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RSK BASES AND KAZHDAN-LUSZTIG CELLS

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ABSTRACT. — From the combinatorial characterizations of the right, left, and two-sided Kazhdan-Lusztig cells of the symmetric group, " RSK bases" are constructed for certain quotients by two-sided ideals of the group ring and the Hecke algebra. Applications to invariant theory, over various base rings, of the general linear group and representation theory, both ordinary and modular, of the symmetric group are discussed.

RÉSUMÉ. — À partir des caractérisations combinatoires des cellules de Kazhdan-Lusztig du groupe symétrique, on construit des bases "RSK" pour certains quotients du l'algèbre du groupe et de l'algèbre de Hecke. On étudie des applications à la théorie des invariants du groupe linéaire général sur divers anneaux de base et à la théorie des réprésentations, soit ordinaire ou modulaire, du groupe symétrique.

1. Introduction: summary and organization of results

The starting point of the work described in this paper is a question in classical invariant theory (§1.1). It leads naturally to questions about representations of the symmetric group over the complex numbers (§1.2, §1.3) and over algebraically closed fields of positive characteristic (§1.5), and in turn to the computation of the determinant of a certain matrix encoding the multiplication of Kazhdan-Lusztig basis elements of the Hecke algebra (§1.6), using which one can recover a well-known criterion for the irreducibility of Specht modules over fields of positive characteristic (§1.7).

For the sake of readability, we have tried, to the extent possible, to keep the proofs of our results independent of each other. So sections 4–6 can be read without reference to one another.

RSK stands for Robinson-Schensted-Knuth.

Keywords: Symmetric group, Hecke algebra, Kazhdan-Lusztig basis, RSK correspondence.
Math. classification: 05E10, 05E15, 20C08, 20C30.

1.1. Motivation from invariant theory

We begin by recalling a basic theorem of classical invariant theory. Let k be a commutative ring with identity and V a free k-module of finite rank d. Let $\mathrm{GL}(V)$ denote the group of k-automorphisms of V, and consider the diagonal action of $\mathrm{GL}(V)$ on $V^{\otimes n}$. Let $\mathfrak{S}_n$ denote the symmetric group of bijections of the set $\{1,\ldots,n\}$ and $k\mathfrak{S}_n$ the group ring of $\mathfrak{S}_n$ with coefficients in k. There is a natural action of $\mathfrak{S}_n$ on $V^{\otimes n}$ by permuting the factors: more precisely, $(v_1\otimes\cdots\otimes v_n)\cdot\sigma := v_{1\sigma}\otimes\cdots\otimes v_{n\sigma}$ (all actions are on the right by convention). This action commutes with the action of $\mathrm{GL}(V)$, and so the k-algebra map $\Theta_n : k\mathfrak{S}_n \to \mathrm{End}_k\, V^{\otimes n}$ defining the action of $\mathfrak{S}_n$ has image in the space $\mathrm{End}_{\mathrm{GL}(V)}\, V^{\otimes n}$ of $\mathrm{GL}(V)$-endomorphisms of $V^{\otimes n}$.

We have the following result (see [4, Theorems 4.1, 4.2]):

> Assume the following: if $f(X)$ is an element of degree n of the polynomial ring $k[X]$ in one variable over k that vanishes as a function on k, then $f(X)$ is identically zero. (This holds for example when k is an infinite field, no matter what n is.) Then the k-algebra homomorphism Θ_n maps onto $\mathrm{End}_{\mathrm{GL}(V)}\, V^{\otimes n}$ and its kernel is the two-sided ideal $J(n,d)$ defined as follows:
>
> - $J(n,d) := 0$ if $d \geqslant n$;
> - if $d < n$, then it is the two-sided ideal generated by the element $y_d := \sum_{\tau\in\mathfrak{S}_{d+1}} (\operatorname{sgn}\tau)\tau$, where $\mathfrak{S}_{d+1}$ is the subgroup of $\mathfrak{S}_n$ consisting of the permutations that fix point-wise the elements $d+2, \ldots, n$, and $\operatorname{sgn}\tau$ denotes the sign of τ.(1)

Thus $k\mathfrak{S}_n/J(n,d)$ gets identified with the algebra of $\mathrm{GL}(V)$-endomorphisms of $V^{\otimes n}$ (under the mild assumption on k mentioned above), and it is of invariant theoretic interest to ask:

> Is there a natural choice of a k-basis for $k\mathfrak{S}_n/J(n,d)$?

Our answer:

Theorem 1. — *Let k be any commutative ring with identity. Those permutations σ of $\mathfrak{S}_n$ such that the sequence $1\sigma, \ldots, n\sigma$ has no decreasing sub-sequence of length more than d form a basis for $k\mathfrak{S}_n/J(n,d)$.*

The proof of the theorem will be given in §4. It involves the Hecke algebra of the symmetric group and its Kazhdan-Lusztig basis. Some further comments on the proof can be found in §1.4.

(1) The subgroup $\mathfrak{S}_{d+1}$ could be taken to be that consisting of the permutations that fix point-wise any arbitrarily fixed set of $n-d-1$ elements.

The theorem enables us to:

- obtain a k-basis, closed under multiplication, for the subring of $\mathrm{GL}(V)$-invariants of the tensor algebra of V (§4.2).
- when k is a field of characteristic 0, to limit the permutations in the well-known description ([27], [29]) of a spanning set for polynomial $\mathrm{GL}(V)$-invariants of several matrices (§4.2); or, more generally, to limit the permutations in the description in [5] of a spanning set by means of "picture invariants" for polynomial $\mathrm{GL}(V)$-invariants of several tensors (§4.3).

1.2. A question about tabloid representations

Let us take the base ring k in §1.1 to be the field $\mathbb{C}$ of complex numbers. Then the ideal $J(n,d)$ has a representation theoretic realization as we now briefly recall (see §5.2 for the justification). Let $\lambda(n,d)$ be the unique partition of n with at most d parts that is smallest in the dominance order (§2.2.1). Consider the linear representation of $\mathfrak{S}_n$ on the free vector space $\mathbb{C}\mathcal{T}_{\lambda(n,d)}$ generated by tabloids of shape $\lambda(n,d)$ (§2.5). The ideal $J(n,d)$ is the kernel of the $\mathbb{C}$-algebra map $\mathbb{C}\mathfrak{S}_n \to \mathrm{End}_{\mathbb{C}}\,\mathbb{C}\mathcal{T}_{\lambda(n,d)}$ defining this representation.

Replacing the special partition $\lambda(n,d)$ above by an arbitrary one λ of n (§2.1) and considering the $\mathbb{C}$-algebra map $\rho_\lambda : \mathbb{C}\mathfrak{S}_n \to \mathrm{End}_{\mathbb{C}}\,\mathbb{C}\mathcal{T}_\lambda$ defining the linear representation of $\mathfrak{S}_n$ on the space $\mathbb{C}\mathcal{T}_\lambda$ generated by tabloids of shape λ, we ask:

> Is there a natural set of permutations that form a $\mathbb{C}$-basis for the group ring $\mathbb{C}\mathfrak{S}_n$ modulo the kernel of the map ρ_λ? Equivalently, one could demand that the images of the permutations under ρ_λ form a basis for the image.

Our answer:

Theorem 2. — *Permutations of μ, as μ varies over partitions that dominate λ, form a $\mathbb{C}$-basis of $\mathbb{C}\mathfrak{S}_n$ modulo the kernel of $\rho_\lambda : \mathbb{C}\mathfrak{S}_n \to \mathrm{End}\,\mathbb{C}\mathcal{T}_\lambda$.*

The dominance order on partitions is the usual one (§2.2). The of a permutation is defined in terms of the RSK-correspondence (§2.4). As follows readily from the definitions, the shape of a permutation σ dominates the partition $\lambda(n,d)$ precisely when $1\sigma, \ldots, n\sigma$ has no decreasing sub-sequence of length exceeding d. Thus, in the case when the base ring is the complex field, Theorem 1 follows from Theorem 2.

The proof of the theorem will be given in §5. Like that of Theorem 1, it too involves the Hecke algebra of the symmetric group and its Kazhdan-Lusztig basis. Some further comments on the proof can be found in §1.4.

The theorem holds also over the integers and over fields of characteristic 0 —as can be deduced easily from the complex case (see §5.3)— but it is not true in general over a field of positive characteristic: see Example 11. A Hecke analogue of the theorem also holds: see §1.4 below.

As pointed out by the referee, the recent paper [9] is concerned with constructing a basis for the annihilator of $\mathbb{C}\mathcal{T}_\lambda$ (and of its Hecke analogue M^λ whose definition is recalled below in §8.2). The answers are in terms of "Murphy basis", which like the Kazhdan-Lusztig basis are known to be "cellular".

1.3. A question regarding the irreducible representations of the symmetric group

The question raised just above (in §1.2) can be modified to get one of more intrinsic appeal. Given a partition λ of n, consider, instead of the action of $\mathfrak{S}_n$ on tabloids of shape λ, the right cell module $R(\lambda)_\mathbb{C}$ in the sense of Kazhdan-Lusztig (§3.5), or, equivalently (see §8.3), the Specht module $S^\lambda_\mathbb{C}$ (§2.6). The right cell modules are irreducible and every irreducible $\mathbb{C}\mathfrak{S}_n$-module is isomorphic to $R(\mu)_\mathbb{C}$ for some μn (§6.2).

The irreducibility of $R(\lambda)_\mathbb{C}$ implies, by a well-known result of Burnside (see, *e.g.*, [3, Chapter 8, §4, No. 3, Corollaire 1]), that the defining $\mathbb{C}$-algebra map $\mathbb{C}\mathfrak{S}_n \to \operatorname{End}_\mathbb{C} R(\lambda)_\mathbb{C}$ is surjective. The dimension of $R(\lambda)_\mathbb{C}$ (equivalently of $S^\lambda_\mathbb{C}$) equals the number $d(\lambda)$ of standard tableaux of shape λ (§2.3.1, §6.2). Thus there exist $d(\lambda)^2$ elements of $\mathbb{C}\mathfrak{S}_n$, even of $\mathfrak{S}_n$ itself, whose images in $\operatorname{End}_\mathbb{C} R(\lambda)_\mathbb{C}$ form a basis (for $\operatorname{End}_\mathbb{C} R(\lambda)_\mathbb{C}$). We ask:

> Is there is a natural choice of such elements of $\mathbb{C}\mathfrak{S}_n$, even of $\mathfrak{S}_n$?

Indeed there is, as the following theorem says. As pointed out to us by Andrew Mathas, the theorem is a consequence of the *cellularity* in the sense of [17] of the Kazhdan-Lusztig basis.

Theorem 3 (Consequence of cellularity of the Kazhdan-Lusztig basis [17]). — *Consider the Kazhdan-Lusztig basis elements of the group ring $\mathbb{C}\mathfrak{S}_n$ indexed by permutations of λ. Their images under the defining $\mathbb{C}$-algebra map $\mathbb{C}\mathfrak{S}_n \to \operatorname{End}_\mathbb{C} R(\lambda)_\mathbb{C}$ form a basis for $\operatorname{End}_\mathbb{C} R(\lambda)_\mathbb{C}$.*

By the Kazhdan-Lusztig basis elements of the group ring $\mathbb{C}\mathfrak{S}_n$, we mean the images in $\mathbb{C}\mathfrak{S}_n$ of the Kazhdan-Lusztig basis elements of the Hecke algebra of $\mathfrak{S}_n$ under the natural map setting the parameter value to 1 (§3). The of a permutation is defined using the RSK-correspondence (§2.4).

The theorem is proved in §6. Some comments on the proof of the theorem can be found in §1.4.

We do not know a natural choice of elements of the group $\mathfrak{S}_n$ itself whose images in $\operatorname{End}_{\mathbb{C}} R(\lambda)_{\mathbb{C}}$ are a basis. Permutations of λ of course suggest themselves, but they do not in general have the desired property (Example 16).

1.4. Comments on the proofs of Theorems 1–3

Properties of the Kazhdan-Lusztig basis of the Hecke algebra associated to the symmetric group are the key to the proofs, although the statements of Theorems 1 and 2 do not involve the Hecke algebra at all. The relevant properties are recalled in two instalments: the first, in §3, is the more substantial; the second, in §4.1, consists of further facts needed more specifically for the proof of Theorem 1.

Theorem 3 follows by combining the Wedderburn structure theory of semisimple algebras, as recalled in §6.1, with the following observation implicit in [15] and explicitly formulated in §3.6:

> A Kazhdan-Lusztig C-basis element C_w kills the right cell (or equivalently Specht) module corresponding to a shape λ unless λ is dominated by the of the indexing permutation w.

The observation in turn follows easily from the combinatorial characterizations of the left, right, and two-sided Kazhdan-Lusztig cells in terms of the RSK-correspondence and the dominance order on partitions (§3.4.1, §3.4.2). Our primary source for these characterizations, which are crucial to our purpose, is [15].

As pointed out to the authors by the referee among others, special properties of the Kazhdan-Lusztig basis of the Hecke algebra of the symmetric group, as the one in the observation above, have been noted and well studied. In fact, they have been axiomatized in [17], where any basis enjoying these properties is termed *cellular* —see also [23, Chapter 2]. Sections 6 and 7.2 below are in effect working out some consequences of cellularity; and 7.1 is in effect proving the cellularity of the Kazhdan-Lusztig basis using results of [15]. Note that establishing cellularity is difficult, there being a reliance on [15] in our case.

Theorem 2 follows by combining the above observation with two well known facts: the isomorphism of the right cell module with the Specht module and the well-known decomposition into irreducibles of $\mathbb{C}\mathcal{T}_\lambda$. A Hecke analogue of Theorem 2 also holds: see Theorem 7 in the earlier version [28] of the present paper. A proof of it parallel to the proof of Theorem 2 as in here can be given using results of [8]. The proof in [28] is different and more in keeping with the ideas developed here.

The main technical point in the proof of Theorem 1 is isolated as Lemma 7, which is a two sided analogue of [24, Lemma 2.11] recalled below as Proposition 6.

1.5. Analogue of Theorem 3 over fields of positive characteristic

The Hecke algebra and its Kazhdan-Lusztig basis make sense over an arbitrary base (§3). The cell modules and Specht modules are also defined and isomorphic over any base (§3.5, §2.6, §8.3). Thus we can ask for the analogue of Theorem 3 over a field of arbitrary characteristic, keeping in mind of course that the cell modules may not be irreducible any longer. We prove:

THEOREM 4. — *Let k be a field of positive characteristic p. Let λ be a partition of a positive integer n no part of which is repeated p or more times. Suppose that the right cell module $R(\lambda)_k$ is irreducible. Consider the Kazhdan-Lusztig basis elements of the group ring $k\mathfrak{S}_n$ indexed by permutations of λ. Their images under the defining k-algebra map $k\mathfrak{S}_n \to \operatorname{End}_k R(\lambda)_k$ form a basis for $\operatorname{End}_k R(\lambda)_k$.*

The theorem is a special case of Theorem 23 proved in §11. Like the proof of Theorem 3, that of Theorem 23 too uses the observation formulated in §3.6, but, the group ring $k\mathfrak{S}_n$ being not necessarily semisimple, we cannot rely on Wedderburn structure theory any more. Instead we take a more head-on approach:

> Choosing a convenient basis of $\operatorname{End}_k R(\lambda)_k$, we express as linear combinations of these basis elements the images in $\operatorname{End}_k R(\lambda)_k$ of the appropriate Kazhdan-Lusztig basis elements of $k\mathfrak{S}_n$. Denoting by $\mathbb{G}(\lambda)$ the resulting square matrix of coefficients, we give an explicit formula for its determinant $\det \mathbb{G}(\lambda)_k$.

In fact, we obtain a formula for $\det \mathbb{G}(\lambda)$, where $\mathbb{G}(\lambda)$ is the analogous matrix of coefficients over an arbitrary base and over the Hecke algebra (rather than the group ring): see §7 for details. We then need only specialize to get $\det \mathbb{G}(\lambda)_k$. Given the formula, it is a relatively easy matter to get a criterion for $\det \mathbb{G}(\lambda)_k$ not to vanish, thereby proving Theorem 23.

1.6. A hook length formula for the determinant of $\mathcal{G}(\lambda)$

To obtain the formula for $\det \mathbb{G}(\lambda)$, we first show that $\mathbb{G}(\lambda)$ has a nice form which enables us to reduce the computation to that of the determinant of a matrix $\mathcal{G}(\lambda)$ of much smaller size. We discuss how this is done.

