) = 0$, which contradicts (3.2); therefore $\mathfrak{z} = [\mathfrak{g}, \mathfrak{g}]$.

(ii) Since $G$ is nilpotent, it must be $r$-step, so that $\mathfrak{g}_{[r]} \subseteq \mathfrak{z}$. Notice that the ideal $\mathfrak{g}_{[r-1]}$ is preserved by every automorphism of $\mathfrak{g}$, therefore it is generated by $\delta_t$-homogeneous elements; since $[\mathfrak{g}, \mathfrak{g}_{[r-1]}] = \mathfrak{g}_{[r]} \neq 0$, then there must exist a $\delta_t$-homogeneous element $y \in \mathfrak{g}_{[r-1]}$ such that, for some $x_0 \in \mathfrak{g}$, $[x_0, y] = z \neq 0$. In particular $y \neq 0$ and moreover, since the ideal $\mathfrak{g}_{[r]}$ is $\delta_t$-homogeneous and 1-dimensional, necessarily $z$ is $\delta_t$-homogeneous.

Since $y \in \mathfrak{g}_{[r-1]}$, the linear map $[\cdot, y] : \mathfrak{g} \to \mathfrak{g}$ takes its values in $\mathfrak{g}_{[r]} = \mathbb{R}z$; therefore, there exists $\omega \in (\mathfrak{g}/\mathfrak{z})^*$ such that $[x, y] = \omega(P(x))z$ for all $x \in \mathfrak{g}$. Notice that $\omega(P(x_0)) = 1$, thus $\omega \neq 0$; moreover, since both $y$ and $z$ are homogeneous, also $\omega$ is homogeneous. Finally

$$J(\bar{x}, \tau)(y) = \omega(\bar{x})\tau(z) \qquad \text{for all } \bar{x} \in \mathfrak{g}/\mathfrak{z},\ \tau \in \mathfrak{z}^*, \tag{3.10}$$

which implies immediately that $G$ is 1-capacious.

(iii) Arguing as in part (ii), but with a complex Lie algebra $\mathfrak{g}$, one finds an identity analogous to (3.10), where now $\omega$ is a $\mathbb{C}$-linear functional on $\mathfrak{g}/\mathfrak{z}$, and $z \in \mathfrak{z}$. The conclusion then follows by taking the $\mathbb{R}$-linearly independent $\mathbb{R}$-linear functionals $\Re\omega, \Im\omega$ on $\mathfrak{g}/\mathfrak{z}$, and the $\mathbb{R}$-linearly independent elements $z, iz \in \mathfrak{z}$.

(iv) Via the canonical identification $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$, we have (with the obvious meaning of the notation) $\mathfrak{z} = \mathfrak{z}_1 \times \mathfrak{z}_2$, $\mathfrak{y} = \mathfrak{y}_1 \times \mathfrak{y}_2$, thus also

$$\mathfrak{z}^* = \mathfrak{z}_1^* \times \mathfrak{z}_2^*, \qquad \mathfrak{y}^* = \mathfrak{y}_1^* \times \mathfrak{y}_2^*, \qquad \mathfrak{g}/\mathfrak{z} = (\mathfrak{g}_1/\mathfrak{z}_1) \times (\mathfrak{g}_2/\mathfrak{z}_2).$$

Moreover clearly $J((\bar{x}_1, \bar{x}_2), (\tau_1, \tau_2)) = (J_1(\bar{x}_1, \tau_1), J_2(\bar{x}_2, \tau_2))$, therefore

$$|J((\bar{x}_1, \bar{x}_2), (\tau_1, \tau_2))| \geqslant \max\{|J_1(\bar{x}_1, \tau_1)|, |J_2(\bar{x}_2, \tau_2)|\}$$

and the conclusion follows immediately. □

Notice that the previous proposition is not sufficient to exhaust all the cases of $h$-capacious groups; an example is shown in § 6.2.

Lemma 3.10. — *Suppose that $G$ is $h$-capacious, and let $\omega_1, \ldots, \omega_h \in (\mathfrak{g}/\mathfrak{z})^*$ be as in the definition. Then the functionals $\omega_j \circ P$ are null on $[\mathfrak{g}, \mathfrak{g}]$. In particular*

$$h \leqslant \min\{\dim \mathfrak{z}, \dim \mathfrak{g} - \dim(\mathfrak{z} + [\mathfrak{g}, \mathfrak{g}])\}.$$

*Moreover, we can find a homogeneous basis of $\mathfrak{g}$ compatible with the descending central series such that the functionals $\omega_1 \circ P, \ldots, \omega_h \circ P$ are part of the dual basis.*

*Proof.* — Notice that $[[\mathfrak{g},\mathfrak{g}],\mathfrak{y}] \subseteq [\mathfrak{g},[\mathfrak{g},\mathfrak{y}]] \subseteq [\mathfrak{g},\mathfrak{z}] = 0$. Then from the definition of $J$ it follows that, for every $x \in [\mathfrak{g},\mathfrak{g}]$,

$$J(P(x),\tau) = 0 \qquad \text{for all } \tau \in \mathfrak{z}^*.$$

Hence, by choosing in (3.8) a $\tau \in \mathfrak{z}^*$ such that $\tau(z_j) \neq 0$, we obtain that the functional $\omega_j \circ P$ is null on $[\mathfrak{g},\mathfrak{g}]$. In particular, the $\omega_j \circ P$ correspond to linearly independent elements of $(\mathfrak{g}/([\mathfrak{g},\mathfrak{g}]+\mathfrak{z}))^*$, and the inequality about $h$ follows.

Let now $W = \ker(\omega_1 \circ P) \cap \dots \cap \ker(\omega_h \circ P)$. Then $W$ is a homogeneous subspace of $\mathfrak{g}$ containing $[\mathfrak{g},\mathfrak{g}]$. Moreover, if $\tilde{\omega}_j$ is the element of $(\mathfrak{g}/W)^*$ corresponding to $\omega_j$, then $\tilde{\omega}_1,\dots,\tilde{\omega}_h$ are a homogeneous basis of $(\mathfrak{g}/W)^*$. We can then choose homogeneous elements $v_1,\dots,v_h \in \mathfrak{g}$ such that the corresponding elements in the quotient $\mathfrak{g}/W$ are the dual basis of $\tilde{\omega}_1,\dots,\tilde{\omega}_h$. Finally, we append to $v_1,\dots,v_h$ a homogeneous basis of $W$ compatible with the descending central series (which, apart from $\mathfrak{g}_{[1]}$, is contained in $W$, and is made of homogeneous ideals), and we are done. □

Here is finally the improvement of Theorem 2.7 for $h$-capacious groups.

Theorem 3.11. — *Suppose that $G$ is $h$-capacious, and let $Q_G$ be its degree of polynomial growth. Let moreover $L_1,\dots,L_n$ be a homogeneous weighted subcoercive system on $G$. If $q \in [1,\infty]$, $\alpha \geqslant 0$ and*

$$\beta > \alpha + \frac{Q_G - h}{2} + \frac{n}{q} - \frac{1}{\max\{2,q\}},$$

*then, for every $K \subseteq \mathbb{R}^n \setminus \{0\}$ compact,*

$$\|\breve{m}\|_{L^1(G,\langle x\rangle_G^\alpha\,dx)} \leqslant C_{K,\alpha,\beta}\|m\|_{B^\beta_{q,q}}$$

*for all $m \in \mathcal{D}(G)$ with $\operatorname{supp} m \subseteq K$.*

*Proof.* — Let $\omega_1,\dots,\omega_h \in (\mathfrak{g}/\mathfrak{z})^*$ and $z_1,\dots,z_h \in \mathfrak{z}$ be as in the definition of $h$-capacious, and set $w_j(\bar{x}) = |\omega_j(\bar{x})|$, $\zeta_j(\tau) = |\tau(z_j)|$. Notice now that, since the $z_j$ are linearly independent, for every choice of $\gamma_1,\dots,\gamma_h \in {]0,1/2[}$, the function $\tilde{\zeta}_{\vec{\gamma}} = \prod_{j=1}^h (1+\zeta_j^{-2\gamma_j})$ is in $L^1_{\mathrm{loc}}(\mathfrak{z}^*)$, so that, by Lemma 3.7, $\sigma_{\tilde{\zeta}_{\vec{\gamma}}}$ is a regular Borel measure; in fact, since the $z_j$ are homogeneous, $\sigma_{\tilde{\zeta}_{\vec{\gamma}}}$ is the sum of $\epsilon_t$-homogeneous regular Borel measures on $\mathbb{R}^n$ (with possibly different degrees of homogeneity), hence $\sigma_{\tilde{\zeta}_{\vec{\gamma}}}$ is locally 1-bounded on $\mathbb{R}^n \setminus \{0\}$.

By Lemma 3.10, we can find a homogeneous basis $v_1,\dots,v_k$ of $\mathfrak{g}$, compatible with the descending central series, such that, if $\hat{v}_1,\dots,\hat{v}_h$ is the dual basis, then $\hat{v}_j = \omega_j \circ P$ for $j = 1,\dots,h$; in particular we have

$\tilde{w}_j(x) = 1 + |\hat{v}_j(x)|$. If we set $\kappa_j = \max\{r : v_j \in \mathfrak{g}_{[r]}\}$, then $Q_G = \sum_{j=1}^k \kappa_j$ and $\langle x \rangle_G \sim 1 + \sum_{j=1}^k |\hat{v}_j(x)|^{1/\kappa_j}$ (*cf.* [26, Proposition 2.1]), thus

$$\langle x \rangle_G^{-2\alpha_j} \leqslant C_{\alpha_j} (1 + |\hat{v}_j(x)|)^{-2\alpha_j/\kappa_j}$$

for $j = 1, \dots, k$ and $\alpha_j \geqslant 0$. Moreover, since the $\omega_j \circ P$ are null on $[\mathfrak{g}, \mathfrak{g}]$, then $\kappa_j = 1$ for $j = 1, \dots, h$.

Notice now that, for fixed $\gamma_1, \dots, \gamma_h \in ]0, 1/2[$, if $\alpha \geqslant 0$ satisfies

$$2\alpha > 2\alpha_{\vec{\gamma}} = \sum_{j=1}^h (1 - 2\gamma_j) + \sum_{j=h+1}^k \kappa_j,$$

then we may choose $\alpha_1, \dots, \alpha_k \geqslant 0$ such that

$$\alpha = \sum_{j=1}^k \alpha_j, \qquad 2\alpha_j > \begin{cases} 1 - 2\gamma_j & \text{for } j = 1, \dots, h, \\ \kappa_j & \text{for } j = h+1, \dots, k, \end{cases}$$

therefore

$$\langle x \rangle_G^{-2\alpha} \prod_{j=1}^h \tilde{w}_j(x)^{-2\gamma_j} \leqslant \prod_{j=1}^h (1 + |\hat{v}_j(x)|)^{-2(\alpha_j + \gamma_j)} \prod_{j=h+1}^k (1 + |\hat{v}_j(x)|)^{-2\alpha_j/\kappa_j},$$

and the right-hand side is clearly integrable over $G$. We can thus apply Proposition 3.8(ii), and the conclusion follows because, if the $\gamma_j$ tend to $1/2$, then $\alpha_{\vec{\gamma}}$ tends to $\sum_{j=h+1}^k \kappa_j = Q_G - h$. ☐

Notice that, if $G$ is a Métivier group, by Proposition 3.9(i) we can take $h = \dim[\mathfrak{g}, \mathfrak{g}]$, so that $Q_G - h = \dim G$; in fact, by Proposition 3.9(iv), the same holds if $G$ is a direct product of Métivier and Euclidean groups.

## 4. Mihlin-Hörmander multipliers

Let $G$ be a homogeneous Lie group, with dimension at infinity $Q_G$, automorphic dilations $\delta_t$ and homogeneous dimension $Q_\delta$; as in [26, §2.1], we suppose that the homogeneity degrees of the elements of the Lie algebra $\mathfrak{g}$ are not less than 1, so that $Q_\delta \geqslant Q_G$. Define $|\cdot|_G$, $\langle \cdot \rangle_G$ as in § 2, and denote by $|\cdot|_\delta$ a subadditive homogeneous norm on $G$ (*cf.* [19]).

Let $L_1, \dots, L_n$ be a homogeneous weighted subcoercive system on $G$, with associated dilations $\epsilon_t$, and Plancherel measure $\sigma$. Denote moreover by $|\cdot|_\epsilon$ an $\epsilon_t$-homogeneous norm on $\mathbb{R}^n$, smooth off the origin.

