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HIDA FAMILIES, p-ADIC HEIGHTS, AND DERIVATIVES

by Trevor ARNOLD

ABSTRACT. — This paper concerns the arithmetic of certain p-adic families of elliptic modular forms. We relate, using a formula of Rubin, some Iwasawa-theoretic aspects of the three items in the title of this paper. In particular, we examine several conjectures, three of which assert the non-triviality of an Euler system, a p-adic regulator, and the derivative of a p-adic L-function. We investigate sufficient conditions for the first conjecture to hold and show that, under additional assumptions, the first conjecture implies the equivalence of the last two.

RÉSUMÉ. — Cet article concerne l'arithmétique de certaines familles p-adiques de formes modulaires elliptiques. En utilisant une formule de Rubin, on examine quelques aspects de la théorie d'Iwasawa pour les objets du titre, dont trois affirment la non-trivialité d'un système d'Euler, d'un régulateur p-adique, et de la dérivée d'une fonction L p-adique. En particulier, on étudie des conditions suffisantes pour que la première conjecture soit vraie et on démontre que, sous des hypothèses supplémentaires, la première conjecture implique que les deux dernières conjectures sont équivalentes.

1. Introduction

1.1. Let f be a normalized new eigenform of even weight $k \geqslant 2$ and level N, choose a prime $p \nmid 2N$ at which f is ordinary, and let ρ_f be the associated 2-dimensional p-adic Galois representation. By work of Hida [5], f belongs to a p-adic family $\mathcal{F}$ of modular forms and the representation ρ_f can be deformed to a continuous representation

$$\rho_{\mathcal{F}}\colon\ \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{Aut}_{\mathbf{H}}(\mathcal{T})$$

on a module $\mathcal{T}$ of rank 2 over a certain ring $\mathbf{H}$ which is a complete local domain, finite and flat over the power series ring $\mathbf{Z}_p[\![Y]\!]$. See 2.5 for a description of some properties of $\rho_{\mathcal{F}}$.

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Let $\mathbf{Q}_\infty$ be the unique $\mathbf{Z}_p$-extension of $\mathbf{Q}$ and let Λ be the cyclotomic Iwasawa algebra $\Lambda = \mathbf{Z}_p[\![\Gamma_C]\!]$, $\Gamma_C = \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$. We can further deform $\rho_{\mathcal{F}}$ in the cyclotomic direction to obtain a representation

$$\widetilde{\rho}_{\mathcal{F}} = \rho_{\mathcal{F}} \otimes \kappa^{\mathrm{univ}} \colon \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{Aut}_{\mathbf{H}[\![\Gamma_C]\!]}(\widetilde{\mathcal{T}}),$$

where $\widetilde{\mathcal{T}} = \mathcal{T} \widehat{\otimes}_{\mathbf{Z}_p} \Lambda$ with $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ acting on Λ via the universal character $\kappa^{\mathrm{univ}} \colon \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \twoheadrightarrow \Gamma_C \hookrightarrow \Lambda^\times$.

1.2. Work of Kitagawa [7] and others associates to $\widetilde{\rho}_{\mathcal{F}}$ a p-adic L-function $\mathcal{L} \in \mathbf{H}[\![\Gamma_C]\!]$ interpolating L-values of modular forms (of varying weight) belonging to the Hida family $\mathcal{F}$. See 4.2 for the interpolation formula satisfied by $\mathcal{L}$.

Assuming that f has Nebentypus ω^{2-k}, where ω is the mod p cyclotomic character, there is a principal ideal $\Theta \subseteq \Lambda_{\mathbf{H}} := \mathbf{H}[\![\Gamma_C]\!]$ (see 2.5) which cuts out the locus where $\widetilde{\rho}_{\mathcal{F}}$ is *self-dual*. The specialization of $\mathcal{L}$ at arithmetic primes (see 2.2 for the definition of arithmetic prime) containing Θ interpolates *central* values of L-functions of (twists of) modular forms in the family $\mathcal{F}$. One expects the order of vanishing of the classical L-functions interpolated by $\mathcal{L}$ at their central points is generically 0 or 1, depending on the common sign in the functional equations for the L-functions of the forms belonging to $\mathcal{F}$.

1.3. Conjecture (Greenberg [3]). — *If $\varepsilon_{\mathcal{F}} = 1$, resp. -1, then all but finitely many of the L-values $L(f_\sigma,\ w(\sigma)/2)$, resp. L-derivatives $L'(f_\sigma,\ w(\sigma)/2)$, are non-zero as σ ranges over all arithmetic characters of $\mathbf{H}$ such that $w(\sigma)/2 \equiv k \pmod{p-1}$ and $\chi_\sigma = 1$.*

Here f_σ denotes the modular form arising as the specialization of $\mathcal{F}$ at σ and $\varepsilon_{\mathcal{F}}$ denotes the *sign* of $\mathcal{F}$, *i.e.*, the common sign of the forms belonging to $\mathcal{F}$. See 2.2 for the definition of χ_σ. The conditions on χ_σ and the class of $w(\sigma)/2$ mod $(p-1)$ ensure that the p-adic L-function $\mathcal{L}$ interpolates the L-value $L(f_\sigma, w(\sigma)/2)$. The following is the natural p-adic analogue of this conjecture.

1.4. Conjecture.

$$\mathrm{ord}_\Theta \mathcal{L} = \begin{cases} 0 & \varepsilon_{\mathcal{F}} = 1 \\ 1 & \varepsilon_{\mathcal{F}} = -1 \end{cases}.$$

In the case that $\varepsilon_{\mathcal{F}} = 1$, Conjectures 1.3 and 1.4 are equivalent by the interpolation property of $\mathcal{L}$. For $\varepsilon_{\mathcal{F}} = -1$, the connection is less clear, as there does not seem to be an easy way to relate derivatives of classical and p-adic L-functions. The goal of this paper is to discuss, in the case $\varepsilon_{\mathcal{F}} = -1$, the relationship between Conjecture 1.4 and the following non-vanishing conjecture for a certain Galois cohomology class $z(1) \in \mathrm{H}^1(\mathbf{Q}, \widetilde{\mathcal{T}}^*)$ arising from Kato's Euler system. (Here, $\widetilde{\mathcal{T}}^* = \mathrm{Hom}_{\Lambda_{\mathbf{H}}}(\widetilde{\mathcal{T}}, \Lambda_{\mathbf{H}})(1)$; see 4.4 for more details concerning this Euler system.)

1.5. In 2.7 below, we define Selmer groups $\mathrm{Sel}(\mathbf{Q}, M)$ attached to various $G_{\mathbf{Q}}$-modules M. Set $T = \widetilde{\mathcal{T}}/\Theta\widetilde{\mathcal{T}}$. The quotient $\widetilde{\mathcal{T}}^* \to T^*$ induces a natural "corestriction" mapping $\mathrm{Sel}(\mathbf{Q}, \widetilde{\mathcal{T}}^*) \to \mathrm{Sel}(\mathbf{Q}, T^*)$.

CONJECTURE. — *The image of* $z(1)$ *under corestriction* $\mathrm{Sel}(\mathbf{Q}, \widetilde{\mathcal{T}}^*) \to \mathrm{Sel}(\mathbf{Q}, T^*)$ *does not lie in the* $\mathbf{H}$*-torsion submodule of* $\mathrm{Sel}(\mathbf{Q}, T^*)$*.*

In the case of sign 1, this conjecture follows easily from Conjecture 1.4 in light of the existence of a Coleman map for $\widetilde{\mathcal{T}}$ (cf. Theorem 4.5, due to Ochiai); we therefore restrict our attention to the case of sign -1 in this paper. As we will discuss in a future paper, this conjecture should imply the main conjecture of Iwasawa theory for T, at least when a related 2-variable main conjecture is known (cf. 4.3). In the case of sign 1, the conjecture should allow the use of an Euler system argument to find a non-trivial (even sharp) upper bound, in terms of $\mathcal{L} \mod \Theta$, for the size of the Selmer group $\mathrm{Sel}(\mathbf{Q}, W)$, where W is the $\mathbf{H}$-divisible Galois module associated to T. Perhaps more interesting is the case of sign -1, where we expect that the conjecture implies that $\mathrm{Sel}(\mathbf{Q}, W)$ has corank 1 over $\mathbf{H}$ and moreover that the maximal cotorsion quotient of this Selmer group can be bounded in terms of a suitable derivative of $\mathcal{L}$. We are therefore interested in finding necessary and sufficient conditions for the conjecture to hold in terms of known conjectures. By way of necessity, we have the following result.

1.6. THEOREM. — *Assume that* $\varepsilon_{\mathcal{F}} = -1$ *and that* $\rho_{\mathcal{F}}$ *is residually irreducible. If* $z(1)$ *does not map to the torsion submodule of* $\mathrm{Sel}(\mathbf{Q}, T^*)$*, then* $\mathrm{ord}_\Theta \mathcal{L} = 1$ *provided that the p-adic height pairing of Plater* [15] *(see §3) is non-degenerate.*

We show that the converse (Theorem 4.14) also holds under the additional assumptions that, roughly, the 2-variable main conjecture and a form of Leopoldt's Conjecture (Greenberg's "Hypothesis L", 4.11) and hold for $\widetilde{\mathcal{T}}$. The proofs of these theorems are based on properties of the p-adic height pairing, the study of which is also of independent interest. As a consequence of our main results, we give conditions which guarantee that this pairing is non-degenerate (Corollary 4.16).

1.7. The main ingredient in the proof of Theorem 1.6 and its converse is Theorem 3.2, which is a generalization of a formula of Rubin [16, Thm.

3.2(b)] relating the p-adic height pairing on Selmer groups with the local (at p) Tate pairing. Our version of the height formula, Theorem 3.2, is new in the sense that formulas similar to that of Theorem 3.2 have only been shown to hold in the case of Galois representations on finitely-generated $\mathbf{Z}_p$-modules. On the other hand, the main ideas in our proof of Theorem 3.2 are not significantly different from those employed by Rubin in [16] in the case of abelian varieties. The general strategy is to use the definition of the p-adic height pairing (reviewed in 3.5) to reduce the formula to a computation (Proposition 3.10) relating the derivative mapping on cohomology (see 3.8) and the coboundary homomorphism.

The formula could also be deduced from Nekovář's treatment of height parings [10] (see especially 11.3.14 of *loc. cit.*, where Nekovář reproves Rubin's formula by a different method). The main contributions of this paper, therefore, are the results of §4, where we apply the height formula to study non-triviality of Euler systems and derivatives of p-adic L-functions associated to Hida families in the case where the Selmer groups are expected to have positive rank.

1.8. We now explain further the main idea behind the proofs in §4. The existence of a Coleman map for T^* allows us, roughly speaking, to treat $\mathcal{L}$ as an element of the local cohomology group $\mathrm{H}^1(\mathbf{Q}_p, \widetilde{\mathcal{T}}^*)$, equal to the localization at p of a global cohomology class $z(1)$ arising from Kato's Euler system. In the case of sign -1, $\mathcal{L}$ restricts to 0 in $\mathrm{H}^1(\mathbf{Q}_p, T^*)$ and the Selmer group $\mathrm{Sel}(\mathbf{Q}, T^*)$ should have positive rank. A key observation is that the representation $\widetilde{\mathcal{T}}^*$ may be viewed as a twist of the cyclotomic deformation of T^*. As a result, the height formula in this situation can then be viewed as relating the height pairing of the corestriction to T^* of the Euler system class $z(1)$ against an arbitrary element b of $\mathrm{Sel}(\mathbf{Q}, T)$ with the local (at p) Tate pairing of the derivative of $\mathcal{L}$ against the localization at p of b. Thus, assuming that the derivative of $\mathcal{L}$ is non-vanishing, the non-degeneracy of the local Tate pairing implies that the corestriction of $z(1)$ is likewise non-vanishing. For the reverse implication, in addition to assuming that the p-adic height pairing is non-degenerate, it is necessary to guarantee somehow that $\mathrm{Sel}(\mathbf{Q}, T^*)$ has the expected rank—this missing global information is provided by assuming 4.11, a form of Leopoldt's Conjecture (which, following Greenberg, we refer to as "Hypothesis L").