The basis of $\operatorname{End} R(\lambda)$ with respect to which the matrix $\mathbb{G}(\lambda)$ is computed suggests itself: $R(\lambda)$ has a basis consisting of classes of Kazhdan-Lusztig elements C_w, where w belongs to a right cell of shape λ of $\mathfrak{S}_n$ (§3.5); considering the endomorphisms which map one of these basis elements to another (possibly the same) and kill the rest, we get the appropriate basis for $\operatorname{End} R(\lambda)$. This means that the matrix $\mathbb{G}(\lambda)$ encodes the multiplication table for Kazhdan-Lusztig basis elements C_w indexed by permutations of λ, modulo those indexed by permutations of lesser shape in the dominance order.

The special (cellularity) properties of the Kazhdan-Lusztig elements now imply that the matrix $\mathbb{G}(\lambda)$, which is of size $d(\lambda)^2 \times d(\lambda)^2$ (where $d(\lambda)$ is the number of standard tableaux of shape λ), is a "block scalar" matrix, *i.e.*, when broken up into blocks of size $d(\lambda) \times d(\lambda)$, only the diagonal blocks are non-zero, and all the diagonal blocks are equal. Denoting by $\mathcal{G}(\lambda)$ the diagonal block, we are thus reduced to computing the determinant of $\mathcal{G}(\lambda)$. The details of this reduction are worked out in §7.

The formula for the determinant of $\mathcal{G}(\lambda)$ is given in Theorem 18, the main ingredients in the proof of which are formulas from [7] and [19]. The relevance of those formulas to the present context is not clear at first sight. They are about the determinant, denoted $\det(\lambda)$, of the matrix of a certain bilinear form, the Dipper-James form, on the Specht module S^λ, computed with respect to the "standard basis" of S^λ; while $\mathcal{G}(\lambda)$ has to do with multiplication of Kazhdan-Lusztig basis elements. The connection between $\det(\lambda)$ and $\det \mathcal{G}(\lambda)$ is established in §9 (see Equation (9.2)) using results of [24].

1.7. On the irreducibility of Specht modules

Finally, we discuss another application of the formula for the determinant of the matrix $\mathcal{G}(\lambda)$ introduced in §1.6. Suppose that the determinant did not vanish when the Hecke algebra is specialized to group ring and the scalars extended to a field k. Then, evidently, the images in $\operatorname{End}_k R(\lambda)_k$ of the Kazhdan-Lusztig basis elements C_w, as w varies over permutations of λ, form a basis for $\operatorname{End}_k R(\lambda)_k$, which means in particular that the defining map $k\mathfrak{S}_n \to \operatorname{End}_k R(\lambda)_k$ is surjective, and so $R(\lambda)_k$ is irreducible.

In other words, the non-vanishing of $\det \mathcal{G}(\lambda)$ in k gives a criterion for the irreducibility of $R(\lambda)_k$ (equivalently, of S_k^λ). The criterion thus obtained matches precisely the one conjectured by Carter and proved in [18, 19]. We thus obtain an independent proof of the Carter criterion. The details are worked out in §11.

1.8. Acknowledgments

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2. Recall of some basic notions

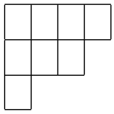
We recall in this section the basic combinatorial and representation theoretic notions that we need. Note that our definition of the RSK-correspondence (§2.4) differs from the standard (as e.g. in [13, Chapter 4]) by a flip.

Throughout n denotes a positive integer.

2.1. Partitions and shapes

By a *partition* λ of n, written λn, is meant a sequence $\lambda_1 \geqslant \cdots \geqslant \lambda_r$ of positive integers such that $\lambda_1 + \cdots + \lambda_r = n$. The integer r is the *number of parts* in λ. We often write $\lambda = (\lambda_1, \cdots, \lambda_r)$; sometimes even $\lambda = (\lambda_1, \lambda_2, \cdots)$. When the latter notation is used, it is to be understood that $\lambda_t = 0$ for $t > r$.

Partitions of n are in bijection with *shapes of Young diagrams* (or simply *shapes*) with n boxes: the partition $\lambda_1 \geqslant \cdots \geqslant \lambda_r$ corresponds to the shape with λ_1 boxes in the first row, λ_2 in the second row, and so on, the boxes being arranged left- and top-justified. Here for example is the shape corresponding to the partition $(4,3,1)$ of 8:



Partitions are thus identified with shapes and the two terms are used interchangeably.

2.2. Dominance order on partitions

Given partitions $\mu = (\mu_1, \mu_2, \ldots)$ and $\lambda = (\lambda_1, \lambda_2, \ldots)$ of n, we say μ *dominates* λ, and write $\mu \trianglerighteq \lambda$, if

$$\mu_1 \geqslant \lambda_1, \quad \mu_1 + \mu_2 \geqslant \lambda_1 + \lambda_2, \quad \mu_1 + \mu_2 + \mu_3 \geqslant \lambda_1 + \lambda_2 + \lambda_3, \quad \cdots .$$

We write $\mu \triangleright \lambda$ if $\mu \trianglerighteq \lambda$ and $\mu \neq \lambda$. The partial order $\trianglerighteq$ on the set of partitions (or shapes) of n will be referred to as the *dominance order*.

2.2.1. The partition $\lambda(n,d)$

Given integers n and d, there exists a unique partition $\lambda(n,d)n$ that has at most d parts and is smallest in the dominance order among those with at most d parts. For example, $\lambda(8,3) = (3,3,2)$.

2.3. Tableaux and standard tableaux

A *Young tableau*, or just *tableau*, of shape λn is an arrangement of the numbers $1, \ldots, n$ in the boxes of shape λ. There are, evidently, $n!$ tableaux of shape λ. A tableau is *row standard* (respectively, *column standard*) if in every row (respectively, column) the entries are increasing left to right (respectively, top to bottom). A tableau is *standard* if it is both row standard and column standard. An example of a standard tableau of shape $(3,3,2)$:

1	3	5
2	6	8
4	7	

2.3.1. The number of standard tableaux

The number of standard tableaux of a given shape λn is denoted $d(\lambda)$. There is a well-known "hook length formula" for it [12]: $d(\lambda) = n!/\prod_\beta h_\beta$, where β runs over all boxes of shape λ and h_β is the *hook length* of the box β which is defined as one more than the sum of the number of boxes to the right of β and the number of boxes below β.

The hook lengths for shapes $(3,3,2)$ and $(4,3,1)$ are shown below:

5	4	2
4	3	1
2	1	

6	4	3	1
4	2	1	
1			

Thus $d(3,3,2) = 8!/(5\cdot 4\cdot 2\cdot 4\cdot 3\cdot 1\cdot 2\cdot 1) = 42$ and $d(4,3,1) = 8!/6\cdot 4\cdot 3\cdot 1\cdot 4\cdot 2\cdot 1\cdot 1 = 70$.

2.4. The RSK-correspondence and the of a permutation

The *Robinson-Schensted-Knuth correspondence* (*RSK correspondence* for short) is a well-known procedure that sets up a bijection between the symmetric group $\mathfrak{S}_n$ and ordered pairs of standard tableaux of the same shape with n boxes. We do not recall here the procedure, referring the reader instead to [13, Chapter 4]. It will be convenient for our purposes to modify slightly the procedure described in [13].

Denoting by $(A(w), B(w)) \leftrightarrow w$ the bijection of [13], what we mean by *RSK correspondence* is the bijection $(B(w), A(w)) \leftrightarrow w$; since $A(w) = B(w^{-1})$ and $A(w^{-1}) = B(w)$ (see [13, Corollary on page 41]), we could equally well define our RSK correspondence as $(A(w), B(w)) \leftrightarrow w^{-1}$. The of a permutation w is defined to be the shape of either of $A(w)$, $B(w)$.

2.4.1. An example

The permutation $(1542)(36)$ (written as a product of disjoint cycles) has $(3,2,1)$. Indeed it is mapped under the RSK correspondence in our sense to the ordered pair (A, B) of standard tableaux, where:

$A =$

1	3	5
2	4	
6		

$B =$

1	2	3
4	6	
5		

2.4.2. Remark

The justification for our modification of the standard definition of RSK-correspondence is that it was the simplest way we could think up of reconciling the notational conflict among the two sets of papers upon which we rely: [6, 24] and [15]. Permutations act on the right in the former —a convention which we too follow— but on the left in the latter. The direction in which they act makes a difference to statements involving the RSK correspondence: most importantly for us, to the characterization of one-sided cells (see §3.4.1 below). This creates a problem: we cannot be quoting literally from both sets of sources without changing something. Altering the definition of RSK correspondence as above is the path of least resistance, and allows us to quote more or less verbatim from both sets.

2.5. Tabloids and tabloid representations

Let $\lambda = (\lambda_1, \lambda_2, \ldots)n$. A *tabloid* of shape λ is a partition of the set $[n] := \{1, \ldots, n\}$ into an ordered r-tuple of subsets, the first consisting of λ_1 elements, the second of λ_2 elements, and so on. Depicted below are two tabloids of shape $(3, 3, 2)$:

1	3	5		3	5	8
7	8	9		1	6	
4	6			2	7	

The members of the first subset are arranged in increasing order in the first row, those of the second subset in the second row, and so on.

Given a tableau T of shape λ, it determines, in the obvious way, a tabloid of shape λ denoted $\{T\}$: the first subset consists of the elements in the first row, the second of those in the second row, and so on.

The defining action of $\mathfrak{S}_n$ on $[n]$ induces, in the obvious way, an action on the set $\mathcal{T}_\lambda$ of tabloids of shape λ. The free $\mathbb{Z}$-module $\mathbb{Z}\mathcal{T}_\lambda$ with $\mathcal{T}_\lambda$ as a $\mathbb{Z}$-basis provides therefore a linear representation of $\mathfrak{S}_n$ over $\mathbb{Z}$. By base change we get such a representation over any commutative ring with unity k: $k\mathcal{T}_\lambda := \mathbb{Z}\mathcal{T}_\lambda \otimes_{\mathbb{Z}} k$. We call it the *tabloid representation* corresponding to the shape λ.

2.6. Specht modules

The Specht module corresponding to a partition λn is a certain $\mathfrak{S}_n$-submodule of the tabloid representation $\mathbb{Z}\mathcal{T}_\lambda$ just defined. For a tableau T

of shape λ, define $\mathfrak{e}_T$ in $\mathbb{Z}\mathcal{T}_\lambda$ by

$$\mathfrak{e}_T := \sum \operatorname{sgn}(\sigma)\{T\sigma\}$$

where the sum is taken over permutations σ of $\mathfrak{S}_n$ in the column stabiliser of T, $\operatorname{sgn}(\sigma)$ denotes the sign of σ, and $\{T\sigma\}$ denotes the tabloid corresponding to the tableau $T\sigma$ in the obvious way (see §2.5). The *Specht module* S^λ is the linear span of the $\mathfrak{e}_T$ as T runs over all tableaux of shape λ. It is an $\mathfrak{S}_n$-submodule of $\mathbb{Z}\mathcal{T}_\lambda$ with $\mathbb{Z}$-basis $\mathfrak{e}_T$, as T varies over standard tableaux (see, for example, [13, §7.2]). By base change we get the Specht module S^λ_k over any commutative ring with identity k: $S^\lambda_k := S^\lambda \otimes_{\mathbb{Z}} k$. Evidently, S^λ_k is a free k-module of rank the number $d(\lambda)$ of standard tableaux of shape λ (§2.3.1).

3. Set up: Hecke algebra and Kazhdan-Lusztig cells

Let n denote a fixed positive integer and $\mathfrak{S}_n$ the symmetric group on n letters. Let S be the subset consisting of the simple transpositions $(1,2)$, $(2,3)$, …, $(n-1,n)$ of the symmetric group $\mathfrak{S}_n$. Then $(\mathfrak{S}_n, S)$ is a Coxeter system in the sense of [3, Chapter 4]. Let $A := \mathbb{Z}[v, v^{-1}]$, the Laurent polynomial ring in the variable v over the integers.

3.1. The Hecke algebra and its T-basis

Let $\mathcal{H}$ be the *Iwahori-Hecke algebra* corresponding to $(\mathfrak{S}_n, S)$, with notation as in [15]. Recall that $\mathcal{H}$ is an A-algebra: it is a free A-module with basis T_w, $w \in \mathfrak{S}_n$, the multiplication being defined by

$$T_sT_w = \begin{cases} T_{sw} & \text{if } \ell(sw) = \ell(w)+1 \\ (v - v^{-1})T_w + T_{sw} & \text{if } \ell(sw) = \ell(w)-1 \end{cases}$$

for $s \in S$ and $w \in \mathfrak{S}_n$ and ℓ is the length function. We put

$$\epsilon(w) := (-1)^{\ell(w)} \quad \text{and} \quad v_w := v^{\ell(w)} \quad \text{for } w \in \mathfrak{S}_n.$$

An induction on length gives (in any case, see [6, Lemma 2.1 (iii)] for (3.2)):

$$T_wT_{w'} = T_{ww'} \qquad \text{if } \ell(w) + \ell(w') = \ell(ww') \tag{3.1}$$

$$T_uT_{u'} = T_{uu'} + \sum_{uu'<w} a_wT_w \quad \text{for } u,\ u' \text{ in } \mathfrak{S}_n. \tag{3.2}$$

Here, as elsewhere, $<$ denotes the Bruhat-Chevalley partial order on $\mathfrak{S}_n$. In particular, the coefficient of T_1 in $T_uT_{u'}$ is non-zero if and only if $u' = u^{-1}$ and equals 1 in that case.

3.1.1. The relation between v and q

We follow the conventions of [15]. In particular, to pass from our notation to that of [20], [6], or [24], we need to replace v by $q^{1/2}$ and T_w by $q^{-\ell(w)/2}T_w$.

3.1.2. Specializations of the Hecke algebra

Let k be a commutative ring with unity and a an invertible element in k. There is a unique ring homomorphism $A \to k$ defined by $v \mapsto a$. We denote by $\mathcal{H}_k$ the k-algebra $\mathcal{H} \otimes_A k$ obtained by extending the scalars to k via this homomorphism. We have a natural A-algebra homomorphism $\mathcal{H} \to \mathcal{H}_k$ given by $h \mapsto h \otimes 1$. By abuse of notation, we continue to use the same symbols for the images in $\mathcal{H}_k$ of elements of $\mathcal{H}$ as for those elements themselves. If M is a (right) $\mathcal{H}$-module, $M \otimes_A k$ is naturally a (right) $\mathcal{H}_k$-module.

An important special case is when we take a to be the unit element 1 of k. We then have a natural identification of $\mathcal{H}_k$ with the group ring $k\mathfrak{S}_n$, under which T_w maps to the permutation w in $k\mathfrak{S}_n$.

Regarding the semisimplicity of $\mathcal{H}_k$, we have this result [7, Theorem 4.3]:

> Assuming k to be a field, $\mathcal{H}_k$ is semi-simple except precisely when
> - either $a^2 = 1$ and the characteristic of k is $\leqslant n$
> - or $a^2 \neq 1$ is a primitive r^{th} root of unity for some $2 \leqslant r \leqslant n$.

3.1.3. Two ring involutions and an A-antiautomorphism

We use the following two involutions on $\mathcal{H}$ both of which extend the ring involution $a \mapsto \overline{a}$ of A defined by $v \mapsto \overline{v} := v^{-1}$:

$$\overline{\sum a_w T_w} := \sum \overline{a_w} T_{w^{-1}}^{-1} \qquad j\left(\sum a_w T_w\right) := \sum \epsilon_w \overline{a_w} T_w \qquad (a_w \in A).$$

These commute with each other and so their composition, denoted $h \mapsto h^\dagger$, is an A-algebra involution of $\mathcal{H}$. The A-algebra anti-automorphism

$$\left(\sum a_w T_w\right)^* = \sum a_w T_{w^{-1}}$$

allows passing back and forth between statements about left cells and orders and those about right ones (§3.4.1).

