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WEAK MIXING AND EIGENVALUES FOR ARNOUX-RAUZY SEQUENCES

by Julien CASSAIGNE, Sébastien FERENCZI & Ali MESSAOUDI (*)

ABSTRACT. — We define by simple conditions two wide subclasses of the so-called Arnoux-Rauzy systems; the elements of the first one share the property of (measure-theoretic) weak mixing, thus we generalize and improve a counter-example to the conjecture that these systems are codings of rotations; those of the second one have eigenvalues, which was known hitherto only for a very small set of examples.

RÉSUMÉ. — Nous définissons par des condition simples deux larges sous-classes des systèmes dits d'Arnoux-Rauzy ; les membres de la première possèdent la propriété de mélange faible (mesurable), ce qui généralise et améliore un contre-exemple à la conjecture que tous ces systèmes sont des codages de rotations ; ceux de la seconde ont des valeurs propres, ce qui n'était connu jusqu'ici que pour un ensemble très restreint d'exemples.

A classical result of Coven, Hedlund and Morse [14], [11] characterizes the *Sturmian systems*, defined as the symbolic systems whose complexity function (the number of words of length n in the language of the system) is $n+1$, as natural codings of irrational rotations of the torus $\mathbb{T}^1$: the Sturmian sequences are codings of the orbits of a rotation by a partition of a fundamental domain such that on each atom the rotation is a translation by a constant. This interaction between word combinatorics and dynamical systems is particularly interesting as one of the possible proofs uses an arithmetic tool, the Euclid algorithm of continued fraction approximation. Thus there have been many attempts at generalizing it to rotations on a torus of higher dimension, especially in view of the difficult problem of simultaneous approximation of several numbers, see [4] for a general survey.

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The first attempt, in dimension 2, led to the definition of the *Arnoux-Rauzy* systems, whose complexity is $2n + 1$ and which satisfy an extra combinatorial condition [3]. One of these systems, the *Tribonacci system*, is indeed a natural coding of a rotation of $\mathbb{T}^2$ ([16], see also [1]), and this gives a good (the best possible in some sense [8]) simultaneous approximation for a pair of algebraic numbers. Every Arnoux-Rauzy system defines an algorithm of simultaneous approximation for some pair of numbers, and was conjectured to be also a natural coding of a rotation of $\mathbb{T}^2$; this conjecture was disproved in [7], by exhibiting an Arnoux-Rauzy system whose language is unbalanced, see Corollary 5 below. This example is quite elaborate, and leaves many open questions, which are asked at the end of [7]; first, we do not know up to which point the conjecture fails: the system considered there could still have some weaker properties than being a natural coding: equipped with its unique invariant probability measure, it could still be measure-theoretically isomorphic to a rotation of $\mathbb{T}^2$ or at least admit a rotation of $\mathbb{T}^2$ as a measure-theoretic factor. Then, we would like to know for which systems the conjecture holds, at least partially: up to now, only Arnoux-Rauzy systems whose generating rules are periodic (see Definition 1 below) are known to satisfy it [2], and no other Arnoux-Rauzy sytems are known to admit rotations of $\mathbb{T}^2$ or $\mathbb{T}^1$ as continuous or measure-theoretic factors (equivalently, to have continuous or measurable *eigenvalues*, see Definition 3 below).

In the present paper, we widen considerably the class of Arnoux-Rauzy systems for which the rotation factors can be described. First, in the negative direction, we prove a stronger result than [7] for a much larger class of systems, by proving that any Arnoux-Rauzy system for which the inverses of some of the *partial quotients* (defined in Definition 1 below and linked to the approximation algorithm) form a convergent series is (measure-theoretically) *weakly mixing*, see Definition 3 below. As a consequence, all these systems (which include the example in [7]) have an unbalanced language and do not satisfy any of the weaker properties mentioned above. Then, in the positive direction, we give some sufficient conditions for the system to have two continuous rotation factors: they correspond to a slow growth of some of the partial quotients; these examples are in general non-periodic and thus completely new, as is an example where the partial quotients are $2n - 1$ and we could find one continuous rotation factor; this means that our conditions are not far from optimal. It is interesting to note that our conditions in both directions involve the same restricted families of partial quotients, while the partial quotients outside these families seem to play no part in the question of weak mixing (though of course they contribute in the exact determination of the eigenvalues). The proofs use first some standard arguments of ergodic theory, related to finite rank systems (which we have tried to keep as short as possible) to give necessary conditions and sufficient conditions for a number to be an eigenvalue, in terms of the quality of its approximation by the Arnoux-Rauzy algorithm; then we make precise estimates to check when these conditions are fulfilled.

1. Arnoux-Rauzy systems

We take here as a definition of Arnoux-Rauzy systems their constructive characterization, derived in [3] from the original definition.

DEFINITION 1. — *An* Arnoux-Rauzy system *is a symbolic system on* $\{1,2,3\}$ *defined by three families of words* A_k, B_k, C_k, *build recursively from* $A_0=1$, $B_0=2$, $C_0=3$, *by using a sequence of combinatorial rules* a, b, c, *such that each one of the three rules is used infinitely many times, where*

- ▷ *by rule* a, $A_{k+1}=A_k$, $B_{k+1}=B_kA_k$, $C_{k+1}=C_kA_k$;
- ▷ *by rule* b, $A_{k+1}=A_kB_k$, $B_{k+1}=B_k$, $C_{k+1}=C_kB_k$;
- ▷ *by rule* c, $A_{k+1}=A_kC_k$, $B_{k+1}=B_kC_k$, $C_{k+1}=C_k$.

The system (X,T) *is the one-sided shift on sequences* $(x_n, n\in\mathbb{N})$ *such that for each* $0\leqslant s\leqslant t$ *there exists* k *such that* $x_s\cdots x_t$ *is a subword of* A_k. *Equipped with the product topology on* $\{1,2,3\}^{\mathbb{N}}$, *it is minimal* [3] *and uniquely ergodic* (*by Boshernitzan's result* [5] *using the fact that the complexity is* $2n+1$) *with a unique invariant probability measure* μ. *We shall consider both the topological system* (X,T) *and the measure-theoretic system* (X,T,μ).

DEFINITION 2. — *If the sequence of rules* a, b, c *is* r_1 *iterated* k_1 *times,..., r_n iterated k_n times,..., with* $r_{n+1}\neq r_n$ *and* $k_n\geqslant 1$, *the* k_n *are called the* partial quotients *of the system.*

We denote by $(n_i, i\geqslant 1)$ *the sequence of indices* $n\geqslant 1$ *such that* $r_n\neq r_{n+2}$. *Since each rule* a, b, c *occurs infinitely often, this sequence is infinite.*

Note that each choice of a sequence of $k_n\geqslant 1$, $n\geqslant 1$, and an infinite sequence of n_i, determines an Arnoux-Rauzy system, which is unique up to a renaming of letters.

We can then define our system in another way, which corresponds to a *multiplicative* form of the approximation algorithm.

Proposition 1. — *Let an Arnoux-Rauzy system* (X, T, μ) *be defined as in Definition* 1*, and* k_n *be its partial quotients.*

Then (X, T) *is the one-sided shift on sequences* $(x_n, n \in \mathbb{N})$ *such that for each* $0 \leqslant s \leqslant t$ *there exists* n *such that* $x_s \dots x_t$ *is a subword of* H_n*, where the three words* H_n*,* G_n*,* J_n *are built from* H_0*,* G_0*,* J_0 *(chosen such that* $\{H_0, G_0, J_0\} = \{1, 2, 3\}$*) by two families of rules:*

- ▷ *if* $n + 1 = n_i$ *for some* i*,*

$$H_{n+1} = G_n H_n^{k_{n+1}}, \quad G_{n+1} = J_n H_n^{k_{n+1}}, \quad J_{n+1} = H_n;$$

we say that the $(n+1)$-th rule is of type 1*;*

- ▷ *otherwise,*

$$H_{n+1} = G_n H_n^{k_{n+1}}, \quad G_{n+1} = H_n, \quad J_{n+1} = J_n H_n^{k_{n+1}};$$

we say that the $(n+1)$-th rule is of type 2.

And rules of type 1 *are used infinitely many times.*

Proof. — Let $p = k_1 + \dots + k_n$. We denote by H_n the word A_p if $r_{n+1} = a$ (in the notations of Definition 1), the word B_p if $r_{n+1} = b$, the word C_p if $r_{n+1} = c$. We denote by G_n the word A_p if $r_{n+2} = a$, the word B_p if $r_{n+2} = b$, the word C_p if $r_{n+2} = c$. Recall that $r_{n+1} \neq r_{n+2}$. Finally, J_n is chosen so that $\{H_n, G_n, J_n\} = \{A_p, B_p, C_p\}$.

