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INTEGRAL REPRESENTATION OF THE n-TH DERIVATIVE IN DE BRANGES-ROVNYAK SPACES AND THE NORM CONVERGENCE OF ITS REPRODUCING KERNEL

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ABSTRACT. — In this paper, we give an integral representation for the boundary values of derivatives of functions of the de Branges–Rovnyak spaces $\mathcal{H}(b)$, where b is in the unit ball of $H^\infty(\mathbb{C}_+)$. In particular, we generalize a result of Ahern–Clark obtained for functions of the model spaces K_b, where b is an inner function. Using hypergeometric series, we obtain a nontrivial formula of combinatorics for sums of binomial coefficients. Then we apply this formula to show the norm convergence of reproducing kernel $k_{\omega,n}^b$ of evaluation of the n-th derivative of elements of $\mathcal{H}(b)$ at the point ω as it tends radially to a point of the real axis.

RÉSUMÉ. — Dans cet article, nous donnons une formule intégrale pour la valeur au bord des dérivées des fonctions de l'espace de de Branges-Rovnyak $\mathcal{H}(b)$, où b est une fonction dans la boule unité de $H^\infty(\mathbb{C}_+)$. En particulier, nous généralisons un résultat d'Ahern-Clark obtenu pour les fonctions de l'espace modèle K_b, où b est une fonction intérieure. En utilisant les séries hypergéométriques, nous obtenons une formule non-triviale de combinatoire concernant la somme de coefficients binômiaux. Puis, nous appliquons cette formule pour démontrer que le noyau reproduisant $k_{\omega,n}^b$, correspondant à l'évaluation de la dérivée n-ième des fonctions de $\mathcal{H}(b)$ au point ω, converge en norme lorsque ω tend radialement vers un point de l'axe réel.

1. Introduction

Let $\mathbb{C}_+$ denote the upper half plane in the complex plane and let $H^2(\mathbb{C}_+)$ denote the usual Hardy space consisting of analytic functions f on $\mathbb{C}_+$

which satisfy

$$\|f\|_2 := \sup_{y>0} \left(\int_{\mathbb{R}} |f(x+iy)|^2 \, dx \right)^{1/2} < +\infty.$$

P. Fatou [12] proved that, for any function f in $H^2(\mathbb{C}_+)$ and for almost all x_0 in $\mathbb{R}$,

$$f^*(x_0) := \lim_{t \to 0^+} f(x_0 + it)$$

exists. Moreover, we have $f^* \in L^2(\mathbb{R})$, $\mathcal{F}f^* = 0$ on $(-\infty, 0)$, where $\mathcal{F}$ is the Fourier–Plancherel transformation, and $\|f^*\|_2 = \|f\|_2$. Of course the boundary points where the radial limit exists depend on the function f. However we cannot say more about the boundary behavior of a typical element of $H^2(\mathbb{C}_+)$. Then many authors, e.g. [16, 1, 2, 14], have studied this question by restricting the class of functions. A particularly interesting class of subspaces of $H^2(\mathbb{C}_+)$ consists of de Branges–Rovnyak spaces.

For $\varphi \in L^\infty(\mathbb{R})$, let T_φ stand for the Toeplitz operator defined on $H^2(\mathbb{C}_+)$ by

$$T_\varphi(f) := P_+(\varphi f), \qquad (f \in H^2(\mathbb{C}_+)),$$

where P_+ denotes the orthogonal projection of $L^2(\mathbb{R})$ onto $H^2(\mathbb{C}_+)$. Then, for $\varphi \in L^\infty(\mathbb{R})$, $\|\varphi\|_\infty \leqslant 1$, the de Branges–Rovnyak space $\mathcal{H}(\varphi)$, associated with φ, consists of those $H^2(\mathbb{C}_+)$ functions which are in the range of the operator $(Id - T_\varphi T_{\overline{\varphi}})^{1/2}$. It is a Hilbert space when equipped with the inner product

$$\langle (Id - T_\varphi T_{\overline{\varphi}})^{1/2} f, (Id - T_\varphi T_{\overline{\varphi}})^{1/2} g \rangle_\varphi = \langle f, g \rangle_2,$$

where $f, g \in H^2(\mathbb{C}_+) \ominus \ker (Id - T_\varphi T_{\overline{\varphi}})^{1/2}$.

These spaces (and more precisely their general vector-valued version) appeared first in L. de Branges and J. Rovnyak [7, 8] as universal model spaces for Hilbert space contractions. As a special case, when b is an inner function (that is $|b| = 1$ a.e. on $\mathbb{R}$), the operator $(Id - T_b T_{\overline{b}})$ is an orthogonal projection and $\mathcal{H}(b)$ becomes a closed (ordinary) subspace of $H^2(\mathbb{C}_+)$ which coincides with the so-called model spaces $K_b = H^2(\mathbb{C}_+) \ominus bH^2(\mathbb{C}_+)$. Thanks to the pioneer works of Sarason, e.g. [18], we know that de Branges-Rovnyak spaces have an important role to be played in numerous questions of complex analysis and operator theory. We mention a recent paper of A. Hartmann, D. Sarason and K. Seip [15] who give a nice characterization of surjectivity of Toeplitz operator and the proof involves the de Branges-Rovnyak spaces. We also refer to works of J. Shapiro [19, 20] concerning the notion of angular derivative for holomorphic self-maps of the unit disk. See

also a paper of J. Anderson and J. Rovnyak [3], where generalized Schwarz-Pick estimates are given and a paper of M. Jury [17], where composition operators are studied by methods based on $\mathcal{H}(b)$ spaces.

In the case where b is an inner function, H. Helson [16] studied the problem of analytic continuation across the boundary for functions in K_b. Then, still when b is an inner function, P. Ahern and D. Clark [1] characterized those points x_0 of $\mathbb{R}$ where every function f of K_b and all its derivatives up to order n have a radial limit. More precisely, if $b = BI_\mu$ is the canonical factorization of the inner function b into Blaschke product B associated with the sequence $(z_k)_k$ and singular inner part I_μ associated with the singular measure μ, then every function $f \in K_b$ and its derivatives up to order n have finite radial limits at x_0 if and only if

$$(1.1) \qquad \sum_k \frac{\Im m(z_k)}{|x_0 - z_k|^{2n+2}} + \int_\mathbb{R} \frac{d\mu(t)}{|t - x_0|^{2n+2}} < +\infty.$$

Recently, we [14] gave an extension of the preceding results of Helson and of Ahern–Clark. See also the paper of E. Fricain [13] where the orthogonal and Riesz basis of $\mathcal{H}(b)$ spaces, which consist of reproducing kernels, are studied.

Now, using Cauchy formula, it is easy to see that if b is inner, $\omega \in \mathbb{C}_+$, n is a non-negative integer and $f \in K_b$, then we have

$$(1.2) \qquad f^{(n)}(\omega) = \int_\mathbb{R} f(t) \, \overline{k^b_{\omega,n}(t)} \, dt,$$

where

$$(1.3) \qquad \frac{k^b_{\omega,n}(z)}{n!} := \frac{i}{2\pi} \frac{1 - b(z) \sum_{p=0}^n \frac{\overline{b^{(p)}(\omega)}}{p!}(z - \overline{\omega})^p}{(z - \overline{\omega})^{n+1}}, \qquad (z \in \mathbb{C}_+).$$

A natural question is to ask if one can extend the formula (1.2) at boundary points x_0. If x_0 is a real point which does not belong to the boundary spectrum of b, then b and all functions of K_b are analytic through a neighborhood of x_0 and then it is obvious that the formula (1.2) is valid at the point x_0. On the other hand, if x_0 satisfies the condition (1.1), then Ahern–Clark [1] showed that the formula (1.2) is still valid at the point $x_0 \in \mathbb{R}$. Recently, K. Dyakonov [10, 11] and then A. Baranov [5] used this formula to get some Bernstein type inequalities in the model spaces K_b.

