ANNALES

DE

L'INSTITUT FOURIER

Nicolas DUTERTRE

Semi-algebraic neighborhoods of closed semi-algebraic sets

Tome 59, nº 1 (2009), p. 429-458.

<http://aif.cedram.org/item?id=AIF_2009__59_1_429_0>

Ann. Inst. Fourier, Grenoble
59, 1 (2009) 429-458

SEMI-ALGEBRAIC NEIGHBORHOODS OF CLOSED SEMI-ALGEBRAIC SETS

by Nicolas DUTERTRE

ABSTRACT. — Given a closed (not necessarly compact) semi-algebraic set X in $\mathbb{R}^n$, we construct a non-negative semi-algebraic $\mathcal{C}^2$ function f such that $X=f^{-1}(0)$ and such that for $\delta > 0$ sufficiently small, the inclusion of X in $f^{-1}([0,\delta])$ is a retraction. As a corollary, we obtain several formulas for the Euler characteristic of X.

RÉSUMÉ. — Étant donné un ensemble semi-algébrique fermé (non nécessairement compact) X de $\mathbb{R}^n$, nous construisons une fonction semi-algébrique f positive et de classe $\mathcal{C}^2$ telle que $X = f^{-1}(0)$ et telle que pour $\delta > 0$ suffisamment petit, l'inclusion de X dans $f^{-1}([0,\delta])$ soit une rétraction. En corollaire, nous obtenons plusieurs formules pour la caractéristique d'Euler de X.

1. Introduction

Let X be a compact algebraic set in $\mathbb{R}^n$. The set X is the set of zeros of a nonnegative polynomial function f. This function f may not be proper as it is explained by the following example due to H. King: let

$$f(x,y) = (x^2+y^2)\big(\big(y(x^2+1)-1\big)^2+y^2\big),$$

then $f^{-1}(0)=\{0\}$ but $f(x,(1+x^2)^{-1}) \to 0$ as $|x| \to +\infty$.

Durfee [8] proved that any compact algebraic set X can be written as the set of zeros of a proper nonnegative polynomial function g. Following Thom's terminology, he called such a function a rug function for X. Then he defined the notion of algebraic neighborhood: a subset T with $X \subset T \subset \mathbb{R}^n$ is an algebraic neighborhood of X in $\mathbb{R}^n$ if $T = g^{-1}([0,\delta])$, where g is a rug function for X and δ is a positive real smaller than all nonzero critical

Keywords: Tubular neighborhood, semi-algebraic sets, retraction, quasiregular approaching semi-algebraic function, quasiregular approaching semi-algebraic neighborhood.
Math. classification: 14P10, 14P25.

values of g. Using the gradient vector field of g, he showed that the inclusion $X \subset T$ is a homotopy equivalence. Thanks to Lojasiewicz's work [19], [20] on the trajectories of a gradient vector field, it is not difficult to see that this homotopy equivalence is actually a retraction. Durfee also proved that two algebraic neighborhoods of a compact algebraic set are isotopic. Here also, this uniqueness result is obtained integrating appropriate gradient vector fields. He extended next these results to the case of a compact semi-algebraic subset X of a semi-algebraic set M of $\mathbb{R}^n$. He defined the notion of a semi-algebraic neighborhood of X in M and proved that the inclusion of X in such a neighborhood is a homotopy equivalence. One should mention that similar results were obtained by Coste and Reguiat [7] in the case of a real closed field using technics of the real spectrum. They obtained a semi-algebraic retraction theorem for any compact semi-algebraic set.

If X is a non-compact algebraic set in $\mathbb{R}^n$ and f is a nonnegative polynomial such that $X = f^{-1}(0)$, then X is not in general a deformation retract of $f^{-1}([0,\delta])$, where δ is a small regular value of f. Let

$$f(x,y) = \big[y(xy-1)\big]^2$$

(f is the square of the Broughton polynomial [4]) and let $X = f^{-1}(0)$. For δ a sufficiently small positive regular value of f, $f^{-1}([0,\delta])$ has one connected components whereas X has three.

Our aim is to extend Durfee's results to the case of closed (not necessarily compact) semi-algebraic sets. More precisely, we consider a closed semi-algebraic set X in $\mathbb{R}^n$ and an open semi-algebraic neighborhood U of X in $\mathbb{R}^n$. We say that $f : U \to \mathbb{R}$ is an approaching function for X in U (Definition 2.3) if

1) f is semi-algebraic, $\mathcal{C}^2$, nonnegative;
2) $X = f^{-1}(0)$;
3) there exists $\delta > 0$ such that $f^{-1}([0,\delta])$ is closed in $\overline{U}$.

However, the notion of approaching function is not enough to get a deformation retract as it is suggested by the Broughton example above. Let $\rho : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ be a proper C^2 semi-algebraic function, let $f : U \to \mathbb{R}$ be a $\mathcal{C}^2$ nonnegative semi-algebraic function such that $X = f^{-1}(0)$ and let $\Gamma_{f,g}$ be the set of points x in $U \setminus X$ where $\nabla f(x)$ and $\nabla \rho(x)$ are colinear (here ∇f denotes the gradient vector field of f). We say that f is ρ-quasiregular (Definition 2.5) if there does not exist a sequence $(x_k)_{k\in\mathbb{N}}$ of points in $\Gamma_{f,\rho}$ such that $\|x_k\| \to +\infty$ and $f(x_k) \to 0$. A ρ-quasiregular approaching semi-algebraic neighborhood of X in U (Definition 3.1) is defined as a set $T = f^{-1}([0,\delta])$ such that:

1) f is a ρ-quasiregular approaching function for X in U;
2) δ is a positive number smaller than all nonzero critical values of f;
3) $f^{-1}([0,\delta])$ is closed in $\overline{U}$;
4) $\Gamma_{f,\rho}$ does not intersect $f^{-1}([0,\delta])$ outside a compact subset K of $\mathbb{R}^n$.

We say that a set is an approaching semi-algebraic neighborhood of X in U if it is a ρ-quasiregular approaching semi-algebraic neighborhood of X in U for some function ρ.

We prove that ρ-quasiregular approaching semi-algebraic neighborhoods always exist (Corollary 2.7) and that if $T = f^{-1}([0,\delta])$ is a ρ-quasiregular approaching semi-algebraic neighborhood of X in U then X is a strong deformation retract of T (Theorem 3.2). In order to construct this retraction, we study a vector field w that is equal to the gradient of f inside a compact subset of $\mathbb{R}^n$ and to the orthogonal projection of the gradient of f onto the levels of ρ outside a compact set. Using the Lojasiewicz inequality with parameters due to Fekak [10] and the usual Lojasiewicz gradient inequality we establish an inequality of “Lojasiewicz's type” for the norm of w. The retraction is then achieved “pushing” $T = f^{-1}([0,\delta])$ along the trajectories of w.

After we show that two ρ-quasiregular approaching semi-algebraic neighborhoods of X are isotopic (Theorem 4.1). As above, the isotopy is obtained integrating a vector field which is equal to a gradient vector field on a compact set of $\mathbb{R}^n$ and to the projection of this gradient vector field onto the levels of ρ at infinity.

As a corollary, this enables us to prove that when X is smooth of class $\mathcal{C}^3$, every approaching semi-algebraic neighborhood of X is isotopic to a tubular neighborhood of X (Theorem 5.7).

Then we prove that two approaching semi-algebraic neighborhoods of X are isotopic (Corollary 6.6).

We end the paper with degree formulas for the Euler-Poincaré characteristic of any closed semi-algebraic set obtained thanks to the machinery developped before (Theorem 7.3, Corollary 7.4 and Corollary 7.5), and with a Petrovskii-Oleinik inequality for the Euler-Poincaré characteristic of any real algebraic set (Proposition 7.8).

The author is very grateful to Zbigniew Szafraniec, Vincent Grandjean, Didier D'Acunto and Andreas Bernig for valuable discussions on this topic.

2. ρ-quasiregular approaching functions

In this section, we define the notion of a ρ-quasiregular approaching function for a closed semi-algebraic set, which generalizes the notion of a rug function introduced by Durfee [8].

Let us consider a closed semi-algebraic set X in $\mathbb{R}^n$. Let U be an open semi-algebraic neighborhood of X. We know that X is the zero set in U of a continuous nonnegative semi-algebraic function $f : U \to \mathbb{R}$ (for example one can take for f the restriction to U of the distance function to X). For any $\delta > 0$, the set $f^{-1}([0,\delta])$ is closed in U for the induced topology. However, even if δ is very small, it is not necessary closed in $\overline{U}$, as it is shown in the following examples.

Example 1. — The set $X = \{0\}$ is a closed semi-algebraic set in $\mathbb{R}$, the set $U =]-1,+\infty[$ is an open semi-algebraic neighborhood of X in $\mathbb{R}$. Let $f : U \to \mathbb{R}$ be defined by $f(x) = x^2(x+1)$. It is clear that for any $\delta > 0$, the set $f^{-1}([0,\delta])$ is not closed in $\overline{U} = [-1,+\infty[$.

Example 2. — The set $X = \{(x,y) \in \mathbb{R}^2 \mid y = 0\}$ is a closed semi-algebraic set in $\mathbb{R}^2$, the set $U = \{(x,y) \in \mathbb{R}^2 \mid x^2y^2 < 1\}$ is an open semi-algebraic neighborhood of X in $\mathbb{R}^2$. Let $f : U \to \mathbb{R}$ be defined by $f(x,y) = y^2$. For any $\delta > 0$, the set $f^{-1}([0,\delta])$ is not closed in $\overline{U} = \{(x,y) \in \mathbb{R}^2 \mid x^2y^2 \leqslant 1\}$.

We would like to avoid this situation. For this we need to put a condition on the tuple (X,U,f).

Definition 2.1. — *Let X be a closed semi-algebraic set in $\mathbb{R}^n$, let U be an open neighborhood of X and let $f : U \to \mathbb{R}$ be a nonnegative continuous semi-algebraic function such that $X = f^{-1}(0)$. We say that (X,U,f) satisfies condition (A) if there does not exist a sequence $(x_k)_{k\in\mathbb{N}}$ of points in U such that $\lim_{k\to+\infty} f(x_k) = 0$ and such that $\lim_{k\to+\infty} x_k$ exists and belongs to $\mathrm{Bd}(U) = \overline{U} \setminus U$.*

It is clear that this condition is satisfied when $U = \mathbb{R}^n$. Let us remark that for any couple (X,U), X being a closed semi-algebraic set in $\mathbb{R}^n$ and U an open semi-algebraic neighborhood of X, there exists a function f such that (X,U,f) satisfies condition (A). If $d : \mathbb{R}^n \to \mathbb{R}$ is the distance function to X then the tuple $(X,U,d_{|U})$ satisfies condition (A).

We will explain how to construct from a function f such that (X,U,f) satisfies condition (A), a nonnegative continuous semi-algebraic function g

such that $X = g^{-1}(0)$ and $g^{-1}([0,\delta])$ is closed in $\overline{U}$ for δ small enough. Actually we will prove a stronger result.

Let us fix a proper $\mathcal{C}^2$ semi-algebraic function $\rho : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$. We will denote by Σ_r the set $\rho^{-1}(r)$, by D_r the set $\rho^{-1}([0,r])$ and by E_r the set $\rho^{-1}([r,+\infty[)$. Note that for r sufficiently big, Σ_r is a non-empty compact $\mathcal{C}^2$-submanifold of $\mathbb{R}^n$. We will call such a ρ a control function.

Lemma 2.2. — *Let X be a closed semi-algebraic set in $\mathbb{R}^n$, let U be an open semi-algebraic neighborhood of X and let $f : U \to \mathbb{R}$ be a continuous nonnegative semi-algebraic function such that $X = f^{-1}(0)$ and (X,U,f) satisfies condition (A). For every integer $q \geqslant 0$, let $f_q : U \to \mathbb{R}$ be defined by $f_q = (1+\rho)^q f$. Let $V \subset U$ be an open semi-algebraic neighborhood of X. There exists an integer q_0 such that for every integer $q \geqslant q_0$, there exists $\delta_q > 0$ such that $f_q^{-1}([0,\delta_q])$ is included in V and closed in $\overline{V}$. Furthermore, if X is compact then one can choose q_0 such that for every integer $q \geqslant q_0$, $f_q^{-1}([0,\delta_q])$ is compact in $\overline{V}$.*

Proof. — Let Z be the closed semi-algebraic set $\overline{U} \setminus V$. Let $d : \mathbb{R}^n \to \mathbb{R}$ be a continuous nonnegative semi-algebraic function such that $X = d^{-1}(0)$ and $Z = d^{-1}(1)$. Let U_1 be the open semi-algebraic neighborhood of X in $\mathbb{R}^n$ defined by $U_1 = d^{-1}([0,\frac{1}{2}[)$ and let V_1 be the open semi-algebraic neighborhood of X in U defined by $V_1 = U_1 \cap U$. It is straightforward to see that $\overline{V_1} \subset V$.