Suppose for example that $r_{n+1} = a$ and $r_{n+2} = b$. Then we have $H_n = A_p$, $G_n = B_p$, and $J_n = C_p$. As $r_{n+1} = a$, for $p' = p + k_{n+1}$ we get by applying k_{n+1} times rule a: $A_{p'} = A_p = H_n$, $B_{p'} = B_p A_p^{k_{n+1}} = G_n H_n^{k_{n+1}}$, and $C_{p'} = C_p A_p^{k_{n+1}} = J_n H_n^{k_{n+1}}$. As $r_{n+2} = b$, we know that $H_{n+1} = B_{p'} = G_n H_n^{k_{n+1}}$.

If $n + 1 = n_i$ for some i, then $r_{n+3} = c$, and we have $G_{n+1} = C_{p'} = J_n H_n^{k_{n+1}}$ and $J_{n+1} = A_{p'} = H_n$. Otherwise, $r_{n+3} = r_{n+1} = a$, and we have $G_{n+1} = A_{p'} = H_n$ and $J_{n+1} = C_{p'} = J_n H_n^{k_{n+1}}$. We check that our formulas are satisfied; other cases are similar.

Since the sequence (n_i) is infinite, rules of type 1 are used infinitely many times. □

We recall:

Definition 3. — *If* (X, T, μ) *is a finite measure-preserving dynamical system, a real number* $0 \leqslant \theta < 1$ *is a* measurable eigenvalue *of* (X, T, μ) *(denoted additively) if there exists a non-constant* f *in* $\mathcal{L}^1(X, \mathbb{R}/\mathbb{Z})$ *such* *that* $f \circ T = f + \theta$ *(in* $\mathcal{L}^1(X, \mathbb{R}/\mathbb{Z})$*);* f *is then an* eigenfunction *for the eigenvalue* θ.

As constants are not eigenfunctions, $\theta = 0$ *is not an eigenvalue if* T *is ergodic.*

If (X, T) *is a topological dynamical system, a real number* $0 \leqslant \theta < 1$ *is a* continuous eigenvalue *of* (X, T) *if* $f \circ T = f + \theta$ *for a non-constant continuous eigenfunction* f.

(X, T, μ) *is* weakly mixing *if it has no measurable eigenvalue.*

An equivalent way to express it, which is less straightforward but more relevant to the conjecture mentioned in the introduction above, is that the system has θ as a measurable (resp continuous) eigenvalue if and only if it admits the rotation of angle θ as a measurable (resp continuous) factor.

We can now state our results:

Theorem 2. — *Let* (X, T, μ) *be an Arnoux-Rauzy system,* k_n *and* n_i *as in Definition 2; we assume that*

(W1) $\quad k_{n_i+2}$ *is unbounded,*

(W2) $$\sum_{i=1}^{+\infty} 1/k_{n_i+1} < +\infty,$$

(W3) $$\sum_{i=1}^{+\infty} 1/k_{n_i} < +\infty.$$

Then (X, T, μ) *is weakly mixing.*

Theorem 3. — *Let* (X, T, μ) *be an Arnoux-Rauzy system,* k_n *and* n_i *as in Definition* 2. *If there exists* $\varepsilon > 0$ *and* $J > 0$ *such that for all* $j > J$

$$\sum_{i=1}^{j} \frac{1}{1 + k_{n_i+1}} \geqslant (12 + \varepsilon) \ln j,$$

then (X, T) *has two rationally independent continuous eigenvalues,* θ_1 *and* θ_2*; all the measurable eigenvalues of* (X, T, μ) *are continuous, and of the form* $a + b\theta_1 + c\theta_2$ *for integers* a*,* b*,* c*.*

Theorem 4. — *Let* (X, T) *be an Arnoux-Rauzy system and* k_n *its partial quotients; if* $k_n = 2n - 1$*,* $n \geqslant 1$ *and all rules are of type* 1*, then* (X, T) *has at least one continuous eigenvalue.*

The sufficient conditions in Theorem 3 could be slightly improved, in particular if we know other partial quotients; but at least when the rules are all of type 1, Theorem 2, Theorem 3 and Theorem 4 seem close enough to suggest some kind of optimality. The example in Theorem 4 is particularly intriguing, as it is possible there might be a second measurable eigenvalue but no continuous one, see remark at the end.

As a consequence of Theorem 2, we get

Corollary 5. — *An Arnoux-Rauzy system satisfying* (W1), (W2) *and* (W3) *is not a natural coding of any rotation of* $\mathbb{T}^k$ *for any* k*; it is not measure-theoretically isomorphic to any rotation of* $\mathbb{T}^k$ *for any* k*; it does not admit any rotation of* $\mathbb{T}^k$ *for any* k *as a measure-theoretic factor. Also the language of* X *is unbalanced: for each positive integer* N *there exist words* U *and* V *of equal length, occurring in sequences of* X*, and* $i \in \{1, 2, 3\}$ *such that the numbers of occurrences of* i *in* U *and* V *differ by at least* N*.*

Proof. — The first three assertions are straightforward consequences of weak mixing. The last one comes from the fact that if the language of X is balanced, then each cylinder $[i]$ is a *bounded remainder set* (see [7]) and such a set cannot exist if there are no eigenvalues [15]. □

Another consequence of this result is to produce new examples of weakly mixing *k-interval exchange transformations*, see for example [13] for definitions and a discussion of the problem; every Arnoux-Rauzy system is a natural coding of such a transformation for $k = 7$ (or $k = 6$ on the circle) [3], while Arnoux-Rauzy systems with the k_n going to infinity fast enough are measure-theoretically isomorphic to 4-interval exchange transformations [13]. Thus every Arnoux-Rauzy system satisfying (W1), (W2) and (W3) gives a weakly mixing 7-interval exchange transformation, and every Arnoux-Rauzy system used in [13] gives a weakly mixing 4-interval exchange transformation.

The class of Arnoux-Rauzy systems is still rich in open questions, such as a complete characterization of those which have measurable eigenvalues, or have continuous eigenvalues, or are measure-theoretically isomorphic to a rotation of $\mathbb{T}^2$, or are a natural coding of a rotation of $\mathbb{T}^2$.

2. Finite rank and spectral properties

We begin now to prove our theorems; before going any further, we give estimates on the lengths of the words, that will be used several times in the sequel:

Corollary 6. — *We denote by* h_n, g_n, j_n *the lengths of* H_n, G_n, J_n*.*

▷ *By a rule of type* 1,

$$h_{n+1} = k_{n+1}h_n + g_n, \quad g_{n+1} = k_{n+1}h_n + j_n, \quad j_{n+1} = h_n.$$

▷ *By a rule of type* 2,

$$h_{n+1} = k_{n+1}h_n + g_n, \quad g_{n+1} = h_n, \quad j_{n+1} = k_{n+1}h_n + j_n.$$

The above Corollary follows immediately from Proposition 1. We shall need also

LEMMA 7. — *For all $n \geqslant 0$ $g_n \leqslant 2h_n$, and $j_n \leqslant 2h_n$.*

Proof. — Let $t_{1,n} \leqslant t_{2,n} \leqslant t_{3,n}$ be the three lengths (h_n, g_n, j_n) after ordering. We shall prove that $t_{2,n} \leqslant h_n \leqslant t_{3,n} \leqslant t_{1,n} + t_{2,n}$, for all $n \geqslant 0$.

The formulas in Corollary 6 imply that h_n is never the smallest of the three lengths, hence $h_n = t_{2,n}$ or $h_n = t_{3,n}$.

We prove by induction that $t_{3,n} \leqslant t_{1,n} + t_{2,n}$: $(t_{1,0}, t_{2,0}, t_{3,0}) = (1,1,1)$ by Definition 1. Then, either $t_{3,n+1} = k_{n+1}t_{3,n} + t_{2,n}$, $t_{2,n+1} = k_{n+1}t_{3,n} + t_{1,n}$, $t_{1,n+1} = t_{3,n}$ and $t_{2,n+1} + t_{1,n+1} - t_{3,n+1} = t_{1,n} + t_{3,n} - t_{2,n} \geqslant 0$, or $t_{3,n+1} = k_{n+1}t_{2,n} + t_{3,n}$, $t_{2,n+1} = k_{n+1}t_{2,n} + t_{1,n}$, $t_{1,n+1} = t_{2,n}$ and $t_{1,n+1} + t_{2,n+1} - t_{3,n+1} = t_{1,n} + t_{2,n} - t_{3,n} \geqslant 0$.

The two desired inequalities follow immediately. □

LEMMA 8. — (i) *For any integers $m \geqslant 0$, $n > 0$,*

$$h_{m+n} \geqslant F_m h_n$$

where F_m is the m-th Fibonacci number *given by $F_0 = F_1 = 1$, $F_{m+1} = F_{m-1} + F_m$.*

(ii) *If we define $\Psi = \sum_{i=0}^{+\infty} 1/F_i$, then $\Psi \leqslant 1 + \sum_{r=0}^{+\infty} \phi^{-r} = \phi + 2$ if $\phi = \frac{1}{2}(1+\sqrt{5}\,)$ is the golden ratio number.*

Proof. — With any type of rules, we have $h_{n+1} \geqslant h_n + h_{n-1}$, and part (i) comes from iterating this estimate. Part (ii) is immediate as $F_i \geqslant \phi^{i-1}$. □

Now we translate the combinatorial definitions into a (well-known but not written completely anywhere) description of Arnoux-Rauzy systems as systems generated by Rokhlin towers: we recall that a (Rokhlin) *tower* of *basis* F is a collection of disjoint *levels* $F, TF, \dots, T^{h-1}F$.

