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## ANALYTIC CONSERVATION LAWS

by A. P. STONE <sup>(1)</sup>

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### 1. Introduction.

Let  $(u^0, \dots, u^n)$  form a (local) coordinate system belonging to the ring  $A$  of germs of analytic functions at some point  $P$  of an analytic manifold, and let  $u^0(P) = \dots = u^n(P) = 0$ . Denote by  $\mathcal{E}$  and  $E$  the corresponding localizations of the modules of differential forms and vector fields on the manifold. These are in fact free  $A$ -modules whose generators are denoted by  $(du^0, \dots, du^n)$  and  $\left(\frac{\partial}{\partial u^0}, \dots, \frac{\partial}{\partial u^n}\right)$  respectively, where  $d$  is the operation of exterior differentiation. If  $h$  is any endomorphism of  $\mathcal{E}$ , one defines a conservation law for  $h$  as any  $\theta \in \mathcal{E}$  for which both  $\theta$  and  $h\theta$  are exact. Conservation laws defined in this manner can be shown to have a relation to conservation laws in the sense of physics. For a given  $h \in \text{Hom}(A, \mathcal{E})$ , the problem investigated in this paper is that of obtaining all conservation laws for  $h$ . It will also be convenient to modify the problem slightly by placing some restrictions on  $h$ . These will be announced in section 3.

We shall denote elements of  $\mathcal{E}$  by  $\theta, \varphi, \dots$  and elements of  $E$  by  $L, M, \dots$ . If  $\mathcal{A}$  is the ring of endomorphisms generated by  $h$  and scalar multiplications, then  $h$  is *cyclic* (i.e., non-derogatory) on  $\mathcal{E}$  if and only if there exists an element  $\theta \in \mathcal{E}$  such that  $\mathcal{A}\theta = \mathcal{E}$ ;  $\theta$  is called a generator of  $\mathcal{E}$  with respect to  $h$ . Equiva-

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lently  $h$  is cyclic on  $\mathcal{E}$  if and only if the minimal and characteristic polynomials of  $h$  are identical. Hence if  $h$  is cyclic and  $\theta$  is a generator of  $\mathcal{E}$ , then  $(\theta, h\theta, \dots, h^n\theta)$  is a basis for  $\mathcal{E}$  and with respect to this basis  $h$  is represented by the matrix

$$(h) = \begin{pmatrix} 0 & \dots & a_0 \\ 1 & & a_1 \\ 0 & \dots & \vdots \\ \vdots & \dots & \vdots \\ 0 & \dots & 1 & a_n \end{pmatrix},$$

and the characteristic equation for  $h$  is

$$h^{n+1} = a_0 I + a_1 h + \dots + a_n h^n.$$

Finally, we note that  $h$  is cyclic on  $\mathcal{E}$  if and only if the adjoint of  $h$  is cyclic on  $E$ . Since we shall write  $Lh$  when the action of  $h$  is on  $L \in E$ , and  $h\theta$  when  $h$  acts on  $\theta \in \mathcal{E}$ , no distinction in notation will be made between  $h$  and its adjoint. That is,  $\langle Lh, \theta \rangle = \langle L, h\theta \rangle$ .

**2. Nijenhuis Torsion.**

An element  $h \in \text{Hom}(\mathcal{E}, \mathcal{E})$  induces homomorphisms of the exterior algebra  $\Lambda^* \mathcal{E}$  generated by  $\mathcal{E}$ . Since  $\mathcal{E}$  is a free module generated by  $du^0, \dots, du^n$ ,

$$\Lambda^* \mathcal{E} = \Lambda^0 \mathcal{E} \oplus \Lambda^1 \mathcal{E} \oplus \dots \oplus \Lambda^{n+1} \mathcal{E}$$

where  $\Lambda^0 \mathcal{E} = A$  and  $\Lambda^1 \mathcal{E} = \mathcal{E}$ . In particular we make use of the induced transformations,

$$\Lambda^2 \mathcal{E} \xleftarrow{h^{(i)}} \Lambda^2 \mathcal{E} \quad i = 0, 1, 2$$

which are given by

$$(2.1) \quad h^{(1)}(\theta \wedge \varphi) = h\theta \wedge \varphi + \theta \wedge h\varphi$$

