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MODULI OF UNIPOTENT REPRESENTATIONS I: FOUNDATIONAL TOPICS

by Ishai DAN-COHEN

Abstract. — With this work and its sequel, *Moduli of unipotent representations II*, we initiate a study of the finite dimensional algebraic representations of a unipotent group over a field of characteristic zero from the modular point of view. Let G be such a group. The stack $\mathcal{M}_n(G)$ of all representations of dimension n is badly behaved. In this first installment, we introduce a nondegeneracy condition which cuts out a substack $\mathcal{M}_n^{\mathrm{nd}}(G)$ which is better behaved, and, in particular, admits a coarse algebraic space, which we denote by $M_n^{\mathrm{nd}}(G)$. We also study the problem of glueing a pair of nondegenerate representations along a common subquotient.

Résumé. — Dans ce travail et sa suite, *Moduli of unipotent representations II*, nous commençons l'étude des représentations algébriques de dimension finie d'un groupe unipotent sur un corps de caractéristique nulle, d'un point de vue modulaire. Soit G un tel groupe. Le champ $\mathcal{M}_n(G)$ de toutes les représentations de dimension n se comporte mal. Dans cette première partie, nous introduisons une condition de non-dégénérescence qui définit un sous-champ $\mathcal{M}_n^{\mathrm{nd}}(G)$ qui se comporte mieux et qui admet en particulier un espace algébrique grossier ; nous le notons $M_n^{\mathrm{nd}}(G)$. Nous étudions aussi le problème de recoller deux représentations non dégénérées le long d'un sous-quotient commun.

Introduction

The purpose of this paper, and its sequel [4], is to develop an approach to the problem of moduli of representations of a unipotent group over a field of characteristic zero. Fix such a field k, a unipotent group G over k, and a positive integer n. The stack $\mathcal{M}_n(G)$ of all representations of a fixed dimension n is badly behaved. It is typically not algebraic(1), and its diagonal, albeit representable, is a positive dimensional group whose fiber

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(1) Thanks are due to Anton Geraschenko for helping me understand this fact.

dimensions can jump in families. We define a nondegeneracy condition (4.1) which cuts out a substack $\mathcal{M}_n^{\mathrm{nd}}(G) \subset \mathcal{M}_n(G)$. This substack is algebraic (7.4) and its diagonal is flat. It then follows from a higher quotient construction known as "rigidification" (see §6) that the fppf sheaf $M_n^{\mathrm{nd}}(G)$ associated to $\mathcal{M}_n^{\mathrm{nd}}(G)$ is an algebraic space (7.5).

A sufficient nondegeneracy condition for the above purpose (at least for representations defined over a field containing k) would simply minimize the dimension of the automorphism group. Unfortunately, this condition is not preserved under taking subquotients (4.11). Our condition includes the recursive assumption that the subquotients of a nondegenerate representation are again nondegenerate. This allows us to fiber the moduli space of representations of dimension $n+1$ over a certain space M_n^{cnd} (9.7) of *compatible pairs* of nondegenerate representations of dimension n. I expect this fibration to play a role in the study of the geometry of our moduli space.

Our recursive condition is motivated also by the results of [4]. There is an invariant w of (the Lie algebra of) G which we call the *width*, which singles out a best-case-scenario for constructing moduli. For $n \leqslant w+1$, our nondegeneracy condition enjoys a very concrete interpretation, and an ensuing construction shows that $M_n^{\mathrm{nd}}(G)$ is a quasi-projective variety. [4] also contains concrete examples of M_n^{nd} as well as of M_n^{cnd} for low n. As a general matter, the present paper is concerned largely with abstract generalities. These abstractions are, in part, justified and motivated by the concrete results and examples of [4]. (Conversely, the more elementary approach taken in [4] is justified, in part, by the conceptual issues discussed here.)

The problem of constructing a coarse space of a moduli stack has a long and rich history, which may help put our problem in context. However, our problem does not seem to fit the rubric suggested by the methods which have emerged from this history. The functor

$$\mathbb{H}\mathrm{om}(G, \mathbb{GL}_n)$$

of homomorphisms to $\mathbb{GL}_n$ is typically not representable, so a direct application of geometric invariant theory to the action of $\mathbb{GL}_n$ by conjugation is not possible; the prospects of a more creative application are unclear. Mumford's theory requires a reductive group as part of the input. But in our context, it is more natural to consider instead the action of the group $\mathbb{B}_n$ of invertible upper triangular matrices on the space

$$X_n^{\mathrm{fl}}(G) \subset \mathbb{H}\mathrm{om}(G, \mathbb{U}_n)$$

of upper triangular representations whose canonically associated filtration is a full flag (2.1). Even if we insist on an action by $\mathbb{GL}_n$ on an appropriate space, the stabilizer subgroups will not be reductive. Partial analogs of Mumford's theory for groups which are not reductive are currently under development in works by A. Asok, B. Doran, and F. Kirwan (c.f. [6], [2]).

Another well known tool is the Keel-Mori theorem (c.f. [11]). This theorem applies to algebraic stacks with finite diagonal and produces an algebraic space. In our context, plagued by a unipotent action with positive dimensional stabilizers, it appears that the only readily available tool is *rigidification*. This technique requires that we restrict attention to representations whose automorphism group is flat, but it has the advantage of producing a sheaf quotient: if we let

$$X_n^{\mathrm{nd}}(G) \subset X_n^{\mathrm{fl}}(G)$$

denote the locus of representations which satisfy our nondegeneracy condition, then

$$M_n^{\mathrm{nd}}(G) = X_n^{\mathrm{nd}}(G)/_{\mathrm{fppf}}\mathbb{B}_n$$

is a quotient of flat sheaves. Since $X_n^{\mathrm{nd}}(G)$ is of finite type, it follows, moreover, that the geometric fibers of the projection $X_n^{\mathrm{nd}}(G) \to M_n^{\mathrm{nd}}(G)$ are orbits.

Our nondegeneracy condition has more in common with Mumford's stability than his semistability. In Mumford's set-up, a reductive algebraic group G acts on a finite-type scheme X. There is then a locus $X^s \subset X$ (depending on the choice of a linearization) of points which are *stable* ([14, §4, Definition 1.7]). In X^s, dimensions of stabilizers are constant on connected components (loc. cit.), and X^s admits a quotient $f : X^s \to Y$ whose geometric fibers are orbits ([14, §4, Theorem 1.10 and §1, Definition 0.6]).

I now give an outline of the paper together with statements of the main theorems and sketches of proofs. If $\mathfrak{g}$ is the Lie algebra of G, then finite dimensional representations of G correspond to finite dimensional nilpotent representations of $\mathfrak{g}$. With this in mind, we begin by studying the problem of moduli of nilpotent representations of a fixed Lie algebra $\mathfrak{g}$ over k. In section 1 we recall the definition and first properties of nilpotent representations. In section 2 we study *flag* representations: those whose associated filtration is a full flag. We give, in particular, a criterion for a nilpotent representation to be a flag representation in coordinates.

In section 3 we develop a technical tool: the scheme-theoretic Lie algebra $\mathfrak{n}(r)$ of the unipotent part $\mathbb{U}(r)$ of the automorphism group of a flag representation $r : \mathfrak{g}_T \to \mathscr{E}nd(\mathcal{E})$ over a general base T over k. We observe that $\mathbb{A}\mathrm{ut}\, r = \mathbb{G}_{m,T} \times \mathbb{U}(r)$, that $\mathbb{U}(r)$ is isomorphic to $\mathfrak{n}(r)$ as a scheme, and that $\mathfrak{n}(r)$ is the total space of a module.

In section 4 we turn to the definition and an initial study of our nondegeneracy condition. Roughly, a nilpotent representation $r : \mathfrak{g} \to \mathrm{End}(E)$ on a vector space E is *nondegenerate*, if it is a flag representation, if every subquotient is (by recursion on the dimension of E) already nondegenerate, and if among representations satisfying the above two conditions, the dimension of the automorphism group is minimal. The dimension of the automorphism group of a nondegenerate representation of dimension n is independent of the choice of nondegenerate representation; it thus defines an invariant of $\mathfrak{g}$ and n which we denote by $A(\mathfrak{g}, n)$. We illustrate the behavior of $A(\mathfrak{g}, n)$ for low n with several examples, showing in particular that the possible triples $(A(\mathfrak{g}, 2), A(\mathfrak{g}, 3), A(\mathfrak{g}, 4))$ are $(2, 2, 2), (2, 2, 3)$ and $(2, 3, 4)$.

Let $\mathfrak{n}_n$ be the Lie algebra of strictly upper triangular $n \times n$ matrices. In section 5 we observe that the locus X_n^{fl} of flag representations of $\mathfrak{g}$ is an open subscheme of $\mathbb{H}\mathrm{om}(\mathfrak{g}, \mathfrak{n}_n)$, we let $X_n^{\mathrm{nd}} \subset X_n^{\mathrm{fl}}$ denote the locus of nondegenerate nilpotent representations, and we prove that $X_n^{\mathrm{nd}} \hookrightarrow X_n^{\mathrm{fl}}$ is an immersion. Here we use $\mathfrak{n}(r)$ to help in showing that X_n^{nd} is compatible with taking infinite unions of rings, and we use $\mathfrak{n}(r)$ again to produce a flattening stratification for the automorphism group of the universal family.

If $\mathcal{X}$ is an algebraic stack whose automorphism groups are flat, then the associated fppf sheaf, $\pi_0^{\mathrm{fppf}}(\mathcal{X})$, is an algebraic space. This is an instance of rigidification, which we review in §6.

We let $\mathcal{M}_n^{\mathrm{nd}}(\mathfrak{g})$ denote the stack of n-dimensional nondegenerate nilpotent representations of $\mathfrak{g}$. The main goal of section 7 is to prove:

Theorem 7.5. — The fppf sheaf $\pi_0^{\mathrm{fppf}}\mathcal{M}_n^{\mathrm{nd}}(\mathfrak{g})$ associated to $\mathcal{M}_n^{\mathrm{nd}}(\mathfrak{g})$ is an algebraic space.

For the proof, we let $\mathbb{B}_n$ denote the group of invertible upper triangular matrices, and we observe that the stack quotient of X_n^{fl} by the action of $\mathbb{B}_n$ by conjugation is equal to the stack of flag representations. In particular, the stack quotient of X_n^{nd} by the action of $\mathbb{B}_n$ is equal to $\mathcal{M}_n^{\mathrm{nd}}(\mathfrak{g})$. It follows that $\mathcal{M}_n^{\mathrm{nd}}(\mathfrak{g})$ is an algebraic stack. Since its inertia is by construction flat, rigidification applies to produce the theorem.

We let

$$M_n^{\mathrm{nd}}(\mathfrak{g}) := \pi_0^{\mathrm{fppf}}\mathcal{M}_n^{\mathrm{nd}}(\mathfrak{g}) \ ,$$

and call it *the moduli space of n-dimensional nondegenerate nilpotent representations.* Next we discuss the functoriality of our moduli spaces. We show that if $\mathfrak{f}, \mathfrak{g}$ are Lie algebras such that for $i = 1, \ldots, n$, $A(\mathfrak{g}, i) = A(\mathfrak{f}, i)$, then any surjection

$$\mathfrak{f} \twoheadrightarrow \mathfrak{g}$$

gives rise to a closed immersion

$$M_n^{\mathrm{nd}}(\mathfrak{g}) \hookrightarrow M_n^{\mathrm{nd}}(\mathfrak{f}) .$$

We end the section with a discussion of a certain variant of the above constructions. We define a *framed nondegenerate nilpotent representation* to be a nondegenerate nilpotent representation equipped with a grading-compatible basis for the associated graded vector space. We let $\mathcal{M}_n^{\mathrm{fnd}}(\mathfrak{g})$ denote the stack of n-dimensional framed nondegenerate nilpotent representations of $\mathfrak{g}$, we prove that $\pi_0^{\mathrm{fppf}} \mathcal{M}_n^{\mathrm{fnd}}(\mathfrak{g})$ is an algebraic space, and we define

$$M_n^{\mathrm{fnd}}(\mathfrak{g}) := \pi_0^{\mathrm{fppf}} \mathcal{M}_n^{\mathrm{fnd}}(\mathfrak{g}) .$$

This gives us a modular interpretation of the sheaf quotient of X_n^{nd} by the action of the group $\mathbb{U}_n$ of upper triangular matrices with 1's on the diagonal.

It is well known that the functor **Lie** induces an equivalence of categories from the category of unipotent groups over k to the category of nilpotent Lie algebras over k, and that given a unipotent group G with Lie algebra $\mathfrak{g}$, **Lie** also induces an isomorphism from the category of finite dimensional representations of G to the category of finite dimensional nilpotent representations of $\mathfrak{g}$. In section 8 we generalize the latter statement to include families of (as well as infinite dimensional) representations. Let $\mathbb{REP}(G)$ denote the fibered category of quasi-coherent representations of G and let $\mathbb{REP}^{\mathrm{nil}}(\mathfrak{g})$ denote the fibered category of locally nilpotent quasi-coherent representations of $\mathfrak{g}$. Then we have

Theorem 8.20. — The functor

$$\mathbf{Lie} : \mathbb{REP}(G) \to \mathbb{REP}^{\mathrm{nil}}(\mathfrak{g})$$

sending a representation to its derivative at the identity is an isomorphism of fibered categories.

The main obstacle is that the functor of automorphisms of a quasi-coherent sheaf may not be representable; it is overcome by a careful analysis of the exponential map. This theorem reduces the problem of moduli of representations of G to the problem of moduli of nilpotent representations of $\mathfrak{g}$, and hence to the context of the previous section. We apply all definitions introduced in sections 2–7 to G through its Lie algebra; in particular we define *the moduli space of n-dimensional nondegenerate representations of* G by

$$M_n^{\mathrm{nd}}(G) := M_n^{\mathrm{nd}}(\mathbf{Lie}\, G)\,.$$

Since a flag representation has two canonically defined subquotients, the moduli spaces of nondegenerate representations form a tower

$$\begin{tikzcd} \vdots \arrow[d, shift left] \arrow[d, shift right] \\ M_{n+1}^{\mathrm{nd}} \arrow[d, shift right, "p_1^n"'] \arrow[d, shift left, "p_2^n"] \\ M_n^{\mathrm{nd}} \arrow[d, shift right, "p_1^{n-1}"'] \arrow[d, shift left, "p_2^{n-1}"] \\ M_{n-1}^{\mathrm{nd}} \arrow[d, shift left] \arrow[d, shift right] \\ \vdots \end{tikzcd}$$

with

$$p_2^{n-1} \circ p_1^n = p_1^{n-1} \circ p_2^n\,.$$

In section 9 we discuss this tower, focussing on a modular answer to the following question: when can two n-dimensional nondegenerate representations be glued along a codimension-one subquotient to produce an $n+1$-dimensional flag representation? The answer is given in the form of a closed algebraic subspace M_n^{cnd} of $M_n^{\mathrm{nd}} \times_{p_2, M_{n-1}^{\mathrm{nd}}, p_1} M_n^{\mathrm{nd}}$ through which the map

$$M_{n+1}^{\mathrm{nd}} \to M_n^{\mathrm{nd}} \underset{p_2, M_{n-1}^{\mathrm{nd}}, p_1}{\times} M_n^{\mathrm{nd}}$$

factors. This generalizes the role played by the diagonal of $\mathbb{P}\mathfrak{g}^{\mathrm{ab}} \times \mathbb{P}\mathfrak{g}^{\mathrm{ab}}$ in the case of three dimensional representations of a Lie algebra of width one ([4]).

This project leads naturally in several directions which I now indicate briefly. Computations performed jointly with Anton Geraschenko reveal that typically $M_n^{\mathrm{nd}}(G)$ has multiple components, many singularities not explained by the multiplicity of components, and sometimes even generically nonreduced components. These geometric features endow $M_n^{\mathrm{nd}}(G)$ with a natural stratification which provides unipotent representations with an intricate discrete invariant and suggests a classification program in the same spirit as classical representation theory. This may lead to a study of representations of an arbitrary algebraic group which mixes classical representation theory with a theory of unipotent representations.

On the other hand, my initial interest in this problem came from the hope to formulate a story somewhat similar to that of [17] for the unipotent fundamental group in a p-adic context. Given a variety over $\mathbb{F}_p$ satisfying certain hypotheses, the theory of the p-adic unipotent fundamental group gives rise to a pair of prounipotent groups U_{cris} and $U_{\text{ét}}$ over $\mathbb{Q}_p$. These groups carry various extra structures, as well as a comparison isomorphism over B_{cris} which together reflect arithmetic properties of the variety and which should in turn be reflected in the structure of the moduli space of representations. For instance, there should be an automorphism whose fixed points single out those unipotent isocrystals which support an F-structure.

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Notations and conventions

0.1. — If T is a scheme and $\mathcal{F}$ is a quasi-coherent sheaf, then the *contravariant total space*, denoted $\mathbb{V}\mathcal{F}$, of $\mathcal{F}$ is defined by

$$\mathbb{V}\mathcal{F}(f : T' \to T) = \mathrm{Hom}_{\mathcal{O}_{T'}}(f^*\mathcal{F}, \mathcal{O}_{T'})$$

([8, 1.7.8]) and if $\phi : \mathcal{E} \to \mathcal{F}$ is a map of quasi-coherent sheaves then $\mathbb{V}\phi$ denotes the induced map $\mathbb{V}\mathcal{F} \to \mathbb{V}\mathcal{E}$. If $\mathcal{E}, \mathcal{F}$ are locally free of finite rank, then the kernel of ϕ *regarded as a map of vector groups* is defined by the Cartesian square

$$\begin{tikzcd} \ker \mathbb{V}\phi^\vee \arrow[r] \arrow[d] & \mathbb{V}\mathcal{E}^\vee \arrow[d, "\mathbb{V}\phi^\vee"] \\ T \arrow[r] & \mathbb{V}\mathcal{F}^\vee \end{tikzcd}$$

where the arrow at the bottom is the zero section. Although $\ker \mathbb{V}\phi^\vee$ may not be a vector group, it is the contravariant total space of a module. Indeed, it suffices to check this under the assumption that $T = \operatorname{Spec} A$ is affine in which case we write E and F for the modules of global sections, we let $Q = \operatorname{cok}(\phi^\vee)$ and we observe that for any A-algebra B we have

$$\operatorname{Hom}_A(Q, B) = \ker(\operatorname{id}_B \otimes \phi)$$

as in the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}(Q, B) \arrow[r] & \operatorname{Hom}(E^\vee, B) \arrow[r] \arrow[d, equal] & \operatorname{Hom}(F^\vee, B) \arrow[d, equal] \\ & & B \otimes E \arrow[r, "\operatorname{id}_B \otimes \phi"'] & B \otimes F \end{tikzcd}$$

0.2. — In general, when working over a scheme T, we use blackboard bold symbols to denote presheaves on the category of affine T-schemes, and calligraphic symbols to denote presheaves on the small Zariski site of T. So, for example, if $r : \mathfrak{g} \to \mathscr{E}nd(\mathcal{F})$ is a representation of a Lie algebra on a quasi-coherent sheaf over T, then $\mathbb{E}\mathrm{nd}(r)$ denotes the functor

$$(f : T' \to T) \mapsto \operatorname{End}(f^* r)$$

and $\mathscr{E}nd(r)$ denotes its restriction to X_{Zar} (0.5). The latter is quasi-coherent but the former may not be.

0.3. — When working over an affine scheme $T = \operatorname{Spec} A$ we use a plain font to denote the module of global sections of a quasi-coherent sheaf; thus $E = \Gamma(T, \mathcal{E})$. On the other hand, when $\mathfrak{g}$ is a Lie algebra over a ring A and $T = \operatorname{Spec} A$, we use $\mathfrak{g}$ again for the sheaf of Lie $\mathcal{O}_T$-algebras associated to $\mathfrak{g}$ when conflating the two poses no danger.

0.4. — Let T be a scheme, $\mathcal{F}$ an $\mathcal{O}_T$-module, and $\mathcal{E} \subset \mathcal{F}$ a submodule. Following Lang, we say that $\mathcal{F}$ is a *vector sheaf* if it is locally free of finite rank. Assuming this to be the case, we say that $\mathcal{E}$ is a *vector subsheaf* if the quotient module $\mathcal{F}/\mathcal{E}$ is a vector sheaf.

0.5. — For generalities on stacks and algebraic stacks, we refer the reader to [13]. An efficient overview can be found in §1 of [15]. When working with algebraic stacks, it is customary to work in the étale topology. We, however, will have occasion to use the fppf topology, which is finer, as well as the Zariski topology, which is coarser. An easy definition of the notion of a (Grothendieck) topology on a category may be found in [18].

A category equipped with a topology is called a *site*. For our purposes, the distinction between a site and its sheaf topos will be unimportant. Thus, if X is a scheme, the reader is invited to think of X_{Zar} as the site whose objects are the (Zariski) open subsets of X, whose only morphisms are inclusions, and whose covering families are the usual coverings by Zariski open subsets. This is often referred to as the *small Zariski site of X*. By contrast, the *big Zariski site of X*, denoted X_{ZAR}, is the site whose underlying category is the category $\mathbf{Aff}(X)$ of all affine X-schemes, and whose coverings are the usual coverings by Zariski open subsets. The *fppf site of X* is again a topology on $\mathbf{Aff}(X)$. Its covering families are families of morphisms $\{f_i : T_i \to T\}_i$ such that $\amalg f_i : \coprod T_i \to T$ if faithfully flat and locally of finite presentation.

0.6. — When working over the category $\mathbf{Aff}(T)$ of affine schemes over a base T, we employ the convention that when no topology is mentioned, the indiscrete topology (that is, the topology whose only coverings are the isomorphisms) is assumed. Notationally, this means the following.

If X is a presheaf on $\mathbf{Aff}(T)$ and G is a group presheaf acting on X, we write X/G for the presheaf quotient and $[X/G]$ for the associated fibered category. Thus for $T' \in \mathbf{Aff}(T)$,

$$(X/G)(T') = X(T')/G(T')$$

and $[X/G](T')$ is the groupoid whose objects are the elements of $X(T')$ and whose morphisms $x \to y$ are those elements of $G(T')$ such that $gx = y$.

Now let τ be a topology on $\mathbf{Aff}(T)$ and suppose X is a τ-sheaf and G is a group τ-sheaf acting on X. Then we write $X/_\tau G$ for the sheaf quotient with respect to τ and $[X/_\tau G]$ for the stack quotient with respect to τ. Thus $X/_\tau G$ is the τ-sheaf associated to X/G and $[X/_\tau G]$ is the τ-stack associated to $[X/G]$.

0.7. — Continuing with the situation of 0.6, suppose $\mathcal{X}$ is a fibered category over $\mathbf{Aff}(T)$. Then $\pi_0(\mathcal{X})$ denotes the presheaf associated to $\mathcal{X}$: $\pi_0(\mathcal{X})(T') = \pi_0(\mathcal{X}(T'))$ is the set of isomorphism classes of objects of $\mathcal{X}(T')$. If $\mathbf{Aff}(T)$ is again endowed with a topology τ then we write $\pi_0^\tau(\mathcal{X})$ for the τ-sheaf associated to $\pi_0(\mathcal{X})$.

0.8. — We remind the reader that a quasi-coherent sheaf on the small Zariski site of a scheme T extends uniquely to a quasi-coherent sheaf on the big Zariski site of T, so there is usually no danger in conflating the two in our notation. Nevertheless, we find it useful to reserve a special notation for the big structure sheaf $o_T : \mathbf{Aff}(T) \to \mathbf{Ring}$ which sends $T' \mapsto \Gamma(T', \mathcal{O}_{T'})$.

