ANNALES

DE

L'INSTITUT FOURIER

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Tome 58, nº 2 (2008), p. 405-428.

<http://aif.cedram.org/item?id=AIF_2008__58_2_405_0>

Ann. Inst. Fourier, Grenoble
58, 2 (2008) 405-428

THE DENJOY-CLARKSON PROPERTY WITH RESPECT TO HAUSDORFF MEASURES FOR THE GRADIENT MAPPING OF FUNCTIONS OF SEVERAL VARIABLES

by Miroslav ZELENÝ (*)

Abstract. — We construct a differentiable function $f : \mathbf{R}^n \to \mathbf{R}$ ($n \geqslant 2$) such that the set $(\nabla f)^{-1}(B(0,1))$ is a nonempty set of Hausdorff dimension 1. This answers a question posed by Z. Buczolich.

Résumé. — On construit une fonction différentiable $f : \mathbf{R}^n \to \mathbf{R}$ ($n \geqslant 2$) telle que l'ensemble $(\nabla f)^{-1}(B(0,1))$ est non vide et sa dimension de Hausdorff est 1. C'est une réponse à une question posée par Z. Buczolich.

1. Introduction

It is well known that if $f : \mathbf{R} \to \mathbf{R}$ is differentiable, then the mapping $f' : \mathbf{R} \to \mathbf{R}$ has the Darboux property and it is a Baire class one function. Such a mapping has also the following interesting property proved by A. Denjoy ([6]) and independently by J. A. Clarkson ([5]): $(f')^{-1}(G)$ is either empty or of positive Lebesgue measure whenever $G \subset \mathbf{R}$ is open, i.e., f' has the *Denjoy-Clarkson property* (see also, e.g., [1]).

In 1990, C. E. Weil posed the problem ([13]) whether the gradient mapping of any Fréchet differentiable function $f : \mathbf{R}^n \to \mathbf{R}$ ($n \geqslant 2$) has the Denjoy-Clarkson property, i.e., whether $(\nabla f)^{-1}(G)$ is either empty

Keywords: Denjoy–Clarkson property, gradient, Hausdorff measure, infinite game.
Math. classification: 26B05, 28A75.
(*) The work is a part of the research project MSM 0021620839 financed by MSMT and partly supported by the grant GAČR 201/06/0198. I thank for the hospitality to Université Bordeaux 1 and Schrödinger institute in Vienna, where some part of this work was done. I also thank J. Malý, E. Matheron, L. Zajíček, and the anonymous referee for their helpful comments and remarks.

or of positive n-dimensional Lebesgue measure whenever $G \subset \mathbf{R}^n$ is open. Z. Buczolich solved the Weil gradient problem in 2002 constructing a differentiable function $f : \mathbf{R}^2 \to \mathbf{R}$ such that $(\nabla f)^{-1}(B(0,1))$ is a nonempty set of Lebesgue 2-dimensional measure zero ([4]). Using ideas from Buczolich's original construction and a two-person game discovered by J. Malý, an alternative proof of Buczolich's result was presented in [11].

Here we continue the work done in [11] being mainly interested in the problem how small the set $(\nabla f)^{-1}(B(0,1))$ can be from the point of view of Hausdorff dimension if it is nonempty. The following theorem gives a lower bound.

Theorem 1.1 (Buczolich [2]). — *Let $f : \mathbf{R}^n \to \mathbf{R}$ be a Fréchet differentiable function and $G \subset \mathbf{R}^n$ be open. Then $(\nabla f)^{-1}(G)$ is either empty or of positive 1-dimensional Hausdorff measure.*

We show that one cannot replace 1-dimensional Hausdorff measure by d-dimensional Hausdorff measure with $d > 1$ in general. Namely, the following result holds true.

Theorem 1.2. — *For every $n \in \mathbf{N}$, $n \geqslant 2$, there exists a Fréchet differentiable function $f : \mathbf{R}^n \to \mathbf{R}$ such that $\nabla f(0) = 0$ and $(\nabla f)^{-1}(B(0,1))$ is a set of Hausdorff dimension 1.*

Actually we prove a little bit more general theorem. To state it we need the following notation. Let $\zeta : [0,\infty) \to [0,\infty)$ be a nondecreasing function and $0 < \beta \leqslant \infty$. For $A \subset \mathbf{R}^n$ we denote

$$\Phi^\zeta_\beta(A) = \inf \Big\{ \sum_{j=1}^\infty \zeta(\operatorname{diam} G_j);\ A \subset \bigcup_{j=1}^\infty G_j,\ G_j \text{ is nonempty, open, and } \operatorname{diam} G_j \leqslant \beta,\ j = 1, 2, \dots \Big\}$$

and $\Phi^\zeta(A) = \lim_{\beta\to 0+} \Phi^\zeta_\beta(A)$. It is well known that Φ^ζ is a Borel measure on $\mathbf{R}^n$.

Theorem 1.3. — *Let $\zeta : [0,\infty) \to [0,\infty)$ be a nondecreasing function with $\lim_{t\to 0+} \zeta(t)/t = 0$. For every $n \in \mathbf{N}$, $n \geqslant 2$, there exists a Fréchet differentiable function $f : \mathbf{R}^n \to \mathbf{R}$ such that $\nabla f(0) = 0$ and $\Phi^\zeta((\nabla f)^{-1}(B(0,1))) = 0$.*

Theorem 1.2 can be inferred as follows. Let $\zeta : [0,\infty) \to [0,\infty)$ be a nondecreasing function such that $\lim_{t\to 0+} \zeta(t)/t = 0$ and $\lim_{t\to 0+} t^\alpha/\zeta(t) = 0$ for every $\alpha > 1$. Applying Theorem 1.3 we get a differentiable function $f : \mathbf{R}^n \to \mathbf{R}$ such that $\nabla f(0) = 0$ and $\Phi^\zeta((\nabla f)^{-1}(B(0,1))) = 0$. It is easy to see that the set $(\nabla f)^{-1}(B(0,1))$ does not have Hausdorff dimension greater than 1 by the definition of ζ and not less than 1 by Theorem 1.1.

Theorem 1.2 answers Buczolich's question posed in [4]. Let us note that some other results related to the Denjoy-Clarkson property of the gradient mapping are presented in [9] and [3]. An interesting result on the Darboux property of the gradient mapping can be found in [10].

The paper is organized as follows. Besides several simple observations Section 2 contains Lemma 2.4 which ensures existence of an auxiliary function having some special properties. Functions of this type will be used as the main building blocks in our construction. Since the proof of this lemma is quite technical and the method of the proof is not used elsewere in the paper, we postpone the proof to the last section. Section 2 is closed by Lemma 2.5 giving sufficient conditions under which a sequence of differentiable functions converges to a differentiable function. In Section 3 we define a two-person game and prove that there is a winning tactic for the second player. The idea of the proof is based on the work of R. Deville and E. Matheron ([7]). This tactic and its properties are essential in the proof of Theorem 1.3, which is presented in Section 4.

2. Auxiliary lemmas

Notation 2.1. — The unit sphere in $\mathbf{R}^n$ centered at 0 is denoted by S_{n-1}. The symbol S_{n-1}^+ stands for the set $\{[x_1,\dots,x_n] \in S_{n-1};\ x_n > 0\}$. The open and closed balls with center x and radius r are denoted by $B(x,r)$ and $\overline{B}(x,r)$ respectively.

SETTING. — *From now on, let n be a natural number greater than 1 and $\zeta : [0,\infty) \to [0,\infty)$ be a fixed nondecreasing function satisfying*

$$\lim_{t\to 0+} \zeta(t)/t = 0.$$

We will write simply Φ and Φ_β instead of Φ^ζ and Φ_β^ζ respectively.

The proofs of the next two lemmas are straightforward and will be omitted.

LEMMA 2.2. — *Each line in $\mathbf{R}^n$ has Φ measure zero.*

LEMMA 2.3. — *Let $\Omega \subset \mathbf{R}^n$ be a nonempty open set.*

(i) *If f and g are lower semicontinuous functions on Ω, then $\min\{f,g\}$ is lower semicontinuous on Ω.*

(ii) *Let* $F : \Omega \times \Omega \to \mathbf{R}$ *be a continuous function and* $c \in \mathbf{R}$. *Then the function*

$$x \mapsto \sup\{r > 0;\ B(x,r) \subset \Omega \text{ and } \forall y \in B(x,r) :\ F(x,y) < c\}$$

is a lower semicontinuous function from Ω *to* $\mathbf{R} \cup \{+\infty\}$.

The following lemma is crucial in our construction. Its proof is quite technical and was postponed to the last section.

LEMMA 2.4. — *Let* $\Omega \subsetneq \mathbf{R}^n$ *be an open set,* $v : \Omega \to S^+_{n-1}$ *be a continuous mapping,* $\eta : \Omega \to (0, +\infty)$ *be a lower semicontinuous function. Let* γ, ε, *and* δ *be positive constants with* $\gamma \leqslant 1$.

Then there exists a differentiable function $h : \mathbf{R}^n \to \mathbf{R}$ *such that*

(a) h *is a* $\mathcal{C}^1$*-function on* Ω,
(b) $\Phi_\infty(\{x \in \Omega;\ ||\nabla h(x)|| \leqslant \Theta\gamma\}) < \varepsilon$, *where* $\Theta \in (0,1)$ *is a constant depending only on* n,
(c) $h(x) = 0$ *and* $\nabla h(x) = 0$ *for* $x \in \mathbf{R}^n \setminus \Omega$,

and for every $x \in \Omega$ *we have*

(d) $|\langle \nabla h(x), v(x)\rangle| < \delta$,
(e) $0 \leqslant h(x) \leqslant \eta(x)$,
(f) $||\nabla h(x)|| \leqslant \gamma$.