$$(2.2) \quad h^{(2)}(\theta \wedge \varphi) = h\theta \wedge h\varphi$$

for any  $\theta, \varphi \in \mathcal{E}$ . The mapping  $h^{(0)}$  is taken to be the identity on  $\mathcal{E} \wedge \mathcal{E}$ . It is easy to verify that these mappings are well-defined. An element  $[h, h]$  of  $\text{Hom}(\mathcal{E}, \mathcal{E} \wedge \mathcal{E})$ , called the *Nijenhuis torsion* of  $h$ , is defined by setting

$$(2.3) \quad [h, h]\theta = -h^{(2)}d\theta + h^{(1)}d(h\theta) - h^{(0)}d(h^2\theta)$$

for any  $\theta \in \mathcal{E}$ . It was originally defined in [2] and it also appears in [1] and [4]. One notes that additivity and homogeneity over the reals is clear, and a short calculation will establish that  $[h, h](a\theta) = a[h, h]\theta$  for any  $a \in A$  and  $\theta \in \mathcal{E}$ .

Since  $E \wedge E \simeq \text{Hom}(\mathcal{E} \wedge \mathcal{E}, A)$ , a dual characterization of  $[h, h]$  may be obtained. This is given for any  $L$  and  $M \in E$  by setting

(2.4)

$$(L \wedge M)[h, h] = [L, M]h^2 + [Lh, Mh] - [Lh, M]h - [L, Mh]h.$$

### 3. The Problem of Conservation Laws.

The problem of conservation laws is to find all forms  $\theta \in \mathcal{E}$  such that both  $\theta$  and  $h\theta$  are exact, for any given  $h \in \text{Hom}(\mathcal{E}, \mathcal{E})$ .

The existence and uniqueness of conservation laws has been established in [4], by Osborn under the conditions that  $h$  be cyclic, that  $[h, h]$  vanish, and that there exists a generator  $L \in E$  such that  $(L, Lh, \dots, Lh^n)$  all commute. It is the purpose of this paper to show that the commutativity condition may be removed if one strengthens the condition that  $h$  is cyclic to the condition that  $h$  possess distinct eigenvalues. While this involves the introduction of the complex field into the problem, the condition of distinct eigenvalues is not unnatural since it appears in the study of hyperbolic systems of partial differential equations. Finally, the condition that  $[h, h]$  vanishes is an integrability condition which in fact guarantees the existence of a basis of conservation laws for  $\mathcal{E}$ .

### 4. Distinct Eigenvalue Case.

Let  $h$  have distinct eigenvalues  $\lambda_0, \dots, \lambda_n$  and denote by  $L_0, \dots, L_n$  a corresponding eigenvector basis for  $E$ . Then the vanishing of the Nijenhuis torsion of  $h$  is a necessary and sufficient condition that there exist coordinates  $\nu^0, \dots, \nu^n$  such that  $\frac{\partial}{\partial \nu^0}, \dots, \frac{\partial}{\partial \nu^n}$  is an eigenvector basis for  $E$ , and that the eigenvalues  $\lambda_i$  are functions of a single variable  $\nu^i$ . As

a result one can always obtain an eigenvector basis of  $\mathfrak{E}$  consisting of closed forms. Another consequence of the vanishing of  $[h, h]$  which is utilized in this paper is the fact that the Lie bracket  $[L_i, L_j]$  of any two eigenvectors  $L_i$  and  $L_j$  can be expressed as a linear combination over  $A$  of  $L_i$  and  $L_j$  alone. The preceding results are obtained in [2].

In the event that all of the eigenvalues are constant, one easily obtains a basis of  $E$  generated by an  $L$  with the property that  $(L, Lh, \dots, Lh^n)$  all commute.

LEMMA 4.1. — *Let  $[h, h] = 0$  and assume that the eigenvalues  $\lambda_0, \dots, \lambda_n$  of  $h$  are distinct and constant in a neighborhood of some point  $P$  of an analytic manifold. Then there exists a generator  $L \in E$  such that  $[Lh^i, Lh^j] = 0$  for any pair  $(i, j)$  of non-negative integers.*

*Proof.* — One merely uses the existence of the eigenvector basis  $\left(\frac{\partial}{\partial \nu^0}, \dots, \frac{\partial}{\partial \nu^n}\right)$  to obtain a generator  $L$  by setting

$$L = \frac{\partial}{\partial \nu^0} + \dots + \frac{\partial}{\partial \nu^n}.$$

Since the eigenvalues are constant one obtains the result very easily.