In an Arnoux-Rauzy system, we build three canonical families of Rokhlin towers; for this, we need the following by-product of the proofs in [3].

PROPOSITION 9. — *Let $\overline{A}_k$, $\overline{B}_k$, $\overline{C}_k$ be the words built from $\overline{A}_0 = A_0$, $\overline{B}_0 = B_0$, $\overline{C}_0 = C_0$, by the* reverse *recursion rules (in rule a, replace B_kA_k by $\overline{A_kB_k}$, C_kA_k by $\overline{A_kC_k}$ and so on); for each k there exists a word V_k such* *that $\overline{A}_k$, $\overline{B}_k$, $\overline{C}_k$ are the three* return words *of V_k, where U is a return word of V if UV occurs in sequences of X and contains exactly two occurrences of V, one as a prefix and one as a suffix.*

Proof. — We use the notations of [3] (in the course of this proof only); then our A_k, B_k, C_k are the *names* of the three *n-segments* in some *graph of words* Γ_n (for an n depending on k) such that there is a *burst* for n. The $\overline{A}_k$, $\overline{B}_k$, $\overline{C}_k$ are the new names of these segments if we replace the *upper* labels of the edges by their *lower* labels (p. 201). The definition of the graphs and n-segments on p. 203 implies that these new names are the three return words of the bispecial word $G_n = D_n$. And the reasoning of p. 208 applied to the names with lower labels shows that they are built with the reverse recursion rules. □

As a consequence, $\overline{H}_n$, $\overline{G}_n$, and $\overline{J}_n$, built by the reverse rules ($\overline{H_n^{k_{n+1}} G_n}$ instead of $G_n H_n^{k_{n+1}}$, $\overline{H_n^{k_{n+1}} J_n}$ instead of $J_n H_n^{k_{n+1}}$ and so on) are the three return words of $W_n = V_{k_1+\cdots+k_n}$ (though this will not be used later, we can check that the words W_n can be constructed inductively as follows: W_0 is the empty word, and for all $n \geqslant 0$, $W_{n+1} = W_n H_n^{k_{n+1}}$). Note that $\overline{H}_n$, resp. $\overline{G}_n$, $\overline{J}_n$, has length h_n, resp. g_n, j_n.

Now, for each n, X is the disjoint union of the towers

$$R_{h,n} = \bigcup_{i=0}^{h_n-1} T^i U_{h,n}, \quad R_{g,n} = \bigcup_{i=0}^{g_n-1} T^i U_{g,n}, \quad R_{j,n} = \bigcup_{i=0}^{j_n-1} T^i U_{j,n},$$

where $U_{h,n} = [\overline{H}_n W_n]$, $U_{g,n} = [\overline{G}_n W_n]$, $U_{j,n} = [\overline{J}_n W_n]$, and we recall that for any word $V = v_0 \cdots v_{r-1}$ the *cylinder* $[V]$ is the subset of X made of sequences with $x_i = v_i$, $0 \leqslant i \leqslant r-1$. Indeed, for fixed n, a given infinite sequence x in the system admits a unique decomposition $Y_0 Y_1 \cdots$ where Y_i is either $\overline{H}_n$, $\overline{G}_n$, or $\overline{J}_n$ for $i > 0$ and Y_0 is a nonempty suffix of $\overline{H}_n$, $\overline{G}_n$, or $\overline{J}_n$, and the choice of one of the three possibilities for Y_0 and the position of x_0 in Y_0 give the atom of the partition containing x; note that these atoms are the cylinders for all the possible words of a given fixed length m, i.e. the vertices of Γ_m in [3] when there is a burst, except for the word $G_m = D_m$ which is cut into the three pieces $U_{h,n}$, $U_{g,n}$, $U_{j,n}$.

The recursion formulas giving the words translate into a construction of the towers by *cutting and stacking* [12]; in particular, the level $T^j U_{h,n}$ is a subset of the level $T^i U_{h,n-1}$ if $j = \ell h_{n-1} + i$, with $0 \leqslant \ell \leqslant k_n - 1$ and $0 \leqslant i \leqslant h_{n-1} - 1$, and of the level $T^i U_{g,n-1}$ if $j = k_n h_{n-1} + i$, with $0 \leqslant i \leqslant g_{n-1} - 1$; and similarly, depending on the type of the recursion rules, for the other towers.

Because the cylinders generate the topology of X, and the measure is defined on Borelian sets, every function in $\mathcal{L}^1(X)$ is the limit of a sequence of functions f_n such that f_n is constant on every level $T^iU_{h,n}$, $0 \leqslant i \leqslant h_n - 1$, $T^iU_{g,n}$, $0 \leqslant i \leqslant g_n - 1$, and $T^iU_{j,n}$, $0 \leqslant i \leqslant j_n - 1$. This means that the Arnoux-Rauzy systems are systems of *rank at most three*, see [12] for a survey.

We give now two conditions for a number to be an eigenvalue of an Arnoux-Rauzy system: a necessary condition to be a measurable eigenvalue and a sufficient condition to be a continuous eigenvalue. These criteria are new, and specific to Arnoux-Rauzy systems, but not surprising as similar conditions apply for some related classes of systems, the *rank one* systems [9] and the linearly recurrent systems [10] [6]; these two propositions constitute the only part of the present paper which does use ergodic theory, albeit in a very elementary way.

PROPOSITION 10. — *If θ is a measurable eigenvalue for an Arnoux-Rauzy system, as described above, then $k_{n+1}\|h_n\theta\| \to 0$ when $n \to +\infty$, where $\| \, . \, \|$ denotes the distance to the nearest integer.*

Proof. — We prove first that, for all $n \geqslant 0$, $\mu(R_{h,n}) \geqslant \frac{1}{3}\cdot$ Indeed, the recursion rules and the cutting and stacking construction allow us to deduce the measures of the towers at stage n from the measures of the towers at stage $n+1$; namely, if the $(n+1)$-th rule (under the multiplicative form) is of type 1, the set $R_{h,n} = \bigcup_{i=0}^{h_n-1} T^iU_{h,n}$ is made of $k_{n+1}h_n$ levels of $R_{h,n+1}$, $k_{n+1}h_n$ levels of $R_{g,n+1}$, and all j_{n+1} levels of $R_{j,n+1}$. Hence

$$\begin{aligned}\mu(R_{h,n}) &= k_{n+1}h_n\mu(U_{h,n+1}) + k_{n+1}h_n\mu(U_{g,n+1}) + j_{n+1}\mu(U_{j,n+1})\\ &= \frac{k_{n+1}h_n}{k_{n+1}h_n + g_n}\mu(R_{h,n+1}) + \frac{k_{n+1}h_n}{k_{n+1}h_n + j_n}\mu(R_{g,n+1}) + \mu(R_{j,n+1});\end{aligned}$$

because of Lemma 7 we have

$$\mu(R_{h,n}) \geqslant \frac{k_{n+1}}{k_{n+1}+2}\big(\mu(R_{h,n+1}) + \mu(R_{g,n+1})\big) + \mu(R_{j,n+1}) \geqslant \frac{1}{3},$$

as $k_{n+1} \geqslant 1$ and X is the disjoint union of all the levels at stage $n+1$.

Similarly, if the $(n+1)$-th rule is of type 2,

$$\mu(R_{h,n}) = k_{n+1}h_n\mu(U_{h,n+1}) + k_{n+1}h_n\mu(U_{j,n+1}) + g_{n+1}\mu(U_{g,n+1})$$

and we get the same estimate.

We can now prove the proposition: let f be an eigenfunction for the eigenvalue θ; for each $\varepsilon > 0$ there exists $N(\varepsilon)$ such that for all $n > N(\varepsilon)$ there exists f_n, which satisfies $\int \|f - f_n\| \mathrm{d}\mu < \varepsilon$ and is constant on each level $T^iU_{h,m}$, $0 \leqslant i \leqslant h_m - 1$, $T^iU_{g,m}$, $0 \leqslant i \leqslant g_m - 1$, $T^iU_{j,m}$, $0 \leqslant i \leqslant j_m - 1$, for $m = n-2$ and hence also for $m = n-1$ and $m = n$. We shall now prove that $k_n\|\theta h_{n-1}\| < 104\varepsilon$ for all $n > N(\varepsilon)$.