3.2. Kazhdan-Lusztig C'- and C-basis

Two types of A-bases for $\mathcal{H}$ are introduced in [20], denoted $\{C'_w \mid w \in \mathfrak{S}_n\}$ and $\{C_w \mid w \in \mathfrak{S}_n\}$. They are uniquely determined by the respective conditions [21, Theorem 5.2]:

$$\begin{aligned} \overline{C'_w} &= C'_w \quad \text{and} \quad C'_w \equiv T_w \mod \mathcal{H}_{<0} \\ \overline{C_w} &= C_w \quad \text{and} \quad C_w \equiv T_w \mod \mathcal{H}_{>0} \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} \mathcal{H}_{<0} &:= \sum_{w \in \mathfrak{S}_n} A_{<0} T_w, \quad A_{<0} := v^{-1}\mathbb{Z}[v^{-1}] \\ \mathcal{H}_{>0} &:= \sum_{w \in \mathfrak{S}_n} A_{>0} T_w, \quad A_{>0} := v\mathbb{Z}[v]. \end{aligned}$$

The anti-automorphism $h \mapsto h^*$ and the ring involution $h \mapsto \overline{h}$ commute with each other, so that, by the characterization (3.3):

$$(C_x)^* = C_{x^{-1}} \qquad (C'_x)^* = C'_{x^{-1}}. \tag{3.4}$$

We have by [20, Theorem 1.1]:

$$C'_w = T_w + \sum_{y \in \mathfrak{S}_n, y < w} p_{y,w} T_y \qquad C_w = T_w + \sum_{y \in \mathfrak{S}_n, y < w} \epsilon_y \epsilon_w \overline{p_{y,w}} T_y \tag{3.5}$$

where $<$ denotes the Bruhat-Chevalley order on $\mathfrak{S}_n$, and $p_{y,w} \in A_{<0}$ for all $y < w$, from which it is clear that

$$C_w = \epsilon_w j(C'_w). \tag{3.6}$$

Combining (3.3) with (3.6), we obtain

$$C_w = \epsilon_w (C'_w)^\dagger \tag{3.7}$$

3.2.1. Notation

For a subset $\mathcal{S}$ of $\mathfrak{S}_n$, denote by $\langle C_y \mid y \in \mathcal{S}\rangle_A$ the A-span in $\mathcal{H}$ of $\{C_y \mid y \in \mathcal{S}\}$. For an A-algebra k, denote by $\langle C_y \mid y \in \mathcal{S}\rangle_k$ the k-span in $\mathcal{H}_k$ of $\{C_y \mid y \in \mathcal{S}\}$. Similar meanings are attached to $\langle T_y \mid y \in \mathcal{S}\rangle_A$ and $\langle T_y \mid y \in \mathcal{S}\rangle_k$.

3.2.2. A simple observation

From (3.5), we get $T_w \equiv C_w \bmod \langle T_x \mid x < w\rangle_A$. From this in turn we get, by induction on the Bruhat-Chevalley order, the following: for a subset $\mathcal{S}$ of $\mathfrak{S}_n$, the (images of) elements T_w, $w \in \mathfrak{S}_n \backslash \mathcal{S}$, form a basis for the A-module $\mathcal{H}/\langle C_x \mid x \in \mathcal{S}\rangle_A$. The same thing holds also in specializations $\mathcal{H}_k$ of $\mathcal{H}$ (§3.1.2): the (images of) elements T_w, $w \in \mathfrak{S}_n \setminus \mathcal{S}$, form a basis for the k-module $\mathcal{H}_k/\langle C_x \mid x \in \mathcal{S}\rangle_k$.

3.3. Kazhdan-Lusztig orders and cells

Let y and w in $\mathfrak{S}_n$. Write $y \leftarrow_{\mathrm{L}} w$ if, for some element s in S, the coefficient of C_y is non-zero in the expression of $C_s C_w$ as a A-linear combination of the basis elements C_x. Replacing all occurrences of "C" by "C'" in this definition would make no difference. The Kazhdan-Lusztig *left pre-order* is defined by: $y \leqslant_{\mathrm{L}} w$ if there exists a chain $y = y_0 \leftarrow_{\mathrm{L}} \cdots \leftarrow_{\mathrm{L}} y_k = w$; the *left equivalence* relation by: $y \sim_{\mathrm{L}} w$ if $y \leqslant_{\mathrm{L}} w$ and $w \leqslant_{\mathrm{L}} y$. Left equivalence classes are called *left cells*. Note that $\sum_{x \leqslant_{\mathrm{L}} w} AC_x$ is a left ideal containing the left ideal $\mathcal{H}C_w$.

Right pre-order, equivalence, and cells are defined similarly. The *two sided pre-order* is defined by: $y \leqslant_{\mathrm{LR}} w$ if there exists a chain $y = y_0, \ldots, y_k = w$ such that, for $0 \leqslant j < k$, either $y_j \leqslant_{\mathrm{L}} y_{j+1}$ or $y_j \leqslant_{\mathrm{R}} y_{j+1}$. Two sided equivalence classes are called *two sided cells*.

3.4. Cells and RSK Correspondence

We now recall the combinatorial characterizations of one and two sided cells in terms of the RSK correspondence (§2.4) and the dominance order on partitions (§2.2). These statements are the foundation on which this paper rests. The ones in §3.4.2, 3.4.2 are used repeatedly, but the more subtle one in §3.4.3 is used only once, namely in the proof of Theorem 1: it is used in the proof of Lemma 7 which is the main ingredient in the proof of that theorem.

Write $(P(w), Q(w))$ for the ordered pair of standard Young tableaux associated to a permutation w by the RSK correspondence (in our sense —see §2.4). Call $P(w)$ the *P-symbol* and $Q(w)$ the *Q-symbol* of w. It will be convenient to use such notation as $(P(w), Q(w))$ for the permutation w, $C_{(P(w),Q(w))}$ or $C(P(w), Q(w))$ for the Kazhdan-Lusztig C-basis element C_w.

3.4.1. Cells in terms of symbols

Two permutations are left equivalent if and only if they have the same Q-symbol; right equivalent if and only if the same P-symbol; two sided equivalent if and only if the same . See [15, Corollary 5.6] (and comments therein about [20, §5], [1]).

3.4.2. The $\leqslant_{\mathrm{LR}}$ relation in terms of dominance

We have $y \leqslant_{\mathrm{LR}} w$ if and only if RSK-shape $(y) \trianglelefteq$ RSK-shape (w), where $\trianglelefteq$ is the usual dominance order on partitions: $\lambda \trianglelefteq \mu$ if $\lambda_1 \leqslant \mu_1$, $\lambda_1 + \lambda_2 \leqslant \mu_1 + \mu_2, \ldots$. See [15, Theorem 5.1] (and comments therein about [10, 2.13.1]). We write $\lambda \triangleleft \mu$ for $\lambda \trianglelefteq \mu$ and $\lambda \neq \mu$.

3.4.3. Unrelatedness of distinct one sided cells in the same two sided cell

If $x \leqslant_{\mathrm{L}} y$ and $x \sim_{\mathrm{LR}} y$, then $x \sim_{\mathrm{L}} y$. See [15, Theorem 5.3] (and comments therein about [22, Lemma 4.1]).

3.5. Cell modules

It follows from the definition of the pre-order $\leqslant_{\mathrm{L}}$ that the A-span $\langle C_y \mid y \leqslant_{\mathrm{L}} w\rangle_A$ of $\{C_y \mid y \leqslant_{\mathrm{L}} w\}$, for w in $\mathfrak{S}_n$ fixed, is a left ideal of $\mathcal{H}$; so is $\langle C_y \mid y <_{\mathrm{L}} w\rangle_A$. The quotient $L(w) := \langle C_y \mid y \leqslant_{\mathrm{L}} w\rangle_A / \langle C_y \mid y <_{\mathrm{L}} w\rangle_A$ is called the *left cell module* associated to w. It is a left $\mathcal{H}$-module. Right cell modules $R(w)$ and two sided cell modules are defined similarly. They are right modules and bimodules respectively.

Let y and w be permutations of the same λ. The left cell modules $L(y)$ and $L(w)$ are then $\mathcal{H}$-isomorphic. In fact, the association $C_{(P,Q(y))} \leftrightarrow C_{(P,Q(w))}$ gives an isomorphism: see [20, §5], [15, Corollary 5.8]. The right cell modules $R(y)$ and $R(w)$ are similarly isomorphic, and we sometimes write $R(\lambda)$ for $R(y) \simeq R(w)$.

When a homomorphism from A to a commutative ring k is specified, such notation as $R(w)_k$ and $R(\lambda)_k$ make sense: see §3.1.2.

3.6. A key observation regarding images of C-basis elements in endomorphisms of cell modules

The image of C_y in End $R(\lambda)$ vanishes unless $\lambda \trianglelefteq (y)$, for, if C_z occurs with non-zero coefficient in $C_x C_y$ (when expressed as an A-linear combinations of the C-basis), where $(x) = \lambda$, and $\lambda \not\trianglelefteq (y)$, then $z \leqslant_{\mathrm{L}} y$ (by definition), so $(z) \trianglelefteq y$ (§3.4.2), which means that $(z) \neq \lambda$, so $z \not\sim_R x$ (§3.4.1).

4. Applications to invariant theory

In §4.1, Theorem 1 stated in §1.1 is proved. In the later subsections, applications of the theorem to rings of multilinear and polynomial invariants are discussed. The base k is an arbitrary commutative ring with unity in §4.1 but in the later subsections it is assumed to satisfy further hypothesis.

4.1. Proof of Theorem 1

Our goal in this subsection is to prove Theorem 1 stated in §1.1. Let n and d be positive integers, $d < n$. (The theorem clearly holds when $d \geqslant n$.) The main ingredient of the proof is Lemma 7 below. Once the lemma is proved, the theorem itself follows easily: see §4.1.2. For the lemma we need some combinatorial preliminaries, beyond those recalled in §2.

4.1.1. Preliminaries to the proof

For λ a partition of n,

- λ' denotes the *transpose* of λ. *E.g.*, $\lambda' = (3,2,2,1)$ for $\lambda = (4,3,1)$.
- t^λ denotes the standard tableau of shape λ in which the numbers 1, 2, …, n appear in order along successive rows; t_λ is defined similarly, with "columns" replacing "rows". *E.g.*, for $\lambda = (4,3,1)$, we have:

$$t^\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 \\ \cline{1-3} 8 \\ \cline{1-1}\end{array} \qquad t_\lambda = \begin{array}{|c|c|c|c|}\hline 1 & 4 & 6 & 8 \\ \hline 2 & 5 & 7 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array}$$

- W_λ denotes the row stabilizer of t^λ. It is a parabolic subgroup of $\mathfrak{S}_n$. *E.g.*, for $\lambda = (4,3,1)$, W_λ is isomorphic to the product $\mathfrak{S}_4 \times \mathfrak{S}_3 \times \mathfrak{S}_1$.
- $w_{0,\lambda}$ denotes the longest element of W_λ. *E.g.*, when $n = 8$ and $\lambda = (4,3,1)$, the sequence $(1w_{0,\lambda}, \ldots, nw_{0,\lambda})$ is $(4,3,2,1,7,6,5,8)$.
- $\mathfrak{D}_\lambda := \{w \in \mathfrak{S}_n \mid t^\lambda w \text{ is row standard}\}$. Clearly $\mathfrak{D}_\lambda$ is a set of right coset representatives of W_λ in $\mathfrak{S}_n$ and w_λ is an element of $\mathfrak{D}_\lambda$.

Proposition 5. — *For λ a partition of n,*

(1) $\ell(wd) = \ell(w) + \ell(d)$, *for $w \in W_\lambda$ and $d \in \mathfrak{D}_\lambda$.*
(2) $d \in \mathfrak{D}_\lambda$ *is the unique element of minimal length in $W_\lambda d$.*

(3) $w_{0,\lambda}\mathfrak{D}_\lambda = \{w \mid w \leqslant_{\mathrm{R}} w_{0,\lambda}\}$. *Thus* $w \leqslant_{\mathrm{R}} w_{0,\lambda}$ *if and only if in every row of* $t^\lambda w$ *the entries are decreasing to the right.*

(4) $w_{0,\lambda}$ *is of shape* λ'*: it corresponds under RSK to* $(t_{\lambda'}, t_{\lambda'})$.

Proof. — (1) and (2) are elementary to see: in any case, see [6, Lemma 1.1 (i), (ii)]. (4) is evident. For (3) see [24, §2.9]. □

Now, fix notation as in §3: A denotes the ring $\mathbb{Z}[v, v^{-1}]$, $\mathcal{H}$ the Hecke algebra, C_w the Kazhdan-Lusztig basis element corresponding to the permutation w, *etc.*

Proposition 6. — ([24, Lemma 2.11]) *The A-span* $\langle C_w \mid w \leqslant_{\mathrm{R}} w_{0,\lambda}\rangle_A$ *of the elements* C_w, $w \leqslant_{\mathrm{R}} w_{0,\lambda}$, *equals the right ideal* $C_{w_{0,\lambda}}\mathcal{H}$. *Similarly* $\langle C_w \mid w \leqslant_{\mathrm{L}} w_{0,\lambda}\rangle_A = \mathcal{H}C_{w_{0,\lambda}}$.

Proof. — It is enough to prove the first equality, the second being a left analogue of the first. The inclusion $\supseteq$ follows immediately from the definition of $\leqslant_{\mathrm{R}}$; the inclusion $\subseteq$ from [24, Lemma 2.11]. □

Setting $x_\lambda := \sum_{w \in W_\lambda} v_w T_w$ and $y_\lambda := \sum_{w\in W_\lambda} \epsilon_w v_w^{-1} T_w$, we have, by [20, Theorem 1.1, Lemma 2.6 (vi)]:

$$x_\lambda = v_{w_{0,\lambda}} C'_{w_{0,\lambda}} \qquad y_\lambda = \epsilon_{w_{0,\lambda}} v^{-1}_{w_{0,\lambda}} C_{w_{0,\lambda}}. \tag{4.1}$$

Lemma 7. — *Let* $\zeta(d)$ *denote the partition* $(d+1, 1, \ldots, 1)$ *of* n. *The two-sided ideal generated by* $C_{w_{0,\zeta(d)}}$ *is a free A-submodule of* $\mathcal{H}$ *with basis* C_x, (x) *has more than d rows (or, equivalently,* $(x) \trianglelefteq \zeta(d)'$*).*

Proof. — Since $w_{0,\zeta(d)}$ has shape $\zeta(d)'$ (see Proposition 5 (4)), it follows from the combinatorial description of $\leqslant_{\mathrm{LR}}$ in §3.4.2 that $x \leqslant_{\mathrm{LR}} w_{0,\zeta(d)}$ if and only if $(x) \trianglelefteq \zeta(d)'$. So it is clear from the definition of the relation $\leqslant_{\mathrm{LR}}$ (§3.3) that the two-sided ideal $\mathcal{H}C_{w_{0,\zeta(d)}}\mathcal{H}$ is contained in $\langle C_x \mid (x) \trianglelefteq \zeta(d)'\rangle_A$. To show the reverse containment, we first observe that C_x belongs to the right ideal $C_{w_{0,\zeta(d)}}\mathcal{H}$ in case $\mu := (x) \trianglelefteq \zeta(d)'$ and $x = w_{0,\mu'}$, the longest element of its shape: it is enough, by Proposition 6, to show that $x \leqslant_{\mathrm{R}} w_{0,\zeta(d)}$; on the other hand, by Proposition 5 (3), $x \leqslant_{\mathrm{R}} w_{0,\zeta(d)}$ is equivalent to $x(1) > x(2) > \cdots > x(d+1)$, which clearly holds for the elements x that we are considering.

Now suppose that x is a general element of $\mu \trianglelefteq \zeta(d)'$. Proceed by induction on the domination order of μ. Let $x \leftrightarrow (P, Q)$ under RSK. Then, on the one hand, the association $C(P, Q) \leftrightarrow C(t_\mu, Q)$ gives an $\mathcal{H}$-isomorphism between the right cell modules $R(x)$ and $R(v)$, where v is the permutation corresponding under RSK to (t_μ, Q) (§3.5); on the other, since $v \leftrightarrow (t_\mu, Q)$ is right equivalent to $w_{0,\mu'} \leftrightarrow (t_\mu, t_\mu)$, there exists, by Proposition 6, an element h in $\mathcal{H}$ such that $C(v) = C_{w_{0,\mu'}}h$; so that, by the definition of right cell modules,

$$C_x \equiv C_u h \pmod{\langle C_y \mid y \lneq_{\mathrm{R}} u\rangle_A}$$

where $u \leftrightarrow (P, t_\mu)$ under RSK. Now, $y \lneq_{\mathrm{R}} u$ implies, by §3.4.3, 3.4.2, $(y) \lhd \mu \trianglelefteq \zeta(d)'$; and, by the induction hypothesis, $C_y \in \mathcal{H}C_{w_{0,\zeta(d)}}\mathcal{H}$. As to $C(P, t_\mu)$, being left equivalent to $C(t_\mu, t_\mu)$, it belongs, once again by Proposition 6, to the left ideal $\mathcal{H}C_{w_{0,\mu'}}$, which as shown in the previous paragraph is contained in $\mathcal{H}C_{w_{0,\zeta(d)}}\mathcal{H}$. Thus $C_x = C(P, Q) \in \mathcal{H}C_{w_{0,\zeta(d)}}\mathcal{H}$, and we are done. □

4.1.2. Proof of Theorem 1 given Lemma 7

As seen in §3.1.2, $k\mathfrak{S}_n$ is the specialization of the Hecke algebra $\mathcal{H}$: $k\mathfrak{S}_n \simeq \mathcal{H}_k := \mathcal{H} \otimes_A k$, where k is an A-algebra via the natural ring homomorphism $A \to k$ defined by $v \mapsto 1$. Under the map $\mathcal{H} \to \mathcal{H} \otimes_A k$ given by $x \mapsto x \otimes 1$, the image of $C_{w_{0,\zeta(d)}}$ is $C_{w_{0,\zeta(d)}} \otimes 1 = y_d$, by Eq. (4.1). Denoting by $\tilde{J}$ the two-sided ideal of $\mathcal{H}$ generated by $C_{w_{0,\zeta(d)}}$, we thus have $\mathcal{H}/\tilde{J} \otimes_A k \simeq k\mathfrak{S}_n/J(n, d)$.