In this paper, our first goal is to obtain an analogue of formula (1.2) for the de Branges–Rovnyak spaces $\mathcal{H}(b)$, where b is an *arbitrary* function in the unit ball of $H^\infty(\mathbb{C}_+)$ (not necessarily inner). We will provide an integral representation for $f^{(n)}(\omega)$, $\omega \in \mathbb{C}_+$, and also show that under certain

conditions the formula remains valid if $\omega = x_0 \in \mathbb{R}$. However, if one tries to generalize techniques used in the model spaces K_b in order to obtain such a representation for the derivative of functions in $\mathcal{H}(b)$, some difficulties appear mainly due to the fact that the evaluation functional in $\mathcal{H}(b)$ (contrary to the model spaces K_b) is not a usual integral operator. Nevertheless, we will overcome this difficulty and provide an integral formula similar to (1.2) for functions in $\mathcal{H}(b)$.

Our second goal is to prove the norm convergence of reproducing kernels of evaluation functional of the n-th derivative as we approach a boundary point. If $n = 0$, for de Branges–Rovnyak spaces of the unit disc, Sarason [18, page 48] showed that

$$\|k_{z_0}^b\|_b^2 = z_0 \overline{b(z_0)} b'(z_0), \qquad (z_0 \in \mathbb{T}).$$

We first obtain

$$\|k_{x_0,n}^b\|_b^2 = \frac{n!^2}{2i\pi} \sum_{p=0}^{n} \frac{\overline{b^{(p)}(x_0)}}{p!} \frac{b^{(2n+1-p)}(x_0)}{(2n+1-p)!}, \qquad (x_0 \in \mathbb{R}),$$

which is an analogue (and generalization) of Sarason's formula for the reproducing kernel of the n-th derivative for de Branges–Rovnyak spaces of the upper half plane. Then we apply this identity to show that $\|k_{\omega,n}^b - k_{x_0,n}^b\|_b \to 0$ as ω tends radially to x_0. Again if $n = 0$, this result is due to Sarason. In establishing the norm convergence we naturally face with the (nontrivial) finite sum

$$(1.4) \quad (-1)^{r+1} \sum_{p=0}^{n} \sum_{\ell=0}^{n} (-2)^{p-\ell} \binom{r}{n-\ell} \binom{2n+1-r}{p} \binom{n-p+\ell}{\ell},$$

with $n, r \in \mathbb{N}$, $0 \leqslant r \leqslant 2n + 1$. Using hypergeometric series we show that this sum is equal to $\pm 2^n$, where the choice of sign depends on r.

We mention a recent and very interesting work of V. Bolotnikov and A. Kheifets [6] who obtained an analogue of the classical Carathéodory–Julia theorem on boundary derivatives. Using different techniques, the authors also obtained a condition which guarantees that we can write an analogue of formula (1.2) for the de Branges–Rovnyak spaces $\mathcal{H}(b)$. More precisely, this condition is

$$(1.5) \qquad \liminf_{\omega \to x_0} \frac{\partial^{2n}}{\partial \omega^n \partial \overline{\omega}^n} \left(\frac{1 - |b(\omega)|^2}{\Im m \, \omega} \right) < +\infty,$$

and it is stated that this is equivalent to the existence of the boundary Schwarz-Pick matrix at point x_0. They also got the norm convergence (under their condition). Comparing condition (1.5) with our condition (2.2) is under further investigation.

The plan of the paper is the following. In the next section, we give some preliminaries concerning the de Branges-Rovnyak spaces. In the third section, we establish an integral formula for the n-th derivative of functions in $\mathcal{H}(b)$. The fourth section contains the part of combinatorics of this paper. In particular, we show how we can compute the sum (1.4) and get an interesting and quite surprising formula. Finally, in the last section, we apply this formula of combinatorics to solve an important problem of norm convergence for the kernels $k_{\omega,n}^b$ corresponding to the n-th derivative at points ω for functions in $\mathcal{H}(b)$. More precisely, we prove that $k_{\omega,n}^b$ tends in norm to $k_{x_0,n}^b$ as ω tends radially to x_0. We also get some interesting relations between the derivatives of the function b at point x_0.

2. Preliminaries

We first recall two general facts about the de Branges-Rovnyak spaces. As a matter of fact, in [18], these results are formulated for the unit disc. However, the same results with similar proofs also work for the upper half plane. The first one concerns the relation between $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$. For $f \in H^2(\mathbb{C}_+)$, we have [18, page 10]

$$f \in \mathcal{H}(b) \Longleftrightarrow T_{\bar{b}}f \in \mathcal{H}(\bar{b}).$$

Moreover, if $f_1, f_2 \in \mathcal{H}(b)$, then

$$(2.1) \qquad \langle f_1, f_2 \rangle_b = \langle f_1, f_2 \rangle_2 + \langle T_{\bar{b}}f_1, T_{\bar{b}}f_2 \rangle_{\bar{b}}.$$

We also mention an integral representation for functions in $\mathcal{H}(\bar{b})$ [18, page 16]. Let $\rho(t) := 1 - |b(t)|^2$, $t \in \mathbb{R}$, and let $L^2(\rho)$ stand for the usual Hilbert space of measurable functions $f : \mathbb{R} \to \mathbb{C}$ with $\|f\|_\rho < \infty$, where

$$\|f\|_\rho^2 := \int_{\mathbb{R}} |f(t)|^2 \rho(t)\, dt.$$

For each $w \in \mathbb{C}_+$, the Cauchy kernel k_w belongs to $L^2(\rho)$. Hence, we define $H^2(\rho)$ to be the span in $L^2(\rho)$ of the functions k_w ($w \in \mathbb{C}_+$). If q is a function in $L^2(\rho)$, then $q\rho$ is in $L^2(\mathbb{R})$, being the product of $q\rho^{1/2} \in L^2(\mathbb{R})$ and the bounded function $\rho^{1/2}$. Finally, we define the operator $C_\rho : L^2(\rho) \longrightarrow H^2(\mathbb{C}_+)$ by

$$C_\rho(q) := P_+(q\rho).$$

Then C_ρ is a partial isometry from $L^2(\rho)$ onto $\mathcal{H}(\bar{b})$ whose initial space equals to $H^2(\rho)$ and it is an isometry if and only if b is an extreme point of the unit ball of $H^\infty(\mathbb{C}_+)$.

In [14], we have studied the boundary behavior of functions of the de Branges–Rovnyak spaces and we mention some parts of [14, Theorem 3.1] that we need here.

THEOREM 2.1. — *Let b be in the unit ball of $H^\infty(\mathbb{C}_+)$ and let*

$$b(z) = \prod_k e^{i\alpha_k} \frac{z - z_k}{z - \overline{z_k}} \, \exp\left(-\frac{1}{i\pi} \int_{\mathbb{R}} \frac{tz + 1}{(t - z)(t^2 + 1)} \, d\mu(t)\right)$$

$$\exp\left(\frac{1}{i\pi} \int_{\mathbb{R}} \frac{tz + 1}{t - z} \frac{\log |b(t)|}{t^2 + 1} \, dt\right)$$

be its canonical factorization. Then, for $x_0 \in \mathbb{R}$ and for a non-negative integer n, the following are equivalent:

(i) *for every function $f \in \mathcal{H}(b)$, $f(x_0 + it)$, $f'(x_0 + it), \ldots f^{(n)}(x_0 + it)$ have finite limits as $t \to 0^+$;*

(ii) *we have*

$$(2.2) \quad \sum_k \frac{\Im m(z_k)}{|x_0 - z_k|^{2n+2}} + \int_{\mathbb{R}} \frac{d\mu(t)}{|x_0 - t|^{2n+2}} + \int_{\mathbb{R}} \frac{|\log |b(t)||}{|x_0 - t|^{2n+2}} \, dt < +\infty.$$

For $f \in \mathcal{H}(b)$, $x_0 \in \mathbb{R}$ and for a non-negative integer n, if $f^{(n)}(x_0 + it)$ has a finite limit as $t \to 0^+$, then we define

$$f^{(n)}(x_0) := \lim_{t \to 0^+} f^{(n)}(x_0 + it).$$

Moreover, under the condition (2.2), we know that for $0 \leqslant j \leqslant 2n + 1$,

$$(2.3) \qquad\qquad \lim_{t \to 0^+} b^{(j)}(x_0 + it)$$

exists (see [2, Lemma 4]) and we denote this limit by $b^{(j)}(x_0)$.