LEMMA 2.5. — *Let* $(f_k)_{k=0}^\infty$ *be a sequence of differentiable functions on* $\mathbf{R}^n$ *and let* $(\delta_k)_{k=0}^\infty$ *be a sequence of positive functions on* $\mathbf{R}^n$ *such that*

(a) (δ_k) *is a decreasing sequence converging pointwisely to* 0,
(b) $|f_k(y) - f_{k-1}(y)| \leqslant 2^{-k} \max\{\delta_k(x), ||x-y||\}$ *whenever* $x, y \in \mathbf{R}^n$,
(c) $|f_{k-1}(y) - f_{k-1}(x) - \nabla f_{k-1}(x) \cdot (y-x)| \leqslant 2^{-k}||y-x||$ *whenever* $x, y \in \mathbf{R}^n$, $||y-x|| \leqslant \delta_k(x)$,
(d) $(\nabla(f_k - f_{k-1}))$ *converges uniformly to* 0,
(e) (∇f_k) *is pointwise convergent.*

Then (f_k) *converges to a differentiable function* f *and*

$$\nabla f(x) = \lim \nabla f_k(x), \qquad x \in \mathbf{R}^n.$$

Proof. — Using conditions (a) and (b) we immediately get that (f_k) converges to a function f on $\mathbf{R}^n$. For $k \in \mathbf{N}$, $x, y \in \mathbf{R}^n$, the same conditions imply the following estimates:

$$
\begin{aligned}
|f(y) - f_k(y)| &\leqslant \sum_{j=k}^{\infty} |f_{j+1}(y) - f_j(y)| \\
&\leqslant \sum_{j=k}^{\infty} 2^{-(j+1)} \max\{\delta_{j+1}(x), ||x-y||\} \qquad (2.1)\\
&\leqslant 2^{-k} \max\{\delta_{k+1}(x), ||x-y||\}.
\end{aligned}
$$

Set $\psi_k = \sup_{\mathbf{R}^n} ||\nabla(f_{k+1} - f_k)||$, $k \in \mathbf{N}$, and $\kappa(x) = \lim \nabla f_k(x)$ for $x \in \mathbf{R}^n$. Fix $x \in \mathbf{R}^n$ and $\varepsilon > 0$. Then we find $k \in \mathbf{N}$ such that $2^{-k} < \varepsilon$ and for all $j \geqslant k$ we have $\psi_j < \varepsilon$ and $||\nabla f_j(x) - \kappa(x)|| < \varepsilon$. Take $x' \in B(x, \delta_{k+1}(x))$, $x' \neq x$. Then we find $j > k$ such that $\delta_{j+1}(x) < ||x' - x|| \leqslant \delta_j(x)$. We have

$$
\begin{aligned}
|f(x') - f(x) - \kappa(x)\cdot(x'-x)| &\leqslant |f(x') - f_j(x')| + |f(x) - f_j(x)| \\
&+ |(f_j(x') - f_{j-1}(x')) - (f_j(x) - f_{j-1}(x))| \\
&+ |f_{j-1}(x') - f_{j-1}(x) - \nabla f_{j-1}(x)\cdot(x'-x)| \\
&+ ||\nabla f_{j-1}(x) - \kappa(x)|| \cdot ||x' - x||.
\end{aligned}
$$

Using the inequality (2.1) we see that the sum of the first two terms is less than $2^{-j+1}||x' - x||$. The fourth term is less than $2^{-j}||x' - x||$ by (c) and the fifth term is clearly less than $\varepsilon||x' - x||$. Using the Mean value theorem we estimate

$$
\begin{aligned}
&|(f_j(x') - f_{j-1}(x')) - (f_j(x) - f_{j-1}(x))| \\
&\leqslant \sup\{||\nabla(f_j - f_{j-1})(z)||;\ z \in \mathbf{R}^n\} \cdot ||x' - x|| \\
&\leqslant \psi_{j-1}||x' - x|| \leqslant \varepsilon||x' - x||.
\end{aligned}
$$

Thus we can conclude $|f(x') - f(x) - \kappa(x)\cdot(x'-x)| \leqslant 5\varepsilon||x' - x||$, which proves that f is differentiable at x with $\nabla f(x) = \kappa(x)$. □

3. The point-hyperplane game PH and how to win it

Let $B = B(0, R)$ be an open ball in $\mathbf{R}^n$. The *point-hyperplane game* PH is a sequence of rounds. The first and the second player play points $\boldsymbol{a}_k$'s in B and hyperplanes p_k's, respectively, obeying the following rules. In the first round, the first player plays a point $\boldsymbol{a}_1 \in B$ and then the second player plays a hyperplane p_1 with $\boldsymbol{a}_1 \in p_1$. In the k-th round, the first player plays a point $\boldsymbol{a}_k \in B \cap p_{k-1}$ and then the second player plays a hyperplane p_k containing $\boldsymbol{a}_k$. The second player wins if the sequence $(\boldsymbol{a}_k)$ converges, otherwise the first player wins.

Because of technical reasons, we suppose the second player plays a unit vector $\boldsymbol{v}_k$ in its k-th move and the corresponding hyperplane p_k is orthogonal to $\boldsymbol{v}_k$ and contains $\boldsymbol{a}_k$.

This game was introduced in the plane by J. Malý and this special case of the above game was used in [11] to give an alternative proof of Buczolich's result on the Weil gradient problem. Variations of Malý's game were further investigated by R. Deville and E. Matheron ([7]). Besides other results (even in the context of Banach spaces), they proved the second player has a winning tactic, i.e., there is a mapping t from $B(0,R)$ to S_{n-1} such that the second player wins the game if he chooses in its k-th move $t(\boldsymbol{a}_k)$, where $\boldsymbol{a}_k$ is the k-th move of the first player. In the next proposition we prove that there is even a winning tactic which is continuous. Although our approach is formally different from that of [7] and [11], the proof is based on ideas which are contained in these papers. Let us note that for our construction the difference between tactic and strategy is not essential but tactic makes the construction simpler.

Proposition 3.1. — *There exists a continuous mapping t from the ball $B(0,R) \subset \mathbf{R}^n$ to S^+_{n-1} such that*

(i) *the second player wins* PH *playing $\boldsymbol{v}_k = t(\boldsymbol{a}_k)$ in its k-th move, where $\boldsymbol{a}_k$ is the k-th move of the first player,*
(ii) *for every $\varepsilon > 0$ there exists $m \in \mathbf{N}$ such that there is no sequence $(\boldsymbol{a}_j)_{j=1}^m$ with $(\boldsymbol{a}_{j+1} - \boldsymbol{a}_j) \perp t(\boldsymbol{a}_j)$, $\boldsymbol{a}_j \in B(0,R)$, and $||\boldsymbol{a}_{j+1} - \boldsymbol{a}_j|| \geqslant \varepsilon$.*

To prove Proposition 3.1 we need the following notation and the next lemma.

Notation 3.2. —

(i) Let f be a k times differentiable function from $\mathbf{R}^m$ to $\mathbf{R}$. Its k-th derivative at a is denoted by $D^k f(a)$ and its norm by $||D^k f(a)||$. For $k = 0$ we set $D^0 f = f$ and $||D^0 f(a)||$ means $|f(a)|$.
(ii) The set of all k-times continuously differentiable functions on an open set G is denoted by $\mathcal{C}^k(G)$. The symbol $\mathcal{C}^1_b(G)$ stands for the set of continuously differentiable functions on G which are bounded on G as well as their derivatives. The space $C^1_b(G)$ is equipped with the norm

$$||f|| = \sup\{|f(x)|;\ x \in G\} + \sup\{||D^1 f(x)||;\ x \in G\}.$$

(iii) The support of a function f is denoted by $\operatorname{spt} f$.

LEMMA 3.3. — *Let* $\alpha > 0$. *Then there exist* $\mathcal{C}^2$*-functions* $\varphi_1, \dots, \varphi_m$ *from* $\mathbf{R}^{n-1}$ *to* $[0,1]$ *such that*

(a) $\operatorname{diam} \operatorname{spt} \varphi_i < \alpha$, $i \in \{1, \dots, m\}$,
(b) $\sum_{i=1}^m \varphi_i(y) = 1$ *for* $y \in B(0,2) \subset \mathbf{R}^{n-1}$,
(c) $||D^p \varphi_i(y)|| < \alpha$ *for every* $y \in \mathbf{R}^{n-1}$, $p \in \{0,1,2\}$, $i \in \{1, \dots, m\}$,
(d) $||D^p \left(\sum_{j \leqslant q} \varphi_j\right)(y)|| < \alpha$ *for every* $y \in B(0,2)$, $p \in \{1,2\}$, $q \in \{1, \dots, m\}$.