On the other hand, combining Lemma 7 with the observation in §3.2.2, we see that $\mathcal{H}/\tilde{J}$ is a free A-module with basis T_x, as x varies over permutations of whose s have at most d rows. The image of T_x in $k\mathfrak{S}_n/J(n, d)$ being the residue class of the corresponding permutation x, the theorem is proved. □

4.2. A "monomial" basis for the GL(V)-invariant sub-algebra of the tensor algebra of End V

Let k be a commutative ring with identity such that no non-zero polynomial in one variable over k vanishes identically as a function on k. Let V be a free module over k of finite rank d. Let $T := T(\operatorname{End} V)$ denote the tensor algebra $\oplus_{n\geqslant 0}(\operatorname{End} V)^{\otimes n}$. The action of the group $\mathrm{GL}(V)$ of units in End V on T preserves the algebra structure, so the ring $T^{\mathrm{GL}(V)}$ of $\mathrm{GL}(V)$-invariants is a sub-algebra. It also preserves degrees, so

$$T^{\mathrm{GL}(V)} \simeq \oplus_{n\geqslant 0}((\operatorname{End} V)^{\otimes n})^{\mathrm{GL}(V)} = \oplus_{n\geqslant 0} \operatorname{End}_{\mathrm{GL}(V)}(V^{\otimes n}).$$

By the classical theorem quoted in §1.1 from [4], we have, for every $n \geqslant 0$, an isomorphism of k-algebras

$$\Theta_n\colon \quad k\mathfrak{S}_n/J(n, d) \simeq \operatorname{End}_{\mathrm{GL}(V)}(V^{\otimes n})$$

where $k\mathfrak{S}_n$ is the group algebra of the symmetric group $\mathfrak{S}_n$, and $J(n,d)$ the two sided ideal as defined in the statement of the quoted theorem.

Now, Theorem 1 gives us a k-basis for $k\mathfrak{S}_n/J(n,d)$. Taking the image under Θ_n gives a basis for $\mathrm{End}_{\mathrm{GL}(V)}(V^{\otimes n})$. Taking the disjoint union over n of these bases gives a basis —call it $\mathfrak{B}$— for $T^{\mathrm{GL}(V)}$, which has an interesting property —see the theorem below— which explains the appearance of term "monomial" in the title of this subsection.

Theorem 8. — *The basis $\mathfrak{B}$ of $T^{\mathrm{GL}(V)}$ defined above is closed under products.*

Proof. — In fact, we get a description of the k-algebra $T^{\mathrm{GL}(V)}$ as follows. Consider the space $\mathfrak{S} := \oplus_{n\geqslant 0} k\mathfrak{S}_n$ with the following multiplication: for π in $\mathfrak{S}_m$ and σ in $\mathfrak{S}_n$, $\pi\cdot\sigma$ is the permutation in $\mathfrak{S}_{m+n}$ that, as a self-map of $[m+n]$, is given by

$$\pi\cdot\sigma(i) := \begin{cases} \pi(i) & \text{if } i \leqslant m \\ \sigma(i-m)+m & \text{if } i \geqslant m+1. \end{cases}$$

For each n, consider the subspace $\mathfrak{P}_n$ of $k\mathfrak{S}_n$ spanned by permutations that have no decreasing sub-sequence of length more than d. The direct sum $\mathfrak{P} := \oplus_{n\geqslant 0}\mathfrak{P}_n$ is a sub-algebra of $\mathfrak{S}$.

The restriction to $\mathfrak{P}_n$ of the canonical map $k\mathfrak{S}_n \to k\mathfrak{S}_n/J(n,d)$ is a vector space isomorphism (Theorem 1). Thus $\oplus_{n\geqslant 0}\Theta_n$ is a vector space isomorphism of the algebra $\mathfrak{P}$ onto $T^{\mathrm{GL}(V)}$. It is evidently also an algebra isomorphism. □

4.3. Application to rings of polynomial invariants

In this subsection, k denotes a field of characteristic 0 and V a k-vector space of finite dimension d. Consider the ring of $\mathrm{GL}_k(V)$-invariant polynomial functions on $(\mathrm{End}_k V)^{\times m}$. It is a direct sum of homogeneous invariant functions, for the action preserves degrees. It is spanned, as a vector space, by products of traces in words (see for example [27, §1], [29]). More precisely, for a fixed degree n, given a permutation σ of n elements and a map ν of $[n]$ to $[m]$, consider the function $f(\sigma,\nu)$ defined as follows: writing σ as a product $(i_1 i_2\cdots)(i_{k+1}i_{k+2}\cdots)\cdots(i_{p+1}i_{p+2}\cdots)$ of disjoint cycles,

$$f(\sigma,\nu) := \mathrm{Trace}\left(A_{\nu(i_1)}A_{\nu(i_2)}\cdots\right)\mathrm{Trace}\left(A_{\nu(i_{k+1})}A_{\nu(i_{k+2})}\cdots\right)\cdots$$
$$\cdots\mathrm{Trace}\left(A_{\nu(i_{p+1})}A_{\nu(i_{p+2})}\cdots\right).$$

As σ and ν vary, the $f(\sigma,\nu)$ span the space of invariants of degree n.

This fact is proved by observing that every polynomial invariant arises as the specialization of a multilinear invariant (by the restitution process). Thus, thanks to Theorem 1, we can restrict the permutation σ to have no decreasing sub-sequence of length more than d, and still the $f(\sigma,\nu)$ would span. We state this formally:

THEOREM 9. — *The invariant functions $f(\sigma,\nu)$, as σ varies over permutations that do not have any decreasing sub-sequence of length exceeding d, form a k-linear spanning set for the ring of $GL_k V$-invariant polynomial functions on $(\mathrm{End}_k V)^{\times n}$.*

4.3.1. Picture invariants

Set $V_b^t := V^{*\otimes b} \otimes V^{\otimes t}$ and consider the ring of polynomial $\mathrm{GL}_k(V)$-invariant functions on the space $V_{b_1}^{t_1} \times \cdots \times V_{b_s}^{t_s}$ of several tensors. In [5, §3], the notion of a "picture invariant" is introduced, generalizing the functions $f(\sigma,\nu)$ defined above. Picture invariants span the space of invariant polynomial functions ([5, Proposition 7]). Just as in the special case of $(\mathrm{End}\, V)^{\times n}$ discussed above, thanks to Theorem 1, we have:

THEOREM 10. — *Only those picture invariants with underlying permutations having no decreasing sub-sequences of length exceeding d suffice to span as a k-vector space the ring of $GL_k(V)$-invariant polynomial functions on the space $V_{b_1}^{t_1} \times \cdots \times V_{b_s}^{t_s}$ of several tensors.*

5. Proof of Theorem 2

In this section, we first prove Theorem 2 (§1.2). We then show that it holds also over the integers and fields of characteristic 0 (§5.3), but not in general over a field of positive characteristic (Example 11). For comments on its Hecke analogue, see §1.4.

As pointed out in §1.2, the results of the recent paper [9] are related to Theorem 2 and its Hecke analogue.

5.1. Proof of Theorem 2

Let n be a positive integer, λ a partition of n, and $\rho_\lambda : \mathbb{C}\mathfrak{S}_n \to \mathrm{End}_{\mathbb{C}}\, \mathbb{C}\mathcal{T}_\lambda$ the map defining the representation of $\mathfrak{S}_n$ on tabloids of shape λ (§2.5). The proof follows by combining the observations in §3.6 and §3.2.2 with the following two facts:

(1) the decomposition into irreducibles of the representation $\mathbb{C}\mathcal{T}_\lambda$ is given by $\mathbb{C}\mathcal{T}_\lambda = \oplus_{\mu \trianglerighteq \lambda} (S^\mu_{\mathbb{C}})^{m(\mu)}$, where $S^\mu_{\mathbb{C}}$ are the Specht modules (§2.6), $\trianglerighteq$ is the domination relation on partitions (§2.2), and the multiplicities $m(\mu)$ are positive.(2)

(2) the Specht module $S^\mu_{\mathbb{C}}$ is $\mathfrak{S}_n$-isomorphic to the right cell module $R(\mu)_{\mathbb{C}}$ (defined in §3.5);

Both facts are well known. For (1), see for example [30, Theorem 2.11.2, Corollary 2.4.7]. For (2), we could refer to [14] or [26]. But in fact we will recall in some detail in §8.3 the following more general fact from [6, 24]: Specht modules can be defined over the Hecke algebra $\mathcal{H}$ and are isomorphic to the corresponding right cell modules.

Since the multiplicities $m(\mu)$ in (1) above are positive, the kernel of ρ_λ is the same as that of the map $\rho'_\lambda : \mathbb{C}\mathfrak{S}_n \to \mathrm{End}_{\mathbb{C}}(\oplus_{\mu \trianglerighteq \lambda} S^\mu_{\mathbb{C}})$. The image of ρ'_λ is clearly contained in $\oplus_{\mu \trianglerighteq \lambda} \mathrm{End}_{\mathbb{C}} S^\mu_{\mathbb{C}}$. Since the $S^\mu_{\mathbb{C}}$ are non-isomorphic for distinct μ,(3) it follows from a density argument (see for example [2, Chapter 8, §4, No. 3, Corollaire 2]) that ρ'_λ maps onto $\oplus_{\mu \trianglerighteq \lambda} \mathrm{End}_{\mathbb{C}} S^\mu_{\mathbb{C}}$. Since $\dim S^\mu_{\mathbb{C}} = d(\mu)$, where $d(\mu)$ is the number of standard tableaux of shape μ, and the $S^\mu_{\mathbb{C}}$ as μ varies over all partitions of n are a complete set of irreducible representations,(4) we obtain, by counting dimensions:

$$\begin{aligned}\dim \ker \rho'_\lambda &= \dim \mathbb{C}\mathfrak{S}_n - \dim \left(\oplus_{\mu \trianglerighteq \lambda} \mathrm{End}_{\mathbb{C}} S^\mu_{\mathbb{C}}\right) \\ &= \sum_{\mu n} d(\mu)^2 - \sum_{\mu \trianglerighteq \lambda} d(\mu)^2 \\ &= \sum_{\mu \ntrianglerighteq \lambda} d(\mu)^2.\end{aligned}$$

Now consider $\mathbb{C}\mathfrak{S}_n$ as the specialization of the Hecke algebra $\mathcal{H}$ as follows (§3.1.2): $\mathbb{C}\mathfrak{S}_n \simeq \mathcal{H} \otimes_A \mathbb{C}$, where $\mathbb{C}$ is an A-algebra via the map $A \to \mathbb{C}$ defined by $v \mapsto 1$. By the observation §3.6, the images $C_w \otimes 1$ in $\mathcal{H} \otimes_A \mathbb{C} \simeq \mathbb{C}\mathfrak{S}_n$ of the Kazhdan-Lusztig basis elements C_w of $\mathcal{H}$ (§3.2) belong to the kernel of ρ'_λ if RSK-shape$(w) \ntrianglerighteq \lambda$. The number of such w being equal to $\sum_{\mu \ntrianglerighteq \lambda} d(\mu)^2$, which as observed above equals $\dim \ker \rho'_\lambda$, we conclude that

$$\ker \rho_\lambda = \ker \rho'_\lambda = \langle C_w \otimes 1 \mid (w) \ntrianglerighteq \lambda \rangle_{\mathbb{C}}. \tag{5.1}$$

(2) In fact, $m(\mu)$ is the number of "semi-standard tableaux of shape μ and content λ".

(3) This is well-known. It also follows from the isomorphism in (2) and the corresponding fact for cell modules proved in §6.2.

(4) Same comment as in footnote 3 applies to both assertions.

By observation §3.2.2, the images of $T_w \otimes 1$, $(w) \trianglerighteq \lambda$, form a basis for $\mathcal{H} \otimes_A \mathbb{C}/\langle C_x \otimes 1 \mid (w) \ntrianglerighteq \lambda\rangle_{\mathbb{C}} \simeq \mathbb{C}\mathfrak{S}_n/\ker\rho'_\lambda$. But the image in $\mathbb{C}\mathfrak{S}_n$ of $T_w \otimes 1$ is the permutation w. This completes the proof of Theorem 2. □

5.2. $J(n,d)$ in $\mathbb{C}\mathfrak{S}_n$ equals $\ker\rho_{\lambda(n,d)}$

We now justify the claim made in §1.2 that the ideal $J(n,d)$ in $\mathbb{C}\mathfrak{S}_n$ equals $\ker\rho_{\lambda(n,d)}$. On the one hand, as is easily seen, the generator y_d of the two sided ideal $J(n,d)$ belongs to $\ker\rho_{\lambda(n,d)}$. Indeed, given a tabloid $\{T\}$ of shape $\lambda(n,d)$, there evidently exist integers a and b, with $1 \leqslant a, b \leqslant d+1$, that appear in the same row of T. This implies that the transposition (a,b) fixes $\{T\}$. Writing $\mathfrak{S}_{d+1}$ as a disjoint union $S \cup S(a,b)$ (for a suitable choice of a subset S), we have $y_d\{T\} = \sum_{\sigma\in\mathfrak{S}_{d+1}} \operatorname{sgn}(\sigma)\sigma\{T\} = \sum_{\sigma\in S}\operatorname{sgn}(\sigma)(\sigma - \sigma(a,b))\{T\} = 0$.

On the other hand, as computed in the proof of Theorem 2 above, $\ker\rho_{\lambda(n,d)}$ as a $\mathbb{C}$-vector space has dimension $\sum_{\mu\ntrianglerighteq\lambda(n,d)} d(\mu)^2$. It suffices therefore to show that $J(n,d)$ too has this same dimension. It follows from Lemma 7 that $J(n,d)$ has dimension $\sum_{\mu\trianglelefteq\zeta(d)'} d(\mu)^2$, where $\zeta(d)$ is the partition of $(d+1,1,\ldots,1)$ of n and $\zeta(d)'$ denotes its transpose (§4.1.1). But $\mu \ntrianglerighteq \lambda(n,d)$ if and only if μ has more than d rows if and only if $\mu \trianglelefteq \zeta(d)'$.□

5.3. Theorem 2 holds over the integers and fields of characteristic 0

We first argue that Theorem 2 holds with $\mathbb{Z}$ coefficients in place of $\mathbb{C}$ coefficients. Let $\rho_{\lambda,\mathbb{Z}}$ be the map $\mathbb{Z}\mathfrak{S}_n \to \operatorname{End}_{\mathbb{Z}}\mathbb{Z}\mathcal{T}_\lambda$ defining the tabloid representation. We claim that Eq. (5.1) holds over $\mathbb{Z}$:

$$\ker\rho_{\lambda,\mathbb{Z}} = \langle C_w \otimes 1 \mid (w) \ntrianglerighteq \lambda\rangle_{\mathbb{Z}}. \tag{5.2}$$

Once this is proved, the rest of the argument is the same as in the complex case: namely, use observation §3.2.2.

We first show the containment $\supseteq$. We have $(C_w\otimes 1)\mathbb{C}\mathcal{T}_\lambda = (C_w\otimes 1)\mathbb{Z}\mathcal{T}_\lambda\otimes_{\mathbb{Z}} \mathbb{C}$ (by flatness of $\mathbb{C}$ over $\mathbb{Z}$). Since $(C_w \otimes 1)\mathbb{Z}\mathcal{T}_\lambda$ is a submodule of the free module $\mathbb{Z}\mathcal{T}_\lambda$, it is free. By Eq. (5.1), $(C_w \otimes 1)\mathbb{C}\mathcal{T}_\lambda = 0$ if $(w) \ntrianglerighteq \lambda$, so $\supseteq$ holds.