Let us study first the case when U is bounded. There exists $\delta > 0$ such that $f^{-1}([0,\delta]) \subset V_1$. Otherwise, we would be able to construct a sequence of points $(x_k)_{k\in\mathbb{N}}$ in $\overline{U} \setminus V_1$ such that $\lim_{k\to+\infty} f(x_k) = 0$. By compactness of $\overline{U} \setminus V_1$, there would exist a subsequence of points $(x_{\varphi(k)})_{k\in\mathbb{N}}$ in $\overline{U} \setminus V_1$ such that $f(x_{\varphi(k)})$ tends to 0 and $x_{\varphi(k)}$ tends to a point y in $\overline{U} \setminus V_1$. If y belongs to U then $f(y) = 0$, which is impossible. So y belongs to $\overline{U} \setminus U$, which is also impossible by condition (A). Since $\overline{V_1}$ is included in V and bounded, the set $f^{-1}([0,\delta])$ is compact in $\overline{V}$.

If U is not bounded and X is not compact, then the semi-algebraic set $F = U \setminus V_1$ is unbounded as well. There exists r_0 such that for every $r \geqslant r_0$, $\Sigma_r \cap F$ is not empty (the set $\{r \in \mathbb{R} \mid \Sigma_r \cap F \neq \emptyset\}$ is an unbounded semi-algebraic set of $\mathbb{R}$). Let $\alpha : [r_0,+\infty[\to \mathbb{R}$ be defined by

$$\alpha(r) = \inf\big\{f(x) \mid x \in \Sigma_r \cap F\big\}.$$

The function α is a semi-algebraic function. Let us show that it is positive. If $\alpha(r) = 0$ then there exists a sequence of points $(x_k)_{k\in\mathbb{N}}$ in $F \cap \Sigma_r$ such that $f(x_k)$ tends to 0. By compactness of Σ_r, we can extract a subsequence $(x_{\varphi(k)})_{k\in\mathbb{N}}$ such that $f(x_{\varphi(k)})$ tends to 0 and $x_{\varphi(k)}$ tends to a point y

in $\Sigma_r \cap \overline{F}$, which is included in $\Sigma_r \cap \overline{U}$. If y belongs to U then $f(y) = 0$ and so y belongs to X, which is impossible for $d(y) \geqslant \frac{1}{2}$. Hence y is in $\mathrm{Bd}(U)$. This is impossible by condition (A). The function α^{-1} is semi-algebraic. From Proposition 2.11 in [6] (see also Proposition 2.6.1 in [2]), there exists $r_1 \geqslant r_0$ and an integer q_0 such that $\alpha(r)^{-1} < r^q$ for every $r \geqslant r_1$ and every integer $q \geqslant q_0$. This implies that for every x in $F \cap E_{r_1}$ and for $q \geqslant q_0$, $f_q(x) = (1 + \rho(x))^q f(x) 1$. It is clear that (X, U, f_q) satisfies condition (A). The same argument as in the case U bounded shows that there exists ϵ_q such that $f_q^{-1}([0, \epsilon_q]) \cap D_{r_1}$ is included in $V_1 \cap D_{r_1}$. We take for δ_q the minimum of 1 and ϵ_q. Since $\overline{V_1} \subset V$, it is easy to see that $f_q^{-1}([0, \delta_q])$ is closed in $\overline{V}$.

It remains to study the case U unbounded but X compact. There exists $r_2 > 0$ such that $X \cap E_{r_2}$ is empty. Let $\beta : [r_2, +\infty[\to \mathbb{R}$ be defined by

$$\beta(r) = \inf \big\{ f(x) \mid x \in U \cap \Sigma_r \big\}.$$

Thanks to condition (A), we can prove that it is a positive semi-algebraic function. There exists $r_3 \geqslant r_2$ and an integer q_1 such that $\beta(r)^{-1} < r^q$ for every $r \geqslant r_3$ and every integer $q \geqslant q_1$. Hence for $x \in U \cap E_{r_3}$ and for $q \geqslant q_1$, $f_q(x) = (1 + \rho(x))^q f(x) > 1$. The tuple (X, U, f_q) satisfies condition (A). As in the previous cases, there exists $\epsilon_q > 0$ such that $f_q^{-1}([0, \epsilon_q]) \cap D_{r_3}$ is included in $V_1 \cap D_{r_3}$.We take for δ_q the minimum of 1 and ϵ_q. The set $f_q^{-1}([0, \delta_q])$ is compact in $\overline{V_1}$ because it is compact in $\mathbb{R}^n$. □

Definition 2.3. — *Let X be a closed semi-algebraic set in $\mathbb{R}^n$ and let U be an open semi-algebraic neighborhood of X in $\mathbb{R}^n$. A function $f : U \to \mathbb{R}$ is called an approaching function for X in U if*

1) *f is semi-algebraic, $\mathcal{C}^2$, nonnegative;*
2) *$X = f^{-1}(0)$;*
3) *there exists $\delta > 0$ such that $f^{-1}([0, \delta])$ is closed in $\overline{U}$. Furthermore if X is compact then $f^{-1}([0, \delta])$ is compact in $\overline{U}$.*

Proposition 2.4. — *Let X be a closed semi-algebraic set in $\mathbb{R}^n$ and let U be an open semi-algebraic neighborhood of X in $\mathbb{R}^n$. There exist approaching functions for X in U.*

Proof. — From [25, Corollary C.12], it is possible to find a $\mathcal{C}^2$ semi-algebraic function $\phi : \mathbb{R}^n \to [0, 1]$ such that $X = \phi^{-1}(0)$ and $\mathrm{Bd}(U) = \phi^{-1}(1)$. Let f be the restriction of ϕ to U. The tuple (X, U, f) satisfies condition (A). Applying Lemma 2.2 to f and U, we can construct approaching functions for X in U. □

We will need a definition. For every open semi-algebraic set U and for every $\mathcal{C}^2$ semi-algebraic function $g : U \to \mathbb{R}$, let $\Gamma_{g,\rho}$ be the semi-algebraic set defined by

$$\Gamma_{g,\rho} = \big\{x \in U \mid \nabla g(x) \text{ and } \nabla\rho(x) \text{ are colinear and } g(x) \neq 0\big\}.$$

Definition 2.5. — *Let $g : U \to \mathbb{R}$ be a $\mathcal{C}^2$ semi-algebraic function. We say that g is ρ-quasiregular if there does not exist a sequence $(x_k)_{k\in\mathbb{N}}$ in $\Gamma_{g,\rho}$ such that $\|x_k\|$ tends to infinity and $|g(x_k)|$ tends to 0.*

This notion of ρ-quasiregularity is a slight modification of the notion of ρ-regularity due to Tibar [24]. Note that our definition does not imply that $g^{-1}(0)$ has only isolated singularities, unlike Tibar's definition.

Proposition 2.6. — *Let X be a closed semi-algebraic set in $\mathbb{R}^n$ and let U be an open semi-algebraic neighborhood of X. Let $f : U \to \mathbb{R}$ be a $\mathcal{C}^2$ semi-algebraic nonnegative function such that $X = f^{-1}(0)$. For every integer q, let $f_q : U \to \mathbb{R}$ be defined by*

$$f_q = (1+\rho)^q f.$$

There exists an integer q_0 such that for every integer $q \geqslant q_0$, the function f_q is ρ-quasiregular.

Proof. — Let r_0 be the greatest critical value of ρ and let $\beta :]r_0, +\infty[\to \mathbb{R}$ be defined by

$$\beta(r) = \inf\big\{f(x) \mid x \in \Sigma_r \cap \Gamma_{f,\rho}\big\}.$$

The function β is semi-algebraic. It is positive since for $r > r_0$, the function $f_{\mid \Sigma_r\cap U}$ admits a finite number of critical values. As in Lemma 2.2, this implies that there exists $r_1 > r_0$ and an integer q_0 such that for $x \in \Gamma_{f,\rho} \cap E_{r_1}$ and for $q \geqslant q_0$, $(1+\rho(x))^q f(x) > 1$. Since $\Gamma_{f,\rho} = \Gamma_{f_q,\rho}$, every function f_q is ρ-quasiregular for $q \geqslant q_0$. □

Corollary 2.7. — *Let X be a closed semi-algebraic set in $\mathbb{R}^n$ and let U be an open semi-algebraic neighborhood of X. Let $f : U \to \mathbb{R}$ be a $\mathcal{C}^2$ semi-algebraic nonnegative function such that $X = f^{-1}(0)$. Assume that (X, U, f) satisfies condition (A). For every integer $q \geqslant 0$, let $f_q : U \to \mathbb{R}$ be defined by $f_q = (1+\rho)^q f$. There exists an integer q_0 such that for every $q \geqslant q_0$, the function f_q is a ρ-quasiregular approaching function for X in U.* □

If X is an algebraic set, it is the zero set of a nonnegative polynomial f. Choosing for ρ a proper nonnegative polynomial and applying the above process, we obtain ρ-quasiregular approaching functions for X that are nonnegative polynomials.

Let us compare our notion of ρ-quasiregular approaching function with the notion of rug function due to Durfee [8]. If X is a compact algebraic set of $\mathbb{R}^n$, a rug function for X is a proper nonnegative polynomial f such that $X = f^{-1}(0)$. It is clear that such a function is a ρ-quasiregular approaching function for X in $\mathbb{R}^n$.

3. Retraction on a closed semi-algebraic set

In this section, we prove that any closed semi-algebraic set is a strong deformation retract of certain closed semi-algebraic neighborhoods of it. First let us specify the closed semi-algebraic neighborhoods that we will consider.

Definition 3.1. — *Let $X \subset \mathbb{R}^n$ be a closed semi-algebraic set, let ρ be a control function and let U be an open semi-algebraic neighborhood of X. A subset T with $X \subset T \subset U$ is a ρ-quasiregular approaching semi-algebraic neighborhood of X in U if $T = f^{-1}([0,\delta])$ where*

1) *f is a ρ-quasiregular approaching function for X in U;*
2) *δ is a positive number smaller than all nonzero critical values of f;*
3) *$f^{-1}([0,\delta])$ is closed in $\overline{U}$ and compact in $\overline{U}$ if X is compact;*
4) *if $\Gamma_{f,\rho}$ is the polar set*

$$\Gamma_{f,\rho} = \big\{x \in U \setminus X \mid \nabla f(x) \text{ and } \nabla \rho(x) \text{ are colinear}\big\},$$

then $\Gamma_{f,\rho}$ does not intersect $f^{-1}([0,\delta])$ outside a compact subset K of $\mathbb{R}^n$.

For short, we will say that such a T is an approaching semi-algebraic neighborhood. By the results of the previous section, it is clear that approaching semi-algebraic neighborhoods always exist.

Theorem 3.2. — *Let X be a closed semi-algebraic set and let T be an approaching semi-algebraic neighborhood of X. Then X is a strong deformation retract of T.*

Proof. — If X is compact, this is already proved by Durfee [8] and Lojaziewicz [19], [20]. So let us assume that X is not compact.

Let us fix f, U, δ, ρ and K which satisfy the conditions of the above definition and such that $T = f^{-1}([0,\delta])$. Furthermore let us assume that $\delta < 1$. We will focus first on the behaviour of f at infinity.

Let $r_0 > 0$ be such that $K \cap E_{r_0}$ is empty and such that Σ_r is a $\mathcal{C}^2$ submanifold for $r \geqslant r_0$. Let $A = T \cap E_{r_0}$. The set A is a closed semi-algebraic set of $\mathbb{R}^n$ and $A \cap \Gamma_{f,\rho}$ is empty. Let us consider the following closed semi-algebraic set Y of $\mathbb{R}^{n+1}$:

$$Y = \big\{(x,t) \in \mathbb{R}^{n+1} \mid x \in A \text{ and } \rho(x) = t\big\}.$$

We will denote by Y_t the fibre $\{x \in A \,|\, (x,t) \in Y\}$. Observe that $Y_t{=}A \cap \Sigma_t$. Let $F : A \to \mathbb{R}$ be defined by

$$F = \Big\| \nabla f - \Big\langle \nabla f, \frac{\nabla\rho}{\|\nabla\rho\|}\Big\rangle \frac{\nabla\rho}{\|\nabla\rho\|}\Big\|.$$

The function F is just the norm of the orthogonal projection of $\nabla f(x)$ on the manifold $\Sigma_{\rho(x)}$. Moreover it is a continuous semi-algebraic function on A. Let $\tilde{f}$ and $\widetilde{F}$ be the semi-algebraic functions defined on Y by $\tilde{f}(x,t) = f(x)$ and $\widetilde{F}(x,t) = F(x)$. They are continuous in x and verify $\widetilde{F}^{-1}(0) \subset \tilde{f}^{-1}(0)$. This inclusion is easy to check since $F(x) = 0$ if and only if $\nabla f(x)$ and $\nabla\rho(x)$ are colinear. On A, this can occur only if x belongs to X.