At the other extreme, if all the eigenvalues are non-constant in a neighborhood of  $P$  then  $(d\lambda_0, \dots, d\lambda_n)$  is a basis of  $\mathfrak{E}$ . If it were not, then  $d\lambda_0 \wedge \dots \wedge d\lambda_n = 0$  and one obtains a contradiction by evaluating the left-hand side of this equation on  $\frac{\partial}{\partial \nu^0} \wedge \dots \wedge \frac{\partial}{\partial \nu^n}$ . We may now obtain the following lemma.

LEMMA 4.2. — *Let  $[h, h] = 0$  and assume that the eigenvalues  $\lambda_0, \dots, \lambda_n$  of  $h$  are distinct and non-constant in a neighborhood of some point  $P$  of an analytic manifold. Then there exists a generator  $L \in E$  such that  $[Lh^i, Lh^j] = 0$  for any pair*

$$(i, j) \in (0, 1, \dots, n).$$

*Proof.* — Since the coefficients  $a_i$  of the characteristic polynomial of  $h$  are symmetric functions of the eigenvalues,

one may easily verify that

$$da_0 \wedge \cdots \wedge da_n = \prod_{i < j}^n (\lambda_i - \lambda_j) d\lambda_0 \wedge \cdots \wedge d\lambda_n$$

and hence that  $(da_0, \dots, da_n)$  is a basis of  $\mathcal{E}$ . It is not hard to verify that the vanishing of the Nijenhuis torsion of  $h$  is sufficient to guarantee that

$$h da_i = da_{i-1} + a_i da_n$$

for  $i = 0, 1, \dots, n$ . One of course defines  $a_{-1}$  to be zero.

Hence with respect to the preceding basis,  $h$  is represented by the companion matrix

$$(h) = \begin{pmatrix} 0 & \cdots & a_0 \\ 1 & & a_1 \\ \vdots & \ddots & \vdots \\ 0 & & 1 & a_n \end{pmatrix}$$

and one can then easily verify that  $\frac{\partial}{\partial a_0}$  is the desired generator, that  $\left(\frac{\partial}{\partial a_0}, \dots, \frac{\partial}{\partial a_n}\right)$  is a cyclic basis of  $E$ , and that  $da_n$  is a generator of the dual space  $\mathcal{E}$ .

### 5. A Decomposition of $\mathcal{E}$ .

With these special cases disposed of, let us assume that  $(\lambda_0, \dots, \lambda_p)$  are locally constant and that  $(\lambda_{p+1}, \dots, \lambda_n)$  are locally non-constant. Let us also assume that all of these eigenvalues are distinct. Let  $(\theta^0, \dots, \theta^p, \theta^{p+1}, \dots, \theta^n)$  be a corresponding basis of eigenforms of  $\mathcal{E}$ , and let us consider the direct sum decomposition  $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$  where  $\mathcal{E}_1$  is spanned by the eigenforms  $(\theta^0, \dots, \theta^p)$  and  $\mathcal{E}_2$  is spanned by the eigenforms  $(\theta^{p+1}, \dots, \theta^n)$ . One may also regard  $h$  as a direct sum  $h_1 \oplus h_2$ , and it is easily verified that  $[h_i, h_i] = 0$  if  $[h, h] = 0$ . We then note that there exist projections  $P_i$  of  $\mathcal{E}$  onto  $\mathcal{E}_i$ ,  $i = 1, 2$ , satisfying

$$\begin{array}{ll} \text{(i)} & P_i^2 = P_i \\ \text{(ii)} & P_1 P_2 = 0 \\ \text{(iii)} & P_1 + P_2 = I. \end{array}$$

A homogeneously generated ideal  $\mathcal{C}$  of  $\Lambda_A^*\mathcal{E}$  is said to be *closed under exterior differentiation*  $d$  if and only if for each  $\varphi \in \mathcal{C}$ ,  $d\varphi \in \mathcal{C}$ . Let  $\mathcal{C}_1$  be the differential ideal generated by

$$(\theta^0, \dots, \theta^p) \quad \text{and} \quad \mathcal{C}_2$$

the differential ideal generated by  $(\theta^{p+1}, \dots, \theta^n)$ . The following results show that if  $[h, h] = 0$ , then  $d\mathcal{C}_i \subset \mathcal{C}_i$  and hence the ideals  $\mathcal{C}_i$  are generated by locally exact 1-forms as a consequence of the Frobenius theorem.