1. Preliminary discussion of nilpotence

Fix a field k and a Lie algebra $\mathfrak{g}$ over k, assumed to be finitely generated. Here we review some of the basics of nilpotent Lie algebras and nilpotent representations, while discussing some initial technicalities pertaining to nilpotence in families.

1.1. — We denote by $\mathfrak{g}^{(i)}$ the i^{th} term in the descending central series: $\mathfrak{g}^{(1)} = \mathfrak{g}$, $\mathfrak{g}^{(i+1)} = [\mathfrak{g}, \mathfrak{g}^{(i)}]$, and by $\mathfrak{g}^{\text{ab}}$ the abelianization: $\mathfrak{g}^{\text{ab}} = \mathfrak{g}^{(1)}/\mathfrak{g}^{(2)}$. The *pronilpotent completion* of $\mathfrak{g}$ is the inverse limit $\varprojlim \mathfrak{g}/\mathfrak{g}^{(n)}$. We say that $\mathfrak{g}$ is *nilpotent* if there exits an n such that $\mathfrak{g}^{(n)} = 0$ and *pronilpotent* if $\mathfrak{g}$ is equal to its pronilpotent completion. If F is a vector space, $\mathfrak{n}(F)$ denotes the free pronilpotent Lie algebra on F. It is characterized by the mapping property $\mathrm{Hom}_{\mathbf{Lie}}(\mathfrak{n}(F), \mathfrak{n}) = \mathrm{Hom}_{\mathbf{Vect}}(F, \mathfrak{n})$ for every nilpotent Lie algebra $\mathfrak{n}$, and may be constructed as the pronilpotent completion of the free Lie algebra on F.

1.2. — The word *filtration* will always refer to an increasing filtration indexed by the natural numbers.

1.3. — Let A be a ring, E an A-module, Fil a filtration of E by submodules, and ϕ an endomorphism of E. We say that ϕ *is nilpotent with respect to* Fil if for each $i \geqslant 1$, $\phi(\mathrm{Fil}_i E) \subset \mathrm{Fil}_{i-1} E$, and we write $\mathfrak{n}_{\mathrm{Fil}} E$ for the space of all such endomorphisms.

More generally, if $(T, \mathcal{O}_T)$ is a ringed space, $\mathcal{E}$ an $\mathcal{O}_T$-module, and Fil a filtration of $\mathcal{E}$ by submodules, we write $\mathfrak{n}_{\mathrm{Fil}}\mathcal{E}$ for the sheaf of endomorphisms of $\mathcal{E}$ nilpotent with respect to Fil: if U is an open subset of T then $\mathfrak{n}_{\mathrm{Fil}}\mathcal{E}(U)$ is the set of those endomorphisms $\mathcal{E}|_U \to \mathcal{E}|_U$ such that for every i, the composite $\mathrm{Fil}_i\mathcal{E}|_U \subset \mathcal{E}|_U \to \mathcal{E}|_U$ factors through $\mathrm{Fil}_{i-1}\mathcal{E}|_U$. We remind the reader, however, that when T is a scheme and $\mathcal{E}$ is quasi-coherent, $\mathscr{E}nd(\mathcal{E})$ may not be quasi-coherent.

1.4. — Let T be a k-scheme, and $r : \mathfrak{g}_T \to \mathscr{E}nd\,\mathcal{E}$ a representation on a quasi-coherent sheaf $\mathcal{E}$. We define the 0*-eigenspace of* r, an $\mathcal{O}_T$-submodule $\mathcal{E}^0$ of $\mathcal{E}$, by

$$\mathcal{E}^0(U) := \{e \in \mathcal{E}(U) \mid (rv)e = 0 \ \forall v \in \mathfrak{g}(U)\} .$$

We remark that $\mathcal{E}^0$ is quasi-coherent. In verifying this, we assume $T = \operatorname{Spec} A$ to be affine, we write E for the module of global sections of $\mathcal{E}$, and we fix an arbitrary element $a \in A$. There is then a natural map $(E^0)_a \to (E_a)^0$ which is injective by exactness of localization, and we are to show that it is surjective. To this end, fix a finite set $v_1, \dots, v_s$ of generators of $\mathfrak{g}$, and consider an arbitrary element $\frac{e}{a^n}$ of $(E_a)^0$. For each $i = 1, \dots, s$, we have $0 = (rv_i)(\frac{e}{a^n}) = \frac{(rv_i)(e)}{a^n}$, from which it follows that there exists an integer l_i such that $a^{l_i}(rv_i)(e) = 0$. Let $l = \max_i l_i$. Then $a^l e \in E^0$, so $\frac{a^l e}{a^{n-l}}$ is an element of $(E^0)_a$ mapping to $\frac{e}{a^n}$, which completes the verification.

Definition 1.5. — Let T be a k-scheme and let $r : \mathfrak{g}_T \to \mathscr{E}nd\,\mathcal{E}$ be a representation of $\mathfrak{g}$ on a quasi-coherent sheaf $\mathcal{E}$ over T. We define the **associated filtration** of $\mathcal{E}$, denoted Fil^r, by

$$\mathrm{Fil}_0^r \mathcal{E} = 0$$

and

$$\mathrm{Fil}_{n+1}^r \mathcal{E} = \tau_n^{-1}((\mathcal{E}/\mathrm{Fil}_n^r \mathcal{E})^0)\,,$$

where

$$\tau_n : \mathcal{E} \to \mathcal{E}/\mathrm{Fil}_n^r \mathcal{E}$$

is the projection, and the 0-eigenspace $(\mathcal{E}/\mathrm{Fil}_n^r \mathcal{E})^0$ is defined as in 1.4. When there is no risk of confusion, we drop the r from the notation, and we sometimes write $\mathcal{E}_n$ instead of $\mathrm{Fil}_n \mathcal{E}$. We also write r_i for the subrepresentation

$$\mathfrak{g}_T \to \mathscr{E}nd(\mathrm{Fil}_i^r \mathcal{E})$$

and r^j or r/r_j for the quotient representation

$$\mathfrak{g}_T \to \mathscr{E}nd(\mathcal{E}/\mathrm{Fil}_j^r \mathcal{E})$$

and finally, for $j \leqslant i$ we write r_i^j for the subquotient

$$\mathfrak{g}_T \to \mathscr{E}nd(\mathrm{Fil}_i^r \mathcal{E}/\mathrm{Fil}_j^r \mathcal{E})\,.$$

Remark 1.6. — Let T be a k-scheme and let $r : \mathfrak{g}_T \to \mathscr{E}nd\,\mathcal{E}$ be a representation of $\mathfrak{g}$ on a quasi-coherent sheaf $\mathcal{E}$ over T. We note the following formula for $0 \leqslant i \leqslant j$ and $0 \leqslant k \leqslant j - i$:

$$\mathrm{Fil}_k^{r_j^i}(\mathcal{E}_j/\mathcal{E}_i) = \mathcal{E}_{i+k}/\mathcal{E}_i\,.$$

Lemma 1.7. — Let $r : \mathfrak{g} \to \operatorname{End} E$ be a representation of $\mathfrak{g}$ on a vector space E over k. Then r factors through $\mathfrak{n}_{\mathrm{Fil}^r} E$ (1.3).

Proof. — Fixing $v \in \mathfrak{g}$, $i \geqslant 1$ and $e \in \mathrm{Fil}^r_i E$, we are to show that $(rv)e \in \mathrm{Fil}^r_{i-1}E$; since, in the notation of 1.5, τ_{i-1} is a morphism of representations, we have

$$\tau_{i-1}((rv)e) = (r^{i-1}v)(\tau_{i-1}e) = 0$$

whence $(rv)e \in \mathrm{Fil}^r_{i-1}E$. □

1.8. — Given a ring B, a module F over B and a filtration Fil, we say that Fil is *exhaustive* if

$$\bigcup_i \mathrm{Fil}_i F = F\,.$$

1.9. — We recall that a representation $r : \mathfrak{g} \to \operatorname{End} E$ on a finite dimensional vector space E is *nilpotent* if either $E = 0$ or E^0 is nonzero and, by recursion on $\dim E$, E/E^0 is nilpotent. (By Engel's theorem ([12, 1.35]), this condition is equivalent to another condition, which is often taken as the definition.)

Proposition 1.10. — *Let $r : \mathfrak{g} \to \operatorname{End} E$ be a representation of $\mathfrak{g}$ on a finite dimensional vector space E. The following conditions are equivalent:*

(i) *r is nilpotent.*
(ii) *Fil^r is exhaustive.*
(iii) *r factors through $\mathfrak{n}_{\mathrm{Fil}}E$ for some exhaustive filtration Fil of E.*

Proof. — (i ⇒ ii) Suppose r is nilpotent. Then r^i is nilpotent for all i; indeed, if we assume for an induction on i that r^i is nilpotent, since

$$E/E_{i+1} = (E/E_i)/((E/E_i)^0)\,,$$

it follows that E/E_{i+1} is nilpotent. This implies that for each i, either $E_i = E$ or $E_i \neq E_{i+1}$. Since E is finite dimensional, there exists an i such that $E_i = E$, from which the conclusion follows.

(ii ⇒ iii) This follows from 1.7 by setting $\mathrm{Fil} := \mathrm{Fil}^r$.

(iii ⇒ i) If $\dim E = 0$, then r is nilpotent by definition. Fix a positive integer n and assume for an induction on n that (iii) implies (i) whenever $\dim E = n$. Let Fil be an exhaustive filtration on E, suppose r factors through $\mathfrak{n}_{\mathrm{Fil}}E$ and suppose $\dim E = n+1$. If i is the smallest number such that $\mathrm{Fil}_i E \neq 0$, then $\mathrm{Fil}_i E \subset E^0$ so $E^0 \neq 0$. It remains to show that the inductive hypothesis may be applied to the quotient representation $r^1 : \mathfrak{g} \to \operatorname{End}(E/E^0)$. Denote by $\tau_1 : E \to E/E^0$ the projection and for each j, let ${}_{\tau_1}\mathrm{Fil}_j(E/E^0)$ be the image of $\mathrm{Fil}_j E$ in E/E^0. Given j an arbitrary natural number, $e \in \mathrm{Fil}_{j+1}E$, and $v \in \mathfrak{g}$, we have

$$(r^1 v)(\tau_1 e) = \tau_1((rv)e)\,;$$

since $(rv)e \in \mathrm{Fil}_j E$, it follows that $(r^1 e)(\tau_1 e) \in {}_{\tau_1}\mathrm{Fil}_j(E/E^0)$. Thus r^1 factors through $\mathfrak{n}_{\tau_1 \mathrm{Fil}}(E/E^0)$; since ${}_{\tau_1}\mathrm{Fil}$ is exhaustive, the inductive hypothesis applies as hoped to conclude that r^1 is nilpotent, and hence that r is nilpotent, concluding the proof. □

COROLLARY 1.11. — Let $r : \mathfrak{g} \to \operatorname{End} E$ be a representation of $\mathfrak{g}$ on a finite-dimensional vector space E. If r is nilpotent then so is every subquotient.

Proof. — It is sufficient to consider subrepresentations and quotient representations separately. Suppose $r : \mathfrak{g} \to \operatorname{End} E$ is nilpotent and let Fil be an exhaustive filtration such that r factors through $\mathfrak{n}_{\mathrm{Fil}} E$ as in 1.10 (iii). If $\iota : E' \hookrightarrow E$ is a subrepresentation, define Fil' by

$$\mathrm{Fil}'_i E' := \iota^{-1} \mathrm{Fil}_i E$$

and if $\pi : E \twoheadrightarrow E'$ is a quotient representation, define Fil' by

$$\mathrm{Fil}'_i E' := \pi(\mathrm{Fil}_i E)$$

as in the proof of (iii ⇒ ii) in Proposition 1.10. Either way, $r' : \mathfrak{g} \to \operatorname{End} E'$ factors through $\mathfrak{n}_{\mathrm{Fil}'} E'$: in the case of a quotient representation this was verified in the proof of (iii ⇒ ii) in 1.10; the case of a subrepresentation is similar. □

DEFINITION 1.12. — If $r : \mathfrak{g} \to \operatorname{End} E$ is a nilpotent representation and Fil is an exhaustive filtration such that r factors through $\mathfrak{n}_{\mathrm{Fil}} E$, we say that r is **nilpotent with respect to** Fil. Note that r is nilpotent with respect to Fil if and only if Fil^r is subordinate to Fil.

2. Flag representations

We continue to work with a finitely generated Lie algebra $\mathfrak{g}$ over a field k. We fix an arbitrary affine k-scheme $T = \operatorname{Spec} A$. We write $\mathfrak{g}_A$ (or $\mathfrak{g}_T$) for the Lie A-algebra (or for the Lie $\mathcal{O}_T$-algebra associated to) $A \otimes \mathfrak{g}$.

DEFINITION 2.1. — Suppose first that A is a field. A nilpotent representation $r : \mathfrak{g}_A \to \operatorname{End} E$ on an A-vector space E is a **flag** representation if Fil^r (1.5) is a (full) flag. Now let A be arbitrary. A representation $r : \mathfrak{g}_T \to \mathscr{E}nd\, \mathcal{E}$ is a **flag** representation if its fibers above field-valued points of T are flag representations in the above sense, and in addition, the following conditions are satisfied:

(i) $\mathcal{E}$ is a vector sheaf (0.4),
(ii) for each $i \geqslant 0$, $\mathrm{Fil}^r_i \mathcal{E}$ is a vector subsheaf (loc. cit.), and
(iii) formation of Fil^r is compatible with base-change.

Counterexamples (2.2-2.4)

We discuss briefly what can go wrong when conditions (ii) and (iii) of definition 2.1 are dropped. Even when the canonical filtration is a full flag by vector subsheaves, the flag can hide a degeneration (2.2); and infinitesimally, the filtration can take on a horizontal flavor (2.3).

2.2. — Let $T = \operatorname{Spec} k[x]$, $E = k[x]^2$, $\mathfrak{g} = k$, and define $r : k[x] \to \mathrm{Mat}_{2\times 2}(k[x])$ by

$$1 \longmapsto \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} .$$

Then $E^0 = \ker \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$, so Fil^r is the standard flag $0 \subset k[x] \subset k[x] \oplus k[x]$. But r_0, the fiber of r above the origin, is the trivial representation.

2.3. — Let $T = \operatorname{Spec} k[\epsilon]/(\epsilon^2)$, $E = k[\epsilon]/(\epsilon^2)$, $\mathfrak{g} = k$, and define $r : k[\epsilon]/(\epsilon^2) \to k[\epsilon]/(\epsilon^2)$ by

$$1 \longmapsto \epsilon .$$

Then Fil^r is given by $0 \subset k\epsilon \subset k \oplus k\epsilon = k[\epsilon]/(\epsilon^2)$.

2.4. — In 2.3, Fil_1 is not locally free; in 2.2, Fil_1 is locally free and co-locally-free, but its formation is not compatible with base change.

We begin our study of flag representations with a discussion of subquotients.

Proposition 2.5. — *Let $r : \mathfrak{g} \to \operatorname{End} E$ be a nilpotent representation of $\mathfrak{g}$ on a vector space E. If r is a flag representation, then every subquotient is of the form r^i_j (1.5) for some $0 \leqslant i \leqslant j$. Moreover, every such subquotient is itself a flag representation.*

The proof follows (2.6–2.8).

Lemma 2.6. — *If $r : \mathfrak{g} \to \mathrm{End}\ E$ is a flag representation and x is an element of E_l not in E_{l-1} then there exists a $v \in \mathfrak{g}$ such that $(rv)x$ is an element of E_{l-1} not in E_{l-2}.*

Proof. — Indeed, every $v \in \mathfrak{g}$ satisfies $(rv)x \in E_{l-1}$ by definition; if, moreover, every $v \in \mathfrak{g}$ satisfies $(rv)x \in E_{l-2}$ then $x \in E_{l-1}$ by definition. □

2.7. — We consider subrepresentations first. Fixing a subrepresentation E', and an element $x \in E'$, it is enough to assume $x \in E_l \setminus E_{l-1}$ and show $E' \supset E_l$. By 2.6, there exists an element $x_i \in (E_i \setminus E_{i-1}) \cap E'$ for all $i \leqslant l$. The sequence of elements $\{x_i\}_i$ then forms a basis for E_l.

2.8. — Since every subrepresentation of a flag representation r is of the form r_j, every quotient representation is of the form r^i. It follows from 1.6 that every subrepresentation is itself a flag representation and hence that every subquotient is of the form r^i_j. Finally, it follows from the same paragraph that each such subquotient is again a flag representation.

We now give a characterization of flag representations in coordinates. We let $\mathfrak{n}_n$ denote the Lie algebra of strictly upper triangular $n \times n$ matrices.

Definition 2.9. — Given a representation $r : \mathfrak{g}_T \to \mathfrak{n}_{n,T}$ of $\mathfrak{g}_T$ on $\mathcal{O}^n_T$, nilpotent with respect to the standard flag, we denote the composite

$$\mathfrak{g}_T \to \mathfrak{n}_{n,T} \to \mathcal{O}_T$$

of r with the $(i,j)^{\text{th}}$ standard projection of $\mathfrak{n}_{n,T}$ by $\lambda^r_{i,j}$ and call it **the full** $(i,j)^{\text{th}}$ **matrix entry of** r. We drop the superscript when there is no danger of confusion. If r is a flag representation and l, m are integers, $0 \leqslant l \leqslant m \leqslant n$, then r^l_m is a representation on $\mathcal{O}^{m-l}_T$, and for $1 \leqslant i < j \leqslant m-l$ we have

$$\lambda^{r^l_m}_{i,j} = \lambda^r_{l+i,l+j} \,.$$

Proposition 2.10. — In the notation of 2.9, a representation of the form $r : \mathfrak{g}_T \to \mathfrak{n}_{n,T}$ is a flag representation if and only if each $\lambda_{i,i+1}$, $i = 1, \ldots, n-1$, is surjective.

The proof follows (2.11–2.13).

2.11. — Let $s : \mathfrak{g} \to \mathfrak{n}_{mT}$ be a representation on $\mathcal{O}^m_T$, nilpotent with respect to the standard flag, suppose $\lambda^s_{i,i+1}$ is surjective for $i = 1, \ldots, m-1$ and let $x = (x_1, \ldots, x_n)$ be a section of the 0-eigenspace of s. I claim that $x_l = 0$ for $l = 2, \ldots, m$. For v a section of $\mathfrak{g}_T$ and $i = 1, \ldots, m$, we have

$$((rv)x)_i = \sum_{i<j\leqslant m} (\lambda^s_{i,j} v) x_j \,. \tag{2.11.1}$$

This family of equations specializes to

$$(\lambda^s_{m-1,m} v) x_m = 0 \tag{2.11.2}$$

when $i = m-1$. Since $\lambda^s_{m-1,m}$ is surjective there exists a section v of $\mathfrak{g}_T$ such that $\lambda^s_{m-1,m} v = 1$; plugging in to 2.11.2 produces $x_m = 0$, which is the base case for a descending induction on l. Suppose $x_j = 0$ for $j = l+1, \ldots, m$. Setting i equal to $l-1$ in 2.11.1 then produces $(\lambda^s_{l-1,l} v) x_l = 0$; plugging in a section v of $\mathfrak{g}$ such that $\lambda^s_{l-1,l} v = 1$ yields $x_l = 0$ and the claim follows. Since the full matrix entries remain surjective after localization (and, indeed, after an arbitrary base change), we conclude that the 0-eigenspace of s is equal to the first step in the standard flag.

2.12. — Returning to the notation of the proposition, suppose $\lambda^r_{i,i+1}$ is surjective for $i = 1, \dots, n-1$. We note first that $\mathrm{Fil}^r_0 \mathcal{O}^n_T = 0$ agrees with step zero of the standard flag $\mathrm{Fil}^{\mathrm{st}}$. Fix a positive integer l and assume for an induction on l that $\mathrm{Fil}^r_l \mathcal{O}^n_T = \mathrm{Fil}^{\mathrm{st}}_l \mathcal{O}^n_T$. Then for $i = 1, \dots, n-l$, $\lambda^{r^l}_{i,i+1}$ is surjective. Hence paragraph 2.11 applies with $s := r^l$ to conclude that $(\mathcal{O}^{n-l}_T)^0 = \mathrm{Fil}^{\mathrm{st}}_1 \mathcal{O}^{n-l}_T$ from which it follows that $\mathrm{Fil}^r_{l+1} \mathcal{O}^n_T = \mathrm{Fil}^{\mathrm{st}}_{l+1} \mathcal{O}^n_T$, as hoped.

2.13. — For the converse, suppose $\lambda_{i,i+1}$ is not surjective, and assume for a contradiction that r is nevertheless a flag representation. After possibly pulling back to a field-valued point $\operatorname{Spec} l \to T$ at which $\lambda_{i,i+1}$ vanishes, we may assume that $T = \operatorname{Spec} l$ is the spectrum of a field and that $\lambda_{i,i+1} = 0$. Since r is nilpotent with respect to the standard flag, Fil^r is subordinate to the standard flag, and since Fil^r is a full flag, it follows that Fil^r is equal to the standard flag. Since r^{i-1} is again a flag representation (2.5), and since $\lambda_{i,i-1} = \lambda^{r^{i-1}}_{1,2}$, after possibly replacing r by r^{i-1} we may assume $\lambda_{1,2} = 0$. It then follows that the 0-eigenspace of r contains step two of the standard flag, a contradiction, as hoped.

Remark 2.14. — We note that every flag representation of $\mathfrak{g}_T$ over T is Zariski locally on T isomorphic to a representation on $\mathcal{O}^n_T$, with associated flag equal to the standard flag.

Definition 2.15. — We denote the s^{th} graded piece $\mathfrak{g}^{(s)}/\mathfrak{g}^{(s+1)}$ of the descending central series of $\mathfrak{g}$ by $\mathfrak{h}_s$. Let $r : \mathfrak{g}_T \to \mathscr{E}nd\, \mathcal{E}$ be a flag representation of $\mathfrak{g}$ on a vector sheaf $\mathcal{E}$ of rank n over T. For $1 \leqslant i \leqslant n$, we set $\mathcal{L}^r_i := \mathrm{gr}^{\mathrm{Fil}^r}_i \mathcal{E}$, and for $1 \leqslant i < j \leqslant n$, we set $\mathcal{L}^r_{i,j} := \mathscr{H}om_{\mathcal{O}_T}(\mathcal{L}^r_j, \mathcal{L}^r_i)$. For each $1 \leqslant i < j \leqslant n$, r defines a map

$$\kappa^r_{i,j} : (\mathfrak{h}_{j-i})_T \to \mathcal{L}^r_{i,j}$$

which we call **the canonical** $(i,j)^{\mathrm{th}}$ **matrix entry of** r. When there is no risk of confusion, we drop the superscript r from the notation.

2.16. — Continuing with the situation and the notation of 2.15, we note that there is a canonical isomorphism

$$\mathcal{L}^{r^l_m}_k = \mathcal{L}^r_{l+k}$$

and a corresponding equality (through the above isomorphism)

$$\kappa^{r^l_m}_{i,j} = \kappa^r_{l+i,l+j}$$

for any $0 \leqslant l \leqslant m \leqslant n$ and $1 \leqslant i < j \leqslant m - l$.

A lengthier discussion of the canonical matrix entries of a flag representation may be found in [4], where they play an important role.

3. Automorphisms of flag representations

We continue to work with a finitely generated Lie algebra $\mathfrak{g}$ over a field k.

3.1. — If $(T, \mathcal{O}_T)$ is a ringed space, $\mathcal{E}$ is an $\mathcal{O}_T$-module, Fil is a filtration by $\mathcal{O}_T$-submodules and ψ is an automorphism of $\mathcal{E}$, we say that ϕ is *unipotent with respect to* Fil if ψ respects Fil and $\mathrm{gr}^{\mathrm{Fil}} \psi = \mathrm{id}_{\mathrm{gr}^{\mathrm{Fil}} \mathcal{E}}$.