We can apply Lojasiewicz's inequality with parameters due to Fekak (see [10, p. 128]). We need some notations: for every t, $\tilde{f}_t$ and $\widetilde{F}_t$ are the functions on Y_t defined by $\tilde{f}_t(x) = \tilde{f}(x,t)$ and $\widetilde{F}_t(x) = \widetilde{F}(x,t)$; for every $S \subset \mathbb{R}$, Y_S denotes the set $Y \cap (\mathbb{R}^n \times S)$. Fekak's Theorem states that there exists a finite partition into semi-algebraic subsets of $\mathbb{R} = \bigcup S_i$, continuous semi-algebraic functions $h_i : Y_{|S_i} \to \mathbb{R}$ and rationnal numbers p_i/q_i such that:

i) $|\tilde{f}(x,t)|^{p_i/q_i} \leqslant h_i(x,t)|\widetilde{F}(x,t)|$ on $Y_{|S_i}$ for $t \in S_i$;

ii) p_i/q_i is the Lojasiewicz exponent of $\tilde{f}_t$ with respect to $\widetilde{F}_t$ for $t \in S_i$.

Since $\bigcup S_i$ is a finite semi-algebraic partition of $\mathbb{R}$, there exist $t_0 \in \mathbb{R}$ and i_0 such that $S_{i_0} = [t_0, +\infty[$. Then for every $t \geqslant t_0$, we have:

i) $|\tilde{f}(x,t)|^{p_{i_0}/q_{i_0}} \leqslant h_{i_0}(x,t)|\widetilde{F}(x,t)|$ for $x \in Y_t$;

ii) p_{i_0}/q_{i_0} is the Lojasiewicz exponent of $\tilde{f}_t$ with respect to $\widetilde{F}_t$.

We know that $\tilde{f}_t = f_{|Y_t}$ and $\widetilde{F}_t = \|\nabla(f_{|Y_t})\|$. By Lojasiewicz's gradient inequality applied to $f_{|Y_t}$, we get $p_{i_0}/q_{i_0} < 1$. Let $\alpha = p_{i_0}/q_{i_0}$ and let $B = T \cap E_{t_0}$. We have proved that there exist $0 \leqslant \alpha < 1$ and a continuous semi-algebraic function $h : B \times [t_0, +\infty[\to \mathbb{R}$ such that for every $x \in B$

$$\big|f(x)\big|^{\alpha} \leqslant h(x,\rho(x))F(x),$$

where F is the norm of the vector field

$$v = \nabla f - \Big\langle \nabla f, \frac{\nabla\rho}{\|\nabla\rho\|}\Big\rangle \frac{\nabla\rho}{\|\nabla\rho\|}.$$

Let C be the compact semi-algebraic set defined by $C = T \cap D_{2t_0}$. By the Lojasiewicz gradient inequality, there exits $d > 0$ and $0 \leqslant \beta < 1$ such that on C

$$|f|^\beta \leqslant d \|\nabla f\|.$$

Here we have applied the Kurdyka-Parusinski version of the Lojasiewicz gradient inequality [18].

We will glue the two vector fields v and ∇f. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^\infty$-function such that:

- $\varphi(x) = 1$ if $\rho(x) \leqslant 1.3 t_0$;
- $\varphi(x) = 0$ if $\rho(x) \geqslant 1.7 t_0$;
- $0 < \varphi(x) < 1$ if $1.3 t_0 < \rho(x) < 1.7 t_0$.

Let w be the following vector field on T:

$$w = (1-\varphi)\, v + \varphi \nabla f.$$

We want to find an inequality of "Lojasiewicz's type" for $\|w\|$. First observe that $\|w\| \geqslant \|v\|$, for

$$w = v + \varphi \Big\langle \nabla f, \frac{\nabla \rho}{\|\nabla \rho\|} \Big\rangle \frac{\nabla \rho}{\|\nabla \rho\|}.$$

Let M be defined by

$$M = \max \big\{ h(x, \rho(x)) \mid x \in T \text{ and } 1.2 t_0 \leqslant \rho(x) \leqslant 1.8 t_0 \big\}.$$

We have $|f(x)|^\alpha \leqslant M \|w(x)\|$ for $x \in T \cap \{x \mid 1.2 t_0 \leqslant \rho(x) \leqslant 1.8 t_0\}$. For $x \in T \cap D_{1.3 t_0}$, we have $|f(x)|^\beta \leqslant d \|\nabla f(x)\|$ and $\nabla f(x) = w(x)$. Calling γ the maximum of α and β and N the maximum of M and d and since $\delta < 1$, we get that for $x \in T \cap D_{1.8 t_0}$,

$$\big|f(x)\big|^\gamma \leqslant N \big\|w(x)\big\|. \tag{1}$$

Now for $x \in T \cap E_{1.7 t_0}$, $w(x) = v(x)$ and then

$$\big|f(x)\big|^\gamma \leqslant h\big(x, \rho(x)\big) \big\|w(x)\big\|. \tag{2}$$

On one hand, we have $\langle \nabla f, w \rangle = (1-\varphi) \langle \nabla f, v \rangle + \varphi \langle \nabla f, \nabla f \rangle$, hence

$$\langle \nabla f, w \rangle = (1-\varphi) \langle v, v \rangle + \varphi \langle \nabla f, \nabla f \rangle,$$

since $\langle v, \nabla f \rangle = \langle v, v \rangle$. On the other hand,

$$\langle w, w \rangle = (1-\varphi^2) \langle v, v \rangle + \varphi^2 \langle \nabla f, \nabla f \rangle.$$

Using the fact that $0 \leqslant \varphi \leqslant 1$, it is easy to see that

$$\langle \nabla f, w \rangle \geqslant \langle w, w \rangle \iff \langle \nabla f, \nabla f \rangle \geqslant \langle v, v \rangle.$$

Since the inequality on the right hand side is verified, we have proved

$$\langle \nabla f, w \rangle \geqslant \langle w, w \rangle. \tag{3}$$

We are going to integrate the vector field $-w/\|w\|$. It is defined on $T \setminus X$. Let ϕ_t be the flow associated with the differential equation:

$$\dot{x} = -\frac{w}{\|w\|}\cdot$$

For every $x \in T$, let

$$b(x) = \sup \big\{ t \mid f(\phi_t(x)) \geqslant 0 \big\} \quad \text{and} \quad \omega(x) = \lim_{t \to b(x)} \phi_t(x).$$

We write $\phi_x(t)$ the trajectory that passes through x. We extend b and ω on T setting $b(x) = 0$ and $\omega(x) = x$ for all $x \in X$. The following facts are proved using inequalities (1), (2) and (3) and adapting to our situation the techniques of Lojasiewicz (see [19], [20], [16], [17] or [22] for details).

Fact 1. — For all $x \in T$, $\{\phi_x(t) \mid 0 \leqslant t \leqslant b(x)\} \subset T$.

Fact 2. — For all $x \in T \cap E_{1.7t_0}$, for all t such that $0 \leqslant t \leqslant b(x)$, $\|\phi_x(t)\| = \|x\|$.

Fact 3. — For all $x \in T \cap D_{1.8t_0}$, for all t such that $0 \leqslant t \leqslant b(x)$, $\|\phi_x(t)\| \leqslant 1.8t_0$.

Fact 4. — For all $x \in T$, $b(x) < +\infty$.

Fact 5. — For all $x \in T$, $f(\omega(x)) = 0$.

Fact 6. — The mapping $\omega : T \to X$, $x \mapsto \omega(x)$ is continuous.

Fact 7. — The mapping $b : T \to X$, $x \mapsto b(x)$ is continuous.

Now we can end the proof of Theorem 3.2. The retraction is given by the mapping: $G : [0,1] \times T \to T$ defined by $G(t,x) = \phi(tb(x), x)$ if $(t,x) \in [0,1[\times T \setminus X$ and $G(t,x) = \omega(x)$ otherwise.

If $\delta \geqslant 1$, we can push $f^{-1}([0,\delta])$ onto $f^{-1}([0,\delta'])$, $\delta' < 1$, along the trajectories of w. □

We end this section with a remark. Using the same method, one can prove the following result. Let $X \subset \mathbb{R}^n$ be a closed semi-algebraic set and let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative semi-algebraic function such that $X = f^{-1}(0)$. Let $\Gamma_{f,\rho}$ be the set

$$\Gamma_{f,\rho} = \big\{ x \in \mathbb{R}^n \mid \nabla f(x) \text{ and } \nabla \rho(x) \text{ are colinear and } f(x) \neq 0 \big\}.$$

Let r be a regular value of ρ. Assume that the following assumption is satisfied: there is no sequence of points (x_k) in $\Gamma_{f,\rho} \cap D_r$ such that $\rho(x_k) \to r$ and $f(x_k) \to 0$. Then for $\delta > 0$ sufficiently small, the inclusion $X \cap D_r \subset f^{-1}([0,\delta]) \cap D_r$ is a deformation retract.

For example, this result can be applied if f has only isolated critical points on its zero level and X intersects Σ_r transversally.

4. Uniqueness of ρ-quasiregular approaching neighborhoods

In this section, we prove that two ρ-quasiregular approaching semi-algebraic neighborhoods of a closed non-compact semi-algebraic set are isotopic. We will prove the following theorem.

Theorem 4.1. — *Let X be a closed non-compact semi-algebraic set and let ρ be a control function. If T_1 and T_2 are two ρ-quasiregular approaching semi-algebraic neighborhoods of X in U_1 and U_2 respectively then there is a continuous family of diffeomorphisms $h_t : \mathbb{R}^n \to \mathbb{R}^n$, $0 \leqslant t \leqslant 1$, such that:*

1) *h_0 is the identity;*
2) *for all t, $h_{t|X}$ is the identity;*
3) *$h_1(T_1) = T_2$.*

Proof. — Let us write $T_i = f_i^{-1}([0, \delta_i])$ where f_i is a ρ-quasiregular approaching function for X in U_i, $i = 1, 2$. We will prove our result adapting the ideas of Durfee [8]. There are three steps.

Let us first consider the case $f_1 = f_2 = f$ and $U_1 = U_2 = U$. We can assume without loss of generality that $\delta_1 < \delta_2$. Thanks to condition 4) in Definition 3.1, we see that $f^{-1}(\delta)$ is ρ-regular at infinity (see [24]) for every δ in $[\delta_1, \delta_2]$. Since $[\delta_1, \delta_2]$ does not contain any critical value of f, Tibar's work implies that T_1 and T_2 are diffeomorphic. Let us be more precise and explain how the family h_t is obtained. As we did in the proof of Theorem 3.2, we can construct a vector field w on $f^{-1}([\delta_1, \delta_2])$ which is equal to the orthogonal projection of ∇f on the levels of ρ outside a set D_R, and equal to ∇f inside a set $D_{R'}$, $R' < R$. Then we extend w to a complete vector field $\widetilde{w}$ defined on $\mathbb{R}^n$ using a smooth function equal to 1 on the closed set $f^{-1}([\delta_1, \delta_2])$ and to 0 on the closed set $X \cup (\mathbb{R}^n \setminus U)$. Integrating this vector field gives the required family h_t.