2. Notation

2.1. Fix a rational prime $p > 2$ and let $\mathbf{Q}_\infty = \bigcup_n \mathbf{Q}_n$ be the $\mathbf{Z}_p$-extension of $\mathbf{Q}$, where $\mathbf{Q}_n$ is the unique extension of $\mathbf{Q}$ contained in $\mathbf{Q}(\mu_{p^n})$ with $\mathrm{Gal}(\mathbf{Q}_n/\mathbf{Q}) \cong \mathbf{Z}/p^n$. Let $\mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}) \cong \Gamma_C \times \Delta$ be the canonical splitting, where $\Gamma_C = \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q}) \cong \mathbf{Z}_p$ and $\Delta = \mathrm{Gal}(\mathbf{Q}(\mu_p)/\mathbf{Q}) \cong \mathbf{Z}/(p-1)\mathbf{Z}$, and denote by $\varepsilon\colon \mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q}) \xrightarrow{\sim} \mathbf{Z}_p^\times$ the p-adic cyclotomic character and $\kappa_C = \varepsilon|_{\Gamma_C} : \Gamma_C \xrightarrow{\sim} 1 + p\mathbf{Z}_p$, resp. $\omega = \varepsilon|_\Delta : \Delta \xrightarrow{\sim} \mu_{p-1}$, its restriction to Γ_C, resp. Δ. Define $\psi\colon \Gamma_C \cong \mathbf{Z}_p$ by composing κ_C with the topological isomorphism $1 + p\mathbf{Z}_p \cong \mathbf{Z}_p$ sending $1 + p$ to 1 and furthermore let γ_C be the topological generator of Γ_C satisfying $\psi(\gamma_C) = 1$.

For any field F, let $\overline{F}$ be a separable closure of F and denote by G_F the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$. If F is a number field and v is a non-archimedean place of F, then we denote by $\mathrm{Frob}_v \in G_F$ a *geometric* Frobenius element.

Set $\Gamma_n = \mathrm{Gal}(\mathbf{Q}_n/\mathbf{Q})$ and let $\Lambda = \mathbf{Z}_p[\![\Gamma_C]\!] = \varprojlim \mathbf{Z}_p[\Gamma_n]$ be the cyclotomic Iwasawa algebra, so κ_C determines an isomorphism $\iota\colon \Lambda \cong \mathbf{Z}_p[\![X]\!]$ satisfying $\iota(\gamma_C) = 1 + X$. More generally, for any complete, local, Noetherian $\mathbf{Z}_p$-algebra R, we set $\Lambda_R = \varprojlim R[\Gamma_n] = \Lambda \,\widehat{\otimes}_{\mathbf{Z}_p} R \cong R[\![X]\!]$. Let $I_R \subseteq \Lambda_R$ be the *augmentation ideal*, *i.e.*, the ideal generated by $\gamma_C - 1$ (or by X). In what follows, we often implicitly identify Λ_R with $R[\![X]\!]$ via the isomorphism $\iota \otimes R$.

If F is a field and M is an $R[G_F]$-module (with R as above), then we denote by $M^\vee = \mathrm{Hom}(M, \mathbf{Q}_p/\mathbf{Z}_p)$ its Pontryagin dual with the usual G_F-action: $(g\phi)(m) = \phi(g^{-1}m)$. If M is a finitely-generated R-module, we denote by $M^* = \mathrm{Hom}(M, R)(1)$ the Tate (or Kummer) dual of M.

The Galois cohomology groups we use are the usual continuous cohomology groups. For every number field F and place v of F, we choose an embedding $\overline{F} \hookrightarrow \overline{F}_v$ and denote by $\mathrm{loc}_v\colon \mathrm{H}^i(F, M) \to \mathrm{H}^i(F_v, M)$ the induced localization (*i.e.*, restriction) map.

2.2. Suppose $\mathbf{H}$ is a 2-dimensional complete local domain, finite and flat over $\mathbf{Z}_p[\![\Gamma_D]\!]$, where Γ_D is the p-part of the group of diamond operators acting on the tower of modular curves $\{X_1(p^k N)\}_k$. Denote by $\kappa_D : \Gamma_D \xrightarrow{\sim} 1 + p\mathbf{Z}_p$ the canonical isomorphism, which gives rise to an isomorphism $\mathbf{Z}_p[\![\Gamma_D]\!] \cong \mathbf{Z}_p[\![Y]\!]$. Inside of $\Lambda_{\mathbf{H}} \cong \mathbf{H}[\![X]\!]$, we define the ideal Θ to be the principal ideal generated by the element $\gamma_C - (1+p)\gamma_D^{1/2}$, where $\gamma_D^{1/2}$ is the unique square root of γ_D in Γ_D (recall $p \neq 2$).

An *arithmetic character* of $\mathbf{H}$ is a continuous ring homomorphism $\sigma\colon \mathbf{H} \to \mathbf{C}_p$ such that $\sigma(\gamma_D) = \chi_\sigma(\gamma_D)\kappa_D(\gamma_D)^{w(\sigma)}$ for some (uniquely-determined) finite-order character χ_σ and integer $w(\sigma)$, which we refer to as the *weight* of σ. Similarly, an *arithmetic character* of Γ_C is a continuous character $\tau\colon \Gamma_C \to \mathbf{C}_p^\times$ such that $\tau(\gamma_C) = \chi_\tau(\gamma_C)\kappa_C(\gamma_C)^{w(\tau)}$ for a finite-order character χ_τ and integer weight $w(\tau)$. An arithmetic character of $\Lambda_{\mathbf{H}}$ is then a continuous ring homomorphism $(\sigma,\tau)\colon \Lambda_{\mathbf{H}} \to \mathbf{C}_p$ whose restriction to $\mathbf{H}$, resp. Γ_C, is equal to σ, resp. τ, for arithmetic characters σ and τ. (Note that a continuous homomorphism $\Lambda_{\mathbf{H}} \to \mathbf{C}_p$ is determined by its restriction to $\mathbf{H}$ and Γ_C.) We call the prime ideals $\mathfrak{p}_\sigma \subseteq \mathbf{H}$ and $\mathfrak{p}_{\sigma,\tau} \subseteq \Lambda_{\mathbf{H}}$ arising as kernels of arithmetic characters *arithmetic primes*. Denote by $\mathcal{O}_\sigma$, resp. $\mathcal{O}_{\sigma,\tau}$, the ring of integers in the finite extension of $\mathbf{Q}_p$ generated by the image of σ, resp. (σ,τ).

2.3. Let $f = \sum_{n\geqslant 1} a_n(f)q^n \in S_k(\Gamma_1(N),\chi_f)$ be a cuspidal newform of even weight $k \geqslant 2$, level N prime to p, and character χ_f. As Deligne [1] has shown, to any such f is attached a 2-dimensional p-adic Galois representation

$$\rho_f\colon G_{\mathbf{Q}} \longrightarrow \mathrm{GL}_2(\mathcal{O}),$$

where $\mathcal{O}$ is the ring of integers in the completion at a place above p of the number field F_f obtained by adjoining the Fourier coefficients of f to $\mathbf{Q}$. This representation satisfies

$$\operatorname{trace}\rho_f(\mathrm{Frob}_\ell) = a_\ell(f), \quad \det\rho_f(\mathrm{Frob}_\ell) = \chi_f(\ell)\ell^{k-1} \quad (\ell \nmid Np).$$

Denoting by $\mathfrak{m} \subseteq \mathcal{O}$ the maximal ideal and $k = \mathcal{O}/\mathfrak{m}$ the residue field, we define the residual representation $\overline{\rho}_f\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(k)$ to be the reduction of ρ_f mod $\mathfrak{m}$. In what follows, we assume that $\overline{\rho}_f$ is *absolutely irreducible*.

2.4. Assume further that the form f is *p-ordinary*, *i.e.*, $|a_p(f)|_v = 1$ for all primes of F_f lying over p. Then work of Hida [5] implies that f belongs to a p-adic family $\mathcal{F}$ of modular forms in the following sense. There is a ring $\mathbf{H}$ as in 2.2 and a formal q-expansion $\mathcal{F} = \sum_{i\geqslant 1} a_i(\mathcal{F})q^i$ such that the specialization $f_\sigma = \sum_{i\geqslant 1} \sigma\big(a_i(\mathcal{F})\big)q^i$ of $\mathcal{F}$ at an arithmetic character σ of $\mathbf{H}$ is the q-expansion of a p-ordinary, new eigenform of weight $w(\sigma)+2$, level Np^r, and character $\chi_{\mathcal{F}}\chi_\sigma\omega^{-w(\sigma)}$, where $\chi_{\mathcal{F}}$ is a Dirichlet character of order prime to p canonically associated to $\mathcal{F}$ and r is defined by $\kappa_D(\ker\chi_\sigma) = 1+p^r\mathbf{Z}_p$. That f *belongs* to $\mathcal{F}$ means the specialization of $\mathcal{F}$ at some arithmetic character of weight $k-2$ is equal to the q-expansion of the *p-stabilization* f_p of f, *i.e.*, the unique newform of level Np which has $a_\ell(f_p) = a_\ell(f)$ for almost all primes ℓ.

2.5. We denote by

$$\rho_{\mathcal{F}}\colon G_{\mathbf{Q}} \longrightarrow \mathrm{Aut}_{\mathbf{H}}(\mathcal{T})$$

the representation attached to the Hida family discussed in 2.4; this representation satisfies

$$\det \rho_{\mathcal{F}}(\mathrm{Frob}_\ell) = \chi_{\mathcal{F}}(\ell)\langle \ell\rangle \ell \quad (\ell \nmid pN),$$

where $\chi_{\mathcal{F}}$ is a Dirichlet character, the *Nebentypus* of $\mathcal{F}$, and $\langle\ \rangle$ is the composition $\mathbf{Z}_p^\times \twoheadrightarrow 1+p\mathbf{Z}_p \cong \Gamma_D \hookrightarrow \mathbf{H}^\times$. Under our assumption (imposed in 2.3) that $\rho_{\mathcal{F}}$ is residually irreducible, we may choose $\mathcal{T}$ to be a *free* $\mathbf{H}$-module by a result of Mazur-Tilouine [8, Cor. 6]. Define $\widetilde{\rho}_{\mathcal{F}}$ to be the cyclotomic deformation of $\rho_{\mathcal{F}}$, *i.e.*, $\widetilde{\rho}_{\mathcal{F}}\colon G_{\mathbf{Q}} \to \mathrm{Aut}_{\Lambda_{\mathbf{H}}}(\widetilde{\mathcal{T}})$, where $\widetilde{\mathcal{T}} = \mathcal{T}\otimes_{\mathbf{H}} \Lambda_{\mathbf{H}}$ with Galois action via $\rho_{\mathcal{F}}\otimes\kappa^{\mathrm{univ}}$.

Note that $\rho_{\mathcal{F}}$ is not self-dual. However, if we define the character

$$\theta\colon \Gamma_C \longrightarrow \mathbf{H}^\times : \gamma_C \longmapsto (1+p)\gamma_D^{1/2},$$

then the twist $\rho_{\mathcal{F}}\otimes\theta$ is isomorphic to its Tate dual, provided that $\chi_{\mathcal{F}}$ is trivial. (The statement that $\chi_{\mathcal{F}}$ is trivial is the same as assuming that the Nebentypus of the original weight k modular form f is ω^{2-k}.) We denote by T the rank-2 $\mathbf{H}$-module serving as a representation space for $\rho_{\mathcal{F}}\otimes\theta$ and set $\widetilde{T} = T\,\widehat{\otimes}_{\mathbf{Z}_p}\Lambda$ with Galois action via $(\rho_{\mathcal{F}}\otimes\theta)\otimes\kappa^{\mathrm{univ}}$, so $T = \widetilde{T}/I_{\mathbf{H}}\widetilde{T}$. Thus we have the identity

$$T \cong \widetilde{\mathcal{T}}/\Theta\widetilde{\mathcal{T}}$$

of $\mathbf{H}[G_{\mathbf{Q}}]$-modules, which allows us to combine information about $\widetilde{\mathcal{T}}$ with the theory of height pairings to study the representation T.