Definition 3.2. — Let $\mathcal{E}$ be a vector sheaf over a k-scheme T, and let Fil be a filtration by vector subsheaves. We call **the group of unipotent automorphism of** $(\mathcal{E}, \mathrm{Fil})$, and denote by $\mathbb{U}_{\mathrm{Fil}}(\mathcal{E})$, the T-group of automorphisms of $\mathcal{E}$ which are unipotent with respect to Fil. The group of unipotent automorphisms of $(\mathcal{E}, \mathrm{Fil})$ is a closed subgroup of $\mathbb{A}\mathrm{ut}\, \mathcal{E}$. Its functor of points is defined as follows. Since Fil is locally split, given $f : T' = \operatorname{Spec} A' \to T$, Fil pulls back to a filtration $f^*\mathrm{Fil}$ on $f^*\mathcal{E}$; $\mathbb{U}_{\mathrm{Fil}}(\mathcal{E})(T')$ is the set of automorphisms of $f^*\mathcal{E}$ which respect $f^*\mathrm{Fil}$ and induce the identity on the associated graded.

Definition 3.3. — Let $r : \mathfrak{g}_T \to \mathscr{E}nd\, \mathcal{E}$ be a flag representation of $\mathfrak{g}$ on a vector sheaf $\mathcal{E}$ over T (2.1). We define **the group of unipotent automorphisms of** r, and denote by $\mathbb{U}(r)$, the subgroup of $\mathbb{A}\mathrm{ut}\, r$ consisting of those automorphisms which are unipotent with respect to Fil^r.

Proposition 3.4. — In the situation and the notation of 3.3, we have

$$\mathbb{A}\mathrm{ut}\, r = \mathbb{G}_{m,T} \times \mathbb{U}(r)$$

Proof. — Let n denote the rank of $\mathcal{E}$. We consider a point ϕ' of $\mathbb{A}\mathrm{ut}\, r$ with values in an arbitrary T-scheme T', adding primes to denote pullback to T'. Then for $i = 1, \ldots, n-1$, $\kappa'_{i,i+1}$ (2.15) is surjective. Thus, for any open $U' \subset T'$ and any morphism

$$\mathcal{L}'_{i+1}|_{U'} \to \mathcal{L}'_i|_{U'} ,$$

$\mathrm{gr}\, \phi'$ induces a commuting square as follows.

$$\begin{tikzcd} \mathcal{L}'_i|_{U'} \arrow[r, "(\mathrm{gr}\,\phi')_i|_{U'}"] & \mathcal{L}'_i|_{U'} \\ \mathcal{L}'_{i+1}|_{U'} \arrow[u] \arrow[r, "(\mathrm{gr}\,\phi')_{i+1}|_{U'}"'] & \mathcal{L}'_{i+1}|_{U'} \arrow[u] \end{tikzcd}$$

Applying this in particular to a family of local isomorphisms, it follows that $(\operatorname{gr}\phi')_i = (\operatorname{gr}\phi')_{i+1}$. This gives us a short exact sequence

$$1 \to \mathbb{U}(r) \to \mathbb{A}\mathrm{ut}(r) \to \mathbb{G}_{m,T} \to 1\ .$$

Finally, there is a natural splitting making $\mathbb{G}_{m,T}$ central, whence the product decomposition. □

Definition 3.5. — Let $r : \mathfrak{g}_T \to \mathscr{E}nd\,\mathcal{E}$ be a flag representation. We let $\mathfrak{n}(r)$ denote the Lie o_T-algebra (0.8) whose points valued in an affine T-scheme T' are given by

$$\mathfrak{n}(r)(T') = \{\phi \in \mathfrak{n}_{\mathrm{Fil}_{T'}}(E_{T'}) \mid r_{T'}(v) \circ \phi = \phi \circ r_{T'}(v) \text{ for all } v \in \mathfrak{g}_{T'}\}\ ,$$

that is, the set of endomorphisms of $E_{T'}$ nilpotent with respect to $\mathrm{Fil}_{T'}$ and equivariant with the action of $r_{T'}$. We call $\mathfrak{n}(r)$ **the Lie algebra of nilpotent infinitesimal automorphisms of** r.

Claim 3.6. — In the situation and the notation of 3.3 and 3.5, the functor $\mathfrak{n}(r)$ is the scheme-theoretic Lie algebra of $\mathbb{U}(r)$. (In the notation of [5, II §4 1.2], $\mathfrak{n}(r) = \mathfrak{Lie}\,\mathbb{U}(r)$.)

Proof. — Fix $T' = \operatorname{Spec} B'$, write $T'[\epsilon] = \operatorname{Spec} B'[\epsilon]/[\epsilon^2]$ and write E' for $\Gamma(T', \mathcal{E}_{T'})$. We are to show that $\mathfrak{n}(r)(T')$ is the kernel of the map

$$\alpha : \mathbb{U}(r)(T'[\epsilon]) \to \mathbb{U}(r)(T')$$

induced by the closed immersion $T'[\epsilon] \hookleftarrow T'$. Consider the (split) short exact sequence of abstract groups

$$1 \longrightarrow \mathfrak{n}_{\mathrm{Fil}'}(E') \xrightarrow{e^{\epsilon\cdot}} \mathbb{U}_{\mathrm{Fil}}(\mathcal{E})(T'[\epsilon]) \xrightarrow{\beta} \mathbb{U}_{\mathrm{Fil}}(\mathcal{E})(T') \longrightarrow 1$$

where for $\phi \in \mathfrak{n}_{\mathrm{Fil}'}(E')$, $e^{\epsilon\phi} := 1 + \epsilon\phi$. Since α is just the restriction of β to the set of automorphisms equivariant with the action, it suffices to check that $\phi \in \mathfrak{n}_{\mathrm{Fil}'}(E')$ is equivariant with the action of $\mathfrak{g}_{T'}$ if and only if $e^{\epsilon\phi}$ is equivariant with the action of $\mathfrak{g}_{T'[\epsilon]}$. This is formal: suppose ϕ is equivariant with the action of $\mathfrak{g}_{T'}$, fix an arbitrary $v \in \mathfrak{g}_{T'[\epsilon]}$, write $v = v_0 + \epsilon v_1$ with $v_0, v_1 \in \mathfrak{g}_{T'}$ and compute

$$\begin{aligned}(1+\epsilon\phi)(v_0+\epsilon v_1) &= v_0 + \epsilon(v_1 + \phi v_0)\\ &= v_0 + \epsilon(v_1 + v_0\phi)\\ &= (v_0+\epsilon v_1)(1+\epsilon\phi);\end{aligned}$$

conversely, suppose $e^{\epsilon\phi}$ is equivariant with the action of $\mathfrak{g}_{T'[\epsilon]}$, fix an element $v \in \mathfrak{g}_{T'}$, regard it as an element of $\mathfrak{g}_{T'[\epsilon]}$ and note that

$$\begin{aligned} v + \epsilon v\phi &= v(1+\epsilon\phi) \\ &= (1+\epsilon\phi)v \\ &= v + \epsilon\phi v \end{aligned}$$

implies $v\phi = \phi v$. □

The Lie algebra of nilpotent infinitesimal automorphisms of r is useful because it is, on the one hand, isomorphic to $\mathbb{U}(r)$ (as a scheme) and, on the other hand, occurs as the kernel of a map of vector groups, hence as the contravariant total space of a module, as I now explain (3.7–3.8).

Claim 3.7. — The exponential power series induces an isomorphism of functors $\mathfrak{n}(r) \to \mathbb{U}(r)$.

Proof. — Given a T-scheme $T' = \operatorname{Spec} B'$ as above, the exponential power series defines a map

$$\mathfrak{n}_{\mathrm{Fil}'}(E') \xrightarrow{\ \exp\ } \mathbb{U}_{\mathrm{Fil}}(\mathcal{E})(T')$$

given by

$$v \longmapsto 1 + v + \tfrac{v^2}{2} + \tfrac{v^3}{3!} + \cdots$$

while the logarithmic power series defines a map

$$\mathfrak{n}_{\mathrm{Fil}'}(E') \longleftarrow \mathbb{U}_{\mathrm{Fil}}(\mathcal{E})(T')$$

given by

$$(u-1) - \tfrac{(u-1)^2}{2} + \tfrac{(u-1)^3}{3} - \cdots \longleftarrow\!\shortmid\; u\,.$$

These are inverse to one another. So it suffices to check that for $v \in \mathfrak{n}_{\mathrm{Fil}'}(E')$, v is equivariant with the action of $\mathfrak{g}_{T'}$ if and only if $\exp v$ is: if $v \in \mathfrak{n}_{\mathrm{Fil}'}(E')$ is equivariant with the action of $\mathfrak{g}_{T'}$ and $w \in \mathfrak{g}_{T'}$ is an arbitrary element then $r_{T'}w$ commutes with the terms of the exponential power series in v and hence with their sum, $\exp v$; conversely, if $\exp v \in \mathbb{U}_{\mathrm{Fil}}(\mathcal{E})(T')$ is equivariant with the action of $\mathfrak{g}_{T'}$ then $r_{T'}w$ commutes with the terms of the logarithmic power series in $\exp v$ hence with their sum, which equals v. □

3.8. — Fix a basis $v_1, \dots, v_m$ for $\mathfrak{g}$, and continuing with our flag representation $r : \mathfrak{g}_T \to \mathscr{E}nd(\mathcal{E})$, define

$$\Psi : \mathfrak{n}_{\mathrm{Fil}}\mathcal{E} \to (\mathfrak{n}_{\mathrm{Fil}}\mathcal{E})^{\oplus m}$$

by
$$\phi \mapsto ([\phi, rv_1], \dots, [\phi, rv_m]) .$$
Then $\mathfrak{n}(r) = \ker \mathbb{V}\Psi^\vee$ (0.1).

Proposition 3.9. — Suppose $r : \mathfrak{g} \to \operatorname{End} E$ is a flag representation of $\mathfrak{g}$ on a vector space E of dimension n. Then
$$2 \leqslant \dim \mathbb{A}\mathrm{ut}\, r \leqslant n .$$

Only the lower bound is needed in the sequel.

Proof. — Fixing a filtered isomorphism $E \cong k^n$, we may replace r with a homomorphism of Lie algebras $r : \mathfrak{g} \to \mathfrak{n}_n$. In the notation of 2.9,
$$\mathbb{U}(r) = \left\{ b \in \mathbb{U}_n \;\middle|\; \sum_{i<k<j} b_{ik}\lambda_{kj} - b_{kj}\lambda_{ik} = 0,\ j - i \geqslant 2 \right\} .$$
For each i, fix $v_i \in \mathfrak{g}$ such that $\lambda_{i,i+1} v_i = 1$. Then by applying the above equation to v_i we solve for $b_{i+1,j}$ in terms of entries of the form $b_{i'+1,j'}$ with either $j' - i' < j - i$ or both of: $j' - i' = j - i$ and $i' < i$. Iterating (and renaming), we can solve for each $b_{i,i+s}$ in terms of entries of the form $b_{1,1+s'}$ with $s' \leqslant s$. This provides the upper bound.

For the lower bound, note that $\mathbb{U}_n$ contains a copy of $\mathbb{G}_a$ whose entries commute with all strictly upper triangular matrices (observe simply that b_{1n} does not intervene in the above equations). Thus
$$\mathbb{A}\mathrm{ut}\, r \supset \mathbb{G}_m \times \mathbb{G}_a .$$
□

Remark 3.10. — Finally, we note that if T is connected and $r : \mathfrak{g}_T \to \mathscr{E}nd\, \mathcal{E}$ is any representation on a vector sheaf, then $\mathbb{A}\mathrm{ut}\, r$ is connected. Indeed, $\mathbb{E}\mathrm{nd}\, r$ is the total space of a module, hence connected with irreducible fibers; and $\mathbb{A}\mathrm{ut}\, r$ is an open subscheme containing a global section.

4. Recursive minimization of automorphism groups

We continue to work with a finitely generated Lie algebra $\mathfrak{g}$ over a field k.

Definition 4.1. — Let $T = \operatorname{Spec} A$ be an affine k-scheme, E an A-module with corresponding $\mathcal{O}_T$-module $\mathcal{E}$, and $r : \mathfrak{g}_T \to \mathscr{E}nd(\mathcal{E})$ a flag representation (2.1). Suppose first that A is a field. Then r is **nondegenerate** if it satisfies the following conditions, recursive on the dimension of E. If $\dim E = 1$, then $r = 0$ is the trivial representation. If, on the other hand, the rank of E is $n \geqslant 2$ then

(i) r_{n-1}, r^1 (1.5) are both nondegenerate, and
(ii) $\dim \mathbb{A}\mathrm{ut}\, r$ is *minimal subject to condition (i)* in the following sense. Let A' be a field containing A, and let r' be a flag representation of $\mathfrak{g}_{A'}$. If r'_{n-1} and r'^1 are both nondegenerate, then

$$\dim \mathbb{A}\mathrm{ut}\, r \leqslant \dim \mathbb{A}\mathrm{ut}\, r' .$$

Now let A be arbitrary. Then r is **nondegenerate** if it satisfies the following conditions, recursive on the rank of $\mathcal{E}$:

(i) r_{n-1}, r^1 are both nondegenerate,
(ii) the fibers of r above field-valued points of T are nondegenerate in the above sense, and
(iii) $\mathbb{A}\mathrm{ut}\, r$ is flat.

Notation 4.2. — We denote the dimension of the automorphism group of some (hence any) nondegenerate n-dimensional representation of $\mathfrak{g}$ by $A(\mathfrak{g}, n)$.

Examples and counterexamples in low dimensions (4.3–4.11)

We consider representations of our fixed Lie algebra $\mathfrak{g}$ on k^n for small n, nilpotent with respect to the standard flag. We begin with the case $n = 2$.

4.3. — Since $\mathfrak{n}_2 = k$ is abelian, every representation $r : \mathfrak{g} \to \mathfrak{n}_2$ factors through the abelianization $\mathfrak{g}^{\mathrm{ab}}$ of $\mathfrak{g}$. Conversely, any linear map $\mathfrak{g} \to \mathfrak{n}_2$ which factors through $\mathfrak{g}^{\mathrm{ab}}$ is a homomorphism of Lie algebras and hence a representation. Thus representations on k^2, nilpotent with respect to the standard flag, correspond canonically to linear functionals $\lambda : \mathfrak{g}^{\mathrm{ab}} \to k$ on the abelianization of $\mathfrak{g}$.

The trivial representation is clearly not a flag representation; conversely, every nontrivial representation is a flag representation. A calculation shows that for r nonzero,

$$\mathbb{A}\mathrm{ut}\, r = \left\{ \begin{pmatrix} b & c \\ 0 & b \end{pmatrix} \right\} = \mathbb{G}_m \times \mathbb{G}_a .$$

In particular, $\dim \mathbb{A}\mathrm{ut}\, r = 2$. Thus every flag representation is nondegenerate and $A(\mathfrak{g}, 2) = 2$ independently of $\mathfrak{g}$.

We now consider three dimensional representations (4.4–4.6).

4.4. — Fix a three dimensional nilpotent representation $r : \mathfrak{g} \to \mathfrak{n}_3$. We use single indices in order to simplify notation: for $v \in \mathfrak{g}$ write

$$rv = \begin{pmatrix} 0 & \lambda_1 v & \lambda_3 v \\ 0 & 0 & \lambda_2 v \\ 0 & 0 & 0 \end{pmatrix}.$$

For each i, λ_i is the composite of r regarded as a map $\mathfrak{g} \to \mathfrak{n}_3$ with one of the three standard projections $\mathfrak{n}_3 \to k$. Thus for each i, λ_i is a linear functional on $\mathfrak{g}$. The equation $r[\cdot,\cdot] = [r\cdot, r\cdot]$ may be rewritten in terms of these linear functionals:

$$\lambda_1[\cdot,\cdot] = 0 \tag{4.4.1}$$

$$\lambda_2[\cdot,\cdot] = 0 \tag{4.4.2}$$

$$\lambda_3[\cdot,\cdot] = \lambda_1 \wedge \lambda_2 . \tag{4.4.3}$$

According to 2.10, r is a flag representation if and only if λ_1, λ_2 are nonzero. For r a flag representation, the automorphism group is given by

$$\mathbb{A}\mathrm{ut}\, r = \left\{ \begin{pmatrix} a & b_1 & c \\ 0 & a & b_2 \\ 0 & 0 & a \end{pmatrix} \,\middle|\, b_1\lambda_2 = b_2\lambda_1 \text{ and } a \neq 0 \right\}. \tag{4.4.4}$$

Thus

$$A(\mathfrak{g},3) = \begin{cases} 3 & \text{if equations 4.4.1–4.4.3 imply } \lambda_1 \wedge \lambda_2 = 0 \\ 2 & \text{otherwise.} \end{cases} \tag{4.4.5}$$

4.5. — Continuing with the notation of 4.4, consider, for a first example in three dimensions, the case $\mathfrak{g} = k^2$. Then the system of equations 4.4.1–4.4.3 becomes simply

$$\lambda_1 \wedge \lambda_2 = 0 . \tag{4.5.1}$$

Hence every flag representation is nondegenerate and we have

$$A(k^2, 3) = 3 .$$

4.6. — Again in the notation of 4.4, consider, for a second example in three dimensions, the case $\mathfrak{g} = \mathfrak{n}_3$. Then flag representations with λ_1, λ_2 linearly independent exist: the natural representation provides an example. So nondegenerate representations are precisely those for which λ_1, λ_2 are linearly independent and $A(\mathfrak{n}_3, 3) = 2$.

We now consider four dimensional representations (4.7–4.10).

4.7. — Given a representation $r : \mathfrak{g} \to \mathfrak{n}_4$, write

$$rv = \begin{pmatrix} 0 & \lambda_1 v & \lambda_4 v & \lambda_6 v \\ 0 & 0 & \lambda_2 v & \lambda_5 v \\ 0 & 0 & 0 & \lambda_3 v \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with $\lambda_i \in \mathfrak{g}^\vee$. In terms of these linear functionals, the equation $r[\cdot,\cdot] = [r\cdot, r\cdot]$ becomes:

$$\lambda_1[\cdot,\cdot] = \lambda_2[\cdot,\cdot] = \lambda_3[\cdot,\cdot] = 0 \tag{4.7.1}$$

and

$$\lambda_4[\cdot,\cdot] = \lambda_1 \wedge \lambda_2 \tag{4.7.2}$$

$$\lambda_5[\cdot,\cdot] = \lambda_2 \wedge \lambda_3 \tag{4.7.3}$$

$$\lambda_6[\cdot,\cdot] = \lambda_1 \wedge \lambda_5 - \lambda_3 \wedge \lambda_4 \,. \tag{4.7.4}$$

The representation r is a flag representation if and only if $\lambda_1, \lambda_2, \lambda_3 \neq 0$ (2.10). For r a flag representation, $\mathbb{A}\mathrm{ut}\, r$ is the set of matrices of the form

$$\begin{pmatrix} a & b_1 & b_4 & b_6 \\ 0 & a & b_2 & b_5 \\ 0 & 0 & a & b_3 \\ 0 & 0 & 0 & a \end{pmatrix}$$

subject to the equations

$$b_1\lambda_2 = b_2\lambda_1 \tag{4.7.5}$$

$$b_2\lambda_3 = b_3\lambda_2 \tag{4.7.6}$$

$$b_1\lambda_5 + b_4\lambda_3 = b_5\lambda_1 + b_3\lambda_4 \,. \tag{4.7.7}$$

The possible pairs of numbers $(A(\mathfrak{g},3), A(\mathfrak{g},4))$ are $(2,2), (2,3)$ and $(3,4)$. Paragraphs 4.8–4.9 explain why this is and paragraph 4.10 gives examples of each of these cases.

4.8. — Suppose $A(\mathfrak{g},3) = 3$, fix a representation $r : \mathfrak{g} \to \mathfrak{n}_4$ and suppose r is nondegenerate. Then according to 4.4.5, in the notation of 4.7, we have

$$\lambda_1, \lambda_2, \lambda_3 \neq 0 \tag{4.8.1}$$

$$\lambda_4[\cdot,\cdot] = \lambda_1 \wedge \lambda_2 = 0 \tag{4.8.2}$$

$$\lambda_5[\cdot,\cdot] = \lambda_2 \wedge \lambda_3 = 0 \,. \tag{4.8.3}$$

Fix elements $a, a' \in k$ such that

$$a\lambda_1 = \lambda_2 \tag{4.8.4}$$

$$a'\lambda_1 = \lambda_3 \,. \tag{4.8.5}$$

Combining 4.7.1 and 4.7.4 with 4.8.2–4.8.5 we get the following three equations:

$$\lambda_1[\cdot,\cdot] = 0 \tag{4.8.6}$$
$$(\lambda_5 - a'\lambda_4)[\cdot,\cdot] = 0 \tag{4.8.7}$$
$$\lambda_1 \wedge (\lambda_5 - a'\lambda_4) = \lambda_6[\cdot,\cdot]\,. \tag{4.8.8}$$

So by 4.4.5 applied with $\lambda_5 - a'\lambda_4$ in place of λ_2 and λ_6 in place of λ_3, we have

$$\lambda_1 \wedge (\lambda_5 - a'\lambda_4) = 0\,. \tag{4.8.9}$$

There is thus an element $a'' \in k$ such that

$$\lambda_5 = a''\lambda_1 + a'\lambda_4\,. \tag{4.8.10}$$

In terms of a, a', a'' the system of equations 4.7.5–4.7.7 is equivalent to

$$ab_1 = b_2 \tag{4.8.11}$$
$$a'b_1 = b_3 \tag{4.8.12}$$
$$a''b_1 + a'b_4 = b_5\,. \tag{4.8.13}$$

This shows that $\dim \mathbb{A}\mathrm{ut}\, r = 4$ and hence that $A(\mathfrak{g}, 4) = 4$.

4.9. — Now suppose instead that $A(\mathfrak{g}, 3) \neq 3$. Fix a nilpotent representation $r : \mathfrak{g} \to \mathfrak{n}_4$, denote its full matrix entries by $\lambda_1, \dots, \lambda_6$ as in 4.7 and suppose r is nondegenerate. Then according to 4.4.5, the pairs $(\lambda_1, \lambda_2), (\lambda_2, \lambda_3)$ are both linearly independent, so the system of equations 4.7.5–4.7.7 becomes

$$b_1 = b_2 = b_3 = 0 \tag{4.9.1}$$
$$b_4\lambda_3 = b_5\lambda_1\,. \tag{4.9.2}$$

If λ_1, λ_3 are linearly dependent then (4.9.2) becomes

$$b_5 = a'b_4 \tag{4.9.3}$$

and $A(\mathfrak{g}, 4) = 3$. If, on the other hand, λ_1, λ_3 are linearly independent, then the system of equations 4.9.1–4.9.2 becomes

$$b_1 = \cdots = b_5 = 0 \tag{4.9.4}$$

and $A(\mathfrak{g}, 4) = 2$.

4.10. — Here, then, are the promised examples. The 4×4 upper triangular Lie algebra $\mathfrak{n}_4$ is of type $(2,2)$, as witnessed by the natural representation and its subquotients. The m-dimensional abelian Lie algebra k^m is of type $(3,4)$: indeed, the system 4.7.1–4.7.4 becomes precisely

$$\lambda_1 \wedge \lambda_2 = 0 \tag{4.10.1}$$

$$\lambda_2 \wedge \lambda_3 = 0 \tag{4.10.2}$$

and, in terms of a, a' such that $a\lambda_1 = \lambda_2$ and $a'\lambda_1 = \lambda_3$,

$$\lambda_1 \wedge (\lambda_5 - a'\lambda_4) = 0\,. \tag{4.10.3}$$

Finally, $\mathfrak{n}_3$ is of type $(2,3)$; this is not hard to check directly but is better understood as a consequence of the fact that the width is bounded by the depth; see [4].