Our starting point is, for some $q \in [1, \infty]$ and $s \in \mathbb{R}$, the following

**hypothesis** $(\mathrm{I}_{q,s})$: for some compact $K_0 \subseteq \mathbb{R}^n \setminus \{0\}$ such that

$$\bigcup_{t>0} \epsilon_t(\mathring{K}_0) = \mathbb{R}^n \setminus \{0\},$$

for all $\beta > s$ and for all $m \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp} m \subseteq K_0$, we have

$$\|\breve{m}\|_{L^1(G)} \leqslant C_\beta \|m\|_{B^\beta_{q,q}(\mathbb{R}^n)}.$$

From Theorems 2.7 and 3.11 we deduce immediately

Proposition 4.1. — *For every $q \in [1,\infty]$, the hypothesis $(\mathrm{I}_{q,s})$ holds in each of the following cases:*

- $s = Q_G/2 + n/q - 1/\max\{2,q\}$;
- *the Plancherel measure $\sigma$ is locally $d$-bounded on $\mathbb{R}^n \setminus \{0\}$ and* $s = Q_G/2 + n/q - d/\max\{2,q\}$;
- *$G$ is $h$-capacious and $s = (Q_G - h)/2 + n/q - 1/\max\{2,q\}$.*

In the rest of this section, we forget how such hypothesis may be checked, and we focus on its consequences.

Proposition 4.2. — *Suppose that $(\mathrm{I}_{q,s})$ holds for some $q \in [1,\infty]$ and $s \in \mathbb{R}$. Then $s \geqslant n/q$. Moreover, for every compact $K \subseteq \mathbb{R}^n \setminus \{0\}$, for every $\alpha \geqslant 0$ and $\beta > \alpha + s$, for every $D \in \mathfrak{D}(G)$, for every $m \in B^\beta_{q,q}(\mathbb{R}^n)$ with $\operatorname{supp} m \subseteq K$, we have*

$$\|\langle\cdot\rangle_G^\alpha D\breve{m}\|_{L^1(G)} \leqslant C_{K,D,\alpha,\beta} \|m\|_{B^\beta_{q,q}(\mathbb{R}^n)}. \tag{4.1}$$

*Proof.* — Let $\lambda \in \mathring{K}_0 \cap \operatorname{supp}\sigma$. For every $m \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp} m \subseteq \mathring{K}_0$, we have $|m(\lambda)| \leqslant \|m\|_{L^\infty(\sigma)} \leqslant \|\breve{m}\|_1 \leqslant C_\beta \|m\|_{B^\beta_{q,q}}$ for all $\beta > s$. Such an inequality gives easily $\|f\|_\infty \leqslant C_\beta \|f\|_{B^\beta_{q,q}}$ for all $f$ in the Schwartz class and $\beta > s$, which however can hold only if $s \geqslant n/q$ (*cf.* [48, § 2.6.2, Theorem 1]).

The hypothesis $(\mathrm{I}_{q,s})$ gives (4.1) in the case $\alpha = 0$, $D = 1$, $m$ smooth, $K = K_0$. The extension to a generic compact $K \subseteq \mathbb{R}^n \setminus \{0\}$ is performed by a partition-of-unity argument and exploiting homogeneity. The full generality is then reached by approximation and interpolation, as in the proof of Theorem 2.7. $\square$

Notice that, by [26, Proposition 2.1], there are constants $a, C > 0$ such that

$$1 + |x|_\delta \leqslant C \langle x\rangle_G^a. \tag{4.2}$$

COROLLARY 4.3. — *Suppose that* $(\mathrm{I}_{q,s})$ *holds. Let* $K \subseteq \mathbb{R}^n \setminus \{0\}$ *be compact,* $\beta > s$. *If* $m \in B^{\beta}_{q,q}(\mathbb{R}^n)$ *and* $\operatorname{supp} m \subseteq K$, *then* $\breve{m} \in L^1(G)$. *Moreover, for* $0 \leqslant \alpha < (\beta - s)/a$,

$$\int_G (1 + |x|_\delta)^\alpha |\breve{m}(x)| \, dx \leqslant C_{K,\alpha,\beta} \|m\|_{B^{\beta}_{q,q}} \tag{4.3}$$

*and, for all* $h \in G$,

$$\begin{aligned} \| \mathrm{R}_h \breve{m} - \breve{m} \|_1 &\leqslant C_{K,\beta} \|m\|_{B^{\beta}_{q,q}} |h|_\delta, \\ \| \mathrm{L}_h \breve{m} - \breve{m} \|_1 &\leqslant C_{K,\beta} \|m\|_{B^{\beta}_{q,q}} |h|_\delta. \end{aligned} \tag{4.4}$$

*Proof.* — Since $\beta > s + a\alpha$, by Proposition 4.2 and (4.2) we have

$$\int_G (1 + |x|_\delta)^\alpha |\breve{m}(x)| \, dx \leqslant C_\alpha \int_G \langle x \rangle_G^{a\alpha} |\breve{m}(x)| \, dx \leqslant C_{K,\alpha,\beta} \|m\|_{B^{\beta}_{q,q}},$$

and in particular $\breve{m} \in L^1(G)$.

Starting from the inequality $\| \mathrm{R}_{\exp(tX)} \breve{m} - \breve{m} \|_1 \leqslant \|X\breve{m}\|_1 |t|$, true for all $X \in \mathfrak{g}$ and $t \in \mathbb{R}$, having chosen a basis $X_1, \dots, X_k$ of $\mathfrak{g}$, with $X_j$ homogeneous of degree $d_j$, we easily obtain $\| \mathrm{R}_h \breve{m} - \breve{m} \|_1 \leqslant C \sum_{j=1}^k \|X_j \breve{m}\|_1 |h|_\delta^{d_j}$, so that also

$$\| \mathrm{R}_h \breve{m} - \breve{m} \|_1 \leqslant C \left( \|\breve{m}\|_1 + \sum_{j=1}^k \|X_j \breve{m}\|_1 \right) |h|_\delta,$$

since $d_j \geqslant 1$. However $\|\breve{m}\|_1 + \sum_{j=1}^k \|X_j \breve{m}\|_1 \leqslant C_{K,\beta} \|m\|_{B^{\beta}_{q,q}}$ by Proposition 4.2, thus we get the former of (4.4); the latter is obtained by replacing $m$ with $\overline{m}$. □

LEMMA 4.4. — *Let* $m$ *be a bounded Borel function on* $\mathbb{R}^n$. *Then we can find bounded Borel functions* $m_j$ *on* $\mathbb{R}^n$ *(for* $j \in \mathbb{Z}$*) such that*

$$\operatorname{supp} m_j \subseteq \{\lambda \,:\, 2^{-1} \leqslant |\lambda|_\epsilon \leqslant 2\}, \qquad \|m_j\|_{B^{\beta}_{q,q}} \leqslant C_{q,\beta} \|m\|_{M_\epsilon B^{\beta}_{q,q}} \tag{4.5}$$

*for all* $q \in [1, \infty]$ *and* $\beta \geqslant 0$, *and moreover*

$$\breve{m} = \sum_{j \in \mathbb{Z}} 2^{-Q_\delta j} \breve{m}_j \circ \delta_{2^{-j}}, \tag{4.6}$$

*in the sense of strong convergence of the corresponding convolution operators.*

*Proof.* — Set $K = \{\lambda \,:\, 2^{-1} \leqslant |\lambda|_\epsilon \leqslant 2\}$. Choose a nonnegative $\eta \in \mathcal{D}(\mathbb{R}^n)$ supported in $K$ and such that $\sum_{j \in \mathbb{Z}} \eta \circ \epsilon_{2^j} = 1$ off the origin, and let $m_j = (m \circ \epsilon_{2^{-j}})\, \eta$. Then clearly (4.5) is satisfied, and moreover $m = \sum_{j \in \mathbb{Z}} m_j \circ \epsilon_{2^j}$ off the origin. In fact, this is locally a finite sum and the convergence is dominated by the constant $\|m\|_\infty$. Since $\sigma(\{0\}) = 0$, by the spectral theorem and (1.1) we then have (4.6), in the sense of strong convergence of the corresponding convolution operators. □

Proposition 4.5. — *Suppose that* $(\mathrm{I}_{q,s})$ *holds. Let* $\beta > s$. *If* $m$ *is a bounded Borel function on* $\mathbb{R}^n$ *such that* $\|m\|_{M_\epsilon B^\beta_{q,q}} < \infty$, *then* $\breve{m}|_{G\setminus\{e\}} \in L^1_{\mathrm{loc}}(G \setminus \{e\})$, *and moreover*

$$\int_{|x|_\delta \geqslant 2|h|_\delta} |\breve{m}(xh) - \breve{m}(x)|\, dx \leqslant C_\beta \|m\|_{M_\epsilon B^\beta_{q,q}}, \tag{4.7}$$

$$\int_{|x|_\delta \geqslant 2|h|_\delta} |\breve{m}(hx) - \breve{m}(x)|\, dx \leqslant C_\beta \|m\|_{M_\epsilon B^\beta_{q,q}} \tag{4.8}$$

*for all* $h \in G \setminus \{e\}$.

*Proof.* — Let the $m_j$ be given by Lemma 4.4 and set $u_j = 2^{-Q_\delta j}\breve{m}_j \circ \delta_{2^{-j}}$.

Firstly we prove that the convergence in (4.6) holds also in $L^1_{\mathrm{loc}}(G\setminus\{e\})$. In fact, let $B_k = \{x \in G : 2^k \leqslant |x|_\delta \leqslant 2^{k+1}\}$; it is sufficient to prove the convergence in each $L^1(B_k)$. We have $\int_{B_k} |u_j|\, d\mu = \int_{B_{k-j}} |\breve{m}_j|\, d\mu$ and, for $j \leqslant k$,

$$\int_{B_{k-j}} |\breve{m}_j(x)|\, dx \leqslant 2^{\alpha(j-k)} \int_{B_{k-j}} |\breve{m}_j(x)||x|^\alpha_\delta\, dx \leqslant C 2^{\alpha(j-k)} \|m\|_{M_\epsilon B^\beta_{q,q}}$$

(where $\alpha > 0$ is as in (4.3)), whereas, for $j \geqslant k$,

$$\int_{B_{k-j}} |\breve{m}_j(x)|\, dx \leqslant \|\breve{m}_j\|_2 \sqrt{\mu(B_{k-j})} \leqslant \sqrt{\sigma(K)\mu(B_0)} \|m\|_\infty\, 2^{Q_\delta(k-j)/2},$$

(here we use a uniform estimate on the $L^2$-norms of the $\breve{m}_j$) so that

$$\sum_j \int_{B_k} |u_j|\, d\mu \leqslant C' \sum_{j\leqslant k} 2^{\alpha(j-k)} + C'' \sum_{j \geqslant k} 2^{Q_\delta(k-j)/2} < \infty.$$

This shows (by uniqueness of limits) that the restriction of the distribution $\breve{m}$ to $G \setminus \{e\}$ coincides with a function in $L^1_{\mathrm{loc}}(G \setminus \{e\})$.

Since $\breve{m} = \sum_{j\in\mathbb{Z}} u_j$ in $L^1_{\mathrm{loc}}(G\setminus\{e\})$, then $\mathrm{R}_h\, \breve{m} - \breve{m} = \sum_{j\in\mathbb{Z}}(\mathrm{R}_h\, u_j - u_j)$ in $L^1_{\mathrm{loc}}(G \setminus \{e, h^{-1}\})$, so that in particular

$$\int_{|x|_\delta \geqslant 2|h|_\delta} |\breve{m}(xh) - \breve{m}(x)|\, dx \leqslant \sum_{j\in\mathbb{Z}} \int_{|x|_\delta \geqslant 2|h|_\delta} |u_j(xh) - u_j(x)|\, dx. \tag{4.9}$$

Let $k \in \mathbb{Z}$. Then, for $j < k$, the $j$-th summand in the right-hand side of (4.9) is not greater than

$$2\int_{|x|_\delta \geqslant |h|_\delta} |u_j(x)|\, dx \leqslant C_\beta \frac{2^{\alpha j}}{|h|^\alpha_\delta} \|m\|_{M_\epsilon B^\beta_{q,q}}$$

by (4.3), whereas, for $j \geqslant k$, it is not greater than

$$\int_G |\breve{m}_j(y\delta_{2^{-j}}(h)) - \breve{m}_j(y)|\,dy \leqslant C_\beta \frac{|h|_\delta}{2^j} \|m\|_{M_\epsilon B^\beta_{q,q}}$$

by (4.4). Putting all together, the left-hand side of (4.9) is majorized by

$$C_\beta \|m\|_{M_\epsilon B^\beta_{q,q}} \left( \frac{2^{k\alpha}}{|h|_\delta^\alpha} \sum_{j<0} 2^{j\alpha} + \frac{|h|_\delta}{2^k} \sum_{j \geqslant 0} 2^{-j} \right)$$

and, in order to obtain an estimate independent of $h$, it is sufficient to choose a $k$ such that $2^k \leqslant |h|_\delta < 2^{k+1}$. Hence we have proved (4.7); the inequality (4.8) is obtained analogously. □

Here is finally the multiplier theorem.