Remark 2.2. — Let $x_0 \in \mathbb{R}$ and suppose that x_0 does not belong to the spectrum $\sigma(b)$ of b, which means (by definition) that, for some $\eta > 0$, b is analytic on $B(x_0, \eta) := \{z \in \mathbb{C} : |z - x_0| < \eta\}$ and $|b(x)| = 1$ on $(x_0 - \eta, x_0 + \eta)$. Denote by $a_p := \dfrac{b^{(p)}(x_0)}{p!}$, $p \geqslant 0$. Since

$$b(x) = \sum_{p=0}^{\infty} a_p(x - x_0)^p, \qquad x \in (x_0 - \eta, x_0 + \eta),$$

we get

$$1 = |b(x)|^2 = b(x)\overline{b(x)} = \sum_{r=0}^{\infty} c_r(x - x_0)^r,$$

where $c_r = \sum_{p=0}^{r} a_p \, \overline{a_{r-p}}$. Hence

$$c_0 = |a_0|^2 = 1 \quad \text{and} \quad \sum_{p=0}^{r} a_p \, \overline{a_{r-p}} = 0, \quad (\forall r \geqslant 1).$$

As we will see in the proof of Theorem 3.3, the condition (2.2) implies that

$$|a_0|^2 = 1 \quad \text{and} \quad \sum_{p=0}^{r} a_p \, \overline{a_{r-p}} = 0, \quad (1 \leqslant r \leqslant n).$$

Therefore, the condition (2.2) is somehow a weaker version of the assumption $x_0 \notin \sigma(b)$.

The next result gives a (standard) Taylor formula at a point on the boundary.

LEMMA 2.3. — *Let h be a holomorphic function in the upper-half plane $\mathbb{C}_+$, let n be a non-negative integer and let $x_0 \in \mathbb{R}$. Assume that $h^{(n)}$ has a radial limit at x_0. Then $h, h', \ldots, h^{(n-1)}$ have radial limits at x_0 and*

$$h(\omega) = \sum_{p=0}^{n} \frac{h^{(p)}(x_0)}{p!} (\omega - x_0)^p + (\omega - x_0)^n \varepsilon(\omega), \qquad (\omega \in \mathbb{C}_+),$$

with $\lim_{t \to 0^+} \varepsilon(x_0 + it) = 0$.

Proof. — The case $n = 1$ is contained in [18, Chap. VI]. To establish the general case one assumes as the induction hypothesis that the property is true for $n - 1$. Applying the induction hypothesis to h', we see that $h', h^{(2)}, \ldots, h^{(n)}$ have a radial limit at x_0 and

$$h'(\omega) = \sum_{p=0}^{n-1} \frac{h^{(p+1)}(x_0)}{p!} (\omega - x_0)^p + (\omega - x_0)^{n-1} \varepsilon_1(\omega),$$

with $\lim_{t \to 0^+} \varepsilon_1(x_0 + it) = 0$. Since h' has a radial limit at x_0, by the case $n = 1$, $h(x_0) = \lim_{t \to 0} h(x_0 + it)$ exists and an application of Cauchy's theorem shows that

$$h(\omega) = h(x_0) + \int_{[x_0, \omega]} h'(u) \, du,$$

for all $\omega = x_0 + it$, $t > 0$. Hence we have

$$h(\omega) = h(x_0) + \int_{[x_0,\omega]} \left(\sum_{p=0}^{n-1} \frac{h^{(p+1)}(x_0)}{p!}(u - x_0)^p + (u - x_0)^{n-1}\varepsilon_1(u) \right) du$$

$$= \sum_{p=0}^{n} \frac{h^{(p)}(x_0)}{p!}(\omega - x_0)^p + \int_{[x_0,\omega]} (u - x_0)^{n-1}\varepsilon_1(u)\, du.$$

Finally, let

$$\varepsilon(\omega) = \frac{1}{(\omega - x_0)^n} \int_{[x_0,\omega]} (u - x_0)^{n-1}\varepsilon_1(u)\, du.$$

It is clear that $\lim_{t \to 0^+} \varepsilon(x_0 + it) = 0$.   □

3. Integral representations

We first begin by proving an integral representation for the derivatives of elements of $\mathcal{H}(b)$ at points ω in the upper half plane. Since ω is away from the boundary, the representation is easy to establish. Let b be a point in the unit ball of $H^\infty(\mathbb{C}_+)$. Recall that for $\omega \in \mathbb{C}_+$, the function

$$k_\omega^b(z) = \frac{i}{2\pi} \frac{1 - \overline{b(\omega)}b(z)}{z - \overline{\omega}}, \qquad (z \in \mathbb{C}_+),$$

is the reproducing kernel of $\mathcal{H}(b)$, that is

$$(3.1) \qquad\qquad f(\omega) = \langle f, k_\omega^b \rangle_b, \qquad (f \in \mathcal{H}(b)).$$

Now let $\omega \in \mathbb{C}_+$ and let n be a non-negative integer. In order to get an integral representation for the nth derivative of f at point ω for functions in the de-Branges-Rovnyak spaces, we need to introduce the following kernels

$$(3.2) \qquad \frac{k_{\omega,n}^b(z)}{n!} := \frac{i}{2\pi} \frac{1 - b(z) \displaystyle\sum_{p=0}^{n} \frac{\overline{b^{(p)}(\omega)}}{p!}(z - \overline{\omega})^p}{(z - \overline{\omega})^{n+1}}, \qquad (z \in \mathbb{C}_+),$$

and

$$(3.3) \qquad \frac{k_{\omega,n}^\rho(t)}{n!} := \frac{i}{2\pi} \frac{\displaystyle\sum_{p=0}^{n} \frac{\overline{b^{(p)}(\omega)}}{p!}(t - \overline{\omega})^p}{(t - \overline{\omega})^{n+1}}, \qquad (t \in \mathbb{R}).$$

For $n = 0$, we see that $k_{\omega,0}^b = k_\omega^b$ and $k_{\omega,0}^\rho = \overline{b(\omega)}k_\omega$. Moreover, we also see that the kernel $k_{\omega,n}^b$ coincides with those of the inner case defined by formula (1.3).

PROPOSITION 3.1. — *Let b be a point in the unit ball of $H^\infty(\mathbb{C}_+)$, let $f \in \mathcal{H}(b)$ and let $g \in H^2(\rho)$ be such that $T_{\bar{b}}f = C_\rho(g)$. Then, for all $\omega \in \mathbb{C}_+$ and for any non-negative integer n, we have $k^b_{\omega,n} \in \mathcal{H}(b)$ and $k^\rho_{\omega,n} \in H^2(\rho)$ and*

$$(3.4) \quad f^{(n)}(\omega) = \langle f, k^b_{\omega,n} \rangle_b = \int_{\mathbb{R}} f(t)\overline{k^b_{\omega,n}(t)}\, dt + \int_{\mathbb{R}} g(t)\rho(t)\overline{k^\rho_{\omega,n}(t)}\, dt.$$

Proof. — According to (3.1) and (2.1), we have

$$f(\omega) = \langle f, k^b_\omega \rangle_b = \langle f, k^b_\omega \rangle_2 + \langle T_{\bar{b}}f, T_{\bar{b}}k^b_\omega \rangle_{\bar{b}}.$$

But using the fact that $k^b_\omega = k_\omega - \overline{b(\omega)}bk_\omega$ and that $T_{\bar{b}}k_\omega = \overline{b(\omega)}k_\omega$, we obtain

$$T_{\bar{b}}k^b_\omega = \overline{b(\omega)}\left(k_\omega - P_+(|b|^2 k_\omega)\right) = \overline{b(\omega)}P_+\left((1 - |b|^2)k_\omega\right) = \overline{b(\omega)}C_\rho(k_\omega),$$

which implies that

$$f(\omega) = \langle f, k^b_\omega \rangle_2 + b(\omega)\langle C_\rho(g), C_\rho(k_\omega) \rangle_{\bar{b}}.$$

Since C_ρ is a partial isometry from $L^2(\rho)$ onto $\mathcal{H}(\bar{b})$, with initial space equals to $H^2(\rho)$, we conclude that

$$f(\omega) = \langle f, k^b_\omega \rangle_2 + b(\omega)\langle g, k_\omega \rangle_\rho = \langle f, k^b_{\omega,0} \rangle_2 + \langle \rho g, k^\rho_{\omega,0} \rangle_2,$$

which gives the representation (3.4) for $n = 0$.