The second case is when $f_2 = (1 + \rho)^q f_1$ and $U_1 = U_2 = U$. Let δ be the minimum of δ_1 and δ_2. Let v_1 (resp. v_2) be the orthogonal projection of ∇f_1 (resp. ∇f_2) on the levels of ρ. By condition 4) in Definition 3.1, there exists $R > 0$ such that v_1 and v_2 do not vanish in $f_1^{-1}(]0, \delta]) \cap E_R$. It is clear that on this set, they do not point in opposite direction. There exists a neighborhood U' of $X \cap D_{2R}$ in D_{2R} such that ∇f_1 and ∇f_2 are nonzero and do not point in opposite direction on $U' \setminus X$. This fact is proved in the same way as Lemma 1.8 in [8]. Hence there exists δ' such that ∇f_1 and ∇f_2 are nonzero and do not point in opposite direction on $f_1^{-1}(]0,\delta']) \cap D_{2R}$. Let δ'' be the minimum of δ and δ'. By the first case, it is enough to prove that $f_2^{-1}([0,\delta''])$ and $f_1^{-1}([0,\delta''])$ are isotopic. Let S be the closed set $f_1^{-1}([0,\delta'']) \setminus f_2^{-1}([0,\delta''[)$ and let $g : S \to [0,1]$ be defined by

$$g = \frac{f_2 - \delta''}{f_2 - f_1}.$$

Note that $g^{-1}(0) = f_2^{-1}(\delta'')$ and $g^{-1}(1) = f_1^{-1}(\delta'')$. The gradient of g is

$$\nabla g = \frac{(f_2 - \delta'')\nabla f_1 + (\delta'' - f_1)\nabla f_2}{(f_2 - f_1)^2}.$$

Let v be its orthogonal projection on the levels of ρ. It is nonzero in $S \cap E_R$. Moreover, ∇g is nonzero in $S \cap D_{2R}$. Gluing these two vector fields, we obtain a C^1 vector field w on S and we proceed as in the first case.

The third case is the general case. Let $U = U_1 \cap U_2$. By Lemma 2.2 and the second case above, we can assume that $T_1 \subset U$, $T_2 \subset U$ and T_1 and T_2 are closed in $\overline{U}$. We need some lemmas.

Lemma 4.2. — *For every integer $q \geqslant 0$, let $f_{1,q} : U \to \mathbb{R}$ be defined by*

$$f_{1,q} = (1+\rho)^q f_1.$$

Let $v_{1,q}$ (resp. v_2) be the orthogonal projection of $\nabla f_{1,q}$ (resp. ∇f_2) on the levels of ρ. There exist $q_0 \in \mathbb{N}$ and $R > 0$ such that for all $q \geqslant q_0$ the vector fields $v_{1,q}$ and v_2 are nonzero and do not point in opposite direction in $f_{1,q}^{-1}(]0,\delta_q]) \cap E_R$, where δ_q is a small regular value of $f_{1,q}$ such that $f_{1,q}^{-1}([0,\delta_q]) \subset U$ and $f_{1,q}^{-1}([0,\delta_q])$ is closed in $\overline{U}$.

Proof. — We know that there exists $R' > 0$ and $U' \subset U$ such that v_1 and v_2 do not vanish in $U' \cap E_{R'}$ since f_1 and f_2 are ρ-quasiregular. Let $\Gamma_{f_1,\rho}$, $\Gamma_{f_2,\rho}$ and $\Gamma_{f_1,f_2,\rho}$ be the semi-algebraic sets

$$\Gamma_{f_1,\rho} = \{x \in U \setminus X \mid v_1(x) = 0\}, \quad \Gamma_{f_2,\rho} = \{x \in U \setminus X \mid v_2(x) = 0\},$$
$$\Gamma_{f_1,f_2,\rho} = \{x \in U \setminus X \mid v_1(x) \text{ and } v_2(x) \text{ point in opposite direction}\},$$

and let Γ be the union of these three sets. Let r_0 be the greatest critical value of ρ and let $\alpha :]r_0, +\infty[\to \mathbb{R}$ be defined by

$$\alpha(r) = \inf\{f_1(x) \mid x \in \Sigma_r \cap \Gamma\}.$$

Then α is a positive semi-algebraic function. To see that it is positive, it is enough to apply Lemma 1.8 of [8] to the semi-algebraic subset $X \cap \Sigma_r$

of the smooth semi-algebraic set Σ_r. As in Lemma 2.2, this implies that there exists $R > r_0$ and an integer q_0 such that for $x \in \Gamma \cap E_R$ and for $q \geqslant q_0$, $(1+\rho(x))^q f_1(x) > 1$. Since $v_{1,q} = (1+\rho)^q v_1$, we see that $\Gamma_{f_1,f_2,\rho} = \Gamma_{f_{1,q};f_2;\rho}$. We take δ_q to be the minimum of δ_1 and 1. This ends the proof of Lemma 4.2. □

Lemma 4.3. — *For every integer $q \geqslant 0$, let $f_{1,q} : U \to \mathbb{R}$ and $f_{2,q} : U \to \mathbb{R}$ be defined by*

$$f_{1,q} = (1+\rho)^q f_1, \quad f_{2,q} = (1+\rho)^q f_2.$$

Let $v_{1,q}$ (resp. $v_{2,q}$) be the orthogonal projection of $\nabla f_{1,q}$ (resp. $\nabla f_{2,q}$) on the levels of ρ. There exist $q_0 \in \mathbb{N}$ and $R > 0$ such that for all $q \geqslant q_0$ and for all $\ell \in \mathbb{N}$ the vector fields $v_{1,q}$ and $v_{2,\ell}$ are nonzero and do not point in opposite direction in $f_{1,q}^{-1}(]0,\delta_q]) \cap E_R$, where δ_q is a small regular value of $f_{1,q}$ such that $f_{1,q}^{-1}([0,\delta_q]) \subset U$ and $f_{1,q}^{-1}([0,\delta_q])$ is closed in $\overline{U}$.

Proof. — It is clear because $v_{2,\ell} = (1+\rho)^\ell v_2$ and $\Gamma_{f_{1,k};f_{2,\ell};\rho} = \Gamma_{f_{1,k};f_2;\rho}$. This ends the proof of Lemma 4.3. □

Let us fix q and δ_q which satisfy the conclusion of Lemma 4.2. Applying Lemma 2.2 to the open semi-algebraic neighborhood $f_{1,q}^{-1}([0,\delta_q[)$ of X and the approaching function f_2, we can find ℓ such that

$$f_{2,\ell}^{-1}\big([0,\epsilon_\ell]\big) \subset f_{1,q}^{-1}\big([0,\delta_q[\big),$$

where ϵ_ℓ is a small regular value of $f_{2\ell}$. Thanks to Lemma 4.3, we can proceed as we did for the second case, namely we consider the closed set $S' = f_{1,q}^{-1}([0,\delta_q]) \setminus f_{2,\ell}^{-1}([0,\epsilon_\ell[)$ and the function $h : S' \to [0,1]$ defined by

$$h = \frac{f_{2,\ell} - \epsilon_\ell}{(f_{2,\ell} - \epsilon_\ell) - (\delta_q - f_{1,q})}.$$

This ends the proof of Theorem 4.1. □

Applying Theorem 4.1 to the case when X is compact and f_1 and f_2 are two rug functions for X, we recover Durfee's uniqueness result.

5. The smooth case

In this section, we assume that X is a closed non-compact semi-algebraic set in $\mathbb{R}^n$ and also a $\mathcal{C}^3$ submanifold of dimension $k < n$. We also assume that ρ is a control function of class $\mathcal{C}^3$. We show that any ρ-quasiregular approaching semi-algebraic neighborhood of X is isotopic to a tubular neighborhood of X. For this, we construct a kind of distance function to X which is $\mathcal{C}^2$ in a semi-algebraic neighborhood of X and ρ-quasiregular.

Let us fix X and ρ satisfying the above assumptions. Let $r_0 > 0$ be such that for all $r \geqslant r_0$, Σ_r is a $\mathcal{C}^3$ submanifold that intersects X transversally. Let F be the following set:

$$F = \big\{(x,v) \in X \times \mathbb{R}^n \mid \rho(x) > r_0,\ \langle v, \nabla\rho(x)\rangle = 0 \\ \text{and } \langle v, w\rangle = 0 \text{ for all } w \in T_x(X \cap \Sigma_{\rho(x)})\big\}.$$

It is a $\mathcal{C}^2$-vector bundle over $X \cap \{x \mid \rho(x) > r_0\}$ whose fibers are $(n-k)$-dimensional. Moreover it is semi-algebraic. We will denote the fiber over x by F_x. Observe that F_x is the normal space of $X \cap \Sigma_{\rho(x)}$ in $\Sigma_{\rho(x)}$.

Let N be the normal bundle over $X \cap \{x \mid \rho(x) < 2r_0\}$:

$$N = \big\{(x,v) \in X \times \mathbb{R}^n \mid \rho(x) < 2r_0 \text{ and } v \perp T_xX\big\}.$$

It is also a $\mathcal{C}^2$ semi-algebraic vector bundle. We denote the fiber over x by N_x.

We will glue these two bundles. By [25, Corollary C.12], it is possible to find a $\mathcal{C}^2$ semi-algebraic function $\phi : X \mapsto [0,1]$ such that $X \cap E_{7/4r_0} = \phi^{-1}(1)$ and $X \cap D_{5/4r_0} = \phi^{-1}(0)$. For each x such that $r_0 < \rho(x) < 2r_0$, let P_x be the restriction to F_x of the orthogonal projection to N_x.

We can define a bundle $H \subset X \times \mathbb{R}^n$ over X in the following way:

- if $\rho(x) < \frac{5}{4}r_0$ then $H_x = N_x$;
- if $r_0 < \rho(x) < 2r_0$ then $H_x = \{v \in \mathbb{R}^n \mid \exists w \in F_x$ such that $v = \phi(x)w + (1-\phi(x))P_x(w)\}$;
- if $\rho(x) > \frac{7}{4}r_0$ then $H_x = F_x$.

It is an exercise of linear algebra to prove that H is a vector bundle whose fibres are $(n-k)$-dimensional planes. Furthermore, it is $\mathcal{C}^2$ semi-algebraic because F and N are $\mathcal{C}^2$ semi-algebraic bundles and ϕ is a $\mathcal{C}^2$ semi-algebraic function. This bundle H will enables us to construct the desired “distance” function to X. Let φ be the following $\mathcal{C}^2$ semi-algebraic mapping:

$$\varphi : H \longrightarrow \mathbb{R}^n, \quad (x,v) \longmapsto x + v.$$

Then there exists a semi-algebraic open neighborhood U of the zero-section $X \times \{0\}$ in H such that the restriction $\varphi_{|U}$ is a $\mathcal{C}^2$ diffeomorphism onto a neighborhood V of X. Moreover, we can take U of the form

$$U = \big\{(x,v) \mid \|v\| < \varepsilon(x)\big\},$$

where ε is a positive $\mathcal{C}^2$ semi-algebraic function on X. The proof of this result is given in [5, Lemma 6.15], for the normal bundle. This proof actually holds in our case. This provides us with a $\mathcal{C}^2$ semi-algebraic retraction

$\pi : V \to X$ and a $\mathcal{C}^2$ semi-algebraic distance function $d' : V \to X$ defined by $\pi(\varphi(x,v)) = x$ and $d'(\varphi(x,v)) = \|v\|^2$.

Lemma 5.1. — *There exists an open semi-algebraic neighborhood W of X in V such that for every $y \in W$, $\rho(y) \leqslant 1.1\,\rho(\pi(y))$. Furthermore, one can choose W of the form*

$$W = \big\{ y \in V \mid d'(y) < \varepsilon'(\pi(y)) \big\},$$

where $\varepsilon' : X \to \mathbb{R}$ is a positive $\mathcal{C}^2$ semi-algebraic function.