Theorem 4.6. — *Suppose that* $(\mathrm{I}_{q,s})$ *holds. If* $m$ *is a bounded Borel function on* $\mathbb{R}^n$ *such that* $\|m\|_{M_\epsilon B^\beta_{q,q}} < \infty$ *for some* $\beta > s$*, then the operator* $m(L)$ *is of weak type* $(1,1)$ *and bounded on* $L^p(G)$ *for* $1 < p < \infty$*, with*

$$\|m(L)\|_{p \to p} \leqslant C_{p,q,\beta} \|m\|_{M_\epsilon B^\beta_{q,q}}.$$

*Proof.* — Notice that $\|m\|_{L^\infty(\sigma)} \leqslant C_{q,\beta} \|m\|_{M_\epsilon B^\beta_{q,q}}$, since $\beta > n/p$ by Proposition 4.2. In view of Proposition 4.5, the conclusion then follows from the Calderón-Zygmund theory of singular integral operators [45, § I.5, Theorem 3 and § I.7.4(iii)]. □

Notice that a compactly supported $m \in B^\beta_{q,q}(\mathbb{R}^n)$ does satisfy an $L^q$ Mihlin-Hörmander condition of order $\beta$, at least for $\beta$ sufficiently large. More precisely, let $\tilde{Q}_\epsilon = \sum_j w_j / \min_j w_j$ denote the *normalized homogeneous dimension* associated with the dilations $\epsilon_t(\lambda) = (t^{w_1}\lambda_1, \dots, t^{w_n}\lambda_n)$; then we have

Proposition 4.7. — *If* $K \subseteq \mathbb{R}^n$ *is compact,* $q \in [1,\infty]$*,* $\beta > \tilde{Q}_\epsilon/q$*, then*

$$\|m\|_{M_\epsilon B^\beta_{q,q}} \leqslant C_{K,q,\beta} \|m\|_{B^\beta_{q,q}}$$

*for all* $m \in B^\beta_{q,q}(\mathbb{R}^n)$ *with* $\operatorname{supp} m \subseteq K$*.*

Recall that $\|m\|_{M_\epsilon B^\beta_{q,q}} = \sup_{t>0} \|(m \circ \epsilon_t)\,\eta\|_{B^\beta_{q,q}}$ for a suitable cut-off function $\eta \in \mathcal{D}(\mathbb{R}^n)$ supported away from the origin. If $\operatorname{supp} m \subseteq K$, then, for some $t_K > 0$ sufficiently large, we have $\|(m \circ \epsilon_t)\,\eta\|_{B^\beta_{q,q}} = 0$ for $t > t_K$; therefore Proposition 4.7 follows immediately from

Lemma 4.8. — *If* $p, q \in [1,\infty]$*,* $\beta > \tilde{Q}_\epsilon/p$ *and* $\eta \in \mathcal{D}(\mathbb{R}^n)$*, then*

$$\sup_{0<t\leqslant 1} \|(f \circ \epsilon_t)\,\eta\|_{B^\beta_{p,q}} \leqslant C_{\eta,p,q,\beta} \|f\|_{B^\beta_{p,q}}.$$

*Proof.* — Without loss of generality, we may suppose that $\beta$ is not an integer (the missing values can be recovered *a posteriori* by interpolation), thus there exists $m \in \mathbb{N}$ such that $m-1<\beta<m$.

Define the $k$-th order difference operator $\Delta_y^k$ and the modulus of continuity $\omega_p^k$ as in [2, § 6.2]. In view of the characterization by differences of the Besov norm given in [2, Theorem 6.2.5], we have to estimate

$$\|(f \circ \epsilon_t)\,\eta\|_p + \left(\int_0^1 \left(\frac{\omega_p^m(r,(f\circ\epsilon_t)\eta)}{r^\beta}\right)^q \frac{dr}{r}\right)^{1/q}. \tag{4.10}$$

The former summand in (4.10) is immediately majorized by Hölder's inequality and embeddings, since $\eta$ is compactly supported and $\beta > n/p$:

$$\|(f\circ\epsilon_t)\,\eta\|_p \leqslant C_{\eta,p}\|f\circ\epsilon_t\|_\infty = C_{\eta,p}\|f\|_\infty \leqslant C_{\eta,p,q,\beta}\|f\|_{B^\beta_{p,q}}.$$

For the latter summand, notice first that

$$\|\Delta_y^k\psi\|_p \leqslant C_{k,p}\|\psi\|_{W_p^k}|y|^k; \tag{4.11}$$

this inequality, together with the Leibniz rule for finite differences, Hölder's inequality and the fact that $\eta \in \mathcal{D}(\mathbb{R}^n)$, gives easily

$$\|\Delta_y^m((f\circ\epsilon_t)\,\eta)\|_p \leqslant C_{\eta,m,p,p_0,\dots,p_m}\sum_{k=0}^m |y|^{m-k}\|\Delta_y^k(f\circ\epsilon_t)\|_{p_k}$$

for any choice of $p_0,\dots,p_k \geqslant p$; since

$$\|\Delta_y^k(f\circ\epsilon_t)\|_{p_k} = t^{-Q_\epsilon/p_k}\|\Delta^k_{\epsilon_t(y)}f\|_{p_k} \quad\text{and}\quad |\epsilon_t(y)|_\infty \leqslant t^{w_*}|y|_\infty$$

for $t\leqslant 1$, where $w_* = \min\{w_1,\dots,w_n\}$, we then get also

$$\omega_p^m(r,(f\circ\epsilon_t)\eta) \leqslant C_{\eta,m,p,p_0,\dots,p_m}\sum_{k=0}^m r^{m-k}t^{-Q_\epsilon/p_k}\omega^k_{p_k}(t^{w_*}r,f).$$

Choose now $p_m = p$, and $p_k = p\beta/k$ for $k<m$. Then, for $k<m$, we have

$$p_k > p, \quad k - \frac{n}{p_k} = \frac{k}{\beta}\left(\beta - \frac{n}{p}\right) < \beta - \frac{n}{p}, \quad w_* k - \frac{Q_\epsilon}{p_k} = \frac{w_* k}{\beta}\left(\beta - \frac{\tilde{Q}_\epsilon}{p}\right) > 0,$$

so that, by (4.11) and the embeddings $B^\beta_{p,q} \subseteq B^k_{p_k,1} \subseteq W^k_{p_k}$,

$$\frac{r^{m-k}t^{-Q_\epsilon/p_k}\omega^k_{p_k}(t^{w_*}r,f)}{r^\beta} \leqslant C_{p,q,\beta}r^{m-\beta}\|f\|_{B^\beta_{p,q}}.$$

For $k=m$, instead,

$$\frac{t^{-Q_\epsilon/p}\omega_p^m(t^{w_*}r,f)}{r^\beta} = t^{w_*\beta - Q_\epsilon/p}\frac{\omega_p^m(t^{w_*}r,f)}{(t^{w_*}r)^\beta} \leqslant \frac{\omega_p^m(t^{w_*}r,f)}{(t^{w_*}r)^\beta}.$$

Putting all together, the latter summand in (4.10) is majorized by

$$C_{\eta,p,q,\beta}\left(\|f\|_{B^\beta_{p,q}}\left(\int_0^1 (r^{m-\beta})^q\,\frac{dr}{r}\right)^{1/q}+\left(\int_0^1\left(\frac{\omega_p^m(r,f)}{r^\beta}\right)^q\frac{dr}{r}\right)^{1/q}\right),$$

and the conclusion follows again by [2, Theorem 6.2.5]. □

## 5. Marcinkiewicz multipliers

Let $G$ be a homogeneous Lie group, with automorphic dilations $\delta_t$ and homogeneous dimension $Q_\delta$. For $w \in L^1(G)$, we define the *maximal operator* $M_w\phi(x) = \sup_{t>0} |\phi * (t^{-Q_\delta} w \circ \delta_{t^{-1}})(x)|$. We say that the function $w$ is *M-admissible* if $M_w$ is bounded on $L^p(G)$ for $1 < p < \infty$.

In terms of maximal operators, we formulate the following hypothesis about the homogeneous group $G$ and a chosen homogeneous weighted subcoercive system $L_1, \dots, L_n$ on it:

**hypothesis** ($\mathrm{J}_{s,d}$)**:** for every $\beta > s$ there exist

- a Borel function $u_\beta$ on $G$ with $u_\beta = u_\beta^*$ and $u_\beta \geqslant c\langle\cdot\rangle_G^{-\theta}$ for some $c, \theta > 0$,
- a positive regular Borel measure $\sigma_\beta$ on $\mathbb{R}^n$, which is locally $d$-bounded on $\mathbb{R}^n \smallsetminus \{0\}$,
- a non-negative real number $\gamma_\beta < 2\beta$,

such that

- the function $\langle\cdot\rangle_G^{-\gamma_\beta} u_\beta$ is M-admissible, and
- for every compact $K \subseteq \mathbb{R}^n \smallsetminus \{0\}$ and every $m \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp} m \subseteq K$, we have

$$\|\breve{m}\|_{L^2(G, u_\beta^{-1}(x)\,dx)} \leqslant C_{K,\beta} \|m\|_{L^2(\sigma_\beta)}. \tag{5.1}$$

Proposition 5.1. — *Let $G$ be a homogeneous group, with degree of polynomial growth $Q_G$, and let $L_1, \dots, L_n$ be a homogeneous weighted subcoercive system on $G$.*

(i) *The hypothesis ($\mathrm{J}_{Q_G/2,1}$) holds. More generally, if the Plancherel measure $\sigma$ is locally $d$-bounded on $\mathbb{R}^n \smallsetminus \{0\}$, then ($\mathrm{J}_{Q_G/2,d}$) holds.*

(ii) *If $G$ is $h$-capacious, then ($\mathrm{J}_{(Q_G-h)/2,1}$) holds.*

*Proof.* — (i) Let $\sigma$ be the Plancherel measure associated with the system $L_1, \dots, L_n$. For $\beta > Q_G/2$, we choose $u_\beta \equiv 1$, $\sigma_\beta = \sigma$. By (1.1), $\sigma$ is $\delta_t$-homogeneous, so that it is locally 1-bounded on $\mathbb{R}^n \smallsetminus \{0\}$. Therefore, by the Plancherel formula, in order to conclude, it is sufficient to show that, for $\gamma_\beta \in \,]Q_G, 2\beta[$, the function $w_\beta = \langle\cdot\rangle_G^{-\gamma_\beta}$ is M-admissible.

The ideals composing the descending central series (3.9) are characteristic and thus $\delta_t$-invariant, hence we can find $\delta_t$-invariant complements $V_k$ of $\mathfrak{g}_{[k+1]}$ in $\mathfrak{g}_{[k]}$. The dilations $\tilde{\delta}_t$ of $\mathfrak{g}$ defined by $\tilde{\delta}_t(x) = t^k x$ for $x \in V_k$ in general are not automorphic, but commute with the $\delta_t$, and moreover, by [26, Proposition 2.1], if $|\cdot|_{\tilde{\delta}}$ is a $\tilde{\delta}_t$-homogeneous norm, then $\langle\cdot\rangle_G \sim 1 + |\cdot|_{\tilde{\delta}}$. We then have

$$\left|\int_G \phi(x\,\delta_t(y)^{-1})\, w_\beta(y)\, dy\right| = \left|\int_{|y|_{\tilde{\delta}}\leqslant 1} + \sum_{h\geqslant 1}\int_{2^{h-1}<|y|_{\tilde{\delta}}\leqslant 2^h}\right|$$
$$\leqslant C_{\gamma_\beta} \sum_{h\geqslant 0} 2^{-h(\gamma_\beta - Q_G)} \int_{|y|_{\tilde{\delta}}\leqslant 1} |\phi(x\,\delta_t(\tilde{\delta}_{2^h}(y))^{-1})|\, dy.$$

Since $\gamma_\beta > Q_G$, if $M_{\mathrm{strong}}$ is the strong maximal function on $G$ associated to a basis of simultaneous eigenvectors of the $\delta_t$ and the $\tilde{\delta}_t$ [5, § 2], we obtain

$$M_{w_\beta}\phi \leqslant C_{\gamma_\beta} M_{\mathrm{strong}}\phi,$$

which gives the conclusion by [5, Theorem 2.1].

(ii) Let $\mathfrak{z}$ be the center of $\mathfrak{g}$, and $P : \mathfrak{g} \to \mathfrak{g}/\mathfrak{z}$ be the canonical projection. Let $\omega_1, \dots, \omega_h \in (\mathfrak{g}/\mathfrak{z})^*$ and $z_1, \dots, z_h \in \mathfrak{z}$ be as in the definition of $h$-capacious. By Lemma 3.10, there exists a homogeneous basis $v_1, \dots, v_k$ of $\mathfrak{g}$ compatible with the descending central series such that, if $\hat{v}_1, \dots, \hat{v}_k$ is the dual basis, then $\omega_j \circ P = \hat{v}_j$ for $j = 1, \dots, h$. Moreover, if we set $\kappa_j = \max\{r : v_j \in \mathfrak{g}_{[r]}\}$, then $\kappa_j = 1$ for $j = 1, \dots, h$ and

$$Q_G = \sum_{j=1}^k \kappa_j, \qquad \langle x\rangle_G \sim 1 + \sum_{j=1}^k |\hat{v}_j(x)|^{1/\kappa_j} \tag{5.2}$$

by [26, Proposition 2.1].