Now straightforward computations show that

$$\frac{\partial^n k^b_{\omega,0}}{\partial \overline{\omega}^n} = k^b_{\omega,n} \quad \text{and} \quad \frac{\partial^n k^\rho_{\omega,0}}{\partial \overline{\omega}^n} = k^\rho_{\omega,n}.$$

Since $k^b_{\omega,0} \in \mathcal{H}(b)$ and $k^\rho_{\omega,0} \in H^2(\rho)$, we thus have $k^b_{\omega,n} \in \mathcal{H}(b)$ and $k^\rho_{\omega,n} \in H^2(\rho)$, $n \geqslant 0$. The representation (3.4) follows now by induction and by differentiating under the integral sign, which is justified by the dominated convergence theorem. □

In the following, we show that (3.4) is still valid at the boundary points x_0 which satisfy (2.2). We will need the boundary analogues of the kernels (3.2) and (3.3), i.e.

$$(3.5) \quad \frac{k^b_{x_0,n}(z)}{n!} := \frac{i}{2\pi} \frac{1 - b(z)\sum_{p=0}^n \dfrac{\overline{b^{(p)}(x_0)}}{p!}(z - x_0)^p}{(z - x_0)^{n+1}}, \quad (z \in \mathbb{C}_+),$$

and

$$(3.6) \quad \frac{k^\rho_{x_0,n}(t)}{n!} := \frac{i}{2\pi} \frac{\sum_{p=0}^n \dfrac{\overline{b^{(p)}(x_0)}}{p!}(t - x_0)^p}{(t - x_0)^{n+1}}, \quad (t \in \mathbb{R} \setminus \{x_0\}).$$

The following result shows that, under condition (2.2), $k_{x_0,n}^b$ is the kernel function in $\mathcal{H}(b)$ for the functional of the n-th derivative at x_0.

LEMMA 3.2. — *Let b be a point in the unit ball of $H^\infty(\mathbb{C}_+)$, let n be a non-negative integer and let $x_0 \in \mathbb{R}$. Assume that x_0 satisfies the condition (2.2). Then $k_{x_0,n}^b \in \mathcal{H}(b)$ and, for every function $f \in \mathcal{H}(b)$, we have*

$$(3.7) \qquad\qquad f^{(n)}(x_0) = \langle f, k_{x_0,n}^b \rangle_b.$$

Proof. — According to Theorem 2.1, the condition (2.2) guarantees that, for every function $f \in \mathcal{H}(b)$, $f^{(n)}(\omega)$ tends to $f^{(n)}(x_0)$, as ω tends radially to x_0. Therefore, an application of the uniform boundedness principle shows that the functional $f \longmapsto f^{(n)}(x_0)$ is bounded on $\mathcal{H}(b)$. Hence, by Riesz' theorem, there exists $\varphi_{x_0,n} \in \mathcal{H}(b)$ such that

$$f^{(n)}(x_0) = \langle f, \varphi_{x_0,n} \rangle_b, \qquad (f \in \mathcal{H}(b)).$$

Since

$$f^{(n)}(\omega) = \langle f, \frac{\partial^n k_{\omega,0}^b}{\partial \overline{\omega}^n} \rangle_b = \langle f, k_{\omega,n}^b \rangle_b, \qquad (f \in \mathcal{H}(b)),$$

we see that $k_{\omega,n}^b$ tends weakly to $\varphi_{x_0,n}$, as ω tends radially to x_0. Thus, for $z \in \mathbb{C}_+$, we can write

$$\varphi_{x_0,n}(z) = \langle \varphi_{x_0,n}, k_z^b \rangle_b = \lim_{t \to 0^+} \langle k_{x_0+it,n}^b, k_z^b \rangle_b = \lim_{t \to 0^+} k_{x_0+it,n}^b(z)$$

$$= \lim_{t \to 0^+} n! \frac{i}{2\pi} \frac{1 - b(z) \sum_{p=0}^n \dfrac{\overline{b^{(p)}(x_0+it)}}{p!}(z - x_0 + it)^p}{(z - x_0 + it)^{n+1}}$$

$$= n! \frac{i}{2\pi} \frac{1 - b(z) \sum_{p=0}^n \dfrac{\overline{b^{(p)}(x_0)}}{p!}(z - x_0)^p}{(z - x_0)^{n+1}},$$

which implies that $\varphi_{x_0,n} = k_{x_0,n}^b$. Hence $k_{x_0,n}^b \in \mathcal{H}(b)$ and for every function $f \in \mathcal{H}(b)$ we have

$$f^{(n)}(x_0) = \langle f, k_{x_0,n}^b \rangle_b.$$

$\square$

For $n = 0$, Lemma 3.2 appears in [18, Chap. V], in the context of the unit disc. The problem with the representation (3.7) is that the inner product in $\mathcal{H}(b)$ is not an explicit integral formula and thus it is not convenient to use it. That is why we prefer to have an integral formula of type (3.4).

If x_0 satisfies the condition (2.2) we also have $k_{x_0,n}^\rho \in L^2(\rho)$. Indeed, according to (3.6), it suffices to prove that $(t - x_0)^{-j} \in L^2(\rho)$, for $1 \leqslant j \leqslant$

$n + 1$. Since $\rho \leqslant 1$, it is enough to verify this fact in a neighborhood of x_0, say $I_{x_0} = [x_0 - 1, x_0 + 1]$. But according to the condition (2.2), we have

$$\int_{I_{x_0}} \frac{1 - |b(t)|^2}{|t - x_0|^{2j}} \, dt \leqslant 2 \int_{I_{x_0}} \frac{|\log |b(t)||}{|t - x_0|^{2j}} \, dt \leqslant 2 \int_{I_{x_0}} \frac{|\log |b(t)||}{|t - x_0|^{2(n+1)}} \, dt < +\infty.$$

THEOREM 3.3. — *Let b be a point in the unit ball of $H^\infty(\mathbb{C}_+)$, let n be a non-negative integer, let $f \in \mathcal{H}(b)$ and let $g \in H^2(\rho)$ be such that $T_{\bar{b}} f = C_\rho(g)$. Then, for every point $x_0 \in \mathbb{R}$ satisfying the condition (2.2), we have*

$$(3.8) \qquad f^{(n)}(x_0) = \int_{\mathbb{R}} f(t) \overline{k_{x_0,n}^b(t)} \, dt + \int_{\mathbb{R}} g(t) \rho(t) \overline{k_{x_0,n}^\rho(t)} \, dt.$$

Proof. — Recall that according to (2.3), the condition (2.2) guarantees that $b^{(j)}(x_0)$ exists for $0 \leqslant j \leqslant 2n + 1$. Moreover, Lemma 3.2 implies that $k_{x_0,p}^b \in \mathcal{H}(b)$, for $0 \leqslant p \leqslant n$. First of all, we prove that

$$h_{x_0,n}(z) := \frac{b(z) - \sum_{p=0}^{n} \frac{b^{(p)}(x_0)}{p!} (z - x_0)^p}{(z - x_0)^{n+1}}, \qquad (z \in \mathbb{C}_+),$$

satisfies

$$(3.9) \qquad h_{x_0,n} = 2i\pi \sum_{p=0}^{n} \frac{b^{(n-p)}(x_0)}{(n-p)!p!} k_{x_0,p}^b.$$