Proof. — Find open balls $C_1, \dots, C_k$ covering $\overline{B}(0,2) \subset \mathbf{R}^{n-1}$ such that $\operatorname{diam} C_i < \alpha$, $i = 1, \dots, k$. It is well known that one can find $\mathcal{C}^\infty$-functions $\psi_1, \dots, \psi_k$ from $\mathbf{R}^{n-1}$ to $[0,1]$ such that $\operatorname{spt} \psi_i \subset C_i$ and $\sum_{i \leqslant k} \psi_i = 1$ on $\overline{B}(0,2)$. Find $s \in \mathbf{N}$ such that

$$\sup\{||D^p \psi_i(y)||;\ i \in \{1, \dots, k\},\ p \in \{0,1,2\},\ y \in \mathbf{R}^{n-1}\} < \alpha s, \tag{3.1}$$

and

$$\sup\{||D^p \left(\sum\nolimits_{j \leqslant i} \psi_j\right)(y)||;\ i \in \{1, \dots, k\},\ p \in \{1,2\},\ y \in \mathbf{R}^{n-1}\} < \alpha s. \tag{3.2}$$

Then we set $m = sk$ and

$$\varphi_{lk+i} = \frac{1}{s}\psi_i,\ l \in \{0, \dots, s-1\},\ i \in \{1, \dots, k\}.$$

Conditions (a) and (b) are obviously satisfied and (c) follows from (3.1). To verify (d) take $q \in \{1, \dots, m\}$. There are $l \in \{0, \dots, s-1\}$ and $i \in \{1, \dots, k\}$ with $lk + i = q$. Then for $y \in B(0,2)$

$$\sum_{j=1}^q \varphi_j = \sum_{j=1}^{lk} \varphi_j + \sum_{j=lk+1}^q \varphi_j = \frac{l}{s} + \sum_{j=1}^i \frac{1}{s}\psi_j$$

and the desired inequality follows immediately from (3.2). □

Proof of Proposition 3.1. — Without any loss of generality we may assume that $R = 1$. Let $\varphi_1^k, \dots, \varphi_{m_k}^k$ be functions from Lemma 3.3 for $\alpha = \frac{1}{n} 8^{-k-1}$. Set

$$X_k = \prod_{i=1}^k \{1, \dots, m_i + 1\}, \qquad X = \prod_{i=1}^\infty \{1, \dots, m_i + 1\};$$

$$\sigma_j^k = \sum_{i<j} \varphi_i^k, \qquad k \in \mathbf{N},\ j \in \{1, \dots, m_k + 1\};$$

$$\tau_s = \sigma_{s_k}^k \varphi_{s_1}^1 \cdots \varphi_{s_{k-1}}^{k-1}, \qquad s = (s_1, \dots, s_k) \in X_k.$$

If $s \in \bigcup_{k=1}^{\infty} X_k$, then $|s|$ stands for the length of s. If $\nu \in X$, then ν_k denotes the k-th coordinate of ν and $\nu|k = (\nu_1, \dots, \nu_k)$. We equip X with the metric ρ defined for $\mu, \nu \in X$, $\mu \neq \nu$, by $\rho(\nu, \mu) = 2^{-k+1}$, where k is the smallest natural number with $\nu_k \neq \mu_k$. Let $<_{\text{lex}}$ be the (strict) lexicographical ordering of X, i.e., $\nu <_{\text{lex}} \mu$ if and only if there is $k \in \mathbf{N}$ with $\nu|(k-1) = \mu|(k-1)$ and $\nu_k < \mu_k$.

The open unit ball in $\mathbf{R}^m$ centered at 0 is denoted by B_m.

To define the desired tactic t we construct functions F_ν, $\nu \in X$, defined on $B(0,2) \subset \mathbf{R}^{n-1}$ satisfying the following properties:

(i) $F_\nu \in \mathcal{C}^2(B(0,2)) \cap \mathcal{C}^1_b(B(0,2))$;
(ii) $D^2 F_\nu(y)(h,h) \leqslant -||h||^2$ for every $y \in B(0,2)$, $h \in \mathbf{R}^{n-1}$, $\nu \in X$;
(iii) the mapping $\nu \mapsto F_\nu$ from X to $\mathcal{C}^1_b(B(0,2))$ is continuous;
(iv) $0 \leqslant F_\mu - F_\nu \leqslant 4\rho(\nu,\mu)$ whenever $\nu <_{\text{lex}} \mu$;
(v) $\operatorname{diam} \operatorname{spt}(F_\mu - F_\nu) \leqslant 4\rho(\nu,\mu)$ whenever $F_\nu \neq F_\mu$;
(vi) for every $y \in B_{n-1}$ and $z \in [-1,1]$ there exists $\nu \in X$ such that $F_\nu(y) = z$.

It is easy to establish the following estimates for $s \in X_k$ and $y \in B(0,2)$:

$$|\tau_s(y)| \leqslant \begin{cases} 1, & k=1, \\ |\varphi^{k-1}_{s_{k-1}}(y)| \leqslant \frac{1}{n} 8^{-(k-1)-1} \leqslant 8^{-k}, & k>1; \end{cases}$$
$$|\partial_i \tau_s(y)| \leqslant k \cdot \frac{1}{n} 8^{-(k-1)-1} \leqslant \frac{1}{n} 2^{-k-2}, \quad i \in \{1, \dots, n-1\};$$
$$|\partial^2_{ij} \tau_s(y)| \leqslant k^2 \cdot \frac{1}{n} 8^{-(k-1)-1} \leqslant \frac{1}{n} 2^{-k-2}, \quad i,j \in \{1, \dots, n-1\}.$$

Then we obtain the following estimates for $s \in \bigcup_{k=1}^{\infty} X_k$ and $y \in B(0,2)$:

$$|\tau_s(y)| \leqslant 8^{-|s|+1}, \tag{3.3}$$
$$||D^1 \tau_s(y)|| \leqslant 2^{-|s|-2}, \tag{3.4}$$
$$||D^2 \tau_s(y)|| \leqslant 2^{-|s|-2}. \tag{3.5}$$

For $\nu \in X$ we define

$$F_\nu(y) = -||y||^2 - 1 + 3 \sum_{k=1}^{\infty} \tau_{\nu|k}(y), \qquad y \in B(0,2).$$

We verify properties (i)–(vi).

(i) By (3.3)–(3.5) we get $F_\nu \in \mathcal{C}^2(B(0,2)) \cap \mathcal{C}^1_b(B(0,2))$.

(ii) By (3.5) we have also

$$
\begin{aligned}
D^2F_\nu(y)(h,h) &\leqslant -2||h||^2 + 3\sum_{k=1}^{\infty} ||D^2\tau_{\nu|k}(y)||\cdot||h||^2 \\
&\leqslant -2||h||^2 + \sum_{k=1}^{\infty} 2^{-k}||h||^2 \\
&= -||h||^2, \qquad y\in B(0,2),\ h\in \mathbf{R}^{n-1},\ \nu\in X.
\end{aligned} \tag{3.6}
$$

(iii) For $\nu,\mu\in X$ with $\nu|k_0=\mu|k_0$, $k_0\in\mathbf{N}\cup\{0\}$, we have

$$
||D^pF_\nu(y)-D^pF_\mu(y)|| \leqslant \sum_{k=k_0+1}^{\infty} 3\cdot||D^p\tau_{\nu|k}(y)|| + \sum_{k=k_0+1}^{\infty} 3\cdot||D^p\tau_{\mu|k}(y)|| \leqslant 12\cdot 2^{-k_0}, \qquad y\in B(0,2),\ p=0,1.
$$

This shows that $\nu\mapsto F_\nu$ is a continuous mapping from X to $\mathcal{C}^1_b(B(0,2))$.

Now observe that for $\nu\in X$, $k\in\mathbf{N}\cup\{0\}$, $N\in\mathbf{N}$, $k<N$, we have

$$
\begin{aligned}
&\sum_{j=k+1}^{N}\tau_{\nu|j} = \varphi^1_{\nu_1}\varphi^2_{\nu_2}\cdots\varphi^k_{\nu_k}\Big(\sigma^{k+1}_{\nu_{k+1}} + \varphi^{k+1}_{\nu_{k+1}}(\sigma^{k+2}_{\nu_{k+2}} + \cdots \\
&+\varphi^{N-2}_{\nu_{N-2}}(\sigma^{N-1}_{\nu_{N-1}} + \varphi^{N-1}_{\nu_{N-1}}\sigma^N_{\nu_N})\dots\Big) \\
&\leqslant \varphi^1_{\nu_1}\varphi^2_{\nu_2}\cdots\varphi^k_{\nu_k}\Big(\sigma^{k+1}_{\nu_{k+1}} + \varphi^{k+1}_{\nu_{k+1}}(\sigma^{k+2}_{\nu_{k+2}} + \cdots + \varphi^{N-2}_{\nu_{N-2}}\sigma^{N-1}_{\nu_{N-1}+1})\dots\Big) \\
&\vdots \\
&\leqslant \varphi^1_{\nu_1}\varphi^2_{\nu_2}\cdots\varphi^k_{\nu_k}\sigma^{k+1}_{\nu_{k+1}+1}
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\varphi^1_{\nu_1}\varphi^2_{\nu_2}\cdots\varphi^k_{\nu_k}\sigma^{k+1}_{\nu_{k+1}} = \tau_{\nu|(k+1)} &\leqslant \sum_{j=k+1}^{\infty}\tau_{\nu|j} \\
&\leqslant \varphi^1_{\nu_1}\varphi^2_{\nu_2}\cdots\varphi^k_{\nu_k}\sigma^{k+1}_{\nu_{k+1}+1}.
\end{aligned} \tag{3.7}
$$

To prove (iv) and (v) suppose that $\mu,\nu\in X$, $k\in\mathbf{N}\cup\{0\}$, $\mu_{k+1}>\nu_{k+1}$, and $\mu|k=\nu|k$.