Case 1: $k_n \geqslant 2$. — Let j be any integer with $0 \leqslant j \leqslant [\frac{1}{2}(k_n+1)]$. Let τ_n be the set $\bigcup_{i=0}^{[\frac{1}{2}k_n]h_{n-1}-1} T^iU_{h,n}$; by construction, for any point x in τ_n, $T^{jh_{n-1}}x$ is in the tower $R_{h,n}$, and in the same level of the tower $R_{h,n-1}$ as x. Thus for μ-almost every $x \in \tau_n$, $f_n(T^{jh_{n-1}}x) = f_n(x)$ while $f(T^{jh_{n-1}}x) = \theta jh_{n-1} + f(x)$; we have

$$\int_{\tau_n} \|f_n \circ T^{jh_{n-1}} - j\theta h_{n-1} - f_n\| \mathrm{d}\mu = \int_{\tau_n} \|j\theta h_{n-1}\| \mathrm{d}\mu = \|j\theta h_{n-1}\|\mu(\tau_n),$$

$$\int_{\tau_n} \|f_n \circ T^{jh_{n-1}} - j\theta h_{n-1} - f_n\| \mathrm{d}\mu \leqslant \int_{\tau_n} \|f_n \circ T^{jh_{n-1}} - f \circ T^{jh_{n-1}}\| \mathrm{d}\mu + \int_{\tau_n} \|f_n - f\| \mathrm{d}\mu < 2\varepsilon.$$

As $\mu(R_{h,n}) \geqslant \frac{1}{3}$, we have

$$\mu(\tau_n) \geqslant \frac{[\frac{1}{2}k_n]h_{n-1}}{3(k_nh_{n-1} + g_{n-1})} \geqslant \frac{k_n - 1}{6(k_n+2)}.$$

So $\mu(\tau_n) \geqslant \frac{1}{25}$. Thus the above estimates imply $\|j\theta h_{n-1}\| < 50\varepsilon$, for $n > N(\varepsilon)$ and any integer $0 \leqslant j \leqslant [\frac{1}{2}(k_n+1)]$. Thus, for $n > N(\varepsilon)$, $\|j\theta h_{n-1}\| < 100\varepsilon$ for any integer $0 \leqslant j \leqslant k_n$.

Let $\varepsilon < \frac{1}{400}$, and suppose $\|k_n\theta h_{n-1}\| \neq k_n\|\theta h_{n-1}\|$: let i be the smallest $0 \leqslant j \leqslant k_n$ such that $\|j\theta h_{n-1}\| \neq j\|\theta h_{n-1}\|$, then one has $i \geqslant 2$ and $\|(i-1)\theta h_{n-1}\| = (i-1)\|\theta h_{n-1}\|$, thus $i\|\theta h_{n-1}\| = (i-1)\|\theta h_{n-1}\| + \|\theta h_{n-1}\| = \|(i-1)\theta h_{n-1}\| + \|\theta h_{n-1}\| < 200\varepsilon < \frac{1}{2}$ thus $\|i\theta h_{n-1}\| = \|(i\|\theta h_{n-1}\|)\| = i\|\theta h_{n-1}\|$, contradiction. Thus we get $k_n\|\theta h_{n-1}\| < 100\varepsilon$ for $n > N(\varepsilon)$.

Case 2: $k_n = 1$ and either $k_{n-1} = 1$ or the $(n-1)$-th rule is of type 1. Hence $g_{n-1} \geqslant k_{n-1}h_{n-2}$. Let τ_n be the set $\bigcup_{i=0}^{k_{n-1}h_{n-2}-1} T^iU_{h,n}$; by construction of the towers, for any point x in τ_n, $T^{h_{n-1}}x$ is in the tower $R_{h,n}$, and in the same level of the tower $R_{h,n-2}$ as x. We have

$$\begin{aligned}\mu(\tau_n) &\geqslant \frac{k_{n-1}h_{n-2}}{3h_n} = \frac{k_{n-1}h_{n-2}}{3(h_{n-1} + g_{n-1})}\\ &\geqslant \frac{k_{n-1}h_{n-2}}{9h_{n-1}} = \frac{k_{n-1}h_{n-2}}{9(k_{n-1}h_{n-2} + g_{n-2})} \geqslant \frac{1}{27}.\end{aligned}$$

As f_n is constant on the levels at stage $n-2$, the same estimates as above give $\|\theta h_{n-1}\| < 54\varepsilon$ for $n > N(\varepsilon)$.

Case 3: $k_n = 1$, $k_{n-1} \geqslant 2$ and the $(n-1)$-th rule is of type 2. — Hence $g_{n-1} = h_{n-2}$ and $h_n = h_{n-1} + h_{n-2}$. By using the reasoning of Case 1 with $\tau_{n-1} = \bigcup_{i=0}^{[\frac{1}{2}k_{n-1}]h_{n-2}-1} T^i U_{h,n-1}$ we get $\|\theta h_{n-2}\| < 50\varepsilon$. By using either (if $k_{n+1} \geqslant 2$) the reasoning of Case 1 with $\tau_{n+1} = \bigcup_{i=0}^{[\frac{1}{2}k_{n+1}]h_n-1} T^i U_{h,n+1}$ or (if $k_{n+1} = 1$) the reasoning of Case 2 with $\tau_{n+1} = \bigcup_{i=0}^{h_n-1} T^i U_{h,n+1}$, we get that $\|\theta h_n\| < 54\varepsilon$. Hence $\|\theta h_{n-1}\| < 104\varepsilon$ for $n > N(\varepsilon)$. □

Proposition 11. — *If for an Arnoux-Rauzy system, as described above, for some* $0 < \theta < 1$, $\sum_{n=0}^{+\infty} k_{n+1}\|h_n\theta\| < +\infty$, *then* θ *is a continuous eigenvalue of* (X, T).

Proof. — We build a function f_n by $f_n(x) = i\theta$ for any x in $T^i U_{h,n}$, $0 \leqslant i \leqslant h_n - 1$, $T^i U_{g,n}$, $0 \leqslant i \leqslant g_n - 1$, $T^i U_{j,m}$, $0 \leqslant i \leqslant j_n - 1$. Then by construction $\sup_{x\in X} \|f_{n+1}(x) - f_n(x)\| = \max_{j=1}^{k_{n+1}} \|jh_n\theta\| \leqslant k_{n+1}\|h_n\theta\|$, and the hypothesis ensures that the f_n converge uniformly to a function f; thus f is continuous, and for every x and n large enough we have $f_n(Tx) = f_n(x) + \theta$ (except for the three points which are on top of all towers, corresponding to the three left extensions of the infinite word beginning by $\overline{H}_n$ for all n); thus $f(Tx) = f(x) + \theta$ (on these three points also as $\|h_n\theta\|$, $\|g_n\theta\|$ and $\|j_n\theta\|$ tend to 0). □

3. Convergents and determinants

We are now going to translate Propositions 10 and 11 in arithmetic terms, before particularizing to special subclasses later. These propositions tell us that the eigenvalues depend on properties of $\|h_n\theta\|$, hence on the quality of the approximation of θ in $1/h_n$; thus we shall have to study the *convergents* h'_n/h_n, and the recursion formulas imply that we have to control two other families of convergents, g'_n/g_n and j'_n/j_n; we shall see then that the quality of the approximation depends on the *determinant* $d_n = h_n g'_n - h'_n g_n$, which generalizes the $p_{n+1}q_n - p_n q_{n+1}$ of the Euclid algorithm; but instead of having a simple value like the latter, our d_n is given by a recursion formula involving two other determinants, e_n and f_n; and a good estimate of the quantity that we need to know, $k_{n+1}\|h_n\theta\|$, is given by $|d_n|/h_n$.

Definition 4. — *Given integers* d_0, e_0, f_0, *we define sequences* (d_n), (e_n), (f_n), $n \geqslant 0$, *by*

▷ *when the* $(n+1)$*-th rule is of type* 1,

$$d_{n+1} = -k_{n+1}d_n + k_{n+1}e_n + f_n, \quad e_{n+1} = -d_n, \quad f_{n+1} = -e_n,$$

▷ *when the $(n+1)$-th rule is of type 2,*

$$d_{n+1} = -d_n, \quad e_{n+1} = -k_{n+1}d_n + k_{n+1}e_n + f_n, \quad f_{n+1} = e_n.$$

Lemma 12. — *If (X,T,μ) is an Arnoux-Rauzy system as described above, and has an eigenvalue θ, there exists a choice of d_0, e_0, f_0, such that $d_n \neq 0$ for infinitely many n and $\lim_{n\to+\infty} d_n/h_n = 0$.*

Proof. — Let θ be an eigenvalue; we apply Proposition 10 to get that $k_{n+1}\|h_n\theta\| \to 0$.

If $\theta = p/q$ for integers p and q, this implies that $\|h_n\theta\| = 0$ ultimately, thus q divides h_n for all $n \geqslant N$; then the recursion formulas imply that q divides also g_n for $n \geqslant N$, thence j_n for $n \geqslant N$ when the $n+1$-st rule is of type 1, thence every j_n for $n \geqslant N$, by backward induction from the next rule of type 1 (we know there exists one); thus q divides every h_n, g_n, j_n for $n \geqslant N$, thus also every h_n, g_n, j_n for $n \geqslant 0$, hence $q = 1$, $\theta = 0$ and this has been excluded. Hence we may assume that *θ is irrational.*

Then there exists a sequence of integers h'_n such that $k_{n+1}|h_n\theta - h'_n| \to 0$. We show now that, provided θ is as well approximated as Proposition 10 requires, *h'_n has to follow the same recursion rules as h_n*, involving two new families g'_n and j'_n:

As $h_{n+1} \leqslant (k_{n+1}+2)h_n$ we get $h_{n+1}|\theta - h'_n/h_n| \to 0$ thus

$$\theta = \lim_{n\to+\infty} \frac{h'_n}{h_n} \quad \text{and} \quad \left|\frac{h'_{n+1}}{h_{n+1}} - \frac{h'_n}{h_n}\right| = \frac{\varepsilon_n}{h_{n+1}}$$

with $\varepsilon_n \to 0$ when $n \to +\infty$.