To simplify a little bit the next computations, we put $a_p := \frac{b^{(p)}(x_0)}{p!}$, $0 \leqslant p \leqslant n$. According to (3.2), we have

$$2i\pi \sum_{p=0}^{n} a_{n-p} \frac{k_{x_0,p}^b(z)}{p!} = \sum_{p=0}^{n} a_{n-p} \left(\frac{\sum_{j=0}^{p} \overline{a_j}(z - x_0)^j b(z) - 1}{(z - x_0)^{p+1}} \right)$$

$$= \frac{\sum_{p=0}^{n} a_{n-p}(z - x_0)^{n-p} \left(b(z) \sum_{j=0}^{p} \overline{a_j}(z - x_0)^j - 1 \right)}{(z - x_0)^{n+1}}$$

$$= \frac{b(z) \left(\sum_{p=0}^{n} \sum_{j=0}^{p} a_{n-p} \overline{a_j}(z - x_0)^{n-p+j} \right) - \sum_{k=0}^{n} a_k(z - x_0)^k}{(z - x_0)^{n+1}}.$$

Therefore, we see that (3.9) is equivalent to

$$(3.10) \qquad \sum_{p=0}^{n}\sum_{j=0}^{p} a_{n-p}\overline{a_j}(z-x_0)^{n-p+j} = 1.$$

But, putting $j = \ell - n + p$, we obtain

$$\sum_{p=0}^{n}\sum_{j=0}^{p} a_{n-p}\overline{a_j}(z-x_0)^{n-p+j} = \sum_{\ell=0}^{n}\left(\sum_{p=n-\ell}^{n} a_{n-p}\overline{a_{\ell-n+p}}\right)(z-x_0)^{\ell}$$

$$= \sum_{\ell=0}^{n}\left(\sum_{q=0}^{\ell} a_{\ell-q}\overline{a_q}\right)(z-x_0)^{\ell}.$$

Consequently, (3.10) is equivalent to

$$(3.11) \qquad |b(x_0)|^2 = 1 \quad \text{and} \quad \sum_{q=0}^{\ell} a_{\ell-q}\overline{a_q} = 0, \qquad (1 \leqslant \ell \leqslant n).$$

Now if we define

$$\varphi(z) := 1 - b(z)\sum_{p=0}^{n}\overline{a_p}(z-x_0)^p, \qquad (z \in \mathbb{C}_+),$$

then φ is holomorphic in $\mathbb{C}_+$ and according to (2.3), φ and its derivatives up to $2n+1$ have radial limits at x_0. An application of Lemma 2.3 shows that we can write

$$\varphi(z) = \sum_{p=0}^{n}\frac{\varphi^{(p)}(x_0)}{p!}(z-x_0)^p + o((z-x_0)^n),$$

as z tends radially to x_0. Assume that there exists $k \in \{0,\dots,n\}$ such that $\varphi^{(k)}(x_0) \neq 0$ and let

$$j_0 := \min\{0 \leqslant p \leqslant n : \varphi^{(p)}(x_0) \neq 0\}.$$

Hence, as $t \to 0^+$,

$$|k_{x_0,n}^b(x_0+it)| \sim \frac{1}{2\pi}\frac{|\varphi^{(j_0)}(x_0)|}{j_0!}t^{j_0-(n+1)},$$

which implies that $\lim_{t\to 0^+}|k_{x_0,n}^b(x_0+it)| = +\infty$. This is a contradiction with the fact that $k_{x_0,n}^b$ belongs to $\mathcal{H}(b)$ and has a finite radial limit at x_0. Therefore we necessarily have $\varphi^{(\ell)}(x_0) = 0$, $0 \leqslant \ell \leqslant n$. But $\varphi(x_0) =$

$1 - b(x_0)\overline{b(x_0)} = 1 - |b(x_0)|^2$ and if we use the Leibniz' rule to compute the derivative of φ, for $1 \leqslant \ell \leqslant n$, we get

$$\varphi^{(\ell)}(x_0) = -\sum_{p=0}^{\ell} \overline{a_p} \binom{\ell}{p} p! b^{(\ell-p)}(x_0) = -\ell! \sum_{p=0}^{\ell} \overline{a_p} a_{\ell-p},$$

which gives (3.11). Hence (3.9) is proved. According to Lemma 3.2, (3.9) implies $h_{x_0,n} \in \mathcal{H}(b)$. Now for almost all $t \in \mathbb{R}$, we have

$$\overline{b(t)} \frac{k_{x_0,n}^b(t)}{n!} = \frac{i}{2\pi} \frac{\overline{b(t)} - |b(t)|^2 \sum_{p=0}^{n} \overline{a_p}(t - x_0)^p}{(t - x_0)^{n+1}}$$

$$= \frac{i}{2\pi}(1 - |b(t)|^2) \frac{\sum_{p=0}^{n} \overline{a_p}(t - x_0)^p}{(t - x_0)^{n+1}} + \frac{i}{2\pi} \frac{\overline{b(t)} - \sum_{p=0}^{n} \overline{a_p}(t - x_0)^p}{(t - x_0)^{n+1}}$$

$$= \rho(t) \frac{k_{x_0,n}^\rho}{n!} + \frac{i}{2\pi} \overline{h_{x_0,n}(t)}.$$

Since $h_{x_0,n} \in \mathcal{H}(b) \subset H^2(\mathbb{C}_+)$, we get that $P_+(\overline{b} k_{x_0,n}^b) = P_+(\rho k_{x_0,n}^\rho)$, which can be written as $T_{\overline{b}} k_{x_0,n}^b = C_\rho k_{x_0,n}^\rho$. It follows from (2.1) and Lemma 3.2 that

$$f^{(n)}(x_0) = \langle f, k_{x_0,n}^b \rangle_b$$
$$= \langle f, k_{x_0,n}^b \rangle_2 + \langle T_{\overline{b}} f, T_{\overline{b}} k_{x_0,n}^b \rangle_{\overline{b}}$$
$$= \langle f, k_{x_0,n}^b \rangle_2 + \langle g, k_{x_0,n}^\rho \rangle_\rho$$
$$= \int_{\mathbb{R}} f(t) \overline{k_{x_0,n}^b(t)} \, dt + \int_{\mathbb{R}} g(t) \rho(t) \overline{k_{x_0,n}^\rho(t)} \, dt,$$

which proves the relation (3.8). $\square$

If b is inner, then it is clear that the second integral in (3.8) is zero and we obtain the formula of Ahern–Clark (1.2).

4. A formula of combinatorics

We first recall some well-known facts concerning hypergeometric series (see [4, 21]).

The $_2F_1$ hypergeometric series is a power series in z defined by

$$(4.1) \qquad _2F_1\left[\begin{matrix} a, b \\ c \end{matrix}; z\right] = \sum_{p=0}^{+\infty} \frac{(a)_p (b)_p}{p!(c)_p} z^p,$$

where $a, b, c \in \mathbb{C}$, $c \neq 0, -1, -2, \ldots$, and

$$(t)_p := \begin{cases} 1, & \text{if } p = 0, \\ t(t+1)\ldots(t+p-1), & \text{if } p \geqslant 1. \end{cases}$$

We see that the hypergeometric series reduces to a polynomial of degree n in z when a or b is equal to $-n$, $(n = 0, 1, 2, \ldots)$. It is clear that the radius of convergence of the $_2F_1$ series is equal to 1. One can show that when $\Re(c - a - b) \leqslant -1$ this series is divergent on the entire unit circle, when $-1 < \Re(c - a - b) \leqslant 0$ this series converges on the unit circle except for $z = 1$ and when $0 < \Re(c - a - b)$ this series is (absolutely) convergent on the entire unit circle (see [4, Theorem 2.1.2]).