4.11. — Finally, we give an example of a flag representation which has minimal automorphism group, and a degenerate subquotient. Let F be a vector space of dimension 2 and let $\mathfrak{g} = \mathfrak{n}(F)$ (1.1). Then

$$\mathrm{Hom}_{\mathbf{Lie}}(\mathfrak{g}, \mathfrak{n}_4) = \mathrm{Hom}_{\mathbf{Vect}}(F, \mathfrak{n}_4) = F^{\vee 6}\,.$$

Denoting the full coordinates of r by $\lambda_1, \dots, \lambda_6$ as in 4.7, suppose λ_1, λ_2 are linearly independent and λ_2, λ_3 are linearly dependent but nonzero. Then $\dim \mathbb{A}\mathrm{ut}\, r^1 = 3$ is not minimal; but λ_1, λ_3 are linearly independent and equations 4.7.5–4.7.7 show that $\dim \mathbb{A}\mathrm{ut}\, r = 2$ is minimal. This example shows that the recursive aspect of the definition of nondegeneracy is not redundant.

5. Parameter spaces

We continue to work with a finitely generated Lie algebra $\mathfrak{g}$ over a field k.

5.1. — Fix an affine k-scheme $T = \mathrm{Spec}\, A$, a vector sheaf $\mathcal{E}$ with module of global sections E, and a full flag by vector subsheaves Fil. We denote by $\mathbb{H}\mathrm{om}_{\mathbf{Mod}(o_T)}(\mathfrak{g}_T, \mathfrak{n}_{\mathrm{Fil}}(\mathcal{E}))$ (or by $\mathbb{H}\mathrm{om}_{\mathbf{Vect}(k)}(\mathfrak{g}, \mathfrak{n}_{\mathrm{Fil}}(\mathcal{E}))$ when $T = \mathrm{Spec}\, k$) the functor $\mathbf{Aff}(A) \to \mathbf{Set}$ sending

$$(f : T' \to T) \mapsto \mathrm{Hom}_{\mathbf{Mod}(\mathcal{O}_{T'})}(f^*\mathfrak{g}, f^*\mathfrak{n}_{\mathrm{Fil}}(\mathcal{E}))$$

and we denote by

$$\mathbb{H}\mathrm{om}_{\mathbf{Lie}(o_T)}(\mathfrak{g}_T, \mathfrak{n}_{\mathrm{Fil}}(\mathcal{E})) = X_{(A,E,\mathrm{Fil})} \supset X^{\mathrm{fl}}_{(A,E,\mathrm{Fil})} \supset X^{\mathrm{nd}}_{(A,E,\mathrm{Fil})}$$

the successively smaller subfunctors whose points are representations, flag representations (2.1), and nondegenerate representations (4.1), respectively.

When $A = k, E = k^n$, and Fil is the standard flag, we write simply $X_n \supset X_n^{\mathrm{fl}} \supset X_n^{\mathrm{nd}}$.

We begin by studying the functors $X_{(A,E,\mathrm{Fil})}, X^{\mathrm{fl}}_{(A,E,\mathrm{Fil})}$ and $X^{\mathrm{nd}}_{(A,E,\mathrm{Fil})}$ (5.2–5.18).

Proposition 5.2. — In the situation and the notation of 5.1

(1) the inclusion $X_{(A,E,\mathrm{Fil})} \hookrightarrow \mathbb{H}\mathrm{om}_{\mathbf{Mod}(o_T)}(\mathfrak{g}_T, \mathfrak{n}_{\mathrm{Fil}}(\mathcal{E}))$ is a closed immersion;

(2) the inclusion $X^{\mathrm{fl}}_{(A,E,\mathrm{Fil})} \hookrightarrow X_{(A,E,\mathrm{Fil})}$ is an open immersion.

Proof. — (1) Preservation of bracket is a closed condition defined by equations depending on the choice of a basis for $\mathfrak{g}$. (2) Let

$$r_{(A,E,\mathrm{Fil})} : \mathfrak{g}_{X_{(A,E,\mathrm{Fil})}} \to \mathfrak{n}_{\mathrm{Fil}}(\mathcal{E}_{X_{(A,E,\mathrm{Fil})}})$$

be the universal family and suppose $\mathcal{E}$ has rank n. Then $X^{\mathrm{fl}}_{(A,E,\mathrm{Fil})}$ is the open locus defined by the nonvanishing of each $\kappa_{i,i+1}^{r_{(A,E,\mathrm{Fil})}}$ (2.15), $i = 1, \dots, n-1$. □

Corollary 5.3. — The functor $X^{\mathrm{fl}}_{(A,E,\mathrm{Fil})}$ is representable by a quasi-projective scheme; in particular, it is locally of finite presentation.

Proposition 5.4. — Let A be a Noetherian k-algebra, (E, Fil) a vector sheaf of rank n filtered by a full flag of vector subsheavs, let B be an A-algebra and let $\mathscr{B}$ denote the directed system of finite type A-subalgebras of B. Then the map

$$\varinjlim_{B' \in \mathscr{B}} X^{\mathrm{nd}}_{(A,E,\mathrm{Fil})}(B') \to X^{\mathrm{nd}}_{(A,E,\mathrm{Fil})}(B)$$

is an isomorphism.

This proposition will be used in a special case in paragraph 5.16 below, and in full generality only in [4, Proposition 7.1]. The proof follows (5.5–5.9).

5.5. — For injectivity, given representations $r' : \mathfrak{g}_{B'} \to \mathfrak{n}_{\mathrm{Fil}}(\mathcal{E}_{B'})$, $r'' : \mathfrak{g}_{B''} \to \mathfrak{n}_{\mathrm{Fil}}(\mathcal{E}_{B''})$ such that

$$\mathrm{id}_B \otimes_{B'} r' = \mathrm{id}_B \otimes_{B''} r'' ,$$

let B''' be the subalgebra generated by B' and B''; then

$$\mathrm{id}_{B'''} \otimes_{B'} r' = \mathrm{id}_{B'''} \otimes_{B''} r'' .$$

We turn to surjectivity. Let $r : \mathfrak{g}_B \to B \otimes \mathfrak{n}_{\mathrm{Fil}} \mathcal{E} = \mathfrak{n}_{\mathrm{Fil}_B} \mathcal{E}_B$ be a nondegenerate nilpotent representation. Then by 5.3 there exists a $B' \in \mathscr{B}$ and a flag representation $r' : \mathfrak{g}_{B'} \to \mathfrak{n}_{\mathrm{Fil}_{B'}}\mathcal{E}_{B'}$ such that $r = \mathrm{id}_B \otimes_{B'} r'$. Assume for an induction on n that after possibly replacing B' by a finite type subalgebra of B containing B', r'_{n-1}, r'^1 are nondegenerate. Fix a basis $v_1, \dots, v_m$ for $\mathfrak{g}$ and define

$$\Psi : \mathfrak{n}_{\mathrm{Fil}_B}\mathcal{E}_B \to (\mathfrak{n}_{\mathrm{Fil}_B}\mathcal{E}_B)^{\oplus m}$$

by

$$\phi \mapsto ([\phi, rv_1], \dots, [\phi, rv_m])$$

and define Ψ' similarly for r' so that writing

$$\begin{tikzcd}
0 & Q \arrow[l] \arrow[r, bend left, "\sigma"] & (\mathfrak{n}_{\mathrm{Fil}_B}\mathcal{E}_B)^\vee \arrow[l, "\chi"] & (\mathfrak{n}_{\mathrm{Fil}_B}\mathcal{E}_B)^{\oplus m\vee} \arrow[l, "\Psi^\vee"'] \\
0 & Q' \arrow[l] \arrow[u, "\alpha"] & (\mathfrak{n}_{\mathrm{Fil}_{B'}}\mathcal{E}_{B'})^\vee \arrow[l, "\chi'"] \arrow[u] & (\mathfrak{n}_{\mathrm{Fil}_{B'}}\mathcal{E}_{B'})^{\oplus m\vee} \arrow[l, "\Psi'^\vee"] \arrow[u]
\end{tikzcd}$$

we have $n(r) = \mathbb{V}Q$, $n(r') = \mathbb{V}Q'$ (3.8). The B-module Q is flat and of finite presentation, hence projective, so that χ splits; fix a splitting σ as in the diagram. Our goal is to show that after possibly replacing B' by a finite type B'-subalgebra of B, χ' splits.

Lemma 5.6. — Disengaging briefly from the notation of the proposition, let B' be a Noetherian ring, let B be a B'-algebra, let N' be a finite B'-module, and consider an element $n' \in N'$. If $1_B \otimes_{B'} n' = 0$, then there exists a finite type subalgebra $B'' \subset B$ such that $1_{B''} \otimes_{B'} n' = 0$.

Proof. — Fix a finite family $\{n'_i\}$ of generators for N', and write

$$1_B \otimes_{B'} n' = \sum_i b_i \otimes_{B'} n'_i$$

Then by [7, Lemma 6.4], there are elements $a_{i,j} \in B'$ and $c_j \in B$ such that

$$\sum_j a_{ij} c_j = b_i \qquad \text{for all } i \tag{5.6.1}$$

and

$$\sum_i a_{ij} n'_i = 0 \qquad \text{for all } j\,. \tag{5.6.2}$$

Let B'' be the subalgebra generated over B' by the (finitely many) c_j. Then again by [7, Lemma 6.4], $1_{B''} \otimes_{B'} n' = 0$ as claimed. $\square$

Lemma 5.7. — In the notation of 5.6, let Q' be a finitely presented B'-module, and let σ be a morphism

$$Q := B \otimes_{B'} Q' \to N := B \otimes_{B'} N'\,.$$

Then after possibly replacing B' by a finite type subalgebra of B there exists a morphism

$$\sigma' : Q' \to N'$$

such that $\sigma = \mathrm{id}_B \otimes_{B'} \sigma'$.

Proof. — Fix a finite presentation

$$F_1' \to F_0' \to Q' \to 0\,,$$

and drop the primes to denote base-change to B:

$$\begin{tikzcd}
0 & & 0 & \\
Q' \arrow[u] \arrow[rr] & & Q \arrow[u] \arrow[dr] & \\
& N' \arrow[rr] & & N \\
F_0' \arrow[u] \arrow[rr] \arrow[ur, dotted] & & F_0 \arrow[uu] & \\
F_1' \arrow[u] \arrow[rr] & & F_1 \arrow[u] &
\end{tikzcd}$$

$$\begin{tikzcd}
B' \arrow[r] & B
\end{tikzcd}$$

Since F_0' is free and finite, after possibly replacing B' by a finite type subalgebra of B, there is a map $\beta' : F_0' \to N'$ commuting with $\sigma\epsilon$ as in the diagram. Now

$$(F_1' \to F_0' \to N' \to N) = 0\,,$$

so by 5.6, after possibly replacing B' by a finite type subalgebra of B,

$$(F_1', \to F_0' \to N') = 0\,.$$

Subsequently, β' factors through Q' to produce the desired morphism. $\square$

5.8. — Returning to the situation of the proposition, after possibly replacing B' by a finite type subalgebra of B containing B', we obtain a candidate $\sigma' : Q' \to (\mathfrak{n}_{\mathrm{Fil}_{B'}}\mathcal{E}_{B'})^\vee$ for our desired splitting. Now since

$$(Q' \to (\mathfrak{n}_{\mathrm{Fil}_{B'}}\mathcal{E}_{B'})^\vee \to Q' \to Q) - (q' \mapsto 1_B \otimes q') = 0$$

and since Q' is of finite type, by 5.6, after possibly replacing B' by a finite type subalgebra of B containing B', we have $\chi'\sigma' = \mathrm{id}_{Q'}$, giving us our desired splitting.

5.9. — Finally, we have that $\mathbb{A}\mathrm{ut}\, r'$ is flat, and also that $\dim(\mathbb{A}\mathrm{ut}\, r'(t)) = \dim Q'(t) + 1$ is locally constant on $T' = \operatorname{Spec} B'$. Since the image of $T' \leftarrow T$ is dense, it follows that $\dim(\mathbb{A}\mathrm{ut}\, r'(t)) = A(\mathfrak{g}, n)$ for all $t \in T'$ which completes the proof.

Proposition 5.10. — The inclusion $X_n^{\mathrm{nd}} \hookrightarrow X_n^{\mathrm{fl}}$ is an immersion. In particular, X_n^{nd} is representable by a quasi-projective scheme.

The proof follows in paragraphs 5.12–5.16. We begin by recalling the theory of Fitting ideals.

5.11. — Let A be a Noetherian ring, F a finite module, and

$$A^r \xrightarrow{\phi} A^s \to F \to 0$$

a free presentation. Fix an integer $0 \leqslant i \leqslant r$. Then by [7, Corollary-Definition 20.4], the ideal $\mathrm{Fitt}_i(F)$ generated by determinants of all $(r-i)\times(r-i)$ minors of ϕ is independent of the choice of presentation. Speaking geometrically, when $X = \operatorname{Spec} A$, we call

$$X_i := Z(\mathrm{Fitt}_i(F)) \setminus Z(\mathrm{Fitt}_{i+1}(F))$$

the i^{th} *Fitting locus of* F. According to Corollary 20.5 of [7], formation of X_i is compatible with base change. Moreover, when A is a field, $\dim F = i$ if and only if $X_i = X$.

Proposition 5.12. — Let X be a Noetherian scheme and $\mathcal{F}$ a coherent sheaf. Then X admits a flattening stratification for $\mathcal{F}$. We recall that this means that there exists a stratification

$$s : \coprod_i X_i \to X$$

of X by immersed subschemes such that given a morphism

$$g : T = \operatorname{Spec} B \to X\,,$$

(i) if B is a field, then g factors through X_i if and only if

$$\dim g^*\mathcal{F} = i\,,$$

and

(ii) if B is Noetherian, then g factors through s if and only if $s^*\mathcal{F}$ is flat.

Proof. — Since formation of the hypothetical stratification is compatible with base change, we may assume $X = \operatorname{Spec} A$ to be affine. Write $F := \Gamma(X, \mathcal{F})$ for the associated A-module, and let X_i be the i^{th} Fitting locus of F (5.11). Fix a $T = \operatorname{Spec} B$-valued point g as in the theorem, and let T_i denote the i^{th} Fitting locus of $B \otimes_k F$.

(i) Suppose B is a field. By 5.11, g factors through X_i if and only if $T_i = T$ if and only if $\dim B \otimes_k F = i$.

(ii) Now suppose only that T is Noetherian. We may assume T is connected. Let s_i denote the immersion

$$X_i \hookrightarrow X\,.$$

Suppose that g factors through s. Then g factors through s_i for some i. Then by compatibility with base-change, $T_i = T$, so by [7, Proposition 20.8], $B \otimes F$ is projective, hence flat. Conversely, suppose $B \otimes F$ is flat. Since $B \otimes F$ is finitely presented, $B \otimes F$ is also projective; and since $\operatorname{Spec} B$ is connected, $B \otimes F$ has constant rank, say i. Then by loc. cit., $T_i = T$, from which it follows that g factors through s_i, hence through s, which concludes the proof of (ii). $\square$

5.13. — Assume for an induction on n that for $i \leqslant n$, the inclusion $\iota : X_i^{\text{nd}} \hookrightarrow X_i^{\text{fl}}$ is an immersion. Let $p_1 : X_i^{\text{fl}} \to X_{i-1}^{\text{fl}}$ denote the map given (on points valued in an arbitrary k-scheme) by $r \mapsto r_{i-1}$ (1.5), and let p_2 denote the map given by $r \mapsto r^1$ (loc. cit.). Assume moreover that the two composites $p_1\iota$, $p_2\iota$ in the following diagram

$$\begin{tikzcd}
X_i^{\text{nd}} \arrow[r] \arrow[d, dotted, shift right] \arrow[d, dotted, shift left] & X_i^{\text{fl}} \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] \\
X_{i-1}^{\text{nd}} \arrow[r] & X_{i-1}^{\text{fl}}
\end{tikzcd}$$

factor through X_{i-1}^{nd} as shown. Define X'_{n+1} by the Cartesian square

$$\begin{tikzcd}
X'_{n+1} \arrow[r] \arrow[d] & X_{n+1}^{\text{fl}} \arrow[d] \\
X_n^{\text{nd}} \times_{X_{n-1}^{\text{nd}}} X_n^{\text{nd}} \arrow[r] & X_n^{\text{fl}} \times_{X_{n-1}^{\text{fl}}} X_n^{\text{fl}}
\end{tikzcd}$$

and let $r'_{n+1} : \mathfrak{g}_{X'} \to \mathfrak{n}_{n+1,X'}$ be its universal family.

5.14. — Write $\mathfrak{n}(r'_{n+1}) = \mathbb{V}Q$, combining 3.8 with 0.1 as above. The Fitting ideals of Q define a flattening stratification $X'_{n+1} = \cup (X'_{n+1})_i$ of $\mathbb{A}\mathrm{ut}\, r'_{n+1}$. Each $(X'_{n+1})_i \subset X'_{n+1}$ is an immersed suscheme; if $T = \operatorname{Spec} l$ is a field then $g : T \to X'_{n+1}$ lands in $(X'_{n+1})_i$ if and only if $\dim g^* \mathbb{A}\mathrm{ut}\, r'_{n+1} = i$; and if T is Noetherian, then $g : T \to X'_{n+1}$ factors through $\coprod (X'_{n+1})_i$ if and only if $g^* \mathbb{A}\mathrm{ut}\, r'_{n+1}$ is flat. We claim that $X^{\mathrm{nd}}_{n+1} = (X'_{n+1})_{A(\mathfrak{g},n+1)}$.

5.15. — Suppose first that $T = \operatorname{Spec} B$ is an affine Noetherian k-scheme, suppose $g : T \to X^{\mathrm{fl}}_{n+1}$ factors through $(X'_{n+1})_{A(\mathfrak{g},n+1)}$ and let r be the corresponding representation over T. Then it is clear that r_n, r^1 are both nondegenerate. Thus if $t \in T$ then $r(t)_n$ and $r(t)^1$ are nondegenerate, and $\dim \mathbb{A}\mathrm{ut}\, r(t) = A(\mathfrak{g}, n+1)$. Hence $r(t)$ is nondegenerate. Finally, it is clear that $\mathbb{A}\mathrm{ut}\, r$ is flat.

Conversely, suppose $r : \mathfrak{g}_T \to \mathfrak{n}_{n+1,T}$ is nondegenerate (T Noetherian) and let $g : T \to X^{\mathrm{fl}}_{n+1}$ be the corresponding map. It is clear that g factors through X'_{n+1}. Flatness of $\mathbb{A}\mathrm{ut}\, r$ implies that g factors through $\coprod_i (X'_{n+1})_i$. Finally, the fiberwise condition implies that set-theoretically g factors though $(X'_{n+1})_{A(\mathfrak{g},n+1)}$, from which it follows that the previous (scheme-theoretic) factorization was actually a factorization through $(X'_{n+1})_{A(\mathfrak{g},n+1)}$.

5.16. — If $T = \operatorname{Spec} B$ is not Noetherian, let $\mathscr{B}$ be the system of finite type subalgebras. Then it follows from 5.4, from 5.15, and finally from the fact that $(X'_{n+1})_{A(\mathfrak{g},n+1)}$ is finite type over a field, hence in particular locally of finite presentation, that

$$\begin{aligned} X^{\mathrm{nd}}_{n+1}(B) &= \varinjlim_{B' \in \mathscr{B}} X^{\mathrm{nd}}_{n+1}(B') \\ &= \varinjlim_{B' \in \mathscr{B}} (X'_{n+1})_{A(\mathfrak{g},n+1)}(B') \\ &= (X'_{n+1})_{A(\mathfrak{g},n+1)}(B) \end{aligned}$$

([9, Ch. 3, Prop. 8.14.2.1]). This completes the proof of 5.10.

Remark 5.17. — It follows from the construction that although $X^{\mathrm{nd}} \hookrightarrow X^{\mathrm{fl}}$ may not be an open immersion, it is close to an open immersion in the sense that it factors as a surjective closed immersion followed by an open immersion.

Remark 5.18. — Although Proposition 5.4 falls short of stating that X^{nd}_n is locally of finite presentation, it follows from the construction of 5.10 that it is locally of finite presentation after all.

6. Review of rigidification

6.1. — Let $\mathcal{C}$ be a site, and $f : \mathcal{X} \to \mathcal{C}$ a stack in groupoids. The *inertia stack* $\mathcal{I}$ may be defined as follows. The object class is the class of all pairs

(x,α), x an object of $\mathcal{X}$ and α an automorphism of x such that $f(\alpha)$ is the identity automorphism of $f(x)$. Given objects (x,α), (y,β), the set of morphisms $(x,\alpha)\to(y,\beta)$ consists of those morphisms $\gamma : x\to y$ such that $\gamma\alpha=\beta\gamma$. There is a morphism $\mathcal{I}\to\mathcal{X}$ given by forgetting the automorphism. For every object x of $\mathcal{X}$ over an object T of $\mathcal{C}$, $x^*\mathcal{I}=\mathbb{A}\mathrm{ut}_{\mathcal{X}(T)}(x)$ is the sheaf of automorphisms of x over the identity automorphism of T. (To make sense of the notation $x^*\mathcal{I}$, we should recall that to an object T of $\mathcal{C}$ there corresponds a stack over $\mathcal{C}$, again denoted by T, and, moreover, that the 2-Yoneda lemma constructs an equivalence of categories $\mathrm{HOM}_{\mathcal{C}}(T,\mathcal{X})=\mathcal{X}(T)$.)

6.2. — Continuing with the situation and the notation of 6.1, let $\mathcal{N}\subset\mathcal{I}$ be a normal group substack. This means that for each $x\in\mathcal{X}$ over $T\in\mathcal{C}$, $x^*\mathcal{N}$ is a normal subgroup of $\mathbb{A}\mathrm{ut}_{\mathcal{X}(T)}(x)$. Then the *rigidification $\mathcal{X}/\mathcal{N}$ of $\mathcal{X}$ by $\mathcal{N}$* may be constructed in two steps as follows. In the first step we construct a prestack $\mathcal{Y}$. Its object class is the same as that of $\mathcal{X}$. Fix an arbitrary $T\in\mathcal{C}$ and $x,y\in\mathcal{X}(T)$. Then precomposition of morphisms $x\to y$ (over the identity of T) with automorphisms of x (over the identity of T) defines a right action of $x^*\mathcal{N}$ on the sheaf $\mathbb{H}\mathrm{om}_{\mathcal{X}(T)}(x,y)$ of morphisms $x\to y$ over the identity of T. We define the set $\mathrm{Hom}_{\mathcal{Y}(T)}(x,y)$ of morphisms $x\to y$ in $\mathcal{Y}$ over the identity of T to be the set of global sections of the sheaf quotient $x^*\mathcal{N}\backslash\mathbb{H}\mathrm{om}_{\mathcal{X}(T)}(x,y)$. To define the set $\mathrm{Hom}_{\mathcal{Y}}(x,y)$ of morphisms between an arbitrary pair of objects $x\in\mathcal{X}(T)$, $y\in\mathcal{X}(U)$, we fix a cleavage of $\mathcal{X}$ and a morphism $f:T\to U$, and we declare the set of morphisms $x\to y$ over f to be $\mathrm{Hom}_{\mathcal{Y}(T)}(x,f^*y)$. Composition of morphisms is defined as follows. Consider three objects x,y,z of $\mathcal{X}$ over an object T of $\mathcal{C}$. Then the composite

$$\begin{tikzcd}
\mathbb{H}\mathrm{om}_{\mathcal{X}(T)}(x,y)\times\mathbb{H}\mathrm{om}_{\mathcal{X}(T)}(y,z) \arrow[r] & \mathbb{H}\mathrm{om}_{\mathcal{X}(T)}(x,z) \arrow[d] \\
 & \mathbb{H}\mathrm{om}_{\mathcal{Y}(T)}(x,z)
\end{tikzcd}$$

factors uniquely through $\mathbb{H}\mathrm{om}_{\mathcal{Y}(T)}(x,y)\times\mathbb{H}\mathrm{om}_{\mathcal{Y}(T)}(y,z)$. Taking global sections gives us our composition law.

