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ON THE FEFFERMAN-PHONG INEQUALITY

by Abdesslam BOULKHEMAIR

ABSTRACT. — We show that the number of derivatives of a non negative 2-order symbol needed to establish the classical Fefferman-Phong inequality is bounded by $\frac{n}{2}+4+\epsilon$ improving thus the bound $2n+4+\epsilon$ obtained recently by N. Lerner and Y. Morimoto. In the case of symbols of type $S^0_{0,0}$, we show that this number is bounded by $n+4+\epsilon$; more precisely, for a non negative symbol a, the Fefferman-Phong inequality holds if $\partial_x^\alpha \partial_\xi^\beta a(x,\xi)$ are bounded for, roughly, $4 \leqslant |\alpha| + |\beta| \leqslant n+4+\epsilon$. To obtain such results and others, we first prove an abstract result which says that the Fefferman-Phong inequality for a non negative symbol a holds whenever all fourth partial derivatives of a are in an algebra $\mathcal{A}$ of bounded functions on the phase space, which satisfies essentially two assumptions : $\mathcal{A}$ is, roughly, translation invariant and the operators associated to symbols in $\mathcal{A}$ are bounded in L^2.

RÉSUMÉ. — Nous montrons que le nombre de dérivées d'un symbole non négatif d'ordre 2, nécessaire pour établir l'inégalité classique de Fefferman-Phong est majoré par $\frac{n}{2}+4+\epsilon$ améliorant ainsi la borne $2n+4+\epsilon$ obtenue récemment par N. Lerner et Y. Morimoto. Dans le cas des symboles de type $S^0_{0,0}$, nous montrons que ce nombre est majoré par $n+4+\epsilon$; plus précisément, pour un symbole non négatif a, on a l'inégalité de Fefferman-Phong si les $\partial_x^\alpha \partial_\xi^\beta a(x,\xi)$ sont bornées en gros pour $4 \leqslant |\alpha| + |\beta| \leqslant n+4+\epsilon$. Pour obtenir ces résultats et d'autres, nous commençons par établir un résultat abstrait qui dit que l'inégalité de Fefferman-Phong pour un symbole non négatif a a lieu pourvu que les dérivées partielles d'ordre 4 de a soient dans une algèbre $\mathcal{A}$ de fonctions bornées sur l'espace des phases, qui vérifie essentiellement deux conditions : $\mathcal{A}$ est, en gros, invariante par translation et les opérateurs associés aux symboles de $\mathcal{A}$ sont bornés dans L^2.

1. Introduction

The classical Fefferman-Phong inequality, [6], states that, if a is a non negative symbol on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying, for all multi-indices α and β,

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leqslant C_{\alpha,\beta} \langle \xi \rangle^{2-|\beta|},$$

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there exists a constant $C > 0$ such that, for all $u \in \mathcal{S}(\mathbb{R}^n)$,

$$\operatorname{Re}(a(x,D)u|u)_{L^2} + C\|u||_{L^2} \geqslant 0, \tag{1.1}$$

where $a(x,D)$ is the standard quantization of the symbol a, that is the pseudodifferential operator defined by

$$a(x,D)u(x) = \int_{\mathbb{R}^n} \mathrm{e}^{2\pi i x\xi} a(x,\xi)\widehat{u}(\xi)d\xi,$$

$\widehat{u}$ being the Fourier transform of u. Thus, it is a great improvement of both the classical and the sharp Gårding inequality. Using the Weyl quantization

$$\operatorname{Op}^w(a)u(x) = \int_{\mathbb{R}^{2n}} \mathrm{e}^{2\pi i(x-y)\xi} a(\tfrac{x+y}{2},\xi)u(y)dyd\xi,$$

the inequality is also equivalent to saying that there exists a constant $C > 0$ such that

$$\operatorname{Op}^w(a) + C \geqslant 0. \tag{1.2}$$

An alternate and recent version of the Fefferman-Phong inequality due to J. M. Bony, [1], says that inequalities (1.1) and (1.2) hold if a is a non negative symbol such that

$$\partial_x^\alpha \partial_\xi^\beta a(x,\xi) \text{ are bounded for } |\alpha| + |\beta| \geqslant 4,$$

which is a remarkable result since it indicates that only the boundedness of derivatives of order larger than or equal to 4 is relevant.

In this paper, we are interested in estimating the number of derivatives of the symbol a needed to obtain the Fefferman-Phong inequality. In [8] (see also the short version [9]), N. Lerner and Y. Morimoto proved that this number is bounded by $4 + 2n + \epsilon$ with an arbitrary $\epsilon > 0$, n being the dimension of the base space. Actually, they established the following more precise result :

Inequalities (1.1) and (1.2) hold if a is a non negative symbol such that

$$\partial_x^\alpha \partial_\xi^\beta a \in \mathcal{A}_0(\mathbb{R}^{2n}) \quad \textit{for } |\alpha| + |\beta| = 4,$$

where $\mathcal{A}_0(\mathbb{R}^{2n})$ is the Wiener type algebra of symbols studied by J. Sjöstrand in [10].

Recall that one way to define the algebra $\mathcal{A}_0(\mathbb{R}^d)$ is as the set of functions that can be written as the sum of a series like

$$\sum_{k\in\mathbb{Z}^d} u_k(x)\mathrm{e}^{ixk},$$

where the functions u_k are bounded, with spectrum in a fixed compact set and

$$\sum_{k\in\mathbb{Z}^d} \|u_k\|_{L^\infty} < \infty\,.$$

Note that the definition of $\mathcal{A}_0$ does not use derivatives. However, in terms of regularity, for a general function on $\mathbb{R}^d$ to be in $\mathcal{A}_0(\mathbb{R}^d)$, it must have $d+1$ bounded derivatives, and this fact is optimal, see [3]. Hence, the result of Lerner-Morimoto.

In this paper, we take back the argument of Lerner-Morimoto and apply more or less known results on L^2 boundedness of pseudodifferential operators to obtain mainly that

For a non negative symbol a, inequalities (1.1) and (1.2) hold

if $\partial_x^\alpha\partial_\xi^\beta a^{(4)}$ *are bounded or locally uniformly square integrable for* $|\alpha| + |\beta| \leqslant n+1$,

or, if $|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)| \leqslant C_{\alpha,\beta}\langle\xi\rangle^{2-|\beta|}$, *for* $|\alpha|+|\beta| \leqslant \left[\frac{n}{2}\right]+5$,

or, if $|\partial_x^\alpha\partial_\xi^\beta a^{(4)}(x,\xi)| \leqslant C_{\alpha,\beta}\langle\xi\rangle^{-|\beta|}$, for $|\alpha|+|\beta| \leqslant \left[\frac{n}{2}\right]+1$, which is equivalent to the more explicit estimates :

$$\text{For } |\alpha|+|\beta| \leqslant \left[\frac{n}{2}\right]+5, \quad |\partial_x^\alpha\partial_\xi^\beta a(x,\xi)| \leqslant C_{\alpha,\beta}\langle\xi\rangle^{4-|\alpha|-|\beta|} \text{ if } |\alpha|<4,$$
$$\text{and } |\partial_x^\alpha\partial_\xi^\beta a(x,\xi)| \leqslant C_{\alpha,\beta}\langle\xi\rangle^{-|\beta|}, \text{ if } |\alpha| \geqslant 4.$$

Here, $a^{(4)}$ stands for the tensor of fourth order derivatives of a. In fact, we shall also use fractional derivatives in such a way that we finally obtain that the number of derivatives of the symbol a needed to obtain the Fefferman-Phong inequality is bounded by $4+n+\epsilon$ (resp. $4+\frac{n}{2}+\epsilon$), where $\epsilon>0$ is arbitrary.

The paper begins with an abstract result. Indeed, we remark that the method of proof of Lerner-Morimoto works if one replaces the algebra $\mathcal{A}_0$ by any subalgebra of L^∞ satisfying few assumptions. See Section 2. Then, we indicate some more or less natural algebras that satisfy the assumptions of Section 2 and prove L^2 boundedness for pseudodifferential operators associated to these algebras in the t-quantization. In the fourth section, we state the result that gives the best bounds for the number of derivatives of the symbol needed to obtain the Fefferman-Phong inequality. We conclude by stating the Fefferman-Phong inequalities in the semi-classical setting.

We are grateful to P. Bolley and N. Lerner for motivating discussions on the subject.

Some notations

"Cst" will always stand for some positive constant which may change from one inequality to the other.

$\|.\|_E$ denotes the norm in the space E.

$\mathcal{L}(E)$ is the space of bounded operators in E.

$(u|v)$ is the scalar product in L^2.

$\widehat{u} = \mathcal{F}(u)$ is the Fourier transform of u.

For a function $a(x,\eta)$, $\mathcal{F}_1(a)(\xi,\eta)$ and $\mathcal{F}_2(a)(x,y)$ denote the Fourier transforms of $x \mapsto a(x,\eta)$ and $\eta \mapsto a(x,\eta)$ respectively.

Sometimes, $\partial_1^\alpha \partial_2^\beta a$ is used for $\partial_x^\alpha \partial_\xi^\beta a(x,\xi)$.

τ_k is the translation operator : $\tau_k u(x) = u(x-k)$.

If $x \in \mathbb{R}^n$, $\langle x \rangle = \sqrt{1+x^2}$.

$[x]$ denotes the integral part of the real number x.

$[\alpha,\beta]$ stands for the compact interval $\{x \in \mathbb{R}; \alpha \leqslant x \leqslant \beta\}$.

$\lambda \sim \mu$ means that $\frac{\lambda}{\mu}$ and $\frac{\mu}{\lambda}$ are bounded.

$H^s(\mathbb{R}^d)$, $s \in \mathbb{R}$, is the usual L^2 Sobolev space.

$B^s(\mathbb{R}^d)$, $s \in \mathbb{N} \cup \{\infty\}$, is the space of bounded functions in $\mathbb{R}^d$ with bounded derivatives up to the order s. For positive non integral s, $B^s(\mathbb{R}^d)$ is the usual Hölder space.