Proof. — Let A be the semi-algebraic set

$$A = \big\{ y \in V \mid \rho(y) > 1.1\rho(\pi(y)) \big\}.$$

Let $\alpha : \pi(A) \to \mathbb{R}$ be the function defined as

$$\alpha(x) = \inf \big\{ d'(y) \mid y \in \pi^{-1}(x) \cap A \big\}.$$

This is a semi-algebraic function on $\pi(A)$. Let us prove that it is positive. The continuity of $\rho \circ \varphi$ implies that for every x in $\pi(A)$, there exists δ_x with $0 < \delta_x < \varepsilon(x)$, such that $\rho(\varphi(x,v)) \leqslant 1.1\rho(\varphi(x,0))$ for every v in H_x with $\|v\| \leqslant \delta_x$. Since $\|v\|^2 = d'(y)$ if $y = \varphi(x,v)$, this proves that $\alpha(x) \geqslant \delta_x > 0$. Let us show that α is locally bounded from below by positive constants, i.e for every $x \in \pi(A)$, there exist $c > 0$ and a neighborhood Ω of x in $\pi(A)$ such that $\alpha > c$ on Ω. If it is not the case, we can find a sequence of points x_n in $\pi(A)$ tending to x such that $\alpha(x_n)$ tends to 0. Hence there exists a sequence of points $y_n = \varphi(x_n, v_n)$ such that v_n tends to 0, x_n tends to x and $\rho(\varphi(x_n,v_n)) > 1.1\rho(\varphi(x_n,0))$. By continuity, we obtain $\rho(\varphi(x,0)) \geqslant 1.1\rho(\varphi(x,0))$, which is impossible. Let $\widetilde{\alpha} : X \to \mathbb{R}$ be defined by $\widetilde{\alpha}(x) = \alpha(x)$ if $x \in \pi(A)$ and $\widetilde{\alpha}(x) = \varepsilon(x)$ if $x \notin \pi(A)$. The function $\widetilde{\alpha}$ is semi-algebraic, positive and locally bounded from below by positive constants. Applying Lemma 6.12 of [5], we can find a positive semi-algebraic $\mathcal{C}^2$ function $\varepsilon' : X \to \mathbb{R}$ such that $\varepsilon' < \widetilde{\alpha}$ on X. □

Let us study the function $d' : W \to \mathbb{R}$ more precisely. Let B be the semi-algebraic set

$$B = \Big\{ y \in W \cap E_{2r_0} \mid \frac{\langle \nabla\rho(y), \nabla\rho(\pi(y)) \rangle}{\|\nabla\rho(y)\| \|\nabla\rho(\pi(y))\|} < 0.9 \Big\}.$$

Let $\beta : \pi(B) \to \mathbb{R}$ be the function defined as

$$\beta(x) = \inf \big\{ d'(y) \mid y \in \pi^{-1}(x) \cap B \big\}.$$

This is a semi-algebraic function on $\pi(B)$ and $\beta(x) \leqslant \varepsilon'(x)$, for every $x \in \pi(B)$. The same argument as in the above lemma shows that β is positive and locally bounded from below by positive constants. Let $\tilde{\beta} : X \to \mathbb{R}$

be defined by $\tilde{\beta}(x) = \beta(x)$ if $x \in \pi(B)$ and $\tilde{\beta}(x) = \varepsilon'(x)$ if $x \notin \pi(B)$. The function $\tilde{\beta}$ is semi-algebraic, positive and locally bounded from below by positive constants. We can find a positive semi-algebraic $\mathcal{C}^2$ function $\varepsilon'' : X \to \mathbb{R}$ such that $\varepsilon'' < \tilde{\beta}$ on X.

Let W' be defined by

$$W' = \big\{y \in V \mid d'(y) < \varepsilon''(\pi(y))\big\}.$$

Note that W' is included in W. For every y in $W' \cap E_{2r_0}$, we have

$$\frac{\langle \nabla\rho(y), \nabla\rho(\pi(y))\rangle}{\|\nabla\rho(y)\|\|\nabla\rho(\pi(y))\|} \geqslant 0.9.$$

Since $\nabla d'(y)$ belongs to $[\nabla\rho(\pi(y))]^{\perp}$, this can be reformulated in the following way: for every y in $W' \cap E_{2r_0}$, we have

$$\frac{\langle \nabla\rho(y), \nabla d'(y)\rangle}{\|\nabla\rho(y)\|\|\nabla d'(y)\|} \leqslant \sqrt{0.19}.$$

Lemma 5.2. — *There exist $q_0 \in \mathbb{N}$ and $r'_0 > 0$ such that for every $q \geqslant q_0$ and for every $x \in X \cap E_{r'_0}$,*

$$\frac{1}{(1+\rho(x))^q} \leqslant \varepsilon''(x).$$

Proof. — Let $h : [0, +\infty[\to \mathbb{R}$ be defined by

$$h(r) = \min\big\{\varepsilon''(x) \mid x \in X \cap \Sigma_r\big\}.$$

Since h is a positive semi-algebraic function, there exists an integer q_0 and a real $r'_0 > 0$ such that $1/h(r) < r^{q_0}$ for every $r \geqslant r'_0$. Hence for every $q \geqslant q_0$ and every $x \in X \cap E_{r'_0}$, we have

$$\frac{1}{(1+\rho(x))^q} \leqslant \varepsilon''(x). \qquad \square$$

Corollary 5.3. — *There exist $q_0 \in \mathbb{N}$ and $r''_0 > 0$ such that for every $q \geqslant q_0$ and for every $y \in W' \cap E_{r''_0}$,*

$$\frac{1}{(1+\rho(\pi(y)))^q} \leqslant \varepsilon''\big(\pi(y)\big).$$

Proof. — By Lemma 5.1, we can find $r''_0 > 0$ such that $\pi(y)$ belongs to $X \cap E_{r'_0}$ if y belongs to $W' \cap E_{r''_0}$. $\square$

Lemma 5.4. — *There exist $q_1 \in \mathbb{N}$ and $r'_1 > 0$ such that for every $q \geqslant q_1$ and for every $x \in X \cap E_{r'_1}$, $\|\nabla\rho(x)\| \leqslant (1+\rho(x))^q$.*

Proof. — Let $c > 0$ be such that $[c, +\infty[$ does not contain any critical value of ρ. Let $\ell : [c, +\infty[\to \mathbb{R}$ be defined by

$$\ell(r) = \max\left\{ \|\nabla\rho(x)\| \mid x \in X \cap \Sigma_r \right\}.$$

Since ℓ is a positive semi-algebraic function, there exits an integer q_1 and a real $r'_1 > 0$ such that $\ell(r) < r^{q_1}$ for every $r \geqslant r'_1$. Hence for every $q \geqslant q_1$ and every $x \in X \cap E_{r'_1}$, we have $\|\nabla\rho(x)\| \leqslant (1+\rho(x))^q$. □

Corollary 5.5. — *There exist $q_1 \in \mathbb{N}$ and $r''_1 > 0$ such that for every $q \geqslant q_1$ and for every $y \in W' \cap E_{r''_1}$, $\|\nabla\rho(\pi(y))\| \leqslant (1+\rho(\pi(y)))^q$.*

Proof. — The proof is the same as Corollary 5.3. □

Proposition 5.6. — *There exists an integer q_2 such that for every $q \geqslant q_2$, the function $d'_q : W' \to \mathbb{R}$ defined by $d'_q = (1+\rho(\pi))^q d'$ is a ρ-quasiregular approaching function for X in W'.*

Proof. — Since $W' = \{y \in V \mid d'(y) < \varepsilon''(\pi(y))\}$ and ε'' is a positive function, (X, W', d') satisfies condition (A). Let

$$W_1 = \left\{ y \in V \mid d'(y) < \tfrac{1}{2}\varepsilon''(\pi(y)) \right\}.$$

We have $\overline{W_1} \subset W'$. By Corollary 5.3, for every $q \geqslant q_0$, the set $E_{r''_0} \cap d_q^{-1}([0, \frac{1}{4}])$ is included in W_1. The tuple (X, W', d'_q) satisfies condition (A). As it has been already explained in Lemma 2.2, there exists $\epsilon_q > 0$ such that $d_q^{-1}([0, \epsilon_q]) \cap D_{r''_0} \subset W_1 \cap D_{r''_0}$. Let δ_q be the minimum of $\frac{1}{4}$ and ϵ_q. The set $d_q'^{-1}([0, \delta_q])$ is included in W_1, hence closed in $\overline{W_1}$ and in $\overline{W'}$. This proves that d'_q is an approaching function for X in W'.

Let us show that it is ρ-quasiregular. Let us fix r greater than r''_0, r''_1 and $2r_0$ and let us fix q_2 greater than q_0 and q_1. For every y in $W \cap E_r$, let P_y be the orthogonal projection onto the space $\nabla\rho(y)^\perp$. We have

$$\nabla d'_q = \big(1+\rho(\pi)\big)^{q-1}\big[\,(1+\rho(\pi))\,\nabla d' + q d' \nabla\rho(\pi)\big],$$

hence,

$$\frac{P_y(\nabla d'_q)}{\big(1+\rho(\pi)\big)^{q-1}} = \big(1+\rho(\pi)\big) P_y(\nabla d') + q d' P_y\big(\nabla\rho(\pi)\big).$$

Let us prove that, for $q \geqslant q_2$ and $R \geqslant r$ sufficiently big, $T(y)$ can not vanish if y belongs to $d_q'^{-1}(]0,1]) \cap E_R$, where

$$T(y) = \big(1+\rho(\pi(y))\big) P_y(\nabla d'(y)) + q d'(y) P_y\big(\nabla\rho(\pi(y))\big).$$

First observe that if y lies in $d_q'^{-1}([0,1]) \cap E_R$, $q \geqslant q_2$ and $R \geqslant r$, then

$$\frac{\langle \nabla\rho(y), \nabla\rho(\pi(y))\rangle}{\|\nabla\rho(y)\|\|\nabla\rho(\pi(y))\|} \geqslant 0.9 \quad \text{and} \quad \frac{\langle \nabla\rho(y), \nabla d'(y)\rangle}{\|\nabla\rho(y)\|\|\nabla d'(y)\|} \leqslant \sqrt{0.19}.$$

This implies that

$$\left\|P_y(\nabla\rho(\pi(y)))\right\| \leqslant \sqrt{0.19}\left\|\nabla\rho(\pi(y))\right\|$$

and

$$\left\|P_y(\nabla d'(y))\right\| \geqslant 0.9\left\|\nabla d'(y)\right\|.$$

Therefore, we have

$$\left\|qd'(y)P_y(\nabla\rho(\pi(y)))\right\| \leqslant \sqrt{0.19}\,qd'(y)\left\|\nabla\rho(\pi(y))\right\|$$

and

$$\left\|\,(1+\rho(\pi(y)))\,P_y(\nabla d'(y))\right\| \geqslant 0.9\big(1+\rho(\pi(y))\big)\left\|\nabla d'(y)\right\|,$$

that is to say

$$\left\|\,(1+\rho(\pi(y)))\,P_y(\nabla d'(y))\right\| \geqslant 0.9\big(1+\rho(\pi(y))\big)2\sqrt{d'(y)}.$$

In order to prove that $T(y)$ does not vanish if $y \in {d'_q}^{-1}([0,1]) \cap E_R$ for $q \geqslant q_2$ and $R \geqslant r$ sufficiently big, it is enough to prove that

$$\frac{1.8}{\sqrt{0.19}} > \frac{q\sqrt{d'(y)}\,\|\nabla\rho(\pi(y))\|}{1+\rho(\pi(y))}.$$

But if $y \in {d'_q}^{-1}(\,]0,1]) \cap E_R$ where $q \geqslant q_2$ and $R \geqslant r$ then we have

$$\sqrt{d'(y)} \leqslant \frac{1}{(1+\rho(\pi(y)))^{\frac{1}{2}q}}.$$

So, if we show that

$$\frac{1.8}{\sqrt{0,19}} > \frac{q\|\nabla\rho(\pi(y))\|}{(1+\rho(\pi(y)))^{\frac{1}{2}q+1}},$$

then the required result is established. Let q be such that $\frac{1}{2}q+1 > q_1$. By Corollary 5.5, we have

$$\frac{q\|\nabla\rho(\pi(y))\|}{(1+\rho(\pi(y)))^{\frac{1}{2}q+1}} \leqslant \frac{q}{(1+\rho(\pi(y)))^{\frac{1}{2}q+1-q_1}},$$

for $y \in {d'_q}^{-1}([0,1]) \cap E_R$, $R \geqslant r$. Lemma 5.1 implies that there exists $R_q \geqslant r$ such that if y belongs to ${d'_q}^{-1}([0,1]) \cap E_R$, with $R \geqslant R_q$, then we have

$$\frac{q}{(1+\rho(\pi(y)))^{\frac{1}{2}q+1-q_1}} < \frac{1.8}{\sqrt{0.19}}.$$

This proves the proposition. □

We can state the main result of this section, which is an application of the uniqueness result stated in Theorem 4.1.

THEOREM 5.7. — *Let X be a closed non-compact semi-algebraic set in $\mathbb{R}^n$ which is a $\mathcal{C}^3$ submanifold. Let ρ be a control function of class $\mathcal{C}^3$. Any ρ-quasiregular approaching semi-algebraic neighborhood of X is isotopic to a tubular neighborhood of X.*

Proof. — We known that there exist ρ-quasiregular approaching functions d'_q for X in W of the form $d'_q = (1+\rho(\pi))^q d'$ by the previous proposition. But for $\nu > 0$ sufficiently small the set $d'^{-1}_q([0,\nu])$ is a tubular neighborhood of X. It is enough to use Theorem 4.1 to conclude. □

6. Uniqueness of approaching semi-algebraic neighborhoods

In this section, we prove that two approaching semi-algebraic neighborhoods of a closed non-compact semi-algebraic set are isotopic. We need first the following proposition.

