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LOCAL RIGIDITY OF ASPHERICAL THREE-MANIFOLDS

by Pierre DERBEZ (*)

ABSTRACT. — In this paper we construct, for each aspherical oriented 3-manifold M , a 2-dimensional class in the l_1 -homology of M whose norm combined with the Gromov simplicial volume of M gives a characterization of those nonzero degree maps from M to N which are homotopic to a covering map. As an application we characterize those degree one maps which are homotopic to a homeomorphism in term of isometries between the bounded cohomology groups of M and N .

RÉSUMÉ. — Dans ce papier nous construisons, pour chaque variété de dimension trois close orientable et asphérique M , une classe d'homologie l_1 de dimension deux dans M dont la norme permet avec le volume simplicial de M de caractériser les applications de degré non-nul de M dans N qui sont homotopes à un revêtement. Comme conséquence, nous donnons un critère d'homéomorphisme pour les applications de degré un en terme d'isométries entre les groupes de cohomologie bornée de M et N .

1. Introduction

Throughout this paper all manifolds are orientable. Given a topological space X we denote by $(C_*(X), \partial)$ its real singular chain complex endowed with the l_1 -norm defined by $\|\sigma\|_1 = \sum_i |a_i|$ if $\sigma = \sum_i a_i \sigma_i$, where σ_i are singular simplices.

Any finite covering map $f: M \rightarrow N$ between closed orientable 3-manifolds induces an isometry $f_{\#}: H_3(M; \mathbf{R}) \rightarrow H_3(N; \mathbf{R})$ with respect to the l_1 (semi) norm induced by the l_1 -norms on the real singular chains of M and N .

For hyperbolic manifolds this condition is sufficient to characterize covering maps by Gromov and Thurston's works. However, since the Gromov

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simplicial volume of a 3-manifold M , which is the l_1 -norm $\|[M]\|_1$ of a generator $[M]$ of $H_3(M; \mathbf{Z}) \subset H_3(M; \mathbf{R})$, does not detect the "non-hyperbolic part" of 3-manifolds one can construct, using pinching maps, many pairwise non-homeomorphic 3-manifolds with the same Gromov simplicial volume related by a degree one map.

When M is a surface bundle over the circle with a fiber of negative Euler characteristic, M. Boileau and S. Wang gave in [3, Theorem 2.1, Corollary 2.3] a characterization of nonzero degree maps $f: M \rightarrow N$ into an irreducible 3-manifold which are homotopic to a covering map in terms of isometry with respect to the Thurston's norm in the second homology group of the manifolds. The purpose of this paper is to extend [3, Theorem 2.1] to aspherical 3-manifolds.

According to the Geometrization Theorem of Perelman, any closed aspherical 3-manifold M admits a JSJ-splitting along a family of characteristic tori \mathcal{T}_M such that each component of $M \setminus \mathcal{T}_M$ either admits a Seifert fibration or has a complete finite volume hyperbolic interior.

We say that M is *orientable** if M is orientable and if each Seifert component of $M \setminus \mathcal{T}_M$ admits a fibration over an orientable surface. This condition is satisfied for example when M contains no embedded Klein bottle or when M is obtained from a holomorphic function $f: (\mathbf{C}^3, 0) \rightarrow (\mathbf{C}, 0)$ with an isolated singularity at 0 by taking the boundary of the singularity of f at 0 defined by $f^{-1}(0) \cap S(\varepsilon)$, where $S(\varepsilon)$ is a Milnor sphere centered at 0 in \mathbf{C}^3 with radius ε (see [18]). Notice that this *orientable** condition is also satisfied when M is a surface bundle with a fiber of negative Euler characteristic ([3]).

In [3, Theorem 2.1], a key point, is that when M is a surface bundle, there exists a class $\alpha_M \in H_2(M) \setminus \{0\}$, namely the class of the fiber, "passing non-trivially through the whole manifold". Of course, such a *fiber class*, does not exist in the homology of a general 3-manifold because if we try to define local classes in M there are often homological obstructions which do not allow to glue them together in order to define a global class. However these obstructions disappear considering the l_1 -completion $H_2^{l_1}(M)$ of $H_2(M)$ and a *fiber class* α_M can be defined in $H_2^{l_1}(M)$ as follows. Let M be a closed orientable aspherical 3-manifold :

When M is a geometric 3-manifold, set $\alpha_M = 0$ excepted when M is a $\widetilde{\text{SL}}_2(\mathbf{R})$ -manifold. In this case, M admits a finite covering $p: \widetilde{M} \rightarrow M$ which is a (true) circle bundle $\xi: \widetilde{M} \rightarrow \widetilde{F}$ over a smooth surface. Then we set

$$\alpha_M = p_{\#} \circ \xi_{\#}^{-1} \left(\left[\widetilde{F} \right]_1 \right)$$

where $[\tilde{F}]_1$ denotes the l_1 -class of the l_1 -cycle \tilde{F} . This makes sense since by [9, Mapping Theorem] ξ induces an isometric isomorphism $\xi_{\sharp}: H_2^{l_1}(\tilde{M}) \rightarrow H_2^{l_1}(\tilde{F})$.

When M is not a geometric 3-manifold, each Seifert component of $M \setminus \mathcal{T}_M$ admits either a Euclidean geometry or a $\mathbf{H}^2 \times \mathbf{R}$ -geometry. For each $\mathbf{H}^2 \times \mathbf{R}$ -component P_i , $i = 1, \dots, l$, of $M \setminus \mathcal{T}_M$ we choose a horizontal properly embedded incompressible surface \mathcal{F}_i in P_i and we set

$$\alpha_M = \sum_{i=1}^l \frac{1}{k_i} \alpha_M(\mathcal{F}_i)$$

where k_i denotes the intersection number between \mathcal{F}_i and the generic fiber of P_i and where $\alpha_M(\mathcal{F}_i)$ denote the l_1 -class of \mathcal{F}_i in M which makes sense since the relative cycle \mathcal{F}_i of P_i can be "filled" in a natural way giving a l_1 -cycle in M (see paragraph 2). If $M \setminus \mathcal{T}_M$ contains no $\mathbf{H}^2 \times \mathbf{R}$ -components we just set $\alpha_M = 0$.

Remark 1.1. — Obviously, it follows from our construction that our fiber class does not need to be unique, as well as the fiber class of a surface bundle when the rank of the homology is distinct from 1, by a result of [17]. On the other hand, it follows from our proof of Theorem 1.2 that our results hold for any choices of a fiber class.

The main result of this paper states as follows

THEOREM 1.2. — *Let $f: M \rightarrow N$ be a nonzero degree map from a closed orientable* aspherical 3-manifold into a closed orientable irreducible 3-manifold such that $\|f_{\sharp}([M])\|_1 = \|[M]\|_1$ and $\|f_{\sharp}(\alpha_M)\|_1 = \|\alpha_M\|_1$ for some fiber class α_M . Then f is homotopic to a $\deg(f)$ -fold covering map.*

To make the hypothesis $\|f_{\sharp}(\alpha_M)\|_1 = \|\alpha_M\|_1$ more concrete one can compare it with a condition given in [6] where we introduce an invariant denoted by $\text{vol}(M)$ and defined as the sum of the absolute value of the orbifolds Euler characteristic of the Seifert pieces of M . This volume is used to state rigidity results, see [6, Theorems 1.3 and 1.6]. Using sections 2 and 3 of this paper and results in [6] one can easily check that $\|\alpha_M\|_1 = \text{vol}(M)$ and if $\|f_{\sharp}([M])\|_1 = \|[M]\|_1$, meaning that $\|M\| = |\deg(f)| \|N\|$, then $\|\alpha_M\|_1 = \text{vol}(M) \geq \|f_{\sharp}\alpha_M\|_1 \geq \|\alpha_N\|_1 = \text{vol}(N)$.

As far as we know, there are no general results to characterize local isometries for aspherical 3-manifolds excepted when the sectional curvature is negative. In the situation we deal with, the best metric we can hope, in many cases, is a metric with non-positive curvature by [14] and

our manifolds contain many totally geodesic surfaces where the curvature vanishes. From the point of view of maps $f: M \rightarrow N$ there are more flexibility when the sectional curvature of M vanishes and so it is more difficult to control the behavior of f . On the other hand, we hope that our results offer an application of the theory of bounded cohomology and l_1 -homology.

Notice that if M and N are both orientable* then the isometry condition is also necessary (see Lemma 2.2 and Proposition 2.4). If N is not orientable* the condition is not necessary. Indeed, consider for N the trivial orientable \mathbf{S}^1 -bundle over the genus -3 surface $\mathbf{R}P(2)\sharp\mathbf{R}P(2)\sharp\mathbf{R}P(2)$ and for M the trivial bundle $\Sigma_2 \times \mathbf{S}^1$ which is a 2-fold covering $p: M \rightarrow N$, where Σ_2 is the genus 2-surface. Let α_M denote the class of Σ_2 in $H_2^{l_1}(M; \mathbf{R})$. Then it follows from the arguments of section 2 that $\|\alpha_M\|_1 > 0$ and $p_{\sharp}(\alpha_M) = 0$.

By the Hahn-Banach Theorem, for each fiber class α_M with $\|\alpha_M\|_1 > 0$, there exists a class β_M in the second bounded cohomology group of M , denoted by $H_b^2(M; \mathbf{R})$ and endowed with the semi-norm $\|\cdot\|_{\infty}$, such that $\langle \beta_M, \alpha_M \rangle = 1$ and $\|\beta_M\|_{\infty} = \frac{1}{\|\alpha_M\|_1}$. When $\|\alpha_M\|_1 = 0$, just set $\beta_M = 0$. Thus we deduce the following

COROLLARY 1.3. — *Let $f: M \rightarrow N$ be a nonzero degree map from a closed orientable* aspherical 3-manifold into a closed orientable irreducible 3-manifold such that $\|f_{\sharp}([M])\|_1 = \|[M]\|_1$. If there exists a class $\beta \in H_b^2(N; \mathbf{R})$ such that $f^{\sharp}(\beta) = \beta_M$ and $\|\beta_M\|_{\infty} = \|\beta\|_{\infty}$ then f is homotopic to a covering map.*

We give the following corollary answering positively to a question of Professor M. Boileau.