We define now, for all $n \geqslant 0$, g'_n by $g'_n = h'_{n+1} - k_{n+1}h'_n$ and d_n by $d_n = h_n g'_n - h'_n g_n$; then we have

$$\frac{h'_{n+1}}{h_{n+1}} - \frac{h'_n}{h_n} = \frac{d_n}{h_n h_{n+1}}$$

and this implies $|d_n| = \varepsilon_n h_n$ and $|g'_n - (h'_n/h_n)g_n| = \varepsilon_n$.

If the $(n+1)$-th rule is of type 2, then $g_{n+1} = h_n$; and

$$\begin{aligned}|g'_{n+1} - h'_n| &= |g_{n+1}\frac{h'_{n+1}}{h_{n+1}} \pm \varepsilon_{n+1} - h'_n| = |g_{n+1}\frac{h'_n}{h_n} \pm \varepsilon_{n+1} \pm \frac{g_{n+1}}{h_{n+1}}\varepsilon_n - h'_n| \\ &= |\pm\varepsilon_{n+1} \pm \frac{\varepsilon_n g_{n+1}}{h_{n+1}}| \leqslant \varepsilon_{n+1} + 2\varepsilon_n\end{aligned}$$

thus $g'_{n+1} = h'_n$ if n is large enough. This implies $d_{n+1} = -d_n$.

We define then j'_n by $j'_n = g'_{n+1} - k_{n+1}h'_n$ when the $(n+1)$-th rule is of type 1; this defines j'_n for infinitely many n, and then we define the j'_n when the $(n+1)$-th rule is of type 2, inductively from the next rule of type 1 by $j'_n = j'_{n+1} - k_{n+1}h'_n$.

We want to estimate $z_n = j'_n - (h'_n/h_n)j_n$. If the $(n+1)$-th rule is of type 1, then $j_n = g_{n+1} - k_{n+1}h_n$ and $j'_n = g'_{n+1} - k_{n+1}h'_n$ thus

$$\begin{aligned}|z_n| = |g'_{n+1} - \frac{h'_n}{h_n}g_{n+1}| &= \Big|g'_{n+1} - \frac{h'_{n+1}}{h_{n+1}}g_{n+1} + g_{n+1}\Big(\frac{h'_{n+1}}{h_{n+1}} - \frac{h'_n}{h_n}\Big)\Big| \\ &\leqslant \varepsilon_{n+1} + \varepsilon_n\frac{g_{n+1}}{h_{n+1}} \leqslant \varepsilon_{n+1} + 2\varepsilon_n.\end{aligned}$$

Otherwise $j_n = j_{n+1} - k_{n+1}h_n$ and $j'_n = j'_{n+1} - k_{n+1}h'_n$, thus

$$z_n = j'_{n+1} - \frac{h'_n}{h_n}j_{n+1} = z_{n+1} + j_{n+1}\Big(\frac{h'_{n+1}}{h_{n+1}} - \frac{h'_n}{h_n}\Big) = z_{n+1} + \frac{j_{n+1}d_n}{h_nh_{n+1}}.$$

Suppose now that the $(p+1)$-th rule is of type 2 for $n \leqslant p \leqslant n+r-1$ while the $(n+r+1)$-th rule is of type 1. Then $z_p = z_{p+1} + j_{p+1}d_p/(h_ph_{p+1})$ for $n \leqslant p \leqslant n+r-1$; but for these p, because the rules are of type 2, we have shown that $d_{p+1} = -d_p$. Hence

$$|z_n| - |z_{n+r}| \leqslant \sum_{p=n}^{n+r-1}\frac{j_{p+1}|d_p|}{h_ph_{p+1}} \leqslant \sum_{p=n}^{n+r-1}\frac{|d_p|}{h_p} = \frac{|d_n|}{h_n}\sum_{p=n}^{n+r-1}\frac{h_n}{h_p}.$$

By Lemma 8 $\sum_{p=n}^{n+r-1} h_n/h_p \leqslant \sum_{i=0}^{r-1} 1/F_i \leqslant \Psi$, thus

$$|z_n| \leqslant |z_{n+r}| + \Psi\frac{|d_n|}{h_n} \leqslant \varepsilon_{n+1+r} + \varepsilon_{n+r} + \Psi\varepsilon_n.$$

And now, if $j_{n+1} = h_n$, then

$$\begin{aligned}|j'_{n+1} - h'_n| = |j_{n+1}\frac{h'_{n+1}}{h_{n+1}} + z_{n+1} - h'_n| &= |h_n\frac{h'_{n+1}}{h_{n+1}} + z_{n+1} - h'_n| \\ &= |\frac{d_n}{h_{n+1}} + z_{n+1}| \leqslant |z_{n+1}| + \varepsilon_n\end{aligned}$$

thus $j'_{n+1} = h'_n$ if n is large enough.

Thus indeed for $n \geqslant N_0$ the quantities h'_n, g'_n, j'_n follow the same recursion rules as the h_n, g_n, j_n; we can now show that *the d_n defined above follow the recursion rules in Definition 4*:

If for $n \geqslant N_0$ we define $e_n = h_nj'_n - h'_nj_n$ and $f_n = g_nj'_n - g'_nj_n$, the recursion formulas on the h_n, g_n, j_n, h'_n, g'_n, j'_n imply that for $n \geqslant N_0$, d_n, e_n, f_n are given by the formulas in Definition 4. For $n < N_0$, we can fix in a unique way the values of e_n, f_n, and modify in a unique way the values of d_n, such that this holds for all $n \geqslant 0$, and all d_n, e_n, f_n are integers.

And if $d_n = 0$ ultimately then h'_n/h_n is constant ultimately and θ is rational, which has been excluded.

As we have seen that $|d_n| = \varepsilon_n h_n$, the lemma is proved. □

Lemma 13. — *If (X,T,μ) is an Arnoux-Rauzy system as described above, and there exists a choice of integers h'_0, g'_0, j'_0 such that, if*

$$d_0 = h_0 g'_0 - h'_0 g_0, \quad e_0 = h_0 j'_0 - h'_0 j_0, \quad f_0 = g_0 j'_0 - g'_0 j_0$$

and the d_n are defined as in Definition 4,

$$\sum_{n=0}^{+\infty} \frac{|d_n|}{h_n} < +\infty; \tag{3.1}$$

then, if we define h'_n, g'_n, j'_n by replacing h, g, j by h', g', j' in Corollary 6, *$\theta = \lim_{n\to+\infty} h'_n/h_n$ is either an integer or a continuous eigenvalue of (X,T).*

Proof. — As in the proof of Lemma 12, we have

$$\frac{h'_{p+1}}{h_{p+1}} - \frac{h'_p}{h_p} = \frac{d_p}{h_p h_{p+1}}.$$

Thus (3.1) ensures that θ exists and $\theta = h'_n/h_n + \sum_{p=n}^{+\infty} d_p/h_p h_{p+1}$, thus

$$k_{n+1}\|h_n\theta\| \leqslant k_{n+1}h_n\Big|\sum_{p=n}^{+\infty} \frac{d_p}{h_p h_{p+1}}\Big| \leqslant \sum_{p=n}^{+\infty} \frac{h_{n+1}}{h_{p+1}}\frac{|d_p|}{h_p} \leqslant \sum_{p=n}^{+\infty} \frac{1}{F_{p-n}}\frac{|d_p|}{h_p},$$

the last estimate using Lemma 8. Thus if we compute $\sum_{n=0}^{N} k_{n+1}\|h_n\theta\|$ it involves terms $|d_n|/h_n$ multiplied by at most $\sum_{p=0}^{n} 1/F_p$. Thus

$$\sum_{n=0}^{+\infty} k_{n+1}\|h_n\theta\| \leqslant \Psi \sum_{n=0}^{+\infty} \frac{|d_n|}{h_n}$$

and we apply Proposition 11. □

4. Weak mixing

Proof of Theorem 2. — We assume that (X,T,μ) has an eigenvalue θ and start from the conclusion of Lemma 12; what we need to prove is that $|d_n|/h_n$ is not small enough.