For the second step, we define $\mathcal{X}/\mathcal{N}$ to be the stack associated to $\mathcal{Y}$. There is a natural map $\mathcal{X}\to\mathcal{X}/\mathcal{N}$ which we denote by p.

Proposition 6.3. — Continuing with the situation and the notation of 6.2, let $\mathcal{Z}$ be a second stack over $\mathcal{C}$. Given a pair of stacks, we write HOM for the category of morphisms. Then the map

$$\mathrm{HOM}(\mathcal{X}/\mathcal{N}, \mathcal{Z}) \to \mathrm{HOM}(\mathcal{X}, \mathcal{Z})$$

induced by p, is fully faithful and has essential image those morphisms $g : \mathcal{X} \to \mathcal{Z}$ which satisfy the following condition. Consider an arbitrary object T of $\mathcal{C}$ and an object x of $\mathcal{X}(T)$. Let z denote its image in $\mathcal{Z}$. Then g induces a map

$$\mathbb{A}\mathrm{ut}_{\mathcal{X}(T)}(x) \to \mathbb{A}\mathrm{ut}_{\mathcal{Z}(T)}(z)\ .$$

We require that $x^*\mathcal{N}$ be contained in its kernel.

Proof. — This amounts to a verification. By [13, Lemme 3.2], the map under consideration factors through an equivalence of categories

$$\mathrm{HOM}(\mathcal{X}/\mathcal{N}, \mathcal{Z}) \xrightarrow{\cong} \mathrm{HOM}(\mathcal{Y}, \mathcal{Z})\ .$$

(Here $\mathcal{Y}$ denotes the prestack that intervened in the construction of $\mathcal{X}/\mathcal{N}$ above.) So in proving the proposition, we may restrict attention to the map

$$\mathrm{HOM}(\mathcal{Y}, \mathcal{Z}) \to \mathrm{HOM}(\mathcal{X}, \mathcal{Z})\ .$$

We begin with the statement concerning the essential image. Let $g : \mathcal{X} \to \mathcal{Z}$ be a morphism satisfying the stated condition. Then a morphism $h : \mathcal{Y} \to \mathcal{Z}$ mapping to g is defined as follows. On the level of objects h is equal to g. Let x_1, x_2 be objects of $\mathcal{X}$ mapping to $T \in \mathcal{C}$ and to z_1, z_2 in $\mathcal{Z}$. Then $\mathbb{A}\mathrm{ut}_{\mathcal{X}(T)}(x_1)$ acts on $\mathbb{H}\mathrm{om}_{\mathcal{X}(T)}(x_1, x_2)$, and $\mathbb{A}\mathrm{ut}_{\mathcal{Z}(T)}(z_1)$ acts on $\mathbb{H}\mathrm{om}_{\mathcal{Z}(T)}(z_1, z_2)$, and the map of sheaves

$$\mathbb{H}\mathrm{om}_{\mathcal{X}(T)}(x_1, x_2) \to \mathbb{H}\mathrm{om}_{\mathcal{Z}(T)}(z_1, z_2)$$

induced by g is equivariant with respect to these actions, hence factors through the quotient $\mathbb{H}\mathrm{om}_{\mathcal{Y}(T)}(x_1, x_2) = x_1^*\mathcal{N}\backslash\mathbb{H}\mathrm{om}_{\mathcal{X}(T)}(x_1, x_2)$.

We now go on to discuss the full faithfulness. Let q denote the projection $\mathcal{X} \to \mathcal{Y}$. Given objects g, h of $\mathrm{HOM}(\mathcal{Y}, \mathcal{Z})$ and a morphism $\gamma : gq \to hq$, we are to construct a morphism $\delta : g \to h$ mapping to γ. To this end we fix an arbitrary object x of $\mathcal{X}(T)$ and, recalling that $\mathcal{X}$ and $\mathcal{Y}$ share the same object class, define $\delta(x) : g(x) \to h(x)$ by $\delta(x) := \gamma(x)$. To check naturality, consider a morphism $\phi : x_1 \to x_2$ in $\mathcal{Y}$ over the identity of T. The desired equality

$$h(\phi)\delta(x_1) = \delta(x_2)g(\phi) \tag{6.3.1}$$

of morphisms $g(x_1) \to h(x_2)$ in $\mathcal{Z}(T)$ may be checked locally on T. So after possibly replacing T by a covering, we may assume that ϕ is the image of a morphism $\psi : x_1 \to x_2$ in $\mathcal{X}$, whereupon 6.3.1 follows from the naturality of γ. □

Corollary 6.4. — In particular, the rigidification of a stack $\mathcal{X}$ by its entire inertia stack

$$\mathcal{X}/\mathcal{I} = \pi_0(\mathcal{X})$$

is canonically isomorphic to the sheaf associated to the functor of isomorphism classes of objects of $\mathcal{X}$.

Proof. — In this case, the rigidified stack $\mathcal{X}/\mathcal{I}$ possesses no nontrivial automorphisms, hence is equivalent to a sheaf. The theorem then specializes to the statement that the projection $\mathcal{X} \to \mathcal{X}/\mathcal{I}$ is the universal map to a sheaf, which is the same as the universal mapping property of $\pi_0(\mathcal{X})$. □

Proposition 6.5. — Continuing with the situation and the notation of 6.2, let T be an object of $\mathcal{C}$, x an object of $\mathcal{X}$, and $\bar{x}$ its image in $\mathcal{X}/\mathcal{N}(T)$. Then

$$\bar{x}^*\mathcal{X} = Bx^*\mathcal{N}$$

is canonically the classifying stack of $x^*\mathcal{N}$.

Proof. — This too is merely a verification. We regard these as stacks over the restricted site $\mathcal{C}|_T$ of objects *over* T. We let $B'x^*\mathcal{N}$ denote the classifying prestack: its objects are the same as those of $\mathcal{C}|_T$, and for $t : T' \to T$ an object of $\mathcal{C}$, $\mathrm{Aut}_{B'x^*\mathcal{N}}(t) = x'^*\mathcal{N}(T)$, where x' denotes the composite $x' = xt$. We set out to construct a morphism

$$\Psi : B'x^*\mathcal{N} \to \bar{x}^*\mathcal{X}$$

and to show that Ψ is fully faithful, and, moreover, that every object of $\bar{x}^*\mathcal{N}$ is locally in the essential image. By the universal mapping property of stackification ([13, Lemme 3.2]), it will then follow that Ψ factors uniquely through an isomorphism as proposed. For simplicity, we restrict attention to the morphism

$$\Psi(T) : B'x^*\mathcal{N}(T) \to \bar{x}^*\mathcal{X}(T)$$

of fibers over T.

An object of $\bar{x}^*\mathcal{X}(T)$ is a pair $(y, \bar{\phi})$, where y is an object of $\mathcal{X}(T)$ and $\bar{\phi} : \bar{y} \to \bar{x}$ is an isomorphism in $\mathcal{X}/\mathcal{N}$. A morphism $(y, \bar{\phi}) \to (z, \bar{\psi})$ in $\bar{x}^*\mathcal{X}(T)$ is a morphism $y \to z$ in $\mathcal{X}$ whose image in $\mathcal{X}/\mathcal{N}$ coincides with $\bar{\psi}^{-1}\bar{\phi}$. In particular, $\mathrm{Aut}_{\bar{x}^*\mathcal{X}(T)}(x, \mathrm{id}_{\bar{x}}) = x^*\mathcal{N}(T)$. So we obtain our hoped-for morphism, together with its full-faithfulness, by sending the unique object of $B'x^*\mathcal{N}(T)$ to $(x, \mathrm{id}_{\bar{x}})$. Finally, consider an object $(y, \bar{\phi})$ of $\bar{x}^*\mathcal{X}(T)$. In showing that $(y, \bar{\phi})$ is *locally* in the essential image, we may, after possibly replacing T by a covering, assume that $\bar{\phi}$ comes from a section ϕ of $\mathbb{H}\mathrm{om}_{\mathcal{X}(T)}(y, x)$. An isomorphism $(y, \bar{\phi}) \to (x, \mathrm{id}_{\bar{x}})$ is then given by ϕ. □

Theorem 6.6. — Let $\mathcal{X}$ be an algebraic stack and let $\mathcal{N} \subset \mathcal{I}$ be a closed group substack of the inertia stack. Suppose that $\mathcal{N}$ is flat and locally of finite presentation over $\mathcal{X}$. Then the rigidification $\mathcal{X}/\mathcal{N}$ with respect to the fppf topology is again an algebraic stack. Moreover, $\mathcal{X}$ is faithfully flat and locally of finite presentation over $\mathcal{X}/\mathcal{N}$.

Proof. — The first statement is Proposition 1.5.4 of [15]. We note that the main ideas behind this theorem are contained in work of Artin (c.f. [1]). As for the second statement, the properties *faithfully flat* and *locally of finite presentation* are local on source and target. It follows from the construction of the stack associated to a prestack that every object $\bar{x}$ of $\mathcal{X}/\mathcal{N}$ comes locally from an object x of $\mathcal{X}$. So by 6.5, $\mathcal{X}$ is locally over $\mathcal{X}/\mathcal{N}$ the classifying stack of a group which is itself faithfully flat and locally of finite presentation. Let G be such a group over a base T. Then the projection $T \to BG$ is an fppf torsor under G, hance faithfully flat and locally of finite presentation. Hence $BG \to T$ is fppf-locally on the source the identity map of T, hence, in particular, faithfully flat and locally of finite presentation, which completes the proof. □

Corollary 6.7. — Let $\mathcal{X}$ be an algebraic stack and let $\mathcal{I}$ denote its inertia stack. Suppose that $\mathcal{I}$ is flat and locally of finite presentation (or, which is the same, that all automorphism groups in $\mathcal{X}$ are flat and locally of finite presentation). Then the fppf sheaf $X := \pi_0^{\mathrm{fppf}}(\mathcal{X})$ associated to $\mathcal{X}$ is an algebraic space. Moreover, the projection $\mathcal{X} \to X$ is faithfully flat and locally of finite presentation.

Proof. — This follows from Theorem 6.6 in view of Corollary 6.4. □

7. Moduli spaces

We now apply the construction of §6 to the parameter spaces of §5. We continue to work with a finitely generated Lie algebra $\mathfrak{g}$ over a field k, and we preserve all notations introduced in §5.

Definition 7.1. — We let $\mathcal{M}_n^{\mathrm{fl}}(\mathfrak{g})$ denote the category fibered in groupoids over $\mathbf{Aff}(k)$ whose objects are flag representations of rank n. Thus an object is a pair (T, r), with $T \in \mathbf{Aff}(k)$ and $r : \mathfrak{g}_T \to \mathscr{E}nd\,\mathcal{E}$ a flag representation. A morphism $(T', r') \to (T, r)$ is a pair (f, ϕ) where $f : T' \to T$ is a map of affine schemes and $\phi : f^*r \to r'$ is an isomorphism of representations. We let $\mathcal{M}_n^{\mathrm{nd}}(\mathfrak{g})$ denote the fibered subcategory of $\mathcal{M}_n^{\mathrm{fl}}(\mathfrak{g})$ whose objects are nondegenerate nilpotent representations. When there is no risk of confusion we write simply $\mathcal{M}_n^{\mathrm{fl}}$ and $\mathcal{M}_n^{\mathrm{nd}}$.

Proposition 7.2. — Both $\mathcal{M}_n^{\mathrm{fl}}(\mathfrak{g})$ and $\mathcal{M}_n^{\mathrm{nd}}(\mathfrak{g})$ are stacks for the fppf topology.

Proof. — Fix arbitrarily a faithfully flat, locally finitely presented morphism $f : T' \to T$ of k-schemes. Let $\mathcal{M}_n^{\mathrm{fl}}(f)$ denote the category of descent data in $\mathcal{M}_n^{\mathrm{fl}}$ relative to f (c.f. [15, 1.2.3]), and let f^{**} denote the functor $\mathcal{M}_n^{\mathrm{fl}}(f) \leftarrow \mathcal{M}_n^{\mathrm{fl}}(T)$ which sends an object over T to its descent datum rel. f. We are to show that f^{**} is an equivalence of categories.

To see that f^{**} is fully faithful, fix arbitrarily flag representations $r_1 : \mathfrak{g}_T \to \mathscr{E}nd\, \mathcal{E}_1$, $r_2 : \mathfrak{g}_T \to \mathscr{E}nd\, \mathcal{E}_2$ over T and a morphism $\phi' : r_2' \to r_1'$ over T'. Then by descent for quasi-coherent sheaves (c.f. [10, Eposé VIII, Theorem 1.1]), there is one and only one morphism $\phi : \mathcal{E}_2 \to \mathcal{E}_1$ of modules whose pullback to T' coincides with ϕ'. We are to verify that ϕ is equivariant with the action of $\mathfrak{g}$. Fix arbitrarily a $v \in \mathfrak{g}$. Then the square

$$\begin{tikzcd} \mathcal{E}_2 \arrow[r, "r_2 v"] \arrow[d, "\phi"'] & \mathcal{E}_2 \arrow[d, "\phi"] \\ \mathcal{E}_1 \arrow[r, "r_1 v"'] & \mathcal{E}_1 \end{tikzcd}$$

commutes after pullback to T', so, by descent for quasi-coherent sheaves, the square itself commutes, indeed.

To see that f^{**} is fully faithful, let $s, t : T'' \rightrightarrows T'$ denote the structural projections of the product $T'' = T' \times_T T'$, let $r' : \mathfrak{g}_{T'} \to \mathscr{E}nd\, \mathcal{E}'$ be a flag representation over T' and let $\psi : s^* r' \to t^* r'$ be an isomorphism obeying the cocycle condition. Then $(\mathcal{E}', \psi)$ is in particular a descent datum for a quasi coherent sheaf $\mathcal{E}$ on T. Equivariance of ψ means that the map $\mathscr{E}nd\, s^*\mathcal{E}' \to \mathscr{E}nd\, t^*\mathcal{E}'$ given by $\alpha \mapsto \psi\alpha\psi^{-1}$ fits into a commutative triangle as follows.

$$\begin{tikzcd} \mathfrak{g}_{T''} \arrow[d, "s^* r'"'] \arrow[dr, "t^* r'"] & \\ \mathscr{E}nd\, s^*\mathcal{E}' \arrow[r] & \mathscr{E}nd\, t^*\mathcal{E}' \end{tikzcd}$$

Since $\mathcal{E}'$ is a vector sheaf (hence, in particular, coherent), the commutativity of this triangle implies that the square

$$\begin{tikzcd} s^*\mathfrak{g}_{T'} \arrow[r] \arrow[d] & t^*\mathfrak{g}_{T'} \arrow[d] \\ s^*\, \mathscr{E}nd\, \mathcal{E}' \arrow[r, "\theta"'] & t^*\, \mathscr{E}nd\, \mathcal{E}' \end{tikzcd}$$

commutes. The pair $(\mathscr{E}nd\,\mathcal{E}', \theta)$ is the descent datum for $\mathscr{E}nd\,\mathcal{E}$ along f. Thus r' is a morphism of descent data. Hence, by descent for quasi coherent sheaves, r' comes from a unique homomorphism $r : \mathfrak{g}_T \to \mathscr{E}nd\,\mathcal{E}$. A straightforward verification shows that r is a flag representation, and, moreover, that if r' was nondegenerate, then so is r. □

We recall that $\mathbb{B}_n$ denotes the group of invertible upper triangular $n \times n$ matrices.

Proposition 7.3. — Let $\mathbb{B}_n$ act on $X_n^{\mathrm{fl}}(\mathfrak{g})$ and on $X_n^{\mathrm{nd}}(\mathfrak{g})$ by conjugation. Then (in the notation of 0.6)

$$[X_n^{\mathrm{fl}}/_{ZAR}\mathbb{B}_n] = \mathcal{M}_n^{\mathrm{fl}}(\mathfrak{g}) \qquad \text{and} \qquad [X_n^{\mathrm{nd}}/_{\mathrm{ZAR}}\mathbb{B}_n] = \mathcal{M}_n^{\mathrm{nd}}(\mathfrak{g})\,.$$

Proof. — Recall our notational convention (0.6) by which $[X_n^{\mathrm{fl}}/\mathbb{B}_n]$ denotes the fibered category whose objects over T are the elements of $X_n^{\mathrm{fl}}(T)$ and whose morphisms $x \to y$ over id_T are those elements b of $\mathbb{B}_n(T)$ such that $bx = y$. There is an obvious map

$$[X_n^{\mathrm{fl}}/\mathbb{B}_n] \to \mathcal{M}_n^{\mathrm{fl}}(\mathfrak{g})\,.$$

Any isomorphism between flag representations of the form $r : \mathfrak{g}_T \to \mathfrak{n}_{nT}$ belongs to $\mathbb{B}_n(T)$. Indeed, any isomorphism of representations must respect the associated filtrations (1.5); the filtration associated to a flag representation of the form $\mathfrak{g}_T \to \mathfrak{n}_{nT}$ is equal to the standard flag; an element of $\mathbb{GL}_n(T)$ which preserves the standard flag is by definition an element of $\mathbb{B}_n(T)$. This shows that the map is fully faithful. Moreover, a general nondegenerate nilpotent representation $r : \mathfrak{g}_T \to \mathscr{E}nd\,\mathcal{E}$ is of the form $\mathfrak{g}_T \to \mathfrak{n}_{nT}$, hence comes from $X_n^{\mathrm{fl}}(\mathfrak{g})(T)$, after possibly replacing T by a Zariski covering of T. That is, every object of the target is Zariski locally in the image. It follows that the map factors through an isomorphism of Zariski stacks as claimed.

The same argument applies to $\mathcal{M}_n^{\mathrm{nd}}$. □

Corollary 7.4. — The stacks $\mathcal{M}_n^{\mathrm{fl}}$ and $\mathcal{M}_n^{\mathrm{nd}}$ are algebraic.

Proof. — We could switch to the étale topology, note that $\mathbb{B}_n$ is smooth, and quote well known results ([13, 4.6.1]). Instead, we stick to the fppf topology, and indicate how this follows from other well known results. By 5.3, X_n^{fl} is represented by a scheme. Denote X_n^{fl} by X, and $\mathbb{B}_n$ by B for short. The *groupoid* $(X, B\times X, s, t, m, e, i)$ *in* $\mathbf{Aff}(k)$ ([15, 1.1.18]) *associated to the action of* B *on* X is defined as follows. Recall that s and t are morphisms

$$B \times X \rightrightarrows X\ ,$$

m is a morphism

$$(B \times X) \underset{s,t}{\times} (B \times X) \to B \times X$$

e is a morphism

$$B \times X \leftarrow X$$

and i is a morphism

$$B \times X \to B \times X\ .$$

We set $s(g,x) = x$, $t(g,x) = gx$, $m((g,x),(g',x')) = (gg',x')$, $e(x) = (I,x)$ (where I denotes the identity matrix), and $i(g,x) = (g^{-1},gx)$. Then the fppf stack quotient $[X/_{\mathrm{fppf}}B]$ is equal to the stack associated to $(X, B \times X, s, t, m, e, i)$ ([15, 1.2.7]) in the fppf topology. By Artin's theorem on stacks in the flat topology (quoted in [15, Theorem 1.3.2] and proved in [1]), the latter is an algebraic stack. On the other hand, by Propositions 7.2 and 7.3,

$$[X/_{\mathrm{ZAR}}B] = [X/_{\mathrm{fppf}}B]$$

by which the assertion concerning $\mathcal{M}_n^{\mathrm{fl}}$ follows.

The assertion concerning $\mathcal{M}_n^{\mathrm{nd}}$ follows similarly from Proposition 5.10. □

Theorem 7.5. — The fppf sheaf $\pi_0^{\mathrm{fppf}}\mathcal{M}_n^{\mathrm{nd}}(\mathfrak{g})$ (0.7) associated to $\mathcal{M}_n^{\mathrm{nd}}(\mathfrak{g})$ is an algebraic space.

Proof. — The automorphism group of a nondegenerate representation is flat and locally of finite presentation. So this follows from 7.4 by Corollary 6.7. □

Definition 7.6. — We define the **moduli space of nondegenerate nilpotent representations** by

$$M_n^{\mathrm{nd}}(\mathfrak{g}) := \pi_0^{\mathrm{fppf}}\mathcal{M}_n(\mathfrak{g})\ .$$

Remark 7.7. — We note that according to Corollary 6.7, the map $\mathcal{M}_n^{\mathrm{nd}} \to M_n^{\mathrm{nd}}$ is faithfully flat and locally of finite presentation.

We now discuss the functoriality of our moduli spaces (7.8–7.14).

Remark 7.8. — Let s be a surjection of Lie algebras

$$\mathfrak{f} \twoheadrightarrow \mathfrak{g}$$

and let $r : \mathfrak{g}_T \to \mathscr{E}nd\,\mathcal{E}$ be a representation of $\mathfrak{g}$ over a k-scheme T. Then the zero eigenspace of rs_T is equal to the zero-eigenspace of r. Since the functor $r \mapsto rs_T$ is exact, it follows that rs_T is a flag representation if and only if r is a flag preresentation. In particular, s gives rise to a fully faithful morphism of stacks

$$\mathcal{M}_n^{\mathrm{fl}}(\mathfrak{g}) \hookrightarrow \mathcal{M}_n^{\mathrm{fl}}(\mathfrak{f}) .$$

PROPOSITION 7.9. — Let s be a surjection of Lie algebras

$$\mathfrak{f} \twoheadrightarrow \mathfrak{g} .$$

Then the morphism

$$\mathcal{M}_n^{\mathrm{fl}}(\mathfrak{g}) \hookrightarrow \mathcal{M}_n^{\mathrm{fl}}(\mathfrak{f})$$

induced by s as in 7.8 is a closed immersion.

The proof follows (7.10–7.11).

LEMMA 7.10. — Let A be a ring and $\theta : E \to F$ a morphism of A-modules with F locally free of finite rank. Then there exists an ideal $I(\theta)$ of A such that for all A-algebras B, $\theta_B := \mathrm{id}_B \otimes \theta = 0$ if and only if $I(\theta)B = 0$.

Proof. — The assertion is local on A, so we may assume F is free. After possibly precomposing θ with a surjection $E' \twoheadrightarrow E$, we may assume E is free as well. To complete the proof, we fix bases and let $I(\theta)$ be the ideal generated by the corresponding matrix entries of θ. □

7.11. — *Completion of proof of* 7.9. Let $r : \mathfrak{f}_T \to \mathscr{E}nd\,\mathcal{E}$ be a flag representation of $\mathfrak{f}$ over an affine k-scheme $T = \operatorname{Spec} A$. Let $i : \mathfrak{a} \to \mathfrak{f}$ be the kernel of $s : \mathfrak{f} \twoheadrightarrow \mathfrak{g}$ and let Z be the closed subscheme of T defined by $I(ri_T)$. Then by 7.10, $g : T' \to T$ factors through Z if and only if g^*r factors through $\mathfrak{g}$. Regarding r as a map $T \to \mathcal{M}_n^{\mathrm{fl}}(\mathfrak{f})$, we have constructed a cartesian square

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] & T \arrow[d] \\ \mathcal{M}_n^{\mathrm{fl}}(\mathfrak{g}) \arrow[r, hook] & \mathcal{M}_n^{\mathrm{fl}}(\mathfrak{f}) \end{tikzcd}$$

with $Z \hookrightarrow T$ a closed immersion, which concludes the proof.