PROPOSITION 6.1. — *Let $X \subset \mathbb{R}^n$ be a closed non-compact semi-algebraic set equipped with a Whitney stratification. There exists a semi-algebraic function $f : \mathbb{R}^n \to \mathbb{R}$ such that:*

1) *$f \geqslant 0$ and $f^{-1}(0) = X$;*
2) *f is of class $\mathcal{C}^3$;*
3) *for every sequence of points $(x_k)_{k\in\mathbb{N}}$ in $\mathbb{R}^n$ tending to a point y in X with $\lim_{k\to+\infty} \nabla f(x_k)/\|\nabla f(x_k)\| = \nu$, one has $\nu \perp T_yS$, where S is the stratum of X containing y and T_yS is its tangent space at y.*

Proof. — We may assume that $0 \notin X$. Let $I : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ be the inversion defined by $I(x) = x/\|x\|^2$ and let Y be the compact semi-algebraic set $I(X) \cup \{0\}$. If $\{S_\alpha\}_{\alpha\in\Lambda}$ is a Whitney semi-algebraic stratification of X then $\{I(S_\alpha)\}_{\alpha\in\Lambda} \cup \{0\}$ is a Whitney stratification of Y. By [3, Theorem 7.1], there exists a continuous semi-algebraic function $g : \mathbb{R}^n \to \mathbb{R}$ such that:

i) $g \geqslant 0$ and $g^{-1}(0) = Y$;
ii) g is of class $\mathcal{C}^3$ on $\mathbb{R}^n \setminus Y$;
iii) for every sequence of points $(z_k)_{k\in\mathbb{N}}$ tending to a point z in Y with $\lim_{k\to+\infty} \nabla g(z_k)/\|\nabla g(z_k)\| = \tau$, one has $\tau \perp T_zR$, where R is the stratum of the stratification $\{I(S_\alpha)\}_{\alpha\in\Lambda} \cup \{0\}$ that contains z.

Let $\tilde{f} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ be defined by $\tilde{f}(x) = g(I(x))$. The function $\tilde{f}$ is clearly semi-algebraic, continuous and nonnegative on $\mathbb{R}^n \setminus \{0\}$. Furthermore it is $\mathcal{C}^3$ on $\mathbb{R}^n \setminus (\{0\} \cup X)$ and $\tilde{f}^{-1}(0) = X$.

Let us consider a sequence of points $(x_k)_{k\in\mathbb{N}}$ tending to a point y in X such that $\lim_{k\to+\infty} \nabla\tilde{f}(x_k)/\|\nabla\tilde{f}(x_k)\| = \nu$. Then the sequence of points $(z_k)_{k\in\mathbb{N}}$ defined by $z_k = I(x_k)$ tends to the point $I(y)$. A computation of partial derivatives gives that

$$\forall k\in\mathbb{N},\quad \nabla g(z_k) = \frac{1}{\|z_k\|^2}\big(-2\langle\nabla\tilde{f}(x_k), x_k\rangle z_k + \nabla\tilde{f}(x_k)\big),$$

which implies that

$$\|\nabla g(z_k)\| = \frac{\|\nabla\tilde{f}(x_k)\|}{\|z_k\|^2} = \|x_k\|^2\cdot\|\nabla\tilde{f}(x_k)\|$$

and that

$$\frac{\nabla g(z_k)}{\|\nabla g(z_k)\|} = -2\Big\langle\frac{\nabla\tilde{f}(x_k)}{\|\nabla f(x_k)\|}, \frac{x_k}{\|x_k\|}\Big\rangle\frac{x_k}{\|x_k\|} + \frac{\nabla\tilde{f}(x_k)}{\|\nabla\tilde{f}(x_k)\|}.$$

Therefore the sequence $\nabla g(z_k)/\|\nabla g(z_k)\|$ tends to $-2\langle\nu, y/\|y\|\rangle y/\|y\| + \nu$. Let us denote this vector by τ. A computation shows that $\tau=\|y\|^2 DI(y)(\nu)$. Let a be a non-zero vector in T_yS (S is the stratum containing y) and let $b = DI(y)(a)$. We have $\langle\tau, b\rangle = 0$ hence $\langle DI(y)(a), DI(y)(\nu)\rangle = 0$, which implies that $\langle a,\nu\rangle = 0$. We have constructed a continuous semi-algebraic function $\tilde{f}$ which satisfies conditions 1) and 3) of the proposition, except that it is not defined at 0. Using [25, Corollary C.12], we can easily obtain a continuous semi-algebraic function $\bar{f}:\mathbb{R}^n\to\mathbb{R}$ satisfying conditions 1) and 3) of the statement. This function is $\mathcal{C}^3$ on $\mathbb{R}^n\setminus X$. In order to get a function $\mathcal{C}^3$ everywhere, we use [25, Corollary C.10]: there exists an odd strictly increasing $\mathcal{C}^3$ semi-algebraic function $\phi:\mathbb{R}\to\mathbb{R}$ such that $\phi\circ\bar{f}$ is $\mathcal{C}^3$ on $\mathbb{R}^n$. The function $\phi\circ\bar{f}$ is the desired function f. □

Let us fix now two control functions ρ_0 and ρ_1. For each $t\in[0,1]$, let $\rho_t:\mathbb{R}^n\to\mathbb{R}$ be defined by $\rho_t = (1-t)\rho_0 + t\rho_1$. The functions ρ_t are also control functions. We will denote by Σ_r^t the set $\rho_t^{-1}(r)$, by D_r^t the set $\rho_t^{-1}([0,r])$ and by E_r^t the set $\rho_t^{-1}([r,+\infty[)$.

Lemma 6.2. — *There exists $r_0 > 0$ such that for all $r\geqslant r_0$ and for all $t\in[0,1]$, the sets Σ_r^t are non-empty compact $\mathcal{C}^2$ hypersurfaces of $\mathbb{R}^n$ that intersect each stratum of X transversally.*

Proof. — As in [8, Lemma 1.8], we can prove using the curve selection lemma at infinity (see [21, Lemma 2]) that there exists a compact set K of $\mathbb{R}^n$ such that $\nabla\rho_0$ and $\nabla\rho_1$ are non-zero and do not point in opposite direction outside K. Furthermore we can find $r_1 > 0$ such that for $r\geqslant r_1$, Σ_r^0 and Σ_r^1 are non-empty $\mathcal{C}^2$ submanifolds lying outside K. This implies that all the sets Σ_r^t lie outside K.

Let $\theta : \mathbb{R}^n \times [0,1] \to \mathbb{R}$ be defined by $\theta(x,t) = \rho_t(x)$. There exists $r_2 \geqslant r_1$ such that for every $r \geqslant r_2$, $\theta^{-1}(r)$ is a $\mathcal{C}^2$ submanifold with boundary $\Sigma_r^0 \cup \Sigma_r^1$ because θ, viewed as a smooth function on a manifold with boundary, admits a finite number of critical values. We see that the function $t_{|\theta^{-1}(r)} : \theta^{-1}(r) \to [0,1]$ is a smooth fibration since on $\theta^{-1}(r)$, $\nabla\rho_t$ can not vanish. This implies that for $t \in [0,1]$, Σ_r^t is a non-empty compact $\mathcal{C}^2$ hypersurface.

To prove the second part of the lemma, we fix a non compact stratum S_α of X. Applying the same method to $\rho_{0|S_\alpha}$ and $\rho_{1|S_\alpha}$ and to the manifold with boundary $S_\alpha \times [0,1]$, we find that there exists $r_\alpha > 0$ such that for each $t \in [0,1]$, Σ_r^t intersects S_α transversally. Finally, we take r_0 to be the minimum of r_2 and the r_α's. □

Let $F : \mathbb{R}^n \times [0,1] \to \mathbb{R}$ be defined by $F(x,t) = f(x)$, where f is the function constructed in Proposition 6.1, and let Γ_F be the semi-algebraic set

$$\Gamma_F = \left\{ (x,t) \in \mathbb{R}^n \times [0,1] \ \middle|\ \operatorname{rank}\begin{bmatrix} \frac{\partial F}{\partial x_1}(x,t) & \cdots & \frac{\partial F}{\partial x_n}(x,t) \\ \frac{\partial \theta}{\partial x_1}(x,t) & \cdots & \frac{\partial \theta}{\partial x_n}(x,t) \end{bmatrix} < 2 \right\},$$

where we recall that θ is defined by $\theta(x,t) = \rho_t(x)$.

LEMMA 6.3. — *There exists $r_3 \geqslant r_0$ and an integer q_0 such that for every $(x,t) \in \theta^{-1}([r_3,+\infty[) \cap \Gamma_F$ and every $q \geqslant q_0$, one has $(1+\theta(x,t))^q f(x) > 1$.*

Proof. — Let $\beta :]0,+\infty[\to \mathbb{R}$ be the semi-algebraic function

$$\beta(R) = \inf\left\{F(x,t) \mid (x,t) \in \theta^{-1}(R) \cap \Gamma_F\right\}.$$

It is a nonnegative semi-algebraic function. Let us prove that it is positive at infinity. If it is not the case, there exists R_0 such that for every $R \geqslant R_0$, $\beta(R) = 0$. This implies that there exists a sequence of points $((x_k^R, t_k^R))_{k\in\mathbb{N}}$ in $\theta^{-1}(R) \cap \Gamma_F$ such that $F(x_k^R, t_k^R) = f(x_k^R)$ tends to 0. Since $\theta^{-1}(R)$ is compact, we can assume that (x_k^R, t_k^R) tends to a point (x^R, t^R) such that $f(x^R) = 0$. We can also assume that $\nabla f(x_k^R)/\|\nabla f(x_k^R)\|$ tends to a unit vector ν^R. We know that $\nu^R \perp T_{x^R}S$ by condition 3) in Proposition 6.1 (S is the stratum containing x^R). Now $\nabla f(x_k^R)/\|\nabla f(x_k^R)\|$ is colinear to $\nabla\rho_{t_k^R}(x_k^R)$, so, taking the limit, we see that ν^R is colinear to $\nabla\rho_{t^R}(x_R)$. Hence $\Sigma_R^{t^R}$ does not intersect S transversally. By the previous lemma, we know that this is not possible if R is big enough. Since β is strictly positive at infinity, there exists $r_3 \geqslant r_0$ and an integer q_0 such that for every $r \geqslant r_0$ and every $q \geqslant q_0$, one has $\beta(r)^{-1} < (1+r)^q$. This implies the result. □

Note that we have proved that for $q \geqslant q_0$, the function g_t defined by $g_t = (1+\rho_t)^q f$ is ρ_t-quasiregular and that, furthermore, the radius r_3 does not depend on t, which is the most important point of the lemma.

Lemma 6.4. — *There exists* $\delta_0 > 0$ *such that for all* $0 < \delta \leqslant \delta_0$ *and all* $t \in [0,1]$, *the set* $g_t^{-1}([0,\delta])$ *is a* ρ_t*-quasiregular approaching semi-algebraic neighborhood of* X *in* $\mathbb{R}^n$.

Proof. — We know that g_t is a ρ_t-quasiregular approaching function for X in U and that Γ_{g_t,ρ_t} does not intersect $g_t^{-1}([0,1])$ outside $D^t_{r_3}$. It remains to show that there exists $0 < \delta_0 < 1$ such that for each $0 < \delta < \delta_0$, δ is a regular value of g_t, $t \in [0,1]$, smaller than all nonzero critical value of g_t. Let $Z = \bigcup_{t\in[0,1]} D^t_{r_3}$. We observe that Z is a compact set and that g_t, $t \in [0,1]$, does not admit any critical point in $g_t^{-1}(]0,1[) \cap \mathbb{R}^n \setminus Z$, because such a point would belong to Γ_{g_t,ρ_t}. Hence it is enough to prove that there exists δ_0, $0 < \delta_0 < 1$, such that g_t does not admit any critical point in $Z \cap g_t^{-1}(]0,\delta_0[)$.