COROLLARY 1.4. — *A degree one map $f: M \rightarrow N$ from a closed orientable* aspherical 3-manifold into a closed orientable irreducible 3-manifold is homotopic to a homeomorphism iff*

(i) $f_{\sharp}: H_3(M; \mathbf{R}) \rightarrow H_3(N; \mathbf{R})$ is an isometry with respect to the l_1 -norms and

(ii) f induces an isometric isomorphism $f^{\sharp}: H_b^2(N; \mathbf{R}) \rightarrow H_b^2(M; \mathbf{R})$, resp. an isometry $f_{\sharp}: H_2^{l_1}(M; \mathbf{R}) \rightarrow H_2^{l_1}(N; \mathbf{R})$.

THEOREM 1.5. — *A nonzero degree map $f: M \rightarrow N$ from a closed orientable aspherical 3-manifold into a closed orientable irreducible 3-manifold is homotopic to a covering map iff it induces a homomorphism $f_*: \pi_1 M \rightarrow \pi_1 N$ with amenable kernel.*

We end this section by mentioning a related result for self maps which is a direct consequence of [24] and [13, Theorem 0.7] using a standard covering space argument suggested by Professor W. Lück:

THEOREM 1.6. — *Any nonzero degree map $f: M \rightarrow M$ from a closed orientable aspherical 3-manifold to itself is homotopic to a $\text{deg}(f)$ -fold covering.*

Organization of the paper. This paper is organized as follows. In Section 2 we collect some technical results which will be used in the proof of the theorem. More precisely we compute the l_1 -norm of certain classes in $H_2^{l_1}(M)$ which come from classical integral homology classes and we study some isometric properties of finite coverings with respect to the l_1 -norms. Section 3 is devoted to the proof of Theorems 1.2 and 1.5.

2. Norm of surfaces in aspherical 3-manifolds

To fix the notations we recall the basic definitions of l_1 -homology and bounded cohomology according to the main papers of [16] and [9]. For a topological space X , denote by $C_*^{l_1}(X)$ the l_1 -completion of the real singular chains $C_*(X)$. Then

$$C_n^{l_1}(X) = \left\{ c = \sum_{i=1}^{\infty} a_i \sigma_i \text{ s.t. } \|c\|_1 = \sum_{i=1}^{\infty} |a_i| < \infty \right\}$$

where $a_i \in \mathbf{R}$ and $\sigma_i: \Delta_n \rightarrow X$ is a singular n -simplex. We will denote by $S_n(X)$ the set of singular n -simplices. The topological dual of $C_*^{l_1}(X)$ is given by the set

$$C_b^n(X) = \left\{ w \in C^n(X) \text{ s.t. } \|w\|_{\infty} = \sup_{\sigma \in S_n(X)} |\langle w, \sigma \rangle| < \infty \right\}$$

Note that the ∂ and δ operators are bounded so that $(C_*^{l_1}(X), \partial)$ and $(C_b^*(X), \delta)$ are chain, resp. cochain, complexes. We denote by $H_*^{l_1}(X)$, resp. by $H_b^*(X)$, the homology, resp. cohomology, of this chain, resp. cochain, complex. The vector spaces $H_*^{l_1}(X)$ and $H_b^*(X)$ are endowed with the quotient semi-norm that we still denote by $\|\cdot\|_1$ and $\|\cdot\|_{\infty}$ respectively. In the same way it is a standard fact that one can define the l_1 -homology and bounded cohomology of a pair of topological spaces (X, A) . Denote by $i: A \rightarrow X$ the natural inclusion and by $j: C_*^{l_1}(X) \rightarrow C_*^{l_1}(X, A)$ the projection. Then we get the classic long exact sequence

$$\dots \rightarrow H_n^{l_1}(A) \xrightarrow{i_{\sharp}} H_n^{l_1}(X) \xrightarrow{j_{\sharp}} H_n^{l_1}(X, A) \xrightarrow{\partial_{\sharp}} H_{n-1}^{l_1}(A) \rightarrow \dots$$

If moreover each component of A has an amenable fundamental group then by [16, Corollary 2.5] we know that $H_n^{l_1}(A) = \{0\}$ for any $n \geq 1$ and thus $j_{\#}$ admits an inverse $j_{\#}^{-1}: H_n^{l_1}(X, A) \rightarrow H_n^{l_1}(X)$ for $n \geq 2$ defined by $j_{\#}^{-1}([z]) = [z + u]$ where z is a relative cycle in (X, A) and u is any l_1 -chain in A such that $\partial u = -\partial z$. It follows from the definition that any continuous map of pairs $f: (X, A) \rightarrow (Y, B)$ induces a bounded homomorphism $f_{\#}: H_n(X, A) \rightarrow H_n(Y, B)$ such that $\|f_{\#}\| \leq 1$. On the other hand, when M is compact orientable n -manifold with (possibly empty) boundary we will denote by $[M]$ its fundamental class in $H_n(M, \partial M)$, by $[M]_1$ the image of $[M]$ under the homomorphism $H_n(M, \partial M) \rightarrow H_n^{l_1}(M, \partial M)$ induced by the completion and by $\|M\|$ its Gromov simplicial volume. For technical reasons we need the following

LEMMA 2.1. — *Let $p: \tilde{X} \rightarrow X$ be a regular covering map with finite Galois group Γ . For any Γ -invariant class $\alpha \in H_n^{l_1}(\tilde{X})$ then $\|p_{\#}(\alpha)\|_1 = \|\alpha\|_1$.*

Proof. — We use the averaging retraction $A: C_b^n(\tilde{X}) \rightarrow C_b^n(X)$ defined in [9] by

$$\langle A(\gamma), \sigma \rangle = \frac{\sum_{g \in \Gamma} \langle g^{\#}\gamma, \tilde{\sigma} \rangle}{\text{Card}(\Gamma)}$$

where $\tilde{\sigma}: \Delta^n \rightarrow \tilde{X}$ denotes a lifting of $\sigma: \Delta^n \rightarrow X$. This definition does not depend on the choice of the lifting $\tilde{\sigma}$ since the covering is regular. By construction, A commutes with the differentials so that it induces a homomorphism $\hat{A}: H_b^n(\tilde{X}) \rightarrow H_b^n(X)$ such that $\|\hat{A}\| \leq 1$. Let $\alpha \in H_n^{l_1}(\tilde{X})$ such that $g_{\#}(\alpha) = \alpha$ for any $g \in \Gamma$. If $\|\alpha\|_1 \neq 0$ then by the Hahn-Banach Theorem, there exists $\beta \in H_b^n(\tilde{X})$ such that $\langle \beta, \alpha \rangle = 1$ and $\|\beta\|_{\infty} = \frac{1}{\|\alpha\|_1}$. Since α is Γ -invariant then by the definition of the averaging we have $\langle \hat{A}(\beta), p_{\#}(\alpha) \rangle = 1$ and thus using the Hölder inequality and the fact that $\|\hat{A}\| \leq 1$ we deduce $\|p_{\#}(\alpha)\|_1 \geq \|\alpha\|_1$. This proves the lemma. \square

2.1. $\widetilde{\text{SL}}_2(\mathbf{R})$ -manifolds

Let M be an orientable* 3-manifold admitting a $\widetilde{\text{SL}}_2(\mathbf{R})$ -geometry. If moreover M is a (true) circle bundle, with projection ξ and base F then by [9, Mapping Theorem] ξ induces an isometric isomorphism $\xi_{\#}: H_2^{l_1}(M) \rightarrow H_2^{l_1}(F)$. Denote by $\alpha_M(F)$ the class $\xi_{\#}^{-1}([F]_1)$.

LEMMA 2.2. — *Let M be an orientable* $\widetilde{\text{SL}}_2(\mathbf{R})$ -manifold.*

(i) *If M is a (true) circle bundle with base F then*

$$\|\alpha_M(F)\|_1 = \|F\|$$

(ii) *Otherwise, for any finite covering $p: \widetilde{M} \rightarrow M$ such that \widetilde{M} is a (true) circle bundle over a surface F and projection $\xi: \widetilde{M} \rightarrow F$ then $\|p_{\#}\alpha_{\widetilde{M}}(F)\| = \|F\|$.*

(iii) *Moreover when \widetilde{M} is a circle bundle, the vector space generated by $p_{\#}\alpha_{\widetilde{M}}(F)$ does not depend on the choice of the finite covering $p: \widetilde{M} \rightarrow M$.*

Proof. — We first check point (i). The inequality $\|\alpha_M(F)\|_1 \leq \|F\|$ follows from the definition. To check the converse inequality we use exactly the same construction as in [2]. Fix a complete hyperbolic metric on F . Since F is orientable we can define a bounded 2-cocycle Ω_F in F in the following way: for each 2-simplex $\sigma: \Delta^2 \rightarrow F$, where Δ^2 denotes the standard 2-simplex, we set $\langle \Omega_F, \sigma \rangle = \mathcal{A}(\text{st}(\sigma))$, where $\text{st}(\sigma)$ denotes the geodesic simplex obtained from σ after straightening and \mathcal{A} denotes the algebraic area with respect to the fixed hyperbolic metric. In particular we get, if z denotes a 2-cycle representing the fundamental class of F

$$\left\langle \xi^{\#}([\Omega_F]), \xi^{\#-1}([F]_1) \right\rangle = \langle [\Omega_F], [F]_{l_1} \rangle = \langle \Omega_F, z \rangle = \text{Area}(F)$$

where $[\Omega_F] \in H_b^2(F)$ and $[F]_{l_1} \in H_2^{l_1}(F)$. Since by the construction $\|\xi^{\#}([\Omega_F])\|_{\infty} = \|[\Omega_F]\|_{\infty} \leq \pi$ then by the Hölder inequality we get $\|\alpha_M(F)\|_1 \geq \|F\|$. This proves point (i). We now prove point (ii). We consider two cases depending on whether the covering is regular or not.