To show the other containment, set $\mathfrak{m} = \langle C_w \otimes 1 \mid (w) \ntrianglerighteq \lambda\rangle_{\mathbb{Z}}$, and consider $\ker\rho_{\lambda,\mathbb{Z}}/\mathfrak{m}$. Since $\mathbb{Z}\mathfrak{S}_n/\mathfrak{m}$ is free, so is its submodule $\ker\rho_{\lambda,\mathbb{Z}}/\mathfrak{m}$, and we have

$$\frac{\ker \rho_{\lambda,\mathbb{Z}}}{\mathfrak{m}} \otimes_{\mathbb{Z}} \mathbb{C} = \frac{\ker \rho_{\lambda,\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}}{\mathfrak{m} \otimes_{\mathbb{Z}} \mathbb{C}} = \frac{\ker \rho_{\lambda,\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}}{\langle C_w \otimes 1 \mid (w) \not\trianglerighteq \lambda \rangle_{\mathbb{C}}}.$$

By the flatness of $\mathbb{C}$ over $\mathbb{Z}$, we have $\ker \rho_{\lambda,\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \ker \rho_\lambda$. The last term in the above display vanishes by Eq. (5.1), and so $\subseteq$ holds (since $\ker \rho_{\lambda,\mathbb{Z}}/\mathfrak{m}$ is free). The proof of Theorem 2 over $\mathbb{Z}$ is complete.

Let now k be a field of characteristic 0 and $\rho_{\lambda,k}$ the map $k\mathfrak{S}_n \to \operatorname{End}_k k\mathcal{T}_\lambda$ defining the representation on tabloids of shape λ. The analogue of Eqs. (5.1) and (5.2) holds over k, since, by the flatness of k over $\mathbb{Z}$, we have $\ker \rho_{\lambda,k} = \ker \rho_{\lambda,\mathbb{Z}} \otimes_{\mathbb{Z}} k$. Now use observation §3.2.2 as in the earlier cases to finish the proof of Theorem 2 over k. □

Example 11. — Theorem 2 does not hold in general over a field k of positive characteristic. We give an example of a non-trivial linear combination of permutations of dominating λ that acts trivially on the tabloid representation space $k\mathcal{T}_\lambda$. Let k be a field of characteristic 2. Let $n = 4$ and $\lambda = (2,2)$. Let us denote a permutation in $\mathfrak{S}_4$ by writing down in sequence the images under it of 1 through 4: *e.g.*, 1243 denotes the permutation σ defined by $1\sigma = 1$, $2\sigma = 2$, $3\sigma = 4$, and $4\sigma = 3$. It is readily seen that the eight permutations in the display below are all of shape $(3,1)$ and that their sum acts trivially on $k\mathcal{T}_\lambda$.

$$2134, \quad 2341, \quad 2314, \quad 1342, \quad 3124, \quad 1243, \quad 4123, \quad 1423.$$

6. Proof of Theorem 3

In this section, k denotes a field, a an invertible element of k, and notation is fixed as in §3. It is assumed throughout this section (but not in the later ones) that $\mathcal{H}_k$ is semisimple (see §3.1.2). Combining the structure theory of semisimple algebras with the results in §3.4–3.6, we derive Theorem 14 as a consequence. Theorem 3 of §1 is the special case of this theorem when k is the complex field and $a = 1$. As mentioned in §1.4, the proof in effect says that Theorem 14 is a consequence of the cellularity in the sense of [17] of the Kazhdan-Lusztig basis.

But first, we need to recall the structure theory, which we do in §6.1 in a form suited to our context, and then the proof of irreducibility of the (right) cell modules, which we do in §6.2 following the argument in [20, §5].

6.1. A recap of the structure theory of semisimple algebras

The facts recalled here are all well known: see, *e.g.*, [2], [16, page 218]. Let V be a simple (right) module for a semisimple algebra $\mathfrak{A}$ of finite dimension over k. Then the endomorphism ring $\operatorname{End}_{\mathfrak{A}} V$ is a division algebra (Schur's Lemma), say E_V. Being a subalgebra of $\operatorname{End}_k V$, it is finite dimensional as a vector space over k, and V is a finite dimensional vector space over it. Set $n_V := \dim_{E_V} V$. The ring $\operatorname{End}_{E_V} V$ of endomorphisms of V as a E_V-vector space can be identified (non-canonically, depending upon a choice of basis) with the ring $\mathcal{M}_{n_V}(D_V)$ of matrices of size $n_V \times n_V$ with entries in the opposite algebra D_V of E_V. The natural ring homomorphism $\mathfrak{A} \to \operatorname{End}_{E_V} V$ is a surjection (density theorem).

There is an isomorphism of algebras (Wedderburn's structure theorem):

$$\mathfrak{A} \simeq \prod_V \operatorname{End}_{E_V} V \simeq \prod_V \mathcal{M}_{n_V}(D_V), \tag{6.1}$$

where the product is taken over all (isomorphism classes of) simple modules. There is a single isomorphism class of simple modules for the simple algebra $\operatorname{End}_{E_V} V$, namely that of V itself, and its multiplicity is n_V in a direct sum decomposition into simples of the right regular representation of $\operatorname{End}_{E_V} V$. Thus n_V is also the multiplicity of V in the right regular representation of $\mathfrak{A}$. And of course

$$\dim_k V = n_V(\dim_k E_V) \geqslant n_V. \tag{6.2}$$

The hypothesis of the following proposition admittedly appears contrived at first sight, but it will soon be apparent (in §3.2) that it is tailor-made for our situation.

Proposition 12. — *Let $W_1, \ldots, W_s$ be $\mathfrak{A}$-modules of respective dimensions $d_1, \ldots, d_s$ over k. Suppose that the right regular representation of $\mathfrak{A}$ has a filtration in which the quotients are precisely $W_1^{\oplus d_1}, \ldots, W_s^{\oplus d_s}$. Then*

1. *$\operatorname{End}_{\mathfrak{A}} W_i = k$ and W_i is absolutely irreducible, $\forall\, i$, $1 \leqslant i \leqslant s$.*
2. *W_i is not isomorphic to W_j for $i \neq j$.*
3. *W_i, $1 \leqslant i \leqslant s$, are a complete set of simple $\mathfrak{A}$-modules.*
4. *$\mathfrak{A} \simeq \prod_{i=1}^s \operatorname{End}_k W_i$.*

Proof. — Let V be a simple submodule of W_i. Then $\dim_k V \leqslant d_i$. The hypothesis about the filtration implies that the multiplicity of V in the right regular representation is at least d_i. From Eq. (6.2), we conclude that

$d_i = n_V$ and $\dim_k E_V = 1$. So $V = W_i$ is simple and $E_V = k$. If $\overline{k}$ denotes an algebraic closure of k, then

$$\mathrm{End}_{\mathfrak{A}\otimes_k \overline{k}}(V \otimes_k \overline{k}) = (\mathrm{End}_{\mathfrak{A}} V) \otimes_k \overline{k} = k \otimes_k \overline{k} = \overline{k}.$$

So V is absolutely irreducible and (1) is proved.

If $W_i \simeq W_j$ for $i \neq j$, then the multiplicity of W_i in the right regular representation would exceed d_i contradicting Eq. (6.2). This proves (2). Since every simple module has positive multiplicity in the right regular representation, (3) is clear. Finally, (4) follows from (1) and Eq. (6.1). □

6.2. Irreducibility and other properties of the cell modules

The proof of Theorem 13 below follows [20, §5]. We give it in detail here for the proof in [20] seems sketchy.

THEOREM 13. — ([20, §5]) *Assume that $\mathcal{H}_k := \mathcal{H} \otimes_A k$ is semisimple. Then*

1. $\mathrm{End}_{\mathcal{H}_k} R(\lambda)_k = k$ *and* $R(\lambda)_k$ *is absolutely irreducible, for all* λn.
2. $R(\lambda)_k \not\simeq R(\mu)_k$ *for partitions* $\lambda \neq \mu$ *of* n.
3. $R(\lambda)_k$, λn, *are a complete set of simple* $\mathcal{H}_k$*-modules.*
4. $\mathcal{H}_k \simeq \prod_{\lambda n} \mathrm{End}_k R(\lambda)_k$.

Proof. — By Proposition 12, it is enough to exhibit a filtration of the right regular representation of $\mathcal{H}_k$ in which the quotients are precisely $R(\lambda)_k^{\oplus d(\lambda)}$, λn, each occurring once. We will in fact exhibit a decreasing filtration $\mathfrak{F} = \{F_i\}$ by right ideals (in fact, two sided ideals) of $\mathcal{H}$ in which the quotients F_i/F_{i+1} are precisely $R(\lambda)^{\oplus d(\lambda)}$, λn, each occurring once. Since $R(\lambda)$ are free A-modules, it will follow that $\mathfrak{F} \otimes_A k$ is a filtration of $\mathcal{H}_k$ whose quotients are $R(\lambda)_k^{\oplus d(\lambda)}$, and the proof will be done.

Let $\succeq$ be a total order on partitions of n that refines the dominance partial order $\trianglerighteq$. Let $\lambda_1 \succ \lambda_2 \succ \cdots$ be the full list of partitions arranged in decreasing order with respect to $\succeq$. Set $F_i := \langle C_w \mid (w) \preceq \lambda_i \rangle_A$. It is enough to prove the following:

1. The F_i are right ideals in $\mathcal{H}$ (they are in fact two sided ideals).
2. $F_i/F_{i+1} \simeq R(\lambda_i)^{\oplus d(\lambda_i)}$.

It follows from the definition in §3.3 of the relation $\leqslant_{\mathrm{LR}}$ that, for any fixed permutation w, $\langle C_x \mid x \leqslant_{\mathrm{LR}} w \rangle_A$ is a two sided ideal of $\mathcal{H}$. But $x \leqslant_{\mathrm{LR}} w$ if and only if RSK-shape$\,(x) \trianglelefteq$ RSK-shape$\,(w)$, by the characterization in §3.4.2. Thus, $\langle C_x \mid \text{RSK-shape}\,(x) \trianglelefteq \lambda \rangle_A$ is a two sided ideal, and F_i

being equal to the sum $\sum_{j\geqslant i}\langle C_x \mid \text{RSK-shape}\,(x) \trianglelefteq \lambda_j\rangle_A$ of two sided ideals is a two sided ideal. This proves (1).

To prove (2), let S_1, S_2, ... be the distinct right cells contained in the two sided cell corresponding to shape λ_i. It follows from the assertions in §3.4.1 that there are $d(\lambda_i)$ of them and the cardinality of each is $d(\lambda_i)$. Fix a permutation w of shape λ_i. Consider the right cell module $R(w)$, which by definition is the quotient of the right ideal $\langle C_x \mid x \leqslant_{\mathrm{R}} w\rangle_A$ by the right ideal $\langle C_x \mid x \lneqq_{\mathrm{R}} w\rangle_A$. If $x \leqslant_{\mathrm{R}} w$ then evidently $x \leqslant_{\mathrm{LR}} w$ and (by §3.4.2) RSK-shape$\,(x) \trianglelefteq \lambda_i$, so RSK-shape$\,(x) \preceq \lambda_i$. Thus we have a map induced by the inclusion: $\langle C_x \mid x \leqslant_{\mathrm{R}} w\rangle_A \to F_i/F_{i+1}$.

We claim that the above map descends to an injective map from the quotient $R(\lambda_i)$. It descends because $x \lneqq_{\mathrm{R}} w$ implies $x \lneqq_{\mathrm{LR}} w$: if $x \sim_{\mathrm{LR}} w$, then $x \sim_{\mathrm{R}} w$ by §3.4.3. To prove that the map from $R(\lambda_i)$ is an injection, let $\sum_{x\,\leqslant_{\mathrm{LR}}\,w} a_x C_x$ belong to F_{i+1} with $a_x \in A$. Suppose that $a_x \neq 0$ for some fixed x. Then, since the C_y form an A-basis of $\mathcal{H}$, we conclude that RSK-shape$\,(x) \preceq \lambda_{i+1}$, so RSK-shape$\,(x) \neq \lambda_i$, and (by §3.4.2) $x \lneqq_{\mathrm{LR}} w$. But this means $x \not\sim_R w$, so $x \lneqq_{\mathrm{R}} w$, and thus the image in $R(\lambda_i)$ of $\sum_{x\,\leqslant_{\mathrm{LR}}\,w} a_x C_x$ vanishes.

The image of $R(w)$ in F_i/F_{i+1} is spanned by the classes $\overline{C_x}$, $x \sim_{\mathrm{R}} w$. Choosing w_1 in S_1, w_2 in S_2, ... we see that the images of $R(w_1)$, $R(w_2)$, ... in F_i/F_{i+1} form a direct sum (for the C_x are an A-basis of $\mathcal{H}$). The $R(w_j)$ are all isomorphic to $R(\lambda)$ (see §3.5). This completes the proof of (2) and also of the theorem. □

6.3. Kazhdan-Lusztig basis in endomorphisms of modules

THEOREM 14. — *Assume that $\mathcal{H}_k := \mathcal{H} \otimes_A k$ is semisimple. For λ a partition of n, the images in $\operatorname{End} R(\lambda)_k$ of the Kazhdan-Lusztig basis elements C_x, $(x) = \lambda$, form a basis (for $\operatorname{End} R(\lambda)_k$).*

Proof. — By Theorem 13 (4), $\mathcal{H}_k \simeq \oplus_{\lambda\vdash n} \operatorname{End} R(\lambda)_k$. The projections to $\operatorname{End} R(\nu)_k$ of C_x, $(x) \trianglelefteq \lambda$, vanish if $\nu \not\trianglelefteq \lambda$ (§3.6). Therefore the projections of the same elements to $\oplus_{\mu\trianglelefteq\lambda} \operatorname{End} R(\mu)_k$ form a basis: note that the number of such elements equals $\sum_{\mu\trianglelefteq\lambda} \dim \operatorname{End}_{R(\mu)_k}$. Again by §3.6, the projections of C_x, $(x) \triangleleft \lambda$, vanish in $\operatorname{End} R(\lambda)_k$. This implies that the projections of C_x, RSK-shape $(x) = \lambda$, in $\operatorname{End} R(\lambda)_k$ form a spanning set. Since the number of such C_x equals $\dim \operatorname{End} R(\lambda)_k$, the theorem follows. □

THEOREM 15. — *Assume that $\mathcal{H}_k := \mathcal{H} \otimes_A k$ is semisimple. Let U be a finite dimensional representation of $\mathcal{H}_k$ and $\mathcal{S}$ the subset of partitions λ of n such that $R(\lambda)_k$ appears in a decomposition of U into irreducibles. Then the images in $\operatorname{End} U$ of C_x, $x \in \mathfrak{S}_n$ such that $(x) \in \mathcal{S}$, form a basis for the image of $\mathcal{H}_k$ (under the map $\mathcal{H}_k \to \operatorname{End} U$ defining U).*

Proof. — It is enough to prove the assertion assuming $U = \oplus_{\lambda\in\mathcal{S}} R(\lambda)_k$. The image of $\mathcal{H}_k$ in $\operatorname{End} U$ is $\oplus_{\lambda\in\mathcal{S}} \operatorname{End} R(\lambda)_k$ (Theorem 13 (1), density theorem, and [2, Corollaire 2, page 39]). Proceed by induction on the cardinality of $\mathcal{S}$. It is enough to show that the relevant images in $\operatorname{End} U$ are linearly independent, for their number equals the dimension of $\oplus_{\lambda\in\mathcal{S}} \operatorname{End} R(\lambda)_k$. Suppose that a linear combination of the images vanishes. Choose $\lambda \in \mathcal{S}$ such that there is no μ in S with $\lambda \triangleleft \mu$. Projections to $\operatorname{End} R(\lambda)_k$ of all C_x, $\lambda \neq (x) \in \mathcal{S}$, vanish (§3.6). So projecting the linear combination to $\operatorname{End} R(\lambda)_k$ and using Theorem 14, we conclude that the coefficients of C_x, $(x) = \lambda$, are all zero. The induction hypothesis applied to $\mathcal{S} \setminus \{\lambda\}$ now finishes the proof. □

Example 16. — The purpose of this example is to show that images in $\operatorname{End} R(\lambda)_{\mathbb{C}}$ of permutations of λ do not in general form a basis of $\operatorname{End} R(\lambda)_{\mathbb{C}}$. Let $n = 4$ and $\lambda = (2,2)$. Then $R(\lambda)_{\mathbb{C}}$ is the unique 2 dimensional complex irreducible representation of $\mathfrak{S}_4$. Consider the action of $\mathfrak{S}_4$ on partitions of $\{1,2,3,4\}$ into two sets of two elements each. There being three such partitions, we get a map $\mathfrak{S}_4 \to \mathfrak{S}_3$, which is surjective and has kernel $\{\text{identity}, (12)(34), (13)(24), (14)(23)\}$. Pulling back the 2-dimensional complex irreducible representation of $\mathfrak{S}_3$ via the above map, we get $R(\lambda)_{\mathbb{C}}$. The permutations of shape λ are (13)(24), (1342), (1243), and (12)(34). The first and last of these act as identity on $R(\lambda)_{\mathbb{C}}$.