We introduce the auxiliary quantity $u_n = \max(|d_n|, |e_n|)$, and *translate our hypotheses into estimates on u_n*: according to the type of the $(n-1)$-th and n-th rules, we have either $d_n - e_{n-2} = k_n(e_{n-1} - d_{n-1})$, or $d_n + e_{n-2} = k_n(e_{n-1} - d_{n-1})$, or $e_n - e_{n-2} = k_n(e_{n-1} - d_{n-1})$, or $e_n + e_{n-2} = k_n(e_{n-1} - d_{n-1})$. This implies

$$\text{(W4)} \qquad \frac{u_n + u_{n-2}}{k_n} \geqslant |e_{n-1} - d_{n-1}| \geqslant u_{n-1} - \min\big(|d_{n-1}|, |e_{n-1}|\big) \geqslant u_{n-1} - u_{n-2},$$

where the last inequality comes from the fact that $|d_{n-2}|$ is either $|d_{n-1}|$ or $|e_{n-1}|$, so in any case $u_{n-2} \geqslant |d_{n-2}| \geqslant \min(|d_{n-1}|, |e_{n-1}|)$.

Let n be in $]n_{i-1}+1, n_i+1]$; if $n \leqslant n_i$, $d_{n-1} = \pm d_{n_{i-1}}$ while if $n = n_i + 1$, $e_{n-1} = \pm d_{n_{i-1}}$, thus in both cases $\min(|d_{n-1}|, |e_{n-1}|) \leqslant u_{n_{i-1}}$ and $|e_{n-1} - d_{n-1}| \geqslant u_{n-1} - u_{n_{i-1}}$, thus $(u_n + u_{n-2})/k_n \geqslant u_{n-1} - u_{n_{i-1}}$. We fix some a and write $u_{n-1} - u_{n_a} = u_{n-1} - u_{n_{i-1}} + \sum_{j=a+1}^{i-1}(u_{n_j} - u_{n_{j-1}})$ thus

$$\text{(W5)} \qquad u_{n-1} - u_{n_a-1} \leqslant \frac{u_n + u_{n-2}}{k_n} + \sum_{j=a+1}^{i-1} \frac{u_{n_j-1} + u_{n_j+1}}{k_{n_j+1}}.$$

We now use (W4) and (W5) to *show that u_n cannot be small compared to h_n* (we shall see at the end that it implies the same for $|d_n|$). We have first to eliminate two possibilities:

▷ If u_n is bounded by M; then by (W4) $|e_{n-1} - d_{n-1}| \leqslant 2M/k_n$. Using (W1), (W2) and (W3), we choose an i such that k_{n_i}, k_{n_i+1} and k_{n_i+2} are all greater than $2M+1$ and put $n = n_i + 1$. Then $e_{n-2} - d_{n-2} = e_{n-1} - d_{n-1} = e_n - d_n = 0$, while the $(n-1)$-th rule is of type 1. Hence $d_{n-2} = e_{n-2} = -d_{n-1} = -e_{n-1} = d_n = e_n$; but also $d_{n-1} = -f_{n-1}$ and, because of the type 1, $e_{n-2} = -f_{n-1}$; thus $e_{n-2} = d_{n-1} = e_{n-1} = 0$ and this is enough to imply $d_n = 0$ ultimately, which is excluded.

▷ If $u_n \geqslant u_{n+1}$ for infinitely many n, but u_n is unbounded. Let E be the set of n such that $u_n = \max_{i \leqslant n+1} u_i$; E is infinite as there are infinitely many m such that $u_m = \max_{i \leqslant m} u_i$, then the first $n \geqslant m$ such that $u_n \geqslant u_{n+1}$ exists because of the hypothesis, and n is in E. u_n is unbounded on E (otherwise u_n would be bounded on all n), and for $n-1 \in E$ (W5) implies $u_{n-1} - u_{n_a-1} \leqslant 2u_{n-1}/k_n + \sum_{j=a+1}^{i-1} 2u_{n-1}/k_{n_j+1}$. Thus, by (W2) for a given ε we can choose a such that $u_{n-1} - u_{n_a-1} \leqslant \varepsilon u_{n-1}$ for all $n \geqslant n_a$, thus $u_{n-1} \leqslant u_{n_a-1}/(1-\varepsilon)$ and is bounded for $n-1 \in E$, which has been excluded.

Hence now we know that, for all n large enough, $u_{n+1} > u_n$. Then we can improve our estimates: for n large enough (W5) implies

$$u_{n-1} - u_{n_a-1} \leqslant \frac{u_n}{k_n} + \frac{u_{n-1}}{k_n} + \sum_{j=a+1}^{i-1} \frac{2u_{n-1}}{k_{n_j+1}}$$

and by (W2) we may choose a such that $\sum_{j=a+1}^{+\infty} \frac{2}{k_{n_j+1}} \leqslant \frac{1}{4}$; hence we get

$$u_n \geqslant u_{n-1}\left(\tfrac{1}{2}k_n - 1\right). \tag{W6}$$

To prove that u_n is not small compared to h_n, we first *re-write the recursion rules showing that they involve the quantities* $e_n - d_n$ *and* $f_n + d_n$; then we *build a sequence* x_n *such that* $|e_n - d_n| \geqslant x_n h_n$ *and* $|f_n + d_n| \geqslant x_n h_n$, and show that x_n is bounded from below and yields a lower bound for u_n/h_n.

Thus we look again at the recursion rules.

If the $(n+1)$-th rule is of type 1, we have:

- $\triangleright$ $u_{n+1} = |d_{n+1}|$,
- $\triangleright$ $e_{n+1} - d_{n+1} = -k_{n+1}(e_n - d_n) - (d_n + f_n)$,
- $\triangleright$ $f_{n+1} + d_{n+1} = k_{n+1}(e_n - d_n) - e_n + f_n$.

If the $(n+1)$-th rule is of type 2, then

- $\triangleright$ $u_{n+1} = |e_{n+1}|$,
- $\triangleright$ $|d_{n+1}| \leqslant u_n < u_{n+1}$,
- $\triangleright$ $e_{n+1} - d_{n+1} = k_{n+1}(e_n - d_n) + (d_n + f_n)$,
- $\triangleright$ $f_{n+1} + d_{n+1} = e_n - d_n$.

Let us prove that, if n is large enough and the $(n+1)$-th rule is of type 2, $e_n - d_n$ *and* $e_{n+1} - d_{n+1}$ *have the same sign*. Indeed,

- if the n-th rule is of type 1, k_n and k_{n+1} are large by (W2), and, using (W6), we get that $|d_n + f_n| \leqslant u_n + u_{n-1} \leqslant k_{n+1}(u_n - u_{n-1}) \leqslant k_{n+1}|e_n - d_n|$, which proves our assertion;
- if the n-th rule is of type 2, $e_{n+1} - d_{n+1} = (k_{n+1} - 1)(e_n - d_n) + e_n + e_{n-1}$, and both $e_n - d_n$ and $e_n + e_{n-1}$ have the same sign as e_n, which proves our assertion.

We are now ready to define the auxiliary sequence x_n:

1) If the n-th and $(n+1)$-the rule are of type 2; then $f_n + d_n = e_{n-1} - d_{n-1}$ and $e_n - d_n$ have the same sign; thus $|e_{n+1} - d_{n+1}| = k_{n+1}|e_n - d_n| + |f_n + d_n|$, while $h_{n+1} = k_{n+1}h_n + g_n$, and $|f_{n+1} + d_{n+1}| = |e_n - d_n|$ while $g_{n+1} = h_n$; this allows us to take $x_{n+1} = x_n$.
2) If the n-th rule is of type 1 and the $(n+1)$-th rule is of type 2, then $|e_{n+1} - d_{n+1}| \geqslant k_{n+1}|e_n - d_n| - |f_n + d_n|$; as k_n is large by (W2), (W6) implies $-|f_n + d_n| \geqslant -u_n - u_{n-1} \geqslant |e_{n-1} - d_{n-1}| - 2|e_n - d_n|$, thus $|e_{n+1} - d_{n+1}| \geqslant k_{n+1}|e_n - d_n| + |e_{n-1} - d_{n-1}| - 2|e_n - d_n|$; as we have $h_{n+1} = k_{n+1}h_n + g_n \leqslant k_{n+1}h_n + h_{n-1} + 2h_n$, if we take

$x_{n+1} = x_n(1 - 4/k_{n+1})$, we satisfy the estimate on $|e_{n+1} - d_{n+1}|$, while the estimate on $|f_{n+1} + d_{n+1}|$ is satisfied as in case 1).

3) If the n-th rule is of type 2 and the $(n+1)$-th rule is of type 1.
   3a) If the $(n-1)$-th rule is of type 1, one has $|f_{n+1} + d_{n+1}| \geqslant k_{n+1}|e_n - d_n| - |f_n| - |e_n|$, and the same estimate on $-u_n - u_{n-1}$ as in case 2) (k_n being large), implies $-|f_n| - |e_n| \geqslant -u_n - u_{n-1} \geqslant |e_{n-1} - d_{n-1}| - 2|e_n - d_n|$.
   3b) If the $(n-1)$-th rule is of type 2, one has $|f_{n+1} + d_{n+1}| \geqslant k_{n+1}|e_n - d_n| - |e_n - e_{n-1}|$, and $|e_n - e_{n-1}| = |e_n| - |e_{n-1}| \leqslant |e_n| - |d_{n-1}| = |e_n| - |d_n| \leqslant |e_n - d_n|$.