Remark 7.12. — Consider again the situation of 7.8 given by a surjection

$$s : \mathfrak{f} \twoheadrightarrow \mathfrak{g}$$

of Lie algebras and a representation $r : \mathfrak{g}_T \to \mathscr{E}nd\,\mathcal{E}$ of $\mathfrak{g}$ over a k-scheme T. Then

$$\mathbb{A}\mathrm{ut}\, rs_T = \mathbb{A}\mathrm{ut}\, r .$$

Suppose $A(\mathfrak{g}, i) = A(\mathfrak{f}, i)$ for $i = 1, \ldots, n$. Then it follows from 7.8 that r is nondegenerate if and only if rs_T is nondegenerate. In particular, s gives rise to a fully faithful morphism of stacks

$$\mathcal{M}_n^{\mathrm{nd}}(\mathfrak{g}) \hookrightarrow \mathcal{M}_n^{\mathrm{nd}}(\mathfrak{f}).$$

Corollary 7.13. — Continuing with the situation of 7.12, the map

$$\mathcal{M}_n^{\mathrm{nd}}(\mathfrak{g}) \hookrightarrow \mathcal{M}_n^{\mathrm{nd}}(\mathfrak{f})$$

is a closed immersion.

Proof. — Indeed, the "if and only if" part of 7.12 implies moreover that the resulting square

$$\begin{tikzcd} \mathcal{M}_n^{\mathrm{fl}}(\mathfrak{g}) \arrow[r, hook] & \mathcal{M}_n^{\mathrm{fl}}(\mathfrak{f}) \\ \mathcal{M}_n^{\mathrm{nd}}(\mathfrak{g}) \arrow[r, hook] \arrow[u, hook] & \mathcal{M}_n^{\mathrm{nd}}(\mathfrak{f}) \arrow[u, hook] \end{tikzcd}$$

is Cartesian, so this follows from 7.8. □

Corollary 7.14. — Continuing with the situation of 7.12, s gives rise to a closed immersion

$$M_n^{\mathrm{nd}}(\mathfrak{g}) \hookrightarrow M_n^{\mathrm{nd}}(\mathfrak{f}).$$

Proof. — The square

$$\begin{tikzcd} \mathcal{M}_n^{\mathrm{nd}}(\mathfrak{g}) \arrow[r, hook] \arrow[d] & \mathcal{M}_n^{\mathrm{nd}}(\mathfrak{f}) \arrow[d] \\ M_n^{\mathrm{nd}}(\mathfrak{g}) \arrow[r, hook] & M_n^{\mathrm{nd}}(\mathfrak{f}) \end{tikzcd}$$

is automatically Cartesian. Since the vertical arrow on the right is fppf (7.7) and the horizontal arrow at the top is a closed immersion (7.13), the proposition follows. □

Recall that $\mathbb{B}_n = \mathbb{T}_n \ltimes \mathbb{U}_n$ is the semidirect product of a torus and a unipotent group. In studying the quotient of X_n^{nd} by $\mathbb{B}_n$ it will be convenient to consider the actions of $\mathbb{U}_n$ on X_n^{nd} and of $\mathbb{T}_n$ on the quotient of X_n^{nd} by $\mathbb{U}_n$ separately. The following variant provides a natural interpretation of the quotient stack $[X_n^{\mathrm{fl}}/_{\mathrm{ZAR}}\mathbb{U}_n]$ in terms of flag representations.

Definition 7.15. — Let T be a k-scheme. A **framed flag representation of $\mathfrak{g}$ over T** is a pair (r, e) where r is a flag representation $\mathfrak{g}_T \to \mathscr{E}nd\,\mathcal{E}$ of $\mathfrak{g}$ over T and $e = (e_1, \ldots, e_n)$ is a basis for $\mathrm{gr}^{\mathrm{Fil}^r}\mathcal{E}$ (1.5) compatible with the grading. If r, r' are framed flag representations, a **framed isomorphism** $r \to r'$ is an isomorphism ϕ of the underlying flag representations such that $\mathrm{gr}(\phi) : \mathrm{gr}\,\mathcal{E} \to \mathrm{gr}\,\mathcal{E}'$ is the isomorphism determined by the given bases. A framed flag representation is **nondegenerate**, if the underlying flag representation is nondegenerate.

We let $\mathcal{M}_n^{\mathrm{ffl}}(\mathfrak{g})$ denote the fibered category of framed flag representations and we let $\mathcal{M}_n^{\mathrm{fnd}}(\mathfrak{g})$ denote the fibered subcategory consisting of framed nondegenerate nilpotent representations.

Proposition 7.16. — The fibered categories $\mathcal{M}_n^{\mathrm{ffl}}(\mathfrak{g})$ and $\mathcal{M}_n^{\mathrm{fnd}}(\mathfrak{g})$ obey fppf descent.

Proof. — We start with $\mathcal{M}_n^{\mathrm{ffl}}$. This is a straightforward verification using the fact that $\mathcal{M}_n^{\mathrm{fl}}$ is an fppf stack (7.2). Let T be an affine k-scheme, $f : T' \to T$ an fppf covering, let $T'' = T' \times_T T'$ and denote by $p_1, p_2 : T'' \rightrightarrows T'$ the two projections. Denote by $\mathcal{M}_n^{\mathrm{ffl}}(f)$ the category of descent data relative to f. We are to show that the functor

$$f^{**} : \mathcal{M}_n^{\mathrm{ffl}}(T) \to \mathcal{M}_n^{\mathrm{ffl}}(f)$$

is an equivalence. To see that f^{**} is fully faithful, fix two framed representations

$$(r : \mathfrak{g}_T \to \mathscr{E}nd(\mathcal{E}), e^r)$$

and

$$(s : \mathfrak{g}_T \to \mathscr{E}nd(\mathcal{F}), e^s)$$

over T. Let α denote the induced isomorphism

$$p_2^* f^* r \to p_1^* f^* r\,,$$

β the induced isomorphism

$$p_2^* f^* s \to p_1^* f^* s\,,$$

and consider a morphism of descent data

$$\phi' : (f^* r, f^* e^r, \alpha) \to (f^* s, f^* e^s, \beta)\,;$$

that is, a morphism

$$\phi' : (f^* r, f^* e^r) \to (f^* s, f^* e^s)$$

such that the square

$$\begin{tikzcd} p_2^* f^* r \arrow[r, "p_2^*\phi'"] \arrow[d, "\alpha"'] & p_2^* f^* s \arrow[d, "\beta"] \\ p_1^* f^* r \arrow[r, "p_1^*\phi'"'] & p_1^* f^* s \end{tikzcd}$$

commutes. Descent for $\mathcal{M}_n^{\mathrm{fl}}$ implies that there is a (necessarily unique) morphism of representations

$$\phi : r \to s$$

such that

$$f^*\phi = \phi' \,;$$

on the other hand

$$f^*\mathrm{gr}\ \phi = \mathrm{gr}\ f^*\phi = \mathrm{gr}\ \phi'$$

sends f^*e^r to f^*e^s which implies that gr ϕ sends e^r to e^s since restriction maps along coverings are injective. This establishes the full faithfulness.

To check essential surjectivity, fix a framed flag representation

$$(r' : \mathfrak{g}_{T'} \to \mathscr{E}nd(\mathcal{E}'), e')$$

and a framed isomorphism

$$\alpha : (p_2^* r', p_2^* e') \to (p_1^* r', p_1^* e')$$

obeying the cocycle condition. Descent for $\mathcal{M}_n^{\mathrm{fl}}$ produces a representation $r : \mathfrak{g}_T \to \mathscr{E}nd(\mathcal{E})$ whose descent data $(f^*r, \alpha^{\mathrm{can.}})$ relative to f is isomorphic to (r', α). Fixing an isomorphism

$$\phi : (r', \alpha) \to (f^*r, \alpha^{\mathrm{can.}})$$

we get a diagram

$$\begin{tikzcd}
 & p_2^*\mathrm{gr}\ \mathcal{E}' \arrow[r, "p_2^*(\mathrm{gr}\ \phi)"] \arrow[dd, "\mathrm{gr}\ \alpha"] & p_2^* f^*\mathrm{gr}\ \mathcal{E} \arrow[dd, "\mathrm{gr}\ \alpha^{\mathrm{can.}}"'] & \\
e'_i \in \mathrm{gr}\ \mathcal{E}' \arrow[ur, "p_2^*"] \arrow[dr, "p_1^*"'] \arrow[rrr, "\mathrm{gr}\ \phi"'] & & & f^*\mathrm{gr}\ \mathcal{E} \ni (\mathrm{gr}\ \phi)(e'_i) \arrow[ul, "p_2^*"'] \arrow[dl, "p_1^*"] \\
 & p_1^*\mathrm{gr}\ \mathcal{E}' \arrow[r, "p_1^*(\mathrm{gr}\ \phi)"'] & p_1^* f^*\mathrm{gr}\ \mathcal{E} &
\end{tikzcd}$$

in which the (small) square at the front and the two trapezoids at the back commute. Since

$$(\mathrm{gr}\ \alpha)(p_2^* e'_i) = p_1^* e'_i \,,$$

it follows that

$$(\mathrm{gr}\ \alpha^{\mathrm{can.}}) p_2^*(\mathrm{gr}\ \phi)(e'_i) = p_1^*(\mathrm{gr}\ \phi)(e'_i) \,,$$

hence that $\{(\mathrm{gr}\ \phi)(e'_i)\}$ descends to a basis $\{e_i\}$ of gr $\mathcal{E}$ making (r, e) into a framed representation. This shows that (r', e', α) is in the essential image of f^* and completes the verification.

Finally, the same argument using the fact that $\mathcal{M}_n^{\mathrm{nd}}$ is an fppf stack shows that $\mathcal{M}_n^{\mathrm{fnd}}$ is an fppf stack. $\square$

Proposition 7.17. — Let $\mathbb{U}_n$ act on $X_n^{\mathrm{fl}}(\mathfrak{g})$ and on $X_n^{\mathrm{nd}}(\mathfrak{g})$ by conjugation. Then

$$[X_n^{\mathrm{fl}}/_{ZAR}\mathbb{U}_n] = \mathcal{M}_n^{\mathrm{ffl}}(\mathfrak{g}) \qquad \text{and} \qquad [X_n^{\mathrm{nd}}/_{\mathrm{ZAR}}\mathbb{U}_n] = \mathcal{M}_n^{\mathrm{fnd}}(\mathfrak{g})\,.$$

Proof. — Consider the map

$$[X_n^{\mathrm{fl}}/\mathbb{U}_n] \to \mathcal{M}_n^{\mathrm{ffl}}(T)$$

which sends $r : \mathfrak{g}_T \to \mathfrak{n}_{n,T}$ to

$$(\mathfrak{g}_T \to \mathfrak{n}_{n,T} \hookrightarrow \mathscr{E}nd\,\mathcal{O}_T^{(n)}, \text{the standard basis of } \mathcal{O}_T^{(n)})\,.$$

If $r, r' \in X_n^{\mathrm{fl}}(T)$, then an isomorphism of representations $b : r \to r'$ is in $\mathbb{U}_n(T)$ if and only if $\operatorname{gr} b = \mathrm{id}_{\mathcal{O}_T^{(n)}}$. The same argument applies to $\mathcal{M}_n^{\mathrm{fnd}}$. $\square$

Corollary 7.18. — The stacks $\mathcal{M}_n^{\mathrm{ffl}}(\mathfrak{g})$ and $\mathcal{M}_n^{\mathrm{fnd}}(\mathfrak{g})$ are algebraic.

Proof. — This is similar to Corollary 7.4. Denote X_n^{fl} by X, and $\mathbb{U}_n$ by U for short. The groupoid $(X, U \times X, s, t, m, e, i)$ in $\mathbf{Aff}(k)$ associated to the action of U on X is defined as in 7.4. Then the fppf stack quotient $[X/_{\mathrm{fppf}}U]$ is equal to the stack associated to $(X, B \times X, s, t, m, e, i)$ ([15, 1.2.7]) in the fppf topology. By Artin's theorem on stacks in the flat topology ([15, Theorem 1.3.2]), the latter is an algebraic stack. On the other hand, by Propositions 7.16 and 7.17,

$$[X/_{\mathrm{ZAR}}U] = [X/_{\mathrm{fppf}}U]$$

by which the assertion concerning $\mathcal{M}_n^{\mathrm{ffl}}$ follows.

The assertion concerning $\mathcal{M}_n^{\mathrm{fnd}}$ follows similarly from Proposition 5.10. $\square$

Corollary 7.19. — The fppf-sheaf $\pi_0^{\mathrm{fppf}}\mathcal{M}_n^{\mathrm{fnd}}(\mathfrak{g})$ associated to $\mathcal{M}_n^{\mathrm{fnd}}(\mathfrak{g})$ is an algebraic space.

Proof. — By construction, the inertia stack of $\mathcal{M}_n^{\mathrm{fnd}}$ is flat. So this follows from 7.18 by Corollary 6.7. $\square$

Definition 7.20. — We define $M_n^{\mathrm{fnd}}(\mathfrak{g})$, **the moduli space of framed nondegenerate nilpotent representations**, by

$$M_n^{\mathrm{fnd}}(\mathfrak{g}) = \pi_0^{\mathrm{fppf}}\mathcal{M}_n^{\mathrm{fnd}}(\mathfrak{g})\,.$$

Remark 7.21. — According to Corollary 6.7, the map $\mathcal{M}_n^{\mathrm{fnd}} \to M_n^{\mathrm{fnd}}$ is faithfully flat and locally of finite presentation.

Remark 7.22. — Remark 7.8 applies without essential change to show that a surjection of Lie algebras

$$s : \mathfrak{f} \twoheadrightarrow \mathfrak{g}$$

gives rise to a fully faithful morphism

$$\mathcal{M}_n^{\mathrm{ffl}}(\mathfrak{g}) \hookrightarrow \mathcal{M}_n^{\mathrm{ffl}}(\mathfrak{f})$$

of stacks, and the proof of 7.9 applies without essential change to show that this map is in fact a closed immersion.

Suppose, moreover, that for $i = 1, \dots, n$, $A(\mathfrak{f}, i) = A(\mathfrak{g}, i)$. Then paragraphs 7.12–7.14 apply without essential change to show that s gives rise to closed immersions

$$\mathcal{M}_n^{\mathrm{fnd}}(\mathfrak{g}) \hookrightarrow \mathcal{M}_n^{\mathrm{fnd}}(\mathfrak{f})$$

and

$$M_n^{\mathrm{fnd}}(\mathfrak{g}) \hookrightarrow M_n^{\mathrm{fnd}}(\mathfrak{f})\,.$$

8. Representations of a unipotent group

In this section we put ourselves in the situation indicated by the title of the paper given by a field k of characteristic zero and a unipotent group G over k. The problem of moduli of representations of G is equivalent to the problem studied in the previous sections applied to the case $\mathfrak{g} := \mathbf{Lie}\, G$. This is in part a matter of reviewing the classical theory. Statements available in the literature, however, focus on representations defined over a field. For our purpose, we need to consider families. It turns out that limiting our investigation to representations on vector bundles is awkward, since this excludes the left regular representation. With a bit more work, we obtain the result we want at the level of arbitrary quasi-coherent representations (Theorem 8.20), as well as a more aesthetically pleasing proof.

8.1. — For proofs of the following facts (as well as a discussion of the definition of a unipotent group), we refer the reader to [5, IV §2]. Let **UG** denote the category of unipotent groups over k and let **NL** denote the category of nilpotent Lie algebras over k. The Lie algebra of a unipotent group is nilpotent. Thus **Lie** is a functor $\mathbf{UG} \to \mathbf{NL}$. On the other hand, if $\mathfrak{g}$ is a nilpotent Lie algebra, its covariant total space may be endowed with a product structure $\star : \mathbb{V}\mathfrak{g}^\vee \times \mathbb{V}\mathfrak{g}^\vee \to \mathbb{V}\mathfrak{g}^\vee$ given by the Baker-Campbell-Hausdorff formula. This makes $(\mathbb{V}\mathfrak{g}^\vee, \star)$ into a unipotent group, and defines a functor $\mathbf{H} : \mathbf{NL} \to \mathbf{UG}$. **Lie** and **H** are quasi-inverse ([5, IV §2 4.5]). In particular, there is a natural isomorphism $\exp : \mathbf{H} \circ \mathbf{Lie} \to \mathrm{id}_{\mathbf{UG}}$, which is called the *exponential map.*

8.2. — For the remainder of this section, we fix a unipotent group G over k and we let $\mathfrak{g}$ denote its Lie algebra. Recall that formation of the Lie algebra is compatible with flat base-change, so for any k-scheme T, $\mathfrak{g}_T$ fits into a split short exact sequence of (abstract) groups

$$\begin{tikzcd} 1 \arrow[r] & \Gamma(T, \mathfrak{g}_T) \arrow[r] & G(T[\epsilon]) \arrow[r] & G(T) \arrow[l, bend right] \arrow[r] & 1 \,. \end{tikzcd}$$

Here $T[\epsilon]$ denotes $\mathrm{spec}_T\, \mathcal{O}_T[\epsilon]/(\epsilon^2)$.

We note a few generalities about quasi-coherent representations of a Lie algebra over a general (affine) base (8.3–8.4).

8.3. — Suppose $T = \operatorname{Spec} B$ is an affine scheme, F is a B-module and $r : B \otimes \mathfrak{g} \to \operatorname{End}(F)$ is a representation. Then for any B-algebra B', r defines a representation $r(B') : B' \otimes \mathfrak{g} \to \operatorname{End}_{B'}(B' \otimes F)$ determined by the commuting square

$$\begin{tikzcd} B \otimes \mathfrak{g} \arrow[r, "r=r(B)"] \arrow[d] & \operatorname{End}_B(F) \arrow[d] \\ B' \otimes \mathfrak{g} \arrow[r, dotted, "r(B')"'] & \operatorname{End}_{B'}(B' \otimes F) \end{tikzcd}$$

and the requirement that $r(B')$ be B'-linear. Thus if $\mathbb{E}\mathrm{nd}(F)$ denotes the functor $B' \mapsto \operatorname{End}_{B'}(B' \otimes F)$, then r extends uniquely to a morphism of Lie o_T-algebras $\mathbb{V}\mathfrak{g}_T^\vee \to \mathbb{E}\mathrm{nd}(F)$, which we denote again by r.

8.4. — Continuing with the situation of 8.3, we remark that any vector in the 0-eigenspace of r is automatically universally in the 0-eigenspace. That is, if $x \in F$ is such that $r(v)(x) = 0$ for all $v \in B \otimes \mathfrak{g}$ then for any B algebra B', and any $v' \in B' \otimes \mathfrak{g}$,

$$r(B')(v')(1 \otimes x) = 0 \,.$$

Indeed, since $\mathfrak{g}$ is nilpotent and finitely generated, it is finite dimensional. Let $v_1, \dots, v_m$ be a basis and write

$$v' = \sum_i b_i' \otimes v_i$$

with $b'_i \in B'$. Then (identifying $B' \otimes_k \mathfrak{g}$ with $B' \otimes_B B \otimes_k \mathfrak{g}$) we have

$$\begin{aligned} r(B')(v')(1 \otimes_B x) &= r(B')(\sum b'_i \otimes_k v_i)(1 \otimes_B x) \\ &= \sum b'_i r(B')(1_{B'} \otimes_B 1_B \otimes_k v_i)(1_{B'} \otimes_B x) \\ &= \sum b'_i \otimes_B (r(B)(1_B \otimes_k v_i)(x)) \\ &= \sum b'_i \otimes_B 0 \\ &= 0\,. \end{aligned}$$

Definition 8.5. — Let B be a k-algebra and let $r : \mathfrak{g}_B \to \mathrm{End}(F)$ be a representation on a B-module F. Then r is **locally nilpotent** if Fil^r (1.5) is exhaustive (1.8).

Definition 8.6. — Let B be a k-algebra and let F be a B-module. We denote by $\mathbb{A}\mathrm{ut}\, F$ the group-valued functor

$$B' \mapsto \mathrm{Aut}_{B'}(B' \otimes F)\,.$$

A representation ρ of G on F (over T) is a morphism of group-valued functors

$$G_T \to \mathbb{A}\mathrm{ut}\, F\,.$$

The submodule $F^{G_T} \subset F$ of *invariants* is then defined to be the set of universally fixed elements of F, that is, those $x \in F$ such that for any B-algebra B' and any $u \in G(B')$,

$$\rho(B')(u)(1_{B'} \otimes_B x) = 1_{B'} \otimes_B x\,.$$

We associate to ρ a filtration Fil^ρ by submodules $F_0 \subset F_1 \subset F_2 \subset \cdots$ of F as for a representation of $\mathfrak{g}$ by setting $F_0 = 0$ and defining F_{i+1} to be the preimage in F of $(F/F_i)^{G_T}$.

Remark 8.7. — Let $\rho : G_T \to \mathbb{A}\mathrm{ut}\, \mathcal{F}$ be a representation of G on a quasi-coherent sheaf $\mathcal{F}$ over an affine k-scheme $T = \operatorname{Spec} B$ with structure morphism $f : \operatorname{Spec} B \to \operatorname{Spec} k$. We denote the B-module associated to $\mathcal{F}$ by F as usual. Then we can define an associated representation

$$f_*\rho : G \to \mathbb{A}\mathrm{ut}\, f_*\mathcal{F}$$

of G on $f_*\mathcal{F}$ by forgetting the B-linearity of the coaction

$$F \to (B \otimes_k A) \otimes_B F = B \otimes_k F\,.$$

This defines a functor

$$f_* : \mathbf{Rep}\, G_T \to \mathbf{Rep}\, G$$

from the category of quasi-coherent representations of G_T to the category of quasi-coherent representations of G which is exact and satisfies

$$f_*(\mathcal{F}^{G_T}) = (f_*\mathcal{F})^G .$$

Indeed, both are equal to the kernel of

$$\alpha - \pi : F \to B \otimes_k F$$

where α is the coaction and π is the projection $x \mapsto 1_B \otimes_k x$. Consequently,

$$\mathrm{Fil}^\rho = \mathrm{Fil}^{f_*\rho} .$$

Proposition 8.8. — Let $T = \operatorname{Spec} B$ be an affine k-scheme, F a B-module, $\rho : G_T \to \mathbb{A}\mathrm{ut}\, F$ a representation and $\mathrm{Fil}^\rho = (F_0 \subset F_1 \subset \cdots)$ the filtration associated to ρ as in 8.6. Then the filtration Fil^ρ is exhaustive.