(iv) By (3.7) we have

$$\begin{aligned}F_\mu - F_\nu &= 3\left(\sum_{j=k+1}^{\infty}\tau_{\mu|j} - \sum_{j=k+1}^{\infty}\tau_{\nu|j}\right)\\ &\geqslant 3\left(\varphi^1_{\mu_1}\varphi^2_{\mu_2}\cdots\varphi^k_{\mu_k}\sigma^{k+1}_{\mu_{k+1}} - \varphi^1_{\nu_1}\varphi^2_{\nu_2}\cdots\varphi^k_{\nu_k}\sigma^{k+1}_{\nu_{k+1}+1}\right)\\ &= 3\varphi^1_{\mu_1}\varphi^2_{\mu_2}\cdots\varphi^k_{\mu_k}(\sigma^{k+1}_{\mu_{k+1}} - \sigma^{k+1}_{\nu_{k+1}+1}) \geqslant 0.\end{aligned}$$

Further, we have

$$\begin{aligned}F_\mu - F_\nu &\leqslant 3\sum_{j=k+1}^{\infty}\tau_{\mu|j} \leqslant 3\sum_{j=k+1}^{\infty}8^{-j+1} = \frac{3}{7}\cdot 8^{-k+1}\\ &< 2^{-k+2} = 4\rho(\mu,\nu).\end{aligned}$$

(v) Here we assume $F_\mu \neq F_\nu$, thus $\operatorname{spt}(F_\mu - F_\nu) \neq \emptyset$. If $k = 0$, then

$$\operatorname{diam}\operatorname{spt}(F_\mu - F_\nu) \leqslant \operatorname{diam} B(0,2) = 4 = 4\rho(\mu,\nu).$$

If $k > 0$, then by (3.7) we have $\operatorname{spt}(F_\mu - F_\nu) \subset \operatorname{spt}\varphi^k_{\mu_k}$. Consequently,

$$\operatorname{diam}\operatorname{spt}(F_\mu - F_\nu) \leqslant \operatorname{diam}\operatorname{spt}\varphi^k_{\mu_k} < \frac{1}{n}8^{-k-1} < 4\rho(\mu,\nu).$$

(vi) For $y \in B_{n-1}$ we have $F_{(1,1,1,\dots)}(y) \leqslant -1$ and $F_{(m_1+1,m_2+1,\dots)}(y) \geqslant 1$. Using this and (iv) we get that $P_y := \{F_\mu(y);\ \mu \in X\} \cap [-1,1]$ is dense in $[-1,1]$. Since the space X is compact and $\mu \mapsto F_\mu$ is continuous, we have $P_y = [-1,1]$. This finishes the construction of the F_ν's and we can define the tactic t.

For $x = [x_1,\dots,x_n] \in \mathbf{R}^n$ we denote $\pi_1(x) = [x_1,\dots,x_{n-1}]$ and $\pi_2(x) = x_n$. Let $x \in B_n$. Then there exists $\mu \in X$ with $F_\mu(\pi_1(x)) = \pi_2(x)$ by (vi). We define $t(x)$ as the vector from S^+_{n-1} orthogonal to the tangent hyperplane to the graph of F_μ at x. The definition does not depend on the choice of μ. Indeed, if $F_\nu(\pi_1(x)) = \pi_2(x)$ for some $\nu \in X$, then $F_\nu \geqslant F_\mu$ or $F_\nu \leqslant F_\mu$ by (iv) and the tangent hyperplanes to the graphs of F_ν and F_μ at x coincide.

To prove continuity of t take a sequence (x^k) with $x^k \in B_n$ converging to $x \in B_n$. For each $k \in \mathbf{N}$ we find $\mu^k \in X$ with $F_{\mu^k}(\pi_1(x^k)) = \pi_2(x^k)$ by (vi). Suppose that $\zeta \in S_{n-1}$ is an accumulation point of the sequence $(t(x^k))$. Then there is a subsequence (u^j) of (x^k) such that $t(u^j) \to \zeta$. For (u^j) there is a corresponding subsequence (α^j) of (μ^k). Going to a subsequence, if necessary, we may assume that (α^j) converges to some $\alpha \in X$. Then we have that $\big(D^1F_{\alpha^j}(\pi_1(u^j))\big)_j$ converges to $D^1F_\alpha(\pi_1(x))$.

This yields $\zeta = t(x)$. Since S_{n-1} is compact we get that the sequence $(t(x^k))$ converges to $t(x)$ as required.

Now we prove that t defines a winning tactic in the game PH for the second player. Let $\boldsymbol{a}_1$, $t(\boldsymbol{a}_1)$, $\boldsymbol{a}_2$, $t(\boldsymbol{a}_2)$, $\ldots$ be a run of the game PH. If $(\boldsymbol{a}_j)$ is eventually constant then the second player wins. So we may assume without any loss of generality that $\boldsymbol{a}_j \neq \boldsymbol{a}_{j+1}$ for all $j \in \mathbf{N}$. Since t is a mapping into S^+_{n-1}, we have even $\pi_1(\boldsymbol{a}_j) \neq \pi_1(\boldsymbol{a}_{j+1})$ for all $j \in \mathbf{N}$.

For every $j \in \mathbf{N}$ find $\nu^j \in X$ with $F_{\nu^j}(\pi_1(\boldsymbol{a}_j)) = \pi_2(\boldsymbol{a}_j)$. Since the F_{ν^j}'s are strictly concave by (ii) and $\pi_1(\boldsymbol{a}_{j+1}) \neq \pi_1(\boldsymbol{a}_j)$, we have

$$\begin{aligned} F_{\nu^{j+1}}(\pi_1(\boldsymbol{a}_{j+1})) &= \pi_2(\boldsymbol{a}_{j+1}) \\ &= F_{\nu^j}(\pi_1(\boldsymbol{a}_j)) + D^1F_{\nu^j}(\pi_1(\boldsymbol{a}_j))(\pi_1(\boldsymbol{a}_{j+1}) - \pi_1(\boldsymbol{a}_j)) \\ &> F_{\nu^j}(\pi_1(\boldsymbol{a}_{j+1})). \end{aligned}$$

Consequently, $\nu^j <_{\mathrm{lex}} \nu^{j+1}$, $F_{\nu^j} \leqslant F_{\nu^{j+1}}$, and $\pi_1(\boldsymbol{a}_{j+1}) \in \operatorname{spt}(F_{\nu^{j+1}} - F_{\nu^j})$. Set $\nu^* = \sup\{\nu^j;\ j \in \mathbf{N}\}$. For all $j \in \mathbf{N}$ we have $\pi_1(\boldsymbol{a}_{j+1}) \in \operatorname{spt}(F_{\nu^*} - F_{\nu^j})$.

By (v) we have $\lim_{j\to\infty} \operatorname{diam} \operatorname{spt}(F_{\nu^*} - F_{\nu^j}) = 0$. Since $(\operatorname{spt}(F_{\nu^*} - F_{\nu^j}))_j$ is a nonincreasing sequence of sets we see that $(\pi_1(\boldsymbol{a}_j))$ converges to some $z \in \overline{B}_{n-1}$. Further, (F_{ν^j}) converges uniformly to F_{ν^*} on $B(0,2)$, hence $F_{\nu^j}(\pi_1(\boldsymbol{a}_j)) \to F_{\nu^*}(z)$. Since $F_{\nu^j}(\pi_1(\boldsymbol{a}_j)) = \pi_2(\boldsymbol{a}_j)$, the sequence $(\pi_2(\boldsymbol{a}_j))$ converges, hence $(\boldsymbol{a}_j)$ is convergent as well.

To prove the second property of t fix $\varepsilon > 0$ and a sequence $(\boldsymbol{a}_j)_{j=1}^p$ inside B_n with $(\boldsymbol{a}_{j+1} - \boldsymbol{a}_j) \perp t(\boldsymbol{a}_j)$ and $||\boldsymbol{a}_{j+1} - \boldsymbol{a}_j|| \geqslant \varepsilon$. Find μ^j with $F_{\mu^j}(\pi_1(\boldsymbol{a}_j)) = \pi_2(\boldsymbol{a}_j)$. Compactness of X and continuity of $\mu \mapsto F_\mu$ give that there exists $c > 0$ such that $||D^1F_\nu(y)|| < c$ for all $\nu \in X$ and $y \in B_{n-1}$. Thus we have

$$||\pi_1(\boldsymbol{a}_{j+1}) - \pi_1(\boldsymbol{a}_j)|| \geqslant (1 + c^2)^{-1/2}\varepsilon.$$

We observe

$$\begin{aligned} F_{\mu^{j+1}}(\pi_1(\boldsymbol{a}_{j+1})) &= \pi_2(\boldsymbol{a}_{j+1}) \\ &= F_{\mu^j}(\pi_1(\boldsymbol{a}_j)) + D^1F_{\mu^j}(\pi_1(\boldsymbol{a}_j))(\pi_1(\boldsymbol{a}_{j+1}) - \pi_1(\boldsymbol{a}_j)). \end{aligned}$$

Using (ii) and Taylor expansion we infer

$$\begin{aligned} F_{\mu^j}(\pi_1(\boldsymbol{a}_{j+1})) &\leqslant F_{\mu^j}(\pi_1(\boldsymbol{a}_j)) + D^1F_{\mu^j}(\pi_1(\boldsymbol{a}_j))(\pi_1(\boldsymbol{a}_{j+1}) - \pi_1(\boldsymbol{a}_j)) \\ &\quad - \frac{1}{2}||\pi_1(\boldsymbol{a}_{j+1}) - \pi_1(\boldsymbol{a}_j)||^2 \\ &\leqslant F_{\mu^j}(\pi_1(\boldsymbol{a}_j)) + D^1F_{\mu^j}(\pi_1(\boldsymbol{a}_j))(\pi_1(\boldsymbol{a}_{j+1}) - \pi_1(\boldsymbol{a}_j)) \\ &\quad - \frac{1}{2}(1 + c^2)^{-1}\varepsilon^2. \end{aligned}$$