Case 1. Assume that p is regular. Denote by Γ the Galois group of the covering. Note that since M is a Seifert bundle with orientable base 2-orbifold then any $g \in \Gamma$ induces an orientation preserving homeomorphism $\bar{g}: F \rightarrow F$ such that $\xi \circ g = \bar{g} \circ \xi$ and thus $\alpha_{\widetilde{M}}(F)$ is Γ -invariant and point (ii) of the lemma follows from Lemma 2.1 and point (i). This completes the proof of point (ii) in Case 1.

Case 2. If p is not regular then consider a finite covering $q: \widehat{M} \rightarrow \widetilde{M}$ such that $p \circ q$ is regular. Since $q_{\#} \left(\left\langle \alpha_{\widehat{M}}(\widehat{F}) \right\rangle \right) = \left\langle \alpha_{\widetilde{M}}(F) \right\rangle$, where $\langle v \rangle$ denotes the vector space generated by the vector v and where \widehat{F} is the base of the bundle \widehat{M} , then point (ii) in Case 2 follows from Case 1.

To check point (iii) it suffices to consider a common covering \widetilde{M} to \widetilde{M}_1 and \widetilde{M}_2 (which corresponds for example to $(p_1)_*(\pi_1 \widetilde{M}_1) \cap (p_2)_*(\pi_1 \widetilde{M}_2)$). This completes the proof of the lemma. □

2.2. Aspherical 3-manifolds

Let M be a closed orientable* aspherical 3-manifold. We fix an orientation on M . In the following we will assume that \mathbf{H}^2 and \mathbf{R} are oriented with the usual convention. Let P denote a component of $M \setminus \mathcal{T}_M$ whose interior admits a $\mathbf{H}^2 \times \mathbf{R}$ -geometry. Since M is orientable* then P admits a Seifert fibration over an orientable basis and we denote by h_P the fiber of P . We orient the fiber h_P in such a way that the universal covering $p: \mathbf{H}^2 \times \mathbf{R} \rightarrow P$ induces an orientation preserving map $\mathbf{R} \rightarrow h_P$. Let \mathcal{F} be an oriented surface and let $f: (\mathcal{F}, \partial\mathcal{F}) \rightarrow (P, \partial P)$ be a proper map. For any $x \in \mathbf{R}$ we denote by $\alpha_M(x\mathcal{F}, f)$ the class defined by $k_{\#}j_{\#}^{-1}f_{\#}(x[\mathcal{F}]_1)$ following the composition of homomorphisms:

$$H_2^{l_1}(\mathcal{F}, \partial\mathcal{F}) \xrightarrow{f_{\#}} H_2^{l_1}(P, \partial P) \xrightarrow{j_{\#}^{-1}} H_2^{l_1}(P) \xrightarrow{k_{\#}} H_2^{l_1}(M)$$

where $k: P \rightarrow M$ denotes the inclusion.

LEMMA 2.3. — We have $\|\alpha_M(x\mathcal{F}, f)\|_1 \leq |x|\|\mathcal{F}\|$ for any $x \in \mathbf{R}$.

Proof. — The proof follows from [9, Equivalence Theorem] combined with [16, Theorem 2.3]. □

Consider now a proper map $f: (\mathcal{F}, \partial\mathcal{F}) \rightarrow (P, \partial P)$ transverse to the fibers of P . We choose always the orientation of each component of F of \mathcal{F} so that so that f is *orientation preserving* which means that the orientation of $f(\mathcal{F})$ followed by the orientation of h_P matches the orientation induced by M . The main purpose of this section is to check the following

PROPOSITION 2.4. — Let M be a closed aspherical orientable* 3-manifold and denote by P_1, \dots, P_l a collection of pairwise distinct Seifert components of $M \setminus \mathcal{T}_M$ whose interior admits a $\mathbf{H}^2 \times \mathbf{R}$ -geometry. For each $i = 1, \dots, l$ assume that we are given an orientation preserving proper embedding $f_i: (\mathcal{F}_i, \partial\mathcal{F}_i) \rightarrow (P_i, \partial P_i)$. Then

(i) *Isometry:* for any $i = 1, \dots, l$ we have the equality

$$\|\alpha_M(\mathcal{F}_i, f_i)\|_1 = \|\mathcal{F}_i\|$$

(ii) *Additivity under JSJ-splitting:*

$$\begin{aligned} \|\alpha_M(x_1\mathcal{F}_1, f_1) + \dots + \alpha_M(x_l\mathcal{F}_l, f_l)\|_1 \\ = \|\alpha_M(x_1\mathcal{F}_1, f_1)\|_1 + \dots + \|\alpha_M(x_l\mathcal{F}_l, f_l)\|_1 \end{aligned}$$

where x_1, \dots, x_l are positive real numbers.

(iii) Let $f: M \rightarrow N$ be a covering map with N orientable*. If $\alpha = \alpha_M(x_1\mathcal{F}_1, f_1) + \dots + \alpha_M(x_l\mathcal{F}_l, f_l)$ then $\|f_{\#}(\alpha)\|_1 = \|\alpha\|_1$.

To prove this proposition we need two intermediate results. Hypothesis are the same as in Proposition 2.4.

LEMMA 2.5. — *Suppose that $\{P_i\}_{i \in I}$ is a family of circle bundles components of $M \setminus \mathcal{T}_M$ admitting a $\mathbf{H}^2 \times \mathbf{R}$ -geometry. For any $i \in I$ there exists a bounded 2-cocycle Ω_{P_i} in M satisfying the following properties:*

- (i) $k_i^*(\Omega_{P_i})$ is a relative 2-cocycle in $(P_i, \partial P_i)$ where $k_i: P_i \hookrightarrow M$ denotes the natural inclusion and $k_i^*(\Omega_{P_j}) = 0$ if $i \neq j$,
- (ii) $|\langle [\Omega_{P_i}], \alpha_M(\mathcal{F}_i, f_i) \rangle| = \text{Area}(\mathcal{F}_i)$ where $\text{Area}(\mathcal{F}_i)$ denotes the area of $\text{int}(\mathcal{F}_i)$ endowed with a complete hyperbolic metric.
- (iii) $\|\sum_{i \in I} [\Omega_{P_i}]\|_\infty = \pi$, where $[\Omega_{P_i}]$ denotes the class of Ω_{P_i} in $H_b^2(M; \mathbf{R})$.

Remark 2.6. — The above result is stated for Seifert pieces which are circle bundles only for convenience. This lemma remains true if we consider a family of Seifert pieces admitting a geometry $\mathbf{H}^2 \times \mathbf{R}$ with an orientable base 2-orbifold. Notice that the bounded class Ω_{P_i} cannot be defined for Seifert pieces with non-orientable basis.

To prove this lemma we need the reduction of singular chains with respect to the JSJ-splitting of aspherical 3-manifolds. This chain map is stated for example in [8]. Since this construction is crucial for our purpose we recall it and fix notation.

Let M be a closed aspherical orientable 3-manifold. Denote by P_1, \dots, P_l the components of $M \setminus \mathcal{T}_M$. As in [8], we consider a chain map $\rho: C_n(M) \rightarrow C_n(M)$ defined as follows:

0-simplices. If $n = 0$ then ρ is the identity.

1-simplices. If $n = 1$ let $\tau: [v_0, v_1] \rightarrow M$ be a 1-simplex. Since \mathcal{T}_M is incompressible, the map τ is homotopic, rel. $\{v_0, v_1\}$, to a reduced 1-simplex i.e. a map $\tau_1: [v_0, v_1] \rightarrow M$ such that either $\tau_1([v_0, v_1]) \subset \mathcal{T}_M$ or $\tau_1|_{(v_0, v_1)}$ is transverse to \mathcal{T}_M and for each component J of $\tau_1^{-1}(P_i)$ then $\tau_1|_J$ is not homotopic rel. ∂J into ∂P_i . Then we set $\rho(\tau) = \tau_1$ and we extend ρ by linearity.

2-simplices. If $n = 2$ let $\sigma: \Delta^2 = [v_0, v_1, v_2] \rightarrow M$ be a 2-simplex. Then σ is homotopic rel. $V(\Delta^2) = \{v_0, v_1, v_2\}$ to a reduced 2-simplex σ_1 such that either $\sigma_1(\Delta^2) \subset \mathcal{T}_M$ or $\sigma_1|_{\text{int}(\Delta^2)}$ is transverse to \mathcal{T}_M , the 1-simplex $\sigma_1|_e$ is reduced for each edge e of Δ^2 and $\sigma_1^{-1}(\mathcal{T}_M)$ contains no loop components. Thus if J is a component of $\sigma_1^{-1}(\mathcal{T}_M)$ such that $J \cap \text{int}(\Delta^2) \neq \emptyset$ then J is a proper arc in Δ^2 connecting two distinct edges (see figure 2.1). Then we set $\rho(\sigma) = \sigma_1$ and we extend ρ by linearity.