2. An abstract result

Let $\mathcal{A}$ be a subalgebra of $L^\infty(\mathbb{R}^{2n})$ satisfying the following properties :

(H1) $\quad \exists\, C_0 > 0, \exists m \geqslant 0, \forall Y \in \mathbb{R}^{2n}, \forall b \in \mathcal{A}, \quad \|\tau_Y b\|_{\mathcal{A}} \leqslant C_0 \langle Y \rangle^m \|b\|_{\mathcal{A}}.$

(H2) $\quad$ The map $b \mapsto \mathrm{Op}^w(b)$ is bounded from $\mathcal{A}$ to $\mathcal{L}(L^2(\mathbb{R}^n))$.

We have then the following :

Theorem 2.1. — *There exists a constant $C > 0$ such that, for any non negative function a on $\mathbb{R}^{2n}$ such that $a^{(4)} \in \mathcal{A}(\mathbb{R}^{2n})$, we have*

$$\mathrm{Op}^w(a) + C\|a^{(4)}\|_{\mathcal{A}} \geqslant 0, \tag{2.1}$$

that is, $\mathrm{Op}^w(a)$ *is semi-bounded.*

The proof follows the same lines as that of [8]. So, we shall be brief and refer to that paper for more details.

Lemma 2.2. — *For any function a defined on $\mathbb{R}^{2n}$ and such that $a^{(4)} \in \mathcal{A}(\mathbb{R}^{2n})$, we have*

$$\mathrm{Op}^w(a) = \mathrm{Op}^{wick}(a - \frac{1}{8\pi}\Delta a) + \mathrm{Op}^w(r),$$

where $r \in \mathcal{A}(\mathbb{R}^{2n})$ is such that $\|r\|_{\mathcal{A}} \leqslant C\|a^{(4)}\|_{\mathcal{A}}$, C being a constant independent of a, and $\Delta a = \sum_{j=1}^{2n} \partial_j^2 a$.

Recall that $\mathrm{Op}^{wick}(a)$, the Wick quantization of a, is the pseudodifferential operator whose Weyl symbol is

$$b(X) = \int a(X+Y)2^n \mathrm{e}^{-2\pi Y^2} dY. \tag{2.2}$$

Proof. — If b is given by (2.2), using Taylor formula, we can write

$$b(X) = a(X) + \frac{1}{8\pi}\Delta a + \frac{1}{6}\iint_0^1 (1-t)^3\ a^{(4)}(X+tY)\ Y^4 \mathrm{e}^{-2\pi Y^2} 2^n dtdY.$$

Another application of Taylor formula allows us to write the Weyl symbol of $\mathrm{Op}^{wick}(\Delta a)$ as

$$\theta(X) = \Delta a(X) + \iint_0^1 (1-t)\ (\Delta a)''(X+tY)\ Y^2 \mathrm{e}^{-2\pi Y^2} 2^n dtdY.$$

Thus, the Weyl symbol of $\mathrm{Op}^{wick}(a - \frac{1}{8\pi}\Delta a)$ is equal to $b - \frac{1}{8\pi}\theta = a - r$ where

$$\begin{aligned} r(X) = &-\frac{1}{6}\iint_0^1 (1-t)^3\ a^{(4)}(X+tY)\ Y^4 \mathrm{e}^{-2\pi Y^2} 2^n dtdY \\ &+\frac{1}{8\pi}\iint_0^1 (1-t)\ (\Delta a)''(X+tY)\ Y^2 \mathrm{e}^{-2\pi Y^2} 2^n dtdY, \end{aligned}$$

that is,

$$r(X) = \iint_0^1 a^{(4)}(X+tY)\ P(t,Y) \mathrm{e}^{-2\pi Y^2} dtdY,$$

where $P(t,Y)$ is a polynomial in (t,Y). Now, by the assumption (H1) on $\mathcal{A}$, we get

$$\|r\|_{\mathcal{A}} \leqslant C_0 \|a^{(4)}\|_{\mathcal{A}} \iint_0^1 \langle Y\rangle^m |P(t,Y)| \mathrm{e}^{-2\pi Y^2} dtdY = C\|a^{(4)}\|_{\mathcal{A}}.$$

□

It follows from the above lemma and assumption (H2) on $\mathcal{A}$ that Theorem 1 is a consequence of the following result of [8] which is independent of the algebra $\mathcal{A}$.

Proposition 2.3. — *There exists a constant $C > 0$ such that for any nonnegative function a defined on $\mathbb{R}^{2n}$ and such that $a^{(4)} \in L^\infty(\mathbb{R}^{2n})$, we have*

$$\mathrm{Op}^{wick}(a - \frac{1}{8\pi}\Delta a) + C\|a^{(4)}\|_{L^\infty} \geqslant 0.$$

The proof of this proposition relies on the Wick pseudodifferential calculus and on a precise result on the decomposition of nonnegative functions as sums of squares. We refer to [8].

This achieves the proof of Theorem 2.1.

Remark 2.4. — It follows from the proof of Lemma 1 that the (H2) assumption can be replaced by the following more general one :

$$(\mathrm{H1})' \qquad \exists C_0 > 0, \exists \delta < 2\pi, \forall Y \in \mathbb{R}^{2n}, \forall b \in \mathcal{A}, \quad \|\tau_Y b\|_{\mathcal{A}} \leqslant C_0\, \mathrm{e}^{\delta Y^2}\, \|b\|_{\mathcal{A}}.$$

We turn now to the Fefferman-Phong inequality in the standard quantization case. To be able to deduce it from the Weyl quantization case, we have to strengthen the assumption (H2). We shall use

(H2)′: The map $a \mapsto \mathrm{Op}_t(a)$ is bounded from $\mathcal{A}$ to $\mathcal{L}(L^2(\mathbb{R}^n))$ for all $t \in [0,1]$ and its norm is uniformly bounded with respect to $t \in [0,1]$.

Recall that $\mathrm{Op}_t(a)$ is the pseudodifferential operator defined by

$$\mathrm{Op}_t(a)u(x) = \int_{\mathbb{R}^{2n}} \mathrm{e}^{2\pi i(x-y)\xi} a((1-t)x+ty,\xi)u(y)dyd\xi, \quad u \in \mathcal{S}(\mathbb{R}^n),$$

so that, $\mathrm{Op}_{1/2} = \mathrm{Op}^w$ and Op_0 is the standard quantization. Recall also that $\mathrm{Op}_t(a) = \mathrm{Op}_0(J^t a)$ where $J^t = \mathrm{e}^{2\pi i t D_x D_\xi}, t \in \mathbb{R}$.

Theorem 2.5. — *Assume that $\mathcal{A}$ satisfies* (H1) *and* (H2)′. *There exists a constant $C > 0$ such that, for any non negative function a on $\mathbb{R}^{2n}$ such that $a^{(4)} \in \mathcal{A}(\mathbb{R}^{2n})$, we have*

$$\mathrm{Re}(a(x,D)u|u)_{L^2} + C\,\|a^{(4)}\|_{\mathcal{A}}\,\|u\|_{L^2}^2 \geqslant 0, \quad u \in \mathcal{S}(\mathbb{R}^n). \tag{2.3}$$

Proof. — We are concerned with the semi-boundedness of the operator $A = [a(x,D)+a(x,D)^\star]/2$. We can write $2A = \mathrm{Op}^w(J^{-1/2}a + J^{1/2}\overline{a})$. Now, by Taylor formula, we have

$$J^{-1/2}a = a - \pi i D_x D_\xi a - \pi^2 \int_0^1 (1-t)\mathrm{e}^{-\pi i t D_x D_\xi}(D_x D_\xi)^2 a dt$$

and

$$J^{1/2}\overline{a} = \overline{a} + \pi i D_x D_\xi \overline{a} - \pi^2 \int_0^1 (1-t)\mathrm{e}^{\pi i t D_x D_\xi}(D_x D_\xi)^2 \overline{a} dt.$$

Since a is real, we get

$$J^{-1/2}a + J^{1/2}\overline{a} = 2a - \pi^2 \int_0^1 (1-t)(J^{-t/2} + J^{t/2})(D_x D_\xi)^2 a\, dt.$$

Hence,

(2.4)

$$A = \mathrm{Op}^w(a) - R \quad \text{where} \quad R = \frac{\pi^2}{2}\int_0^1 (1-t)(\mathrm{Op}_{(1-t)/2}(b) + \mathrm{Op}_{(1+t)/2}(b))dt$$

and $b = (D_x D_\xi)^2 a$. Since $b \in \mathcal{A}$, it follows from assumption (H2)$'$ that R is bounded in $L^2(\mathbb{R}^n)$ with an operator norm bounded by $C\,\|a^{(4)}\|_{\mathcal{A}}$ and therefore the result follows from Theorem 1. □

3. On algebras of symbols and boundedness of operators

We present here some more or less known algebras of symbols which give rise to L^2-bounded pseudodifferential operators and to which we are going to apply the results of the preceding section.

3.1. Uniformly local Sobolev algebras

If E is a Banach space of functions or distributions on $\mathbb{R}^d$ (for example, containing $\mathcal{D}(\mathbb{R}^d)$, to avoid trivial cases), we shall denote by E_{ul} the set of functions or distributions u which are locally uniformly in E, that is, the set of u such that $u\,\tau_y\chi$ is in E uniformly in $y \in \mathbb{R}^d$ for some $\chi \in \mathcal{D}(\mathbb{R}^d)$ with non zero integral. The space E_{ul} is then naturally normed by $\|u\|_{E_{ul}} = \sup_{y\in\mathbb{R}^{2n}} \|u\,\tau_y\chi\|_E$.

We shall apply this procedure to the usual Sobolev space $H^s(\mathbb{R}^{2n})$ as well as to its anisotropic analogues $H^{s,s'}(\mathbb{R}^n \times \mathbb{R}^n)$ and $H^\sigma(\mathbb{R}^{2n})$, $s, s' \in \mathbb{R}$, $\sigma = (\sigma_1, \dots, \sigma_{2n}) \in \mathbb{R}^{2n}$. Recall that these are also Hilbert spaces and are defined by :

- $u \in H^{s,s'}(\mathbb{R}^n \times \mathbb{R}^n)$ iff u is a tempered distribution such that the integral
$$\int_{\mathbb{R}^n\times\mathbb{R}^n} \Big|\langle\xi\rangle^s \langle\xi'\rangle^{s'}\,\widehat{u}(\xi,\xi')\Big|^2 d\xi d\xi'$$
is finite.
- $u \in H^\sigma(\mathbb{R}^{2n})$ iff u is a tempered distribution such that the integral
$$\int_{\mathbb{R}^{2n}} \Big|\prod_{i=1}^{2n} \langle\xi_i\rangle^{\sigma_i}\,\widehat{u}(\xi)\Big|^2 d\xi$$
is finite.