This gives

$$F_{\mu^{j+1}}(\pi_1(\boldsymbol{a}_{j+1})) - F_{\mu^j}(\pi_1(\boldsymbol{a}_{j+1})) \geqslant \frac{1}{2}(1+c^2)^{-1}\varepsilon^2. \tag{3.8}$$

Find $q \in \mathbf{N}$ with $2^{-q} < \frac{1}{8}(1+c^2)^{-1}\varepsilon^2$. According to (iv) and (3.8) we have $\rho(\mu^j,\mu^i) > 2^{-q}$ for all $j,i \in \{1,\ldots,p\}$, $j \neq i$. This implies $p \leqslant (m_1+1)\cdot(m_2+1)\cdots(m_q+1)$ and we are done. □

4. Proof of Theorem 1.3

4.1. Auxiliary objects

Let $\Theta \in (0,1)$ be the constant from Lemma 2.4 and $t : B(0,2) \to S^+_{n-1}$ be the mapping from Proposition 3.1 for $R = 2$. Thus for every $q \in \mathbf{N}$ there exists $m_q \in \mathbf{N}$ such that there is no sequence $(\boldsymbol{a}_j)_{j=1}^{m_q}$ such that

- $(\boldsymbol{a}_{j+1} - \boldsymbol{a}_j) \perp t(\boldsymbol{a}_j)$,
- $\boldsymbol{a}_j \in B(0,2) \subset \mathbf{R}^n$,
- $||\boldsymbol{a}_{j+1} - \boldsymbol{a}_j|| \geqslant \Theta/(q+1)$.

We set

$$\gamma_i = \frac{1}{q+1} + 2^{-i}$$

for $1+m_1+\cdots+m_{q-1} \leqslant i \leqslant m_1+\cdots+m_{q-1}+m_q$. Observe that $\lim \gamma_i = 0$ and $\gamma_i \leqslant 1$ for every $i \in \mathbf{N}$.

We will construct a sequence $(f_k)_{k=0}^{\infty}$ of differentiable functions on $\mathbf{R}^n$ and a sequence $(\delta_k)_{k=0}^{\infty}$ of positive functions on $\mathbf{R}^n$ such that (b) and (c) of Lemma 2.5 are satisfied. We set $f_0(x) = 0$ and $\delta_0(x) = 1$ on $\mathbf{R}^n$, $G_0 = \mathbf{R}^n \setminus \{0\}$, and

$$G_k = \{x \in G_{k-1};\ ||\nabla f_k(x)|| < 1\} \quad \text{for } k > 0.$$

Further we will define continuous mappings $\varphi_k : G_k \to B(0,2) \subset \mathbf{R}^n$ starting with $\varphi_0(x) = 0$ on G_0. For $k \geqslant 1$, we require

(p1) $||\nabla(f_k - f_{k-1})(x)|| \leqslant \gamma_k$, $x \in \mathbf{R}^n$,
(p2) $||\nabla(f_k - f_{k-1})(x) - (\varphi_k(x) - \varphi_{k-1}(x))|| < \Theta 2^{-k}$, $x \in G_k$,
(p3) G_k is open and ∇f_k is continuous on G_k,
(p4) δ_k is lower semicontinuous on G_{k-1} and $\delta_k \leqslant \frac{1}{2}\delta_{k-1}$ on $\mathbf{R}^n$,
(p5) $f_k = f_{k-1}$ and $\nabla f_k = \nabla f_{k-1}$ on $\mathbf{R}^n \setminus G_{k-1}$,
(p6) $\Phi_\infty(\{x \in G_{k-1};\ ||\nabla(f_k - f_{k-1})(x)|| \leqslant \Theta\gamma_k\}) < 2^{-k}$,
(p7) $(\varphi_k(x) - \varphi_{k-1}(x)) \perp t(\varphi_{k-1}(x))$, $x \in G_k$.

4.2. Construction of the auxiliary objects

Now suppose that we have constructed functions $f_0, \dots, f_{k-1}$, $\delta_0, \dots, \delta_{k-1}$, and mappings $\varphi_0, \dots, \varphi_{k-1}$ together with the corresponding $G_0, \dots, G_{k-1}$. We set

$$\tau(x) = \sup\{r > 0;\ B(x,r) \subset G_{k-1} \text{ and } \forall y \in B(x,r) : \|\nabla f_{k-1}(y) - \nabla f_{k-1}(x)\| < 2^{-k}\} \quad \text{for } x \in G_{k-1},$$

and

$$\tau(x) = \sup\{r > 0;\ \forall y \in B(x,r) :\ |f_{k-1}(y) - f_{k-1}(x) - \nabla f_{k-1}(x) \cdot (y-x)| \leqslant 2^{-k}\|y - x\|\} \quad \text{for } x \in \mathbf{R}^n \setminus G_{k-1}.$$

Since f_{k-1} is a $\mathcal{C}^1$-function on G_{k-1} and differentiable on $\mathbf{R}^n$, it is easy to see that τ is positive on $\mathbf{R}^n$. According to Lemma 2.3 the function τ is lower semicontinuous on G_{k-1}. We set $\delta_k = \min\{\frac{1}{2}\delta_{k-1}, \tau\}$. Clearly, δ_k is positive on $\mathbf{R}^n$, lower semicontinuous on G_{k-1}, and $\delta_k \leqslant \frac{1}{2}\delta_{k-1}$. Thus (p4) holds. Moreover, condition (c) of Lemma 2.5 is satisfied for $x \in \mathbf{R}^n \setminus G_{k-1}$, $y \in \mathbf{R}^n$ by the definition of τ and, for $x \in G_{k-1}$, $y \in \mathbf{R}^n$, (c) follows by the Mean Value Theorem.

Claim. — *The function $\psi : G_{k-1} \to \mathbf{R}$ defined by*

$$\psi(x) = \inf_{z \in \mathbf{R}^n} \max\{\delta_k(z), \|x - z\|\}$$

is lower semicontinuous and positive on G_{k-1}.

Proof of Claim. — Take $x \in G_{k-1}$ and $\alpha \in \mathbf{R}$ such that $\psi(x) > \alpha$. Find $0 < \varepsilon < \psi(x) - \alpha$. For $y \in B(x, \varepsilon) \cap G_{k-1}$ we have

$$\psi(y) = \inf_{z \in \mathbf{R}^n} \max\{\delta_k(z), \|y - z\|\} \geqslant \inf_{z \in \mathbf{R}^n} \max\{\delta_k(z), \|x - z\| - \varepsilon\} \geqslant \psi(x) - \varepsilon > \alpha,$$

which proves lower semicontinuity of ψ on G_{k-1}.

For each $x \in G_{k-1}$ there exists $\varepsilon > 0$ such that $\delta_k(z) > \varepsilon$ for every $z \in B(x, \varepsilon)$ since δ_k is lower semicontinuous and positive on G_{k-1}. Then we have $\psi(x) \geqslant \varepsilon$. Hence ψ is positive on G_{k-1}. $\square$

We apply Lemma 2.4 to

$$\Omega := G_{k-1},\ v := t \circ \varphi_{k-1},\ \eta := 2^{-k}\psi,$$
$$\gamma := \gamma_k,\ \varepsilon := 2^{-k},\ \text{and } \delta := \Theta 2^{-k}.$$

We get a differentiable function $h : \mathbf{R}^n \to \mathbf{R}$ such that

- h is a $\mathcal{C}^1$-function on G_{k-1},
- $\Phi_\infty(\{x \in G_{k-1};\ ||\nabla h(x)|| \leqslant \Theta\gamma_k\}) < 2^{-k}$,
- $h(x) = 0$ and $\nabla h(x) = 0$ for $x \in \mathbf{R}^n \setminus G_{k-1}$,

and for every $x \in G_{k-1}$ we have

- $|\langle \nabla h(x), t(\varphi_{k-1}(x))\rangle| < \Theta 2^{-k}$,
- $0 \leqslant h(x) \leqslant 2^{-k}\psi(x)$,
- $||\nabla h(x)|| \leqslant \gamma_k$.

We set $f_k = f_{k-1} + h$. Thus also the set G_k is defined and the conditions (p1), (p3), (p5), and (p6) are satisfied.

We verify (b) of Lemma 2.5. For $x \in \mathbf{R}^n$, $y \in G_{k-1}$ we have

$$|f_k(y) - f_{k-1}(y)| = h(y) \leqslant 2^{-k}\psi(y) \leqslant 2^{-k}\max\{\delta_k(x), ||y - x||\}.$$

If $y \in \mathbf{R}^n \setminus G_{k-1}$, then $|f_k(y) - f_{k-1}(y)| = 0$ and the desired inequality is satisfied.

We define $\varphi_k(x)$, $x \in G_k$, as the orthogonal projection of $\varphi_{k-1}(x) + \nabla h(x)$ onto $\varphi_{k-1}(x) + t(\varphi_{k-1}(x))^\perp$ and we get (p7). Since ∇h and φ_{k-1} are continuous on G_{k-1}, hence also on G_k, the mapping φ_k is continuous on G_k as well. We have

$$\begin{aligned}
&||(\nabla f_k(x) - \nabla f_{k-1}(x)) - (\varphi_k(x) - \varphi_{k-1}(x))|| \\
&= ||\nabla h(x) - (\varphi_k(x) - \varphi_{k-1}(x))|| \\
&= |\langle \nabla h(x), t(\varphi_{k-1}(x))\rangle| < \Theta 2^{-k}
\end{aligned}$$

and (p2) is verified. Using (p2) for $x \in G_k$ we infer

$$\begin{aligned}
||\varphi_k(x)|| &\leqslant ||\nabla f_k(x)|| + ||\nabla f_k(x) - \varphi_k(x)|| \\
&\leqslant ||\nabla f_k(x)|| + \sum_{j=1}^{k} ||(\nabla f_j(x) - \nabla f_{j-1}(x)) - (\varphi_j(x) - \varphi_{j-1}(x))|| \\
&< 1 + \sum_{j=1}^{k} \Theta 2^{-j} < 2.
\end{aligned}$$

Therefore $\varphi_k(x) \in B(0,2)$ for every $x \in G_k$. This finishes the construction of the auxiliary objects.