The representation $\rho_{\mathcal{F}}$ satisfies the following interpolation property with respect to specialization at arithmetic primes: For any arithmetic character (σ,τ) satisfying $w(\sigma)\geqslant 0$, the composition

$$G_{\mathbf{Q}} \xrightarrow{\rho_{\mathcal{F}}} \mathrm{Aut}_{\Lambda_{\mathbf{H}}}(\widetilde{\mathcal{T}}) \xrightarrow{(\sigma,\tau)} \mathrm{GL}_2(\mathbf{C}_p)$$

is isomorphic to the 2-dimensional p-adic Galois representation $\rho_{f_\sigma}\otimes\tau$: $G_{\mathbf{Q}}\to\mathrm{GL}_2(\mathbf{C}_p)$, where ρ_{f_σ} is (the extension of scalars to $\mathbf{C}_p$ of) the p-adic representation attached to the newform f_σ determined by σ.

2.6. By work of Wiles [18, Thm. 2.2.2], the representation $\mathcal{T}$ admits a filtration $\mathcal{T}\supseteq \mathrm{F}^+\mathcal{T}\supseteq 0$ by $G_{\mathbf{Q}_p}$-submodules such that $\mathrm{F}^+\mathcal{T}$ is free of rank 1 over $\mathbf{H}$ and $G_{\mathbf{Q}_p}$ acts on $\mathrm{F}^+\mathcal{T}$ by the unramified character $\alpha\colon G_{\mathbf{Q}_p}\to\mathbf{H}^\times$ satisfying $\sigma\big(\alpha(\mathrm{Frob}_p)\big) = a_p(f_\sigma)$ for every arithmetic character σ of $\mathbf{H}$ of weight $w(\sigma)\geqslant 0$. This filtration gives in an obvious way a filtration $\widetilde{\mathcal{T}}\supseteq\mathrm{F}^+\widetilde{\mathcal{T}}\supseteq 0$ on $\widetilde{\mathcal{T}}$ satisfying $\mathrm{F}^+\widetilde{\mathcal{T}} = (\mathrm{F}^+\mathcal{T})\otimes_{\mathbf{H}}\Lambda_{\mathbf{H}}$.

For $R=\mathbf{H}$ or $\Lambda_{\mathbf{H}}$, denote by $R^\vee = \mathrm{Hom}_{\mathbf{Z}_p}(R,\mathbf{Q}_p/\mathbf{Z}_p)$ the Pontryagin dual of R. Define $W = \mathrm{Hom}_{\mathbf{Z}_p}(T^*,\mathbf{Q}_p/\mathbf{Z}_p)(1)$ and $\widetilde{W} = \mathrm{Hom}_{\mathbf{Z}_p}(\widetilde{T}^*,\mathbf{Q}_p/\mathbf{Z}_p)(1)$ and similarly define $W^* = \mathrm{Hom}_{\mathbf{Z}_p}(T,\mathbf{Q}_p/\mathbf{Z}_p)(1)$ and $\widetilde{W}^* =$

$\mathrm{Hom}_{\mathbf{Z}_p}(\widetilde{T}, \mathbf{Q}_p/\mathbf{Z}_p)(1)$. By freeness of T as an $\mathbf{H}$-module, we have a natural identification $W = T \otimes_{\mathbf{H}} \mathbf{H}^\vee$ of $\mathbf{H}[G_{\mathbf{Q}}]$-modules. Note, then, that the $G_{\mathbf{Q}_p}$-filtration on T gives rise in an obvious way to a filtration $W \supseteq \mathrm{F}^+ W \supseteq 0$ on W. Similar remarks apply to $\widetilde{W}$, W^*, and $\widetilde{W}^*$.

2.7. We now define, following Plater [15, §5] and others (see Greenberg [2], e.g.), local conditions and Selmer groups for the representations discussed above. Fix a number field F. Let R be a complete, local, Noetherian $\mathbf{Z}_p$-algebra and M a finitely-generated R-module. If $N \subseteq M$ is an R-submodule, then define N^{sat} to be the *R-saturation* of N in M, *i.e.*, $N^{\mathrm{sat}} = \{m \in M \mid rm \in N \text{ for some } r \in R\}$.

Suppose given an $R[G_F]$-module M which admits a filtration $M \supseteq \mathrm{F}^+ M \supseteq 0$ such that $\mathrm{F}^+ M$ is G_{F_v}-invariant for all places v of F lying over p; set $\mathrm{F}^- M = M/\mathrm{F}^+ M$. If M is finitely generated over R and $v \nmid p$, then we define the local conditions to be the R-saturation of the submodule of unramified classes:

$$\mathrm{H}^1_{\mathrm{f}}(F_v, M) = \mathrm{H}^1_{\mathrm{ur}}(F_v, M)^{\mathrm{sat}} = \ker\big(\mathrm{H}^1(F_v, M) \longrightarrow \mathrm{H}^1(I_v, M)\big)^{\mathrm{sat}},$$

where $I_v \subseteq G_{F_v}$ is the inertia subgroup. For $v \mid p$, set

$$\mathrm{H}^1_{\mathrm{f}}(F_v, M) = \ker\big(\mathrm{H}^1(F_v, M) \longrightarrow \mathrm{H}^1(I_v, \mathrm{F}^- M)\big)^{\mathrm{sat}}.$$

For cofinitely generated R-modules M, we define the local condition for M at v to be dual to the local condition at v for the finitely generated R-module $M^\vee(1) = \mathrm{Hom}_{\mathbf{Z}_p}(M, \mathbf{Q}_p/\mathbf{Z}_p)(1)$ with respect to the (perfect) local Tate pairing

$$\mathrm{H}^1(F_v, M) \times \mathrm{H}^1\big(F_v, M^\vee(1)\big) \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p.$$

In other words, $\mathrm{H}^1_{\mathrm{f}}(F_v, M)$ is defined to be the exact orthogonal complement of $\mathrm{H}^1_{\mathrm{f}}\big(F_v, M^\vee(1)\big)$ under this pairing. Finally, for any v and M (finitely or cofinitely generated), set $\mathrm{H}^1_{\mathrm{s}}(F_v, M) = \mathrm{H}^1(F_v, M)/\mathrm{H}^1_{\mathrm{f}}(F_v, M)$.

Assume that M is unramified outside of a finite set of places of F and let Σ be any finite set containing the places of F lying over p, the archimedean places, and the places at which M is ramified. We then define, for M finitely or cofinitely generated over R,

$$\mathrm{Sel}^\Sigma(F, M) = \ker\Big(\mathrm{H}^1(F, M) \longrightarrow \bigoplus_{v \notin \Sigma} \mathrm{H}^1_{\mathrm{s}}(F_v, M)\Big)$$
$$\mathrm{Sel}(F, M) = \ker\Big(\mathrm{Sel}^\Sigma(F, M) \longrightarrow \bigoplus_{v \in \Sigma} \mathrm{H}^1_{\mathrm{s}}(F_v, M)\Big)$$

$$\mathrm{Sel}_\Sigma(F,M)=\ker\Big(\mathrm{Sel}(F,M)\longrightarrow\bigoplus_{v\in\Sigma}\mathrm{H}^1_{\mathrm{f}}(F_v,M)\Big).$$

We apply these definitions, in particular, for $R=\mathbf{H}$, resp. $\Lambda_{\mathbf{H}}$, and $M=T$, T^*, W, or W^*, resp. $\widetilde{T}$, $\widetilde{T}^*$, $\widetilde{W}$, or $\widetilde{W}^*$. When M and v are clear from the context, we denote the localization maps $\mathrm{H}^1(F,M)\to\mathrm{H}^1(F_v,M)$, $\mathrm{H}^1(F,M)\to\mathrm{H}^1_{\mathrm{s}}(F_v,M)$, and $\mathrm{Sel}(F,M)\to\mathrm{H}^1_{\mathrm{f}}(F_v,M)$ by loc_v, $\mathrm{loc}_{\mathrm{s}}$, and $\mathrm{loc}_{\mathrm{f}}$, respectively.

2.8. By Lemma 2.10 below, for $M=T$ or T^*, we have that $\mathrm{Sel}^\Sigma(F,M)=\mathrm{H}^1(\mathbf{Q}_\Sigma/\mathbf{Q},M)$, where $\mathbf{Q}_\Sigma$ is the maximal extension of $\mathbf{Q}$ unramified outside of Σ. We also have the following interaction of the local conditions with the cup product pairing

$$\mathrm{H}^1(\mathbf{Q}_p,T)\times\mathrm{H}^1(\mathbf{Q}_p,T^*)\longrightarrow\mathrm{H}^2\big(\mathbf{Q}_p,\mathbf{H}(1)\big)\cong\mathbf{H}.$$

From the definitions, we have that $\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_p,T)$ is the $\mathbf{H}$-saturation of the image of the map $\mathrm{H}^1(\mathbf{Q}_p,\mathrm{F}^+T)\to\mathrm{H}^1(\mathbf{Q}_p,T)$ induced by inclusion. Thus, given that $(\mathrm{F}^+T)^*\cong\mathrm{F}^-(T^*)$, the diagram

$$\begin{array}{ccccc}\mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_p,T^*) & \times & \mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_p,T) & \to & \mathbf{H}\\ \uparrow & & \downarrow & & \|\\ \mathrm{H}^1(\mathbf{Q}_p,T^*) & \times & \mathrm{H}^1(\mathbf{Q}_p,T) & \to & \mathbf{H}\end{array}\tag{2.8.1}$$

commutes, where the vertical arrows represent, from left to right, the quotient, inclusion, and identity maps.

2.9. Lemma. — *For $M=T$ or T^*, we have*

$$\mathrm{rk}_{\mathbf{H}}\,\mathrm{H}^1(\mathbf{Q}_v,M)=\begin{cases}0 & v\neq p\\ 2 & v=p\end{cases}$$

Moreover, $\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_p,M)$ and $\mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_p,M)$ have rank 1 over $\mathbf{H}$.

Proof. — This lemma can be found in Perrin-Riou [14, Prop. 2.1.3] in the p-adic case. Because $\mathbf{H}$ is free over $\Lambda_D:=\mathbf{Z}_p[\![\Gamma_D]\!]\cong\mathbf{Z}_p[\![Y]\!]$, the rank of $\mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_p,M)$ as an $\mathbf{H}$-module is determined by its rank as a Λ_D-module. We have that $\mathrm{H}^1(\mathbf{Q}_v,M)=\varprojlim\mathrm{H}^1(\mathbf{Q}_v,M/Y^nM)$, and M/Y^nM is a finitely generated $\mathbf{Z}_p$-module of rank $2\,\mathrm{rk}_{\mathbf{Z}_p}\mathbf{H}/Y^n\mathbf{H}$. Set $V_n=(M/Y^nM)\otimes\mathbf{Q}_p$. By the classical Euler-Poincaré characteristic formula (Milne [9, Thm. I.2.8], e.g.), we have

$$\sum_{i=0}^{2}(-1)^i\dim_{\mathbf{Q}_p}\mathrm{H}^i(\mathbf{Q}_v,V_n)=\begin{cases}0 & v\neq p\\ -\dim_{\mathbf{Q}_p}V_n & v=p.\end{cases}$$

By local duality, $\dim_{\mathbf{Q}_p} \mathrm{H}^2(\mathbf{Q}_v, V_n) = \dim_{\mathbf{Q}_p} \mathrm{H}^0(\mathbf{Q}_v, V_n^*)$, where we have $V_n^* \cong (M^*/Y^n M^*) \otimes \mathbf{Q}_p$. Because $M(\mathbf{Q}_v) = M^*(\mathbf{Q}_v) = 0$, both $\dim_{\mathbf{Q}_p} \mathrm{H}^0(\mathbf{Q}_v, V_n)$ and $\dim_{\mathbf{Q}_p} \mathrm{H}^0(\mathbf{Q}_v, V_n^*)$ are bounded as n varies, which gives the first statement.