We shall consider the spaces $H^s_{ul}(\mathbb{R}^{2n})$ for $s > n$, $H^{s,s'}_{ul}(\mathbb{R}^n \times \mathbb{R}^n)$ for $s > \frac{n}{2}$, $s' > \frac{n}{2}$, and $H^{\sigma}_{ul}(\mathbb{R}^{2n})$ for $\sigma_i > \frac{1}{2}$, $1 \leqslant i \leqslant 2n$. These are Banach subalgebras of $L^\infty(\mathbb{R}^{2n})$ and we have the inclusions

$$H^s_{ul}(\mathbb{R}^{2n}) \subset H^{\frac{s}{2},\frac{s}{2}}_{ul}(\mathbb{R}^n \times \mathbb{R}^n) \subset H^{(\frac{s}{2n},\dots,\frac{s}{2n})}_{ul}(\mathbb{R}^{2n})$$

with continuous injections, of course.

Clearly, these algebras satisfy trivially the assumption (H1). They also satisfy assumption (H2)$'$, although this is much less trivial. In fact, we are going to show that, if $a \in H^{\sigma}_{ul}(\mathbb{R}^{2n})$, $\sigma_i > \frac{1}{2}$, $1 \leqslant i \leqslant 2n$, then, for all $t \in \mathbb{R}$, $\mathrm{Op}_t(a)$ is bounded in $L^2(\mathbb{R}^n)$ and its operator norm can be estimated by $C(1+t^2)^N \|a\|_{H^{\sigma}_{ul}}$. Using dyadic decompositions with respect to each variable, this is equivalent to the following :

Theorem 3.1. — *There exist positive constants C and M such that, for any $a \in L^\infty(\mathbb{R}^{2n})$ with $\mathrm{supp}(\widehat{a}) \subset \prod_{i=1}^{2n}[-R_i, R_i]$, $R_i \geqslant 1$, $1 \leqslant i \leqslant 2n$, and any $t \in \mathbb{R}$, the following inequality holds :*

$$\|\mathrm{Op}_t(a)\|_{\mathcal{L}(L^2)} \leqslant C(1+t^2)^M (R_1 R_2 \dots R_{2n})^{\frac{1}{2}} \, \|a\|_{L^2_{ul}} \, .$$

Proof. — In fact, the cases $t = 0$ and $t = \frac{1}{2}$ are proved in [2] and [4] respectively. Unfortunately, we have not been able to deduce this theorem from these cases (is it possible ?). However, the proof is very similar to that of Theorem 2.1 in [4] since it consists roughly in replacing the “$\frac{1}{2}$” by t (or by $1-t$) and then to supervise the dependence on t of the estimates. So, we shall be brief and refer to [4] for missing details.

Note also that, in this paper, we only need the case $0 \leqslant t \leqslant 1$. However, for $t \in \mathbb{R}$, the proof is the same and this more general case may be useful elsewhere.

We have to study

$$I = (\mathrm{Op}_t(a)v|u), \quad u, v \in \mathcal{S}(\mathbb{R}^n),$$

and we can assume that $a \in \mathcal{S}(\mathbb{R}^{2n})$. The first step is to write

(3.1)
$$\begin{aligned} I &= \int \mathcal{F}_2(a)(x,y)\, v(x+(1-t)y)\, \overline{u}(x-ty)\, dxdy \\ &= \int \chi(x)\mathcal{F}_2(a)(x+k,y)\, v(x+(1-t)y+k)\, \overline{u}(x-ty+k)\, dxdydk, \end{aligned}$$

where $\chi \in \mathcal{S}(\mathbb{R}^n)$ and $\int \chi(x)dx = 1$. Next, introducing the weight $\omega(x) = \prod_{i=1}^n \langle x_i \rangle^{s_i}$, $s_i > 1$, $1 \leqslant i \leqslant n$, and applying Cauchy-Schwarz and Peetre inequalities, we obtain

$$(3.2)\quad |I| \leqslant \text{Cst } \langle t\rangle^{s} \sup_{k\in\mathbb{R}^n}\left(\int \left|\chi(x)\,\omega(x)\,\omega(y)\,\mathcal{F}_2(a)(x+k,y)\right|^2 dxdy\right)^{\frac{1}{2}} \|u\|_{L^2}\|v\|_{L^2},$$

where $s = s_1 + ... + s_n$ and the constant does not depend on (u, v, a, t). The $\langle t\rangle^s$ appears when one applies Peetre inequality.

The second step in the proof consists in improving the last estimate. Note that we have not used yet the spectral property of a. We can write

$$I = \int \mathcal{F}_2(a_R)(Rx, y)\, v_R(x + (1-t)y)\, \overline{u}_R(x - ty)\, dxdy,$$

where we have set

$$a_R(x,\eta) = a(x, R_1^{-1}\eta_1, \dots, R_n^{-1}\eta_n), \quad u_R(x) = (R_1 \dots R_n)^{\frac{1}{2}}\, u(Rx)$$

and $v_R(x) = (R_1 \dots R_n)^{\frac{1}{2}}\, v(Rx)$, $Rx = (R_1x_1, \dots, R_nx_n)$, and we have applied some obvious changes of variables. Now, applying the estimate (3.2) to $(\mathrm{Op}_t[a_R(Rx,\eta)])v_R|u_R)$ (with $s_i = 2$) and using the fact that the support of $y \mapsto \mathcal{F}_2(a_R)(x,y)$ is contained in $[-1,1]^n$, we obtain

$$(3.3)\quad |I| \leqslant \text{Cst } \langle t\rangle^{2n}(R_1 \dots R_n)^{\frac{1}{2}} \sup_{k\in\mathbb{R}^n}\left(\int \left|\chi(x)\,\omega(x)\,a(Rx+k,\eta)\right|^2 dxd\eta\right)^{\frac{1}{2}} \|u\|_{L^2}\|v\|_{L^2},$$

where $\omega(x) = \prod_{i=1}^n \langle x_i\rangle^2$ and $\chi \in \mathcal{S}(\mathbb{R}^n)$, $\int \chi(x)dx = 1$.

For the third and last step in the proof of Theorem 3, we write

(3.4)

$$\begin{aligned} I &= \int \mathcal{F}_1(b)(\xi,\eta)\, \widehat{V}(\eta - t\xi)\, \overline{\widehat{U}}(\eta + (1-t)\xi)\, d\xi d\eta, \\ &= \int \psi(\eta)\, \mathcal{F}_1(b)(\xi, \eta+\ell)\, \widehat{V}(\eta - t\xi + \ell)\, \overline{\widehat{U}}(\eta + (1-t)\xi + \ell)\, d\xi d\eta d\ell \end{aligned}$$

where $b(x,\eta) = a(R^{-1}x, R\eta)$, $U(x) = u(R^{-1}x)/\sqrt{R_1 \dots R_n}$, $V(x) = v(R^{-1}x)\ /\sqrt{R_1 \dots R_n}$ $R^{-1}x = (R_1^{-1}x_1, \dots, R_n^{-1}x_n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$, $\int \psi(\eta)d\eta = 1$. Next, by mutiplying and dividing, we introduce in the last integral the weight (with $s \in 2\mathbb{N}$)

$$\langle \eta + (1-t)\xi\rangle^s \langle \eta - t\xi\rangle^s = \sum_{|\alpha|,|\beta|\leqslant 2s} c_{\alpha,\beta}(t)\, \xi^\alpha\, \eta^\beta,$$

and obtain the expression

$$I = \sum_{|\alpha|,|\beta|\leqslant 2s} c_{\alpha,\beta}(t) \int \Big(\mathrm{Op}_t(b_{\alpha,\beta,\ell})V_\ell|U_\ell\Big) d\ell,$$

where $b_{\alpha,\beta,\ell}(x,\eta) = D_x^\alpha b(x,\eta+\ell)\psi(\eta)\eta^\beta$, $\widehat{U}_\ell(\xi) = \langle\xi\rangle^{-s}\widehat{U}(\xi+\ell)$, $\widehat{V}_\ell(\eta) = \langle\eta\rangle^{-s}\widehat{V}(\eta+\ell)$ and, of course, the $c_{\alpha,\beta}(t)$ are polynomial in t of degree $|\alpha| \leqslant 2s$.