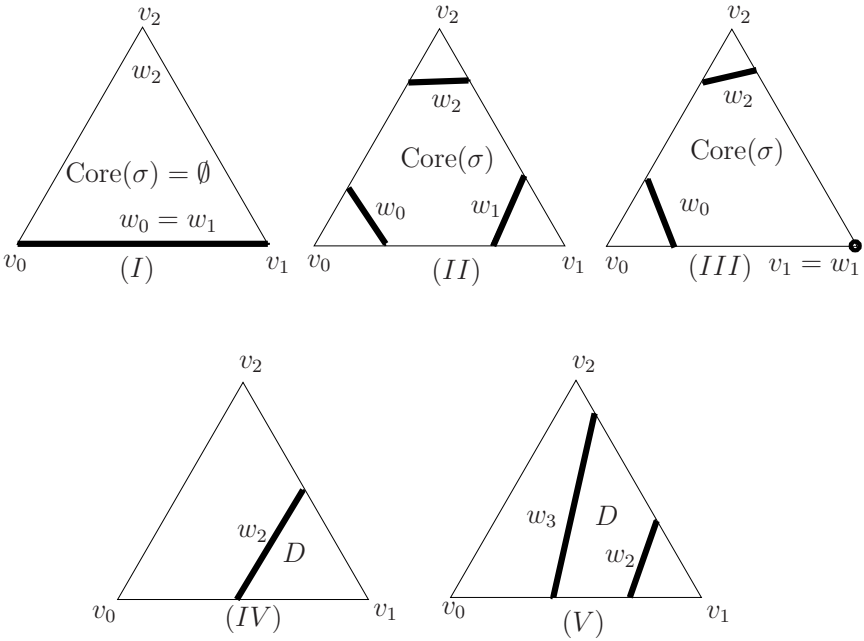


Figure 2.1

Remark 2.7. — Suppose that $\sigma: \Delta^2 \rightarrow M$ is a reduced 2-simplex. If $\sigma(e)$ is not contained in \mathcal{T}_M for any edge e of Δ^2 then there exists a unique component, denoted by $\text{Core}(\sigma)$, of $\Delta^2 \setminus \sigma^{-1}(\mathcal{T}_M)$ which meets the three edges of Δ^2 (see [8]).

3-simplices. If $n = 3$ let $\sigma: \Delta^3 = [v_0, v_1, v_2, v_3] \rightarrow M$ be a 3-simplex. Then σ is homotopic rel. $V(\Delta^3) = \{v_0, v_1, v_2, v_3\}$ to a reduced 3-simplex σ_1 such that either $\sigma_1(\Delta^3) \subset \mathcal{T}_M$ or $\sigma_1|\text{int}(\Delta^3)$ is transverse to \mathcal{T}_M , the 2-simplex $\sigma_1|\Delta_i^2$ is reduced for each face Δ_i^2 of Δ^3 and if D is a component of $\sigma_1^{-1}(\mathcal{T}_M)$ such that $D \cap \text{int}(\Delta^3) \neq \emptyset$ then D is either a normal triangle or a normal rectangle (see figure 2.2). Then we set $\rho(\sigma) = \sigma_1$ and we extend ρ by linearity. Notice that the reduction is stable under the ∂ -operator.

Proof of Lemma 2.5. — We use here the technique developed in [1].

Step 1: Crushing M into P_i . Denote by $p_i: \widetilde{M}_i \rightarrow M$ the covering map corresponding to the subgroup $(k_i)_*(\pi_1 P_i)$ of $\pi_1 M$, fix a lifting $\widetilde{k}_i: P_i \rightarrow \widetilde{M}_i$ of $k_i: P_i \rightarrow M$ and denote by \widetilde{P}_i the image of \widetilde{k}_i . There exists a retraction $r_i: \widetilde{M}_i \rightarrow P_i$ crushing each component of $\widetilde{M}_i \setminus \widetilde{P}_i$ to the corresponding component of $\partial \widetilde{P}_i$ such that $r_i|\widetilde{P}_i = \widetilde{k}_i^{-1}$. Denote by F_i the base surface

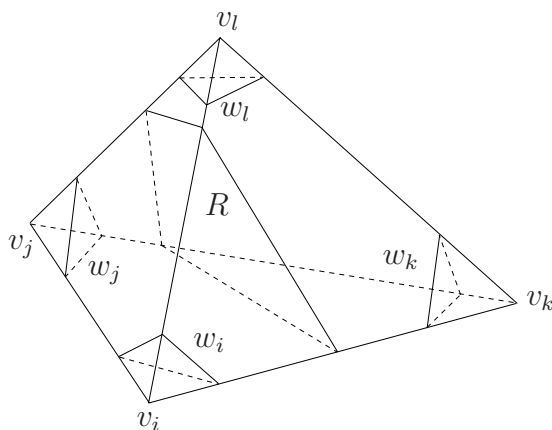


Figure 2.2. Normal triangles and rectangles

of the circle bundle P_i and by $\xi_i: P_i \rightarrow F_i$ the projection. Fix a complete hyperbolic metric on $\text{int}(F_i)$, crush each component of ∂F_i to a point, denote by \widehat{F}_i the new surface and by $q_i: F_i \rightarrow \widehat{F}_i$ the natural crushing map. This construction is equivalent to adding a limit parabolic point to each component C of ∂F_i . This parabolic point corresponds to the fixed point of the parabolic isometry generating $\pi_1 C$.

$$\begin{array}{ccccc}
 & & \widetilde{M}_i & & \\
 & \nearrow \widetilde{k}_i & \downarrow p & \searrow r_i & \\
 P_i & \xrightarrow{k_i} & M & \xrightarrow{\xi_i} & F_i \xrightarrow{q_i} \widehat{F}_i
 \end{array}$$

Step 2: Straightening simplices on surfaces with boundary. Let $\sigma: \Delta^2 = [v_0, v_1, v_2] \rightarrow \widehat{F}_i$ be a (singular) 2-simplex. Consider an edge $\tau = \sigma|[v_i, v_j]: [v_i, v_j] \rightarrow \widehat{F}_i$ and denote by $\widetilde{\tau}: [v_i, v_j] \rightarrow \overline{\mathbf{H}^2}$ a lifting of τ in the hyperbolic space union its boundary. Then $\widetilde{\tau}$ is homotopic by a homotopy fixing the end points to the unique geodesic arc (which may be constant) connecting the end points of $\widetilde{\tau}$. Denote by $\text{st}(\widetilde{\tau})$ the new straight 1-simplex and by $\text{st}(\tau)$ the projection of $\text{st}(\widetilde{\tau})$ into \widehat{F}_i . We straighten each edge of σ and finally we homotop σ to a straight 2-simplex $\text{st}(\sigma)$. As in the proof of Lemma 2.2 we define a bounded 2-cocycle $\widehat{\omega}_i$ on \widehat{F}_i by setting $\langle \widehat{\omega}_i, \sigma \rangle = \mathcal{A}(\text{st}(\sigma))$, the algebraic area of $\text{st}(\sigma)$. Thus $q_i^\#(\widehat{\omega}_i)$ defines a relative 2-cocycle on $(F_i, \partial F_i)$ such that $\langle q_i^\#(\widehat{\omega}_i), z_i \rangle = \text{Area}(F_i)$, where z_i is a relative 2-cycle representing the fundamental class of F_i .

Step 3: Lifting the singular chains. Let $\mu = \sum_l a_l \mu_l$ be a n -chain for $n = 2, 3$ where $a_l \in \mathbf{R}$ and $\mu_l: \Delta^n \rightarrow M$ is a singular n -simplex. We choose a decomposition of each component of $\Delta^n \setminus \rho(\mu_l)^{-1}(\mathcal{T}_M)$ into n -simplices $\nabla_l^j, j = 1, \dots, n_l$ (recall that ρ denotes the reduction operator). Next we replace μ by the n -chain $\sigma = \sum_{l,j} a_l \rho(\mu_l) | \nabla_l^j$. Denote by $\tilde{\sigma}$ the preimage of σ in \widetilde{M}_i . Then $\tilde{\sigma}$ is a locally finite n -chain in \widetilde{M}_i . Since \widetilde{P}_i is compact then we define a finite n -chain $\tilde{\sigma}_i$ in \widetilde{M}_i by taking only the simplices of $\tilde{\sigma}$ which meet \widetilde{P}_i .

Step 4: Definition of a bounded cocycle satisfying the conclusion of the lemma. Keeping the same notation as in Step 3 we define a 2-cochain Ω_{P_i} in M by setting

$$\langle \Omega_{P_i}, \mu \rangle = \left\langle g_i^\# \widehat{w}_i, \tilde{\sigma}_i \right\rangle$$

where μ is a singular 2-simplex and where $g_i = q_i \circ \xi_i \circ r_i$. By construction $\|\Omega_{P_i}\|_\infty \leq \pi$. Indeed let $\sigma: \Delta^2 \rightarrow M$ be a singular 2-simplex. By construction of Ω_{P_i} we may assume that σ is reduced. First note that it follows from the construction that for each triangle Δ of $\Delta^2 \setminus \sigma^{-1}(\mathcal{T}_M)$ (given in the decomposition of Step 3) whose an edge is a component of $\sigma^{-1}(\mathcal{T}_M)$ then $\langle \Omega_{P_i}, \sigma | \Delta \rangle = 0$ (the simplices of $\sigma | \Delta$ are sent into a point or a geodesic arc after straightening in \widehat{F}_i). On the other hand there exist at most one triangle Δ_σ of $\Delta^2 \setminus \sigma^{-1}(\mathcal{T}_M)$ whose no edge is a component of $\sigma^{-1}(\mathcal{T}_M)$. This triangle necessarily lives in $\text{Core}(\sigma)$. Since there exists at most one component of $\widetilde{\sigma | \Delta_\sigma}$ which meets \widetilde{P}_i then the inequality $\|\Omega_{P_i}\|_\infty \leq \pi$ follows.