By the same reasoning as above, $\mathrm{H}^1(\mathbf{Q}_p, \mathrm{F}^+ M)$ has rank 1 over $\mathbf{H}$. Thus, the second statement follows from the facts that $(\mathrm{F}^- M)(\mathbf{Q}_p) = 0$ and that $\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_p, M)$ is the saturation of the image of $\mathrm{H}^1(\mathbf{Q}_p, \mathrm{F}^+ M)$ in $\mathrm{H}^1(\mathbf{Q}_p, M)$. □

2.10. Lemma. — *For $M = T$ or T^*, if $v \notin \Sigma$, then* $\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_v, M) = \mathrm{H}^1_{\mathrm{ur}}(\mathbf{Q}_v, M)$.

Proof. — The argument is sketched on p. 110 of Plater [15] (following Perrin-Riou [14, Lemme 2.2.1]); we review it here. Inflation-restriction and the fact that $G_{\mathbf{Q}_v}/I_v$ has cohomological dimension 1 imply that the sequence

$$0 \longrightarrow \mathrm{H}^1_{\mathrm{ur}}(\mathbf{Q}_v, M) \longrightarrow \mathrm{H}^1(\mathbf{Q}_v, M) \longrightarrow \mathrm{H}^1(I_v, M)^{G_{\mathbf{Q}_v}} \longrightarrow 0$$

is exact. By definition of $\mathrm{H}^1_{\mathrm{f}}$ (see 2.7), the quotient $\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_v, M)/\mathrm{H}^1_{\mathrm{ur}}(\mathbf{Q}_v, M)$ is thus isomorphic to $(\mathrm{H}^1(I_v, M)^{G_{\mathbf{Q}_v}})_{\mathbf{H}\text{-tors}}$. For $v \notin \Sigma$, M is unramified at v, so $\mathrm{H}^1(I_v, M) \cong M(-1)$ by an exercise in tame inertia. In particular, $\mathrm{H}^1(I_v, M)$ is $\mathbf{H}$-torsion-free, so $\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_v, M)/\mathrm{H}^1_{\mathrm{ur}}(\mathbf{Q}_v, M) = 0$. □

2.11. If M is a finitely-generated $\mathbf{H}$-module and K is *any* Galois extension (possibly infinite) of a local or global field F, then we set

$$\mathrm{H}^1(K, M) = \varprojlim_{L \subseteq K' \subseteq K} \mathrm{H}^1(K', M),$$

the inverse limit taken with respect to corestriction between finite extensions K' of L contained in K; we make the analogous definition for Selmer groups. Thus, an element of $\mathrm{H}^1(K, M)$ can be viewed as a projective system of elements of $\mathrm{H}^1(K', M)$. With this definition, Shapiro's Lemma continues to hold for infinite extensions of L: There is a canonical isomorphism $\mathrm{H}^1(K, M) \cong \mathrm{H}^1(L, M \otimes \mathbf{Z}_p[\![\mathrm{Gal}(K/L)]\!])$. In particular, we have canonical isomorphisms of $\Lambda_{\mathbf{H}}$-modules $\mathrm{H}^1(\mathbf{Q}_\infty, T) \cong \mathrm{H}^1(\mathbf{Q}, \widetilde{T})$, $\mathrm{H}^1(\mathbf{Q}_{\infty,p}, T) \cong \mathrm{H}^1(\mathbf{Q}, \widetilde{T})$, $\mathrm{Sel}(\mathbf{Q}_\infty, T) \cong \mathrm{Sel}(\mathbf{Q}, \widetilde{T})$, etc. The maps on local and global Galois cohomology groups and Selmer groups induced by the quotient $\Lambda_{\mathbf{H}} \to \Lambda_{\mathbf{H}}/I_{\mathbf{H}}$, *i.e.*, the maps $\mathrm{H}^1(\mathbf{Q}, \widetilde{T}) \to \mathrm{H}^1(\mathbf{Q}, T)$, $\mathrm{Sel}(\mathbf{Q}, \widetilde{T}) \to \mathrm{Sel}(\mathbf{Q}, T)$, etc., which correspond via Shapiro's lemma to corestriction from $\mathbf{Q}_\infty$ to $\mathbf{Q}$, are denoted below by cor_∞ (the particular incarnation of cor_∞ we mean is determined by context).

3. Heights and the height formula

3.1. In this section, we first recall the definition of the height pairing, due to Plater [15], and then show that the height formula, Theorem 3.2, holds in our situation. The height pairing is an $\mathbf{H}$-bilinear map

$$\langle\ ,\ \rangle\colon\ \mathrm{Sel}(\mathbf{Q},T^*)\times\mathrm{Sel}(\mathbf{Q},T)\longrightarrow\mathrm{Frac}(\mathbf{H})$$

depending on $\kappa_C\colon\Gamma_C\xrightarrow{\sim}1+p\mathbf{Z}_p$. We review the definition of $\langle\ ,\ \rangle$ in 3.3–3.5. As we discussed in 2.8, the cup product $\mathrm{H}^1(\mathbf{Q}_p,T^*)\times\mathrm{H}^1(\mathbf{Q}_p,T)\to\mathrm{H}^2\big(\mathbf{Q}_p,\mathbf{H}(1)\big)$ induces another pairing

$$(\ ,\ )_p\colon\mathrm{H}^1_{\mathrm{s}}(\mathbf{Q},T^*)\times\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q},T)\longrightarrow\mathbf{H}.$$

The height formula compares these two pairings.

Nothing in this section depends in any essential way on the fact that our representation T arises from a Hida family, so our treatment should apply to any "geometric" 2-dimensional p-ordinary representation ρ satisfying Plater's hypotheses (H1)–(H3) [15, p. 107]:

(H1) $\rho|_{G_{\mathbf{Q}_p}}$ is indecomposable and of the form

$$\rho|_{G_{\mathbf{Q}_p}}\cong\begin{pmatrix}\chi_1 & *\\ 0 & \chi_2\end{pmatrix}$$

for characters $\chi_i\colon G_{\mathbf{Q}_p}\to\mathbf{H}^\times$.

(H2) For any finite extension $F/\mathbf{Q}$ and any prime v of F, $T(F_v)=T^*(F_v)=(\mathrm{F}^+T)^*(F_v)=0$.

(H3) $W(\mathbf{Q}_\infty)$ is a cotorsion $\mathbf{H}$-module.

Note that these hypotheses are satisfied by the representation T introduced in §2. (This follows from the properties of T enumerated in 2.5–2.6.)

3.2. Theorem (cf. Rubin [17, Thm. 3.2(ii)]). — *Suppose that $a\in\mathrm{Sel}(\mathbf{Q},T)$ satisfies $a=\mathrm{cor}_\infty a^{(\infty)}$ for some $a^{(\infty)}\in\mathrm{H}^1(\mathbf{Q},\widetilde{T})$ such that $\mathrm{loc}_{\mathrm{s}}\,a^{(\infty)}\in\mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_p,\widetilde{T})$ is divisible by γ_C-1, say $\mathrm{loc}_{\mathrm{s}}\,a^{(\infty)}=(\gamma_C-1)\alpha^{(\infty)}$. Then for any $b\in\mathrm{Sel}(\mathbf{Q},T^*)$, we have*

$$\langle a,b\rangle=(\mathrm{cor}_\infty\,\alpha^{(\infty)},\mathrm{loc}_{\mathrm{f}}\,b)_p.$$

We give a proof of this theorem following the basic outline of Rubin's proof in [17] for the Tate module of an abelian variety. The idea is to find an expression which is local at p for the height pairing in the case considered in the theorem and then compare this expression to the local Tate pairing by a computation. Consequently, it is necessary to review the definition of the height pairing; we do this in 3.3–3.5, which are essentially a summary of §6 in Plater [15]. In §4, we use this theorem to relate classes arising from

Kato's Euler system (which we substitute for a on the left hand side of the formula) to the derivative of the associated p-adic L-function (which is related to $\operatorname{cor}_\infty \alpha^{(\infty)}$ on the right hand side via Ochiai's "Coleman map" for $\widetilde{\mathcal{T}}$, cf. 4.5).

3.3. For any prime v and any $G_{\mathbf{Q}}$ module for which the local conditions have been defined, denote by $\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_v, M)^{\mathrm{u}}$ the submodule of $\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_v, M)$ consisting of *universal norms* from the cyclotomic direction, *i.e.*,

$$\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_v, M)^{\mathrm{u}} = \operatorname{cor}_\infty\big(\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_{\infty,v}, M)\big),$$

where $\operatorname{cor}_\infty$ denotes corestriction from $\mathbf{Q}_{\infty,p}$ to $\mathbf{Q}_p$. Globally, let $\operatorname{Sel}(\mathbf{Q}, M)^{\mathrm{u}} \subseteq \operatorname{Sel}(\mathbf{Q}, M)$ be the submodule consisting of elements whose localizations belong to $\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_v, M)^{\mathrm{u}}$ for all v. The following proposition shows that it suffices to define the height pairing on elements which are locally everywhere universal norms.

3.4. Proposition. — *For* $M = T$ *or* T^*, *the quotient* $\operatorname{Sel}(\mathbf{Q}, M)/\operatorname{Sel}(\mathbf{Q}, M)^{\mathrm{u}}$ *is a torsion* $\mathbf{H}$*-module.*

Proof. — By definition, $\operatorname{Sel}(\mathbf{Q}, M)/\operatorname{Sel}(\mathbf{Q}, M)^{\mathrm{u}}$ injects into the direct sum

$$\bigoplus_v \mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_v, M)/\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_v, M)^{\mathrm{u}}.$$

We show that each summand is $\mathbf{H}$-torsion, from which the proposition follows. For $v \neq p$, the local conditions $\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_v, M)$ are *a fortiori* $\mathbf{H}$-torsion by Lemma 2.9. For $v = p$, $\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_p, M)/\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_p, M)^{\mathrm{u}}$ is $\mathbf{H}$-torsion by Lemma 5.8 of Plater [15]. □

3.5. We now give the definition, originally due to Perrin-Riou [14] in the p-adic case and later generalized by Plater [15] to the case considered here, of the height pairing. Starting with $a \in \operatorname{Sel}(\mathbf{Q}, T^*)^{\mathrm{u}}$ and $b \in \operatorname{Sel}(\mathbf{Q}, T)^{\mathrm{u}}$, we define $\langle a, b\rangle$ by the following procedure. Corresponding to $b \in \mathrm{H}^1(\mathbf{Q}, T)$ is an extension of $\mathbf{H}[G_{\mathbf{Q}}]$-modules

$$0 \longrightarrow T \longrightarrow X \longrightarrow \mathbf{H} \longrightarrow 0$$

which dualizes to a short exact sequence

$$0 \longrightarrow \mathbf{H}(1) \longrightarrow X^* \longrightarrow T^* \longrightarrow 0. \tag{3.5.1}$$

Because the connecting homomorphism $\delta\colon \mathrm{H}^1(\mathbf{Q}, T^*) \to \mathrm{H}^2\big(\mathbf{Q}, \mathbf{H}(1)\big)$ is given by $\cup\, b$ and because the the diagram (2.8.1) shows that $a \cup b = 0$, a is the image of some element $\widetilde{a} \in \mathrm{H}^1(\mathbf{Q}, X^*)$.

Since a is a universal norm locally everywhere, we may choose, for every rational prime v, elements $a_v^{(\infty)} \in \mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_{\infty,v}, T^*)$ satisfying $\mathrm{cor}_\infty a_v^{(\infty)} = \mathrm{loc}_v a$, where cor_∞ denotes corestriction from $\mathbf{Q}_{\infty,v}$ to $\mathbf{Q}_v$. The same argument as above shows that $a_v^{(\infty)}$ can be lifted to an element $\widetilde{a}_v^{(\infty)} \in \mathrm{H}^1(\mathbf{Q}_{\infty,v}, X^*)$. Examining the cohomology sequence associated to (3.5.1), we see that $\mathrm{loc}_v \widetilde{a} - \mathrm{cor}_\infty \widetilde{a}_v^{(\infty)}$ is the image of some $w_v \in \mathrm{H}^1\big(\mathbf{Q}_v, \mathbf{Z}_p(1)\big)$.