Now, it remains to remark that the support of $\widehat{b}_{\alpha,\beta,\ell}$ is contained in

$$[-1,1]^n \times \prod_{i=1}^n [-1 - R_i R_{n+i}, 1 + R_i R_{n+i}]$$

(if $\mathrm{supp}(\widehat{\psi}) \subset B(0,1)$) and then to apply the estimate (3.3) to $(\mathrm{Op}_t(b_{\alpha,\beta,\ell})V_\ell|U_\ell)$. This yields the estimate of Theorem 3 with $M = n+s$ (and s is an even integer greater than $\frac{n}{2}$) since

$$\int \|U_\ell\|_{L^2}\|V_\ell\|_{L^2} d\ell \leqslant \Big(\int \|U_\ell\|_{L^2}^2 d\ell\Big)^{1/2}\Big(\int \|V_\ell\|_{L^2}^2 d\ell\Big)^{1/2}$$
$$= \mathrm{Cst}\ \|U\|_{L^2}\|V\|_{L^2} = \mathrm{Cst}\ \|u\|_{L^2}\|v\|_{L^2}.$$

□

3.2. Hölder type algebras

A well known algebra of bounded functions in $\mathbb{R}^{2n}$ which also satisfies (H1) and (H2)$'$ is the Hölder algebra $B^s(\mathbb{R}^{2n})$ for $s > n$. Here, to be simple, when $s \in \mathbb{N}$, this will be the Sobolev space $W^{s,\infty}$ and not the Zygmund class even if many of our statements hold with the latter. One can obtain somewhat more general algebras by considering Hölder anisotropic regularity in the same spirit as in the case of the uniformly local Sobolev spaces. The more general one is defined by means of a $2n$-dyadic partition of unity in $\mathbb{R}^{2n}$, $1 = \sum_{j\in\mathbb{N}^{2n}} \varphi_j$, where $\varphi_j(\zeta) = \varphi_{j_1}(\zeta_1)\dots\varphi_{j_n}(\zeta_{2n})$, $j = (j_1,\dots,j_{2n})$, based on a dyadic partition of unity in $\mathbb{R}$, $1 = \sum_{k\in\mathbb{N}}\varphi_k$, (for example, $\varphi_0 \in \mathcal{D}(\mathbb{R})$, $\varphi_1 \in \mathcal{D}(\mathbb{R}\setminus 0)$, $\varphi_{k+1}(t) = \varphi_k(\frac{t}{2})$, $t\in\mathbb{R}$, $k \geqslant 1$). If $\sigma \in (\mathbb{R}_+^*)^{2n}$, we have, by definition,

$u \in B^\sigma(\mathbb{R}^{2n})$ if and only if $u \in L^\infty(\mathbb{R}^{2n})$ and $(2^{j\sigma}\|\varphi_j(D)u\|_{L^\infty})_{j\in\mathbb{N}^{2n}}$ is bounded,

where $j\sigma = j_1\sigma_1 + \dots + j_{2n}\sigma_{2n}$. One can define similarly $B^{s,s'}(\mathbb{R}^n\times\mathbb{R}^n)$, $s>0$, $s'>0$, if one wants to take derivatives only in the directions of the subspaces $\mathbb{R}^n\times\{0\}$ or $\{0\}\times\mathbb{R}^n$, by using a dyadic partition of unity in $\mathbb{R}^n\times\mathbb{R}^n$ which is a tensor product of standard dyadic partitions of unity in $\mathbb{R}^n$.

These spaces have natural normed structures and, in fact, are Banach algebras of bounded continuous functions. Note also the following inclusions, for $s > 0$, which are similar to those with the uniformly local Sobolev spaces,

$$B^s(\mathbb{R}^{2n}) \subset B^{\frac{s}{2},\frac{s}{2}}(\mathbb{R}^n\times\mathbb{R}^n) \subset B^{(\frac{s}{2n},\dots,\frac{s}{2n})}(\mathbb{R}^{2n}), \tag{3.5}$$

and that the associated injections are continuous. The fact that $B^s(\mathbb{R}^{2n})$, for $s > n$, $B^{s,s'}(\mathbb{R}^n \times \mathbb{R}^n)$, for $s, s' > \frac{n}{2}$, and $B^\sigma(\mathbb{R}^{2n})$, for $\sigma_i > \frac{1}{2}$, $1 \leqslant i \leqslant 2n$, satisfy the (H2)$'$ assumption is a consequence of Theorem 3 and the above inclusions. The fact that they satisfy (H1) is trivial and holds with arbitrary exponents.

3.3. $S^0_{1,0}$ type algebras

Another algebra which will be important for us is defined as follows. The idea is that of an $S^0_{1,0}$ type algebra with a limited regularity. To any reasonable Banach space E of functions in $\mathbb{R}^{2n}$, one can associate the space denoted by $S^m_{1,0}E$, $m \in \mathbb{R}$, and defined as the set of functions $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ such that

(*i*) $a(x,\xi)\chi(\xi)$ is in E, for all $\chi \in \mathcal{D}(\mathbb{R}^n)$.

(*ii*) $\{\lambda^{-m} a(x,\lambda\xi)\chi(\xi); \lambda \geqslant 1\}$ is a bounded subset of E, for all $\chi \in \mathcal{D}(\mathbb{R}^n \setminus 0)$.

The reason for such a definition is that when E is formally the space $B^\infty(\mathbb{R}^{2n})$, we obtain in fact the usual Hörmander space $S^m_{1,0}$.

The space $S^m_{1,0}E$ is at least a normed space since it can be equipped with the norm

$$\|a(x,\xi)\varphi(\xi)\|_E + \sup\{2^{-jm}\|a(x,2^j\xi)\varphi_0(\xi)\|_E \ ; \ j \in \mathbb{N}\},$$

where $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and $\varphi_0 \in \mathcal{D}(\mathbb{R}^n \setminus 0)$ are such that they define a dyadic partition of unity in $\mathbb{R}^n$:

$$\varphi(\xi) + \sum_{j \geqslant 0} \varphi_0(2^{-j}\xi) = 1. \tag{3.6}$$

Here, we are essentially interested by the cases where E is one of the Banach algebras defined in the preceding subsection. In these cases, we obtain spaces $S^m_{1,0}E$ which are Banach spaces and, when $m = 0$, even Banach algebras, as one can check easily. The fact that these Banach algebras satisfy the (H1) assumption is not completely trivial and we state it as

Proposition 3.2. — *There exist constants $C > 0$ and $M > 0$ such that, for any $(y,\eta) \in \mathbb{R}^{2n}$ and any $a \in S^0_{1,0}E$, $\tau_{(y,\eta)}a$ is in $\in S^0_{1,0}E$ and*

$$\|\tau_{(y,\eta)}a\|_{S^0_{1,0}E} \leqslant C \langle\eta\rangle^M \|a\|_{S^0_{1,0}E}.$$

Here, E stands for one of the three algebras of the preceding subsection : $B^s(\mathbb{R}^{2n})$, $B^{s,s'}(\mathbb{R}^n \times \mathbb{R}^n)$ or $B^\sigma(\mathbb{R}^{2n})$, with $s, s' > 0$, $\sigma \in (\mathbb{R}^*_+)^{2n}$.

Proof. — We shall treat the case of $E = B^\sigma(\mathbb{R}^{2n})$, the others being similar.

Let $a \in E$ and $\chi \in \mathcal{D}(\mathbb{R}^n)$. We can write, using the dyadic partition of unity (3.6),

$$\begin{aligned} &a(x-y,\xi-\eta)\chi(\xi) = \\ &= a(x-y,\xi-\eta)\varphi(\xi-\eta)\chi(\xi) + \sum_{j\geqslant 0} a(x-y,\xi-\eta)\varphi_0(2^{-j}(\xi-\eta))\chi(\xi) \\ &= \tau_{(y,\eta)}(a\varphi)(x,\xi)\chi(\xi) + \sum_{j\geqslant 0} \tau_{(y,\eta)}[a_j(x,2^{-j}\xi)]\chi(\xi) \end{aligned}$$

where we have set $a_j(x,\xi) = a(x,2^j\xi)\varphi_0(\xi)$. By definition, $a\varphi \in E$ and (a_j) is a bounded sequence of E. It follows from the translation invariance of E, from the lemma below and from the fact that the sum above has a number of non vanishing terms which is finite and does not depend on (y,η), that $\chi\tau_{(y,\eta)}a \in E$ and $\|\chi\tau_{(y,\eta)}a\|_E \leqslant \mathrm{Cst}\ \|a\|_{S^0_{1,0}E}$. Note that we also used the fact that E is an algebra.

Lemma 3.3. — *Given $\sigma \in (\mathbb{R}^*_+)^d$, there exists a constant $C > 0$ such that, for any $h \in (\mathbb{R}^*_+)^d$ and any $u \in B^\sigma(\mathbb{R}^d)$, the function $x \mapsto u(hx) = u(h_1x_1,...,h_dx_d)$ is in $B^\sigma(\mathbb{R}^d)$ and*

$$\|u(hx)\|_{B^\sigma} \leqslant C\,\widetilde{h}^\sigma \|u\|_{B^\sigma},$$

where $\widetilde{h}^\sigma = \widetilde{h}_1^{\sigma_1}...\widetilde{h}_d^{\sigma_d}$, $\widetilde{h}_i = \max\{1,h_i\}$, $i \in \{1,...,d\}$.

The proof of this lemma is easy and is left to the reader.

Assume now that $\chi \in \mathcal{D}(\mathbb{R}^n \setminus 0)$ and let $\lambda \geqslant 1$. As before, write

(3.7)
$$\begin{aligned} &a(x-y,\lambda\xi-\eta)\chi(\xi) = \\ &= a(x-y,\lambda\xi-\eta)\varphi(\xi-\lambda^{-1}\eta)\chi(\xi) + \sum_{j\geqslant 0} a(x-y,\lambda\xi-\eta)\varphi_0 \\ &\qquad\qquad (2^{-j}(\xi-\lambda^{-1}\eta))\chi(\xi) \\ &= a(x-y,\lambda\xi-\eta)\varphi(\xi-\lambda^{-1}\eta)\chi(\xi) + \sum_{j\geqslant 0} \tau_{(y,\lambda^{-1}\eta)}[a_{\lambda,j}(x,2^{-j}\xi)]\chi(\xi) \end{aligned}$$

where $a_{\lambda,j}(x,\xi) = a(x,2^j\lambda\xi)\varphi_0(\xi)$. The number of non vanishing terms in the last sum is finite and does not depend on (y,η,λ), and, clearly, we can estimate that sum in the E norm as before by $\mathrm{Cst}\ \|a\|_{S^0_{1,0}E}$. It remains to treat the first term in (3.7). On the support of this term, we have $|\xi-\lambda^{-1}\eta| \leqslant 1$ and $\gamma_1 \leqslant |\xi| \leqslant \gamma_2$ with some positive constants γ_1 and γ_2, so that, $\lambda^{-1}|\eta| \leqslant 1+\gamma_2$. Now, if $\lambda^{-1}|\eta|$ is small enough, for example,