Let $\daleth_{\vec{t}}$ be the $k$-parameter family of dilations on $\mathfrak{g}$ given by $\daleth_{\vec{t}}(v_j) = t_j v_j$. Clearly the $\daleth_{\vec{t}}$ are in general not automorphisms, but the automorphic dilations $\delta_t$ can be obtained as a particular case: $\delta_t = \daleth_{(t^{b_1},\dots,t^{b_k})}$, where $b_j$ is the $\delta_t$-homogeneous degree of $v_j$.

If $\beta > (Q_G - h)/2$, then $2\beta > Q_G - h = \sum_{j=h+1}^k \kappa_j$, so that we can find $\eta_{\beta,1}, \dots, \eta_{\beta,h} \in [0,1[$ and $\gamma_{\beta,1}, \dots, \gamma_{\beta,k} > 0$ such that

$$2\beta > \gamma_\beta = \sum_{j=1}^k \gamma_{\beta,j}, \qquad \gamma_{\beta,j} > \begin{cases} 1 - \eta_{\beta,j} & \text{for } j = 1, \dots, h, \\ \kappa_j & \text{for } j = h+1, \dots, k. \end{cases}$$

Let now $\sigma'$ be the Plancherel measure on $\mathbb{R}^n \times \mathfrak{z}^*$ associated to the system $L_1, \dots, L_n$ extended with the central derivatives, as in § 3, and let $\sigma_\beta$ be the push-forward of the measure $\prod_{j=1}^h (1 + |\tau(z_j)|^{-\eta_{\beta,j}})\, d\sigma'(\lambda,\tau)$ via the canonical projection on the first factor of $\mathbb{R}^n \times \mathfrak{z}^*$. By Lemma 3.7, since $\eta_{\beta,1}, \dots, \eta_{\beta,h} < 1$, the measure $\sigma_\beta$ is a regular Borel measure on $\mathbb{R}^n$; moreover, since the $z_j$ are $\delta_t$-homogeneous, $\sigma_\beta$ is the sum of $\epsilon_t$-homogeneous regular Borel measures of different degrees (where $\epsilon_t$ are the dilations associated with the system $L_1, \dots, L_n$), and consequently $\sigma_\beta$ is locally 1-bounded on $\mathbb{R}^n \setminus \{0\}$. Finally, if we set $u_\beta(x) = \prod_{j=1}^h (1+|\omega_j(P(x))|)^{-\eta_{\beta,j}} = \prod_{j=1}^h (1 + |\hat{v}_j(x)|)^{-\eta_{\beta,j}}$, then $u_\beta = u_\beta^*$ and, by (5.2), $u_\beta^{-1}$ is dominated by some power of $\langle\cdot\rangle_G$; moreover, by Corollary 3.6, for every compact $K \subseteq \mathbb{R}^n \setminus \{0\}$ and every $m \in \mathcal{D}(\mathbb{R}^n)$ with $\operatorname{supp} m \subseteq K$, we have (5.1).

In order to conclude, we must show that $w_\beta = \langle\cdot\rangle_G^{-\gamma_\beta} u_\beta$ is M-admissible. In fact, again by (5.2),

$$w_\beta(x) \leqslant C_\beta \prod_{j=1}^h (1 + |\hat{v}_j(x)|)^{-(\gamma_{\beta,j}+\eta_{\beta,j})} \prod_{j=h+1}^k (1 + |\hat{v}_j(x)|)^{-\gamma_{\beta,j}/\kappa_j},$$

and the exponents $\gamma_{\beta,j} + \eta_{\beta,j}$, $\gamma_{\beta,j}/\kappa_j$ are all greater than 1 by construction. The conclusion then follows as in part (i), but with a multi-variate decomposition, by [5, Theorem 2.1] applied to the multi-parameter dilations $\daleth_{\vec{t}}$. $\square$

Suppose now that, for $l = 1, \dots, \varrho$, $G_l$ is a homogeneous Lie group, with dilations $(\delta_{l,t})_{t>0}$, and that $L_{l,1}, \dots, L_{l,n_l}$ is a homogeneous weighted subcoercive system on $G_l$. Set $G^\times = G_1 \times \dots \times G_\varrho$, and let $L_{l,j}^\times \in \mathfrak{D}(G^\times)$ be defined by

$$L_{l,j}^\times(f_1 \otimes \dots \otimes f_\varrho) = f_1 \otimes \dots \otimes (L_{l,j} f_j) \otimes \dots \otimes f_\varrho$$

for $l = 1, \dots, \varrho$, $j = 1, \dots, n_l$. By [26, Corollary 5.5], we know that

$$L_{1,1}^\times, \dots, L_{1,n_1}^\times, \dots, L_{\varrho,1}^\times, \dots, L_{\varrho,n_\varrho}^\times$$

is a homogeneous weighted subcoercive system on $G^\times$.

We then show how the hypotheses on the factor groups $G_l$ can be put together in order to obtain weighted estimates on the product group $G^\times$. In the following, inequalities involving vectors are to be read componentwise.

Proposition 5.2. — *Suppose that, for $l = 1, \dots, \varrho$, the homogeneous group $G_l$, with the system $L_{l,1}, \dots, L_{l,n_l}$, satisfies $(\mathrm{J}_{s_l,d_l})$. For $q \in [1, \infty]$, if*

$$\vec{\beta} > \vec{s} + \frac{\vec{n}}{q} - \frac{\vec{d}}{\max\{2, q\}},$$

*where $\vec{s} = (s_1, \dots, s_\varrho)$, $\vec{d} = (d_1, \dots, d_\varrho)$, then there exists*

$$w_{\vec{\beta}} = w_{\vec{\beta},1} \otimes \dots \otimes w_{\vec{\beta},\varrho} \in L^1(G^\times),$$

*with $w_{\vec\beta} > 0$, $w_{\vec\beta}^* = w_{\vec\beta}$, such that $w_{\vec\beta,l}$ is M-admissible on $G_l$ for $l = 1, \dots, \varrho$, and moreover, for every compact $K = \prod_{l=1}^{\varrho} K_l \subseteq \prod_{l=1}^{\varrho}(\mathbb{R}^{n_l} \setminus \{0\})$, and for every $m \in S^{\vec\beta}_{q,q}B(\mathbb{R}^{\vec n})$ with $\operatorname{supp} m \subseteq K$, we have*

$$\|\mathcal{K}_{L^\times} m\|_{L^2(G^\times, w_{\vec\beta}^{-1}(x)\,dx)} \leqslant C_{K,\vec\beta,p,q} \|m\|_{S^{\vec\beta}_{q,q}B(\mathbb{R}^{\vec n})}.$$

*Proof.* — Take $\vec\alpha$ such that $\vec\alpha > \vec s$, $\vec\beta > \vec\alpha + \vec n/q - \vec d/\max\{2,q\}$. For $l = 1, \dots, \varrho$, since $\alpha_l > s_l$, by $(\mathrm{J}_{s_l,d_l})$ we can find a function $u_{\vec\alpha,l} = u_{\vec\alpha,l}^* > 0$ on $G_l$ such that $u_{\vec\alpha,l} \geqslant c_l \langle\cdot\rangle_{G_l}^{-\theta_l}$ for some $c_l, \theta_l > 0$, a positive regular Borel measure $\sigma_{\vec\alpha,l}$ on $\mathbb{R}^{n_l}$ locally $d_l$-bounded on $\mathbb{R}^{n_l} \setminus \{0\}$, and a positive real number $\gamma_{\vec\alpha,l} < 2\alpha_l$ such that the function $w_{\vec\beta,l} = \langle\cdot\rangle_{G_l}^{-\gamma_{\vec\alpha,l}} u_{\vec\alpha,l}$ is M-admissible on $G_l$ and

$$\|\mathcal{K}_{L_l} m_l\|_{L^2(G_l, u_{\vec\alpha,l}^{-1}(x_l)\,dx_l)} \leqslant C_{K_l,\alpha_l} \|m_l\|_{L^2(\sigma_{\vec\alpha,l})} \tag{5.3}$$

for every compact $K_l \subseteq \mathbb{R}^{n_l} \setminus \{0\}$ and every $m_l \in \mathcal{D}(\mathbb{R}^{n_l})$ with $\operatorname{supp} m_l \subseteq K_l$.

Set $u_{\vec\alpha} = u_{\vec\alpha,1} \otimes \dots \otimes u_{\vec\alpha,\varrho}$, $\sigma_{\vec\alpha} = \sigma_{\vec\alpha,1} \times \dots \times \sigma_{\vec\alpha,\varrho}$. By "taking the Hilbert tensor product" of the inequalities (5.3), from [26, Corollary 5.5] we deduce that

$$\|\mathcal{K}_{L^\times} m\|_{L^2(G^\times, u_{\vec\alpha}^{-1}(x)\,dx)} \leqslant C_{K,\vec\alpha} \|m\|_{L^2(\sigma_{\vec\alpha})}$$

for every compact $K = \prod_{l=1}^{\varrho} K_l \subseteq \prod_{l=1}^{\varrho}(\mathbb{R}^{n_l} \setminus \{0\})$ and every $m \in \mathcal{D}(\mathbb{R}^{\vec n})$ with $\operatorname{supp} m \subseteq K$.

Notice now that, again by taking tensor products, Lemma 2.6 gives

$$\|m\|_{L^2(\sigma_{\vec\alpha})} \leqslant C_{K,\vec\alpha,\vec\eta} \|m\|_{S^{\vec\eta}_{2,2}B(\mathbb{R}^{\vec n})}$$

for $\vec\eta > (\vec n - \vec d)/2$, whereas trivially

$$\|m\|_{L^2(\sigma_{\vec\alpha})} \leqslant C_{K,\vec\alpha} \|m\|_\infty \leqslant C_{K,\vec\alpha} \|m\|_{S^{0}_{\infty,1}B(\mathbb{R}^{\vec n})},$$

so that, by embeddings and interpolation (*cf.* the proof of Theorem 2.7),

$$\|m\|_{L^2(\sigma_{\vec\alpha})} \leqslant C_{K,\vec\alpha,\vec\eta,q} \|m\|_{S^{\vec\eta}_{q,q}B(\mathbb{R}^{\vec n})}$$

for $\vec\eta > \vec n/q - \vec d/\max\{2,q\}$.

Putting all togehter, we have

$$\|\mathcal{K}_{L^\times} m\|_{L^2(G^\times, u_{\vec\alpha}^{-1}(x)\,dx)} \leqslant C_{K,\vec\alpha,\vec\eta,q} \|m\|_{S^{\vec\eta}_{q,q}B(\mathbb{R}^{\vec n})}$$

for $\vec\eta > \vec n/q - \vec d/\max\{2,q\}$. On the other hand, by Theorem 2.7,

$$\|\mathcal{K}_{L_l} m_l\|_{L^2(G_l, \langle x_l\rangle_{G_l}^{\gamma_l} u_{\vec\alpha,l}^{-1}(x_l)\,dx_l)} \leqslant C_{K_l,\vec\alpha,\gamma_l,\eta_l} \|m_l\|_{B^{\eta_l}_{2,2}(\mathbb{R}^{n_l})}$$

for $\eta_l > \gamma_l/2 + \theta_l/2 + n_l/2$, so that, by tensor products and embeddings,

$$\|\mathcal{K}_{L^\times} m\|_{L^2(G^\times, \langle x_1\rangle_{G_1}^{\gamma_1} \cdots \langle x_\varrho\rangle_{G_\varrho}^{\gamma_\varrho} u_{\vec{\alpha}}^{-1}(x)\,dx)} \leqslant C_{K,\vec{\alpha},\vec{\gamma},\vec{\eta},q} \|m\|_{S^{\vec{\eta}}_{q,q} B(\mathbb{R}^{\vec{n}})}$$

for $\vec{\eta} > \vec{\gamma}/2 + \vec{\theta}/2 + \vec{n}$. By interpolation, we obtain that

$$\|\mathcal{K}_{L^\times} m\|_{L^2(G^\times, \langle x_1\rangle_{G_1}^{\gamma_1} \cdots \langle x_\varrho\rangle_{G_\varrho}^{\gamma_\varrho} u_{\vec{\alpha}}^{-1}(x)\,dx)} \leqslant C_{K,\vec{\alpha},\vec{\gamma},\vec{\eta},q} \|m\|_{S^{\vec{\eta}}_{q,q} B(\mathbb{R}^{\vec{n}})}$$

for $\vec{\eta} > \vec{\gamma}/2 + \vec{n}/q - \vec{d}/\max\{2,q\}$.