For any prime v, the restriction ψ_v of ψ to a decomposition group at v can be viewed as an element of $\mathrm{H}^1(\mathbf{Q}_v, \mathbf{Z}_p)$, and thereby cup product with ψ_v yields a homomorphism

$$\cup\,\psi_v \colon \mathrm{H}^1\big(\mathbf{Q}_v, \mathbf{H}(1)\big) \longrightarrow \mathrm{H}^2\big(\mathbf{Q}_v, \mathbf{H}(1)\big) \cong \mathbf{H}.$$

The p-adic height pairing is then defined by the formula

$$\langle a, b\rangle = \sum_v w_v \cup \psi_v, \tag{3.5.2}$$

which one can show (as in [14, 1.2.4]) exists and is independent of all the choices made. One expects the following conjecture to hold when, as here, the representation under consideration is attached to a Hida family.

3.6. Conjecture. — *The kernel on either side of the pairing* (3.5.2) *consists of the* $\mathbf{H}$*-torsion submodule.*

3.7. As we now show, when the class a is *globally* a norm from $\mathrm{H}^1(\mathbf{Q}_\infty, T^*)$, the definition (3.5.2) can be given a simpler form by making suitable choices of $\widetilde{a}$ and $\widetilde{a}_v^{(\infty)}$. Thus in what follows we assume that there is an element $a^{(\infty)} \in \mathrm{H}^1(\mathbf{Q}_\infty, T^*)$ such that $\mathrm{cor}_\infty a^{(\infty)} = a$. As above, we may choose an element $\widetilde{a}^{(\infty)} \in \mathrm{H}^1(\mathbf{Q}_\infty, X^*)$ mapping to $a^{(\infty)}$; then $\widetilde{a} = \mathrm{cor}_\infty \widetilde{a}^{(\infty)}$ maps to a.

For $v = p$, choose as in 3.5 an element $a_p^{(\infty)} \in \mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_{\infty,p}, T^*)$ which satisfies $\mathrm{cor}_\infty a_p^{(\infty)} = \mathrm{loc}_p a$. Moreover, choose an element $w_p^{(\infty)} \in \mathrm{H}^1(\mathbf{Q}_{\infty,p}, X^*)$ mapping to $\mathrm{loc}_p a^{(\infty)} - a_p^{(\infty)}$ in $\mathrm{H}^1(\mathbf{Q}_{\infty,p}, T^*)$. If we set $\widetilde{a}_p^{(\infty)} = \mathrm{loc}_p \widetilde{a}^{(\infty)} - w_p^{(\infty)}$, then we have $\mathrm{loc}_p \widetilde{a} - \mathrm{cor}_\infty \widetilde{a}_p^{(\infty)} = \mathrm{cor}_\infty w_p^{(\infty)} \in \mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_p, X^*)$.

In the case that $v \nmid p$, Lemma 2.9 shows that $\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_{\infty,v}, T^*) = \mathrm{H}^1(\mathbf{Q}_{\infty,v}, T^*)$. Thus for $v \in \Sigma$ not equal to p, we can set $\widetilde{a}_v^{(\infty)} = \mathrm{loc}_v \widetilde{a}^{(\infty)}$: 2.9 guarantees that the image of $\widetilde{a}_v^{(\infty)}$ lies in $\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_{\infty,v}, T^*)$. By inspecting (3.5.2), we get the following simpler formula for the height pairing in the case under consideration.

Proposition. — *Suppose* $a \in \mathrm{Sel}(\mathbf{Q}, T^*)^{\mathrm{u}}$, $b \in \mathrm{Sel}(\mathbf{Q}, T)^{\mathrm{u}}$, *and* $a = \mathrm{cor}_\infty a^{(\infty)}$ *for some* $a^{(\infty)} \in \mathrm{H}^1(\mathbf{Q}_\infty, T^*)$. *Then with the choices made in*

3.7, we have

$$\langle a,b\rangle = \operatorname{cor}_\infty w_p^{(\infty)} \cup \psi_p.$$

3.8. Recall that the ideal $I_{\mathbf{H}} \subseteq \Lambda_{\mathbf{H}}$ is defined as the kernel of the $\mathbf{H}$-linear map $\Lambda_{\mathbf{H}} \to \mathbf{H}\colon \gamma_C \mapsto 1$, *i.e.*, $I_{\mathbf{H}}$ is generated by X when $\Lambda_{\mathbf{H}}$ is identified with $\mathbf{H}[\![X]\!]$ via the isomorphism determined by κ_C. For a free $\mathbf{H}$-module M, define the *derivative map*

$$\mathrm{Der} = \mathrm{Der}_M\colon M \otimes_{\mathbf{H}} I_{\mathbf{H}} \longrightarrow M$$

by the formula $m \otimes f \mapsto f'(0)m$, where by $f'(0) \in \mathbf{H}$ we mean the coefficient a_1 in the power series expansion $f = \sum_{i>0} a_i X^i$ obtained by viewing f as an element of $\mathbf{H}[\![X]\!]$. Note that Der is $\mathbf{H}$-linear and $G_{\mathbf{Q}}$-equivariant and therefore induces maps on cohomology $\mathrm{Der} = \mathrm{Der}^i_M\colon \mathrm{H}^i(\mathbf{Q}_p, M \otimes_{\mathbf{H}} I_{\mathbf{H}}) \to \mathrm{H}^i(\mathbf{Q}_p, M)$ which are functorial in M. The maps Der^i_M are compatible with the quotient $\Lambda_{\mathbf{H}} \to \mathbf{H}\colon f \mapsto f(0)$, *i.e.*, $\mathrm{Der}^i_M(fx) = f(0)\,\mathrm{Der}^i_M(x)$ for any $x \in \mathrm{H}^1(\mathbf{Q}_p, M \otimes_{\mathbf{H}} I_{\mathbf{H}})$ and $f \in \Lambda_{\mathbf{H}}$.

If we define $\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_p, T^* \otimes_{\mathbf{H}} I_{\mathbf{H}})$ to be the $\Lambda_{\mathbf{H}}$-saturation of

$$\ker\big(\mathrm{H}^1(\mathbf{Q}_p, T^* \otimes_{\mathbf{H}} I_{\mathbf{H}}) \longrightarrow \mathrm{H}^1(I_p, (\mathrm{F}^-\,T^*) \otimes_{\mathbf{H}} I_{\mathbf{H}})\big),$$

then Der similarly induces a map $\mathrm{Der}_{\mathrm{s}}\colon \mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_p, T^* \otimes_{\mathbf{H}} I_{\mathbf{H}}) \to \mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_p, T^*)$ making the diagram

$$\begin{array}{ccc} \mathrm{H}^1(\mathbf{Q}_p, T^* \otimes_{\mathbf{H}} I_{\mathbf{H}}) & \longrightarrow & \mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_p, T^* \otimes_{\mathbf{H}} I_{\mathbf{H}}) \\ \Big\downarrow{\scriptstyle \mathrm{Der}^1_{T^*}} & & \Big\downarrow{\scriptstyle \mathrm{Der}_{\mathrm{s}}} \\ \mathrm{H}^1(\mathbf{Q}_p, T^*) & \longrightarrow & \mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_p, T^*) \end{array}$$

commute. $\mathrm{Der}_{\mathrm{s}}$ is well-defined because $\mathrm{H}^1(I_p, (\mathrm{F}^-\,T^*) \otimes_{\mathbf{H}} I_{\mathbf{H}})[I_{\mathbf{H}}] = 0$, which follows from the fact that $(\mathrm{F}^-\,T^*)^{I_p} = 0$.

3.9. Lemma. — *The map*

$$\mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_p, T^* \otimes_{\mathbf{H}} I_{\mathbf{H}}) \longrightarrow \mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_p, T^* \otimes_{\mathbf{H}} \Lambda_{\mathbf{H}})$$

induced by inclusion $T^* \otimes_{\mathbf{H}} I_{\mathbf{H}} \hookrightarrow T^* \otimes_{\mathbf{H}} \Lambda_{\mathbf{H}}$ is injective.

Proof. — From the definitions, there are natural inclusions

$$\mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_p, T^* \otimes_{\mathbf{H}} I_{\mathbf{H}}) \subseteq \mathrm{H}^1(I_p, (\mathrm{F}^-\,T^*) \otimes_{\mathbf{H}} I_{\mathbf{H}})/(\Lambda_{\mathbf{H}}\text{-torsion})$$

and

$$\mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_p, T^* \otimes_{\mathbf{H}} \Lambda_{\mathbf{H}}) \subseteq \mathrm{H}^1(I_p, (\mathrm{F}^-\,T^*) \otimes_{\mathbf{H}} \Lambda_{\mathbf{H}})/(\Lambda_{\mathbf{H}}\text{-torsion}).$$

The lemma thus follows from the fact that $(\mathrm{F}^-\,T^*)^{I_p} = 0$. □

3.10. Proposition (cf. Rubin [17, Prop. 4.3]). — *Suppose that*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence of $\mathbf{H}[G_{\mathbf{Q}_p}]$*-modules finitely-generated over* $\mathbf{H}$ *and choose* $c^{(\infty)} \in \mathrm{H}^1(\mathbf{Q}_p, C \otimes_{\mathbf{H}} I_{\mathbf{H}})$ *which is the image of some* $b^{(\infty)} \in \mathrm{H}^1(\mathbf{Q}_p, B \otimes_{\mathbf{H}} \Lambda_{\mathbf{H}})$*. Further assume that* $\mathrm{H}^0(\mathbf{Q}_p, C) = 0$*. Then* $\mathrm{cor}_\infty\, b^{(\infty)} \in \mathrm{H}^1(\mathbf{Q}_p, B)$ *lies in* $\mathrm{H}^1(\mathbf{Q}_p, A) \subseteq \mathrm{H}^1(\mathbf{Q}_p, B)$ *and*

$$\mathrm{cor}_\infty\, b^{(\infty)} \cup \psi_p = \delta_p(\mathrm{Der}\, c^{(\infty)}) \in \mathrm{H}^2(\mathbf{Q}_p, A),$$

where $\delta_p \colon \mathrm{H}^1(\mathbf{Q}_p, C) \to \mathrm{H}^2(\mathbf{Q}_p, A)$ *is the connecting homomorphism.*

Although the corresponding proposition in [17] deals with finite Galois modules, our proof is similar. The formula in this proposition continues to hold for higher cohomology groups, with essentially the same proof, but for simplicity we have limited ourselves to the case of H^1, as that is all that is needed in the sequel.

Proof. — The assertion that $\mathrm{cor}_\infty\, b^{(\infty)}$ lies in $\mathrm{H}^1(\mathbf{Q}_p, A)$ is clear, so we only need check the cup product formula, which we do, as in [17], by a computation.

For $\mathbf{H}[G_{\mathbf{Q}_p}]$-modules M, elements of $\mathrm{H}^i(\mathbf{Q}_p, M \otimes_{\mathbf{H}} \Lambda_{\mathbf{H}})$, resp. $\mathrm{H}^i(\mathbf{Q}_p, M \otimes_{\mathbf{H}} I_{\mathbf{H}})$, can be represented as formal power series $\sum_{k\geqslant 0} m_k X^k$, resp. $\sum_{k\geqslant 1} m_k X^k$, where each m_k is a function $(G_{\mathbf{Q}_p})^i \to M$. (This representation depends, of course, on the isomorphism $\Lambda_{\mathbf{H}} \cong \mathbf{H}[\![X]\!]$ determined by κ_C.) Thus, we can write $b^{(\infty)} = \sum_{k\geqslant 0} b_k X^k$ and $c^{(\infty)} = \sum_{k\geqslant 1} c_k X^k$ and we have $\mathrm{cor}_\infty\, b^{(\infty)} = b_0 \colon G_{\mathbf{Q}_p} \to B$ and $\mathrm{Der}\, c^{(\infty)} = c_1 \colon G_{\mathbf{Q}_p} \to C$.