There exists a neighborhood U of X in Z such that ∇g_0 and ∇g_1 do not vanish and do not point in opposite direction in $U \setminus X$. Let δ_0, $0 < \delta_0 \ll 1$, be a regular value of g_0 and g_1, smaller than all nonzero critical value of g_0 and g_1 such that $g_0^{-1}([0,\delta_0]) \cap Z$ and $g_1^{-1}([0,\delta_0]) \cap Z$ are included in U. We claim that for each $t \in [0,1]$, $g_t^{-1}(]0,\delta_0]) \cap Z$ does not contain any critical point. Let us remark first that $g_t^{-1}([0,\delta_0]) \cap Z$ is included in U. This is an easy consequence of the following implication:

$$1 + g_t(x) \leqslant \Big(\frac{\delta_0}{f(x)}\Big)^{\frac{1}{q}} \Longrightarrow 1 + g_0(x) \leqslant \Big(\frac{\delta_0}{f(x)}\Big)^{\frac{1}{q}} \text{ or } 1 + g_1(x) \leqslant \Big(\frac{\delta_0}{f(x)}\Big)^{\frac{1}{q}}.$$

Now if g_t admits a critical point x in $g_t^{-1}(]0,\delta_0]) \cap Z$ then

$$\big(1+\rho_t(x)\big)^q \nabla f(x) + q\big(1+\rho_t(x)\big)^{q-1} f(x) \nabla \rho_t(x)$$

vanishes which implies that $\nabla g_0(x)$ and $\nabla g_1(x)$ point in opposite direction. This is impossible and δ_0 is the required common regular value. □

Let $G : \mathbb{R}^n \times [0,1] \to \mathbb{R}$ be defined by $G(x,t) = g_t(x)$. Let δ be a positive regular value of G smaller than δ_0. The set $T_0 = g_0^{-1}([0,\delta])$ (resp. $T_1 = g_1^{-1}([0,\delta])$) is a ρ_0-quasiregular (resp. ρ_1-quasiregular) approaching semi-algebraic neighborhood of X in $\mathbb{R}^n$.

Theorem 6.5. — *There exists a continuous family of diffeomorphisms* $h_s : \mathbb{R}^n \to \mathbb{R}^n$, $0 \leqslant s \leqslant 1$, *such that:*

1) h_0 *is the identity;*
2) *for all* s, $h_{s|X}$ *is the identity;*
3) $h_1(T_0) = T_1$.

Proof. — Let δ' be a positive regular value of G strictly smaller than δ. Let W be the following semi-algebraic set of $\mathbb{R}^n \times \mathbb{R}$:

$$W = \{(x,t) \in \mathbb{R}^n \times [0,1] \mid \delta' \leqslant G(x,t) \leqslant \delta\}.$$

It is a $\mathcal{C}^2$-manifold with corners of dimension $n+1$. Changing r_3 into a greater value if necessary, we can assume that for $r \geqslant r_3$, the compact sets $\theta^{-1}(r)$ are smooth manifolds that intersect W transversally. Let e_{n+1} be the unit vector in $\mathbb{R}^{n+1}$ equal to $(0,\ldots,0,1)$, this the gradient of the function t. The restriction of the function t does not admit any critical point on the manifolds $G^{-1}(\delta'')$, $\delta'' \in [\delta',\delta]$, for otherwise one of the g_t's would have a critical point on $g_t^{-1}(\delta'')$. By Lemmas 6.2 and 6.3, the restriction of the function t does not admit any critical point on the manifolds $\theta^{-1}(r) \cap G^{-1}(\delta'')$, with $\delta'' \in [\delta',\delta]$ and $r \geqslant r_3$. Proceeding as in the previous sections, we define a vector field $\widetilde{w}$ on W which is equal to the projection of e_{n+1} on the levels of G in a compact set of W and which is equal to the projection of e_{n+1} on the manifolds $\theta^{-1}(r) \cap G^{-1}(\delta'')$ at infinity. Let U be an open neighborhood of W disjoint from $G^{-1}(0)$. Using a function equal to 1 on W and 0 on the closed set $\mathbb{R}^n \times [0,1] \setminus U$, we extend $\widetilde{w}$ to a vector field $\overline{w}$ equal to e_{n+1} on $\mathbb{R}^n \times [0,1] \setminus U$. Integrating $\overline{w}$ gives a family of diffeomorphisms $H_s : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n \times [0,1]$ such that H_0 is the identity, $H_1(T_0 \times \{0\}) = T_1 \times \{1\}$ and $H_{s|X\times\{0\}} = \mathrm{id}_{|X} \times \{s\}$ for $s \in [0,1]$ (here $\mathrm{id}_{|X}$ is the identity on X). Let $h_s : \mathbb{R}^n \to \mathbb{R}^n$, $s \in [0,1]$, be defined by $\forall x \in \mathbb{R}^n$, $H_s(x,0) = (h_s(x),s)$. The family h_s is the required family of diffeomorphisms. □

Corollary 6.6. — *Two approaching semi-algebraic neighborhoods of a closed non-compact semi-algebraic set are isotopic.*

Proof. — This is a consequence of Theorems 4.1 and 6.5. □

Corollary 6.7. — *Let X be a closed semi-algebraic set in $\mathbb{R}^n$ and let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a $\mathcal{C}^2$ semi-algebraic diffeomorphism whose inverse is also semi-algebraic. Then an approaching semi-algebraic neighborhood of X and an approaching semi-algebraic neighborhood of $\phi(X)$ are diffeomorphic.*

Proof. — Let ρ be a control function and let T be a ρ-quasiregular approaching semi-algebraic neighborhood of X of the form $f^{-1}([0,\delta])$. The function $\rho \circ \phi^{-1}$ is a control function and $\phi(T) = (f \circ \phi^{-1})^{-1}([0,\delta])$ is a $(\rho \circ \phi^{-1})$-quasiregular approaching semi-algebraic neighborhood of $\phi(X)$ diffeomorphic to T. □

7. Degree formulas for the Euler-Poincaré characteristic of a closed semi-algebraic set

In this section, we give degree formulas for the Euler-Poincaré characteristic of a closed semi-algebraic set X included in $\mathbb{R}^n$. When X is algebraic, we deduce from these formulas a Petrovskii-Oleinik inequality for $|1-\chi(X)|$.

Let $X \subset \mathbb{R}^n$ be a closed semi-algebraic set and let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative $\mathcal{C}^2$ semi-algebraic function such that $X = f^{-1}(0)$, i.e f is an approaching function for X in $\mathbb{R}^n$. Let ρ be a control function. For every $q \in \mathbb{N}$, we will denote by f_q the function defined by $f_q = (1+\rho)^q f$. We will also denote by $\Gamma_{f,\rho}$ (resp. $\Gamma_{f_q,\rho}$) the polar set

$$\Gamma_{f,\rho} = \big\{x \in \mathbb{R}^n \setminus X \mid \nabla f(x) \text{ (resp. } \nabla f_q(x)) \text{ and } \nabla\rho(x) \text{ are colinear}\big\}.$$

Note that $\Gamma_{f,\rho} = \Gamma_{f_q,\rho}$ for each $q \in \mathbb{N}$. The following proposition is similar to Proposition 2.6 and is proved in the same way.

Proposition 7.1. — *There exists an integer q_0 such that for every $q \geqslant q_0$, the following property holds: for any sequence $(x_k)_{k\in\mathbb{N}} \subset \Gamma_{f_q,\rho}$ such that $\lim_{k\to+\infty} \|x_k\| = +\infty$, we have $\lim_{k\to+\infty} f_q(x_k) = +\infty$.*

Let us fix an integer q satisfying the property of the previous proposition. Let $\Sigma(f_q)$ be the set of critical points of f_q and let $\Sigma^*(f_q)$ be the set of critical points of f_q lying in $\mathbb{R}^n \setminus X$.

Corollary 7.2. — *The set $\Sigma^*(f_q)$ is compact.*

Proof. — It is clearly closed as an union of connected components of the closed set $\Sigma(f_q)$. If it is not bounded, there exists a sequence of points $(x_k)_{k\in\mathbb{N}}$ such that $x_k \notin X$, $\nabla f_q(x_k) = 0$ and $\lim_{k\to+\infty} \|x_k\| = +\infty$. Since for each $k \in \mathbb{N}$, x_k also belongs to $\Gamma_{f_q,\rho}$, this gives a contradiction. □

Let us decompose $\Sigma^*(f_q)$ into the finite union of its connected components $K_1^q, \dots, K_{m_q}^q$:

$$\Sigma^*(f_q) = \bigcup_{i=1}^{m_q} K_i^q.$$

Before stating the main results of this section, we need to introduce some notations. For each $i \in \{1, \dots, m_q\}$, let U_i be a relatively compact neighborhood of K_i^q such that ∂U_i is a smooth hypersurface and $U_i \cap \Sigma^*(f_q) = K_i^q$. For any mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ such that $F^{-1}(0) \cap U_i = K_i^q$ or $F^{-1}(0) \cap U_i$ is empty, we will denote by $\deg_{K_i^q} F$ the topological degree of the mapping

$$\frac{F}{\|F\|} : \partial U_i \longrightarrow S^{n-1}, \quad x \longmapsto \frac{F(x)}{\|F(x)\|}.$$

It is well known that this topological degree does not depend on the choice of the relatively compact neighborhood U_i.

THEOREM 7.3. — *The Euler-Poincaré characteristic of the closed semi-algebraic set X is related to ∇f_q by the formula*

$$\chi(X) = 1 - \sum_{i=1}^{m_q} \deg_{K_i^q} \nabla f_q.$$

Proof. — By Proposition 7.1, f_q is a ρ-quasiregular approaching function for X in $\mathbb{R}^n$. Theorem 3.2 implies that for $\varepsilon > 0$ sufficiently small

$$\chi(X) = \chi\big(\{f_q \leqslant \varepsilon\}\big).$$

By the Mayer-Vietoris sequence, we have

$$1 = \chi\big(\{f_q \leqslant \varepsilon\}\big) + \chi\big(\{f_q \geqslant \varepsilon\}\big) - \chi\big(\{f_q = \varepsilon\}\big). \tag{1}$$

We will apply Morse theory to the manifold with boundary D_R and to the function f_q. We will follow the terminology of [9], Section 2, pp. 46–47. Let us first show that f_q does not admit any inward critical point on $\Sigma_R \cap \{f_q \geqslant \varepsilon\}$ for R sufficiently big and ε sufficiently small (an inward critical point p is a critical point p of $f_{q|\Sigma_R}$ such that $\nabla f_q(p)$ is a negative multiple of $\nabla \rho(p)$). If it is not the case, then we can find a sequence of points $(x_k)_{k\in\mathbb{N}}$ in $\Gamma_{f_q,\rho}$ such that $\nabla f_q(x_k)$ is a negative multiple of $\nabla\rho(x_k)$. Using the version at infinity of the Curve Selection Lemma (see [21, Lemma 2]), we obtain that $\lim_{k\to+\infty} f_q(x_k)$ exists and belongs to $[0,+\infty[$, which contradicts the property of Proposition 7.1.

Let us fix R sufficiently big and ε sufficiently small so that $\Sigma^*(f_q) \subset D_R$, f_q does not have inward critical points in $\Sigma_R \cap \{f_q \geqslant \varepsilon\}$ and

$$\chi\big(\{f_q \geqslant \varepsilon\}\big) = \chi\big(\{f_q \geqslant \varepsilon\} \cap D_R\big) \quad \text{and} \quad \chi\big(\{f_q = \varepsilon\}\big) = \chi\big(\{f_q = \varepsilon\} \cap D_R\big).$$

Since f_q does not have inward critical points in $\Sigma_R \cap \{f_q \geqslant \varepsilon\}$, Morse theory for manifolds with boundary implies that

$$\chi\big(\{f_q \geqslant \varepsilon\} \cap D_r\big) - \chi\big(\{f_q = \varepsilon\} \cap D_r\big) = \sum_{i=1}^{m_q} \deg_{K_i^q} \nabla f_q. \tag{2}$$

The final result is just a combination of equalities (1) and (2). □

Let $F_q : \mathbb{R}^n \to \mathbb{R}^n$ be the mapping defined by

$$F_q = qf\,\nabla\rho + (1+\rho)\nabla f.$$

Note that $\nabla f_q = (1+\rho)^{q-1} F_q$. Hence ∇f_q and F_q admit the same zeros in $\mathbb{R}^n$.

Corollary 7.4. — *The Euler-Poincaré characteristic of X is related to F_q by the formula*

$$\chi(X) = 1 - \sum_{i=1}^{m_q} \deg_{K_i^q} F_q.$$

Proof. — It is enough to prove that $\deg_{K_i^q} F_q = \deg_{K_i^q} \nabla f_q$, for every $i \in \{1, \dots, m_q\}$. Let us choose a relatively compact neighborhood U_i of K_i^q such that ∂U_i is a smooth manifold, $F_q^{-1}(0) \cap U_i = K_i^q = \nabla f_q^{-1}(0) \cap U_i$. The result is clear since on ∂U_i, we have $\nabla f_q / \|\nabla f_q\| = F_q / \|F_q\|$. □

Corollary 7.5. — *Let $G_q : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the mapping defined by $G_q(\lambda; x) = (f(x)\lambda - 1, F_q(x))$. The set $G_q^{-1}(0)$ is compact and if $R > 0$ is such that $G_q^{-1}(0) \subsetneq B_R^{n+1}$, then*

$$\chi(X) = 1 - \deg_{S_R^n} G_q.$$

Here B_R^{n+1} and S_R^n are the ball and the sphere of radius R in $\mathbb{R}^{n+1}$.