In particular, if we take $\vec{\gamma} = (\gamma_{\vec{\alpha},1}, \dots, \gamma_{\vec{\alpha},\varrho})$, $\vec{\eta} = \vec{\alpha} + \vec{n}/q - \vec{d}/\max\{2,q\}$ and set $w_{\vec{\beta}} = w_{\vec{\beta},1} \otimes \dots \otimes w_{\vec{\beta},\varrho}$, we get

$$\|\mathcal{K}_{L^\times} m\|_{L^2(G^\times, w_{\vec{\beta}}^{-1}(x)\,dx)} \leqslant C_{K,\vec{\beta},q} \|m\|_{S^{\vec{\eta}}_{q,q} B(\mathbb{R}^{\vec{n}})},$$

for every compact $K = \prod_{l=1}^{\varrho} K_l \subseteq \prod_{l=1}^{\varrho} (\mathbb{R}^{n_l} \setminus \{0\})$ and every $m \in \mathcal{D}(\mathbb{R}^{\vec{n}})$ with $\operatorname{supp} m \subseteq K$. The conclusion then follows by approximation. □

Notice that, in the particular case $\varrho = 1$, the previous proposition, together with Hölder's inequality and Proposition 4.2, gives the following

Corollary 5.3. — *Suppose that a homogeneous weighted subcoercive system $L_1, \dots, L_n$ on a homogeneous Lie group $G$ satisfies $(\mathrm{J}_{s,d})$. Then, for $q \in [1,\infty]$, it satisfies also $(\mathrm{I}_{q,s+n/q-d/\max\{2,q\}})$. In particular, $s \geqslant d/2$.*

The weighted estimate on $G^\times$ given by Proposition 5.2 are the starting point for the following multi-variate multiplier results. In fact, we are going to consider a setting which is more general than the product group $G^\times$.

Let $G$ be a connected Lie group, endowed with Lie group homomorphisms

$$\upsilon_l : G_l \to G \qquad \text{for } l = 1, \dots, \varrho.$$

Then, for $l = 1, \dots, \varrho$, the operators $L_{l,1}, \dots, L_{l,n_l}$ correspond (via the derivative $\upsilon_l'$ of the homomorphism) to operators $L^\flat_{l,1}, \dots L^\flat_{l,n_l} \in \mathfrak{D}(G)$, which are essentially self-adjoint. Since we want to give a meaning to joint functions of these operators on $G$, we suppose in the following that $L^\flat_{1,1}, \dots L^\flat_{1,n_1}, \dots, L^\flat_{\varrho,1}, \dots, L^\flat_{\varrho,n_\varrho}$ commute strongly, *i.e.*, they admit a joint spectral resolution $E^\flat$ on $L^2(G)$.

In order to obtain multiplier results on $G$, we would like to "transfer" to $G$ the estimates obtained on the product group $G^\times$. However, we cannot apply directly the classical transference results (*cf.* [6, 3, 7]), since the map

$$\upsilon^\times : G^\times \ni (x_1, \dots, x_n) \mapsto \upsilon_1(x_1) \cdots \upsilon_\varrho(x_\varrho) \in G$$

in general is not a group homomorphism and consequently it does not yield an action of $G^\times$ on $L^p(G)$ by translations. Nevertheless, under the sole assumption of (strong) commutativity of the differential operators $L^\flat_{l,j}$ on

$G$, we are able to express the operator $m(L^\flat)$ on $G$ by a sort of convolution with the kernel $\mathcal{K}_{L^\times} m$ of the operator $m(L^\times)$ on $G^\times$.

Proposition 5.4. — (i) *For every $m \in \mathcal{D}(\mathbb{R}^{\vec{n}})$ and $\phi \in L^2 \cap C_0(G)$,*

$$m(L^\flat)\phi(x) = \int_{G^\times} \phi(x\, v^\times(y)^{-1})\, \mathcal{K}_{L^\times} m(y)\, dy.$$

(ii) *Under the hypotheses of Proposition 5.2, the previous identity holds for every $m \in S^{\vec{\beta}}_{q,q}B(\mathbb{R}^{\vec{n}})$ with compact support* $\operatorname{supp} m \subseteq \prod_{l=1}^{\varrho} (\mathbb{R}^{n_l} \setminus \{0\})$.

*Proof.* — (i) If $m \in \mathcal{D}(\mathbb{R}^{\vec{n}})$, then we can decompose $m = \sum_{k\in\mathbb{N}} g_{k,1} \otimes \dots \otimes g_{k,\varrho}$, where $g_{k,l} \in \mathcal{D}(\mathbb{R}^{n_l})$ for $k \in \mathbb{N}$, $l = 1, \dots, \varrho$, and the convergence is in $\mathcal{D}(\mathbb{R}^{\vec{n}})$. In particular, by applying [26, Corollary 5.5] and Theorem 2.7 to the group $G^\times$, we obtain that

$$\mathcal{K}_{L^\times} m = \sum_{k\in\mathbb{N}} \mathcal{K}_{L_1} g_{k,1} \otimes \dots \otimes \mathcal{K}_{L_\varrho} g_{k,\varrho}$$

in $L^1(G^\times)$. On the other hand, for all $\phi \in L^2 \cap C_0(G)$, we have

$$g_{k,l}(L_l^\flat)\phi(x) = \int_{G_l} \phi(x\, v_l(y_l)^{-1})\, \mathcal{K}_{L_l} g_{k,l}(y_l)\, dy_l$$

by [26, Proposition 3.7], and in particular (being $\mathcal{K}_{L_l} g_{k,l} \in L^1(G_l)$) also $g_{k,l}(L_l^\flat)\phi \in L^2 \cap C_0(G)$, so that, by iterating,

$$(g_{k,1} \otimes \dots \otimes g_{k,\varrho})(L^\flat)\phi(x) = \int_{G^\times} \phi(x\, v^\times(y)^{-1}) \prod_{l=1}^{\varrho} \mathcal{K}_{L_l} g_{k,l}(y_l)\, dy.$$

Summing over $k \in \mathbb{N}$, the left-hand side converges in $L^2(G)$ to $m(L^\flat)\phi$, whereas (since $y \mapsto \phi(x\, v^\times(y)^{-1})$ is bounded) the right-hand side converges pointwise to $\int_{G^\times} \phi(x\, v^\times(y)^{-1})\, \mathcal{K}_{L^\times} m(y)\, dy$, and the conclusion follows.

(ii) Choose $\vec{\beta}'$ such that $\vec{\beta} > \vec{\beta}' > \vec{s} + \vec{n}/q - \vec{d}/\max\{2, q\}$. Take a compact $K = \prod_{l=1}^{\varrho} K_l \subseteq \prod_{l=1}^{\varrho} (\mathbb{R}^{n_l} \setminus \{0\})$ and a sequence $m_k \in \mathcal{D}(\mathbb{R}^{\vec{n}})$ with $\operatorname{supp} m_k \subseteq K$ such that $m_k \to m$ in $S^{\vec{\beta}'}_{q,q}(\mathbb{R}^{\vec{n}})$. By Proposition 5.2 and Hölder's inequality, we then have $\mathcal{K}_{L^\times} m_k \to \mathcal{K}_{L^\times} m$ in $L^1(G^\times)$; moreover, by Corollary 5.3, $\beta_l' > n_l/q$ for $l = 1, \dots, \varrho$, so that $m_k \to m$ uniformly. Therefore the conclusion follows by applying (i) to the functions $m_k$ and passing to the limit. □

We are now going to exploit the Littlewood-Paley theory. An important tool will be the following result, which summarizes a well-known argument for proving properties of square functions.

Lemma 5.5. — *Let $(X, \mu)$ be a $\sigma$-finite measure space, $1 \leqslant p < \infty$, $T_{\vec{k}}$ ($\vec{k} \in \mathbb{N}^\varrho$) bounded linear operators on $L^p(X, \mu)$. Let $A > 0$ be such that,* *for all choices of* $\varepsilon_k^i \in \{-1,1\}$ $(1 \leqslant i \leqslant \varrho,\ k \in \mathbb{N})$ *and of a finite subset* $I \subseteq \mathbb{N}^\varrho$*, we have*

$$\left\| \sum_{\vec{k}\in I} \varepsilon^1_{k_1} \cdots \varepsilon^\varrho_{k_\varrho} T_{\vec{k}} \right\|_{p\to p} \leqslant A. \tag{5.4}$$

*Then, for all* $f \in L^p(X,\mu)$*,*

$$\left\| \left( \sum_{\vec{k}\in\mathbb{N}^\varrho} |T_{\vec{k}} f|^2 \right)^{1/2} \right\|_p \leqslant C_{\varrho,p} A \|f\|_p. \tag{5.5}$$

*Moreover, if* $p > 1$*, for all* $\{f_{\vec{k}}\}_{\vec{k}\in\mathbb{N}^\varrho} \subseteq L^p(X,\mu)$*, if* $\left(\sum_{\vec{k}} |f_{\vec{k}}|^2\right)^{1/2} \in L^p(X,\mu)$*, then*

$$\left\| \sum_{\vec{k}\in\mathbb{N}^\varrho} T_{\vec{k}} f_{\vec{k}} \right\|_p \leqslant C_{\varrho,p'} A \left\| \left( \sum_{\vec{k}\in\mathbb{N}^\varrho} |f_{\vec{k}}|^2 \right)^{1/2} \right\|_p$$

*where the series on the left-hand side converges unconditionally in* $L^p$*.*

*Proof.* — For $n \in \mathbb{N}$, let $r_n : [0,1] \to \mathbb{R}$ be the $n$-th Rademacher function, $r_n(t) = (-1)^{\lfloor 2^n t \rfloor}$, and set $r_{\vec{k}} = r_{k_1} \otimes \cdots \otimes r_{k_\varrho}$ for $\vec{k} \in \mathbb{N}^\varrho$. Then $(r_{\vec{k}})_{\vec{k}}$ is an (incomplete) orthonormal system in $L^2([0,1]^\varrho)$, and Khinchin's inequalities hold: for $1 \leqslant p < \infty$, there exist $c_{\varrho,p}, C_{\varrho,p} > 0$ such that

$$c_{\varrho,p}^{-1} \|f\|_p \leqslant \|f\|_2 \leqslant C_{\varrho,p} \|f\|_p \qquad \text{for all } f \in \operatorname{span}\{r_{\vec{k}} \,:\, \vec{k} \in \mathbb{N}^\varrho\}.$$

(see [44, Appendix D] or [16, Appendix C]).

Consequently, for all finite $I \subseteq \mathbb{N}^\varrho$ and $f \in L^p(X,\mu)$, we have

$$\begin{aligned} \left\| \left( \sum_{\vec{k}\in I} |T_{\vec{k}} f|^2 \right)^{1/2} \right\|_p^p &= \int_X \left( \sum_{\vec{k}\in I} |T_{\vec{k}} f(x)|^2 \right)^{p/2} d\mu(x) \\ &\leqslant C_{\varrho,p}^p \int_X \int_{[0,1]^\varrho} \left| \sum_{\vec{k}\in I} T_{\vec{k}} f(x) r_{\vec{k}}(t) \right|^p dt\, d\mu(x) \\ &= C_{\varrho,p}^p \int_{[0,1]^\varrho} \left\| \left( \sum_{\vec{k}\in I} r_{\vec{k}}(t) T_{\vec{k}} \right) f \right\|^p dt \leqslant C_{\varrho,p}^p A^p \|f\|_p^p. \end{aligned}$$

Since $I \subseteq \mathbb{N}^\varrho$ was arbitrary, (5.5) follows by monotone convergence.

Notice now that the vector-valued Lebesgue space $V_p = L^p(X,\mu; l^2(\mathbb{N}^\varrho))$ can be thought of as a space of sequences of $L^p(X,\mu)$-functions:

$$V_p = \left\{ (f_{\vec{k}})_{\vec{k}\in\mathbb{N}^\varrho} \in L^p(X,\mu)^{\mathbb{N}^\varrho} \,:\, \left( \sum_{\vec{k}} |f_{\vec{k}}|^2 \right)^{1/2} \in L^p(X,\mu) \right\},$$

with norm $\|(f_{\vec{k}})_{\vec{k}\in\mathbb{N}^\varrho}\|_{V_p} = \|(\sum_{\vec{k}} |f_{\vec{k}}|^2)^{1/2}\|_p$. The inequality (5.5) therefore means that the operator $f \mapsto (T_{\vec{k}} f)_{\vec{k}\in\mathbb{N}^\varrho}$ is bounded $L^p(X,\mu) \to V_p$, with norm not greater than $C_{\varrho,p}A$.