The fact that $b^{(\infty)}$ is a cocycle gives the following formula relating the coefficients in the power series representation of $b^{(\infty)}$:

$$\sum b_k(gh)X^k = (X+1)^{\psi_p(g)} \sum g b_k(h) X^k + \sum b_k(g) X^k,$$

where we are viewing ψ_p as a homomorphism $G_{\mathbf{Q}_p} \twoheadrightarrow \Gamma_C \xrightarrow{\sim} \mathbf{Z}_p$. Comparing linear terms, this gives

$$b_1(gh) = \psi_p(g) g b_0(h) + g b_1(h) + b_1(g).$$

Thus, by definition of the cup product, we have

$$(\psi_p \cup \mathrm{cor}_\infty\, b^{(\infty)})(g,h) = \psi_p(g) g b_0(h) = -g b_1(h) + b_1(gh) - b_1(g).$$

But, by assumption, b_1 is a lift of the cocycle $c_1 = \mathrm{Der}\, c^{(\infty)}$, so the proposition follows from the definition of δ_p. □

3.11. Proof of 3.2. By Proposition 3.7, we have

$$\langle a, b\rangle = \mathrm{cor}_\infty\, w_p^{(\infty)} \cup \psi_p,$$

where $w_p^{(\infty)} \in \mathrm{H}^1\big(\mathbf{Q}, \mathbf{H}(1)\big)$ maps to $\mathrm{loc}_p\, \widetilde{a} - \mathrm{cor}_\infty\, \widetilde{a}_p^{(\infty)} \in \mathrm{H}^1(\mathbf{Q}_p, X^*)$, so Proposition 3.10 gives

$$\mathrm{cor}_\infty\, w_p^{(\infty)} \cup \psi_p = \delta_p\big(\mathrm{Der}(\mathrm{loc}_p\, a^{(\infty)} - a_p^{(\infty)})\big) = \mathrm{Der}(\mathrm{loc}_p\, a^{(\infty)} - a_p^{(\infty)}) \cup \mathrm{loc}_p\, b.$$

By definition, $a_p^{(\infty)}$ lies in $\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_p, T^* \otimes_{\mathbf{H}} \Lambda_{\mathbf{H}})$, so $\mathrm{loc}_p\, a^{(\infty)} - a_p^{(\infty)}$ maps to $\mathrm{loc}_{\mathrm{s}}\, a^{(\infty)}$ in $\mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_p, T^* \otimes_{\mathbf{H}} \Lambda_{\mathbf{H}})$. As $\mathrm{loc}_p\, b$ lies in $\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_p, T)$, Lemma 3.9 thus shows that

$$\langle a, b\rangle = \mathrm{Der}_{\mathrm{s}}(\mathrm{loc}_{\mathrm{s}}\, a^{(\infty)}) \cup \mathrm{loc}_{\mathrm{f}}\, b,$$

which gives Theorem 3.2 in view of (2.8.1).

4. Euler systems and derivatives

4.1. Theorem 1.6, now follows as a mostly formal consequence of the height formula, Theorem 3.2, given the existence of a Coleman map for $\widetilde{\mathcal{T}}$, Theorem 4.5 (due to Ochiai). Under the additional assumption that the 2-variable main conjecture, Conjecture 4.3, holds in our situation, we show that the non-degeneracy of the height pairing follows from the non-triviality of the derivative of $\mathcal{L}$ with respect to Θ and statements concerning the ranks of certain Selmer groups associated to $\mathcal{T}$. Hence, under these conditions Conjecture 1.5 implies the equivalence of Conjectures 1.4 and 3.6. We also discuss the validity of Conjecture 1.5 in the context of Greenberg's "Hypothesis L" ([4, p. 339]; the "L" stands for "Leopoldt").

Throughout §4, we assume the following about the Hida family $\mathcal{F}$: The residual representation $\overline{\rho}_{\mathcal{F}}$ is irreducible, the Nebentypus $\chi_{\mathcal{F}}$ is trivial, and the sign of $\mathcal{F}$ is -1. Before proceeding to the proof of our main theorems, we first give precise statements of some of the ingredients. Theorem 4.2 (due to Kitagawa) gives the existence of a 2-variable p-adic L-function for $\widetilde{\mathcal{T}}$, Conjecture 4.3 states the conjectural relationship between this p-adic L-function and the Selmer group $\mathrm{Sel}(\mathbf{Q}, \widetilde{\mathcal{W}})$, and Theorem 4.5 (due to Ochiai) relates the p-adic L-function to Kato's Euler system.

4.2. Theorem (Kitagawa [7, Thm. 1.1]). — *There is an element* $\mathcal{L} = \mathcal{L}(X) \in \Lambda_{\mathbf{H}} \cong \mathbf{H}[\![X]\!]$ *such that for every arithmetic character* (σ, τ) *with* $1 \leqslant w(\tau) \leqslant w(\sigma) + 1$ *the interpolation formula*

$$(\sigma, \tau)(\mathcal{L}) = d(\sigma, \tau) L(f_\sigma, \omega^{-w(\tau)}\tau\kappa_C^{-w(\tau)}, w(\tau))$$

holds, where $d(\sigma,\tau)$ is an explicit non-zero constant (involving complex and p-adic periods).

Recall that f_σ is the modular form arising as the specialization at σ of the Hida family under consideration; see 2.5 for more details. This p-adic L-function is related to the Selmer group of $\widetilde{\mathcal{W}} = \widetilde{W} \otimes \theta^{-1}$ by the following conjecture.

4.3. Conjecture (2-variable main conjecture). — $\mathrm{Sel}(\mathbf{Q}, \widetilde{\mathcal{W}})$ *is a cotorsion $\Lambda_{\mathbf{H}}$-module and for every height 1 prime $\mathfrak{p} \subseteq \Lambda_{\mathbf{H}}$,*

$$\mathrm{ord}_{\mathfrak{p}}\, \mathcal{L} = \mathrm{length}_{(\Lambda_{\mathbf{H}})_{\mathfrak{p}}} (\mathrm{Sel}(\mathbf{Q}, \widetilde{\mathcal{W}})^{\vee})_{\mathfrak{p}}.$$

We should explain what we mean by this conjecture in the case that $\Lambda_{\mathbf{H}}$ is not regular in codimension 1. Let $\overline{\Lambda}_{\mathbf{H}}$ denote the integral closure of $\Lambda_{\mathbf{H}}$ and denote by $\overline{\mathcal{L}}$ the image of $\mathcal{L}$ in $\overline{\Lambda}_{\mathbf{H}}$. We make the convention that

$$\mathrm{ord}_{\mathfrak{p}}\, \mathcal{L} = \mathrm{length}_{(\Lambda_{\mathbf{H}})_{\mathfrak{p}}} (\mathrm{Sel}(\mathbf{Q}, \widetilde{\mathcal{W}})^{\vee})_{\mathfrak{p}}$$

means

$$\mathrm{ord}_{\overline{\mathfrak{p}}}\, \overline{\mathcal{L}} = \mathrm{length}_{(\overline{\Lambda}_{\mathbf{H}})_{\overline{\mathfrak{p}}}} (\mathrm{Sel}(\mathbf{Q}, \widetilde{\mathcal{W}})^{\vee} \otimes \overline{\Lambda}_{\mathbf{H}})_{\overline{\mathfrak{p}}}$$

for every height 1 prime $\overline{\mathfrak{p}}$ of $\overline{\Lambda}_{\mathbf{H}}$ lying over $\mathfrak{p}$. Regardless, in the sequel we are only concerned with this conjecture for the case $\mathfrak{p} = \Theta$, which is a *principal* height 1 prime, so $(\Lambda_{\mathbf{H}})_\Theta$ is a DVR and there is no difficulty with the notation as it stands in the conjecture.

Under some assumptions, the "$\geqslant$" inequality of this conjecture was proved by Ochiai [12, Thm. 2.6] using Kato's Euler system. However, it is the "$\leqslant$" inequality that we use below, which we do not expect to follow from an Euler system argument. Skinner-Urban have announced a proof of this conjecture in some cases, though we know of no reference.

4.4. We do not give here a complete definition of or even a complete list of the properties satisfied by Kato's Euler system, as for us the main significance of this Euler system is that it gives a collection of *global* cohomology classes related (via Ochiai's Coleman map, 4.5) to Kitagawa's p-adic L-function described in 4.2. The key difficulty in establishing the non-triviality of the restriction of this Euler system to $\mathcal{T}$ is that there does not seem to be any obvious way to relate the essentially local information provided by the p-adic L-function to the global behavior of the Euler system without assuming at least some statement about ranks of Selmer groups; this is why Greenberg's "Hypothesis L" arises.

The Euler system we use here is a modification, due to Ochiai [13, §6], of Kato's original construction and consists of a collection of cohomology classes $z(r) \in \mathrm{Sel}^{\Sigma}(\mathbf{Q}(\mu_r), \widetilde{\mathcal{T}}^*)$, one for each squarefree integer r prime to p, which satisfy certain norm-compatibility relations. The specializations of this Euler system at arithmetic primes are related to L-values of the corresponding modular forms via the dual exponential map. See, for example, [6], [11], or [13] for the precise formulas. By interpolating the dual exponential maps of the specializations of $\mathcal{F}$ at arithmetic primes, Ochiai was able to relate this Euler system to the p-adic L-function of Theorem 4.2.

4.5. Theorem (Ochiai [13, Cor. 6.17]). — *There is an $\Lambda_{\mathbf{H}}$-linear injection ("Coleman map")*

$$\mathrm{Col}\colon \mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_p, \widetilde{\mathcal{T}}^*) \longrightarrow \Lambda_{\mathbf{H}}$$

with pseudo-null cokernel such that $\mathrm{Col}\big(\mathrm{loc}_{\mathrm{s}}\, z(1)\big) = u\mathcal{L}$ *for a unit* $u \in \Lambda_{\mathbf{H}}^{\times}$.

Recall that a $\Lambda_{\mathbf{H}}$-module is *psuedo-null* if it is annihilated by a height 2 ideal.

4.6. In order to apply the height formula, we need to elaborate somewhat on the notion of twisting introduced in 2.5 and in particular examine how it interacts with the Coleman map and p-adic L-function introduced above. For any continuous character $\eta\colon \Gamma_C \to \mathbf{H}^{\times}$, we define an $\mathbf{H}$-algebra isomorphism

$$\mathrm{Tw}_{\eta}\colon \Lambda_{\mathbf{H}} \longrightarrow \Lambda_{\mathbf{H}}$$

by the formula $\mathrm{Tw}_{\eta}(g) = \eta(g)g$ for all $g \in \Gamma_C \subseteq \Lambda_{\mathbf{H}}^{\times}$. Taking $\eta = \theta$, the character defined in 2.5, and identifying $\Lambda_{\mathbf{H}} \cong \mathbf{H}[\![X]\!]$ via the isomorphism determined by κ_C, we have that $\big(\mathrm{Tw}_{\theta}(\Theta)\big) = (X)$ (equality of ideals in $\Lambda_{\mathbf{H}}$), where Θ is as in 2.4. In particular $\Theta^n \mid \mathcal{L}$ if and only if $X^n \mid \mathrm{Tw}_{\theta}\,\mathcal{L}$.