We note that a power series $\sum_p \alpha_p z^p$ $(\alpha_0 = 1)$ can be written as a hypergeometric series $_2F_1\left[{a,\,b \atop c}; z\right]$ if and only if

$$(4.2) \qquad \frac{\alpha_{p+1}}{\alpha_p} = \frac{(p+a)(p+b)}{(p+1)(p+c)}.$$

Finally we recall two useful well-known formulas [4, page 68] for the hypergeometric series:

$$(4.3) \quad _2F_1\left[{a,\,b \atop c}; z\right] = (1-z)^{c-a-b} {}_2F_1\left[{c-a,\,c-b \atop c}; z\right] \qquad \text{(Euler's formula)},$$

and

$$(4.4)$$
$$_2F_1\left[{a,\,b \atop c}; \tfrac{1}{2}\right] = 2^a {}_2F_1\left[{a,\,c-b \atop c}; -1\right], \qquad \Re(b-a) > -1, \quad \text{(Pfaff's formula)}.$$

Now we state the result which we use in the last section.

PROPOSITION 4.1. — Let $n, r \in \mathbb{N}$, $0 \leqslant r \leqslant 2n + 1$ and define

$$(4.5) \quad A_{n,r} := (-1)^{r+1} \sum_{p=0}^{n} \sum_{\ell=0}^{n} (-2)^{p-\ell} \binom{r}{n-\ell} \binom{2n+1-r}{p} \binom{n-p+\ell}{\ell}.$$

Then

$$A_{n,r} = \begin{cases} -2^n, & 0 \leqslant r \leqslant n \\ 2^n, & n+1 \leqslant r \leqslant 2n+1. \end{cases}$$

For the proof of this result, we need the following lemma.

LEMMA 4.2. — For $m \in \mathbb{N}$, we have
$$(4.6)$$
$$\sum_{k=0}^{m} \binom{m}{k} (z-1)^{-k} {}_2F_1\left[{a,\,b-k \atop c}; z\right] = \frac{(c-a)_m}{(c)_m} \left(\frac{z}{z-1}\right)^m {}_2F_1\left[{a,\,b \atop c+m}; z\right].$$

Proof. — First note that (4.6) is equivalent to

(4.7)
$$\sum_{k=0}^{m}\binom{m}{k}(1-z)^{-k}(-1)^{m-k}{}_2F_1\left[{a,\,b-k \atop c};z\right] = \frac{(c-a)_m}{(c)_m}\left(\frac{z}{1-z}\right)^m {}_2F_1\left[{a,\,b \atop c+m};z\right],$$

and denote by LH the left hand side of the inequality (4.7). Applying transformation (4.3), we obtain

$$LH = \sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}(1-z)^{c-a-b}{}_2F_1\left[{c-a,\,c-b+k \atop c};z\right].$$

Now we introduce the operator of difference Δ defined by $\Delta f(x) = f(x+1) - f(x)$. Then it is well-known and easy to verify that

$$\Delta^m f(x) = \sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}f(x+k).$$

Using this formula, we see that $LH = (1-z)^{c-a-b}\Delta^m f(c-b)$, with

$$f(x) := {}_2F_1\left[{c-a,\,x \atop c};z\right].$$

But now we can compute $\Delta^m f(x)$. Indeed, we have

$$\Delta f(x) = \sum_{k=0}^{+\infty}\frac{(c-a)_k}{(c)_k}\left((x+1)_k - (x)_k\right)\frac{z^k}{k!}$$

$$= \sum_{k=1}^{+\infty}\frac{(c-a)_k}{(c)_k}(x+1)_{k-1}\frac{z^k}{(k-1)!}$$

$$= \frac{(c-a)}{c}\,z\,{}_2F_1\left[{c-a+1,\,x+1 \atop c+1};z\right],$$

and by induction, it follows that

$$\Delta^m f(x) = \frac{(c-a)_m}{(c)_m}z^m {}_2F_1\left[{c-a+m,\,x+m \atop c+m};z\right].$$

Therefore, we get

$$LH = (1-z)^{c-a-b}\frac{(c-a)_m}{(c)_m}z^m {}_2F_1\left[{c-a+m,\,c-b+m \atop c+m};z\right].$$

Applying once more Euler's formula, we obtain (4.7). $\qquad\square$

Proof of Proposition 4.1: Changing ℓ into $n-\ell$ in the second sum of (4.5), we see that

$$(4.8) \quad A_{n,r} = (-1)^{r+1}\sum_{p=0}^{n}\sum_{\ell=0}^{n}(-2)^{p+\ell-n}\binom{r}{\ell}\binom{2n+1-r}{p}\binom{2n-p-\ell}{n-\ell}.$$

Hence

$$A_{n,2n+1-r} = (-1)^{2n+1-r+1} \sum_{p=0}^{n} \sum_{\ell=0}^{n} (-2)^{p+\ell-n} \binom{2n+1-r}{\ell} \binom{r}{p} \binom{2n-p-\ell}{n-\ell}$$

$$= -(-1)^{r+1} \sum_{p=0}^{n} \sum_{\ell=0}^{n} (-2)^{p+\ell-n} \binom{2n+1-r}{\ell} \binom{r}{p} \binom{2n-p-\ell}{n-p}$$

$$= -A_{n,r}.$$

Therefore, it is sufficient to show $A_{n,r} = -2^n$ for $0 \leqslant r \leqslant n$ and then the result for $n+1 \leqslant r \leqslant 2n+1$ will follow immediately.

We will now assume that $0 \leqslant r \leqslant n$. Changing p to $n-p$ in the first sum of (4.8) and permuting the two sums, we get

$$A_{n,r} = (-1)^{r+1} \sum_{\ell=0}^{n} (-2)^{l} \binom{r}{\ell} \sum_{p=0}^{n} (-2)^{-p} \binom{2n+1-r}{n-p} \binom{n+p-\ell}{n-\ell}.$$

According to (4.1) and (4.2), we see that

$$\sum_{p=0}^{n} (-2)^{-p} \binom{2n+1-r}{n-p} \binom{n+p-\ell}{n-\ell} = \binom{2n+1-r}{n} {}_2F_1\left[\begin{matrix} n-\ell+1, -n \\ n+2-r \end{matrix}; \tfrac{1}{2} \right],$$

which implies

$$A_{n,r} = (-1)^{r+1} \binom{2n+1-r}{n} \sum_{\ell=0}^{n} (-2)^{\ell} \binom{r}{\ell} {}_2F_1\left[\begin{matrix} n-\ell+1, -n \\ n+2-r \end{matrix}; \tfrac{1}{2} \right].$$

Since $r \leqslant n$ and since $\binom{r}{\ell} = 0$ if $r < \ell$, the sum (in the last equation) ends at $\ell = r$ and we can apply Lemma 4.2 with $m = r$, $a = -n$, $b = n+1$, $c = n+2-r$ and $z = \frac{1}{2}$. Therefore

$$A_{n,r} = (-1)^{r+1} \binom{2n+1-r}{n} \frac{(2n+2-r)_r}{(n+2-r)_r} (-1)^r {}_2F_1\left[\begin{matrix} -n, n+1 \\ n+2 \end{matrix}; \tfrac{1}{2} \right]$$

$$= -\frac{(2n+1)!}{(n+1)!n!} \, {}_2F_1\left[\begin{matrix} -n, n+1 \\ n+2 \end{matrix}; \tfrac{1}{2} \right].$$

We now use formula (4.4) which gives

$$_2F_1\left[\begin{matrix} -n, n+1 \\ n+2 \end{matrix}; \tfrac{1}{2} \right] = 2^{-n} \, {}_2F_1\left[\begin{matrix} -n, 1 \\ n+2 \end{matrix}; -1 \right]$$

$$= 2^{-n} \sum_{i=0}^{\infty} \frac{(-n)_i (1)_i}{(n+2)_i} \frac{(-1)^i}{i!}$$

$$= 2^{-n} \sum_{i=0}^{n} \binom{n}{i} \frac{i!(n+1)!}{(n+i+1)!},$$

where we have used $(1)_i = i!$, $\dfrac{(-n)_i(-1)^i}{i!} = \dbinom{n}{i}$ and $(n+2)_i = \dfrac{(n+i+1)!}{(n+1)!}$.