If $p > 1$, the hypothesis (5.4) is equivalent to

$$\left\| \sum_{\vec{k}\in I} \varepsilon^1_{k_1} \cdots \varepsilon^n_{k_n} T^*_{\vec{k}} \right\|_{p'\to p'} \leqslant A;$$

consequently we have that $S : f \mapsto (T^*_{\vec{k}} f)_{\vec{k}\in\mathbb{N}^\varrho}$ is bounded $L^{p'}(X,\mu) \to V_{p'}$, with norm not greater than $C_{\varrho,p'}A$. This means that the transpose operator $S^* : V_p \to L^p(X,\mu)$ is bounded too, with the same norm; since

$$S^*\left((f_{\vec{k}})_{\vec{k}}\right) = \sum_{\vec{k}} T_{\vec{k}} f_{\vec{k}},$$

where the series on the right-hand side converges unconditionally in $L^p$, the remaining part of the conclusion follows. □

For $l = 1,\dots,\varrho$, let $\epsilon_{l,t}$ be the dilations on $\mathbb{R}^{n_l}$ associated to the weighted subcoercive system $L_{l,1},\dots,L_{l,n_l}$, and fix an $\epsilon_l$-homogeneous norm $|\cdot|_{\epsilon_l}$ on $\mathbb{R}^{n_l}$, smooth off the origin. Choose a non-negative $\xi \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp}\xi \subseteq [1/2, 2]$ and such that, if $\xi_k(t) = \xi(2^{-k}t)$, then

$$\sum_{k\in\mathbb{Z}} \xi_k^2(t) = 1 \qquad \text{for } t > 0, \tag{5.6}$$

and set, for $l = 1,\dots,\varrho$ and $k \in \mathbb{Z}$, $\chi_{l,k}(\lambda) = \xi(|\epsilon_{l,2^{-k}}(\lambda)|_{\epsilon_l}) = \xi_k(|\lambda|_{\epsilon_l})$ for $\lambda \in \mathbb{R}^{n_l}$. Moreover, for $\vec{k} = (k_1,\dots,k_\varrho) \in \mathbb{Z}^\varrho$, let $\chi_{\vec{k}} = \chi_{1,k_1} \otimes \cdots \otimes \chi_{\varrho,k_\varrho}$, $T_{\vec{k}} = \chi_{\vec{k}}(L^\flat)$, and define the square function

$$g(\phi) = \left( \sum_{\vec{k}\in\mathbb{Z}^\varrho} |T_{\vec{k}}\phi|^2 \right)^{1/2}.$$

Finally, set $X_{\vec{n}} = \{\lambda \in \mathbb{R}^{\vec{n}} : |\lambda_1|_{\epsilon_1} \cdots |\lambda_\varrho|_{\epsilon_\varrho} = 0\}$.

Lemma 5.6. — *For $1 < p < \infty$ and for all $\phi \in L^2 \cap L^p(G)$,*

$$c_p \|E^\flat(\mathbb{R}^{\vec{n}} \setminus X_{\vec{n}})\phi\|_p \leqslant \|g(\phi)\|_p \leqslant C_p\|\phi\|_p.$$

*Proof.* — Using the characterization (1.4) of $L^\infty$ Mihlin-Hörmander conditions, it is not difficult to prove, for $l = 1,\dots,\varrho$, $s \in \mathbb{N}$, $(\varepsilon^l_k)_{k\in\mathbb{Z}} \in \{-1,0,1\}^{\mathbb{Z}}$, $N \in \mathbb{N}$, that

$$\left\| \sum_{|k|\leqslant N} \varepsilon^l_k \chi_{l,k} \right\|_{M_{\epsilon_l} B^s_{\infty,\infty}} \leqslant C_{l,s},$$

where $C_{l,s} > 0$ does not depend on $(\varepsilon^l_k)_k$ or $N$.

By Theorem 4.6 applied to the group $G_l$, and by transference to the group $G$ (see [3, Theorem 2.7] and [26, Proposition 3.7]), we then have

$$\left\| \sum_{|k|\leqslant N} \varepsilon^l_k \chi_{l,k}(L^\flat_l) \right\|_{p\to p} \leqslant \left\| \sum_{|k|\leqslant N} \varepsilon^l_k \chi_{l,k}(L_l) \right\|_{p\to p} \leqslant C_{l,p}$$

for $1 < p < \infty$, $l = 1, \dots, \varrho$, where $C_{l,p} > 0$ does not depend on $(\varepsilon^l_k)_k$ or $N$, and consequently also

$$\left\| \sum_{|k_1|,\dots,|k_\varrho|\leqslant N} \varepsilon^1_{k_1} \cdots \varepsilon^\varrho_{k_\varrho} T_{\vec{k}} \right\|_{p\to p} \leqslant C_{1,p} \cdots C_{\varrho,p}.$$

Moreover, by (5.6) and the properties of the spectral integral, $\sum_{\vec{k}\in\mathbb{Z}^\varrho} T^2_{\vec{k}}$ converges strongly to $E^\flat(\mathbb{R}^{\vec{n}} \smallsetminus X_{\vec{n}})$. The conclusion follows then immediately by Lemma 5.5. □

In the following, we will consider Marcinkiewicz conditions on $\mathbb{R}^{\vec{n}}$ adapted to the system $\daleth_{\vec{t}} = \epsilon_{1,t_1} \times \cdots \times \epsilon_{\varrho,t_\varrho}$ of multi-variate dilations.

Theorem 5.7. — *Suppose that, for $l = 1, \dots, \varrho$, the homogeneous group $G_l$, with the system $L_{l,1}, \dots, L_{l,n_l}$, satisfies $(\mathrm{J}_{s_l,d_l})$. If $q \in [1,\infty]$ and*

$$\vec{\beta} > \vec{s} + \frac{\vec{n}}{q} - \frac{\vec{d}}{\max\{2,q\}},$$

*then, for every Borel $m : \mathbb{R}^{\vec{n}} \to \mathbb{C}$ with $m|_{X_{\vec{n}}} = 0$ and $\|m\|_{M_\daleth S^{\vec{\beta}}_{q,q}B} < \infty$, the operator $m(L^\flat)$ is bounded on $L^p(G)$ for $1 < p < \infty$ and*

$$\|m(L^\flat)\|_{p\to p} \leqslant C_{\vec{\beta},p,q} \|m\|_{M_\daleth S^{\vec{\beta}}_{q,q}B}.$$

*Proof.* — Choose a non-negative $\zeta \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp}\zeta \subseteq [1/4, 4]$ and such that $\zeta \equiv 1$ on $[1/2, 2]$. For $l = 1, \dots, \varrho$, set $\eta_l(\lambda) = \zeta(|\lambda|_{\epsilon_l})$ and $\eta = \eta_1 \otimes \cdots \otimes \eta_\varrho$. If we set $m_{\vec{k}} = (m \circ \daleth_{(2^{k_1},\dots,2^{k_\varrho})})\eta$, $f_{\vec{k}} = m_{\vec{k}} \circ \daleth_{(2^{-k_1},\dots,2^{-k_\varrho})}$ for $\vec{k} \in \mathbb{Z}^\varrho$, then we have $\chi_{\vec{k}} m = f_{\vec{k}} \chi_{\vec{k}}$, so that $T_{\vec{k}} m(L^\flat) = f_{\vec{k}}(L^\flat) T_{\vec{k}}$.

Let $w_{\vec{\beta}} = w_{\vec{\beta},1} \otimes \cdots \otimes w_{\vec{\beta},\varrho} \in L^1(G^\times)$ be given by Proposition 5.2. Set $w_{\vec{\beta},l,k} = 2^{kQ_{\delta_l}} w_{\vec{\beta},l} \circ \delta_{l,2^k}$ for $k \in \mathbb{Z}$, $l = 1, \dots, \varrho$, and let $w_{\vec{\beta},\vec{k}} = w_{\vec{\beta},1,k_1} \otimes \cdots \otimes w_{\vec{\beta},\varrho,k_\varrho}$ for $\vec{k} \in \mathbb{Z}^\varrho$. For $l = 1, \dots, \varrho$, if $\pi_l$ denotes the unitary representation of $G_l$ on $L^2(G)$ induced by the homomorphism $\upsilon_l$, since $w_{\vec{\beta},l}$ is M-admissible on $G_l$, then the maximal function $M_{\vec{\beta},l}$ on $G$ defined by $M_{\vec{\beta},l}\phi(x) = \sup_{k\in\mathbb{Z}} |\pi_l(w_{\vec{\beta},l,k})\phi(x)|$ is bounded on $L^p(G)$ for $1 < p < \infty$, by transference [3, Theorem 2.11].

If $\phi \in L^2 \cap C_0(G)$, then we have, by Proposition 5.4(ii) and Hölder's inequality,

$$\begin{aligned}
|f_{\vec{k}}(L^\flat) T_{\vec{k}}\phi(x)|^2 &\leqslant \left( \int_{G^\times} |T_{\vec{k}}\phi(x\, \upsilon^\times(y)^{-1})| |\mathcal{K}_{L^\times} f_{\vec{k}}(y)| \, dy \right)^2 \\
&\leqslant \int_{G^\times} |T_{\vec{k}}\phi(x\, \upsilon^\times(y)^{-1})|^2 w_{\vec{\beta},\vec{k}}(y)\, dy \int_{G^\times} |\mathcal{K}_{L^\times} m_{\vec{k}}(y)|^2 w_{\vec{\beta}}^{-1}(y)\, dy \\
&\leqslant C_{\vec{\beta},q} \|m_{\vec{k}}\|^2_{S^{\vec{\beta}}_{q,q}B(\mathbb{R}^{\vec{n}})} \pi_1(w_{\vec{\beta},1,k_1}) \cdots \pi_\varrho(w_{\vec{\beta},\varrho,k_\varrho})(|T_{\vec{k}}\phi|^2)
\end{aligned}$$

thus

$$\begin{aligned}
&\|g(m(L^\flat)\phi)\|_p \\
&\qquad \leqslant C_{\vec{\beta},q} \|m\|_{M_\daleth S^{\vec{\beta}}_{q,q}B} \left\| \sum_{\vec{k}\in\mathbb{Z}^\varrho} \pi_1(w_{\vec{\beta},1,k_1}) \cdots \pi_\varrho(w_{\vec{\beta},\varrho,k_\varrho})(|T_{\vec{k}}\phi|^2) \right\|_{p/2}^{1/2}
\end{aligned}$$

for $2 \leqslant p < \infty$.

On the other hand, since $w_{\vec{\beta}} = w_{\vec{\beta}}^*$, for every $\psi \in L^{(p/2)'}(G)$ we have

$$\begin{aligned}
&\left| \int_G \left( \sum_{\vec{k}\in\mathbb{Z}^\varrho} \pi_1(w_{\vec{\beta},1,k_1}) \cdots \pi_\varrho(w_{\vec{\beta},\varrho,k_\varrho})(|T_{\vec{k}}\phi|^2) \right) \psi \, d\mu_G \right| \\
&\qquad \leqslant \sum_{\vec{k}\in\mathbb{Z}^\varrho} \int_G \left( \pi_1(w_{\vec{\beta},1,k_1}) \cdots \pi_\varrho(w_{\vec{\beta},\varrho,k_\varrho})(|T_{\vec{k}}\phi|^2) \right) |\psi| \, d\mu_G \\
&\qquad \leqslant \int_G \left( \sum_{\vec{k}\in\mathbb{Z}^\varrho} |T_{\vec{k}}\phi|^2 \right) M_{\vec{\beta},\varrho} \cdots M_{\vec{\beta},1}(|\psi|) \, d\mu_G \\
&\qquad \leqslant C_{\vec{\beta},p} \left\| \sum_{\vec{k}\in\mathbb{Z}^\varrho} |T_{\vec{k}}\phi|^2 \right\|_{p/2} \|\psi\|_{(p/2)'},
\end{aligned}$$

that is,

$$\left\| \sum_{\vec{k}\in\mathbb{Z}^\varrho} \pi_1(w_{\vec{\beta},1,k_1}) \cdots \pi_\varrho(w_{\vec{\beta},\varrho,k_\varrho})(|T_{\vec{k}}\phi|^2) \right\|_{p/2} \leqslant C_{\vec{\beta},p} \|g(\phi)\|_p^2.$$

Putting all together, and applying Lemma 5.6, we get the conclusion for $2 \leqslant p < \infty$ (notice that $E^\flat(\mathbb{R}^{\vec{n}} \setminus X_{\vec{n}}) m(L^\flat) = m(L^\flat)$ because $m|_{X_{\vec{n}}} = 0$). Thus we are done when $m(L^\flat)$ is self-adjoint, *i.e.*, when $m$ is real-valued; in the general case, one can decompose $m$ in its real and imaginary parts and then apply the previous result to each part. □

The hypothesis $m|_{X_{\vec{n}}} = 0$ in Theorem 5.7 does not have an analogue in Theorem 4.6, because the spectral measure of the origin for a homogeneous weighted subcoercive system on a homogeneous group is zero. On the other hand, if $E^\flat_l$ is the joint spectral resolution of $L^\flat_{l,1},\dots,L^\flat_{l,n_l}$, then $E^\flat_l(\{0\})$ need not be zero. However we have the following

Proposition 5.8. — *$E^\flat_l(\{0\})$ is bounded on $L^p(G)$ for $1 \leqslant p \leqslant \infty$. If moreover $\overline{\upsilon_l(G_l)}$ is not compact in $G$, then $E^\flat_l(\{0\}) = 0$.*

*Proof.* — Choose $\psi \in \mathcal{D}(\mathbb{R}^{n_l})$ with $\psi(0) = 1$, so that $\psi_t = \psi \circ \epsilon_{l,t} \to \chi_{\{0\}}$ pointwise for $t \to +\infty$, and then $\psi_t(L^\flat_l) \to E^\flat_l(\{0\})$ strongly as operators on $L^2(G)$. By [26, Proposition 3.7] we have $\psi_t(L^\flat_l) = \pi_l(\mathcal{K}_{L_l}\psi_t)$, thus

$$\|\psi_t(L^\flat_l)\|_{L^p(G)\to L^p(G)} \leqslant \|\mathcal{K}_{L_l}\psi_t\|_{L^1(G_l)} = \|\mathcal{K}_{L_l}\psi\|_{L^1(G_l)} < \infty$$

by (1.1) and Theorem 2.7. For every $f \in L^2 \cap L^p(G)$ and $g \in L^2 \cap L^{p'}(G)$, we then have

$$|\langle E^\flat_l(\{0\})f, g\rangle| = \lim_{t\to+\infty} |\langle \psi_t(L^\flat_l)f, g\rangle| \leqslant \|\mathcal{K}_{L_l}\psi\|_{L^1(G_l)} \|f\|_{L^p(G)} \|g\|_{L^{p'}(G)},$$

which gives the required boundedness of $E^\flat_l(\{0\})$.