7. The matrix $\mathcal{G}(\lambda)$ and a formula for its determinant

Let λ be a fixed partition of n. Our goal in this section is to study the action of the elements C_w, $(w) = \lambda$, on the right cell module $R(\lambda)$. More specifically, it is to state Theorem 18. The motivation for this was already indicated (see §1.5): it is to prove analogues of Theorem 3 over fields of positive characteristic. As too was already indicated (in §1.5), there is a bonus to be had: our study enables a different approach to questions about irreducibility of Specht modules.

As we observe in §7.2, all information about the action can conveniently be gathered together into a matrix $\mathbb{G}(\lambda)$ which breaks up nicely into blocks of the same size (Proposition 17). The non-zero blocks all lie along the diagonal and are all equal to a certain matrix $\mathcal{G}(\lambda)$ defined in §7.1. This matrix encodes the multiplication table modulo lower cells of the C_w of λ. Theorem 18 gives a formula for its determinant.

As remarked in §1.4, the argument in §7.1 is in effect deducing the cellularity of the Kazhdan-Lusztig basis from results of [15] recalled in §3; and the argument in §7.2 is in effect deducing Proposition 17 as a consequence of cellularity.

7.1. On products of C-basis elements

Let $P_1, \ldots, P_m$ be the complete list of standard tableaux of shape λ. We claim:

$$C(P_i,P_j)\cdot C(P_k,P_l) = g_j^k C(P_i,P_l) \mod \langle C_y \mid (y) \lhd \lambda,\ y \lneqq_{\mathrm{L}} (P_k,P_l),\ y \lneqq_{\mathrm{R}} (P_i,P_j)\rangle_A \tag{7.1}$$

the coefficient g_j^k being independent of i and l.

To prove the claim, consider the expression of the left hand side as a linear combination of the C-basis elements. For any C_y occurring with non-zero coefficient, we have $y \leqslant_{\mathrm{R}} (P_i,P_j)$ and $y \leqslant_{\mathrm{L}} (P_k,P_l)$, by the definition of the pre-orders (§3.3). By §3.4.2, $(y) \trianglelefteq \lambda$; and if $(y) \neq \lambda$, then $y \lneqq_{\mathrm{R}} (P_i,P_j)$ and $y \lneqq_{\mathrm{L}} (P_k,P_l)$. If $(y) = \lambda$, then, by §3.4.1, $y \sim_{\mathrm{LR}} (P_k,P_l)$; by §3.4.3, $y \sim_{\mathrm{L}} (P_k,P_l)$; by §3.4.1, the Q-symbol of y is P_l; and, analogously, the P-symbol of y is P_i. That g_j^k doesn't depend upon i and l follows from the description of the $\mathcal{H}$-isomorphisms between one sided cells of the same as recalled in §3.5, and the claim is proved. We set

$$\mathcal{G}(\lambda) := (g_j^k)_{1 \leqslant j,k \leqslant m}. \tag{7.2}$$

7.2. Relating the matrix $\mathcal{G}(\lambda)$ to the action on $R(\lambda)$

Enumerate as $P_1, \ldots, P_m$ all the standard Young tableaux of shape λ. Let us write $C(k,l)$ for the C-basis element $C(P_k, P_l)$. Consider the ordered basis $C(1,1), C(1,2), \ldots, C(1,m)$ of $R(\lambda)$. Denote by $\mathfrak{e}_i^j$ the element of $\operatorname{End} R(\lambda)$ that sends $C(1,i)$ to $C(1,j)$ and kills the other basis elements. Any element of $\operatorname{End} R(\lambda)$ can be written uniquely as $\sum \alpha_i^j \mathfrak{e}_i^j$. Arrange the coefficients as a row matrix like this:

$$\left(\alpha_1^1 \ \alpha_2^1 \ \cdots \ \alpha_m^1 \mid \alpha_1^2 \ \alpha_2^2 \ \cdots \ \alpha_m^2 \mid \ \cdots \ \mid \alpha_1^m \ \alpha_2^m \ \cdots \ \alpha_m^m\right).$$

Now consider such row matrices for $\rho_\lambda(C(k,l))$. Arrange them one below the other, the first row corresponding to the value $(1,1)$ of (k,l), the second to $(2,1), \ldots$, the m^{th} row to $(m,1)$, the $(m+1)^{\text{th}}$ row to $(1,2), \ldots$, and the last to (m,m). We thus get a matrix —denote it $\mathbb{G}(\lambda)$—of size $d(\lambda)^2 \times d(\lambda)^2$, where $d(\lambda) := \dim R(\lambda)$.

Let us compute $\mathbb{G}(\lambda)$ in the light of (7.1). Setting

$$\alpha_i^j(k,l) := \alpha_i^j(\rho_\lambda(C(k,l))),$$

we have (mind the abuse of notation: this equation holds in $R(\lambda)$, not in $\mathcal{H}$):

$$C(1,i)C(k,l) = \sum_j \alpha_i^j(k,l)C(1,j).$$

Applying (7.1) to the left hand side and reading the result as an equation in $R(\lambda)$, we see that it equals $g_i^k C(1,l)$. Thus

$$\alpha_i^j(k,l) = \begin{cases} g_i^k & \text{if } j = l \\ 0 & \text{otherwise.} \end{cases}$$

which means the following:

Proposition 17. — *The matrix $\mathbb{G}(\lambda)$ (defined earlier in this section) is of block diagonal form, with uniform block size $d(\lambda) \times d(\lambda)$, and each diagonal block equal to the matrix $\mathcal{G}(\lambda) = (g_i^k)$ of §7.1, where the row index is k and the column index i.*

7.3. A formula for the determinant of $\mathcal{G}(\lambda)$

Theorem 18 below gives a formula for the determinant of the matrix $\mathcal{G}(\lambda)$ of (7.1). In order to state it, we need some notation. Set

- $[\lambda]$:= the set of nodes in the Young diagram of shape λ;
- h_{ab}:= hook length of the node $(a,b) \in [\lambda]$ (see §2.3.1).

- For a positive integer m,

$$[m]_v := v^{1-m} + v^{3-m} + \cdots + v^{m-3} + v^{m-1}$$
$$[m]_q := 1 + v^2 + v^4 + \cdots + v^{2(m-1)}.$$

Assuming λ has r rows, we can associate to λ a decreasing sequence —called the *β-sequence*— of positive integers, the hook lengths of the nodes in the first column of λ. The shape can be recovered from the sequence, so the association gives a bijection between shapes and decreasing sequences of positive integers. Given such a sequence $\beta_1 > \cdots > \beta_r$, write $d(\beta_1, \ldots, \beta_r)$ for the number $d(\lambda)$ of standard tableaux of shape λ (§2.3.1). Extend the definition of $d(\beta_1, \ldots, \beta_r)$ to an arbitrary sequence of $\beta_1, \ldots, \beta_r$ of non-negative integers at most one of which is zero as follows: if the integers are not all distinct, then it is 0; if the integers are all distinct and positive, then it is $\operatorname{sgn}(w)\, d(\beta_{w(1)}, \ldots, \beta_{w(r)})$ where w is the permutation of the symmetric group $\mathfrak{S}_r$ such that $\beta_{w(1)} > \cdots > \beta_{w(r)}$; if the integers are distinct and one of them —say β_k— is zero, then it is $d(\beta_1 - 1, \beta_2 - 1, \ldots, \beta_{k-1} - 1, \beta_{k+1} - 1, \ldots, \beta_r - 1)$, which is defined by induction on r.

Theorem 18. — (Hook Formula) *For a partition λ of n,*

$$\det \mathcal{G}(\lambda) = \epsilon_{w_o,\lambda'}^{d(\lambda)} \prod \left(\frac{[h_{ac}]_v}{[h_{bc}]_v} \right)^{d(\beta_1, \ldots, \beta_a + h_{bc}, \ldots, \beta_b - h_{bc}, \ldots, \beta_r)} \tag{7.3}$$

with notation as above, where $\beta_1 > \cdots > \beta_r$ is the β-sequence of λ and the product runs over $\{(a, b, c) \mid (a, c), (b, c) \in [\lambda]$ and $a < b\}$.

The proof of the theorem will be given in §9 and §10. Some comments about it may be found at the beginning of §9.

8. Preliminaries about permutation and Specht modules

We recall some notation and results about permutation modules and Specht modules needed in the sequel. The isomorphism θ recalled from [24] in §8.3 plays a fundamental role.

We keep the set up of §3. We will also use freely the notation and results recalled in §4.1.1. We will be frequently referring to [20, 6, 24]. The reader should be alert to the difference, pointed out in §3.1.1, between our notation and of these papers.

8.1. Some notation

Fix a partition λn. Let λ', t^λ, t_λ, W_λ, $w_{0,\lambda}$, $\mathfrak{D}_\lambda$, x_λ, and y_λ be as defined in §4.1.1.

- w_λ denotes the element of $\mathfrak{D}_\lambda$ that takes t^λ to t_λ. By a *prefix* of w_λ we mean an element of the form $s_{i_1} \cdots s_{i_j}$ for some j, $1 \leqslant j \leqslant k$, where $s_{i_1} \cdots s_{i_k}$ is some reduced expression of w_λ.
- Set $z_\lambda := v_{w_\lambda} x_\lambda T_{w_\lambda} y_{\lambda'}$.

It is elementary to see the following:

$$(8.1) \qquad w_\lambda w \text{ belongs to } \mathfrak{D}_\lambda \text{ and } \ell(w_\lambda w) = \ell(w_\lambda) + \ell(w) \quad \text{for } w \in W_{\lambda'}.$$

8.2. Permutation modules M^λ and Specht modules S^λ

Following [6] —see §3, 4 of that paper— we define the *permutation module* M^λ to be the right ideal $x_\lambda \mathcal{H}$, the *Specht module* S^λ to be the right ideal $z_\lambda \mathcal{H}$. They are $\mathcal{H}$-analogues respectively of the $\mathfrak{S}_n$-representations on tabloids of shape λ and its sub-representation the Specht module of shape λ (defined respectively in §2.5 and §2.6): the corresponding $\mathfrak{S}_n$-modules are recovered on setting $v = 1$ (see [24, Page 143]).

8.2.1. Bases for M^λ and S^λ

- [6, Lemma 3.2 (1)] $\{x_\lambda T_d \mid d \in \mathfrak{D}_\lambda\}$ is a basis for M^λ.
- [6, Theorem 5.6] The elements $v_d z_\lambda T_d$, d a prefix of $w_{\lambda'}$, form an A-basis for the Specht module S^λ called the "standard basis".

8.2.2. The bilinear form $\langle \ , \ \rangle$ on M^λ

As in [6, page 34], define a bilinear form $\langle \ , \ \rangle$ on M^λ by setting $\langle x_\lambda T_d, x_\lambda T_e \rangle$ equal to 1 or 0 accordingly as elements d, e of $\mathfrak{D}_\lambda$ are equal or not: as just recalled in §8.2.1, $x_\lambda T_d$, $d \in \mathfrak{D}_\lambda$, form a basis for M^λ. The form is evidently symmetric. We have, by [6, Lemma 4.4]:

$$(8.2) \qquad \langle m_1 h, m_2 \rangle = \langle m_1, m_2 h^* \rangle \quad \text{for } m_1, m_2 \text{ in } M^\lambda \text{ and } h \text{ in } \mathcal{H}$$

where $h \mapsto h^*$ is the A-anti-automorphism of $\mathcal{H}$ given by $T_w \mapsto T_{w^{-1}}$.

8.3. McDonough-Pallikaros isomorphism between right cell and Specht modules

There is described in [24, Theorem 3.5] a map from the right $\mathcal{H}$-ideal $\langle C_w \mid w \leqslant_{\mathrm{R}} w_{0,\lambda}\rangle_A$ to $M^{\lambda'}$: it is denoted θ, defined by $m \mapsto v_{w_{\lambda'}} x_{\lambda'} T_{w_{\lambda'}} m$, and evidently a map of $\mathcal{H}$-modules.(5) Using Proposition 6 and (4.1), we can determine the image of θ: $v_{w_{\lambda'}} x_{\lambda'} T_{w_{\lambda'}} C_{w_{0,\lambda}} \mathcal{H} = x_{\lambda'} T_{w_{\lambda'}} y_\lambda \mathcal{H} = S^{\lambda'}$. As to the kernel of θ, it equals $\langle C_w \mid w <_{\mathrm{R}} w_{0,\lambda}\rangle_A$ as proved in [24, Theorem 3.5]. Thus θ gives an isomorphism from the right cell module $R(w_{0,\lambda})$ to $S^{\lambda'}$. Since $w_{0,\lambda}$ is of shape λ' (Proposition 5 (2)) we conclude that $R(\lambda) \cong S^\lambda$.

9. The first part of the proof of Theorem 18: relating $\det \mathcal{G}(\lambda)$ to the Gram determinant $\det(\lambda)$

Towards the proof of Theorem 18, we relate, using results from [6, 24], the determinant of the matrix $\mathcal{G}(\lambda)$ (defined in (7.1)) to the *Gram determinant* $\det(\lambda)$, namely, the determinant of the matrix of the restriction to the Specht module S^λ of the bilinear form $\langle\ ,\ \rangle$ on M^λ (defined in §8.2.2), with respect to the "standard basis" as in the second item in §8.2.1. The Gram determinant being well studied and results about it being readily available in the literature, we are thus lead to conclusions about $\mathcal{G}(\lambda)$.

9.1. The Dipper-James bilinear form on $R(\lambda)$ computed in terms of its C-basis

Pulling back via the isomorphism θ of §8.3 the restriction to S^λ of the bilinear form on M^λ defined in §8, we get a bilinear form on $R(\lambda)$ (which we continue to denote by $\langle\ ,\ \rangle$). Let us compute the matrix of this form with respect to the basis $C(1,1), \ldots, C(1,m)$, where, as in §7.2, $P_1, \ldots, P_m$ is an enumeration of all standard tableaux of shape λ, and $C(k,l)$ is short hand notation for $C(P_k, P_l)$. We further assume that $P_1 = t_\lambda$, so that the right cell with P-symbol P_1 is the one containing $w_{0,\lambda'}$ (which under RSK

(5) The factor $v_{w_{\lambda'}}$ in the definition of θ appears only in deference to [24]. If it were omitted: the resulting θ would still be a $\mathcal{H}$-module map; the powers of v_{w_λ} in Equations (9.1) and (9.2) (and in the calculations leading up to them) would have to be omitted; and a suitable power of $v_{w_{\lambda'}}$ would have to be added to Equation (10.3); no other changes would have to be made.

corresponds to the pair (P_1, P_1) —see Proposition 5 (2)). The explanations for the steps in the following calculation appear below:

$$\begin{aligned}
\langle C(1,i), C(1,j)\rangle &= \langle C(1,i)\theta, C(1,j)\theta\rangle \\
&= \langle x_\lambda T_{w_\lambda} v_{w_\lambda} C(1,i), x_\lambda T_{w_\lambda} v_{w_\lambda} C(1,j)\rangle \\
&= v_{w_\lambda}^2 \langle x_\lambda T_{w_\lambda}, x_\lambda T_{w_\lambda} C(1,j)C(1,i)^*\rangle \\
&= v_{w_\lambda}^2 \langle x_\lambda T_{w_\lambda}, x_\lambda T_{w_\lambda} C(1,j)C(i,1)\rangle \\
&= v_{w_\lambda}^2 \langle x_\lambda T_{w_\lambda}, x_\lambda T_{w_\lambda} g_j^i C(1,1)\rangle \\
&= v_{w_\lambda}^2 g_j^i \, \langle x_\lambda T_{w_\lambda}, x_\lambda T_{w_\lambda} \epsilon_{w_{0,\lambda'}} v_{w_{0,\lambda'}} y_{\lambda'}\rangle \\
&= \epsilon_{w_{0,\lambda'}} v_{w_{0,\lambda'}} v_{w_\lambda}^2 g_j^i \sum_{w\in W_{\lambda'}} \epsilon_w v_w^{-1} \langle x_\lambda T_{w_\lambda}, x_\lambda T_{w_\lambda} T_w\rangle \\
&= \epsilon_{w_{0,\lambda'}} v_{w_{0,\lambda'}} v_{w_\lambda}^2 g_j^i.
\end{aligned}$$

The first equality follows from definition of the form on $R(\lambda)$; the second from the definition of θ; the third from (8.2); the fourth from (3.4). For the fifth, substitute for $C(1,j)C(i,1)$ using (7.1) and observe that the "smaller terms" on the right hand side belong to the kernel of θ (§8.3). The sixth follows by substituting for $C(1,1) = C_{w_{0,\lambda'}}$ from (4.1); the seventh from the definition of $y_{\lambda'}$; and the final equality by combining the definition of the form with (8.1) (observe that $T_{w_\lambda} T_w = T_{w_\lambda w}$ since $\ell(w_\lambda) + \ell(w) = \ell(w_\lambda w)$ and that $w_\lambda w$ belongs to $\mathfrak{D}_\lambda$).