$\lambda^{-1}|\eta| \leqslant \gamma_1/2$, then, we also have $|\xi - \lambda^{-1}\eta| \geqslant \gamma_1/2$, so that we can replace φ by some $\widetilde{\varphi} \in \mathcal{D}(\mathbb{R}^n \setminus 0)$ and we can then estimate the term

$$a(x-y, \lambda\xi - \eta)\widetilde{\varphi}(\xi - \lambda^{-1}\eta)\chi(\xi)$$

as before. Finally, we are left with the term $a(x-y, \lambda\xi-\eta)\varphi(\xi-\lambda^{-1}\eta)\chi(\xi)$ under the condition that $\lambda \sim |\eta|$. We can now write it as $(\tau_{(y,\eta)}a_\lambda)(x, \lambda\xi)\,\chi(\xi)$ with $a_\lambda(x,\xi) = a(x,\xi)\varphi(\xi/\lambda)$, and then, apply Lemma 3.3 and the translation invariance of E to obtain

$$\|(\tau_{(y,\eta)}a_\lambda)(x, \lambda\xi)\,\chi(\xi)\|_E \leqslant \mathrm{Cst}\ \lambda^{|\sigma'|}\|a_\lambda\|_E$$

where $|\sigma'| = \sigma_{n+1} + ... + \sigma_{2n}$. The last thing we do is to restrict ourselves to $\lambda = 2^k$, to rewrite a_λ and then to estimate it as follows

$$a_\lambda(x,\xi) = a(x,\xi)\varphi(\xi) + \sum_{j=0}^{k-1} a(x,\xi)\varphi_0(2^{-j}\xi),$$

$$\|a_\lambda\|_E \leqslant \mathrm{Cst}\ \ k\|a\|_{S^0_{1,0}E} \leqslant \mathrm{Cst}\ \ \ln\lambda\,\|a\|_{S^0_{1,0}E} \leqslant \mathrm{Cst}\ \ \ln\langle\eta\rangle\,\|a\|_{S^0_{1,0}E}.$$

Hence,

$$\|(\tau_{(y,\eta)})a_\lambda(x,\lambda\xi)\,\chi(\xi)\|_E \leqslant \mathrm{Cst}\ \langle\eta\rangle^{|\sigma'|}\ln\langle\eta\rangle\,\|a\|_{S^0_{1,0}E}.$$

This finishes the proof of the proposition. □

Concerning the L^2 boundedness of operators associated with symbols in $S^0_{1,0}E$, one can prove the following result :

THEOREM 3.4. — *Let E stands for $B^s(\mathbb{R}^{2n})$ with $s > \frac{n}{2}$, or $B^{s,s'}(\mathbb{R}^n \times \mathbb{R}^n)$ with $s > 0, s' > \frac{n}{2}$, or $B^\sigma(\mathbb{R}^{2n})$ with $\sigma_i > 0$ if $1 \leqslant i \leqslant n$, and $\sigma_i > \frac{1}{2}$ if $n+1 \leqslant i \leqslant 2n$. There exist positive constants C and M ($M > 2n$ works) such that, for any function $a \in S^0_{1,0}E$ and any $t \in \mathbb{R}$, the operator $\mathrm{Op}_t(a)$ is bounded in $L^2(\mathbb{R}^n)$ with an operator norm estimated by $C\,\langle t\rangle^M\|a\|_{S^0_{1,0}E}$.*

Proof. — In view of the inclusions (3.5), it is sufficient to treat the case of $E = B^\sigma(\mathbb{R}^{2n})$.

We follow ideas of [5], [2] and [4]. Unfortunately, the theorem is not a consequence of the results obtained in these papers.

Let $a \in S^0_{1,0}E$. Since the usual regularized functions of a are bounded in $S^0_{1,0}E$ by a constant times $\|a\|_{S^0_{1,0}E}$, we can assume that $a \in \mathcal{S}(\mathbb{R}^{2n})$. Write

$$a(x,\eta) = a(x,\eta)\varphi(\eta) + \sum_{k\geqslant 0} a(x,\eta)\varphi_0(2^{-k}\eta),$$

where $\varphi(\eta) + \sum_{k\geqslant 0}\varphi_0(2^{-k}\eta) = 1$ is a standard dyadic partition of unity in $\mathbb{R}^n$. In order to treat the terms of this decomposition, we need the following

Lemma 3.5. — *Let K be a compact set in $\mathbb{R}^n$. Then, there exists a constant $C_K > 0$ such that, for any $a \in \mathcal{S}(\mathbb{R}^{2n})$ with $\operatorname{supp}(\eta \mapsto a(x,\eta)) \subset K$ and for any $t \in \mathbb{R}$,*

$$\|\mathrm{Op}_t(a)\|_{\mathcal{L}(L^2)} \leqslant C_K \langle t\rangle^{2n} \|a\|_E.$$

To prove the lemma, write the $2n$-dyadic decomposition of a, $a = \sum_{j\in\mathbb{N}^{2n}} a_j$, and apply (3.3) to each a_j. We get

$$\|\mathrm{Op}_t(a_j)\|_{\mathcal{L}(L^2)} \leqslant C \langle t\rangle^{2n} 2^{\frac{|j'|}{2}} \sup_{z\in\mathbb{R}^n} \left(\int \left|\chi_0(x)\,\omega(x)\,a_j(2^{j'}x+z,\eta)\right|^2 dxd\eta\right)^{\frac12},$$

where $j' = (j_{n+1}, ..., j_{2n})$, $2^{j'}x = (2^{j_{n+1}}x_1, \ldots, 2^{j_{2n}}x_n)$, $\omega(x) = \prod_{i=1}^n \langle x_i\rangle^2$ and $\chi_0 \in \mathcal{S}(\mathbb{R}^n)$ with $\int \chi_0(x)dx = 1$. Now, it follows from the fact that $a(x,\eta)$ has a compact support in η that each $a_j(x,\eta)$ is rapidly decreasing in η and that, for all $\alpha \in \mathbb{N}^n$,

$$\|\eta^\alpha a_j(x,\eta)\|_{L^\infty} \leqslant C_\alpha 2^{-j\sigma}\|a\|_E.$$

Hence,

$$\int \left|\chi_0(x)\,\omega(x)\,a_j(2^{j'}x+z,\eta)\right|^2 dxd\eta \leqslant \mathrm{Cst}\ 2^{-2j\sigma}\|a\|_E^2,$$

so that,

$$\|\mathrm{Op}_t(a)\|_{\mathcal{L}(L^2)} \leqslant \mathrm{Cst}\ \langle t\rangle^{2n} \sum_j 2^{\frac{|j'|}{2}-j\sigma}\|a\|_E = C_K \langle t\rangle^{2n}\|a\|_E,$$

which proves the lemma.

Of course, Lemma 3.5 applies to the term $a(x,\eta)\varphi(\eta)$. Now, let us consider the terms $a(x,\eta)\varphi_0(2^{-k}\eta) = a_k(x, 2^{-k}\eta)$, $k \geqslant 0$, where we have set $a_k(x,\eta) = a(x, 2^k\eta)\varphi_0(\eta)$. By definition of $S^0_{1,0}E$, the sequence (a_k) is bounded in E. Write $a_k = b_k + r_k$ where b_k is given by

$$b_k(x,\eta) = 2^{kn}\int \chi(2^k y)\,a_k(x-y,\eta)\,dy\,.$$

with $\chi \in \mathcal{S}(\mathbb{R}^n)$, $\widehat{\chi} = 1$ near 0, and $\operatorname{supp}(\widehat{\chi})$ is small enough (for example, $\widehat{\chi}(\xi) = \widehat{\chi}_1(\xi/\epsilon)$ with $\operatorname{supp}(\widehat{\chi}_1)$ in the unit ball and ϵ small enough). Clearly, (b_k) is also a bounded sequence in E and $\|b_k\|_E \leqslant \|\chi\|_{L^1}\|a_k\|_E = \|\chi_1\|_{L^1}\|a_k\|_E$, since E is translation invariant.

Set $b(x,\eta) = \sum_k b_k(x, 2^{-k}\eta)$ and let us estimate $I = (\mathrm{Op}_t(b)v|u)$ for $u, v \in \mathcal{S}(\mathbb{R}^n)$. We have

$$I = \sum_k \int \mathcal{F}_1(b_k)(\xi, 2^{-k}\eta)\,\widehat{v}(\eta - t\xi)\,\overline{\widehat{u}}(\eta + (1-t)\xi)\,d\xi d\eta.$$

Since $|\eta| \sim 2^k$ and $|\xi| \leqslant \epsilon 2^k$ on the support of integration, with a small enough ϵ, we also have $|\eta - t\xi| \sim |\eta + (1-t)\xi| \sim |\eta| \sim 2^k$. Here, $\alpha \sim \beta$ means that the ratio $\frac{\alpha}{\beta}$ has (positive) upper and lower bounds. Note that ϵ depends on t (in fact, $\epsilon \sim \frac{1}{\langle t \rangle}$). However, one can choose bounds on $2^{-k}|\eta - t\xi|$ and $2^{-k}|\eta + (1-t)\xi|$ that do not. Therefore, we can take a $\psi \in \mathcal{D}(\mathbb{R}^n \setminus 0)$ such that we can write

$$I = \sum_k \Big(\mathrm{Op}_t(b_k(x, 2^{-k}\eta))\psi(2^{-k}D)v \Big| \psi(2^{-k}D)u \Big);$$

so that,

$$\begin{aligned} |I| &\leqslant \sum_k \|\mathrm{Op}_t(b_k(x, 2^{-k}\eta))\|_{\mathcal{L}(L^2)} \|\psi(2^{-k}D)v\|_{L^2} \|\psi(2^{-k}D)u\|_{L^2} \\ &\leqslant \mathrm{Cst}\ \sup_k \|\mathrm{Op}_t(b_k(x, 2^{-k}\eta))\|_{\mathcal{L}(L^2)} \|v\|_{L^2} \|u\|_{L^2} \\ &= \mathrm{Cst}\ \sup_k \|\mathrm{Op}_t(b_k(2^{-k}x, \eta))\|_{\mathcal{L}(L^2)} \|v\|_{L^2} \|u\|_{L^2} \end{aligned}$$

To obtain the last equality, we have applied the following lemma whose proof is easy and left out :

Lemma 3.6. — *For any $a \in \mathcal{S}'(\mathbb{R}^n \times \mathbb{R}^n)$, $t \in \mathbb{R}$ and $\lambda > 0$, $\mathrm{Op}_t(a)$ is bounded in $L^2(\mathbb{R}^n)$ if and only if $\mathrm{Op}_t(a(x/\lambda, \lambda\eta))$ is, and we have*

$$\|\mathrm{Op}_t(a(x/\lambda, \lambda\eta))\|_{\mathcal{L}(L^2)} = \|\mathrm{Op}_t(a)\|_{\mathcal{L}(L^2)}.$$

Now, it remains to apply Lemma 3.5 in conjunction with Lemma 3.3 to obtain

$$|I| \leqslant \mathrm{Cst}\ \langle t \rangle^{2n} \sup_k \|b_k\|_E \|v\|_{L^2} \|u\|_{L^2} \leqslant C\, \langle t \rangle^{2n} \sup_k \|a_k\|_E \|v\|_{L^2} \|u\|_{L^2},$$

so that, since $u, v \in \mathcal{S}(\mathbb{R}^n)$ are arbitrary, $\|\mathrm{Op}_t(b)\|_{\mathcal{L}(L^2)} \leqslant C\, \langle t \rangle^{2n} \sup_k \|a_k\|_E$.