Suppose now that $\overline{\upsilon_l(G_l)}$ is not compact and that $E^\flat_l(\{0\})f = f$ for some $f \in L^2(G)$. This means that $d\pi_l(L_{l,1})f = \dots = d\pi_l(L_{l,n_l})f = 0$, and proceeding analogously as in the proof of [26, Theorem 5.2] one gets that $\pi_l(y)f = f$ for every $y \in G_l$. If $f \neq 0$, we can find a compact $K \subseteq G$ such that $\int_K |f(x)|^2\,dx \neq 0$; on the other hand, since $\overline{\upsilon_l(G_l)}$ is not compact, it is easy to construct inductively a sequence $\{y_k\}_{k\in\mathbb{N}}$ in $G_l$ such that the sets $K\upsilon_l(y_k)$ for $k \in \mathbb{N}$ are pairwise disjoint, but then

$$\int_G |f(x)|^2\,dx \geqslant \sum_{k\in\mathbb{N}} \int_{K\upsilon_l(y_k)} |f(x)|^2\,dx = \sum_{k\in\mathbb{N}} \int_K |f(x)|^2\,dx = \infty,$$

contradiction. Hence $f = 0$, and then $E^\flat_l(\{0\}) = 0$ by arbitrariness of $f$. $\square$

Proposition 5.8 allows to relax the hypothesis $m|_{X_{\vec{n}}} = 0$ in Theorem 5.7. Namely, for $I \subseteq \{1,\dots,\varrho\}$, let $\vec{n}_I = (n_l)_{l\in I}$, so that $\mathbb{R}^{\vec{n}_I} = \prod_{l\in I}\mathbb{R}^{n_l}$; let moreover $\iota_I : \mathbb{R}^{\vec{n}_I} \to \mathbb{R}^{\vec{n}}$ be the canonical linear embedding, and define on $\mathbb{R}^{\vec{n}_I}$ the system of multi-variate dilations $\daleth_{I,(t_l)_{l\in I}} = \prod_{l\in I}\epsilon_{l,t_l}$. Then the decomposition

$$m(L^\flat) = \sum_{I\subseteq\{1,\dots,\varrho\}} \prod_{l\notin I} E^\flat_l(\{0\}) \prod_{l\in I} E^\flat_l(\mathbb{R}^{n_l}\setminus\{0\})(m\circ\iota_I)(L^\flat_I),$$

where $L^\flat_I = (L^\flat_{l,j})_{l\in I, 1\leqslant j\leqslant n_l}$, shows that the $L^p$-boundedness of $m(L^\flat)$ can be obtained by applying Theorem 5.7 to the subsystems $L^\flat_I$ of $L^\flat$:

Corollary 5.9. — *Suppose that, for $l = 1, \dots, \varrho$, the homogeneous group $G_l$, with the system $L_{l,1}, \dots, L_{l,n_l}$, satisfies $(\mathrm{J}_{s_l,d_l})$. If $q \in [1,\infty]$ and*

$$\vec\beta > \vec s + \frac{\vec n}{q} - \frac{\vec d}{\max\{2,q\}},$$

*then, for every Borel $m : \mathbb{R}^{\vec n} \to \mathbb{C}$ we have*

$$\|m(L^\flat)\|_{p\to p} \leqslant C_{\vec\beta,p,q} \sum_{I\in\mathcal{I}} \|m \circ \iota_I\|_{M_{\daleth_I} S^{\vec\beta_I}_{q,q}B}, \tag{5.7}$$

*where $\mathcal{I}$ is the set of the $I \subseteq \{1,\dots,\varrho\}$ such that $\prod_{l\notin I} E^\flat_l(\{0\}) \neq 0$, and where $\|m\circ\iota_I\|_{M_{\daleth_I} S^{\vec\beta_I}_{q,q}B} = |m(0)|$ for $I = \emptyset$.*

In particular, if all the $\overline{\upsilon_l(G_l)}$ are not compact, then the hypothesis $m|_{X_{\vec n}} = 0$ in Theorem 5.7 can be dropped.

We conclude the section with a comparison of the Mihlin-Hörmander and Marcinkiewicz conditions, which constitute the hypotheses of Theorems 4.6 and 5.7 respectively: we obtain that, under suitable hypotheses on the orders of smoothness, a Marcinkiewicz condition is weaker than the corresponding Mihlin-Hörmander condition.

Proposition 5.10. — *If $q \in [1,\infty]$ and $\beta_l > \tilde Q_{\epsilon_l}/q$ for $l = 1,\dots,\varrho$, then*

$$\|m\|_{M_\daleth S^{\vec\beta}_{q,q}B} \leqslant C_{q,\vec\beta} \|m\|_{M_\epsilon B^{\beta_1+\dots+\beta_\varrho}_{q,q}},$$

*where $\epsilon_t = \daleth_{(t,\dots,t)} = \epsilon_{1,t} \times \dots \times \epsilon_{\varrho,t}$.*

*Proof.* — For $q < \infty$, in view of the characterization of $S^{\vec\beta}_{q,q}B(\mathbb{R}^{\vec n})$ as a tensor product of the $B^{\beta_l}_{q,q}(\mathbb{R}^{n_l})$ (*cf.* [41, Theorem 2.2]), from Lemma 4.8 we immediately get

$$\sup_{\vec t>0,\, |\vec t|_\infty \leqslant 1} \|(f\circ\daleth_{\vec t})\,\eta\|_{S^{\vec\beta}_{q,q}B} \leqslant C_{\eta,q,\vec\beta}\|f\|_{S^{\vec\beta}_{q,q}B} \tag{5.8}$$

for $\eta = \eta_1 \otimes \dots \otimes \eta_\varrho \in \mathcal{D}(\mathbb{R}^{\vec n})$; the same holds also for $q = \infty$, as it is easily proved via the characterization by differences of the $S^{\vec\beta}_{\infty,\infty}B$-norm (*cf.* [40, § 2.3.4]).

Suppose now that $\operatorname{supp}\eta_l \subseteq \{\lambda_l \in \mathbb{R}^{n_l} : a \leqslant |\lambda_l|_\infty \leqslant b\}$ for some $b > a > 0$ and $l = 1,\dots,\varrho$, and take $\tilde\eta \in \mathcal{D}(\mathbb{R}^{\vec n})$ such that $\tilde\eta|_{\{\lambda\,:\,a\leqslant|\lambda|_\infty\leqslant b\}} \equiv 1$. If $\vec t > 0$ and $|\vec t|_\infty = 1$, then $(\tilde\eta\circ\daleth_{\vec t})\eta = \eta$, thus from (5.8) we get

$$\sup_{\vec t>0,\, |\vec t|_\infty = 1} \|(f\circ\daleth_{\vec t})\,\eta\|_{S^{\vec\beta}_{q,q}B} \leqslant C_{\eta,q,\vec\beta}\|f\tilde\eta\|_{S^{\vec\beta}_{q,q}B}. \tag{5.9}$$

For an arbitrary $\vec{t} > 0$, set $r = |\vec{t}|_\infty$, so that $|r^{-1}\vec{t}|_\infty = 1$; then we have $m \circ \daleth_{\vec{t}} = (m \circ \epsilon_r) \circ \daleth_{r^{-1}\vec{t}}$, so that, by (5.9) applied to $f = m \circ \epsilon_r$,

$$\|(m \circ \daleth_{\vec{t}})\, \eta\|_{S^{\vec{\beta}}_{q,q}B} \leqslant C_{\eta,q,\vec{\beta}} \|(m \circ \epsilon_r)\, \tilde{\eta}\|_{S^{\vec{\beta}}_{q,q}B} \leqslant C_{\eta,q,\vec{\beta}} \|(m \circ \epsilon_r)\, \tilde{\eta}\|_{B^{\beta_1+\cdots+\beta_\varrho}_{q,q}},$$

and the conclusion follows by a suitable choice of $\eta$ and $\tilde{\eta}$. □

Notice that a Mihlin-Hörmander condition on $m : \mathbb{R}^{\vec{n}} \to \mathbb{C}$ gives some control also on the restriction of $m$ to $X_{\vec{n}} \setminus \{0\}$, so that it can be used to satisfy the more involved hypothesis of Corollary 5.9. Namely, by the trace theorem for Besov spaces, under the hypothesis on $\beta_1, \dots, \beta_\varrho$ of Proposition 5.10, if $m$ satisfies an $L^q$ Mihlin-Hörmander condition of order $\beta_1 + \cdots + \beta_\varrho$, then $m \circ \iota_I$ satisfies an $L^q$ Mihlin-Hörmander condition of order $\sum_{l \in I} \beta_l$ for $\emptyset \neq I \subseteq \{1, \dots, \varrho\}$; therefore, by Proposition 5.10, all the summands in the right-hand side of (5.7), except possibly for $|m(0)|$, are majorized by $\|m\|_{M_\epsilon B^{\beta_1+\cdots+\beta_\varrho}_{q,q}}$.

## 6. Examples and applications

### 6.1. Multipliers for a single operator

Although the present work focuses on $L^p$ multipliers for systems of multiple operators, some results can be deduced also for single operators.

In view of the characterization stated in § 1, a homogeneous weighted subcoercive system made of a single operator $L$ is simply a self-adjoint Rockland operator. Hence from Theorem 4.6 and Proposition 4.1 we get

Corollary 6.1. — *Let $L$ be a self-adjoint Rockland operator on a homogeneous Lie group $G$. Suppose that $G$ is $h$-capacious, and let $Q_G$ be its dimension at infinity. If $m : \mathbb{R} \to \mathbb{C}$ satisfies an $L^2$ Mihlin-Hörmander condition of order $s > (Q_G - h)/2$, then $m(L)$ is of weak type* $(1,1)$ *and bounded on $L^p(G)$ for $1 < p < \infty$.*

This corollary summarizes several results present in the literature. For a general (positive) Rockland operator, this result is stated in the unpublished paper [18] with regularity threshold $Q_G/2$ (see also [23, 46]); the improvement on the threshold is proved in [20] for products of Euclidean and H-type groups. By restricting to the case of a homogeneous sublaplacian $L$ on a stratified Lie group $G$, we recover the result of [28, 4], where the threshold is half the homogeneous dimension of $G$.

Notice that, when a homogeneous Lie group $G$ is stratified (*i.e.*, when the elements of degree 1 generate the whole Lie algebra), then the homogeneous dimension coincides with $Q_G$. However, on a nilpotent Lie group $G$ there may be multiple homogeneous structures, and the homogeneous dimension $Q_\delta$ depends on the chosen automorphic dilations $\delta_t$ (for instance, if $X, Y, T$ is a basis of the Lie algebra of the Heisenberg group with $[X, Y] = T$, then we can set $\delta_t(X) = tX$, $\delta_t(Y) = t^a Y$, $\delta_t(T) = t^{1+a} T$ for all $a \geqslant 1$, and we have $Q_\delta = 2 + 2a$), whereas $Q_G$ is intrinsic of the Lie group structure of $G$. In fact (under the hypothesis that all the homogeneity degrees are not less than 1) we always have $Q_\delta \geqslant Q_G$, with equality if and only if $G$ is stratified [26, Proposition 2.2].