Proof. — Since $G_q(\lambda; x) = 0$ if and only if $F_q(x) = 0$, $f(x) \neq 0$ and $\lambda = 1/f(x)$, it is straightforward to see that $G_q^{-1}(0)$ is compact. The rest of the proof is easy. □

These formulas are global versions of a result due to Khimshiasvili [13] on the Euler characteristic of the real Milnor fibre. It states that, if $g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ is an analytic function-germ with an isolated critical point at the origin, then

$$\chi\big(g^{-1}(\delta) \cap B_\varepsilon^n\big) = 1 - \operatorname{sign}(-\delta)^n \deg_0 \nabla g,$$

for any regular value δ of g, $0 < |\delta| \ll \varepsilon \ll 1$. Here $\deg_0 \nabla g$ is the topological degree of $\nabla g / \|\nabla g\| : S_\varepsilon^{n-1} \to S^{n-1}$.

In their fundamental paper [23], Petrovskii and Oleinik estimated the Euler characteristic of some real projective algebraic sets. More precisely they gave an upper bound for the quantities

- $|\chi(Y) - 1|$ where Y is a real projective hypersurface of even dimension;
- $|2\chi(Z_-) - 1|$ where Z_- is the subset of $\mathbb{R}P^n$ that is bounded by a real projective hypersurface Y of odd dimension and even degree and corresponds to the negative values of the polynomial that determines Y.

These results were generalized by Kharlamov [11], [12]. In [1], Arnol'd found a new proof, based on Khimshiashvili's formula, and an equivalent formulation of the original Petrovskii-Oleinik inequalities. Let us state

Arnol'd's version of these inequalities. We need some notations. With every n-tuple of positive integers $\mathbf{m} = (m_1, \ldots, m_n)$ and with every positive integer m_0, we will associate the objects:

- $\Delta_n(\mathbf{m})$ is the parallelepiped in $\mathbb{R}^n$ defined by the inequalities
$$0 \leqslant x_1 \leqslant m_1 - 1, \ldots, 0 \leqslant x_n \leqslant m_n - 1;$$
- $\mu = m_1 \cdots m_n$ is the number of integral points in $\Delta_n(\mathbf{m})$;
- $\nu = \frac{1}{2}(m_1 + \cdots + m_n - n)$ is the mean value of the sum of the coordinates of the points in $\Delta_n(\mathbf{m})$,
- $\Pi_n(\mathbf{m})$ is the number of integral points on the central section $x_1 + \cdots + x_n = \nu$ of the parallelepiped $\Delta_n(\mathbf{m})$;
- $\Pi_n(\mathbf{m}, m_0)$ is the number of integral points in $\Delta_n(\mathbf{m})$ that lie in the strip
$$\nu - \tfrac{1}{2}m_0 \leqslant x_1 + \cdots + x_n \leqslant \nu + \tfrac{1}{2}m_0;$$
- $O_n(\mathbf{m}, m_0)$ is the number of integral points in $\Delta_n(\mathbf{m})$ that satisfy the inequalities
$$\nu - \tfrac{1}{2}m_0 \leqslant x_1 + \cdots + x_n \leqslant \nu.$$

Arnol'd [1] proved the following theorem.

Theorem 7.6. — *Let f be a homogeneous polynomial of degree d in $\mathbb{R}^n$ defining a non-singular hypersurface Y in $\mathbb{R}P^{n-1}$. If n is even, we have*
$$\left|1 - \chi(Y)\right| \leqslant \Pi_n(\mathbf{d} - \mathbf{1}), \quad \text{where } \mathbf{d} - \mathbf{1} = (d-1, \ldots, d-1) \text{ in } \mathbb{N}^n.$$
If n is odd and d is even, let Z_- be the subset of $\mathbb{R}P^n$ that is bounded by Y and corresponds to the negative values of the polynomial f. We have
$$\left|1 - 2\chi(Z^-)\right| \leqslant \Pi_n(\mathbf{d} - \mathbf{1}).$$

Khovanskii [14] (see also [15]), gave an affine version of this theorem.

Proposition 7.7. — *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a polynomial of degree d such that the surface $\{f = 0\}$ is nonsingular and the domains $\{f \leqslant c\}$ are compact for every $c \in \mathbb{R}$. Then the Euler-Poincaré of the domain $\{f \leqslant 0\}$ satisfies the inequality*
$$\left|1 - 2\chi(\{f \leqslant 0\})\right| \leqslant \Pi_n(\mathbf{d} - \mathbf{1}, d-1),$$
where $\mathbf{d} - \mathbf{1} = (d-1, \ldots, d-1)$ in $\mathbb{N}^n$.

Our aim is to give a Petrovskii-Oleinik inequality for the Euler-Poincaré characteristic of any algebraic set in $\mathbb{R}^n$. Let X be an algebraic set in $\mathbb{R}^n$ defined as the zero set of the polynomials $f_1, \ldots, f_k$, each f_i having degree d_i. Hence $X = \{x \in \mathbb{R}^n \mid f(x) = 0\}$ where $f = f_1^2 + \cdots + f_k^2$. The degree of the polynomial f is $d = 2\max\{d_1, \ldots, d_k\}$. The following proposition gives an upper bound for $|1 - \chi(X)|$ in terms of d.

PROPOSITION 7.8. — *Let X be an algebraic set in $\mathbb{R}^n$ defined as the set of zeros of a nonnegative polynomial f of even degree d. We have*

$$\left|1 - \chi(X)\right| \leqslant O_{n+1}(\mathbf{d}+\mathbf{1}, 2),$$

where $\mathbf{d}+\mathbf{1} = (d+1, \ldots, d+1)$ in $\mathbb{N}^{n+1}$.

Proof. — Let $\omega : \mathbb{R}^n \to \mathbb{R}$ be defined by $\omega(x) = x_1^2 + \cdots + x_n^2$. Applying the argument described above to the functions f and ω, we find that there exists an integer q sufficiently big and a real $R > 0$ sufficiently big such that

$$\chi(X) = 1 - \deg_{S_R^n} G_q.$$

Let δ be a small positive regular value of G_q and let $\{p_1, \ldots, p_l\}$ be the set of preimages of δ by G_q lying in B_R^{n+1}. We have

$$1 - \chi(X) = \deg_{S_R^n}(G_q - \delta) = \sum_{j=1}^{\ell} \deg_{p_j}(G_q - \delta).$$

Since each component of $G_q - \delta$ has a degree not exceeding $d+1$, the square of the euclidian distance function in $\mathbb{R}^{n+1}$ has degree 2 and $2+(n+1)(d+1) \equiv n+1 \bmod 2$; Theorem 2 of [14] applied to the vector field $G_q - \delta$ and the function $R - (x_1^2 + \cdots + x_n^2 + \lambda^2)$ gives

$$\left|\sum_{j=1}^{\ell} \deg_{p_j}(G_q - \delta)\right| \leqslant O_{n+1}(\mathbf{d}+\mathbf{1}, 2),$$

where $\mathbf{d}+\mathbf{1} = (d+1, \ldots, d+1)$ in $\mathbb{N}^{n+1}$. □

BIBLIOGRAPHY

[1] V. I. ARNOLD, "Index of a singular point of a vector field, the Petrovski-Oleinik inequality, and mixed Hodge structures", *Funct. Anal. Appl.* **12** (1978), p. 1-14.

[2] J. BOCHNAK, M. COSTE & M. F. ROY, *Géométrie algébrique réelle*, Ergebnisse der Mathematik, vol. 12, Springer-Verlag, 1987.

[3] L. BROECKER & M. KUPPE, "Integral geometry of tame sets", *Geom. Dedicata* **82** (2000), p. 285-323.

[4] S. A. BROUGHTON, "On the topology of polynomial hypersurfaces, Singularities, Part 1 (Arcata, Calif., 1981), pp. 167–178", in *Proc. Sympos. Pure Math.*, vol. 40, Amer. Math. Soc., Providence, RI, 1983.

[5] M. COSTE, "An introduction to o-minimal geometry, in Dottorato di Recerca in Matematica", PhD Thesis, Dip. Mat. Univ. Pisa. Istituti Editoriali e Poligrafici Internazionali, Pisa, 2000.

[6] M. COSTE, "An introduction to semi-algebraic geometry, in Dottorato di Recerca in Matematica", PhD Thesis, Dip. Mat. Univ. Pisa. Istituti Editoriali e Poligrafici Internazionali, Pisa, 2000.

[7] M. Coste & M. Reguiat, Trivialités en famille, in *Real algebraic geometry* (Rennes, 1991), pp. 193–204, Lecture Notes in Math., vol. 1524, Springer, Berlin, 1992.

[8] A. H. Durfee, "Neighborhoods of algebraic sets", *Trans. Amer. Math. Soc.* **276** (1983), p. 517-530.

[9] N. Dutertre, "Geometrical and topological properties of real polynomial fibres", *Geom. Dedicata* **105** (2004), p. 43-59.

[10] A. Fekak, "Exposants de Lojasiewicz pour les fonctions semi-algébriques", *Ann. Polon. Math.* **56** (1992), p. 123-131.

[11] V. M. Kharlamov, "A generalized Petrovskii inequality", *Funct. Anal. Appl.* **8** (1974), p. 50-56.

[12] ———, "A generalized Petrovskii inequality II", *Funct. Anal. Appl.* **9** (1975), p. 93-94.

[13] G. M. Khimshiashvili, "On the local degree of a smooth map", *Soobshch. Akad. Nauk Gruz. SSR* **85** (1977), p. 309-311.

[14] A. G. Khovanskii, "Index of a polynomial vector field", *Funct. Anal. Appl.* **13** (1978), p. 38-45.

[15] ———, "Boundary indices of polynomial 1-forms with homogeneous components", *St. Petersburg Math. J.* **10** (1999), p. 553-575.

[16] K. Kurdyka, "On gradients of functions definable in o-minimal structures", *Ann. Inst. Fourier* **48** (1998), p. 769-783.

[17] K. Kurdyka, T. Mostowski & A. Parusinski, "Proof of the gradient conjecture of R. Thom", *Ann. of Math. (2)* **152** (2000), p. 763-792.

[18] K. Kurdyka & A. Parusinski, "w_f-stratification of subanalytic functions and the Lojasiewicz inequality", *C. R. Acad. Sci. Paris Sér. I Math.* **318** (1994), p. 129-133.

[19] S. Lojasiewicz, Une propriété topologique des sous-ensembles analytiques réels, *Colloques Internationaux du CNRS, Les équations aux dérivées partielles*, éd. B. Malgrange (Paris 1962), vol. 117, Publications du CNRS, Paris, 1963.

[20] ———, "Sur les trajectoires du gradient d'une fonction analytique réelle", *Seminari di Geometria 1982–1983, Bologna* (1984), p. 115-117.

[21] A. Nemethi & A. Zaharia, "Milnor fibration at infinity", *Indag. Math.* **3** (1992), p. 323-335.

[22] A. Nowel & Z. Szafraniec, "On trajectories of analytic gradient vector fields", *J. Differential Equations* **184** (2002), p. 215-223.

[23] O. A. Oleinik & I. G. Petrovskii, *On the topology of real algebraic surfaces*, Amer. Math. Soc. Transl., vol. 70, Amer. Math. Soc., 1952.

[24] M. Tibăr, "Regularity at infinity of real and complex polynomial functions", in *Singularity theory (Liverpool, 1996)*, London Math. Soc. Lecture Note Ser., vol. 263, Cambridge Univ. Press, 1999, p. xx, 249-264.

[25] L. Van den Dries & C. Miller, "Geometric categories and *o*-minimal structures", *Duke Math. J.* **84** (1996), p. 497-540.

Manuscrit reçu le 7 mars 2007,
accepté le 7 avril 2008.

Nicolas DUTERTRE
Université de Provence
Centre de Mathématiques et Informatique
39 rue Joliot-Curie
13453 Marseille Cedex 13 (France)
dutertre@cmi.univ-mrs.fr