We turn now to the study of $r(x,\eta) = \sum_k r_k(x, 2^{-k}\eta)$. We need here to use the space $F = B^{\sigma'}(\mathbb{R}^{2n})$ with $0 < \sigma'_i < \sigma_i$ for $1 \leqslant i \leqslant n$, and $\sigma'_i = \sigma_i$ for $n+1 \leqslant i \leqslant 2n$. Applying as above Lemma 3.3, Lemma 3.5 and Lemma 3.6, we can write

$$\|\mathrm{Op}_t(r_k(x, 2^{-k}\eta))\|_{\mathcal{L}(L^2)} \leqslant \mathrm{Cst}\ \langle t \rangle^{2n} \|r_k\|_F.$$

It suffices now to show that $\|r_k\|_F \leqslant \delta_k \|r_k\|_E$ with $(\delta_k) \in \ell^1$ to finish the proof of the theorem. Write the $2n$-dyadic decomposition of r_k, $r_k = \sum_{j \in \mathbb{N}^{2n}} r_{k,j}$. On the support of $\mathcal{F}_1(r_{k,j})(\xi, \eta)$, we have

$$|\xi_i| \sim 2^{j_i}, 1 \leqslant i \leqslant n, \text{ and } |\xi| \geqslant \epsilon 2^k.$$

This implies that there exists (a rather large) $k_0 \in \mathbb{N}$ (more precisely, one can check that $2^{k_0} \sim \frac{1}{\epsilon}$) such that

$$r_{k,j} = 0 \text{ if } j_i < k - k_0, \forall i \in \{1, ..., n\}.$$

Therefore, given $j \in \mathbb{N}^{2n}$ such that $r_{k,j} \neq 0$, there exists $i \in \{1, ..., n\}$ such that $j_i \geqslant k - k_0$, which allows us to estimate $\|r_{k,j}\|_{L^\infty}$ as follows :

$$2^{j\sigma'}\|r_{k,j}\|_{L^\infty} = 2^{-j(\sigma-\sigma')}2^{j\sigma}\|r_{k,j}\|_{L^\infty} \leqslant 2^{-j_i(\sigma_i-\sigma'_i)}\|r_k\|_E,$$

so that,

$$\|r_k\|_F \leqslant 2^{-(k-k_0)\tau}\|r_k\|_E \leqslant \text{Cst } \epsilon^{-\tau}2^{-k\tau}\|r_k\|_E \leqslant \text{Cst } \langle t\rangle^\tau 2^{-k\tau}\|r_k\|_E,$$

where $\tau = \min\{\sigma_i - \sigma'_i; 1 \leqslant i \leqslant n\}$. Hence,

$$\|\text{Op}_t(r)\|_{\mathcal{L}(L^2)} \leqslant \text{Cst } \langle t\rangle^{2n+\tau} \sup_k \|r_k\|_E \leqslant \text{Cst } \langle t\rangle^{2n+\tau} \sup_k \|a_k\|_E.$$

This achieves the proof of the theorem. □

The conclusion of this subsection is that, if E is one of the algebras $B^s(\mathbb{R}^{2n})$ with $s > \frac{n}{2}$, $B^{s,s'}(\mathbb{R}^n \times \mathbb{R}^n)$ with $s > 0, s' > \frac{n}{2}$, or $B^\sigma(\mathbb{R}^{2n})$ with $\sigma_i > 0$ if $1 \leqslant i \leqslant n$, and $\sigma_i > \frac{1}{2}$ if $n+1 \leqslant i \leqslant 2n$, then, the algebras $S^0_{1,0}E$ satisfy (H1) and (H2)$'$.

4. Estimates on the needed number of derivatives

In view of the results of the preceding sections, we can state the following theorem which gives bounds on the number of derivatives needed for the Fefferman-Phong inequality to hold.

Theorem 4.1. — *Let a be a non negative function defined on $\mathbb{R}^n \times \mathbb{R}^n$. Then, the Fefferman-Phong inequalities (1.1) and (1.2) hold if a satisfies one of the following conditions, with a constant C that depends linearly on the norm of a in the considered space :*

(*i*) *$\partial_1^\alpha \partial_2^\beta a$ is in $L^\infty(\mathbb{R}^{2n})$ or $L^2_{ul}(\mathbb{R}^{2n})$ for $4 \leqslant |\alpha| + |\beta| \leqslant n + 5$.*

(*ii*) *For $|\alpha| + |\beta| = 4$, $\partial_1^\alpha \partial_2^\beta a$ is in $B^{n+\epsilon}(\mathbb{R}^{2n})$ or $H^{n+\epsilon}_{ul}(\mathbb{R}^{2n})$, $\epsilon > 0$, or in one of the other algebras of subsections 3.1 and 3.2.*

(*iii*) *For $|\alpha| + |\beta| = 4$, $\partial_1^\alpha \partial_2^\beta a$ is in $S^0_{1,0}E$ where E is $B^{\frac{n}{2}+\epsilon}(\mathbb{R}^{2n})$ or $B^{\epsilon,\frac{n}{2}+\epsilon}(\mathbb{R}^n \times \mathbb{R}^n)$ or $B^\sigma(\mathbb{R}^{2n})$ with $\sigma = (\epsilon, ..., \epsilon; \frac{1}{2}+\epsilon, ..., \frac{1}{2}+\epsilon)$, ϵ being an arbitrary positive number (see subsection 3.3).*

(*iv*) *$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leqslant C_{\alpha,\beta}\langle\xi\rangle^{2-|\beta|}$ for $|\alpha| + |\beta| \leqslant [\frac{n}{2}] + 5$.*

(*v*) *$a \in S^2_{1,0}E$ with $E = B^{\frac{n}{2}+4+\epsilon}(\mathbb{R}^{2n})$.*

(*vi*) *For* $|\alpha|+|\beta|\leqslant 4$, $\partial_1^\alpha\partial_2^\beta a$ *is in* $S^{2-|\beta|}_{1,0}E$ *where* E *is* $B^{\epsilon,\frac{n}{2}+\epsilon}(\mathbb{R}^n\times\mathbb{R}^n)$ *or* $B^\sigma(\mathbb{R}^{2n})$ *with* $\sigma=(\epsilon,...,\epsilon;\frac12+\epsilon,...,\frac12+\epsilon)$, ϵ *being an arbitrary positive number (see subsection 3.3).*

Proof. — Parts (i), (ii) and (iii) are consequences of Theorem 2.1 and Theorem 2.5 since all the considered algebras satisfy the assumptions (H1) and (H2)$'$.

Here, we shall prove (v) and (vi), (iv) being a consequence of (v). The proof is an adaptation of that of Corollary 1.3.2 (i) of [8], and we refer to that paper for missing details. Let E stands for either $B^{\frac{n}{2}+4+\epsilon}(\mathbb{R}^{2n})$, $B^{\epsilon,\frac{n}{2}+\epsilon}(\mathbb{R}^n\times\mathbb{R}^n)$ or $B^\sigma(\mathbb{R}^{2n})$ with $\sigma=(\epsilon,...,\epsilon;\frac12+\epsilon,...,\frac12+\epsilon)$.

First, let us treat the case of Weyl quantization. If $\varphi(\xi)+\sum_{k\geqslant 0}\varphi_0(2^{-k}\xi)=1$ is a dyadic partition of unity in $\mathbb{R}^n$, we can write

$$\mathrm{Op}^w(a)=\mathrm{Op}^w(a\varphi)+\sum_{j\geqslant 0}\mathrm{Op}^w[a_j(x,2^{-j}\xi)],$$

where $a_j(x,\xi)=a(x,2^j\xi)\varphi_0(\xi)$. Since $a\varphi$ is in $S^0_{1,0}E$, it follows from Theorem 3.4 that $\mathrm{Op}^w(a\varphi)$ is bounded in $L^2(\mathbb{R}^n)$. So, let us consider $I_j=(\mathrm{Op}^w[a_j(x,2^{-j}\xi)]v|v)$, $j\geqslant 0$, $v\in\mathcal{S}(\mathbb{R}^n)$. By performing a simple change of variables in the integral defining I_j, we can write

$$I_j=(\mathrm{Op}^w(b_j)v_j|v_j),$$

where $b_j(x,\xi)=a_j(2^{-j/2}(x,\xi))$ and $v_j(x)=v(2^{-j/2}x)2^{-jn/4}$. It follows from the assumptions that, for $|\alpha|+|\beta|=4$, the functions $\partial_1^\alpha\partial_2^\beta b_j(x,\xi)=2^{-2j}\partial_1^\alpha\partial_2^\beta a_j(2^{-j/2}(x,\xi))$ are bounded in E. This is not sufficient a priori for our goal. However, since they are supported in the annuli $c_1 2^{j/2}\leqslant|\xi|\leqslant c_2 2^{j/2}$, they are in fact bounded in $S^0_{1,0}E$. Indeed, if $\chi\in\mathcal{D}(\mathbb{R}^n\setminus 0)$ and $\lambda\geqslant 1$, we have $\lambda\sim 2^{j/2}$ on the support of the functions $\partial_1^\alpha\partial_2^\beta b_j(x,\lambda\xi)\chi(\xi)$; so that, applying Lemma 4 yields

$$\begin{aligned}\|\partial_1^\alpha\partial_2^\beta b_j(x,\lambda\xi)\chi(\xi)\|_E &\leqslant \mathrm{Cst}\sup_{j\geqslant 0}2^{-2j}\|\partial_1^\alpha\partial_2^\beta a_j\|_E\\ &\leqslant \mathrm{Cst}\sum_{\beta'\leqslant\beta}\sup_{j\geqslant 0}2^{(|\beta'|-2)j}\|(\partial_1^\alpha\partial_2^{\beta-\beta'}a)(x,2^j\xi)\,\partial^{\beta'}\varphi_0(\xi)\|_E.\end{aligned}$$

It follows from this and from the fact that b_j is non negative (we use a non negative φ_0, of course) that the Fefferman-Phong inequality holds for $\mathrm{Op}^w(b_j)$. However, this is not sufficient to conclude since we have to sum constants and the v_j are only bounded with respect to j. So, consider the operator $B_j=\mathrm{Op}^w(\psi_j)\mathrm{Op}^w(b_j)\mathrm{Op}^w(\psi_j)$ where $\psi_j(\xi)=\psi_0(2^{-j/2}\xi)$, $\psi_0\in\mathcal{D}(\mathbb{R}^n\setminus 0)$ and $\psi_0=1$ in a (large enough) neighborhood of $\mathrm{supp}(\varphi_0)$.