Proof. — Formation of the associated filtration is compatible with taking subrepresentations. By [16, II 2.2.2.2], every element of $f_*\mathcal{F}$ is contained in a finite dimensional subrepresentation; by [5, IV 2.5], the filtration associated to a finite dimensional representation over a field is strictly increasing, hence exhaustive. □

We recall the definition and a first property of the derivative of a representation:

8.9. — Let $T = \operatorname{Spec} B$ be an affine k-scheme and let $\rho : G_T \to \mathbb{A}\mathrm{ut}\, F$ be a representation of a unipotent group G on a B-module F. Then $\mathbf{Lie}(\rho)$ is the representation $B \otimes \mathfrak{g} \to \mathrm{End}(F)$ of $\mathfrak{g}$ induced by $\rho(B)$ and $\rho(B[\epsilon])$, forming a morphism of split short exact sequences of abstract groups as in the following diagram.

$$\begin{tikzcd}
1 \arrow[r] & B \otimes \mathfrak{g} \arrow[r] \arrow[d, dotted, "\mathbf{Lie}(\rho)"'] & G(B[\epsilon]/(\epsilon^2) \arrow[r] \arrow[d, "\rho(B[\epsilon]/(\epsilon^2))"'] & G(B) \arrow[r] \arrow[d, "\rho(B)"'] \arrow[l, bend right] & 1 \\
1 \arrow[r] & \mathrm{End}(F) \arrow[r] & \mathrm{Aut}(B[\epsilon]/(\epsilon^2) \otimes F) \arrow[r] & \mathrm{Aut}(F) \arrow[r] \arrow[l, bend left] & 1
\end{tikzcd}$$

Formation of $\mathbf{Lie}(\rho)$ is compatible with arbitrary base-change; that is, given any B-algebra B', $\mathbf{Lie}(\rho)(B')$ fits into a morphism of split short exact sequences of abstract groups, as follows.

$$\begin{tikzcd}
1 \arrow[r] & B' \otimes \mathfrak{g} \arrow[r] \arrow[d, "\mathbf{Lie}(\rho)(B')"'] & G(B'[\epsilon]/(\epsilon^2) \arrow[r] \arrow[d, "\rho(B'[\epsilon]/(\epsilon^2))"'] & G(B') \arrow[r] \arrow[d, "\rho(B')"'] \arrow[l, bend right] & 1 \\
1 \arrow[r] & \mathrm{End}(B' \otimes F) \arrow[r] & \mathrm{Aut}(B'[\epsilon]/(\epsilon^2) \otimes F) \arrow[r] & \mathrm{Aut}(B' \otimes F) \arrow[r] \arrow[l, bend left] & 1
\end{tikzcd}$$

Proposition 8.10. — Let T be an affine k-scheme, $\rho : G_T \to \mathbb{A}\mathrm{ut}\, F$ a quasi-coherent representation of a unipotent group G, $r = \mathbf{Lie}(\rho)$ the associated representation of the Lie algebra $\mathfrak{g}$ of G. As explained in 8.3, r extends uniquely to a morphism of Lie-algebra-valued functors

$$\mathbb{V}\mathfrak{g}_T^\vee \to \mathbb{E}\mathrm{nd}(F)\,,$$

hence corresponds to a point ϕ of

$$\mathbb{E}\mathrm{nd}(F)(S^\bullet\mathfrak{g}_T^\vee) = \mathrm{End}_{S^\bullet\mathfrak{g}_T^\vee}(S^\bullet\mathfrak{g}_T^\vee \otimes F)\,.$$

On the other hand,

$$\rho \circ \exp : \mathbb{V}\mathfrak{g}_T^\vee \to G_T \to \mathbb{A}\mathrm{ut}\, F$$

corresponds to a point

$$\psi \in \mathrm{Aut}_{S^\bullet\mathfrak{g}_T^\vee}(S^\bullet\mathfrak{g}_T^\vee \otimes F)\,.$$

Then ϕ is locally nilpotent and

$$\psi = 1 + \phi + \frac{\phi^2}{2} + \frac{\phi^3}{3!} + \cdots$$

This is standard when F is a vector sheaf. The present situation requires a more careful argument since the functors $\mathbb{A}\mathrm{ut}\, F$ and $\mathbb{U}_{\mathrm{Fil}}F$ may not be representable. The proof follows (8.11–8.14). We avoid any mention of $\mathbb{U}_{\mathrm{Fil}}F$.

Lemma 8.11. — Let B be a ring, F a module, $n \in \mathbb{N}$. Let $B_n = B[\tau]/\tau^{n+1}$ and $C_n = B[\epsilon_1, \dots, \epsilon_n]/(\epsilon_1^2, \dots, \epsilon_n^2)$. Then the map

$$\alpha : \mathrm{Aut}_{B_n}(B_n \otimes F) \to \mathrm{Aut}_{C_n}(C_n \otimes F)$$

induced by

$$\tau \mapsto \epsilon_1 + \cdots + \epsilon_n$$

is injective.

Proof. — The map $B_n \to C_n$ is injective with image the subring of invariants for the action of S_n which permutes the variables. The Reynolds operator ([14, Ch. 1, Definition 1.5]) for this action provides a splitting of the injection regarded as a map of B-modules. It is thus universally injective. Now given an automorphism ϕ of $B_n \otimes F$, $\phi, \alpha(\phi)$ form a commuting square

$$\begin{tikzcd} B_n \otimes F \arrow[r, hook] \arrow[d, "\phi"'] & C_n \otimes F \arrow[d, "\alpha(\phi)"] \\ B_n \otimes F \arrow[r, hook] & C_n \otimes F \end{tikzcd}$$

in which the horizontal maps are injective, from which it follows that ϕ is uniquely determined by $\alpha(\phi)$. □

8.12. — Let B be a ring containing $\mathbb{Q}$, $G = \operatorname{Spec} A$ an algebraic group over B, and let $\mathfrak{g}$ denote the Lie algebra of G. We let *exp* denote the *formal exponential map*

$$\mathfrak{g} \to G(B[[\tau]])$$

as defined in [5, II §6 no. 3]. Following [5], we denote $exp(v)$ by $e^{\tau v}$ and given a map $B[[\tau]] \to B'$ sending $\tau \mapsto t \in B'$, we denote the image of $e^{\tau v}$ in $G(B')$ by e^{tv}. We recall that *exp* satisfies the following two properties:

(1) the element $e^{\epsilon v}$ of $G(B[\epsilon]/(\epsilon^2))$ determined by the map $\tau \mapsto \epsilon$ is also the image of v under

$$\mathfrak{g} \to G(B[\epsilon]/(\epsilon^2))\,;$$

and

(2) $e^{(\tau+\tau')v} = e^{\tau v}e^{\tau' v}$ in $G(B[[\tau, \tau']])$.

Lemma 8.13. — Continuing with the notation and the situation of 8.12, let $\rho : G \to \mathbb{A}\mathrm{ut}\, F$ be a quasi-coherent representation over $T = \operatorname{Spec} B$ and $r = \mathbf{Lie}(\rho) : \mathfrak{g} \to \operatorname{End}(F)$ its derivative. Fix a vector $v \in \mathfrak{g}$ and write

$$\phi := r(v)\,,$$

$$\psi := \rho(B[[\tau]])(e^{\tau v})\,.$$

Then

$$\psi = \sum \frac{\tau^i \phi^i}{i!}\,.$$

Proof. — The map

$$\operatorname{End}(F) \to \operatorname{Aut}(B[\epsilon]/(\epsilon^2) \otimes F)$$

defined by

$$\sigma \mapsto 1 + \epsilon\sigma$$

is injective with cokernel equal to $\operatorname{Aut}(F)$. Since r is defined by the map of short exact sequences

$$\begin{tikzcd}
 & v \arrow[rr, mapsto, bend left] \arrow[dd, mapsto, bend right] & & e^{\epsilon v} & & \\
0 \arrow[r] & \mathfrak{g} \arrow[rr] \arrow[d, "r"] & & G(B[\epsilon]/(\epsilon^2)) \arrow[r] \arrow[d, "\rho(B[\epsilon]/(\epsilon^2))"] & G(B) \arrow[r] \arrow[d, "\rho(B)"] & 0 \\
0 \arrow[r] & \operatorname{End}(F) \arrow[rr] & & \operatorname{Aut}(B[\epsilon]/(\epsilon^2) \otimes F) \arrow[r] & \operatorname{Aut}(F) \arrow[r] & 0 \\
 & \phi \arrow[rr, mapsto, bend right] & & 1 + \epsilon\phi & &
\end{tikzcd}$$

induced by ρ, it follows that

$$\rho(B[\epsilon]/(\epsilon^2))(e^{\epsilon v}) = 1 + \epsilon\phi\,.$$

Notation as in 8.11, the map

$$B[\epsilon]/(\epsilon^2) \to C_n \text{ defined by } \epsilon \mapsto \epsilon_i$$

gives rise to a commuting square

$$\begin{tikzcd}
e^{\epsilon v} \arrow[rrr, mapsto, bend left=8] \arrow[dd, mapsto, bend right] & & & 1+\epsilon\phi \arrow[dd, mapsto, bend left] \\
& G(B[\epsilon]/(\epsilon^2)) \arrow[r] \arrow[d] & \mathrm{Aut}(B[\epsilon]/(\epsilon^2) \otimes F) \arrow[d] & \\
e^{\epsilon_i v} & G(C_n) \arrow[r] & \mathrm{Aut}(C_n \otimes F) & 1+\epsilon_i\phi
\end{tikzcd}$$

from which it now follows that $\rho(C_n)(e^{\epsilon_i v}) = 1 + \epsilon_i\phi$.

Property (2) of *exp* implies that given nilpotent elements t, t' in a B-algebra B',

$$e^{(t+t')v} = e^{tv}e^{t'v}\,.$$

So the map

$$\tau \mapsto \epsilon_1 + \cdots + \epsilon_n$$

gives rise to a commuting square

$$\begin{tikzcd}
e^{\tau v} \arrow[rr, mapsto, bend left=8] \arrow[ddd, mapsto, bend right] & & \psi \\
& G(B[[\tau]]) \arrow[r] \arrow[d] & \mathrm{Aut}(B[[\tau]] \otimes F) \arrow[d] \\
& G(C_n) \arrow[r] & \mathrm{Aut}(C_n \otimes F) \\
e^{\epsilon_1 v} \cdots e^{\epsilon_n v} \arrow[rr, mapsto] & & (1+\epsilon_1\phi)\cdots(1+\epsilon_n\phi)
\end{tikzcd}$$

in which $e^{\tau v}$ maps to ψ on the upper right and to $(1+\epsilon_1\phi)\cdots(1+\epsilon_n\phi)$ on the lower right as shown. On the other hand, in the notation of 8.11, the vertical map on the right factors through

$$\mathrm{Aut}(B_n \otimes F) \to \mathrm{Aut}(C_n \otimes F)\,.$$

By 8.11, this map is injective. Since in $\mathrm{Aut}(C_n \otimes F)$,

$$\begin{aligned}(1+\epsilon_1\phi)\cdots(1+\epsilon_n\phi) \\ &= \sum_i (\text{sum of } i\text{-fold products of distinct } \epsilon_j\text{'s})\phi^i \\ &= \sum_i \frac{(\epsilon_1\cdots\epsilon_n)^i}{i!}\phi^i \end{aligned} \tag{8.13.1}$$

this map sends

$$\sum \frac{\tau^i\phi^i}{i!} \text{ to } (1+\epsilon_1\phi)\cdots(1+\epsilon_n\phi)\,.$$

It follows that ψ maps to $\sum \frac{\tau^i\phi^i}{i!}$ in $\mathrm{Aut}(B_n \otimes F)$ which concludes the proof of the lemma. □

8.14. — Returning to the situation of the proposition, let

$$B' := B \otimes S^{\bullet}\mathfrak{g}^{\vee}$$

and let $v \in \mathfrak{g}_{B'}$ be the universal section. Then $r(B')(v) = \phi$ as defined in the proposition. By 8.13,

$$\rho(B'[[\tau]])(e^{\tau v}) = \sum \frac{\tau^i\phi^i}{i!}\,.$$

By [5, IV §2 4.1],

$$exp : \mathfrak{g}_{B'} \to G(B'[[\tau]])$$

factors through $G(B'[\tau])$. The situation is summarized in the following diagram.

$$\begin{tikzcd}
v \arrow[rr, mapsto, bend left] \arrow[dddd, mapsto, bend right] & & \phi \\
& \mathfrak{g}_{B'} \arrow[r] \arrow[d] & \mathrm{End}(B' \otimes F) \\
& G(B'[\tau]) \arrow[r] \arrow[d] & \mathrm{Aut}(B'[\tau] \otimes F) \arrow[d] \\
& G(B'[[t]]) \arrow[r] & \mathrm{Aut}(B'[[\tau]] \otimes F) \\
e^{\tau v} \arrow[rr, mapsto] & & \sum \frac{\tau^i\phi^i}{i!}
\end{tikzcd}$$

This implies that $\sum \frac{\tau^i\phi^i}{i!}$ is in $\mathrm{Aut}(B'[\tau] \otimes F)$ and in particular that ϕ is locally nilpotent. Let exp denote the global exponential map

$$\mathbb{V}\mathfrak{g}_B^{\vee} \to G_B\,.$$

Then by definition, $\exp(B')$ is the composite below.

$$\begin{tikzcd}
\mathfrak{g}_{B'} \arrow[r] & G(B'[\tau]) \arrow[r] & G(B') \\
& \tau \arrow[r, mapsto] & 1
\end{tikzcd}$$

Finally, ψ, as defined in the proposition, equals $\rho(B')(\exp(B')(v))$. So we consider the commuting square

$$\begin{tikzcd}
e^{\tau v} \arrow[rr, mapsto, bend left] \arrow[ddd, mapsto, bend right] & & \sum \frac{\tau^i \phi^i}{i!} & \\
& G(B'[\tau]) \arrow[r] \arrow[d] & \mathrm{Aut}(B'[\tau] \otimes F) \arrow[d] & \tau \arrow[d, mapsto, bend left] \\
& G(B') \arrow[r] & \mathrm{Aut}(B' \otimes F) & 1 \\
e^{v} \arrow[rr, mapsto, bend right] & & \psi &
\end{tikzcd}$$

from which it follows that

$$\psi = \sum \frac{\phi^i}{i!}$$

as claimed.

Corollary 8.15. — Let $T = \operatorname{Spec} B$ be an affine k-scheme, $\rho : G_T \to \mathbb{A}\mathrm{ut}\, F$ a quasi-coherent representation, $r = \mathbf{Lie}(\rho)$. Let F^{G_T} denote the module of invariants of ρ (8.6) and let F^0 denote the 0-eigenspace of r (1.4). Then

$$F^{G_T} = F^0 .$$

The proof follows (8.16–8.17).

Lemma 8.16. — Let B be a ring, F a B-module, $\phi \in \operatorname{End}(F)$ and suppose ϕ is locally nilpotent. Let $\psi = \sum \frac{\phi^i}{i!}$ and let $x \in F$. Then $\phi x = 0$ if and only if $\psi x = x$.

Proof. — If $\phi x = 0$ then

$$\begin{aligned}
\psi x &= x + \phi(x) + \frac{\phi^2(x)}{2} + \cdots \\
&= x .
\end{aligned}$$

If $\psi(x) = x$ then

$$\begin{aligned}
\phi x &= (\log \psi) x \\
&= (\log(1 + (\psi - 1)))x \\
&= ((\psi - 1) - \frac{(\psi - 1)^2}{2} + \frac{(\psi - 1)^3}{3} - \cdots)x \\
&= 0 .
\end{aligned}$$

□

8.17. — Returning to the proof of the corollary, suppose $x \in F^{G_T}$, let $v \in B \otimes \mathfrak{g}$ and let $u = \exp(B)(v) \in G(B)$. Then

$$\rho(B)(u)(x) = x$$

and

$$\rho(B)(u) = \sum \frac{r(v)^i}{i!}$$

so

$$r(v)(x) = 0$$

by the lemma.

Conversely, suppose $x \in F^0$, let B' be a B-algebra, let $u \in G(B')$ and let $v = \log(B')(u) \in B' \otimes \mathfrak{g}$. As explained in 8.8, $r(B')(v)(x) = 0$. Thus

$$\begin{aligned}\rho(B')(u)(x) &= \sum \frac{r(B')(v)^i}{i!}(x)\\ &= x\,.\end{aligned}$$

Corollary 8.18. — Let T be an affine k-scheme, $\rho : G_T \to \mathbb{A}\mathrm{ut}\, F$ a quasi-coherent representation, $r = \mathbf{Lie}(\rho)$. Then r is locally nilpotent.

Proof. — It follows from the fact that **Lie** is exact, from 8.8 and from 8.15 that the canonical filtration associated to ρ witnesses the local nilpotence of r. □

Definition 8.19. — Let $\mathbb{REP}(G)$ denote the full stack of quasi-coherent representations over the category of affine k-schemes. Thus an object is a triple (T, F, ρ), $T = \operatorname{Spec} B$ an affine k-scheme, F a B-module, $\rho : G_T \to \mathbb{A}\mathrm{ut}\, F$ a representation; and a morphism

$$(T', F', \rho') \to (T, F, \rho)$$

is a pair (f, ϕ), f a morphism $T' \to T$ and ϕ a morphism of representations $f^*\rho \to \rho'$.

Let $\mathbb{REP}^{\mathrm{nil}}(\mathfrak{g})$ denote the fibered category of locally nilpotent representations of $\mathfrak{g}$: an object is a triple (T, F, r) with $T = \operatorname{Spec} B$ an affine k-scheme, F a B-module, $r : B \otimes \mathfrak{g} \to \operatorname{End}(F)$ a locally nilpotent representation; and a morphism

$$(T', F', r') \to (T, F, r)$$

is a pair (f, ϕ), $f : T' \to T$ a morphism of k-schemes and $\phi : f^*r \to r'$ a morphism of representations.

Theorem 8.20. — The functor

$$\mathbf{Lie} : \mathbb{REP}(G) \to \mathbb{REP}^{\mathrm{nil}}(\mathfrak{g})$$

sending a representation to its derivative at the identity is an isomorphism of fibered categories.

The proof is in paragraphs 8.21-8.23.

8.21. — Compatibility with Cartesian morphisms is clear; so it is enough to fix an affine k-scheme $T = \operatorname{Spec} B$ and show that

$$\mathbf{Lie}(T) : \mathbb{REP}(G)(T) \to \mathbb{REP}^{\mathrm{nil}}(\mathfrak{g})(T)$$

is an isomorphism. We begin by constructing an inverse

$$\Psi : \mathrm{Ob}(\mathbb{REP}^{\mathrm{nil}}(\mathfrak{g})(T)) \to \mathrm{Ob}(\mathbb{REP}(G)(T))$$

to $\mathrm{Ob}(\mathbf{Lie}(T))$.

Let Fil be an exhaustive increasing filtration indexed by $\mathbb{N}$ on a B-module F and let $\mathfrak{n}_{\mathrm{Fil}}(F)$ denote the submodule of $\mathrm{End}(F)$ consisting of those endomorphisms which preserve the filtration and induce 0 on the associated graded. (Note, however, that if F is not finitely presented, formation of $\mathfrak{n}_{\mathrm{Fil}}(F)$ may not be compatible with flat base-change; so it is better not to think of it as a quasi-coherent sheaf.) Given $v_1, v_2 \in \mathfrak{n}_{\mathrm{Fil}}(F)$ and $s \in \mathbb{N}$, all but finitely many terms of $v_1 \star v_2$ are in $(\mathfrak{n}_{\mathrm{Fil}}(F))^{(s)}$; hence $v_1 \star v_2$ is a locally finite sum. Moreover,

$$\sum \frac{(v_1 \star v_2)^i}{i!} = (\sum \frac{v_1^i}{i!})(\sum \frac{v_2^i}{i!}) \tag{8.21.2}$$

([3, §6.4]).

8.22. — Let $r : \mathfrak{g}_T \to \mathrm{End}(F)$ be a locally nilpotent representation on a B-module F and let $T' = \operatorname{Spec} B'$ be an arbitrary affine T-scheme. Then by 8.3, $r(T')$ is locally nilpotent. Let $\mathrm{Fil}(B')$ denote the associated filtration on $B' \otimes F$. There is thus a factorization of $r(T')$ as

$$B' \otimes \mathfrak{g} \to \mathfrak{n}_{\mathrm{Fil}(B')}(B' \otimes F) \subset \mathrm{End}_{B'}(B' \otimes F)\,.$$

Now given $u \in G(B')$, define

$$\Psi(r) : G_T \to \mathbb{A}\mathrm{ut}\, F$$

by

$$\Psi(r)(B')(u) = \sum_i \frac{(r(B') \log(B')(u))^i}{i!}\,.$$

By 8.21, $\Psi(r)$ is a morphism of group-valued functors. To check that

$$\mathbf{Lie}(T) \circ \Psi = \mathrm{id}_{\mathrm{Ob}(\mathbb{REP}^{\mathrm{nil}}(\mathfrak{g})(T))}$$

fix a locally nilpotent representation $r : \mathfrak{g}_T \to \mathrm{End}(F)$ and consider the following diagram.

$$\begin{tikzcd}
 & v \arrow[rr, mapsto] \arrow[ddd, bend right] & & e^{\epsilon v} \arrow[ddddl, bend left] & \\
0 \arrow[r] & B \otimes \mathfrak{g} \arrow[r] \arrow[d, "r"] & G(B[\epsilon]/(\epsilon^2)) \arrow[r] \arrow[d, "\Psi(r)(B[\epsilon])"] & G(B) \arrow[r] \arrow[d, "\Psi(r)(B)"] \arrow[l, bend right] & 0 \\
0 \arrow[r] & \mathrm{End}(F) \arrow[r] & \mathrm{Aut}(B[\epsilon]/(\epsilon^2) \otimes F) \arrow[r] & \mathrm{Aut}(F) \arrow[r] \arrow[l, bend left] & 0 \\
 & r(v) \arrow[r, mapsto] & 1 + \epsilon r(v) & & \\
 & & \sum \frac{(r(B[\epsilon])(\log(B[\epsilon])(e^{\epsilon v}))^i}{i!} & &
\end{tikzcd}$$

Our goal being to check that the square on the left commutes, we compute:

$$\sum \frac{(r(B[\epsilon])(\log(B[\epsilon])(e^{\epsilon v}))^i}{i!} = \sum \frac{(\epsilon r(B)(v))^i}{i!} = 1 + \epsilon r(v)\ .$$

To check that

$$\Psi \circ \mathbf{Lie}(T) = \mathrm{id}_{\mathrm{Ob}(\mathbb{REP}(G)(T))}\ ,$$

fix a representation $\rho : G_T \to \mathbb{A}\mathrm{ut}\, F$, an affine T-scheme $T' = \operatorname{Spec} B'$ and a point $u \in G(B')$. We then have

$$\Psi(\mathbf{Lie}(T)(\rho))(T')(u) = \sum \frac{((\mathbf{Lie}\ \rho)(B') \log(B')(u))^i}{i!} = \rho(T')(u)$$

by 8.10.

8.23. — Let inv denote the functor which takes a quasi-coherent representation of G to its module of invariants (8.6), and let null denote the functor which takes a locally nilpotent quasi coherent representation (8.5) of $\mathfrak{g}$ to its 0-eigenspace (1.4). Let $T = \operatorname{Spec} B$ denote an arbitrary affine k-scheme, as above, and let $\mathbb{QCOH}(T)$ denote the category of B-modules. Then in the notation of 8.19, Corollary 8.15 states that the triangle of functors

$$\begin{tikzcd}
\mathbb{REP}(G)(T) \arrow[rr, "\mathbf{Lie}(T)"] \arrow[dr, "\mathrm{inv}"'] & & \mathbb{REP}^{\mathrm{nil}}(\mathfrak{g})(T) \arrow[dl, "\mathrm{null}"] \\
 & \mathbb{QCOH}(T) &
\end{tikzcd}$$

commutes. We've shown that the map of object classes $\mathrm{Ob}(\mathbf{Lie}(T))$ admits an inverse. Finally, note that if one fixes a particular construction of duals and tensor products inside the category of quasi-coherent sheaves on T, it makes sense to say that $\mathrm{Ob}(\mathbf{Lie}(T))$ respects duals and tensor products.