Because $\widetilde{T}^* = \widetilde{\mathcal{T}}^* \otimes \theta^{-1}$, there are (abusing notation slightly) $\mathbf{H}$-linear isomorphisms

$$\mathrm{Tw}_{\theta}^*\colon \mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}, \widetilde{\mathcal{T}}^*) \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}, \widetilde{T}^*), \quad \mathrm{Tw}_{\theta}^*\colon \mathrm{Sel}(\mathbf{Q}, \widetilde{\mathcal{T}}^*) \xrightarrow{\sim} \mathrm{Sel}(\mathbf{Q}, \widetilde{T}^*)$$

satisfying $\mathrm{Tw}_{\theta}^*(\lambda x) = \mathrm{Tw}_{\theta}(\lambda)\,\mathrm{Tw}_{\theta}^*(x)$ for all $\lambda \in \Lambda_{\mathbf{H}}$. Thus, there is a unique $\Lambda_{\mathbf{H}}$-linear injection Col_{θ} making the diagram

$$\begin{array}{ccccc}
\mathrm{Sel}^{\Sigma}(\mathbf{Q}, \widetilde{\mathcal{T}}^*) & \xrightarrow{\mathrm{loc}_{\mathrm{s}}} & \mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}, \widetilde{\mathcal{T}}^*) & \xrightarrow{\mathrm{Col}} & \Lambda_{\mathbf{H}} \\
\downarrow{\scriptstyle \mathrm{Tw}_{\theta}^*} & & \downarrow{\scriptstyle \mathrm{Tw}_{\theta}^*} & & \downarrow{\scriptstyle \mathrm{Tw}_{\theta}} \\
\mathrm{Sel}^{\Sigma}(\mathbf{Q}, \widetilde{T}^*) & \xrightarrow[\mathrm{loc}_{\mathrm{s}}]{} & \mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}, \widetilde{T}^*) & \xrightarrow[\mathrm{Col}_{\theta}]{} & \Lambda_{\mathbf{H}}
\end{array}$$

commute. By definition, we have $\mathrm{Col}_{\theta}\big(\mathrm{Tw}_{\theta}^*(\mathrm{loc}_{\mathrm{s}}\, z(1))\big) = \mathrm{Tw}_{\theta}(\mathcal{L})$.

For ease of notation, we define $z = \mathrm{Tw}_\theta^* z(1) \in \mathrm{Sel}^\Sigma(\mathbf{Q}, \widetilde{T}^*)$ and $\overline{z} = \mathrm{cor}_\infty \mathrm{Tw}_\theta^* z(1) \in \mathrm{Sel}^\Sigma(\mathbf{Q}, T^*)$ for what follows. Given the above, the following is equivalent to Theorem 1.6.

4.7. Theorem. — *Assume that Conjecture 3.6 (non-degeneracy of the height pairing) holds. If $\overline{z} \notin \mathrm{Sel}(\mathbf{Q}, T^*)_{\mathbf{H}\text{-tors}}$, then $X^2 \nmid \mathrm{Tw}_\theta \mathcal{L}$.*

Proof. — By 4.5, $\mathrm{coker}(\mathrm{Tw}_\theta \circ \mathrm{Col})$ is annihilated by an idea of height 2, so we may choose $\lambda \in \Lambda_{\mathbf{H}}$ prime to X annihilating this cokernel. Recall we are assuming that $\varepsilon_{\mathcal{F}} = -1$, so the interpolation property of $\mathcal{L}$, 4.2, implies that $X \mid \mathrm{Tw}_\theta \mathcal{L}$. As $\mathrm{loc}_\mathrm{s}\, z$ maps to $\mathrm{Tw}_\theta \mathcal{L}$ under Col_θ, we thus have that $\mathrm{loc}_\mathrm{s}\, \lambda z = X\alpha^{(\infty)} \in \mathrm{H}^1_\mathrm{s}(\mathbf{Q}_p, \widetilde{T}^*)$ for some $\alpha^{(\infty)}$. The height formula, Theorem 3.2, then gives

$$\langle \mathrm{cor}_\infty \lambda z, b\rangle = (\mathrm{cor}_\infty \alpha^{(\infty)}, \mathrm{loc}_p b)_p \tag{4.7.1}$$

for any $b \in \mathrm{Sel}(\mathbf{Q}, T)$. Under our assumption that the height pairing is non-degenerate, the left hand side of (4.7.1) is non-zero for some choice of b, which shows that $\mathrm{cor}_\infty \alpha^{(\infty)} \neq 0$ by the non-degeneracy of the local Tate pairing. It remains to show how this implies that $X^2 \nmid \mathrm{Tw}_\theta \mathcal{L}$, which is part of the following lemma. □

4.8. Lemma. — *Suppose that $\mathrm{loc}_\mathrm{s}\, \lambda z = X\alpha^{(\infty)}$ for some $\alpha^{(\infty)} \in \mathrm{H}^1_\mathrm{s}(\mathbf{Q}_p, \widetilde{T}^*)$ and $\lambda \in \mathrm{Ann}_{\Lambda_{\mathbf{H}}}(\mathrm{coker}\,\mathrm{Col}_\theta)$ prime to X. Then $X^2 \mid \mathrm{Tw}_\theta \mathcal{L}$ if and only if $\mathrm{cor}_\infty \alpha^{(\infty)} = 0$.*

Proof. — Suppose first that $X^2 \mid \mathrm{Tw}_\theta \mathcal{L}$, so there exists $\beta^{(\infty)} \in \mathrm{H}^1_\mathrm{s}(\mathbf{Q}_p, \widetilde{T}^*)$ such that $\mathrm{Col}_\theta X^2\beta^{(\infty)} = \lambda \mathrm{Tw}_\theta \mathcal{L}$ (*viz.* $\beta^{(\infty)} = \mathrm{Col}_\theta^{-1} \lambda X^{-2} \mathrm{Tw}_\theta \mathcal{L}$). Both $X\beta^{(\infty)}$ and $\alpha^{(\infty)}$ map under Col_θ to $\lambda X^{-1} \mathrm{Tw}_\theta \mathcal{L}$, so $X\beta^{(\infty)} = \alpha^{(\infty)}$, which shows that $\mathrm{cor}_\infty \alpha^{(\infty)} = 0$.

Conversely, suppose that $\mathrm{cor}_\infty \alpha^{(\infty)} = 0$. We first show that the cokernel of $\mathrm{cor}_\infty \colon \mathrm{H}^1_\mathrm{s}(\mathbf{Q}_p, \widetilde{T}^*) \to \mathrm{H}^1_\mathrm{s}(\mathbf{Q}_p, T^*)$ is a torsion $\mathbf{H}$-module, which follows if we show that the cokernel of $\mathrm{cor}_\infty \colon \mathrm{H}^1(\mathbf{Q}_p, \widetilde{T}^*) \to \mathrm{H}^1(\mathbf{Q}_p, T^*)$ is $\mathbf{H}$-torsion. As $T^*(\mathbf{Q}_p) = 0$, the kernel of the latter map is $X\mathrm{H}^1(\mathbf{Q}_p, \widetilde{T}^*)$. By an Euler-Poincaré characteristic argument similar to that in the proof of Lemma 2.9, one can show that $\mathrm{H}^1(\mathbf{Q}_p, \widetilde{T}^*)$ has rank at least 2 over $\Lambda_{\mathbf{H}}$, which gives what we want in view of the fact (Lemma 2.9) that $\mathrm{rk}_{\mathbf{H}}\, \mathrm{H}^1(\mathbf{Q}_p, T^*) = 2$. The fact that Col_θ is injective with pseudo-null cokernel implies that $\mathrm{H}^1_\mathrm{s}(\mathbf{Q}_p, \widetilde{T}^*)/X\mathrm{H}^1_\mathrm{s}(\mathbf{Q}_p, \widetilde{T}^*)$ has rank 1 over $\mathbf{H}$. Consequently, the kernel of the induced map $\mathrm{cor}_\infty \colon \mathrm{H}^1_\mathrm{s}(\mathbf{Q}_p, \widetilde{T}^*)/X\mathrm{H}^1_\mathrm{s}(\mathbf{Q}_p, \widetilde{T}^*) \to \mathrm{H}^1_\mathrm{s}(\mathbf{Q}_p, T^*)$ is annihilated by some $\mu \in \Lambda_{\mathbf{H}}$ prime to X. In particular, $\mu\alpha^{(\infty)} \in X\mathrm{H}^1_\mathrm{s}(\mathbf{Q}_p, \widetilde{T}^*)$, so $X^2 \mid \mu\, \mathrm{Col}_\theta \alpha^{(\infty)} = \mu\lambda \mathrm{Tw}_\theta \mathcal{L}$, which gives the lemma, as both λ and μ are prime to X. □

4.9. We now analyze some sufficient conditions for the converse of Theorem 4.7 to hold. The underlying philosophy is that the rank of the Selmer group $\mathrm{Sel}(\mathbf{Q}, T^*)$ should be governed by the Euler system class $z(1)$: Regardless of the sign of $\mathcal{F}$, $\mathrm{Sel}^{\Sigma}(\mathbf{Q}, T^*)$ should have rank 1 over $\mathbf{H}$, with $\mathrm{cor}_\infty \mathrm{Tw}_\theta^* z(1)$ providing a non-torsion class, and $\mathrm{Sel}(\mathbf{Q}, T^*)$ has rank 1 or 0 according as a multiple of $\mathrm{cor}_\infty \mathrm{Tw}_\theta^* z(1)$ belongs or does not belong to $\mathrm{Sel}(\mathbf{Q}, T^*)$. Moreover, the question of whether $\mathrm{cor}_\infty \mathrm{Tw}_\theta^* z(1)$ belongs to $\mathrm{Sel}(\mathbf{Q}, T^*)$ or not is related via the Coleman map, 4.5, to the vanishing or non-vanishing along Θ of the p-adic L-function $\mathcal{L}$ of 4.2. Thus statements about the ranks of Selmer groups attached to T and T^* are closely related to Conjectures 1.4 and 1.5.

4.10. The two 5-term sequences

$$(4.10.1)\quad 0 \longrightarrow \mathrm{Sel}(\mathbf{Q}, T^*) \longrightarrow \mathrm{Sel}^{\Sigma}(\mathbf{Q}, T^*) \longrightarrow \bigoplus_{v\in\Sigma} \mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_v, T^*) \longrightarrow \\ \longrightarrow \mathrm{Sel}(\mathbf{Q}, W)^\vee \longrightarrow \mathrm{Sel}_{\Sigma}(\mathbf{Q}, W)^\vee \longrightarrow 0$$

and

$$(4.10.2)\quad 0 \longrightarrow \mathrm{Sel}_{\Sigma}(\mathbf{Q}, T^*) \longrightarrow \mathrm{Sel}(\mathbf{Q}, T^*) \longrightarrow \bigoplus_{v\in\Sigma} \mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_v, T^*) \longrightarrow \\ \longrightarrow \mathrm{Sel}^{\Sigma}(\mathbf{Q}, W)^\vee \longrightarrow \mathrm{Sel}(\mathbf{Q}, W)^\vee \longrightarrow 0$$

are exact by the orthogonality of local conditions under the local Tate pairing. The analogous sequences obtained by replacing T^* with $\widetilde{T}^*$ and W with $\widetilde{W}$ are likewise exact. In addition, we define the *Shafarevich-Tate group*

$$Ш^1(\mathbf{Q}, W) = \ker\Big(\mathrm{Sel}(\mathbf{Q}, W) \longrightarrow \bigoplus_{v\in\Sigma} \mathrm{H}^1(\mathbf{Q}_v, W)\Big).$$

4.11. Hypothesis L (Greenberg [4, p. 339]). — $Ш^1(\mathbf{Q}, W)^\vee$ *is a torsion* $\mathbf{H}$-*module.*

The statement given here is what Greenberg would refer to as "Hypothesis L for the representation T^*". Note that, in our context, the validity of Hypothesis L is independent of the finite set Σ of primes containing p in view of the fact (Lemma 2.9) that $\mathrm{H}^1(\mathbf{Q}_v, T^*)$ is a torsion $\mathbf{H}$-module for $v \nmid p$. For the same reason, Hypothesis L is equivalent to the statement that $\mathrm{Sel}_{\Sigma}(\mathbf{Q}, W)^\vee$ is a torsion $\mathbf{H}$-module.

Hypothesis L can fail in general, even for representations over "large" ($\geqslant$ 2-dimensional) rings (see [4, p. 386ff] for a discussion of this), but we expect it to hold for many "naturally-occurring" representations and, in particular, for the representation T^* considered here. In general, the existence of a non-trivial Euler system for T^* should imply that $\mathrm{Sel}_\Sigma(\mathbf{Q}, W)^\vee$ is a torsion $\mathbf{H}$-module, so in our context we expect that Hypothesis L follows from Conjecture 1.5; this implication will be discussed in a future paper.