We check the cocycle condition $\langle \delta \Omega_{P_i}, \sigma \rangle = 0$ for each 3-simplex $\sigma: \Delta^3 \rightarrow M$. Since $\langle \delta \Omega_{P_i}, \sigma \rangle = \langle \Omega_{P_i}, \partial \sigma \rangle$ then we may assume that σ is reduced. Consider the 3-chain $\sum_j \sigma | \nabla_j$, where ∇_j is the decomposition (given in Step 3) of $\Delta^3 \setminus \sigma^{-1}(\mathcal{T}_M)$ into 3-simplices. The 2-faces of ∇_j are made of *interior triangles* which consist of the triangles whose interiors are in the interior of Δ^3 and of triangles which define the 2-simplices of a decomposition of $\partial \Delta^3 \setminus (\sigma^{-1}(\mathcal{T}_M) \cap \partial \Delta^3)$. Since each interior triangle is the face of two adjacent tetrahedra then one can replace σ by $\sum_j \sigma | \nabla_j$. Denote still $\sum_j \sigma | \nabla_j$ by σ . The 2-chain of \widetilde{M}_i defined by

$$\partial \tilde{\sigma}_i - \left(\partial \sigma \right)_i \tag{*}$$

consists of an alternate sum of 2-simplices of $\partial \tilde{\sigma}$ which does not meet \widetilde{P}_i . Since the retraction r_i crush each component of $\widetilde{M}_i \setminus \widetilde{P}_i$ to $\partial \widetilde{P}_i$ then by construction

$$\left\langle g_i^\# \widehat{w}_i, \partial \tilde{\sigma}_i - \left(\partial \sigma \right)_i \right\rangle = 0 \tag{**}$$

On the other hand by the definition

$$\langle \delta\Omega_{P_i}, \sigma \rangle = \left\langle g_i^\# \widehat{w}_i, \left(\widetilde{\partial\sigma} \right)_i \right\rangle$$

Thus using (*) and (**) we get, since $g_i^\# \widehat{w}_i$ is a cocyle by construction,

$$\langle \delta\Omega_{P_i}, \sigma \rangle = \left\langle g_i^\# \widehat{w}_i, \widetilde{\partial\sigma}_i \right\rangle = 0$$

On the other hand it is easily checked from the construction that $k_i^* \Omega_{P_i}$ is a relative cocycle of $(P_i, \partial P_i)$ and $k_i^*(\Omega_{P_j}) = 0$ for any $i \neq j$.

We check point (ii). First note that $\alpha_M(\mathcal{F}_i, f_i) = [(k_i)_\#((f_i)_\#(\mu_i) + u)]$ where μ_i is a relative 2-cycle representing the fundamental class of \mathcal{F}_i and u is a l_1 -chain in ∂P_i such that $\partial u = -\partial f_i(\mu_i)$. Thus the construction yields

$$\langle [\Omega_{P_i}], \alpha_M(\mathcal{F}_i, f_i) \rangle = \left\langle g_i^\# \widehat{w}_i, (\widetilde{k}_i)_\#(f_i)_\#(\mu_i) \right\rangle = \left\langle q_i^\# \widehat{w}_i, (\xi_i \circ f_i)_\#(\mu_i) \right\rangle$$

But since $\xi_i \circ f_i$ is a finite covering, with positive degree denoted by d_i then $(\xi_i \circ f_i)_\#([\mathcal{F}_i]) = d_i[F_i]$ and thus we get (see Step 2)

$$\langle [\Omega_{P_i}], \alpha_M(\mathcal{F}_i, f_i) \rangle = d_i \left\langle q_i^\# \widehat{w}_i, z_i \right\rangle = d_i \text{Area}(F_i) = \text{Area}(\mathcal{F}_i)$$

To complete the proof of the lemma it remains to compute the norm of the classes defined by Ω_{P_i} . Denote by Ω the sum $\sum_i \Omega_i$. We first check that $\|\sum_i \Omega_{P_i}\|_\infty \leq \pi$. Let $\sigma: \Delta^2 \rightarrow M$ be a singular 2-simplex. If there exists an edge e of Δ^2 such that $\rho\sigma(e) \subset \mathcal{T}_M$ then $\langle \sum_{i \in I} \Omega_{P_i}, \sigma \rangle = 0$. If for any edge e of Δ^2 we have $\rho\sigma(e) \not\subset \mathcal{T}_M$ then there exists a unique component $\text{Core}(\sigma)$ of $(\rho\sigma)^{-1}(M \setminus \mathcal{T}_M)$ which meets the three edges of Δ^2 . Denote by P_ν the component of $M \setminus \mathcal{T}_M$ such that $\rho\sigma(\text{Core}(\sigma)) \subset \text{int}(P_\nu)$. If $\nu \in I$ then we have $|\langle \sum_{i \in I} \Omega_{P_i}, \sigma \rangle| = |\langle \Omega_{P_\nu}, \sigma \rangle| \leq \pi$ and if $\nu \notin I$ then $|\langle \sum_{i \in I} \Omega_{P_i}, \sigma \rangle| = 0$. This proves that $\|\sum_{i \in I} \Omega_{P_i}\|_\infty \leq \pi$. Using lemma 2.3 and points (i) and (ii) of the Lemma, we get the following equalities

$$|\langle [\Omega], \alpha_M(\mathcal{F}_i, f_i) \rangle| = \text{Area}(\mathcal{F}_i) \leq \|[\Omega]\|_\infty \|\alpha_M(\mathcal{F}_i, f_i)\|_1 \leq \|[\Omega]\|_\infty \|\mathcal{F}_i\|$$

this completes the proof of Lemma 2.5 since $\text{Area}(\mathcal{F}_i) = \pi \|\mathcal{F}_i\|$. □

LEMMA 2.8. — *Let M be a closed aspherical orientable* 3-manifold and let $p: \widetilde{M} \rightarrow M$ denote a finite regular covering whose each Seifert piece is a circle bundle with $\mathbf{H}^2 \times \mathbf{R}$ -geometry. Assume that we are given orientation preserving proper embeddings $f_i: (\widetilde{\mathcal{F}}_i, \partial\widetilde{\mathcal{F}}_i) \rightarrow (\widetilde{P}_i, \partial\widetilde{P}_i)$ where $\{\widetilde{P}_i\}_{i \in I}$ is a collection of Seifert pieces of \widetilde{M} . Then we have the equality*

$$\left\| p_\# \left(\sum_i \alpha_{\widetilde{M}}(x_i \widetilde{\mathcal{F}}_i, f_i) \right) \right\|_1 = \left\| \sum_i \alpha_{\widetilde{M}}(x_i \widetilde{\mathcal{F}}_i, f_i) \right\|_1$$

where the x_i are positive real numbers.

Proof. — Denote by Γ the automorphism group of $p: \widetilde{M} \rightarrow M$. Let $\tilde{\alpha}$ be the element $\sum \alpha_{\widetilde{M}}(x_i \tilde{\mathcal{F}}_i, f_i)$ and denote by $\text{Av}(\tilde{\alpha})$ the class obtained by averaging $\tilde{\alpha}$ defined by $\text{Av}(\tilde{\alpha}) = \sum_{g \in \Gamma} g_{\#}(\tilde{\alpha})$. For a Seifert piece \tilde{P} of \widetilde{M} denote by $\Omega_{\tilde{P}}$ the bounded 2-cocycle of \widetilde{M} constructed in Lemma 2.5 and denote by Ω the sum $\sum_{\tilde{P}} \Omega_{\tilde{P}}$. Notice that each $g \in \Gamma$ acts on \widetilde{M} as an orientation preserving homeomorphism which preserves the JSJ-splitting. In particular for each Seifert piece \tilde{P} of \widetilde{M} then there exists a unique Seifert piece \tilde{P}' such that $g(\tilde{P}) = \tilde{P}'$ and $g|_{\tilde{P}}: (\tilde{P}, \partial\tilde{P}) \rightarrow (\tilde{P}', \partial\tilde{P}')$ is a homeomorphism. Moreover since each Seifert piece of M has an orientable basis then $g|_{\tilde{P}}: (\tilde{P}, \partial\tilde{P}) \rightarrow (\tilde{P}', \partial\tilde{P}')$ induces an orientation preserving homeomorphism between the bases of \tilde{P} and \tilde{P}' . Then we get

$$\langle [\Omega], \text{Av}(\tilde{\alpha}) \rangle = \text{Card}(\Gamma) \sum_{i \in I} x_i \text{Area}(\tilde{\mathcal{F}}_i) \leq \pi \|\text{Av}(\tilde{\alpha})\|_1$$

which proves that

$$\|\text{Av}(\tilde{\alpha})\|_1 \geq \text{Card}(\Gamma) \sum_{i \in I} x_i \|\tilde{\mathcal{F}}_i\|$$

Since $\text{Av}(\tilde{\alpha})$ is Γ -invariant then by Lemma 2.1 $\|p_{\#}(\text{Av}(\tilde{\alpha}))\|_1 = \|\text{Av}(\tilde{\alpha})\|_1$. Moreover using the definitions and Lemma 2.3

$$\|p_{\#}(\text{Av}(\tilde{\alpha}))\|_1 \leq \sum_{g \in \Gamma} \|p_{\#}g_{\#}(\tilde{\alpha})\|_1 \leq \sum_{g \in \Gamma} \|g_{\#}(\tilde{\alpha})\|_1 \leq \text{Card}(\Gamma) \sum_{i \in I} x_i \|\tilde{\mathcal{F}}_i\|$$

We deduce that $\sum_{g \in \Gamma} \|p_{\#}g_{\#}(\tilde{\alpha})\|_1 = \sum_{g \in \Gamma} \|g_{\#}(\tilde{\alpha})\|_1$. On the other hand, since we know that $\|p_{\#}g_{\#}(\tilde{\alpha})\|_1 \leq \|g_{\#}(\tilde{\alpha})\|_1$ for any $g \in \Gamma$ then we get in particular $\|p_{\#}(\tilde{\alpha})\|_1 = \|\tilde{\alpha}\|_1$. \square

Proof of Proposition 2.4. — To complete the proof of Proposition 2.4 it remains to check the following points

- (i) $\|\alpha_M(\mathcal{F}_i, f_i)\|_1 \geq \|\mathcal{F}_i\|$ for $i = 1, \dots, l$,
- (ii) $\|\sum \alpha_M(x_i \mathcal{F}_i, f_i)\| \geq \sum \|\alpha_M(x_i \mathcal{F}_i, f_i)\|$, and
- (iii) the covering property.