This gives us, for each ρ, ρ', an equality of sets

$$\begin{aligned}\mathrm{Hom}(\rho,\rho') &= (\rho^\vee \otimes \rho')^{G_T} \\ &= \mathbf{Lie}(\rho^\vee \otimes \rho')^0 \\ &= ((\mathbf{Lie}\ \rho)^\vee \otimes \mathbf{Lie}\ \rho')^0 \\ &= \mathrm{Hom}(\mathbf{Lie}\ \rho, \mathbf{Lie}\ \rho')\end{aligned}$$

compatible with identity elements and composition, hence an isomorphism of categories as claimed.

9. Glueing of flag representations

The $(n+1)^{\mathrm{st}}$ moduli space M^{nd}_{n+1} is naturally fibered over a certain closed subscheme M^{cnd}_n of $M^{\mathrm{nd}}_n \times_{p_2, M_{n-1}, p_1} M^{\mathrm{nd}}_n$. In this section we give a construction as well as a modular interpretation of M^{cnd}_n.

We work over a field k of characteristic zero and work interchangeably with representations of a fixed unipotent group G and with nilpotent representations of its Lie algebra $\mathfrak{g}$.

9.1. — There are two natural maps

$$p_1, p_2 : \mathcal{M}^{\mathrm{fl}}_n \rightrightarrows \mathcal{M}^{\mathrm{fl}}_{n-1}$$

given by

$$p_1(r) = r_{n-1}\ ,\ p_2(r) = r^1\ .$$

Recall that

$$\mathcal{M}^{\mathrm{fl}}_n \underset{p_2, \mathcal{M}^{\mathrm{fl}}_{n-1}, p_1}{\times} \mathcal{M}^{\mathrm{fl}}_n$$

may be described as the stack whose objects are 4-ples (T, r, r', ϕ), T an affine k-scheme, r, r' flag representations $\mathfrak{g}_T \to \mathscr{E}nd\,\mathcal{E}$, ϕ an isomorphism $r^1 \to r'_{n-1}$. A morphism

$$(U, s, s', \chi) \to (T, r, r', \phi)$$

consists of a morphism $f : U \to T$ and isomorphisms $f^*r \to s, f^*r' \to s'$ such that the square

$$\begin{tikzcd} f^*(r^1) \arrow[r, equal] \arrow[d, "f^*\phi"'] & (f^*r)^1 \arrow[r] & s^1 \arrow[d, "\chi"] \\ f^*(r'_{n-1}) \arrow[r, equal] & (f^*r')_{n-1} \arrow[r] & s'_{n-1} \end{tikzcd}$$

commutes. Then there is a natural map

$$p = (p_1, p_2) : \mathcal{M}^{\mathrm{fl}}_{n+1} \to \mathcal{M}^{\mathrm{fl}}_n \underset{\mathcal{M}^{\mathrm{fl}}_{n-1}}{\times} \mathcal{M}^{\mathrm{fl}}_n$$

sending r over T to (T, r_n, r^1, ϕ) where ϕ is the canonical isomorphism $(r_n)^1 \to (r^1)_{n-1}$.

Definition-Proposition 9.2. — In the notation of 9.1, the image of p is the same in the indiscrete and fppf topologies, and is a closed substack of $\mathcal{M}^{\mathrm{fl}}_n \times_{\mathcal{M}^{\mathrm{fl}}_{n-1}} \mathcal{M}^{\mathrm{fl}}_n$. Denote it by $\mathcal{M}^{\mathrm{cfl}}_n$. We call an object in the essential image of p *a compatible pair of flag representations.*

The proof is in paragraphs 9.3–9.6.

9.3. — Consider first the indiscrete topology in which the image is just the essential image of the functor. An object (T, r, r', ϕ) of $\mathcal{M}^{\mathrm{fl}}_n \times_{\mathcal{M}^{\mathrm{fl}}_{n-1}} \mathcal{M}^{\mathrm{fl}}_n$ gives rise to a two step extension

$$0 \to r_1 \to r \to r' \to r'^{n-1} \to 0$$

hence to a class $o(T, r, r', \phi) \in \mathrm{Ext}^2_{\mathrm{Rep}\, G_T}(r'^{n-1}, r_1)$. Then (T, r, r', ϕ) is in the essential image of p if and only if $o(T, r, r', \phi) = 0$. Indeed, (T, r, r', ϕ) is in the essential image of p if and only if there exists a quasi-coherent (necessarily $n+1$-dimensional, flag) representation r'' over T and isomorphisms $\alpha : r \to r''_n$, $\beta : r''^1 \to r'$ forming a commuting pentagon:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & & \\
 & r_1 \arrow[rr, equal] \arrow[d] & & r_1 \arrow[d] & & \\
0 \arrow[r] & r \arrow[rr, "\alpha"] \arrow[d] & & r'' \arrow[r] \arrow[dd, "\beta"'] & r'^{n-1} \arrow[r] \arrow[dd, equal] & 0 \\
 & r^1 \arrow[dr, "\sim", "\phi"'] \arrow[dd] & & & & \\
0 \arrow[rr] & & r'_{n-1} \arrow[r] & r' \arrow[r] \arrow[d] & r'^{n-1} \arrow[r] & 0 \\
 & 0 & & 0 & &
\end{tikzcd}$$

(by which the rest of the (commutative) diagram is uniquely determined). The vertical extension on the left gives rise to a long exact sequence

$$\cdots \to \mathrm{Ext}^1(r'^{n-1}, r) \xrightarrow{\pi_*} \mathrm{Ext}^1(r'^{n-1}, r^1) \xrightarrow{\delta} \mathrm{Ext}^2(r'^{n-1}, r_1) \to \cdots$$

Under π_* followed by δ the class of

$$0 \to r \to r'' \to r'^{n-1} \to 0$$

goes to

$$0 \to r^1 \to r' \to r'^{n-1} \to 0$$

goes to

$$0 \to r_1 \to r \to r' \to r'^{n-1} \to 0\,.$$

Thus (r'', α, β) as above exists if and only if the class of the two-step extension is zero.

9.4. — Suppose now that $(r, r', \phi) \in (\mathcal{M}_n^{\mathrm{fl}} \times_{\mathcal{M}_{n-1}^{\mathrm{fl}}} \mathcal{M}_n^{\mathrm{fl}})(T)$ is fppf-locally in the essential image of p. Then there is an fppf map $f : U \to T$ such that $f^*(r, r', \phi)$ is compatible. Hence the class of

$$0 \to (f^*r)_1 \to f^*r \to f^*r' \to (f^*r')^{n-1} \to 0$$

in $\mathrm{Ext}^2((f^*r')^{n-1}, (f^*r)_1)$ is zero. This corresponds to the class of

$$0 \to f^*(r_1) \to f^*r \to f^*r' \to f^*(r'^{n-1}) \to 0$$

in $\mathrm{Ext}^2(f^*(r'^{n-1}), f^*(r_1))$ which is the image of $o(r, r', \phi)$ under

$$\mathrm{Ext}^2(f^*(r'^{n-1}), f^*(r_1)) \xleftarrow{\cong} f^*\mathrm{Ext}^2(r'^{n-1}, r_1) \leftarrow \mathrm{Ext}^2(r'^{n-1}, r_1)\,.$$

It follows that $o(r, r', \phi) = 0$, hence that (T, r, r', ϕ) is in the essential image of p by 9.3. This shows that the image is the same in the indiscrete and fppf topologies.

9.5. — Suppose $(r, r', \phi) \in (\mathcal{M}_n^{\mathrm{fl}} \times_{\mathcal{M}_{n-1}^{\mathrm{fl}}} \mathcal{M}_n^{\mathrm{fl}})(T)$ and denote by L'_n, L_1 the line sheaves corresponding to r'^{n-1}, r_1 respectively. We claim that

$$\mathrm{Ext}^2(r'^{n-1}, r_1) = \mathrm{H}^2(G, k) \otimes L_n'^{\vee} \otimes L_1.$$

Proof. — We denote by inv_* the functor which takes a quasi-coherent representation to its sheaf of invariants, and by inv^* the functor which endows a quasi-coherent sheaf with the trivial group action. We have

$$\begin{aligned}\mathrm{Ext}^2(r'^{n-1}, r_1) &= \mathrm{H}^2(G_T, (r'^{n-1})^{\vee} \otimes r_1) \\ &= R^2\mathrm{inv}_*\mathrm{inv}^*(L_n'^{\vee} \otimes L_1)\end{aligned}$$

since both representations are trivial

$$= R^2\mathrm{inv}_*(1_T) \otimes L_n'^{\vee} \otimes L_1$$

by the projection formula (here 1_T denotes the trivial representation on $\mathcal{O}_T$)

$$= \mathrm{H}^2(G,k) \otimes L_n'^{\vee} \otimes L_1$$

by compatibility with flat base change. □

9.6. — We continue to work with an arbitrary affine k-scheme $T = \operatorname{Spec} B$ and an object (r, r', ϕ) of $\mathcal{M}_n^{\mathrm{fl}} \times_{\mathcal{M}_{n-1}^{\mathrm{fl}}} \mathcal{M}_n^{\mathrm{fl}}(T)$ as above. By 9.5, $\mathrm{Ext}^2({r'}^{n-1}, r_1)$ is locally free of finite rank. So we can apply 7.10 to the map of B-modules $B \to \mathrm{Ext}^2({r'}^{n-1}, r_1)$ given by $1 \mapsto o(T, r, r', \phi)$ to obtain a closed subscheme $Z \subset T$ representing the vanishing locus of $o(T, r, r', \phi)$. We claim that there is a Cartesian square as follows.

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & \mathcal{M}_n^{\mathrm{cfl}} \arrow[d] \\ T \arrow[r] & \mathcal{M}_n^{\mathrm{fl}} \underset{\mathcal{M}_{n-1}^{\mathrm{fl}}}{\times} \mathcal{M}_n^{\mathrm{fl}} \end{tikzcd}$$

Indeed, given $f : T' \to T$, $f^*(r, r', \phi)$ forms a compatible pair if and only if

$$0 = o(f^*(r, r', \phi)) = f^*(o(r, r', \phi))$$

if and only if f factors through Z. We've shown that the pullback of our map $\mathcal{M}_n^{\mathrm{cfl}} \to \mathcal{M}_n^{\mathrm{fl}} \times_{\mathcal{M}_{n-1}^{\mathrm{fl}}} \mathcal{M}_n^{\mathrm{fl}}$ along an arbitrary morphism $T \to \mathcal{M}_n^{\mathrm{fl}} \times_{\mathcal{M}_{n-1}^{\mathrm{fl}}} \mathcal{M}_n^{\mathrm{fl}}$ from an affine k-scheme is a closed immersion. It follows that the former is itself a closed immersion, which completes the proof of 9.2.

Definition 9.7. — We define $\mathcal{M}_n^{\mathrm{cnd}}$, **the stack of compatible pairs of nondegenerate nilpotent representations of dimension** n, by the Cartesian square below.

$$\begin{tikzcd} \mathcal{M}_n^{\mathrm{cnd}} \arrow[r, hook] \arrow[d, hook] & \mathcal{M}_n^{\mathrm{cfl}} \arrow[d, hook] \\ \mathcal{M}_n^{\mathrm{nd}} \underset{\mathcal{M}_{n-1}^{\mathrm{nd}}}{\times} \mathcal{M}_n^{\mathrm{nd}} \arrow[r, hook] & \mathcal{M}_n^{\mathrm{fl}} \underset{\mathcal{M}_{n-1}^{\mathrm{fl}}}{\times} \mathcal{M}_n^{\mathrm{fl}} \end{tikzcd}$$

We define M_n^{cnd}, **the moduli space of compatible pairs of nondegenerate nilpotent representations of dimension** n, to be the subfunctor of $M_n^{\mathrm{nd}} \times_{M_{n-1}^{\mathrm{nd}}} M_n^{\mathrm{nd}}$ whose T-valued points are those pairs (r, r') such that given any square as follows,

$$\begin{tikzcd} T' \arrow[r, "(s,s',\phi)"] \arrow[d] & \mathcal{M}_n^{\mathrm{nd}} \underset{\mathcal{M}_{n-1}^{\mathrm{nd}}}{\times} \mathcal{M}_n^{\mathrm{nd}} \arrow[d] \\ T \arrow[r, "(r,r')"'] & M_n^{\mathrm{nd}} \underset{M_{n-1}^{\mathrm{nd}}}{\times} M_n^{\mathrm{nd}} \end{tikzcd}$$

(s, s', ϕ) is in (the essential image of) $\mathcal{M}_n^{\mathrm{cnd}}$.

Proposition 9.8. — The functor M_n^{cnd} is a closed algebraic subspace of $M_n^{\mathrm{nd}} \times_{M_{n-1}^{\mathrm{nd}}} M_n^{\mathrm{nd}}$. The natural map $M_{n+1}^{\mathrm{nd}} \to M_n^{\mathrm{nd}} \times_{M_{n-1}^{\mathrm{nd}}} M_n^{\mathrm{nd}}$ factors through M_n^{cnd} as shown.

$$\begin{tikzcd} M_{n+1}^{\mathrm{nd}} \arrow[d] & \\ M_n^{\mathrm{cnd}} \arrow[r, hook] & M_n^{\mathrm{nd}} \underset{M_{n-1}^{\mathrm{nd}}}{\times} M_n^{\mathrm{nd}} \end{tikzcd}$$

The proof follows (9.9–9.12)

9.9. — The morphism

$$\mathcal{M}_n^{\mathrm{nd}} \times_{\mathcal{M}_{n-1}^{\mathrm{nd}}} \mathcal{M}_n^{\mathrm{nd}} \to \mathcal{M}_n^{\mathrm{nd}} \times_{M_{n-1}^{\mathrm{nd}}} \mathcal{M}_n^{\mathrm{nd}}$$

is represented by $\mathbb{I}\mathrm{som}(r^1, r'_{n-1})$. That is, suppose $T \to \mathcal{M}_n^{\mathrm{nd}} \times_{M_{n-1}^{\mathrm{nd}}} \mathcal{M}_n^{\mathrm{nd}}$ corresponds to the pair of representations (r, r') with r^1, r'_{n-1} fppf-locally isomorphic and consider the fibered product below.

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \mathcal{M}_n^{\mathrm{nd}} \underset{\mathcal{M}_{n-1}^{\mathrm{nd}}}{\times} \mathcal{M}_n^{\mathrm{nd}} \arrow[d] \\ T \arrow[r] & \mathcal{M}_n^{\mathrm{nd}} \underset{M_{n-1}^{\mathrm{nd}}}{\times} \mathcal{M}_n^{\mathrm{nd}} \end{tikzcd}$$

Objects of Y are 6-tuples $(T', f, s, s', \phi, \psi)$, T' a k-scheme, f a map $T' \to T$, $(s, s', \phi) \in (\mathcal{M}_n^{\mathrm{nd}} \times_{\mathcal{M}_{n-1}^{\mathrm{nd}}} \mathcal{M}_n^{\mathrm{nd}})(T')$ and ψ an isomorphism $f^*(r, r') \to (s, s')$. Given two objects

$$a_1 = (T', f_1, s_1, s_1', \phi_1, \psi_1)\,,\ a_2 = (T', f_2, s_2, s_2', \phi_2, \psi_2)$$

over T', there is exactly one isomorphism $a_2 \to a_1$ over $\mathrm{id}_{T'}$ if $f_1 = f_2$ and the induced isomorphism

$$(s_2, s_2') \to (s_1, s_1')$$

respects ϕ_1, ϕ_2 and no isomorphisms otherwise. Then, on the one hand, Y is equivalent to the full subcategory consisting of those objects such that

$$(s, s') = f^*(r, r') \text{ and } \phi = \mathrm{id}_{f^*(r,r')}\,,$$

which, on the other hand, is clearly equivalent to $\mathbb{I}\mathrm{som}_{T'}(r^1, r_{n-1}')$.

Since r^1, r_{n-1}' are fppf-locally isomorphic, $\mathbb{I}\mathrm{som}_{T'}(r^1, r_{n-1}')$ is an fppf-torsor under $\mathbb{A}\mathrm{ut}\, r^1$. Hence, in particular, the morphism

$$\mathcal{M}_n^{\mathrm{nd}} \times_{\mathcal{M}_{n-1}^{\mathrm{nd}}} \mathcal{M}_n^{\mathrm{nd}} \to \mathcal{M}_n^{\mathrm{nd}} \times_{M_{n-1}^{\mathrm{nd}}} \mathcal{M}_n^{\mathrm{nd}}$$

is flat and locally of finite presentation.

9.10. — The inertia stack of $\mathcal{M}_n^{\mathrm{nd}} \times_{M_{n-1}^{\mathrm{nd}}} \mathcal{M}_n^{\mathrm{nd}}$ is flat and locally of finite presentation. Indeed, if $(r, r') \in (\mathcal{M}_n^{\mathrm{nd}} \times_{M_{n-1}^{\mathrm{nd}}} \mathcal{M}_n^{\mathrm{nd}})(T)$ then

$$\mathbb{A}\mathrm{ut}(r, r') = \mathbb{A}\mathrm{ut}\, r \times_T \mathbb{A}\mathrm{ut}\, r'$$

is flat and locally of finite presentation. Moreover,

$$M_n^{\mathrm{nd}} \times_{M_{n-1}^{\mathrm{nd}}} M_n^{\mathrm{nd}} = \pi_0^{\mathrm{fppf}}(\mathcal{M}_n^{\mathrm{nd}} \underset{M_{n-1}^{\mathrm{nd}}}{\times} \mathcal{M}_n^{\mathrm{nd}})\,.$$

It follows from Corollary 6.7 that the map

$$\mathcal{M}_n^{\mathrm{nd}} \times_{M_{n-1}^{\mathrm{nd}}} \mathcal{M}_n^{\mathrm{nd}} \to M_n^{\mathrm{nd}} \times_{M_{n-1}^{\mathrm{nd}}} M_n^{\mathrm{nd}}$$

is flat and locally of finite presentation.

CLAIM 9.11. — There exists a map $\mathcal{M}_n^{\mathrm{cnd}} \to M_n^{\mathrm{cnd}}$ which forms a Cartesian square as follows.

$$\begin{tikzcd}
\mathcal{M}_n^{\mathrm{cnd}} \arrow[r] \arrow[d, hook] & M_n^{\mathrm{cnd}} \arrow[d, hook] \\
\mathcal{M}^{\mathrm{nd}} \underset{\mathcal{M}_{n-1}^{\mathrm{nd}}}{\times} \mathcal{M}_n^{\mathrm{nd}} \arrow[r] & M_n^{\mathrm{nd}} \underset{M_{n-1}^{\mathrm{nd}}}{\times} M_n^{\mathrm{nd}}
\end{tikzcd}$$

Proof. — Let $\mathcal{Y}$ denote the fibered product below.

$$\begin{tikzcd}
\mathcal{Y} \arrow[r] \arrow[d, hook] & M_n^{\mathrm{cnd}} \arrow[d, hook] \\
\mathcal{M}^{\mathrm{nd}} \underset{\mathcal{M}_{n-1}^{\mathrm{nd}}}{\times} \mathcal{M}_n^{\mathrm{nd}} \arrow[r] & M_n^{\mathrm{nd}} \underset{M_{n-1}^{\mathrm{nd}}}{\times} M_n^{\mathrm{nd}}
\end{tikzcd}$$

$\mathcal{Y}$ is the full subcategory of $\mathcal{M}_n^{\mathrm{nd}} \times_{\mathcal{M}_{n-1}^{\mathrm{nd}}} \mathcal{M}_n^{\mathrm{nd}}$ consisting of those objects whose image in $M_n^{\mathrm{nd}} \times_{M_{n-1}^{\mathrm{nd}}} M_n^{\mathrm{nd}}$ lies in M_n^{cnd}. Thus $\mathcal{M}_n^{\mathrm{cnd}}, \mathcal{Y}$ are both full subcategories, and we claim that they are in fact equal, the inclusion $\mathcal{Y} \subset \mathcal{M}_n^{\mathrm{cnd}}$ being clear.

For the reverse inclusion we are to consider a square

$$\begin{tikzcd}
T' \arrow[rrrr, "{(s,s',\chi)}"] \arrow[d] \arrow[dr, "{(r,r',\phi)}"] & & & & \mathcal{M}_n^{\mathrm{nd}} \underset{\mathcal{M}_{n-1}^{\mathrm{nd}}}{\times} \mathcal{M}_n^{\mathrm{nd}} \arrow[d] \\
T \arrow[r] & \mathcal{M}_n^{\mathrm{cnd}} \arrow[rr] & & \mathcal{M}_n^{\mathrm{nd}} \underset{\mathcal{M}_{n-1}^{\mathrm{nd}}}{\times} \mathcal{M}_n^{\mathrm{nd}} \arrow[r] & M_n^{\mathrm{nd}} \underset{M_{n-1}^{\mathrm{nd}}}{\times} M_n^{\mathrm{nd}}
\end{tikzcd}$$

and to show that $(s, s', \chi) \in \mathcal{M}_n^{\mathrm{cnd}}(T')$. After possibly replacing T' by an fppf cover, we may assume $s \cong r, s' \cong r'$. Fixing isomorphisms as in the diagram below,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & & & \\
 & s_1 \arrow[r, "\cong"] \arrow[d] & r_1 \arrow[d] & & & & \\
 & s \arrow[r, "\cong"] \arrow[d] & r \arrow[d] & & & & \\
 & s^1 \arrow[r, "\cong"] \arrow[ddd] & r^1 \arrow[dr, "\phi"] \arrow[ddd] & & & & \\
0 \arrow[rrr] & & & r'_{n-1} \arrow[r] \arrow[d, "\cong"] & r' \arrow[r] \arrow[d, "\cong"] & r'^{n-1} \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[rrr] & & & s'_{n-1} \arrow[r] & s' \arrow[r] & s'^{n-1} \arrow[r] & 0 \\
 & 0 & 0 & & & &
\end{tikzcd}$$

we claim that $(s, s', s^1 \to r^1 \to r'_{n-1} \to s'_{n-1})$ is a compatible pair. Indeed, the square

$$\begin{tikzcd}
\mathrm{Ext}^1(s'^{n-1}, r^1) \arrow[r, "\cong"] \arrow[d, "\delta"'] & \mathrm{Ext}^1(r'^{n-1}, r^1) \arrow[d, "\delta"] \\
\mathrm{Ext}^2(s'^{n-1}, r_1) \arrow[r, "\cong"'] & \mathrm{Ext}^2(r'^{n-1}, r_1)
\end{tikzcd}$$

commutes and the arrow at the top sends the class of

$$0 \to r^1 \to s' \to s'^{n-1} \to 0$$

to the class of

$$0 \to r^1 \to r' \to r'_{n-1} \to 0$$

from which it follows that $(r, s', r^1 \to r'_{n-1} \to s'_{n-1})$ is a compatible pair; the next step follows similarly from commutativity of the square below.

$$\begin{tikzcd}
\mathrm{Ext}^1(s'^{n-1}, s^1) \arrow[r, "\cong"] \arrow[d] & \mathrm{Ext}^1(s'^{n-1}, r^1) \arrow[d] \\
\mathrm{Ext}^2(s'^{n-1}, s_1) \arrow[r, "\cong"'] & \mathrm{Ext}^2(s'^{n-1}, r_1)
\end{tikzcd}$$

Finally, $s^1 \to r^1 \to r'_{n-1} \to s'_{s-1}$ differs from χ by an automorphism of s^1 from which it follows that (s, s', χ) is compatible, again by a similar argument. □

9.12. — We've shown that the inclusion

$$M_n^{\mathrm{cnd}} \hookrightarrow M_n^{\mathrm{nd}} \times_{M_{n-1}^{\mathrm{nd}}} M_n^{\mathrm{nd}}$$

is a closed immersion by checking fppf locally on the target. Finally, the factorization follows from the universal mapping property of the fppf sheaf associated to a stack.

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