4.12. LEMMA. — *The natural corestriction map*

$$\mathrm{cor}_\infty\colon\ \mathrm{Sel}^\Sigma(\mathbf{Q}, \widetilde{T}^*)/I_{\mathbf{H}}\,\mathrm{Sel}^\Sigma(\mathbf{Q}, \widetilde{T}^*) \longrightarrow \mathrm{Sel}^\Sigma(\mathbf{Q}, T^*)$$

is injective.

Proof. — Recall from 2.8 that $\mathrm{Sel}^\Sigma(\mathbf{Q}, \widetilde{T}^*) = \mathrm{H}^1(\mathbf{Q}_\Sigma/\mathbf{Q}, \widetilde{T}^*)$ and $\mathrm{Sel}^\Sigma(\mathbf{Q}, T) = \mathrm{H}^1(\mathbf{Q}_\Sigma/\mathbf{Q}, T)$. The long exact sequence in cohomology associated to the exact sequence of $\mathrm{Gal}(\mathbf{Q}_\Sigma/\mathbf{Q})$-modules

$$0 \longrightarrow \widetilde{T}^* \xrightarrow{\cdot X} \widetilde{T}^* \longrightarrow T^* \longrightarrow 0$$

gives the statement of the lemma. □

4.13. LEMMA. — *The natural maps (dual to restriction)*

$$\mathrm{Sel}^\Sigma(\mathbf{Q}, \widetilde{W})^\vee/I_{\mathbf{H}}\,\mathrm{Sel}^\Sigma(\mathbf{Q}, \widetilde{W})^\vee \longrightarrow \mathrm{Sel}^\Sigma(\mathbf{Q}, W)^\vee$$
$$\mathrm{Sel}(\mathbf{Q}, \widetilde{W})^\vee/I_{\mathbf{H}}\,\mathrm{Sel}(\mathbf{Q}, \widetilde{W})^\vee \longrightarrow \mathrm{Sel}(\mathbf{Q}, W)^\vee$$
$$\mathrm{Sel}_\Sigma(\mathbf{Q}, \widetilde{W})^\vee/I_{\mathbf{H}}\,\mathrm{Sel}_\Sigma(\mathbf{Q}, \widetilde{W})^\vee \longrightarrow \mathrm{Sel}_\Sigma(\mathbf{Q}, W)^\vee$$

are injective with $\mathbf{H}$*-torsion cokernel.*

Proof. — In light of the fact that $\mathrm{Sel}^\Sigma(\mathbf{Q}, W) = \mathrm{H}^1(\mathbf{Q}_\Sigma/\mathbf{Q}, W)$, we have that the restriction map

$$\mathrm{H}^1(\mathbf{Q}_\Sigma/\mathbf{Q}, W) \cong \mathrm{Sel}^\Sigma(\mathbf{Q}, W) \longrightarrow \mathrm{Sel}^\Sigma(\mathbf{Q}, \widetilde{W})[I_{\mathbf{H}}] \cong \mathrm{H}^1(\mathbf{Q}_\Sigma/\mathbf{Q}_\infty, W)[I_{\mathbf{H}}]$$

is surjective, because Γ_C has cohomological dimension 1, and has kernel isomorphic to $\mathrm{H}^1\big(\mathbf{Q}_\infty/\mathbf{Q}, W(\mathbf{Q}_\infty)\big)$, by the inflation-restriction sequence. The statement for Sel^Σ thus follows from the facts that $\mathrm{H}^1\big(\mathbf{Q}_\infty/\mathbf{Q}, W(\mathbf{Q}_\infty)\big) \cong W(\mathbf{Q}_\infty)/(\gamma_C - 1)W(\mathbf{Q}_\infty)$ and that $W(\mathbf{Q}_\infty)$ is a cotorsion $\mathbf{H}$-module.

For $v \in \Sigma$, the local condition $\mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_v, W)$ is the maximal $\mathbf{H}$-divisible submodule of $\mathrm{H}^1(\mathbf{Q}_v, W)$ by [15, Prop. 5.1]. The kernel of the restriction map

$$\mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_p, W) \longrightarrow \mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_p, \widetilde{W})[I_{\mathbf{H}}]$$

is therefore *a fortiori* a cotorsion $\mathbf{H}$-module.

Now consider the commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Sel}(\mathbf{Q}, W) & \longrightarrow & \mathrm{Sel}^{\Sigma}(\mathbf{Q}, W) & \longrightarrow & \bigoplus_{v\in\Sigma} \mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_v, W) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & \mathrm{Sel}(\mathbf{Q}, \widetilde{W})[I_{\mathbf{H}}] & \longrightarrow & \mathrm{Sel}^{\Sigma}(\mathbf{Q}, \widetilde{W})[I_{\mathbf{H}}] & \longrightarrow & \bigoplus_{v\in\Sigma} \mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_v, \widetilde{W}) \end{array}$$

with vertical arrows given by restriction. By the above, the cokernel of the middle vertical map is trivial and the kernels of both the middle and the righthand vertical maps are $\mathbf{H}$-cotorsion. The statement for Sel now follows from the snake lemma.

One applies the same argument to the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathrm{Sel}_{\Sigma}(\mathbf{Q}, W) & \longrightarrow & \mathrm{Sel}(\mathbf{Q}, W) & \longrightarrow & \bigoplus_{v\in\Sigma} \mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_v, W) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & \mathrm{Sel}_{\Sigma}(\mathbf{Q}, \widetilde{W})[I_{\mathbf{H}}] & \longrightarrow & \mathrm{Sel}(\mathbf{Q}, \widetilde{W})[I_{\mathbf{H}}] & \longrightarrow & \bigoplus_{v\in\Sigma} \mathrm{H}^1_{\mathrm{f}}(\mathbf{Q}_v, \widetilde{W}) \end{array}$$

to prove the statement for Sel_{Σ}. □

4.14. Theorem. — *Assume Hypothesis L and the* 2*-variable main conjecture, 4.3. If Conjecture 1.4 holds, then Conjecture 1.5 holds.*

This theorem leaves something to be desired in that, ideally, one would like to find sufficient conditions for Conjecture 1.5 to hold which are related only to the non-vanishing of (classical or p-adic) L-functions. On the other hand, a proof of the 2-variable main conjecture should in many cases be contained in recent work of Skinner-Urban and, as we remarked in 4.11, we expect that Hypothesis L is a consequence of Conjecture 1.5 (and is therefore necessary for 1.5 to hold).

Proof. — Suppose by way of contradiction that $\overline{z} \in \mathrm{Sel}^{\Sigma}(\mathbf{Q}, T^*)_{\mathbf{H}\text{-tors}}$. Then we may choose $\lambda \in \Lambda_{\mathbf{H}}$ prime to X such that $\lambda\overline{z} = 0$. (We are of course viewing here $\mathbf{H}$-modules as $\Lambda_{\mathbf{H}}$-modules via the natural quotient $\Lambda_{\mathbf{H}} \twoheadrightarrow \mathbf{H}$.) By Lemma 4.12, it follows that $\lambda z \in I_{\mathbf{H}} \mathrm{Sel}^{\Sigma}(\mathbf{Q}, \widetilde{T}^*)$, say $\lambda z = Xy$ with $y \in \mathrm{Sel}^{\Sigma}(\mathbf{Q}, \widetilde{T}^*)$ mapping to $\overline{y} = \mathrm{cor}_{\infty}\, y \in \mathrm{Sel}^{\Sigma}(\mathbf{Q}, T^*)$. A computation on cocycles shows that $\mathrm{loc}_s\, \overline{y} = \lambda\, \mathrm{Der}_{\mathrm{s}} \mathrm{loc}_{\mathrm{s}}\, z \in \mathrm{H}^1_{\mathrm{s}}(\mathbf{Q}_p, T^*)$, which we claim does not lie in the $\mathbf{H}$-torsion submodule. If this claim does not hold, then, replacing y by a suitable $\mathbf{H}$-multiple, we may assume that $\mathrm{loc}_{\mathrm{s}}\, \overline{y} = 0$. The claim follows by applying Lemma 4.8 with $\alpha^{(\infty)} = \mathrm{loc}_{\mathrm{s}}\, y$.

Thus, Hypothesis L, Lemma 2.9, and the exact sequence (4.10.1) imply that $\mathrm{Sel}(\mathbf{Q}, W)^{\vee}$ is a torsion $\mathbf{H}$-module. On the other hand, we know that

$X \mid \mathrm{Tw}_\theta \mathcal{L}$ from the assumption that $\varepsilon_{\mathcal{F}} = -1$, so the 2-variable main conjecture, after applying Tw_θ, implies that

$$\mathrm{length}_{(\Lambda_{\mathbf{H}})_{(X)}}(\mathrm{Sel}(\mathbf{Q}, \widetilde{W})^\vee)_{(X)} > 0.$$

Lemma 4.13 then implies that $\mathrm{rk}_{\mathbf{H}} \mathrm{Sel}(\mathbf{Q}, W)^\vee > 0$, a contradiction. □

4.15. THEOREM. — *Under Hypothesis L and the 2-variable main conjecture, if Conjectures 1.4 (non-vanishing of the derivative of $\mathcal{L}$) and 1.5 (non-vanishing of Euler system) hold, then Conjecture 3.6 (non-degeneracy of height pairing) holds.*

Proof. — Under our assumptions, we have that T is self-dual, so the height pairing is a self-pairing

$$\langle\ ,\ \rangle\colon\ \mathrm{Sel}(\mathbf{Q}, T^*) \times \mathrm{Sel}(\mathbf{Q}, T^*) \longrightarrow \mathbf{H}$$

and the height formula gives the equation

$$\langle \overline{z}, \overline{z} \rangle = \big(\mathrm{Der}\,\mathrm{loc}_{\mathrm{s}}\, z, \mathrm{loc}_{\mathrm{f}}\, \overline{z}\big)_p.$$

The theorem follows if we can show that $\mathrm{loc}_{\mathrm{f}}\, \overline{z}$ is not a torsion class. By the exact sequences (4.10.1) and (4.10.2) applied to $\widetilde{T}^*$ and $\widetilde{W}$, Lemma 2.9 applied to $\widetilde{T}^*$, and the 2-variable main conjecture (which implies that $\mathrm{Sel}(\mathbf{Q}, \widetilde{W})^\vee$ is a torsion $\Lambda_{\mathbf{H}}$-module), we have that $\mathrm{Sel}^\Sigma(\mathbf{Q}, \widetilde{W})^\vee$ has rank 1 over $\Lambda_{\mathbf{H}}$. Also note that the 2-variable main conjecture and Lemma 4.13 imply that $\mathrm{Sel}(\mathbf{Q}, W)^\vee$ has rank 1 over $\mathbf{H}$.

Suppose by way of contradiction that $\mathrm{loc}_{\mathrm{f}}\, \overline{z}$ is a torsion class. Lemma 2.9 and the exact sequence (4.10.2) show that $\mathrm{Sel}^\Sigma(\mathbf{Q}, T^*)^\vee$ has rank 2 over $\mathbf{H}$, which by Lemma 4.13 implies that $\mathrm{length}_{(\Lambda_{\mathbf{H}})_{(X)}}(\mathrm{Sel}^\Sigma(\mathbf{Q}, \widetilde{W})^\vee_{\mathrm{tors}})_{(X)} > 0$. But an examination of the exact sequences (4.10.1) and (4.10.2) (for $\widetilde{T}^*$ and $\widetilde{W}$) then shows that $\mathrm{length}_{(\Lambda_{\mathbf{H}})_{(X)}}(\mathrm{Sel}_\Sigma(\mathbf{Q}, \widetilde{W})^\vee)_{(X)} > 0$, contradicting Hypothesis L by Lemma 4.13. □

4.16. COROLLARY. — *Under Hypothesis L, the 2-variable main conjecture, and Conjecture 1.5, Conjectures 1.4 and 3.6 are equivalent.*

Proof. — Combine Theorems 4.7 and 4.15. □

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