In particular, the determinant of the matrix of the form $\langle\ ,\ \rangle$ on $R(\lambda)$ with respect to the basis $C(1,1), \ldots, C(1,m)$ equals

$$\epsilon_{w_{0,\lambda'}}^{d(\lambda)} v_{w_{0,\lambda'}}^{d(\lambda)} v_{w_\lambda}^{2d(\lambda)} \det \mathcal{G}(\lambda). \tag{9.1}$$

9.2. The "T-basis" of $R(\lambda')$ and its relationship to the C-basis

Following [24, §2], we define the "T-basis" of the right cell module $R(\lambda')$ and show that it has a uni-triangular relationship with the C-basis. We do this by means of the "C-basis" and "T-basis" of the right $\mathcal{H}$-module $C_{w_{0,\lambda}}\mathcal{H}$, which are defined respectively by:

- C_w, $w \leqslant_{\mathrm{R}} w_{0,\lambda}$
- $C_{w_{0,\lambda}} T_d$, $d \in \mathfrak{D}_\lambda$.

That the "C-basis" is an A-basis follows from Proposition 6. That the "T-basis" is an A-basis is item (2) in the following:

PROPOSITION 19.

(1) $C_{w_{0,\lambda}} T_y = \epsilon(y) v_y^{-1} C_{w_{0,\lambda}}$ *for* $y \in W_\lambda$.
(2) $C_{w_{0,\lambda}} T_d$, $d \in \mathfrak{D}_\lambda$, *form an* A*-basis for the right ideal* $C_{w_{0,\lambda}} \mathcal{H}$.
(3) $w \sim_{\mathrm{R}} w_{0,\lambda}$ *if and only if* $w = w_{0,\lambda} d$ *for a prefix* d *of* w_λ. *In particular, prefixes of* w_λ *belong to* $\mathfrak{D}_\lambda$.

Proof. — (1) follows from [20, Equation (2.3.d)]; (2) from item (1), Proposition 5 (2), and (3.5) —see [24, Page 136]; (3) from [24, Lemma 3.3 (iv)]. □

The elements $w \leqslant_{\mathrm{R}} w_{0,\lambda}$ are precisely $w_{0,\lambda} d$, $d \in \mathfrak{D}_\lambda$ (Proposition 5 (2)). Let $d_1, \ldots, d_M$ be the elements of $\mathfrak{D}_\lambda$ ordered so that $i \leqslant j$ if $d_i \leqslant d_j$ in the Bruhat order. By [24, Proposition 2.13] and its proof, the two bases above are related by a uni-triangular matrix with respect to an ordering as above (keeping in mind that T_y in the notation of [24] equals $v_y T_y$ in ours):

$$\begin{pmatrix} C_{w_{0,\lambda}} T_{d_1} \\ \vdots \\ C_{w_{0,\lambda}} T_{d_M} \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ \star & & 1 \end{pmatrix} \begin{pmatrix} C_{w_{0,\lambda} d_1} \\ \vdots \\ C_{w_{0,\lambda} d_M} \end{pmatrix}.$$

Let us now read this equation in the quotient $R(\lambda')$ of $C_{w_{0,\lambda}} \mathcal{H}$. Let $d_{i_1}, \ldots, d_{i_m}$ with $1 \leqslant i_1 < \cdots < i_m \leqslant M$ be such that they are all the prefixes of w_λ —see Proposition 19 (3)— so that $w_{0,\lambda} d_{i_1}, \ldots, w_{0,\lambda} d_{i_m}$ are all the elements right equivalent to $w_{0,\lambda}$. Writing $e_1, \ldots, e_m$ in place of $d_{i_1}, \ldots, d_{i_m}$, and noting that $C_{w_{0,\lambda} d_j}$ vanishes in $R(\lambda')$ unless $w_{0,\lambda} d_j \sim_{\mathrm{R}} w_{0,\lambda}$, we have:

$$\begin{pmatrix} C_{w_{0,\lambda}} T_{e_1} \\ \vdots \\ C_{w_{0,\lambda}} T_{e_m} \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ \star & & 1 \end{pmatrix} \begin{pmatrix} C_{w_{0,\lambda} e_1} \\ \vdots \\ C_{w_{0,\lambda} e_m} \end{pmatrix}.$$

We conclude that $C_{w_{0,\lambda}} T_{e_1}, \ldots, C_{w_{0,\lambda}} T_{e_m}$ form an A-basis for $R(\lambda')$. It is called the T*-basis* and is in uni-triangular relationship with the C-basis $C_{w_{0,\lambda} e_1}, \ldots, C_{w_{0,\lambda} e_m}$. In particular, the determinants of the matrices of the form $\langle\ ,\ \rangle$ on $R(\lambda')$ (defined in §9.1) with respect to the T- and C-bases are the same.

9.3. $\det \mathcal{G}(\lambda)$ and the Gram determinant $\det(\lambda)$

Continuing towards our goal of relating $\det \mathcal{G}(\lambda)$ to the Gram determinant $\det(\lambda)$, let us compute the image under the map θ of the T-basis elements of $R(\lambda)$. Given a prefix e of $w_{\lambda'}$, we have

$$\begin{aligned} C_{w_{0,\lambda'}} T_e \theta &= v_{w_\lambda} x_\lambda T_{w_\lambda} C_{w_{0,\lambda'}} T_e && \text{(by the definition of } \theta \text{ in §8.3)} \\ &= \epsilon_{w_{0,\lambda'}} v_{w_{0,\lambda'}} \left(v_{w_\lambda} x_\lambda T_{w_\lambda} y_{\lambda'} \right) T_e && \text{(by (4.1))} \\ &= \epsilon_{w_{0,\lambda'}} v_{w_{0,\lambda'}} z_\lambda T_e && \text{(by the definition of } z_\lambda \text{ in §8.1)} \\ &= \epsilon_{w_{0,\lambda'}} v_{w_{0,\lambda'}} v_e^{-1} (v_e z_\lambda T_e). \end{aligned}$$

Noting that $v_e z_\lambda T_e$ is a standard basis element of S^λ (see §8.2.1), we conclude that the determinant of the matrix of the bilinear form $\langle\ ,\ \rangle$ on $R(\lambda)$ with respect to the T-basis equals $v_{w_{0,\lambda'}}^{2d(\lambda)} \left(\prod_e v_e^{-1}\right)^2 \det(\lambda)$. Combined with (9.1) and the conclusion of §9.2, this gives

$$\det \mathcal{G}(\lambda) = (\epsilon_{w_{0,\lambda'}} v_{w_{0,\lambda'}} v_{w_\lambda}^{-2})^{d(\lambda)} (\prod_e v_e)^{-2} \det(\lambda) \tag{9.2}$$

where the product is taken over all prefixes e of $w_{\lambda'}$.

10. Conclusion of the proof of Theorem 18

In this section, we complete the proof of Theorem 18 by using results from [7, 19]. Both sides of Equation (7.3) are elements of A. To prove they are equal, we may pass to the quotient field $K := \mathbb{Q}(v)$ of A. We do this tacitly in the sequel. Observe that $\mathcal{H}_K$ is semisimple (see 3.1.2).

10.1. Second half of the proof of Theorem 18

We set things up to be able to use a formula from [7] for the Gram determinant $\det(\lambda)$. Let $S_1, \ldots, S_m$ be an enumeration of all the standard tableaux of shape λ. For i, u such that $1 \leqslant i \leqslant m$, $1 \leqslant u \leqslant n$, let S_i^u denote the standard tableau obtained from S_i by deleting all nodes with entries exceeding u; set $\gamma_{ui} := \prod_{j=1}^{a-1} [h_{jb}]_q / [h_{jb} - 1]_q$ where (a, b) is the position of the node in S_i^u containing u, h_{jb} is the hook length in S_i^u of the node in position (j, b), and $[s]_q := 1 + v^2 + v^4 + \cdots + v^{2(s-1)}$ for a positive integer s. By [7, Theorem 4.11], the Gram determinant $\det(\lambda)$ is given by

$$\det(\lambda) = v^{2r} \prod_{i=1}^{m} \prod_{u=1}^{n} \gamma_{ui} \qquad \text{for some integer } r. \tag{10.1}$$

We now apply the equation in [19, Corollary 2.30, page 251]. Computing $\Delta_\mu(\lambda')$ (in the notation of [19]) with $\mu = 1^n$, and re-indexing the product in the right side of that equation over nodes of λ rather than of λ', we get

$$\prod_{i=1}^{m}\prod_{u=1}^{n}\gamma_{ui} = \prod\left(\frac{[h_{ac}]_q}{[h_{bc}]_q}\right)^{d(\beta_1,\ldots,\beta_a+h_{bc},\ldots,\beta_b-h_{bc},\ldots,\beta_r)} \tag{10.2}$$

where the product on the right hand side runs over triples (a, b, c) as in the statement of the theorem.

Combining Equations (9.2), (10.1), (10.2), and (10.4), we get

$$\epsilon_{w_{0,\lambda'}}^{d(\lambda)} v_{w_{0,\lambda'}}^{d(\lambda)} \det \mathcal{G}(\lambda) = \prod\left(\frac{[h_{ac}]_q}{[h_{bc}]_q}\right)^{d(\beta_1,\ldots,\beta_a+h_{bc},\ldots,\beta_b-h_{bc},\ldots,\beta_r)}. \tag{10.3}$$

The left hand side is an element of A. As to the right hand side, it is regular with value 1 at $v = 0$, since the same is true for $[s]_q$ for every positive integer s. Thus both sides of the equation belong to $1 + v\mathbb{Z}[v]$ and

$$\det \mathcal{G}(\lambda) = \epsilon_{w_{0,\lambda'}}^{d(\lambda)} v_{w_{0,\lambda'}}^{-d(\lambda)} + \text{higher degree terms.}$$

The "bar-invariance" of the C-basis elements (§3.1) means that:

$$\overline{g_j^k} = g_j^k \text{ for } g_j^k \text{ as in (7.1) and so also } \overline{\det \mathcal{G}(\lambda)} = \det \mathcal{G}(\lambda).$$

Thus $\det \mathcal{G}(\lambda)$ has the form:

$$\epsilon_{w_{0,\lambda'}}^{d(\lambda)} v_{w_{0,\lambda'}}^{-d(\lambda)} + \cdots + \epsilon_{w_{0,\lambda'}}^{d(\lambda)} v_{w_{0,\lambda'}}^{d(\lambda)}$$

the terms represented by $\cdots$ being of v-degree strictly between $-d(\lambda)\ell(w_{0,\lambda'})$ and $d(\lambda)\ell(w_{0,\lambda'})$. Equating the v-degrees on both sides of (10.3) gives

$$d(\lambda)\ell(w_{0,\lambda'}) = \sum d(\beta_1, \ldots, \beta_a + h_{bc}, \ldots, \beta_b - h_{bc}, \ldots, \beta_r)\,(h_{ac} - h_{bc}).$$

Using this and substituting $v^{h_{ac}}[h_{ac}]_v$, $v^{h_{bc}}[h_{bc}]_v$, respectively for $[h_{ac}]_q$, $[h_{bc}]_q$ into (10.3), we arrive at the theorem.

□

Lemma 20. — *The integer r in the exponent of v in Equation (10.1) is given by*

$$r = d(\lambda)\left(\ell(w_\lambda) - \ell(w_{0,\lambda'})\right) + \sum_{i=1}^{m}\ell(d_i) \tag{10.4}$$

where $d_1, \ldots, d_m$ are all the prefixes of $w_{\lambda'}$.

Proof. — We essentially work through the proof of [7, Theorem 4.11] to calculate the exponent of v appearing in 10.1.

Let $d_1, \ldots, d_m$ be ordered so that $i < j$ if $\ell(d_i) < \ell(d_j)$. Let e_i and f_i, $1 \leqslant i \leqslant m$, be bases of S^λ as in [7]. The $e_i := v_{d_i} z_\lambda T_{d_i}$ are just the standard basis (see §8.2.1). The f_i are an orthogonal basis in uni-triangular relationship with the e_i [7, Theorem 4.7]. Thus $\det(\lambda) = \prod_{i=1}^{m} \langle f_i, f_i \rangle$.

Let the enumeration $S_1, \ldots, S_m$ of standard tableaux of shape λ be such that $S_i = t_\lambda d_i$. Then, from [7, Lemma 4.10], $\langle f_i, f_i \rangle = v^{2r_i} \prod_{u=1}^{n} \gamma_{ui}$. We claim:

(1) $r_1 = \ell(w_\lambda) - \ell(w_{0,\lambda'})$
(2) for $i > 1$, $r_i = r_j + 1$ where $j < i$ such that $e_i = v e_j T_{(k-1,k)}$.

The lemma being clear given the claim, it remains only to prove the claim.

Item (2) of the claim follows from the following two observations made in the course of the proof of [7, Lemma 4.10]: $\langle f_i, f_i \rangle = c_j \langle f_j, f_j \rangle$, and

$$\gamma_{ui} = \begin{cases} \gamma_{uj} & \text{if } u \neq k-1, k \\ \gamma_{kj} & \text{if } u = k-1 \\ v^{-2} c_j \gamma_{k-1,j} & \text{if } u = k. \end{cases}$$

(The definition of c_j is irrelevant for our purposes.)

To prove item (1) of the claim, we compute $\langle f_1, f_1 \rangle$. We have $f_1 = e_1 = z_\lambda$. Substituting for z_λ and in turn for y_λ from their definitions in §8.1 and §4.1.1, we get

$$\begin{aligned} \langle f_1, f_1 \rangle &= \langle z_\lambda, z_\lambda \rangle \\ &= \langle v_{w_\lambda} x_\lambda T_{w_\lambda} y_{\lambda'}, v_{w_\lambda} x_\lambda T_{w_\lambda} y_{\lambda'} \rangle \\ &= v_{w_\lambda}^2 \sum_{u,u' \in W_{\lambda'}} \epsilon_u \epsilon_{u'} v_u^{-1} v_{u'}^{-1} \langle x_\lambda T_{w_\lambda} T_u, x_\lambda T_{w_\lambda} T_{u'} \rangle. \end{aligned}$$

Using in order (8.2), (3.2), (8.1) and the definition of $\langle\ ,\ \rangle$, we get:

$$\begin{aligned} \langle x_\lambda T_{w_\lambda} T_u, x_\lambda T_{w_\lambda} T_{u'} \rangle &= \langle x_\lambda T_{w_\lambda}, x_\lambda T_{w_\lambda} T_{u'} T_{u^{-1}} \rangle \\ &= \langle x_\lambda T_{w_\lambda}, x_\lambda T_{w_\lambda} (T_{u'u^{-1}} + \sum_{w \in W_{\lambda'}; w > u'u^{-1}} c_w T_w) \rangle \\ &= \langle x_\lambda T_{w_\lambda}, x_\lambda T_{w_\lambda u' u^{-1}} + \sum_{w \in W_{\lambda'}; w > u'u^{-1}} c_w T_{w_\lambda w} \rangle \\ &= \begin{cases} 1 & \text{if } u = u' \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

so that

$$\langle f_1, f_1 \rangle = v_{w_\lambda}^2 \sum_{u \in W_{\lambda'}} v_u^{-2} = v_{w_\lambda}^2 v_{w_{0,\lambda'}}^{-2} \sum_{u \in W_{\lambda'}} v_{w_{0,\lambda'}}^2 v_u^{-2} = v_{w_\lambda}^2 v_{w_{0,\lambda'}}^{-2} \sum_{u \in W_{\lambda'}} v_u^2.$$

Routine calculations show:

$$\sum_{u \in \mathfrak{S}_n} v_u^2 \quad = \quad [n]_q^! \qquad \text{and} \qquad \sum_{u \in W_{\lambda'}} v_u^2 \quad = \quad [\lambda'_1]_q^! \cdots [\lambda'_r]_q^!$$

where $[n]_q^! := [n]_q[n-1]_q \cdots [1]_q$ and $\lambda' = (\lambda'_1, \ldots, \lambda'_r)$. Finally, a pleasant verification, given the fact that $S_1 = t_\lambda$, shows:

$$[\lambda'_1]_q^! \cdots [\lambda'_r]_q^! = \prod_{u=1}^{n} \gamma_{u1}.$$

The proof of the claim (and so also of the lemma) is complete. □

11. On the irreducibility of Specht modules

Let k be a field and a a non-zero element of k. Fix notation as in §3, and consider k as an A-module via the map $A \to k$ defined by $v \mapsto a$.