4.3. The desired function and its properties

Using (p2), for $k > p$ and $x \in G_k$, we estimate

$$\begin{aligned} ||\nabla f_k(x) - \nabla f_p(x)|| &\leqslant ||\varphi_k(x) - \varphi_p(x)|| \\ &+ \sum_{j=p+1}^{k} ||(\nabla f_j(x) - \nabla f_{j-1}(x)) - (\varphi_j(x) - \varphi_{j-1}(x))|| \\ &\leqslant ||\varphi_k(x) - \varphi_p(x)|| + \sum_{j=p+1}^{k} \Theta 2^{-j} \\ &\leqslant ||\varphi_k(x) - \varphi_p(x)|| + 2^{-p}. \end{aligned} \tag{4.1}$$

We check that (f_k) satisfies also conditions (a), (d), and (e) of Lemma 2.5. Conditions (a) and (d) immediately follow from (p4) and (p1) respectively.

To prove (e) fix $x \in \mathbf{R}^n$. If $x \notin G_k$ for some $k \in \mathbf{N}$, then the sequence $(\nabla f_k(x))$ is eventually constant by (p5) hence convergent. If $x \in \bigcap_{k=0}^{\infty} G_k$, then the sequence $(\varphi_k(x))$ is a well defined sequence of elements of $B(0,2)$. Since t determines a winning tactic for the second player in PH, we have that $(\varphi_k(x))$ is convergent by (p7). This and (4.1) give that $(\nabla f_k(x))$ is also convergent.

Set $f = \lim f_k$. According to Lemma 2.5 the function f is well defined and differentiable. Moreover, we have $\nabla f(x) = \lim \nabla f_k(x)$.

The set $(\nabla f)^{-1}(B(0,1))$ is nonempty, since $0 \notin G_0$ and we have $\nabla f(0) = \nabla f_0(0) = 0$.

Finally, we compute the Φ measure of $(\nabla f)^{-1}(B(0,1))$. Denote

$$S = (\nabla f)^{-1}(B(0,1)) \setminus \{0\}.$$

Since we have $\nabla f(x) = \nabla f_k(x)$ whenever $k \in \mathbf{N}$ and $x \notin G_k$, we obtain $S \subset \bigcap_{k=0}^{\infty} G_k$. Set

$$M_k = \{x \in G_{k-1};\ ||\nabla f_k(x) - \nabla f_{k-1}(x)|| \leqslant \Theta\gamma_k\}.$$

Suppose that $x \in S$ belongs only to finitely many M_j's. Then eventually $||\varphi_k(x) - \varphi_{k-1}(x)|| > \Theta\gamma_k - \Theta 2^{-k}$ by (p2) and according to Proposition 3.1(ii), (p7), and the choice of the γ_k's we have that all $\varphi_k(x)$ cannot be contained in $B(0,2)$, a contradiction. So $S \subset \bigcap_{k=1}^{\infty} \bigcup_{j=k}^{\infty} M_j$. We have $\Phi_\infty(M_j) \leqslant 2^{-j}$ by (p6). For $k \geqslant 1$ we obtain

$$\Phi_\infty(S) \leqslant \Phi_\infty\Big(\bigcup_{j=k}^{\infty} M_j\Big) \leqslant \sum_{j=k}^{\infty} \Phi_\infty(M_j) \leqslant 2^{-k+1}.$$

This implies $\Phi_\infty(S) = 0$. Since the families of zero sets for Φ and Φ_∞ coincide we have $\Phi(S) = 0$. We get $\Phi((\nabla f)^{-1}(B(0,1))) = 0$ by Lemma 2.2.

5. Proof of Lemma 2.4

Notation 5.1. —

(i) Let $M \subset \mathbf{R}^n$. The symbols $\operatorname{lin} M$, $\operatorname{aff} M$, and $\operatorname{conv} M$ stand for the linear span of M, the affine span of M, and the convex hull of M respectively.

(ii) If A is a nonempty subset of $\mathbf{R}^n$ then $B(A, r)$ denotes the set of all points $y \in \mathbf{R}^n$ with $\operatorname{dist}(y, A) < r$.

(iii) Let $S \subset \mathbf{R}^n$ be an n-dimensional simplex, i.e., the convex hull of a set of $n+1$ affinely independent points. Then the set of all $(n-1)$-dimensional faces of S is denoted by $\mathcal{F}(S)$.

(iv) The symbol $\langle \cdot, \cdot \rangle$ (or just "$\cdot$") stands for the scalar product on $\mathbf{R}^n$. If $\boldsymbol{u} \in \mathbf{R}^n$ and $M \subset \mathbf{R}^n$, then we denote

$$A(\boldsymbol{u}, M) := \sup\{|\langle \boldsymbol{u}, (\boldsymbol{b} - \boldsymbol{a})/||\boldsymbol{b} - \boldsymbol{a}||\rangle|;\ \boldsymbol{a}, \boldsymbol{b} \in M, \boldsymbol{a} \neq \boldsymbol{b}\}.$$

Lemma 5.2. — *Let $S \subset \mathbf{R}^n$ be an n-dimensional simplex. Then there exists a finite family $\mathfrak{H}$ of hyperplanes with the following property: If $\alpha \in (0, 1]$ and $\varepsilon > 0$, then there exists a nonnegative $\mathcal{C}^1$-function $\psi : \mathbf{R}^n \to \mathbf{R}$ such that*

(a) $\operatorname{spt} \psi \subset S$,

(b) $\sup_S \psi < \operatorname{diam} S$,

and, for any $\delta > 0$ and any $\boldsymbol{u} \in S_{n-1}$ with $A(\boldsymbol{u}, F) < \delta$ for every $F \in \mathcal{F}(S)$, ψ further satisfies

(c) $\forall x \in \mathbf{R}^n :\ \operatorname{dist}(\nabla\psi(x), \operatorname{conv}\{\pm\alpha\boldsymbol{u}\}) < 2\delta$,

(d) $\forall x \in S \setminus B(\bigcup \mathfrak{H}, \varepsilon) :\ \operatorname{dist}(\nabla\psi(x), \{\pm\alpha\boldsymbol{u}\}) < 2\delta$.

Proof. — For $F_1, F_2 \in \mathcal{F}(S)$, $F_1 \neq F_2$, we set

$$D(F_1, F_2) = \{x \in \mathbf{R}^n;\ \operatorname{dist}(x, \operatorname{aff}(F_1)) = \operatorname{dist}(x, \operatorname{aff}(F_2))\}.$$

Clearly, $D(F_1, F_2)$ can be covered by two affine subspaces. We denote them by $D^1(F_1, F_2)$ and $D^2(F_1, F_2)$. We define the desired family $\mathfrak{H}$ by

$$\mathfrak{H} = \{\operatorname{aff}(F);\ F \in \mathcal{F}(S)\} \cup \{D^i(F_1, F_2);\ F_1, F_2 \in \mathcal{F}(S), F_1 \neq F_2, i \in \{1, 2\}\}.$$

Fix $\alpha \in (0, 1]$ and $\varepsilon > 0$. We will construct the desired function ψ. We set $d(x) = \operatorname{dist}(x, \mathbf{R}^n \setminus S)$, $x \in \mathbf{R}^n$. Let τ be a nonnegative $\mathcal{C}^1$-function supported by $B(0, \varepsilon/2)$ with $\int_{\mathbf{R}^n} \tau = 1$. We set $\omega(x) = \alpha \max\{d(x) - \varepsilon/2, 0\}$ for $x \in \mathbf{R}^n$, and $\psi = \omega * \tau$. Using well known properties of convolution, we easily verify that ψ is a nonnegative $\mathcal{C}^1$-function satisfying (a) and (b).

Now suppose that $\delta > 0$ and $\boldsymbol{u} \in S_{n-1}$ satisfies $A(\boldsymbol{u}, F) < \delta$ for every $F \in \mathcal{F}(S)$. It is easy to observe that, for all $x \in S \setminus \bigcup \mathfrak{H}$, $\nabla d(x)$ exists,

$||\nabla d(x)|| = 1$, and $\nabla d(x)$ is orthogonal to $\mathrm{aff}(F)$ for some $F \in \mathcal{F}(S)$. These facts, $A(\boldsymbol{u}, F) < \delta$ for $F \in \mathcal{F}(S)$, and a quick computation give $\nabla d(x) \in B(\{\pm\boldsymbol{u}\}, 2\delta)$ for $x \in S \setminus \bigcup \mathfrak{H}$. Since $\nabla\omega(x) \in B(\{\alpha\boldsymbol{u}, -\alpha\boldsymbol{u}, 0\}, 2\delta)$ a.e., we get (c).

If $x \in S \setminus B(\bigcup \mathfrak{H}, \varepsilon)$, then $\nabla\omega(y) = \alpha\nabla d(x)$ for $y \in B(x, \varepsilon/2)$ and, consequently, $\nabla\psi(x) = \alpha\nabla d(x)$. This proves (d). □

Notation 5.3. — To each n-dimensional simplex S we associate a family of hyperplanes $\mathfrak{H}(S)$ which satisfies the conclusion of Lemma 5.2.