   Thus in both subcases

$$|f_{n+1} + d_{n+1}| \geqslant k_{n+1}|e_n - d_n| + |e_{n-1} - d_{n-1}| - 2|e_n - d_n|.$$

   As we have $g_{n+1} = k_{n+1}h_n + j_n \leqslant k_{n+1}h_n + h_{n-1} + 2h_n$, if we take $x_{n+1} = x_n(1 - 4/k_{n+1})$, we satisfy the estimate on $|f_{n+1} + d_{n+1}|$, while the estimate on $|e_{n+1} - d_{n+1}|$ is satisfied as in case 1).

4) If the n-th and $(n+1)$-the rule are of type 1; then, if we take $x_{n+1} = x_n(1 - 4/k_{n+1})$, we satisfy the estimate on $|f_{n+1} + d_{n+1}|$ as in case 3a) and the estimate on $|e_{n+1} - d_{n+1}|$ as in case 2).

Thus, using (W2) and (W3), we get that $x_n \geqslant c$, and $|e_n - d_n| \geqslant ch_n$ for all n. And for infinitely many n, because of rules of type 1, $u_n = |d_n|$ while $|e_n| \leqslant u_{n-1}$, hence because of (W6) we have $|d_n| \geqslant \frac{1}{3}ch_n$ and we have contradicted the conclusion of Lemma 12. □

5. Eigenvalues

Proof of Theorem 3. — Let (h'_0, g'_0, j'_0) be any choice of integers; we shall show that (3.1) is satisfied; we build h'_n, g'_n, j'_n, d_n, e_n, f_n as in Lemma 13.

We want to show that $|d_n|$ is small compared to h_n; for this, we *build two sequences w_n and ℓ_n*, such that w_n gives an upper bound for $|d_n|$, with an easier recursion formula than for d_n, and then $1/\ell_n$ gives an upper bound for w_n/h_n.

We define w_n by $w_{-1} = |f_0|$, $w_0 = \max\{|d_0|, |e_0|, |d_0 - e_0|\}$, $w_{n+1} = k_{n+1}w_n + w_{n-1}$. Then we have

$$w_n \geqslant \max\big\{|d_n|, |e_n|, |d_n - e_n|\big\}.$$

Indeed

$$\begin{aligned}&\max\left\{|d_{n+1}|,|e_{n+1}|,|d_{n+1}-e_{n+1}|\right\}\\&\quad=\max\left\{|k_{n+1}(d_n-e_n)-f_n|,|d_n|,|(k_{n+1}-1)(d_n-e_n)-e_n-f_n|\right\},\end{aligned}$$

and $|k_{n+1}(d_n-e_n)-f_n|\leqslant k_{n+1}w_n+w_{n-1}$ because $|f_0|=w_{-1}$ and $|f_n|=|e_{n-1}|\leqslant w_{n-1}$ if $n>0$, while for the same reason $|(k_{n+1}-1)(d_n-e_n)-e_n-f_n|\leqslant(k_{n+1}-1)w_n+w_n+w_{n-1}$. Thus in particular $w_n\geqslant|d_n|$.

In the forthcoming construction of ℓ_n, the rules of type 1 will be the ones which make $1/\ell_n$ small. But there will be a technical problem if *if there is no rule of type* 2 *for* n *large enough*; that is why in that case we select an infinite sequence S such that:

- ▷ two consecutive elements of S differ at least by 2,
- ▷ for $n\in S$ the n-th rule is of type 1 but we do not have simultaneously $k_n\geqslant 2$ and $k_{n+1}=1$,
- ▷ for some J' and all $j>J'$, $\sum_{1\leqslant i\leqslant j,n_i\notin S}1/(1+k_{n_i+1})\geqslant(12+\frac{1}{2}\varepsilon)\ln j$.

For $n\in S$ we say that the n-th rule is a *downgraded* rule of type 1.

We can now build ℓ_n such that $h_n\geqslant\ell_n w_n$.

We start from some $0<\ell_0\leqslant h_0/w_0$, $0<\ell_1\leqslant h_1/w_1$. Then:

1) If the n-th rule is of type 2, then $h_{n+1}=k_{n+1}h_n+h_{n-1}$ and

$$\frac{h_{n+1}}{w_{n+1}}\geqslant\frac{k_{n+1}\ell_n w_n+\ell_{n-1}w_{n-1}}{k_{n+1}w_n+w_{n-1}};$$

as $w_n\geqslant w_{n-1}$, this is an average between ℓ_n and ℓ_{n-1} which is closer to ℓ_n, hence we can take

$$\ell_{n+1}=\min\left(\ell_n,\tfrac{1}{2}(\ell_n+\ell_{n-1})\right).$$

2) If the n-th rule is a downgraded rule of type 1, then $h_{n+1}\geqslant k_{n+1}h_n+h_{n-1}$ and the estimate in case 1) gives still a lower bound; we take

$$\ell_{n+1}=\min\left(\ell_n,\tfrac{1}{2}(\ell_n+\ell_{n-1})\right).$$

3) If the n-th rule is of type 1 and $k_n \geqslant 2$; then $h_{n+1} \geqslant k_{n+1}h_n + k_n h_{n-1}$ and

$$\begin{aligned}\frac{h_{n+1}}{w_{n+1}} &\geqslant \frac{k_{n+1}\ell_n w_n + k_n \ell_{n-1} w_{n-1}}{k_{n+1}w_n + w_{n-1}} \\ &\geqslant \frac{k_{n+1}\ell_n w_n + l_{n-1}w_{n-1}}{k_{n+1}w_n + w_{n-1}} + \frac{(k_n-1)\ell_{n-1}w_{n-1}}{k_{n+1}w_n + w_{n-1}} \\ &\geqslant \min(\ell_n, \frac{\ell_n + \ell_{n-1}}{2}) + (k_n - 1)\ell_{n-1}\frac{w_{n-1}}{(k_{n+1}+1)w_n} \\ &\geqslant \min(\ell_n, \frac{\ell_n + \ell_{n-1}}{2}) + (k_n - 1)\ell_{n-1}\frac{1}{(k_n+1)(k_{n+1}+1)}\end{aligned}$$

(as $w_n \leqslant (k_n+1)w_{n-1}$). Thus, if the n-th rule is a non-downgraded rule of type 1 and $k_n \geqslant 2$, we take

$$\ell_{n+1} = \min\left(\ell_n, \tfrac{1}{2}(\ell_n + \ell_{n-1})\right) + \frac{\ell_{n-1}}{3(k_{n+1}+1)}.$$

4) If the n-th rule is of type 1, $k_n = 1$ and the $(n-1)$-th rule is of type 2; then $h_{n+1} = k_{n+1}h_n + g_n = k_{n+1}h_n + h_{n-1} + j_{n-1}$, $j_{n-1} = k_{n-1}h_{n-2} + j_{n-2}$, $h_{n-1} = k_{n-1}h_{n-2} + g_{n-2} \leqslant (k_{n-1}+2)h_{n-2}$, thus $j_{n-1} \geqslant k_{n-1}h_{n-2} \geqslant (k_{n-1}/(k_{n-1}+2))h_{n-1} \geqslant \frac{1}{3}h_{n-1}$; hence

$$\begin{aligned}\frac{h_{n+1}}{w_{n+1}} &\geqslant \frac{k_{n+1}\ell_n w_n + \frac{4}{3}\ell_{n-1}w_{n-1}}{k_{n+1}w_n + w_{n-1}} \\ &\geqslant \frac{k_{n+1}\ell_n w_n + \ell_{n-1}w_{n-1}}{k_{n+1}w_n + w_{n-1}} + \frac{1}{3}\frac{\ell_{n-1}w_{n-1}}{k_{n+1}w_n + w_{n-1}},\end{aligned}$$

and $w_n \leqslant 2w_{n-1}$. Thus, if the n-th rule is a non-downgraded rule of type 1, $k_n = 1$ and the $(n-1)$-th rule is of type 2, we take

$$\ell_{n+1} = \min\left(\ell_n, \tfrac{1}{2}(\ell_n + \ell_{n-1})\right) + \frac{\ell_{n-1}}{6(k_{n+1}+1)}.$$

5) If the n-th rule is of type 1, the $(n-1)$-th rule is of type 1 and $k_n = k_{n-1} = 1$; then $h_{n+1} = k_{n+1}h_n + g_n = k_{n+1}h_n + h_{n-1} + j_{n-1}$, $j_{n-1} = h_{n-2}$, $h_{n-1} = h_{n-2} + g_{n-2} \leqslant 3h_{n-2}$, thus again $j_{n-1} \geqslant \frac{1}{3}h_{n-1}$ and we have the same estimate as in case 4). Thus, if the n-th rule is a non-downgraded rule of type 1, the $(n-1)$-th rule is of type 1 and $k_n = k_{n-1} = 1$, we take $\ell_{n+1} = \min(\ell_n, \frac{1}{2}(\ell_n + \ell_{n-1})) + \ell_{n-1}/(6(k_{n+1}+1))$.