One can write, using the Weyl pseudodifferential calculus,

$$B_j = \mathrm{Op}^w(b_j\psi_j^2) + R_j = \mathrm{Op}^w(b_j) + R_j,$$

with some remainder operator R_j. The reason for introducing the operator B_j is that

$$\sum_j (B_j v_j|v_j) = \sum_j \Big(\mathrm{Op}^w(b_j)\psi_j(D)v_j \Big| \psi_j(D)v_j\Big)$$
$$\geqslant \sum_j -\mathrm{Cst}\ \|\psi_j(D)v_j\|_{L^2}^2 = \sum_j -\mathrm{Cst}\ \|\psi_0(2^{-j}D)v\|_{L^2}^2 \geqslant -\mathrm{Cst}\ \|v\|_{L^2}^2,$$

and it is thus sufficient to prove that R_j is bounded in $L^2(\mathbb{R}^n)$ and that $(\|R_j\|_{\mathcal{L}(L^2)})_j$ is a summable sequence. It follows from the Weyl calculus that $R_j = \mathrm{Op}^w(r_j)$ where r_j is given by the following expression (see [8]) :

$$r_j(x,\xi) = \frac{2^n}{8\pi^2}\sum_{|\alpha|=2}\frac{1}{\alpha!}$$
$$\iiint_0^1 (1-t)\,\mathrm{e}^{-4\pi y\eta}\,\partial_\eta^\alpha[\psi_j(\xi+\eta)\psi_j(\xi-\eta)]\,\partial_1^\alpha b_j(x+ty,\xi)\,dyd\eta dt.$$

Clearly, on the support of $r_j(x,\xi)$, we have $|\xi| \sim 2^{j/2}$. Moreover, since the function $\psi_j(\xi+\eta)\psi_j(\xi-\eta)$ is differentiated, we also have $|\eta| \sim 2^{j/2}$ on the support of integration. Setting $\widetilde{\psi}_0(\xi,\eta) = \psi_0(\xi+\eta)\psi_0(\xi-\eta)$, we see that $r_j(x,\xi)$ is a finite combination of the integrals

$$2^{-2j}\iiint_0^1 (1-t)\,\mathrm{e}^{-4\pi y\eta}\,\partial_2^\alpha\widetilde{\psi}_0[2^{-j/2}(\xi,\eta)]\,\partial_1^\alpha a_j[2^{-j/2}(x+ty,\xi)]\,dyd\eta dt,$$

or, after some integrations by parts whose gains are some negative powers of 2^j, of the integrals

$$r_{j,\alpha}(x,\xi) = 2^{-4j}\iiint_0^1 (1-t)\,\mathrm{e}^{-4\pi y\eta}\,|2^{-j/2}\eta|^{-2}\,\partial_2^\alpha\widetilde{\psi}_0[2^{-j/2}(\xi,\eta)]$$
$$\Delta_1\partial_1^\alpha a_j[2^{-j/2}(x+ty,\xi)]\,dyd\eta dt.$$

Now, the fact that $\|\mathrm{Op}^w(r_{j,\alpha})\|_{\mathcal{L}(L^2)} = \|\mathrm{Op}^w[r_{j,\alpha}(2^{-j/2}x, 2^{j/2}\xi)]\|_{\mathcal{L}(L^2)}$, (see Lemma 3.6), suggests that we consider

$$r_{j,\alpha}(2^{-j/2}x, 2^{j/2}\xi) =$$
$$= 2^{-4j}\iiint_0^1 (1-t)\,\mathrm{e}^{-4\pi y\eta}\,|\eta|^{-2}\partial_2^\alpha\widetilde{\psi}_0(\xi,\eta)\Delta_1\partial_1^\alpha a_j$$
$$(2^{-j}x + 2^{-j}ty,\xi)\,dyd\eta dt$$
$$= 2^{-4j}\iiint_0^1 (1-t)\,\mathrm{e}^{-4\pi y\eta}\,\langle 4\pi y\rangle^{-2n}\langle D_\eta\rangle^{2n}[|\eta|^{-2}\partial_2^\alpha\widetilde{\psi}_0(\xi,\eta)]\Delta_1\partial_1^\alpha a_j$$
$$(2^{-j}x + 2^{-j}ty,\xi)\,dyd\eta dt,$$

where we have performed the change of variables $(y,\eta)\mapsto(2^{-j/2}y,2^{j/2}\eta)$ and, then, integrations by parts. Hence, using the fact that E is an algebra which is translation invariant and Lemma 3.3, we obtain that $r_{j,\alpha}(2^{-j/2}x,2^{j/2}\xi)$ is in E and that

$$\begin{aligned}\|r_{j,\alpha}(2^{-j/2}x,2^{j/2}\xi)\|_E &\leqslant \mathrm{Cst}\ 2^{-4j}\|\Delta_1\partial_1^\alpha a_j(2^{-j}x,\xi)\|_E\\ &\leqslant \mathrm{Cst}\ 2^{-4j}\|\Delta_1\partial_1^\alpha a_j\|_E,\end{aligned}$$

and, consequently,

$$\|r_j(2^{-j/2}x,2^{j/2}\xi)\|_E\leqslant \mathrm{Cst}\ 2^{-4j}\|a_j^{(4)}\|_E\leqslant \mathrm{Cst}\ 2^{-2j}\sup_{k\geqslant 0}2^{-2k}\|a_k^{(4)}\|_E.$$

It remains to note that $r_j(2^{-j/2}x,2^{j/2}\xi)$ has a support in ξ which is contained in fixed compact set and then to apply Lemma 3.5. The result is that

$$\|R_j\|_{\mathcal{L}(L^2)}=\|\mathrm{Op}^w[r_j(2^{-j/2}x,2^{j/2}\xi)]\|_{\mathcal{L}(L^2)}\leqslant \mathrm{Cst}\ 2^{-2j}\sup_{k\geqslant 0}2^{-2k}\|a_k^{(4)}\|_E,$$

and this achieves the proof of the Fefferman-Phong inequality in the Weyl quantization case.

The case of the standard quantization can be seen to be a consequence of Weyl quantization case. In fact, if $A=[a(x,D)+a(x,D)^\star]/2$, it follows from (2.4) that we can write

$$A=\mathrm{Op}^w(a)-R\quad\text{where}\quad R=\frac{\pi^2}{2}\int_0^1(1-t)(\mathrm{Op}_{(1-t)/2}(b)+\mathrm{Op}_{(1+t)/2}(b))dt$$
$$\text{and}\quad b=(D_xD_\xi)^2a.$$

Clearly, $b\in S^0_{1,0}E$ and applying Theorem 3.4 yields the fact that R is a bounded operator in $L^2(\mathbb{R}^n)$ and that its operator norm is estimated by $\mathrm{Cst}\,\|b\|_{S^0_{1,0}E}$. This establishes the Fefferman-Phong inequality in the standard quantization case and, at the same time, completes the proof of Theorem 4.1. □

5. Semi-classical estimates

One can also prove semi-classical Fefferman-Phong inequalities using an abstract setting. However, here, the algebra $\mathcal{A}$ has to satisfy the following additional assumption :

(H 3) $\qquad \exists C_1>0,\forall h\in[0,1],\forall b\in\mathcal{A},\quad \|b(hX)\|_{\mathcal{A}}\leqslant C_1\|b\|_{\mathcal{A}}.$

The following results are consequences of those of section 2.

Corollary 5.1. — *Assume that $\mathcal{A}$ satisfies the assumptions (H1), (H2) (resp. (H2)$'$) and (H3).*

(i) *There exists a positive constant C such that, for any non negative function a on $\mathbb{R}^{2n}$ such that $a^{(4)} \in \mathcal{A}(\mathbb{R}^{2n})$, and for any $h \in [0,1]$, we have*

$$\mathrm{Op}^w[a(x,h\xi)] + C\,h^2\,\|a^{(4)}\|_{\mathcal{A}} \geqslant 0,$$

$$(\text{resp.}\quad \mathrm{Re}(a(x,hD)u|u)_{L^2} + C\,h^2\,\|a^{(4)}\|_{\mathcal{A}}\,\|u\|^2_{L^2} \geqslant 0, \quad u \in \mathcal{S}(\mathbb{R}^n)).$$

(ii) *There exists a positive constant C such that, for any non negative function $a_h(x,\xi)$ on $\mathbb{R}^{2n}$, $h \in]0,1]$, such that the functions $(\partial_1^\alpha \partial_2^\beta a_h)(x,\xi/h)h^{-|\beta|}$ are bounded in $\mathcal{A}$ for $|\alpha|+|\beta|=4$, we have*

$$\forall h \in]0,1], \quad \mathrm{Op}^w(a_h) + C\,M\,h^2 \geqslant 0,$$

$$(\text{resp.}\quad \mathrm{Re}(a_h(x,D)u|u)_{L^2} + C\,M\,h^2\,\|u\|^2_{L^2} \geqslant 0, \quad u \in \mathcal{S}(\mathbb{R}^n)),$$

where $M = \sup\{\|(\partial_1^\alpha \partial_2^\beta a_h)(x,\xi/h)h^{-|\beta|}\|_{\mathcal{A}}\,;\ 0 < h \leqslant 1, |\alpha|+|\beta|=4\}$.