Remark 5.4. — Observe that we may and do require

$$\mathfrak{H}(S + \boldsymbol{w}) = \{H + \boldsymbol{w};\ H \in \mathfrak{H}(S)\}$$

whenever S is an n-dimensional simplex and $\boldsymbol{w} \in \mathbf{R}^n$.

Lemma 5.5. — *Let $\varepsilon > 0$. Let $\Omega \subset \mathbf{R}^n$ be an open set, and $M_j \subset \Omega$, $j = 1, \dots, k$, be closed sets in Ω such that $\Phi(\bigcap_{j=1}^k M_j) = 0$. Then there are open sets $G_1, \dots, G_k$ such that $\Phi_\infty(\bigcap_{j=1}^k G_j) < \varepsilon$ and $M_j \subset G_j$, $j = 1, \dots, k$.*

Proof. — Find an open set G such that $\bigcap_{j=1}^k M_j \subset G \subset \Omega$ and $\Phi_\infty(G) < \varepsilon$. The sets $M_j \setminus G$ are closed in Ω and their intersection is empty. Thus we can find open sets $H_j \subset \Omega$ such that $M_j \setminus G \subset H_j$ and $\bigcap_{j=1}^k H_j = \emptyset$ (cf. [8, 1.5.18]). We set $G_j = G \cup H_j$ and we are done. □

Lemma 5.6. — *Let $B \subset \mathbf{R}^n$ be a closed ball, $\boldsymbol{u} \in S_{n-1}$, $\delta > 0$, $\omega > 0$, and let G be an open set containing B. Let $\mathfrak{V}$ be a countable family of affine subspaces of $\mathbf{R}^n$. Then there exists a finite family $\mathcal{S}$ of n-dimensional simplicies with disjoint interiors such that*

- $\forall S \in \mathcal{S}\ \forall F \in \mathcal{F}(S):\ A(\boldsymbol{u}, F) < \delta$,
- $B \subset \bigcup \mathcal{S} \subset G$,
- $\forall S \in \mathcal{S}:\ \operatorname{diam} S < \omega$,
- *no element of $\mathfrak{V}$ is contained in a hyperplane of $\bigcup\{\mathfrak{H}(S);\ S \in \mathcal{S}\}$.*

Proof. — Let $B = \overline{B}(x, r)$. Find $\varepsilon \in (0, \omega)$ with $\overline{B}(x, r + 3\varepsilon) \subset G$ and a finite family $\mathcal{P}$ of n-dimensional simplicies with disjoint interiors such that $\bigcup \mathcal{P} = [0, 1]^n$. Moreover we may require that $\operatorname{diam} S < \varepsilon$ for every $S \in \mathcal{P}$. Denote $\mathcal{L} = \{\mathrm{aff}(F) - \mathrm{aff}(F);\ F \in \mathcal{F}(S), S \in \mathcal{P}\}$. The set $\mathcal{L}$ is a finite family of $(n-1)$-dimensional subspaces. Thus we can find $\tilde{\boldsymbol{u}} \in S_{n-1} \setminus \bigcup \mathcal{L}$ with $||\tilde{\boldsymbol{u}} - \boldsymbol{u}|| < \delta/2$. Set

$$c = \sup\{|\langle \tilde{\boldsymbol{u}}, \boldsymbol{a}/||\boldsymbol{a}||\rangle|;\ \boldsymbol{a} \in \bigcup \mathcal{L} \setminus \{0\}\}.$$

According to the choice of $\tilde{\boldsymbol{u}}$ we have $c < 1$. Let $\alpha \in (0,1)$ to be chosen later. Denote π the orthogonal projection on $\operatorname{lin}\{\tilde{\boldsymbol{u}}\}$. We define an invertible linear mapping $T : \mathbf{R}^n \to \mathbf{R}^n$ by $T(x) = x - \pi(x) + \alpha\pi(x)$. For $\boldsymbol{a} \in \bigcup \mathcal{L} \setminus \{0\}$ we have

$$\begin{aligned} ||\pi(\boldsymbol{a})|| &= |\langle \boldsymbol{a}, \tilde{\boldsymbol{u}}\rangle| \leqslant c||\boldsymbol{a}||, \\ ||T(\boldsymbol{a})|| &= \sqrt{||\boldsymbol{a}||^2 - (1-\alpha^2)||\pi(\boldsymbol{a})||^2} \geqslant ||\boldsymbol{a}||\sqrt{1-(1-\alpha^2)c^2}, \\ |\langle \tilde{\boldsymbol{u}}, T(\boldsymbol{a})/||T(\boldsymbol{a})||\rangle| &= \frac{\alpha||\pi(\boldsymbol{a})||}{||T(\boldsymbol{a})||} \leqslant \frac{\alpha c}{\sqrt{1-(1-\alpha^2)c^2}}. \end{aligned}$$

Choosing α sufficiently small we get

$$\sup\{|\langle \tilde{\boldsymbol{u}}, T(\boldsymbol{a})/||T(\boldsymbol{a})||\rangle|;\ \boldsymbol{a} \in \bigcup \mathcal{L} \setminus \{0\}\} < \delta/4. \tag{5.1}$$

We set

$$\mathcal{S}^* = \{T(S+\boldsymbol{z});\ \boldsymbol{z} \in \mathbf{Z}^n, S \in \mathcal{P}, T(S+\boldsymbol{z}) \cap \overline{B}(x, r+\varepsilon) \neq \emptyset\},$$

where $\mathbf{Z}$ denotes the set of all integers. If $S^* \in \mathcal{S}^*$, $F \in \mathcal{F}(S^*)$, and $\boldsymbol{a}, \boldsymbol{b} \in F$, $\boldsymbol{a} \neq \boldsymbol{b}$, then $T^{-1}(\boldsymbol{a}) - T^{-1}(\boldsymbol{b}) \in \bigcup \mathcal{L} \setminus \{0\}$ and using (5.1) we obtain

$$|\langle \tilde{\boldsymbol{u}}, (\boldsymbol{a}-\boldsymbol{b})/||\boldsymbol{a}-\boldsymbol{b}||\rangle| = |\langle \tilde{\boldsymbol{u}}, T(T^{-1}(\boldsymbol{a}-\boldsymbol{b}))/||T(T^{-1}(\boldsymbol{a}-\boldsymbol{b}))||\rangle| < \delta/4.$$

Then we have

$$\begin{aligned} |\langle \boldsymbol{u}, (\boldsymbol{a}-\boldsymbol{b})/||\boldsymbol{a}-\boldsymbol{b}||\rangle| &\leqslant |\langle \boldsymbol{u}-\tilde{\boldsymbol{u}}, (\boldsymbol{a}-\boldsymbol{b})/||\boldsymbol{a}-\boldsymbol{b}||\rangle| \\ &\quad + |\langle \tilde{\boldsymbol{u}}, (\boldsymbol{a}-\boldsymbol{b})/||\boldsymbol{a}-\boldsymbol{b}||\rangle| < \delta/2 + \delta/4 = 3\delta/4. \end{aligned}$$

Thus we see that $\mathcal{S}^*$ satisfies:

$$\forall S^* \in \mathcal{S}^*\ \forall F \in \mathcal{F}(S^*) :\ A(\boldsymbol{u}, F) < \delta.$$

Since $||T|| = 1$ and $\operatorname{diam} S < \varepsilon < \omega$ for every $S \in \mathcal{P}$, we have

- $\overline{B}(x, r+\varepsilon) \subset \bigcup \mathcal{S}^* \subset \overline{B}(x, r+2\varepsilon)$,
- $\forall S^* \in \mathcal{S}^* :\ \operatorname{diam} S^* < \omega$.

If Y is an affine subspace of $\mathbf{R}^n$ and $H \subset \mathbf{R}^n$ is a hyperplane, then there exist at most meagerly many $\boldsymbol{v} \in \mathbf{R}^n$ with $Y \subset H + \boldsymbol{v}$. Using this fact and Remark 5.4, we find $\boldsymbol{w} \in B(0, \varepsilon)$ such that no hyperplane of $\bigcup\{\mathfrak{H}(S^* + \boldsymbol{w});\ S^* \in \mathcal{S}^*\}$ contains an element of $\mathfrak{V}$. Now it is easy to check that $\mathcal{S} := \{S^* + \boldsymbol{w};\ S^* \in \mathcal{S}^*\}$ satisfies all required properties. □

To prove the next lemma we use Besicovitch covering theorem (see, e.g., [12]) : *Let $A \subset \mathbf{R}^n$ and $\mathcal{W}$ be a system of closed balls in $\mathbf{R}^n$. Assume that diameters of balls from $\mathcal{W}$ are uniformly bounded and that for each point x of A there exists a ball from $\mathcal{W}$ centered at x. Then there exist a constant c_B depending only on n and systems $\mathcal{L}_1, \dots, \mathcal{L}_{c_B}$, such that $\mathcal{L}_i \subset \mathcal{W}$, each $\mathcal{L}_i$ is a disjoint system, and $A \subset \bigcup \bigcup_{i=1}^{c_B} \mathcal{L}_i$.*

Lemma 5.7. — *Let $\Omega \subset \mathbf{R}^n$ be an open set, $\delta > 0$, $u : \Omega \to S_{n-1}$ be a continuous mapping, $\eta : \Omega \to (0, +\infty)$ be a lower semicontinuous function, and $\mathfrak{V}$ be a countable family of affine subspaces of $\mathbf{R}^n$. Then there exists a family $\mathcal{T}$ of n-dimensional simplicies such that*

(i) $\bigcup \mathcal{T} = \Omega$ *and $\mathcal{T}$ is locally finite in Ω, i.e., for every $x \in \Omega$ there exists $r > 0$ such that $B(x, r)$ intersects at most finitely many elements of $\mathcal{T}$,*

(ii) $\mathcal{T} = \bigcup_{j=1}^{c_B} \mathcal{T}_j$ *and the simplicies of $\mathcal{T}_j$ have disjoint interiors, $j = 1, \dots, c_B$,*

(iii) $\forall S \in \mathcal{T}\ \forall F \in \mathcal{F}(S)\ \forall y \in S :\ A(u(y), F) < \delta$,

(iv) $\forall S \in \mathcal{T} :\ \operatorname{diam} S < \inf_S \eta$,

(v) *no element of $\mathfrak{V}$ is contained in a hyperplane of $\bigcup\{\mathfrak{H}(S);\ S \in \mathcal{T}\}$.*

Proof. — Using Lemma 2.3 we may and do assume that, for every $x \in \Omega$, we have

- $\overline{B}(x, 2\eta(x)) \subset \Omega$,
- $\sup\{||u(x) - u(y)||; y \in B(x, 2\eta(x))\} < \delta/2$.