LEMMA 5.1. — *If  $h$  has distinct eigenvalues and  $[h, h] = 0$ , then  $[P_i, P_i] = 0$ .*

*Proof.* — Let  $(L_0, \dots, L_p, L_{p+1}, \dots, L_n)$  be a dual basis of eigenvectors, and let  $E_i$  be the dual space of  $\mathcal{E}_i$ . Then any  $M$  and  $N$  in  $E$  may be written in the form  $M = M_1 + M_2$  and  $N = N_1 + N_2$ , where  $M_i, N_i$  are in  $E_i$ . Since  $[h, h] = 0$ ,

$$[N_1, M_1] = \sum_{i=0}^p \alpha_i L_i \quad \text{and} \quad [N_2, M_2] = \sum_{j=p+1}^n \beta_j L_j$$

for  $\alpha_i, \beta_j \in A$ . A short computation will then establish that  $(N \wedge M)[P_i, P_i] = 0$ , for  $i = 1, 2$ .

LEMMA 5.2. — *Let  $P$  be any projection. Then*

- (i)  $2P^{(2)} = P^{(1)}P^{(1)} - P^{(1)}$
- (ii)  $2P^{(2)} = P^{(1)}P^{(2)} = P^{(2)}P^{(1)}$

The proof of Lemma 5.2 is a direct verification from the definitions of  $P^{(i)}$  which are given in equations 2.1 and 2.2. This lemma may then be applied to obtain the following theorem.

THEOREM 5.3. — *For any  $\theta \in \mathcal{E}$ ,*

- (i)  $-2P_2^{(2)} d(P_1\theta) = P_2^{(1)}[P_2, P_2]\theta$
- (ii)  $-2P_1^{(2)} d(P_2\theta) = P_1^{(1)}[P_1, P_1]\theta$ .

Hence, if  $h$  has distinct eigenvalues and  $[h, h] = 0$  then the preceding theorem implies  $d\mathcal{C}_i \subset \mathcal{C}_i$ ,  $i = 1, 2$ .

6. Cyclic Case.

Since  $h_1$  and  $h_2$  are cyclic, one may regard  $\mathcal{C}_1$  and  $\mathcal{C}_2$  as being generated by elements of the form

$$\mathcal{C}_1 = \{\varphi^0, \dots, \varphi^p\}, \quad \mathcal{C}_2 = \{\varphi^{p+1}, \dots, \varphi^n\}$$

where  $\varphi^0$  generates  $\mathcal{E}_1$  and  $\varphi^{p+1}$  generates  $\mathcal{E}_2$ . That is,

$$\begin{aligned} h_1 \varphi^j &= \varphi^{j+1}, & 0 \leq j < p \\ h_1 \varphi^p &= b_0 \varphi^0 + \dots + b_p \varphi^p \\ h_2 \varphi^k &= \varphi^{k+1}, & p+1 \leq k < n \\ h_2 \varphi^n &= c_0 \varphi^{p+1} + \dots + c_{n-p-1} \varphi^n, \end{aligned}$$

where  $b_j$  and  $c_k \in A$ . The  $b_j$  in fact are symmetric functions of the eigenvalues  $\lambda_0, \dots, \lambda_p$  and the  $c_k$  are symmetric functions of the eigenvalues  $\lambda_{p+1}, \dots, \lambda_n$ . Since  $d\mathcal{C}_i \subset \mathcal{C}_i$  there exist coordinates  $(y^0, \dots, y^p; z^{p+1}, \dots, z^n)$  such that  $\varphi^j = dy^j$  where  $0 \leq j \leq p$  and also such that  $\varphi^k = dz^k$  where  $p+1 \leq k \leq n$ .

It is easy to verify that  $\frac{\partial}{\partial y^p}$  is a generator with respect to  $h_1$  of the space  $E_1$  which is dual to  $\mathcal{E}_1$ . It is somewhat harder to verify that  $\frac{\partial}{\partial c_0}$  is a generator with respect to  $h_2$  of  $E_2$ . The idea of the proof, which is similar to that contained in Lemma 4.2, is as follows. If  $\Lambda$  is the Vandermonde determinant of the eigenvalues  $\lambda_{p+1}, \dots, \lambda_n$ , one finds that the relations

$$\begin{aligned} h_2 dc_j &= dc_{j-1} + c_j dc_{n-p-1} \\ dc_0 \wedge \dots \wedge dc_{n-p-1} &= (-1)^{(n-p-1)(n-p)/2} dc_{n-p-1} \wedge \dots \wedge h_2^{n-p-1} dc_{n-p-1} \\ &= (-1)^{(n-p-1)(n-p)/2} \Lambda d\lambda_{p+1} \wedge \dots \wedge d\lambda_n \end{aligned}$$