6) If the n-th and $(n-1)$-th rules are of type 1 and $k_n = 1$, $k_{n-1} \geqslant 2$; then we take $\ell_{n+1} = \min(\ell_n, \frac{1}{2}(\ell_n + \ell_{n-1}))$ but the estimates of

case 3) are valid for the step before, hence

$$\begin{aligned}\ell_n &= \min\left(\ell_{n-1}, \tfrac{1}{2}(\ell_{n-2}+\ell_{n-1})\right) + \frac{\ell_{n-2}}{3(k_n+1)}\\ &\geqslant \min\left(\ell_{n-1}, \tfrac{1}{2}(\ell_{n-2}+\ell_{n-1})\right) + \frac{\ell_{n-2}}{6(k_n+1)} + \frac{\ell_{n-2}}{6(k_n+1)}\\ \ell_n &\geqslant \min\left(\ell_{n-1}, \tfrac{1}{2}(\ell_{n-2}+\ell_{n-1})\right) + \frac{\ell_{n-2}}{6(k_n+1)} + \frac{\ell_{n-1}}{6(k_{n+1}+1)}.\end{aligned}$$

It remains now to show that ℓ_n *is large enough.*

If we have found L such that $\ell_{m-1} \geqslant L$, $\ell_m \geqslant L$, and if the n-th rule is of type 2 or a downgraded rule of type 1 for $m \geqslant n \geqslant p$, then we get $\ell_p \geqslant L$, $\ell_{p+1} \geqslant L$.

If we have found L such that $\ell_{m-1} \geqslant L$, $\ell_m \geqslant L$, and if the n-th rule is a non-downgraded rule of type 1 for $m \leqslant n \leqslant m+p-1$, then we get by induction on $1 \leqslant q \leqslant p$ that:

▷ $$\ell_{m+q} \geqslant L + \frac{L}{12(1+k_{m+1})} + \cdots + \frac{L}{12(1+k_{m+q-1})} + \frac{L}{12(1+k_{m+q})}$$

if $q \geqslant 2$, $k_{m+q-1} = 1$, $k_{m+q-2} \geqslant 2$;

▷ $$\begin{aligned}\ell_{m+q} &\geqslant L + \frac{L}{12(1+k_{m+1})} + \cdots + \frac{L}{12(1+k_{m+q-1})} + \frac{L}{3(1+k_{m+q})}\\ &\geqslant L + \frac{L}{12(1+k_{m+1})} + \cdots + \frac{L}{6(1+k_{m+q})} + \frac{L}{6(1+k_{m+q+1})}\end{aligned}$$

if $k_{m+q} = 1$, $k_{m+q-1} \geqslant 2$;

▷ $$\ell_{m+q} \geqslant L + \frac{L}{12(1+k_{m+1})} + \cdots + \frac{L}{12(1+k_{m+q-1})} + \frac{L}{6(1+k_{m+q})}$$

in all other cases.

At the end

$$\min(\ell_{m+p}, \ell_{m+p+1}) \geqslant L + \frac{L}{12(1+k_{m+1})} + \cdots + \frac{L}{12(1+k_{m+p})}.$$

Hence, if the n-th rule is a non-downgraded rule of type 1 for $m_i \leqslant n \leqslant m_i + p_i - 1$, $i = 1, 2, \ldots$, $1 < m_1$, $1 \leqslant p_i$, $m_i + p_i - 1 < m_{i+1}$, we have

$$\begin{aligned}\ell_{m_i+p} &\geqslant \ell'_{m_i+p}\\ &= \min(\ell_0, \ell_1) \prod_{j=1}^{i-1} \left(1 + \frac{1}{12(1+k_{m_j+1})} + \cdots + \frac{1}{12(1+k_{m_j+p_j})}\right)\\ &\qquad \left(1 + \frac{1}{12(1+k_{m_i+1})} + \cdots + \frac{1}{12(1+k_{m_i+p})}\right).\end{aligned}$$

Thus, for the sequence ℓ'_n which will yield upper bounds for ℓ_n, we have

$$\ln \ell'_{m_i+p} \geqslant r + \sum_{j=1}^{i-1}\sum_{r=0}^{p_i} \frac{1}{12(1+k_{m_j+r})} + \sum_{r=0}^{p} \frac{1}{12(1+k_{m_i+r})}$$

for a constant r.

The $m_i + p$, $i \geqslant 1$, $1 \leqslant p \leqslant p_i$ are just the $n_j + 1$, $j \geqslant 1$ except if the n_j-th rule is a downgraded rule of type 1; in this last case, $n_j + 1$ is an $m_i + p_i + 1$ and we have proved that $\ell_{n_j+1} \geqslant \ell'_{m_i+p_i}$. And we have

$$\ln \ell'_{n_j+1} \geqslant r + \sum_{1\leqslant i\leqslant j, i\notin S} \frac{1}{12(1+k_{n_i+1})} \geqslant r + \left(1+\frac{\varepsilon}{24}\right)\ln j$$

for $j > J'$. Thus we get $\sum_{j=1}^{+\infty} 1/\ell'_{n_j+1} < +\infty$.

We use now our estimates to conclude.

If ultimately there is no rule of type 2, then all the n are n_i; for isolated ones, n is in S, and for these $|d_{n+1}|/h_{n+1} \leqslant 1/\ell_{n+1} \leqslant 1/\ell'_n$; hence

$$\sum_{n=1}^{+\infty} \frac{|d_n|}{h_n} \leqslant 2\sum_{j=1}^{+\infty} \frac{1}{\ell'_{n_j+1}} < +\infty.$$

If there are infinitely many rules of type 2, then S is empty. If the $(m+1)$-th, $(m+2)$-th, $\dots$, $(m+p)$-th rules are of type 2, and the m-th and $(m+p+1)$-th rules are of type 1, then: for all $1 \leqslant r \leqslant p$, $|d_{m+r}| = |d_m|$ and $h_{m+r} \geqslant F_r h_m$; $|d_{m+r}|/h_{m+r} \leqslant 1/F_{r-1} \times 1/\ell_{m+1}$ for $1 \leqslant r \leqslant p$ and $|d_{m+p+1}|/h_{m+p+1} \leqslant 1/\ell_{m+1}$. Hence

$$\sum_{n=1}^{+\infty} \frac{|d_n|}{h_n} \leqslant (\Psi+1)\sum_{j=1}^{+\infty} \frac{1}{\ell'_{n_j+1}} < +\infty.$$

Hence *in every case we can apply Lemma 13*: each choice of (h'_0, g'_0, j'_0) gives a $\theta = \lim h'_n/h_n$, which is either an integer or a continuous eigenvalue; let $(h'_0(i), g'_0(i), j'_0(i))$, $i = 0, 1, 2$, be the vectors $(1,1,1)$, $(1,0,0)$, $(0,1,0)$; for $(h'_0, g'_0, j'_0) = (h'_0(i), g'_0(i), j'_0(i))$, the corresponding $h'_n(i)/h_n(i)$ converge to real numbers $\theta_0 = 1, \theta_1$ and θ_2. A rational relation $\sum_{i=0}^{2} a_i\theta_i = 0$ can be written with a_i integers; because of our estimates on the approximation, we get that $|\sum_{i=0}^{2} a_i h'_n(i)| < 1$ for n large enough, thus for n large enough $\sum_{i=0}^{2} a_i h'_n(i) = 0$; and this gives a rational relation between the vectors $(h'_0(i), g'_0(i), j'_0(i))$ which is excluded. Hence θ_1 and θ_2 are not integers and are continuous eigenvalues of (X,T).

And, by the proof of Lemma 12, any measurable eigenvalue θ is a limit of h'_n/h_n where, for $n \geqslant q$, the h'_n and auxiliary quantities g'_n and j'_n are given by the rules in Lemma 13; we can extend these rules backwards in a unique way to $n=0$, and (h'_0, g'_0, j'_0) is an integer and can be written as $a(1,1,1)+b(1,0,0)+c(0,1,0)$, hence θ has the required expression and thus is a continuous eigenvalue. □

Proof of Theorem 4. — Let $(h'_0, g'_0, j'_0) = (1,0,1)$; we check that $d_0 = -1$, $d_1 = 2$, $d_2 = -3$, and prove by induction that $d_n = (-1)^{n+1}(n+1)$; we have $h_n \geqslant \Pi_{p=1}^{n}(2n-1)$, hence we can apply Lemma 13; and θ, the limit of h'_n/h_n, is not an integer as it is between $h'_1/h_1 = \frac{1}{2}$ and $h'_2/h_2 = \frac{5}{8}$. □

In this last context, for the directions (h'_0, g'_0, j'_0) out of the space generated by $(1,0,1)$ and $(1,1,1)$ we can prove that d_n/h_n grows like $\frac{1}{n}$, right in the gap between our necessary and sufficient conditions; the results in [6], though they do not apply in this case, suggest the possibility that there might be a second measurable eigenvalue but no other continuous one.

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