We first check points (i) and (ii). To this purpose we consider two cases.

Case 1. Assume that each $P_i, i = 1, \dots, l$ is homeomorphic to a circle bundle. By Lemma 2.5 we know that there exists a bounded 2-cocycle Ω_{P_i} such that $\|[\Omega_{P_i}]\|_{\infty} = \pi$ and $|\langle [\Omega_{P_i}], \alpha_M(\mathcal{F}_i, f_i) \rangle| = \text{Area}(\mathcal{F}_i)$. Then point (i) follows from Hölder inequality.

We check point (ii). Again, applying Lemma 2.5 we know that for each $i \in \{1, \dots, l\}$ there exists a bounded 2-cocycle Ω_i in M such that

$$\langle [\Omega_i], \alpha_M(x_j \mathcal{F}_j, f_j) \rangle = \delta_{ij} x_j \text{Area}(\mathcal{F}_j)$$

for any i, j in $\{1, \dots, l\}$. Thus we get

$$\left\langle \sum_i [\Omega_i], \sum_j \alpha_M(x_j \mathcal{F}_j, f_j) \right\rangle = \sum_i x_i \text{Area}(\mathcal{F}_i) \leq \pi \left\| \sum_j \alpha_M(x_j \mathcal{F}_j, f_j) \right\|_1$$

Hence

$$\left\| \sum_i \alpha_M(x_i \mathcal{F}_i, f_i) \right\|_1 \geq \sum_i x_i \|\mathcal{F}_i\| \geq \sum_i \|\alpha_M(x_i \mathcal{F}_i, f_i)\|_1$$

This proves point (ii) in Case 1.

Case 2. We consider now the general case. Let $p: \widetilde{M} \rightarrow M$ be a finite regular covering of M whose each Seifert piece (in particular each component \widetilde{P}_i over P_i for $i = 1, \dots, l$) is a circle bundle (such a covering exists by [15, Proposition 4.4]). For each $i = 1, \dots, l$ consider a covering $\widetilde{f}_i: (\widetilde{\mathcal{F}}_i, \partial \widetilde{\mathcal{F}}_i) \rightarrow (\widetilde{P}_i, \partial \widetilde{P}_i)$ of $f_i: (\mathcal{F}_i, \partial \mathcal{F}_i) \rightarrow (P_i, \partial P_i)$ (obtained by considering the group $(f_i)_*^{-1}(p_*(\pi_1 \widetilde{P}_i))$). By construction \widetilde{f}_i is an orientation preserving embedding. Denote by $d_i > 0$ the degree of the covering $p_i: \widetilde{\mathcal{F}}_i \rightarrow \mathcal{F}_i$. By Case 1 we know that $\|\alpha_{\widetilde{M}}(\widetilde{\mathcal{F}}_i, \widetilde{f}_i)\|_1 = \|\widetilde{\mathcal{F}}_i\|$ for $i = 1, \dots, l$. On the other hand by Lemma 2.8 we know that $\|p_{\#}(\alpha_{\widetilde{M}}(\widetilde{\mathcal{F}}_i, \widetilde{f}_i))\|_1 = \|\alpha_M(\mathcal{F}_i, f_i)\|_1$. Since any continuous map induces a chain map then

$$p_{\#}(\alpha_{\widetilde{M}}(\widetilde{\mathcal{F}}_i, \widetilde{f}_i)) = d_i \alpha_M(\mathcal{F}_i, f_i)$$

which implies that

$$\|d_i \alpha_M(\mathcal{F}_i, f_i)\|_1 = \|\widetilde{\mathcal{F}}_i\|$$

and thus $\|\alpha_M(\mathcal{F}_i, f_i)\|_1 = \|\mathcal{F}_i\|$ for $i = 1, \dots, l$. To check point (ii) we know from Case 1 that

$$\left\| \sum \alpha_{\widetilde{M}} \left(\frac{x_i}{d_i} \widetilde{\mathcal{F}}_i, \widetilde{f}_i \right) \right\| = \sum \left\| \alpha_{\widetilde{M}} \left(\frac{x_i}{d_i} \widetilde{\mathcal{F}}_i, \widetilde{f}_i \right) \right\|$$

Then using Lemma 2.8 in the right and left hand side, we get

$$\left\| \sum \alpha_M(x_i \mathcal{F}_i, f_i) \right\| = \sum \|\alpha_M(x_i \mathcal{F}_i, f_i)\|.$$

We check point (iii). Let $f: M \rightarrow N$ denote a finite covering map and let $\alpha = \sum \alpha_M(x_i \mathcal{F}_i, f_i)$. Using the construction of Case 2 with the same notations then $\alpha = p_{\#}(\widetilde{\alpha})$ where $\widetilde{\alpha} = \sum \alpha_{\widetilde{M}} \left(\frac{x_i}{d_i} \widetilde{\mathcal{F}}_i, \widetilde{f}_i \right)$. Possibly passing to some finite covering there are no loss of generality assuming $f \circ p$ is regular. Hence we get using Lemma 2.8

$$\|f_{\#}(\alpha)\| = \|f_{\#} p_{\#}(\widetilde{\alpha})\| = \|\widetilde{\alpha}\| \geq \|\alpha\|$$

This completes the proof of Proposition 2.4. □

3. Characterizations of covering maps

Given a closed irreducible orientable 3-manifold M we denote by $\mathcal{H}(M)$ (resp. $\mathcal{S}(M)$) the disjoint union of the hyperbolic (resp. Seifert) components of $M \setminus \mathcal{T}_M$ (see [11], [12] and [21]). In order to prove Theorem 1.2 we first check the following technical result.

PROPOSITION 3.1. — *Let M be a closed aspherical orientable 3-manifold. Any π_1 -surjective nonzero degree map $f: M \rightarrow N$ into a closed irreducible orientable 3-manifold satisfying the following conditions*

- (i) *Each Seifert component of $M \setminus \mathcal{T}_M$, resp. of $N \setminus \mathcal{T}_N$, is homeomorphic to a product, resp. to a \mathbf{S}^1 -bundle over an orientable surface, each Seifert component of $M \setminus \mathcal{T}_M$ has at least two boundary components (if $\mathcal{T}_M \neq \emptyset$) and each component of \mathcal{T}_M is shared by two distinct components of $M \setminus \mathcal{T}_M$,*
- (ii) *$\|f_{\#}[M]\|_1 = \|[M]\|_1$, where $[M] \in H_3(M; \mathbf{R})$ is the fundamental class*
- (iii) *$\|f_{\#}\alpha_M(\mathcal{F}, g)\|_1 = \|\alpha_M(\mathcal{F}, g)\|_1$ for each orientation preserving proper embedding $g: \mathcal{F} \rightarrow P$ when P runs over the Seifert pieces of M is homotopic to a homeomorphism.*

3.1. Proof of Proposition 3.1

Throughout this section we always assume that the map $f: M \rightarrow N$ and the manifolds M, N satisfy the hypothesis of Proposition 3.1. Notice that we may assume in addition that M is not a virtual torus bundle by [23]. Thus since each Seifert piece P of M is homeomorphic to a product we may assume that P is a $\mathbf{H}^2 \times \mathbf{R}$ -manifold. Hence this implies, using hypothesis (ii) and (iii), that either $\|N\| \neq 0$ or $H_2^{l_1}(N; \mathbf{R})/\ker \|\cdot\|_1 \neq \{0\}$. Hence either N is non-geometric or admits a geometry $\mathbf{H}^3, \mathbf{H}^2 \times \mathbf{R}$ or $\widetilde{\mathrm{SL}}(2, \mathbf{R})$. The proof of Proposition 3.1 will come from the following sequence of claims.

CLAIM 3.2. — *The map $f|_T: T \rightarrow N$ is π_1 -injective for any characteristic torus T in M . Moreover, $f_*(\pi_1 P)$ is a non-abelian group for each Seifert piece P of M .*

Proof. — Let T be a characteristic torus of M . From the Rigidity Theorem of Soma [20] and from hypothesis (ii) it is sufficient to consider the case when T is shared by two distinct Seifert components P and P' of M . Denote by h and h' the \mathbf{S}^1 -fiber of P and of P' respectively. If $f|_T: T \rightarrow N$ is not π_1 -injective then we may assume, since h and h' generate a rank 2

subgroup of $\pi_1 T$ (by minimality of the JSJ-decomposition), that P (for example) contains a simple closed curve c distinct from the fiber h such that $[c] \in \ker(f|T)_*$.

Moreover since ∂P is not connected then there exists an orientation preserving proper embedding $j: (F, \partial F) \rightarrow (P, \partial P)$ where F is a connected surface such that c is a boundary component of $j(F)$.