Hence

$$
{}_2F_1\left[\begin{matrix} -n, n+1 \\ n+2 \end{matrix}; \tfrac{1}{2}\right] = 2^{-n}(n+1)!n! \sum_{i=0}^{n} \frac{1}{(n+i+1)!(n-i)!},
$$

which implies

$$
A_{n,r} = -2^{-n} \sum_{i=0}^{n} \binom{2n+1}{i}.
$$

But

$$
\sum_{i=0}^{n} \binom{2n+1}{i} = \frac{1}{2} \sum_{i=0}^{2n+1} \binom{2n+1}{i} = 2^{2n},
$$

which ends the proof. $\square$

5. Norm convergence for the reproducing kernels.

In Section 3, we saw that if $x_0 \in \mathbb{R}$ satisfies (2.2), then $k_{\omega,n}^b$ tends *weakly* to $k_{x_0,n}^b$ in $\mathcal{H}(b)$ as ω approaches radially to x_0. It is natural to ask if this weak convergence can be replaced by norm convergence. In other words, is it true that $\|k_{\omega,n}^b - k_{x_0,n}^b\|_b \to 0$ as ω tends radially to x_0?

In [1], Ahern and Clark said that they can prove this result for the case where b is inner and $n = 0$. For general functions b in the unit ball of H^∞, Sarason [18, Chap. V] got this norm convergence for the case $n = 0$. In this section, we prove the general case.

Since we already have weak convergence, to prove the norm convergence, it is sufficient to prove that $\|k_{\omega,n}^b\|_b \to \|k_{x_0,n}^b\|_b$ as ω tends radially to x_0. Therefore we need to compute $\|k_{x_0,n}^b\|_b$. For $n = 0$, in the context of the unit disc, Sarason [18, Chap. V] proved that $\|k_{z_0}^b\|_b^2 = z_0 \overline{b(z_0)} b'(z_0)$, $z_0 \in \mathbb{T}$. We can give an analogue of this formula showing that the norm of $k_{x_0,n}^b$ can be expressed in terms of the derivatives of b at x_0.

PROPOSITION 5.1. — *Let b be a point in the unit ball of $H^\infty(\mathbb{C}_+)$, let n be a non-negative integer and let $x_0 \in \mathbb{R}$ satisfying the condition (2.2). Then*

$$
\|k_{x_0,n}^b\|_b^2 = \frac{n!^2}{2i\pi} \sum_{p=0}^{n} \frac{\overline{b^{(p)}(x_0)}}{p!} \frac{b^{(2n+1-p)}(x_0)}{(2n+1-p)!}.
$$

Proof. — Following notations of Section 3, we define

$$\varphi(z) = 1 - b(z) \sum_{p=0}^{n} \frac{\overline{b^{(p)}(x_0)}}{p!} (z - x_0)^p.$$

Then, by (2.3) and Lemma 2.3, as z tends radially to x_0, we have

$$k_{x_0,n}^b(z) = \frac{in!}{2\pi} (z - x_0)^{-n-1} \left(\sum_{p=0}^{2n+1} \frac{\varphi^{(p)}(x_0)}{p!} (z - x_0)^p + o((z - x_0)^{2n+1}) \right).$$

As we have shown in the proof of Theorem 3.3, $\varphi^{(k)}(x_0) = 0$ if $0 \leqslant k \leqslant n$. Hence

$$k_{x_0,n}^b(z) = \frac{in!}{2\pi} \sum_{p=0}^{n} \frac{\varphi^{(p+n+1)}(x_0)}{(p+n+1)!} (z - x_0)^p + o((z - x_0)^n).$$

Using once more Lemma 2.3, we can also write

$$k_{x_0,n}^b(z) = \sum_{p=0}^{n} \frac{(k_{x_0,n}^b)^{(p)}(x_0)}{p!} (z - x_0)^p + o((z - x_0)^n),$$

which implies

$$(k_{x_0,n}^b)^{(p)}(x_0) = \frac{in!}{2\pi} \frac{p!}{(p+1+n)!} \varphi^{(p+n+1)}(x_0).$$

But, according to Lemma 3.2, we have $\|k_{x_0,n}^b\|_b^2 = (k_{x_0,n}^b)^{(n)}(x_0)$ and we get

$$\|k_{x_0,n}^b\|_b^2 = \frac{in!^2}{2\pi} \frac{\varphi^{(2n+1)}(x_0)}{(2n+1)!}.$$

Finally, the result follows by Leibniz' rule.   □

The next result provides an affirmative answer to the question of norm convergence.

THEOREM 5.2. — *Let b be a point in the unit ball of $H^\infty(\mathbb{C}_+)$, let n be a non-negative integer and let $x_0 \in \mathbb{R}$ satisfying the condition (2.2). Then*

$$\left\| k_{\omega,n}^b - k_{x_0,n}^b \right\|_b \longrightarrow 0, \quad \text{as } \omega \text{ tends radially to } x_0.$$

Proof. — We denote by $a_p(\omega) := \frac{b^{(p)}(\omega)}{p!}$ and $a_p := a_p(x_0)$. We recall that

$$k_{\omega,n}^b(z) = \frac{in!}{2\pi} \left(\frac{1}{(z - \overline{\omega})^{n+1}} - \sum_{p=0}^{n} \overline{a_p(\omega)} (z - \overline{\omega})^{p-n-1} b(z) \right).$$

We have

$$\frac{\partial^n}{\partial z^n} \left(\frac{1}{(z - \overline{\omega})^{n+1}} \right) = (-1)^n \frac{(2n)!}{n!} \frac{1}{(z - \overline{\omega})^{2n+1}},$$

and by Leibniz' rule

$$\frac{\partial^n}{\partial z^n}\left((z-\overline{\omega})^{p-n-1}b(z)\right) = \sum_{\ell=0}^{n}\binom{n}{\ell}(-1)^{\ell}\frac{(n-p+\ell)!}{(n-p)!}(z-\overline{\omega})^{p-n-\ell-1}b^{(n-\ell)}(z).$$

According to Proposition 3.4, we have $\|k_{\omega,n}^b\|_b^2 = (k_{\omega,n}^b)^{(n)}(\omega)$, which implies
(5.1)

$$\|k_{\omega,n}^b\|_b^2 = \frac{in!}{2\pi}\frac{(-1)^n\frac{(2n)!}{n!} - \sum_{p=0}^{n}\sum_{\ell=0}^{n}\binom{n}{\ell}(-1)^{\ell}\frac{(n-p-\ell)!}{(n-p)!}(\omega-\overline{\omega})^{n+p-\ell}\overline{a_p(\omega)}b^{(n-\ell)}(\omega)}{(\omega-\overline{\omega})^{2n+1}}.$$

For $0 \leqslant s \leqslant n$, the function $b^{(s)}$ is analytic in the upper-half plane and its derivative of order $2n+1-s$, which coincides with $b^{(2n+1)}$, has a radial limit at x_0. According to Lemma 2.3, as ω tends radially to x_0, we have

$$b^{(s)}(\omega) = \sum_{r=s}^{2n+1}a_r\frac{r!}{(r-s)!}(\omega-x_0)^{r-s} + o((\omega-x_0)^{2n+1-s}).$$

Hence if we put $\omega = x_0 + it$, we get

$$(\omega-\overline{\omega})^s b^{(s)}(\omega) = 2^s\sum_{r=s}^{2n+1}a_r\frac{r!}{(r-s)!}i^r t^r + o(t^{2n+1}),$$

and thus

$$(\omega-\overline{\omega})^{n+p-\ell}\overline{a_p(\omega)}b^{(n-\ell)}(\omega)$$
$$= \frac{(-1)^p}{p!}2^{n+p-\ell}\left(\sum_{r=n-\ell}^{2n+1}a_r\frac{r!}{(r-n+\ell)!}i^r t^r + o(t^{2n+1})\right)$$
$$\times\left(\sum_{j=p}^{2n+1}\overline{a_j}\frac{j!}{(j-p)!}(-i)^j t^j + o(t^{2n+1})\right).$$