The purpose of this section is to revisit the question of when the Specht module $S_k^\lambda := S^\lambda \otimes_A k$ (which, by §8.3, is isomorphic to the right cell module $R(\lambda) \otimes_A k$) is irreducible (as a module over $\mathcal{H}_k := \mathcal{H} \otimes_A k$). As we will see, the results of §7–10 afford us a fresh approach to this question. In addition, they allow us to prove generalizations of Theorems 14, 15 to situations when $\mathcal{H}_k$ is not necessarily semisimple.

The form $\langle\ ,\ \rangle$ defined on M^λ by Dipper-James has been recalled in §8.2.2. Let $\langle\ ,\ \rangle_k$ denote the form on $M^\lambda \otimes_A k$ obtained by extension of scalars and also its restriction to S_k^λ. Note that $\langle\ ,\ \rangle_k$ is symmetric (since $\langle\ ,\ \rangle$ on M^λ is).

11.1. Preliminaries

We first establish some notation and make a few observations. Thus equipped, we recall some results from [25] in a form that is convenient or us.

For an automorphism $\ddagger$ of $\mathcal{H}$ and a (right) $\mathcal{H}$-module M, we denote by $M^\ddagger$ the (right) $\mathcal{H}$-module whose underlying A-module is the same as that of M —it is convenient to write $m^\ddagger$ for an element m of M thought of in $M^\ddagger$— and with the action of $\mathcal{H}$ being given by $m^\ddagger h^\ddagger := (mh)^\ddagger$.

- For a (right) $\mathcal{H}$-ideal $\mathfrak{I}$, the symbol $\mathfrak{I}^\ddagger$ can be interpreted without conflict as either the image of $\mathfrak{I}$ under $\ddagger$ or the module $M^\ddagger$ defined as above taking M to be $\mathfrak{I}$.
- If $\ddagger$ is an involution, then $M \simeq (M^\ddagger)^\ddagger$ naturally.
- For an anti-automorphism $\ddagger$ of $\mathcal{H}$, $M^\ddagger$ defined similarly would naturally be a left $\mathcal{H}$-module: $h^\ddagger m^\ddagger := (mh)^\ddagger$.

Now consider the involution $\dagger$ on $\mathcal{H}$ defined in §3.1.3 and the permutation module M^λ defined in §8.2. From equations (3.3), (3.6), and (4.1), it follows that $(C_{w_{0,\lambda}}\mathcal{H})^\dagger = M^\lambda$; from Proposition 6 that

$$C'_w,\ w \leqslant_{\mathrm{R}} w_{0,\lambda}, \text{ form an } A\text{-basis for } M^\lambda. \tag{11.1}$$

Set $N^\lambda := \langle C'_w \mid (w) \trianglelefteq \lambda' \rangle_A$ and $\hat{N}^\lambda := \langle C'_w \mid (w) \triangleleft \lambda' \rangle_A$. From §3.4.2 and Proposition 5 (4) it follows that $M^\lambda \subseteq N^\lambda$. Set $\tilde{S}^\lambda := M^\lambda / M^\lambda \cap \hat{N}^\lambda$. From §§3.4.1–3.4.3 it follows that $\{w \mid w \leqslant_{\mathrm{R}} w_{0,\lambda} \text{ and } (w) \triangleleft \lambda'\} = \{w \mid w \lneq_{\mathrm{R}} w_{0,\lambda}\}$, so that $M^\lambda \cap \hat{N}^\lambda = \langle C'_w \mid w \lneq_{\mathrm{R}} w_{0,\lambda} \rangle_A$. Thus the images in $\tilde{S}^\lambda$ of C'_w, $w \sim_{\mathrm{R}} w_{0,\lambda}$, form a basis for $\tilde{S}^\lambda$. And we get

$$R(\lambda')^\dagger \simeq \tilde{S}^\lambda \qquad R(\lambda) \simeq (\tilde{S}^{\lambda'})^\dagger. \tag{11.2}$$

For a (right) $\mathcal{H}$-module M, the dual $M^{\mathrm{dual}} := \mathrm{Hom}_A(M, A)$ is naturally a left $\mathcal{H}$-module: $(m)(h\phi) := (mh)\phi$, for $\phi \in M^{\mathrm{dual}}$, $m \in M$, and $h \in \mathcal{H}$. We use the anti-automorphism $*$ defined in §3.1.3 to switch between right and left module structures: $(M^{\mathrm{dual}})^*$ becomes a right $\mathcal{H}$-module.

- The process $M \mapsto M^{\mathrm{dual}}$ commutes with that of $M \mapsto M^\ddagger$ defined earlier in this section: in particular, $(M^{\mathrm{dual}})^* \simeq (M^*)^{\mathrm{dual}}$ naturally.
- If M is free as an A-module, then $(M^{\mathrm{dual}})^{\mathrm{dual}} \simeq M$ naturally.

Proposition 21 ([25], Theorem 5.2). — *We have an isomorphism $((S^\lambda)^{\mathrm{dual}})^* \simeq (S^{\lambda'})^\dagger$. In particular S_k^λ is irreducible if and only if $S_k^{\lambda'}$ is so.*

Proof. — It is proved in [25, Theorem 5.2] that $(\tilde{S}^{\lambda'})^\dagger \simeq ((\tilde{S}^\lambda)^{\mathrm{dual}})^*$: a perfect pairing $(\ ,\ ) : \tilde{S}^\lambda \times (\tilde{S}^{\lambda'})^\dagger \to A$ with the property that $(m, nh) = (mh^*, n)$ is given. Combining this statement with the isomorphisms (11.2) and the isomorphism $R(\lambda) \simeq S^\lambda$ of §8.3, the proposition follows. ☐

A shape λ is called *e-regular* if the number of rows in it of any given length is less than e. Let now e be the smallest positive integer such that $1 + a^2 + \cdots + a^{2(e-1)} = 0$; if there is no such integer, then $e = \infty$.

Proposition 22. — ([25, Theorem 6.9]) *If λ is e-regular, the bilinear form $\langle\ ,\ \rangle_k$ on S_k^λ is non-zero.*

Proof. — Consider the form $\langle\ ,\ \rangle_\lambda$ on M^λ defined in [25, page 114]. Using (3.4), this can be expressed in our notation as follows in terms of the basis (11.1) of M^λ: for w, x such that $w \leqslant_{\mathrm{R}} w_{0,\lambda}$, $x \leqslant_{\mathrm{R}} w_{0,\lambda}$, $\langle C'_w, C'_x \rangle_\lambda$ is the coefficient of $C'_{w_{0,\lambda}}$ in $C'_w C'_{x^{-1}}$. It follows readily from the definition in §3.3 of the relation $\leqslant_{\mathrm{R}}$ that if either $w \lneq_{\mathrm{R}} w_{0,\lambda}$ or $x \lneq_{\mathrm{R}} w_{0,\lambda}$ (which is equivalent to $x^{-1} \lneq_{\mathrm{L}} w_{0,\lambda}^{-1} = w_{0,\lambda}$), then $\langle C'_w, C'_x \rangle_\lambda = 0$. Thus $\langle\ ,\ \rangle_\lambda$ descends to $\tilde{S}^\lambda$.

From [25, Theorem 6.9 —see also its proof] it follows that $\langle\ ,\ \rangle_\lambda$ does not vanish on $\tilde{S}^\lambda$ if λ' is e-regular. This means that there exist $w \sim_{\mathrm{R}} w_{0,\lambda}$,

$x \sim_{\mathrm{R}} w_{0,\lambda}$ such that the coefficient of $C'_{w_{0,\lambda}}$ in $C'_w C'_{x^{-1}}$ is non-zero. Applying the involution †, we conclude that the coefficient of $C_{w_{0,\lambda}}$ in $C_w C_{x^{-1}}$ is non-zero (see (3.7)).

Consider the ordered pairs of standard tableaux associated to $w_{0,\lambda}$, w, and x by the RSK-correspondence: $w_{0,\lambda} \leftrightarrow (t_{\lambda'}, t_{\lambda'})$ by Proposition 5 (4); let Q_w and Q_x be the standard tableaux of shape λ' such that $w \leftrightarrow (t_{\lambda'}, Q_w)$ and $x \leftrightarrow (t_{\lambda'}, Q_x)$ (see §3.4). Then $x^{-1} \leftrightarrow (Q_x, t_{\lambda'})$ (see §2.4) and, by (7.1),

$$C_w C_{x^{-1}} = C(t_{\lambda'}, Q_w) C(Q_x, t_{\lambda'}) \equiv g_w^x C(t_{\lambda'}, t_{\lambda'}) \text{ mod "lower terms".}$$

The conclusion of the last paragraph translated to this notation says that the coefficient g_w^x is non-zero.

Consider the pull-back to $R(\lambda')_k$ via the isomorphism θ of §8.3 of the form $\langle\ ,\ \rangle_k$ on $S_k^{\lambda'}$. Denoting it too by $\langle\ ,\ \rangle_k$, the big display in §9.1 says that $\langle C_w, C_x\rangle_k$ equals the coefficient g_w^x up to sign and a power of v. Thus $\langle C_w, C_x\rangle_k \neq 0$, which means that the form $\langle\ ,\ \rangle_k$ on $S_k^{\lambda'}$ is non-zero. □

11.2. Analogues of Theorems 14, 15 for not necessarily semi-simple $\mathcal{H}_k$

Theorem 23. — *For an e-regular shape λ such that S_k^λ is irreducible, the Kazhdan-Lusztig basis elements C_w, w of λ, thought of as operators on S_k^λ form a basis for* $\operatorname{End} S_k^\lambda$.

Proof. — By (8.2), the radical of the form $\langle\ ,\ \rangle_k$ on S_k^λ is a $\mathcal{H}_k$-submodule. Since S_k^λ is assumed irreducible, the form is either identically zero or non-degenerate. But, as shown in Proposition 22 above, it is non-zero under the assumption of e-regularity of λ. Thus its matrix with respect to any basis of S_k^λ has non-zero determinant. By (9.1), $\det \mathcal{G}(\lambda)|_{v=a}$ is such a determinant (up to a sign and power of a), so it is non-zero. It now follows from Proposition 17 that the operators C_w, w of λ, form a basis for $\operatorname{End} S_k^\lambda$. □

Corollary 24. — *Suppose that λ' is e-regular and that S_k^λ is irreducible. Then the elements C'_w, $(w) = \lambda'$, as operators on S_k^λ form a basis for* $\operatorname{End} S_k^\lambda$.

Proof. — By Theorem 23, the C_w, $(w) = \lambda'$, as operators on $S_k^{\lambda'}$ form a basis for $\operatorname{End} S_k^{\lambda'}$ ($S_k^{\lambda'}$ is irreducible by Proposition 21). Since $S_k^{\lambda'} \simeq (((S_k^\lambda)^{\mathrm{dual}})^*)^\dagger$ (Proposition 21 again), and ${C_w}^\dagger = \epsilon_w C'_w$ by (3.7), it follows that the C'_w, $(w) = \lambda'$, as operators on $((S_k^\lambda)^{\mathrm{dual}})^*$ form a basis for $\operatorname{End}((S_k^\lambda)^{\mathrm{dual}})^*$. Since $(C'_w)^* = C'_{w^{-1}}$ by (3.4) and the s of w and w^{-1} are the same, the result follows. □

Theorem 25. — *Let $\mathcal{S}$ be the set of e-regular shapes λ such that the Specht module S_k^λ is irreducible. Let U be a finite dimensional semisimple $\mathcal{H}_k$-module, every irreducible component of which is of the form S_k^λ, $\lambda \in \mathcal{S}$. Let $\mathcal{T}$ be the subset of $\mathcal{S}$ consisting of those shapes λ such that S_k^λ appears as a component of U. Then the images in $\operatorname{End} U$ of C_x, $x \in \mathfrak{S}_n$ such that (x) belongs to $\mathcal{T}$, form a basis for the image of $\mathcal{H}_k$ in $\operatorname{End} U$ (under the map $\mathcal{H}_k \to \operatorname{End} U$ defining U).*

Proof. — The proof is similar to that of Theorem 15. □

11.3. A criterion for irreducibility of S_k^λ

We first observe that Proposition 17 gives us a criterion for irreducibility of Specht modules (Theorem 26). We then deduce from this criterion a conjecture of Carter [18, Conjecture 1.2] about irreducibility of Specht modules (Corollary 27). Of course the conjecture has long been proved [18, 19], but our approach is new.

Theorem 26. — *If $\det \mathcal{G}(\lambda)|_{v=a}$ does not vanish in k, then S_k^λ is irreducible.*

Proof. — Suppose that $\det \mathcal{G}(\lambda)|_{v=a}$ does not vanish in k. Then, by Proposition 17, the matrix $\mathbb{G}$ is invertible (in k, after specializing to $v = a$). Thus the elements C_w, w of λ, are linearly independent (and so form a basis) as operators on S_k^λ. In particular, S_k^λ is irreducible, and the assertion is proved. □

11.3.1. A new proof of Carter's conjecture

Let p denote the smallest positive integer such that $p = 0$ in k; if no such integer exists, then $p = \infty$. For an integer h, define $\nu_p(h)$ as the largest power of p (possibly 0) that divides h in case p is positive, and as 0 otherwise. Recall that e denotes the smallest positive integer such that $1 + a^2 + \cdots + a^{2(e-1)} = 0$; if there is no such integer, then $e = \infty$. For an integer h, define

$$\nu_{e,p}(h) := \begin{cases} 0 & \text{if } e = \infty \text{ or } e \nmid h \\ 1 + \nu_p(h/e) & \text{otherwise.} \end{cases}$$

The (e,p)-*power diagram* of shape λ is the filling up of the nodes of the shape λ by the $\nu_{e,p}$'s of the respective hook lengths.

Observe that $e = p$ if $a = 1$.

Corollary 27. — [18, 19] *If the* (e,p)*-power diagram of* λ *has either no column or no row containing different numbers, then* S_k^λ *is irreducible.*

Proof. — It is enough to do the case when no column of the (e,p)-power diagram has different numbers: if the condition is met on rows and not on columns, we can pass to λ' and use the observation ([11, Corollary 3.3] or Proposition 21 above) that S_k^λ is irreducible if and only if $S_k^{\lambda'}$ is.

So assume that in every column of the (e,p)-power diagram the numbers are all the same. We claim that each of the factors $[h_{ac}]_v/[h_{bc}]_v$ on the right hand side of (7.3) makes sense as an element of k and is non-zero. Combining the claim with Theorems 18 and 26 yields the assertion.

To prove the claim, we need the following elementary observations, where h denotes a positive integer:

- $[h]_v$ vanishes in k if and only if e is finite and divides h.
- if e is finite and divides h, then $[h]_v = ([h/e]_v)\,|_{v=v^e}\,[e]_v$.
- $a^{2e} = 1$ if e is finite.

If either $e = \infty$ or e does not divide any of the hook lengths in shape λ, then the claim follows from the the first of the above observations. So now suppose that e is finite and divides either h_{ac} or h_{bc}. By our hypothesis, e then divides both h_{ac} and h_{bc}; moreover both h_{ac}/e and h_{bc}/e are divisible by p to the same extent. Using the second and third observations above, we conclude that the image in k of $[h_{ac}]_v/[h_{bc}]_v$ is the same as that of the rational number h_{ac}/h_{bc} (written in reduced form), and so is non-zero. □

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