Indeed, denote by $T_1 = T$ the component of ∂P which contains c and by T_2, \dots, T_r the other components of ∂P with $r \geq 2$. For each $i = 1, \dots, r$ fix a basis $\langle s_i, h \rangle$, where s_i is a section of T_i with respect to the \mathbf{S}^1 -fibration of P such that $s_1 + \dots + s_r$ is nul-homologous in P and where h denotes the fiber of P . Denote by (α, β) the coprime integers with $\alpha \neq 0$ such that $c = \alpha[s_1] + \beta[h]$. Then

$$[c] + \alpha[s_2] + \dots + \alpha[s_r] - \beta[h] = 0 \text{ in } H_1(P; \mathbf{Z})$$

Thus there exists a nontrivial class η in $H_2(P, \partial P; \mathbf{Z})$ such that

$$\partial\eta = ((\alpha, \beta), (\alpha, 0), \dots, (\alpha, 0), (\alpha, -\beta))$$

in $H_1(\partial P) = H_1(T_1) \oplus H_1(T_2) \oplus \dots \oplus H_1(T_{r-1}) \oplus H_1(T_r)$. Since P is aspherical, it follows from [22] that each class in $H_2(P, \partial P; \mathbf{Z})$ can be represented by a properly embedded incompressible surface. This can be argued as follows. By the Poincaré Duality, $H_2(P, \partial P; \mathbf{Z}) \simeq H^1(P; \mathbf{Z})$, there exists a homomorphism $\rho: \pi_1 P \rightarrow \mathbf{Z} = \pi_1 S^1$ corresponding to η . Since the spaces are aspherical the homomorphism is induced by a map $f_\eta: P \rightarrow S^1$. Taking the pre-image of a regular value $\theta \in S^1$ and using the construction given in [10, Chapter 6] we may arrange f_η by a homotopy so that the components of $f_\eta^{-1}(\theta)$ are properly embedded incompressible surfaces. Denote by F such a surface. Then F is a horizontal surface and c is parallel to some components of ∂F .

Denote by $T \times [-1, 1]$ a regular neighborhood of T such that $T = T \times \{0\}$ and parametrize $T = \mathbf{S}^1 \times \mathbf{S}^1$ such that $c = \mathbf{S}^1 \times \{*\}$. As in [19], consider the relation \sim on M defined by $z \sim z'$ iff $z = z'$ or $z = (x, y, t) \in T \times I$, $z' = (x', y', t') \in T \times I$ and $y = y', t = t'$. Denote by $X = M / \sim$ the quotient space and by $\pi: M \rightarrow X$ the quotient map. Then the map f factors through X . Denote by $g: X \rightarrow N$ the map such that $f = g \circ \pi$. Denote by \hat{P} the image of P under π . Topologically \hat{P} is obtained from P after Dehn filling along T , identifying the meridian of a solid torus V to c so that the Seifert fibration of P extends to a Seifert fibration of \hat{P} and the image \hat{F} of F is a surface in \hat{P} obtained from F after gluing a 2-disk along each component of ∂F parallel to c . Consider the following commutative

diagram

$$\begin{array}{ccccc}
 F & \xrightarrow{j} & P & \xrightarrow{k} & M & \xrightarrow{f} & N \\
 \pi|_F \downarrow & & \pi|_P \downarrow & & \pi \downarrow & \nearrow g & \\
 \widehat{F} & \xrightarrow{\widehat{j}} & \widehat{P} & \xrightarrow{\widehat{k}} & X & &
 \end{array}$$

where \widehat{j} is induced by j and where $k: P \rightarrow M$ is the inclusion and $\widehat{k}: \widehat{P} \rightarrow X$ denotes the induced inclusion. Note that it follows from our construction, using standard homological arguments, that

$$\pi_{\#}(\alpha_M(F, j)) = \alpha_X(\widehat{F}, \widehat{j}) \in H_2^{l_1}(X; \mathbf{R}) \tag{*}$$

where $\alpha_X(\widehat{F}, \widehat{j})$ is defined by $\widehat{k}_{\#}\alpha_{\widehat{P}}(\widehat{F}, \widehat{j})$. We deduce, using hypothesis (iii) of Proposition 3.1, the following equalities:

$$\|\alpha_M(F, j)\|_1 \geq \|\pi_{\#}\alpha_M(F, j)\|_1 \geq \|f_{\#}\alpha_M(F, j)\|_1 = \|\alpha_M(F, j)\|_1$$

Thus using Lemma 2.3, equality (*) and Proposition 2.4(i) we get :

$$\|\widehat{F}\| \geq \|\alpha_X(\widehat{F}, \widehat{j})\|_1 = \|F\|$$

A contradiction. This proves the π_1 -injectivity of the map $f|_T$. It remains to check that $f_*(\pi_1 P)$ is a non-abelian group for each Seifert piece P . Assume that $f_*(\pi_1 P)$ is an abelian subgroup of $\pi_1 N$. Then the map $f|_P: P \rightarrow N$ factors through a space X with abelian fundamental group. Since $H_2^{l_1}(X)$ is trivial then we get a contradiction with hypothesis (iii) of Proposition 3.1 using point (i) of Proposition 2.4. □

CLAIM 3.3. — *There is a map g homotopic to f such that for each Seifert piece Σ of N then each component of $g^{-1}(\Sigma)$ is a Seifert piece of M .*

Proof. — By hypothesis (ii) one can apply [20, Rigidity Theorem]. Thus one may assume that f induces a $\text{deg}(f)$ -covering map from $\mathcal{H}(M)$ to $\mathcal{H}(N)$. Next, by Claim 3.2 one can apply [11, Mapping Theorem] which implies that one can arrange f by a homotopy so that for each canonical torus U of N then $f^{-1}(U)$ is a disjoint union of canonical tori of M . Hence for each Seifert piece Σ of N the space $f^{-1}(\Sigma)$ is a canonical graph submanifold of M (i.e. a submanifold which is the union of some Seifert pieces of M). If a component G of $f^{-1}(\Sigma)$ is not a Seifert manifold then there exists a canonical torus T in the interior of G which is shared by two distinct Seifert pieces Σ_1 and Σ_2 of G . Since by Claim 3.2 the group $f_*(\pi_1 \Sigma_i)$ is non-abelian, for $i = 1, 2$, then using [11, Addendum to Theorem VI.I.6] we know that $f|_{\Sigma_i}: \Sigma_i \rightarrow \Sigma$ is homotopic to a fiber preserving map.

Since $f|T$ is π_1 -injective we get a contradiction by the minimality of the JSJ-decomposition. This proves the claim. \square

Since f is π_1 -surjective then to complete the proof of Proposition 3.1 it remains to check the following

CLAIM 3.4. — *There is a map g homotopic to f , rel. to $\mathcal{H}(M)$, such that for each Seifert piece Σ of N and for each component G of $g^{-1}(\Sigma)$ then $g|G: G \rightarrow \Sigma$ is a covering map.*

Proof. — First of all we know that for each component G of $f^{-1}(\Sigma)$ then $f|G: G \rightarrow \Sigma$ is fiber preserving and non-degenerate in the sense of [11]. On the other hand, notice that Σ is necessarily homeomorphic to a product. Indeed if $\partial\Sigma \neq \emptyset$ this is obvious and if $\partial\Sigma = \emptyset$ this comes from the following argument: first note that in this case $\Sigma = N$ and $G = M$, thus if Σ is not homeomorphic to a product then the bundle has a non-zero Euler number and using the Seifert volume in [5, Theorem 3 and Lemma 3] and in [4, Theorem 4] we get a contradiction (since G has a zero Euler number and $\deg(f) \neq 0$). Thus after choosing appropriate sections we identify G with $K \times \mathbf{S}^1$, resp. Σ with $F \times \mathbf{S}^1$, where K , resp. F , is a hyperbolic surface.

Let \mathcal{F} denote a component of $(f|G)^{-1}(F)$. Arrange f so that \mathcal{F} is incompressible in G . Since f is non-degenerate and fiber preserving then the inclusion $i: \mathcal{F} \rightarrow G$ is necessarily an orientation preserving proper embedding and $f|\mathcal{F}: \mathcal{F} \rightarrow F$ descends to a map $\pi: K \rightarrow F$. Therefore we get

$$f_{\#}(\alpha_M(\mathcal{F}, i)) = \deg(f|\mathcal{F}: \mathcal{F} \rightarrow F)\alpha_N(F, j)$$

where $j: F \rightarrow \Sigma$ is the inclusion. This implies that

$$\|\mathcal{F}\| = |\deg(f|\mathcal{F}: \mathcal{F} \rightarrow F)| \times \|F\|$$

Thus we get the equality

$$\|K\| = \deg(\pi) \times \|F\|$$

Hence π is homotopic to a covering map which implies that $f|G$ is also homotopic to a covering map. This proves the claim and completes the proof of Proposition 3.1. \square

3.2. Proof of Theorem 1.2

We first check that the condition is necessary.

When $\|\alpha_M\|_1 = 0$ there is nothing to prove. So let's assume $\|\alpha_M\|_1 > 0$

Then either M is a $\widetilde{\text{SL}}_2(\mathbf{R})$ -manifold and Lemma 2.2 applies or M is not a $\widetilde{\text{SL}}_2(\mathbf{R})$ -manifold and Proposition 2.4 applies.

We verify now that the condition is sufficient. First of all note that according to [23] we may assume that M is not a virtual torus bundle. In the sequel it will be convenient to consider the following commutative diagram

$$\begin{array}{ccc}
 M_2 & \xrightarrow{f_2} & N_2 \\
 q \downarrow & & \downarrow p \\
 M_1 & \xrightarrow{f_1} & N_1 \\
 s \downarrow & & \downarrow r \\
 M & \xrightarrow{f} & N
 \end{array}$$

obtained as follows. The map $s: M_1 \rightarrow M$ is a finite covering such that each Seifert piece of M_1 is a circle bundle over an orientable surface with at least two boundary components if $\mathcal{T}_{M_1} \neq \emptyset$, and each canonical torus of M_1 is shared by two distinct components of $M_1 \setminus \mathcal{T}_{M_1}$ (for the existence of such a covering see [7, Lemmas 3.2 and 3.5]), the map $r: N_1 \rightarrow N$ is a finite covering corresponding to the subgroup $f_*s_*(\pi_1 M_1)$ in $\pi_1 N$, which is of finite index since $\deg(f) \neq 0$, the map $f_1: M_1 \rightarrow N_1$ is a lifting of $f \circ s$ which exists by our construction, the map $p: N_2 \rightarrow N_1$ is a finite covering such that each Seifert piece of N_2 is a \mathbf{S}^1 -bundle over an orientable surface and $f_2: M_2 \rightarrow N_2$ is the finite covering of f_1 corresponding to p , and $q: M_2 \rightarrow M_1$ is the covering corresponding to the subgroup $(f_1)_*^{-1}(p_*\pi_1 N_2)$. Notice that it follows from the construction that f_1 and f_2 are π_1 -surjective.