We deduce from (5.1) that

$$\|k_{\omega,n}^b\|_b^2 = \frac{(-1)^n n!}{2^{2n+2}\pi}t^{-2n-1}\left[(-1)^n\frac{(2n)!}{n!} - n!\sum_{p=0}^{n}\sum_{\ell=0}^{n}(-1)^{p+\ell}2^{n+p-\ell}\binom{n-p+\ell}{\ell}\right.$$
$$\left.\times\left(\sum_{r=n-\ell}^{2n+1}a_r\binom{r}{n-\ell}i^r t^r\right)\left(\sum_{j=p}^{2n+1}\overline{a_j}\binom{j}{p}(-i)^j t^j\right) + o(t^{2n+1})\right],$$

and denoting by $c_n = \dfrac{(-1)^n n!}{2^{2n+2}\pi}$, we can write

$$\|k_{\omega,n}^b\|_b^2 = c_n t^{-2n-1} \left[(-1)^n \frac{(2n)!}{n!} - n! \sum_{s=0}^{2n+1} \lambda_{s,n} t^s + o(t^{2n+1}) \right],$$

with

$$\lambda_{s,n} := i^s \sum_{p=0}^{n} \sum_{\ell=0}^{n} (-1)^{p+\ell} 2^{n+p-\ell} \binom{n-p+\ell}{\ell} \sum_{r=0}^{s} \binom{r}{n-\ell} \binom{s-r}{p} (-1)^{s-r} a_r \overline{a_{s-r}},$$

where we assume that $\binom{a}{b} = 0$ if $a < b$ or $b < 0$.

Now we recall that $k_{\omega,n}^b$ is weakly convergent as ω tends radially to x_0 and thus $\|k_{\omega,n}^b\|_b$ remains bounded. Therefore we necessarily have

$$(5.2) \qquad (-1)^n \frac{(2n)!}{n!} - n! \lambda_{0,n} = 0 \qquad \text{and} \qquad \lambda_{s,n} = 0, \quad (1 \leqslant s \leqslant 2n),$$

which implies that

$$(5.3) \qquad \|k_{\omega,n}^b\|_b^2 = -n! c_n \lambda_{2n+1,n} + o(1),$$

as ω tends radially to x_0. But

$$\lambda_{2n+1,n} = (-1)^n i \sum_{r=0}^{2n+1} (-1)^{r+1} a_r \overline{a_{2n+1-r}} \sum_{p=0}^{n} \sum_{\ell=0}^{n} (-1)^{p+\ell} 2^{n+p-\ell}$$

$$\times \binom{n-p+\ell}{\ell} \binom{r}{n-\ell} \binom{2n+1-r}{p}$$

$$= (-1)^n i 2^n \sum_{r=0}^{2n+1} A_{n,r} a_r \overline{a_{2n+1-r}}.$$

According to Proposition 4.1, we have $A_{n,r} = -2^n$ if $0 \leqslant r \leqslant n$ and $A_{n,r} = 2^n$ if $n+1 \leqslant r \leqslant 2n+1$. Then we obtain

$$\lambda_{2n+1,n} = (-1)^n i 2^{2n} \left(\sum_{r=n+1}^{2n+1} a_r \overline{a_{2n+1-r}} - \sum_{r=0}^{n} a_r \overline{a_{2n+1-r}} \right).$$

Now note that

$$\sum_{r=n+1}^{2n+1} a_r \overline{a_{2n+1-r}} = \sum_{r=0}^{n} \overline{a_r} a_{2n+1-r},$$

which means

$$\lambda_{2n+1,n} = (-1)^{n+1} 2^{2n+1} \Im m \left(\sum_{r=0}^{n} \overline{a_r} a_{2n+1-r} \right).$$

But Proposition 5.1 implies that

$$\lambda_{2n+1,n} = (-1)^{n+1} 2^{2n+2} \frac{\pi}{n!^2} \|k_{x_0,n}^b\|_b^2,$$

and finally using (5.3) and the definition of c_n, we obtain

$$\|k_{\omega,n}^b\|_b^2 = \|k_{x_0,n}^b\|_b^2 + o(1),$$

which proves that $\|k_{\omega,n}^b\|_b \longrightarrow \|k_{x_0,n}^b\|_b$ as ω tends radially to x_0. Since $k_{\omega,n}^b$ tends also weakly to $k_{x_0,n}^b$ in $\mathcal{H}(b)$ as ω tends radially to x_0, we get the desired conclusion. □

Remark 5.3. — We have already seen in the proof of Theorem 3.3 that if x_0 satisfies the condition (2.2), then $|a_0| = 1$ and

$$(5.4) \qquad \sum_{p=0}^{k} a_p \overline{a_{k-p}} = 0, \qquad (1 \leqslant k \leqslant n),$$

where $a_p := \frac{b^{(p)}(x_0)}{p!}$. In fact, we can prove that the relation (5.4) is also valid for $n+1 \leqslant k \leqslant 2n$, k even. Indeed, according to (5.2), for $n+1 \leqslant s \leqslant 2n$, we have

$$0 = \lambda_{s,n} := i^s \sum_{p=0}^{n} \sum_{\ell=0}^{n} (-1)^{p+\ell} 2^{n+p-\ell} \binom{n-p+\ell}{\ell} \sum_{r=0}^{s} \binom{r}{n-\ell} \binom{s-r}{p}$$
$$\times (-1)^{s-r} a_r \overline{a_{s-r}}$$

$$= (-i)^s 2^n \sum_{r=0}^{s} (-1)^r a_r \overline{a_{s-r}} \sum_{p=0}^{n} \sum_{\ell=0}^{n} (-1)^{p+\ell} 2^{p-\ell}$$
$$\times \binom{n-p+\ell}{\ell} \binom{r}{n-\ell} \binom{s-r}{p}$$

$$(5.5) \qquad = (-i)^s 2^n \sum_{r=0}^{s} a_r \overline{a_{s-r}} A_{n,r,s},$$

with

$$(5.6) \quad A_{n,r,s} := (-1)^r \sum_{p=0}^{n} \sum_{\ell=0}^{n} (-1)^{p+\ell} 2^{p-\ell} \binom{n-p+\ell}{\ell} \binom{r}{n-\ell} \binom{s-r}{p}.$$

Using similar arguments as in the proof of Proposition 4.1, we show that for every $0 \leqslant r \leqslant n < s$, we have

$$(5.7) \qquad A_{n,r,s} = \binom{s}{n} \frac{\Gamma\left(\frac{s-n+1}{2}\right) \Gamma\left(\frac{s-n+2}{2}\right)}{\Gamma\left(\frac{s-2n+1}{2}\right) \Gamma\left(\frac{s+2}{2}\right)}.$$

In the proof of this identity, we use Bayley's Theorem [4] which says that

$${}_2F_1\left[{a, 1-a \atop b}; \frac{1}{2}\right] = \frac{\Gamma\left(\frac{b}{2}\right) \Gamma(\frac{1+b}{2})}{\Gamma\left(\frac{a+b}{2}\right) \Gamma\left(\frac{1-a+b}{2}\right)}.$$

Now using (5.6) it is easy to see that $A_{n,s-r,s} = (-1)^s A_{n,r,s}$, and with (5.5) and (5.7), we obtain

$$\binom{s}{n} \frac{\Gamma\left(\frac{s-n+1}{2}\right)\Gamma\left(\frac{s-n+2}{2}\right)}{\Gamma\left(\frac{s-2n+1}{2}\right)\Gamma\left(\frac{s+2}{2}\right)} \left(\sum_{r=0}^{n} a_r\overline{a_{s-r}} + (-1)^s \sum_{r=n+1}^{s} a_r\overline{a_{s-r}}\right) = 0.$$

Now recall that the Gamma function is a meromorphic function in the complex plane without zeros and with poles at zero and the negative integers. Therefore we see that if s is even $(n+1 \leqslant s \leqslant 2n)$, then

$$\sum_{r=0}^{s} a_r\overline{a_{s-r}} = 0.$$

But if s is odd $(n+1 \leqslant s \leqslant 2n)$, then $\frac{s-2n+1}{2}$ is zero or a negative integer and, using this argument, we are not able to conclude that $\sum_{r=0}^{s} a_r\overline{a_{s-r}} = 0.$ This still remains as an open question.

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