The existence of a homogeneous sublaplacian forces $G$ to be stratified (modulo rescaling the homogeneity degrees). On the other hand, on non-stratified homogeneous groups $G$ there might exist higher-order self-adjoint Rockland operators $L$ (for instance, in the previous example of the Heisenberg group, one can take $L = (-iX)^{2a} - Y^2$ when $a \in \mathbb{N}$), to which Corollary 6.1 applies, with threshold (at most) $Q_G/2$. Therefore our result is also an improvement of [9, Corollary 7.1], where the required threshold is $Q_\delta/2$.

Multiplier results for a single operator can be deduced from Theorem 5.7 too, through a sort of "spectral mapping"; some examples in the context of a non-nilpotent Lie group are presented in the following § 6.3.

### 6.2. Plancherel measure and capacity map

In order to obtain the sharpest results from the previous multiplier theorems, properties of the Plancherel measure associated with a weighted subcoercive system and of the capacity map of a group must be investigated.

If $L_1, \dots, L_n$ is a weighted subcoercive system on a nilpotent group $G$, then the associated Plancherel measure $\sigma$ is related to the group Plancherel measure, defined on the set $\widehat{G}$ of (equivalence classes of) irreducible (unitary) representations of $G$. In fact, for every irreducible representation $\pi$ of $G$ on a Hilbert space $\mathcal{H}_\pi$, we can find[(2)] a complete orthonormal system $\{v_{\pi,\alpha}\}_\alpha$ of $\mathcal{H}_\pi$ made of joint eigenvectors of $d\pi(L_1), \dots, d\pi(L_n)$. If $\lambda_{\pi,\alpha} \in \mathbb{R}^n$ denotes the eigenvalue of $d\pi(L_1), \dots, d\pi(L_n)$ corresponding to

[(2)] If $\Delta = p(L_1, \dots, L_n)$ is a positive weighted subcoercive operator, then $\pi(\mathcal{K}_L(e^{-p}))$ is compact (since $G$ is CCR), thus $d\pi(\Delta)$ has discrete spectrum and finite-dimensional eigenspaces, and moreover it commutes with $d\pi(L_1), \dots, d\pi(L_n)$.

the eigenvector $v_{\pi,\alpha}$, then, for every $m \in \mathcal{D}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |m(\lambda)|^2 \, d\sigma(\lambda) = \|\breve{m}\|^2_{L^2(G)} = \int_{\widehat{G}} \|\pi(\breve{m})\|^2_{\mathrm{HS}} \, d\pi = \int_{\widehat{G}} \sum_\alpha |m(\lambda_{\pi,\alpha})|^2 \, d\pi$$

(*cf.* [26, Proposition 3.7]). If one is able to determine both the group Plancherel measure and the eigenvectors $v_{\pi,\alpha}$ in such a way that the function $(\pi, \alpha) \mapsto \lambda_{\pi,\alpha}$ is sufficiently regular, then the measure $\sigma$ on $\mathbb{R}^n$ is determined by the previous identity as the push-forward of the product of the group Plancherel measure times a counting measure.

This route can be followed, *e.g.*, for the free 2-step nilpotent group on 3 generators $N_{3,2}$, which is determined by the relations

$$[X_1, X_2] = T_3, \qquad [X_2, X_3] = T_1, \qquad [X_3, X_1] = T_2,$$

where $X_1, X_2, X_3, T_1, T_2, T_3$ is a basis of its Lie algebra $\mathfrak{n}_{3,2}$; the dilations $\delta_t$ given by $\delta_t(X_j) = tX^j$, $\delta_t(T_j) = t^2 T_j$ define a stratification of $N_{3,2}$, so that $Q_{N_{3,2}} = Q_\delta = 9$.

If $L = -(X_1^2 + X_2^2 + X_3^2)$ is the sublaplacian, $\Delta = -(T_1^2 + T_2^2 + T_3^2)$ is the central Laplacian and $D = -(X_1 T_1 + X_2 T_2 + X_3 T_3)$, then $L, \Delta, D$ is a homogeneous weighted subcoercive system, with Plancherel measure $\sigma$ given by

$$\int_{\mathbb{R}^3} f \, d\sigma = \sum_{\alpha \in 2\mathbb{N}+1} \int_0^\infty \int_{-1}^1 f\left(\lambda, \lambda^{3/2} \frac{\theta(1-\theta^2)}{\alpha}, \lambda^2 \frac{(1-\theta^2)^2}{\alpha}\right) \frac{(1-\theta^2)^3}{8\pi^4 \alpha^4} \, d\theta \, \lambda^{7/2} \, d\lambda$$

(*cf.* [11]). This measure is supported on a countable family of surfaces accumulating on the axis $\mathbb{R} \times \{0\} \times \{0\}$, and it is not difficult to show that $\sigma$ is locally 2-bounded on $\mathbb{R}^3 \setminus \{0\}$, so that the system $L, \Delta, D$ satisfies the hypothesis $(\mathrm{J}_{9/2,2})$.

On the group $N_{3,2}$ we can also consider the system $L, -iT_1, -iT_2, -iT_3$; in this case the Plancherel measure $\sigma$ is given by

$$\int_{\mathbb{R}^4} f \, d\sigma = \sum_{\alpha \in 2\mathbb{N}+1} \int_0^\infty \int_{S^2} \int_0^1 f\left(\lambda, \lambda \frac{\nu\omega}{\alpha}\right) \frac{\nu^3}{32\pi^5 \alpha^4 \sqrt{1-\nu}} \, d\nu \, d\omega \, \lambda^{7/2} \, d\lambda.$$

One can show that $\sigma$ is locally $\frac{7}{2}$-bounded on $\mathbb{R}^4 \setminus \{0\}$, so that the system $L, -iT_1, -iT_2, -iT_3$ satisfies $(\mathrm{J}_{9/2,7/2})$.

As an example of computation of the capacity map $J$, we consider instead the 3-step group $G_{6,23}$ of [34], which is defined by the relations

$$[X_6, X_5] = X_4, \qquad [X_6, X_4] = X_2, \qquad [X_6, X_3] = -X_1,$$
$$[X_5, X_4] = X_1, \qquad [X_5, X_3] = X_2,$$

where $X_6, X_5, X_4, X_3, X_2, X_1$ is a basis of its Lie algebra $\mathfrak{g} = \mathfrak{g}_{6,23}$. It can be shown that the unique automorphic dilations $\delta_t$ (modulo automorphisms and rescaling) are given by $\delta_t(X_j) = t^{w_j} X_j$ with $w_6 = w_5 = 1$, $w_4 = w_3 = 2$, $w_2 = w_1 = 3$, so that this is an example of a non-stratifiable group, with $12 = Q_\delta > Q_{G_{3,2}} = 11$.

With the notation of § 3, we have

$$\mathfrak{z} = \operatorname{span}\{X_2, X_1\}, \qquad \mathfrak{y} = \operatorname{span}\{X_4, X_3, X_2, X_1\}.$$

If we denote by $\bar{X}_6, \bar{X}_5, \bar{X}_4, \bar{X}_3$ and $X_2^*, X_1^*$ the bases induced by $X_6, \dots, X_1$ on $\mathfrak{g}/\mathfrak{z}$ and $\mathfrak{z}^*$ respectively, then we have

$$|J(x_6\bar{X}_6 + x_5\bar{X}_5 + x_4\bar{X}_4 + x_3\bar{X}_3, t_2X_2^* + t_1X_1^*)|^2 = (x_6^2 + x_5^2)(t_2^2 + t_1^2)$$

with respect to a suitable norm on $\mathfrak{y}^*$, therefore the dual elements $\bar{X}_6^*, \bar{X}_5^* \in (\mathfrak{g}/\mathfrak{z})^*$ and $X_2, X_1 \in \mathfrak{z}$ attest that $G_{6,23}$ is 2-capacious (despite the fact that Proposition 3.9 does not apply to this group). Consequently, every homogeneous weighted subcoercive system on $G_{6,23}$, such as

$$(-iX_6)^{4k} + (-iX_5)^{4k} + (-iX_3)^{2k}, \quad -iX_2, \quad -iX_1$$

for $k \in \mathbb{N} \setminus \{0\}$, satisfies the hypothesis $(\mathrm{J}_{9/2,1})$.

Further examples and details may be found in [25].

### 6.3. Non-nilpotent groups

Theorem 5.7 allows one to obtain multiplier theorems also on groups which are not homogeneous, even not nilpotent. An interesting class of examples comes by considering an action of a torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ on a homogeneous group $N$ by automorphisms which commute with dilations, and the corresponding semidirect product $N \rtimes \mathbb{T}^d$ (or alternatively its universal covering group $N \rtimes \mathbb{R}^d$).

Take for instance a diamond group $G = H_n \rtimes \mathbb{T}^d$ (see [24]). If $L$ is a $\mathbb{T}^d$-invariant homogeneous sublaplacian on $H_n$ and $U_1, \dots, U_d$ are the partial derivatives on the torus $\mathbb{T}^d$, then $L, -iU_1, \dots, -iU_d$ is a weighted subcoercive system on $G$, since these operators commute and they generate an algebra containing the sublaplacian $\Delta = L + (-iU_1)^2 + \dots + (-iU_d)^2$. In fact, each of the operators $L, -iU_1, \dots, -iU_d$ can be considered as a homogeneous weighted subcoercive system in itself: $L$ is Rockland on $H_n$, and therefore satisfies $(\mathrm{J}_{(\dim H_n)/2,1})$, whereas $-iU_j$ comes from the corresponding derivative on the $j$-th factor of $\mathbb{R}^d$, which satisfies $(\mathrm{J}_{1/2,1})$. By applying Theorem 5.7, we then obtain that, if $m : \mathbb{R}^{1+d} \to \mathbb{C}$ vanishes on the coordinate hyperplanes and

$$\|m\|_{M_\daleth S^{\vec{s}}_{2,2}B(\mathbb{R}^{1+d})} < \infty \qquad \text{for } \vec{s} > \left(\frac{\dim H_n}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right),$$

then $m(L, -iU_1, \dots, -iU_d)$ is bounded on $L^p(G)$ for $1 < p < \infty$.

Thanks to Corollary 5.9 and Proposition 5.10, this result in turn yields a multiplier theorem for the sublaplacian $\Delta$: if $m : \mathbb{R} \to \mathbb{C}$ satisfies

$$\|m\|_{M_\epsilon B^s_{2,2}(\mathbb{R})} < \infty \qquad \text{for } s > \frac{\dim H_n + d}{2} = \frac{\dim G}{2},$$

then $m(\Delta)$ is bounded on $L^p(G)$ for $1 < p < \infty$. We remark that:

- this condition is sharper than the one following by the general result of Alexopoulos [1], which instead requires an $L^\infty$ condition of order $s > \frac{\dim G+1}{2}$;
- this is an example of a group in which the regularity threshold in a multiplier theorem can be lowered to half the topological dimension, which is neither a Heisenberg or related group, nor $SU_2$;
- the sublaplacian $\Delta$ can be replaced by any operator of the form

$$L^{k_0} + -(iU_1)^{2k_1} + \dots + (-iU_d)^{2k_d} \qquad \text{or} \qquad L^{k_0}(-iU_1)^{k_1} \cdots (-iU_d)^{k_d}$$

  for some $k_0, k_1, \dots, k_d \in \mathbb{N} \setminus \{0\}$, obtaining an analogous multiplier result with identical smoothness requirement.

Spectral multipliers for operators such as the complete Laplacian

$$\Delta_c = L + (-iT)^2 + (-iU_1)^2 + \dots + (-iU_d)^2,$$

where $T$ is the central derivative on $H_n$, can also be studied. By considering $L, -iT$ together as a homogeneous system on $H_n$, and each of the $-iU_j$ separately as before, one obtains a multiplier theorem for $\Delta_c$, with an $L^\infty$ condition of order $s > \frac{\dim G}{2}$.

Analogous considerations hold if one replaces $H_n$ by any Métivier group, and also if one takes the universal covering group $H_n \rtimes \mathbb{R}^d$; this last case comprises, for $d = 1$, the oscillator groups. Notice that the previous result about the Laplacian $\Delta_c$, when stated on the universal covering group, is sharper than [1], since the degree of growth of the group is greater than its topological dimension.

Further examples include the plane motion group $\mathbb{R}^2 \rtimes \mathbb{T}$, and the semidirect product $N_{2,3} \rtimes \mathbb{T}$ determined by the action of $SO_2$ on the free 3-step nilpotent group $N_{2,3}$ considered, *e.g.*, in [26, § 5.3]. In these last cases, for some distinguished sublaplacians, we still get a sharpening of the result by Alexopoulos: although the required order of smoothness is the same, our condition is expressed in terms of an $L^2$ instead of an $L^\infty$ Besov norm.

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