(iii) *There exists a positive constant C such that, for any non negative function $a_h(x,\xi)$ on $\mathbb{R}^{2n}$, $h \in]0,1]$, such that the functions $(\partial_1^\alpha \partial_2^\beta a_h)(xh^{1/2},\xi h^{-1/2})h^{-|\beta|}$ are bounded in $\mathcal{A}$ for $|\alpha|+|\beta|=4$, we have*

$$\forall h \in]0,1], \quad \mathrm{Op}^w(a_h) + C\,M\,h^2 \geqslant 0,$$

$$(\text{resp.}\quad \mathrm{Re}(a_h(x,D)u|u)_{L^2} + C\,M\,h^2\,\|u\|^2_{L^2} \geqslant 0, \quad u \in \mathcal{S}(\mathbb{R}^n)),$$

where $M = \sup\{\|(\partial_1^\alpha \partial_2^\beta a_h)(xh^{1/2},\xi h^{-1/2})h^{-|\beta|}\|_{\mathcal{A}}\,;\ 0 < h \leqslant 1, |\alpha|+|\beta|=4\}$.

Proof. — One can check easily, using the (H3) assumption, that (iii) implies (ii) which implies (i). The proof of (iii) is formally the same as that of [8]. In fact, one has just to replace the Wiener-Sjöstrand algebra $\mathcal{A}_0$ by the algebra $\mathcal{A}$ and, of course, to apply Theorem 2.1 (resp. Theorem 2.5). So, we refer to [8]. □

Taking, for example, $\mathcal{A} = B^{n+1}(\mathbb{R}^{2n})$, we obtain the following

Corollary 5.2. — *There exists a positive constant C such that, for any non negative function a_h on $\mathbb{R}^{2n}$, $h \in]0,1]$, such that*

$$|\partial_1^\alpha \partial_2^\beta a_h(x,\xi)| \leqslant C_{\alpha,\beta}\,h^{|\beta|} \quad \text{for} \quad h \in]0,1] \quad \text{and} \quad 4 \leqslant |\alpha|+|\beta| \leqslant n+5,$$

we have

$$\forall h \in]0,1], \quad \mathrm{Op}^w(a_h) + C\,M\,h^2 \geqslant 0,$$

$$(\text{resp.}\quad \mathrm{Re}(a_h(x,D)u|u)_{L^2} + C\,M\,h^2\,\|u\|^2_{L^2} \geqslant 0, \quad u \in \mathcal{S}(\mathbb{R}^n)),$$

where $M = \sup\{C_{\alpha,\beta};\ 4 \leqslant |\alpha|+|\beta| \leqslant n+5\}$.

Proof. — It is easily seen that a_h satisfies the conditions of Corollary 1(*iii*). Moreover, the algebra $B^{n+1}(\mathbb{R}^{2n})$ satisfies the assumptions (H1), (H2)′ and (H3). □

6. Remarks and further results

1. A natural question that can be raised is whether these upper bounds on the number of derivatives needed for the Fefferman-Phong inequality to hold are optimal. In fact, this is not quite clear for us. However, we can at least say that the bounds $n+\epsilon$ and $\frac{n}{2}+\epsilon$ concerning the number of derivatives needed for the L^2 boundedness of the involved pseudodifferential operators are optimal. See [5], [2], [4]. Furthermore, the "4" number of derivatives is reputed "to be optimal", and it would be a great achievement if one can reduce it. Roughly, one can say that the bounds are optimal with respect to the method of proof.

2. By using the same ideas, one can estimate the number of derivatives needed for the sharp Gårding inequality to hold. The usual argument for proving this inequality is simpler of course than that needed to establish the Fefferman-Phong inequality, and even works for systems, that is, for matrices of symbols and operators. For example, if a is some function on the phase space such that $a'' \in L^\infty$, one can write $\mathrm{Op}^{wick}(a) = \mathrm{Op}^w(a) + \mathrm{Op}^w(r)$ where

$$r(X) = 2^n \int_{\mathbb{R}^{2n}} \int_0^1 (1-t)a''(X+tY)Y^2 \mathrm{e}^{-2\pi Y^2} dtdY.$$

See [8]. Clearly, if a is non negative, $a'' \in \mathcal{A}$ and $\mathcal{A}$ is a subalgebra of L^∞ satisfying the assumptions (H1) and (H2), it follows from the positivity of the Wick quantization that $\mathrm{Op}^w(a) + C\|a^{(4)}\|_{\mathcal{A}} \geqslant 0$. Of course, the same is true for the standard quantization if $\mathcal{A}$ satisfies (H1) and (H2)′. Here, one can even show that (H2)′ is not necessary and that it is enough to assume instead that $a \mapsto a(x,D)$ is bounded from $\mathcal{A}$ to $\mathcal{L}(L^2(\mathbb{R}^n))$. Taking back the arguments developped in the preceding sections, one can prove, for example, that for a non negative symbol a on $\mathbb{R}^{2n}$, we have

$$\mathrm{Op}^w(a) + C \geqslant 0,$$

and $$\mathrm{Re}(a(x,D)u|u)_{L^2} + C\|u||_{L^2} \geqslant 0, \quad u \in \mathcal{S}(\mathbb{R}^n)),$$

if $\partial_1^\alpha \partial_2^\beta a''$ *are bounded or locally uniformly in* L^2 *for* $|\alpha|+|\beta| \leqslant n+1$ *or* $|\alpha|,|\beta| \leqslant \left[\frac{n}{2}\right]+1$,

or, $\langle\xi\rangle^{|\beta|-1}\partial_1^\alpha\partial_2^\beta a(x,\xi)$ *are bounded for* $|\alpha|+|\beta| \leqslant \left[\frac{n}{2}\right]+3.$

3. The Fefferman-Phong inequalities (that is, Theorem 2.1 and Theorem 2.5) also hold when $\mathcal{A} = S^0_{\varrho,\varrho}E$ where $0 < \varrho < 1$ and E is one of the spaces $B^s(\mathbb{R}^{2n})$ with $s > n$, $B^{s,s'}(\mathbb{R}^n \times \mathbb{R}^n)$ with $s > \frac{n}{2}, s' > \frac{n}{2}$, or $B^\sigma(\mathbb{R}^{2n})$ with $\sigma_i > \frac{1}{2}$, $1 \leqslant i \leqslant 2n$, (or even the spaces obtained when B is replaced by H_{ul}). Here, the space $S^m_{\varrho,\varrho}E$, $m \in \mathbb{R}$, is defined as the set of functions $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ such that

(*i*) $a(x,\xi)\chi(\xi)$ is in E, for all $\chi \in \mathcal{D}(\mathbb{R}^n)$.

(*ii*) $\{\lambda^{-m} a(\lambda^{-\varrho}x, \lambda^{\varrho}\xi)\, \chi(\lambda^{\varrho-1}\xi); \lambda \geqslant 1\}$ is a bounded subset of E, for all $\chi \in \mathcal{D}(\mathbb{R}^n \setminus 0)$.

In fact, one can apply the same argument as that used above to check that the algebra $\mathcal{A} = S^0_{\varrho,\varrho}E$ satisfies the assumption (H1). The fact that $\mathcal{A}$ satisfies (H2) is already proven in [4]. The same property is proven in [2] in the case of the standard quantization. Now, the case of the t-quantization can be handled similarly by the same methods. One obtains, for example, that the Fefferman-Phong inequalities (1.1) and (1.2) hold for the non negative function a on the phase space if it satisfies the estimates

$$|\partial_x^\alpha \partial_\xi^\beta a^{(4)}(x,\xi)| \leqslant C_{\alpha,\beta} \langle \xi \rangle^{\varrho(|\alpha|-|\beta|)} \quad \text{for} \quad |\alpha|, |\beta| \leqslant \left[\frac{n}{2}\right] + 1.$$

Such inequalities with symbols in the classes $S^0_{\varrho,\varrho}E$ are to be compared with similar ones obtained by J.-M. Bony, [1], and D. Tataru, [11], under more or less different assumptions. However, those authors do not consider the limited regularity of the symbols as we do.

4. In the case of symbols of type (1,0), D. Tataru proved in [11] the sharp Gårding and the Fefferman-Phong inequalities with a limited regularity in the x variables by means of the FBI transform. He used 2 derivatives for the first one and 4 for the other one but did not limit the regularity with respect to the frequency variables. These are to be compared with Theorem 5 (*vi*) above which says that the Fefferman-Phong inequality holds if one uses 4 derivatives in (x,ξ) plus "ϵ" derivative in x and $\frac{n}{2} + \epsilon$ derivatives in ξ. It is likely that, by doing a paradifferential decomposition of the symbol as did Tataru, one can remove the "ϵ" in the case of the x regularity.

5. Consider the class $S_N(1,g)$ of Hörmander with order weight $m = 1$, a slowly varying and temperate metric g which also satisfies the uncertainty principle $g \leqslant g^\sigma$, and with limited regularity $N \in \mathbb{N}$. Then, it is known that $\mathcal{A} = S_N(1,g)$ is an algebra of bounded functions and, that, if N is large enough, it satisfies the (H2) assumption. See Hörmander's book, [7], chapter 18. Unfortunately, we do not know whether the (H1) (or (H1)$'$) assumption is satisfied by this algebra. We only know that this is so if g satisfies what we call the "naive" temperance property : $g_X \leqslant \mathrm{Cst}\, g_Y \langle X - Y\rangle^M$, $X, Y \in \mathbb{R}^{2n}$, with some M. Such a property is, for example, satisfied by the usual $S^m_{\varrho,\delta}$ classes. Thus, we obtain another case of application of Theorem 2.1 although here we are not able to give some information on the number N.

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