are consequences of the vanishing of  $[h_2, h_2]$  and the distinctness of the eigenvalues. Thus  $dc_{n-p-1}$  generates  $\mathcal{E}_2$  with respect to  $h_2$ , and with respect to the basis  $(dc_0, \dots, dc_{n-p-1})$  of  $\mathcal{E}_2$ ,  $h_2$  is represented by the companion matrix

$$(h_2) = \begin{pmatrix} 0 & & & c_0 \\ 1 & & & \vdots \\ \vdots & \ddots & & \vdots \\ 0 & & 1 & c_{n-p-1} \end{pmatrix}$$

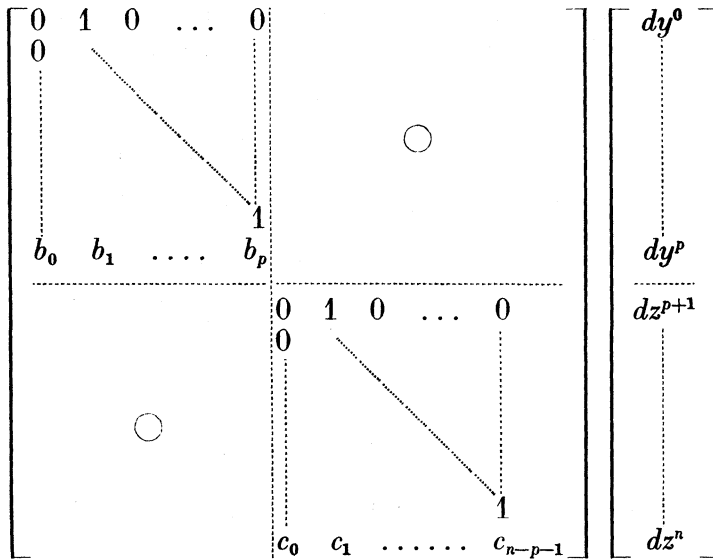


One then concludes that  $\frac{\partial}{\partial c_0}$  generates  $E_2$  and that

$$\left( \frac{\partial}{\partial c_0}, \dots, \frac{\partial}{\partial c_{n-p-1}} \right)$$

is a cyclic basis of  $E_2$ . These results may be summarized in the following theorem.

**THEOREM 6.1.** — *If  $[h, h] = 0$  and if the eigenvalues of  $h$  are distinct, with  $p \leq n$  eigenvalues constant in a neighborhood of a point  $P$  of an analytic manifold, then there exist generators  $N_i$  of  $E_i$  such that the bases  $(N_1, \dots, N_1 h_1^p)$  and  $(N_2, \dots, N_2 h_2^{n-p-1})$  of  $E_1$  and  $E_2$  respectively all commute. That is, there exist coordinates  $(y^0, \dots, y^p; z^{p+1}, \dots, z^n)$  such that  $(dy^0, \dots, dy^p)$  and  $(dz^{p+1}, \dots, dz^n)$  are bases of  $\mathcal{E}_1$  and  $\mathcal{E}_2$  respectively. Moreover, the action of  $h$  on the basis  $(dy^0, \dots, dy^p) \cup (dz^{p+1}, \dots, dz^n)$  of  $\mathcal{E}$  is given (schematically) by*



The last statement then guarantees that if one solves the problem of conservation laws in  $\mathcal{E}_1$  and  $\mathcal{E}_2$  separately, then the problem is also solved in  $\mathcal{E}$ . Since the last theorem also removes the commutativity restriction imposed in [4], at least for the case of distinct eigenvalues, one is now in a position to obtain conservation laws.

## BIBLIOGRAPHY

- [1] A. FROLICHER and A. NIJENHUIS, Theory of vector-valued differential forms, I. *Proc. Ned. Akad. Wet. Amsterdam*, 59 (1956), 338-359.
- [2] A. NIJENHUIS,  $X_{n-1}$ -forming sets of eigenvectors. *Proc. Ned. Akad. Wet. Amsterdam*, 54 (1951).
- [3] H. OSBORN, The Existence of Conservation Laws, I. *Ann. of Math.*, 69 (1959), 105-118.
- [4] H. OSBORN, Les lois de conservation, *Ann. Inst. Fourier, Grenoble*, 14 (1964), 71-82.

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