CLAIM 3.5. — *The map f_2 is homotopic to a homeomorphism.*

Proof. — Assume that M is a $\widetilde{\text{SL}}(2, \mathbf{R})$ -manifold. Since f has nonzero degree then f is homotopic to a non degenerate fiber preserving map and N is also a $\widetilde{\text{SL}}(2, \mathbf{R})$ -manifold. Thus f_2 is a π_1 -surjective nonzero degree map between circle bundle with nonzero Euler numbers. Denote by F_2 , resp. G_2 , the base of M_2 , resp. N_2 . It follows from the hypothesis of the theorem combined with Lemma 2.2 that $\|(f_2)_\#(\alpha_{M_2})\|_1 = \|\alpha_{M_2}\|_1$. Thus f_2 induces a map $g: F_2 \rightarrow G_2$ such that $\xi \circ f_2 = g \circ \pi$ where $\pi: M_2 \rightarrow F_2$ and $\xi: N_2 \rightarrow G_2$ denote the bundle projections. Since by definition $\alpha_{M_2} = \pi_\#^{-1}([F_2]_{l_1})$ then condition $\|(f_2)_\#(\alpha_{M_2})\|_1 = \|\alpha_{M_2}\|_1$ implies

$$\|[F_2]_{l_1}\|_1 = \|g_\#([F_2]_{l_1})\|_1 = \deg(g)\|[G_2]_{l_1}\|_1$$

and thus $\|[F_2]\| = \deg(g)\|[G_2]\|$. This proves that g and hence f_2 is homotopic to a homeomorphism (recall that f_2 is π_1 -surjective).

Assume now that M is not a $\widetilde{\text{SL}}(2, \mathbf{R})$ -manifold. Then using point (ii) of Proposition 2.4 (additivity property) and the isometry hypothesis we have $\|f_{\#}\alpha_M(\mathcal{F}_i, f_i)\|_1 = \|\alpha_M(\mathcal{F}_i, f_i)\|_1$ for any $i = 1, \dots, l$.

Indeed, by hypothesis we know that $\|f_{\#}\alpha_M\|_1 = \|\alpha_M\|_1$ then by point (ii) of Proposition 2.4 (additivity property) and using the definition of α_M we have

$$\|f_{\#}\alpha_M\|_1 = \|\alpha_M\|_1 = \sum_i \left\| \alpha_M \left(\frac{1}{k_i} \mathcal{F}_i, f_i \right) \right\|_1$$

Since, by paragraph 2, any continuous map induces a contraction with respect to the l_1 -norm we get

$$\begin{aligned} \|f_{\#}\alpha_M\|_1 &= \|\alpha_M\|_1 \geq \sum_i \left\| f_{\#}\alpha_M \left(\frac{1}{k_i} \mathcal{F}_i, f_i \right) \right\|_1 \\ &\geq \left\| \sum_i f_{\#}\alpha_M \left(\frac{1}{k_i} \mathcal{F}_i, f_i \right) \right\|_1 = \|f_{\#}\alpha_M\|_1 \end{aligned}$$

Hence we get

$$\sum_i \left(\left\| \alpha_M \left(\frac{1}{k_i} \mathcal{F}_i, f_i \right) \right\|_1 - \left\| f_{\#}\alpha_M \left(\frac{1}{k_i} \mathcal{F}_i, f_i \right) \right\|_1 \right) = 0$$

Again, since $f_{\#}$ is a contraction, then each term of the sum is non-negative and thus $\|f_{\#}\alpha_M(\mathcal{F}_i, f_i)\|_1 = \|\alpha_M(\mathcal{F}_i, f_i)\|_1$ for any $i = 1, \dots, l$.

Note that if $g_i: \mathcal{G}_i \rightarrow P_i$ is any orientation preserving proper map of a surface \mathcal{G}_i then

$$\|f_{\#}\alpha_M(\mathcal{G}_i, g_i)\|_1 = \|\alpha_M(\mathcal{G}_i, g_i)\|_1$$

This comes from the following observation: by [25, Lemma 6] there are rational numbers r_i, s_i and a vertical surface W_i in P_i (i.e. an incompressible properly embedded surface in P_i which is fibered by the \mathbf{S}^1 -fibers of P_i) such that

$$(g_i)_{\#}[\mathcal{G}_i] = r_i(f_i)_{\#}[\mathcal{F}_i] + s_i[W_i] \in H_2(P_i, \partial P_i)$$

and since W_i has zero simplicial volume the equality follows. In order to apply Proposition 3.1 to the map f_2 it remains to check hypothesis (iii). Let $g: F_2 \rightarrow P_2$ be an orientation preserving embedding of a surface into a Seifert piece P_2 of M_2 . Denote by P the Seifert piece of M such that P_2 is over P . Then by the above equality, applied to $s \circ q \circ g: F_2 \rightarrow P$ we have

$$\|f_{\#}\alpha_M(F_2, s \circ q \circ g)\|_1 = \|\alpha_M(F_2, s \circ q \circ g)\|_1$$

On the other hand, using point (iii) of Proposition 2.4 we know that

$$\|\alpha_M(F_2, s \circ q \circ g)\|_1 = \|\alpha_{M_2}(F_2, f_2)\|_1$$

By the commutativity of the diagram we have

$$f_{\#}(\alpha_M(F_2, s \circ q \circ g)) = r_{\#}p_{\#}(f_2)_{\#}(\alpha_{M_2}(F_2, f_2))$$

Therefore, this yields

$$\begin{aligned} \|\alpha_{M_2}(F_2, f_2)\|_1 &= \|r_{\#}p_{\#}(f_2)_{\#}(\alpha_{M_2}(F_2, f_2))\|_1 \leq \|(f_2)_{\#}(\alpha_{M_2}(F_2, f_2))\|_1 \\ &\leq \|\alpha_{M_2}(F_2, f_2)\|_1 \end{aligned}$$

Accordingly we deduce that f_2 satisfies hypothesis of Proposition 3.1 which implies that f_2 is homotopic to a homeomorphism. \square

Since M is an aspherical 3-manifolds then it has a torsion free fundamental group ([10]). Since p, q, r, s are finite covering maps then they induce injective homomorphisms at the π_1 -level and since f_2 induces an isomorphism f must be π_1 -injective. Consider the finite covering $\tilde{N} \rightarrow N$ corresponding to $f_*(\pi_1 M)$. Then f lifts to a map $\tilde{f}: M \rightarrow \tilde{N}$ inducing an isomorphism at the π_1 -level. We deduce from this point using [13, Theorem 0.7] that \tilde{f} is a homeomorphism. This implies that f is a covering map and completes the proof of Theorem 1.2.

3.3. Proof of Theorem 1.5

By the Mapping Theorem of [9] the map f induces an isometry $f_{\#}: H_3^{l_1}(M) \rightarrow H_3^{l_1}(N)$. On the other hand, using the same construction as in the proof of Lemma 2.5 in dimension three (instead of dimension 2) one deduces that the natural map $H_3(M) \rightarrow H_3^{l_1}(M)$ is an isometry. Indeed, if $\|M\| = 0$ there is nothing to prove and if $\|M\| > 0$ this means that M contains some hyperbolic pieces $\mathcal{H}_1, \dots, \mathcal{H}_l$ in its geometric decomposition. Thus by the straightening technique used in the proof of Lemma 2.5 one can in the same way construct an element $\Omega \in H_b^3(M)$ such that $\langle \Omega, [M] \rangle = \text{vol}(\mathcal{H}_1) + \dots + \text{vol}(\mathcal{H}_l)$ with $\|\Omega\|_{\infty} \leq V_3$, where V_3 denotes the supremum of the volume of geodesic 3-simplices in the hyperbolic 3-space. Hence the l_1 -norm of $[M]$ is $\|M\|$ in $H_3^{l_1}(M)$, proving that $H_3(M) \rightarrow H_3^{l_1}(M)$ is an isometry. This implies that $f_{\#}: H_3(M; \mathbf{R}) \rightarrow H_3(N; \mathbf{R})$ is an isometry.

Using the same covering argument as above one can assume, without loss of generality, that f is π_1 -surjective. If M is orientable* then Corollary 1.5 follows from Theorem 1.2 by the Mapping Theorem of [9]. If M is not orientable* then there exists a 2-fold finite covering $p: M_2 \rightarrow M$ such that M_2 is orientable*. Note that the composition $g = f \circ p_2$ is not π_1 -surjective. Indeed if g is π_1 -surjective then $f \circ p_2$ is homotopic to a homeomorphism because since f_* has an amenable kernel then so is $\ker(g_*)$ and thus g

induces an isometric isomorphism $g_{\sharp}: H_2^{l_1}(M_2) \rightarrow H_2^{l_1}(N)$. Since moreover $\|M_2\| = 2\|N\|$ one can apply Theorem 1.2. A contradiction. Since g is not π_1 -surjective then there exists a 2-fold covering $f': M_2 \rightarrow N_2$ of the map f . Again, since f_* has an amenable kernel then so is $\ker(f'_*)$. Moreover f' is π_1 -surjective by construction and thus it induces an isometric isomorphism $f'_{\sharp}: H_2^{l_1}(M_2) \rightarrow H_2^{l_1}(N_2)$ and $\|M_2\| = \deg(f')\|N_2\|$. Hence by Theorem 1.2 the f' is homotopic to a homeomorphism. Hence f is homotopic to a homeomorphism. This completes the proof of the corollary.

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