Set $\mathcal{W} = \{\overline{B}(x, \eta(x));\ x \in \Omega\}$. Using Besicovitch theorem we obtain systems $\mathcal{L}_1, \dots, \mathcal{L}_{c_B}$ such that $\bigcup\bigcup_{j=1}^{c_B} \mathcal{L}_j = \Omega$, $\mathcal{L}_j \subset \mathcal{W}$, and each $\mathcal{L}_j$ is a disjoint system.

Let $x \in \Omega$ and $j \in \{1, \dots, c_B\}$. Since η is a positive lower semicontinuous function on Ω there exist $r > 0$ and $c > 0$ such that $\eta(y) > c$ for $y \in B(x, 2r)$. If a ball $\overline{B}(z, \eta(z)) \in \mathcal{W}$ intersects $B(x, r)$, then $\eta(z) > \min\{r, c\}$. Indeed, if $z \notin B(x, 2r)$ then $\eta(z) > r$, and if $z \in B(x, 2r)$ then $\eta(z) > c$. Using this observation and disjointness of the family $\mathcal{L}_j$ we see that there are only finitely many balls from $\mathcal{L}_j$ intersecting $B(x, r)$. Then it is easy to find $s \in (0, r)$ such that at most one ball of $\mathcal{L}_j$ intersects $B(x, s)$. Thus we have that $\mathcal{L}_j$ is discrete in Ω, i.e., for every $x \in \Omega$ there exists $s > 0$ such that $B(x, s)$ intersects at most one element of $\mathcal{L}_j$.

For each $j \in \{1, \dots, c_B\}$ and $B = \overline{B}(x, \eta(x)) \in \mathcal{L}_j$ we find an open set $G_j(B)$ such that $\overline{B}(x, \eta(x)) \subset G_j(B) \subset B(x, 2\eta(x))$ and $\{G_j(B);\ B \in \mathcal{L}_j\}$ is a discrete family in Ω. Using Lemma 5.6 we find for every $B = \overline{B}(x, \eta(x)) \in \mathcal{L}_j$ a finite family $\mathcal{T}_j(B)$ of n-dimensional simplicies with disjoint interiors such that

- $\forall S \in \mathcal{T}_j(B)\ \forall F \in \mathcal{F}(S) :\ A(u(x), F) < \delta/2$,
- $B \subset \bigcup \mathcal{T}_j(B) \subset G_j(B)$,
- $\forall S \in \mathcal{T}_j(B) :\ \operatorname{diam} S < \inf_{\overline{B}(x, 2\eta(x))} \eta$,
- no element of $\mathfrak{V}$ is contained in a hyperplane of $\bigcup\{\mathfrak{H}(S);\ S \in \mathcal{T}_j(B)\}$.

We set $\mathcal{T}_j = \bigcup\{\mathcal{T}_j(B);\ B \in \mathcal{L}_j\}$ and $\mathcal{T} = \bigcup_{j=1}^{c_B} \mathcal{T}_j$. Thus (i) and (ii) are satisfied.

Take $S \in \mathcal{T}_j$, $F \in \mathcal{F}(S)$, and $y \in S$. Let $B = \overline{B}(x, \eta(x)) \in \mathcal{L}_j$ be a (uniquely determined) ball with $S \in \mathcal{T}_j(B)$. Take $\boldsymbol{a}, \boldsymbol{b} \in F$, $\boldsymbol{a} \neq \boldsymbol{b}$. Set $\boldsymbol{u} = (\boldsymbol{b}-\boldsymbol{a})/||\boldsymbol{b}-\boldsymbol{a}||$. We have $y \in B(x, 2\eta(x))$ and therefore $||u(y)-u(x)|| < \delta/2$. Consequently, we have

$$|\langle u(y), \boldsymbol{u}\rangle| = |\langle u(y) - u(x), \boldsymbol{u}\rangle| + |\langle u(x), \boldsymbol{u}\rangle| < \frac{\delta}{2}||\boldsymbol{u}|| + \delta/2 = \delta$$

and (iii) follows. Conditions (iv) and (v) are obviously satisfied. $\square$

Proof of Lemma 2.4. — It is well known and easy to see that there exist continuous tangent vector fields $t^1, \dots, t^{n-1}$ on S_{n-1}^+ such that the set $\{z, t^1(z), \dots, t^{n-1}(z)\}$ forms an orthonormal basis of $\mathbf{R}^n$ whenever $z \in S_{n-1}^+$. We set $u^i(x) = t^i(v(x))$, $x \in \Omega$, $i = 1, \dots, n-1$. Then $u^1, \dots, u^{n-1}$ are continuous mappings from Ω to S_{n-1} so that $\{v(x), u^1(x), \dots, u^{n-1}(x)\}$ forms an orthonormal basis of $\mathbf{R}^n$ for every $x \in \Omega$.

We set $\Theta := (nc_B 2^{c_B+5})^{-1}$. Choose a positive number $\tilde{\delta}$ such that $2nc_B\tilde{\delta} < \min\{\Theta\gamma, \delta\}$. Using Lemma 2.3 we may assume without any loss of generality that η satisfies the following conditions:

$$\forall x \in \Omega:\ \eta(x) \leqslant \operatorname{dist}^2(x, \mathbf{R}^n \setminus \Omega), \tag{5.2}$$

$$\forall x \in \Omega\ \forall y \in B(x, \eta(x))\ \forall i \in \{1, \dots, n-1\}:\ ||u^i(x) - u^i(y)|| < \tilde{\delta}. \tag{5.3}$$

Applying Lemma 5.7 to each u^i we find families $\mathcal{T}^i$ of n-dimensional simplicies, $i = 1, \dots, n-1$, such that

(i) $\bigcup \mathcal{T}^i = \Omega$ and $\mathcal{T}^i$ is locally finite in Ω,

(ii) $\mathcal{T}^i = \bigcup_{j=1}^{c_B} \mathcal{T}_j^i$ and the simplicies of $\mathcal{T}_j^i$ have disjoint interiors, $j = 1, \dots, c_B$,

(iii) $\forall S \in \mathcal{T}^i\ \forall F \in \mathcal{F}(S)\ \forall y \in S:\ A(u^i(y), F) < \tilde{\delta}/2$,

(iv) $\forall S \in \mathcal{T}^i:\ \operatorname{diam} S < \frac{1}{(n-1)c_B} \inf_S \eta$,

(v) if $1 < k < n$ and $H_1 \in \bigcup\{\mathfrak{H}(S);\ S \in \mathcal{T}^1\}, \dots, H_{k-1} \in \bigcup\{\mathfrak{H}(S);\ S \in \mathcal{T}^{k-1}\}$, then $H_1 \cap \dots \cap H_{k-1}$ is contained in no hyperplane of $\bigcup\{\mathfrak{H}(S);\ S \in \mathcal{T}^k\}$.

Set

$$M^i = \bigcup\{S \cap H;\ S \in \mathcal{T}^i,\ H \in \mathfrak{H}(S)\} \quad \text{and} \quad M = \bigcap_{i=1}^{n-1} M^i.$$

The family $\mathcal{T}^i$ is locally finite in Ω by (i). Thus M^i is closed in Ω. Further, the set M is covered by a countable family of lines. Indeed, each set of the form $H_1 \cap \dots \cap H_{n-1}$, where $H_1 \in \bigcup\{\mathfrak{H}(S);\ S \in \mathcal{T}^1\}$, ..., $H_{n-1} \in \bigcup\{\mathfrak{H}(S);\ S \in \mathcal{T}^{n-1}\}$, is a line, a point or empty set by (v). Thus $\Phi(M) = 0$ by Lemma 2.2. Applying Lemma 5.5 we find open sets G^i, $i = 1, \dots, n-1$, such that $M^i \subset G^i \subset \Omega$ and $\Phi_\infty(\bigcap_{i=1}^{n-1} G^i) < \varepsilon$.

Now fix i, j, and $S \in \mathcal{T}_j^i$. Pick a point $x_{j,S}^i \in S$. By (iii) we have $A(u^i(x_{j,S}^i), F) < \tilde{\delta}/2$ for every $F \in \mathcal{F}(S)$. Using the properties of $\mathfrak{H}(S)$ guaranteed by Lemma 5.2 for $\alpha := \Theta\gamma 2^{j+4}$ ($\leqslant 1$),

$$\varepsilon := \operatorname{dist}\left(S \cap \bigcup \mathfrak{H}(S), \mathbf{R}^n \setminus G^i\right),$$

$\delta := \tilde{\delta}/2$, and $\boldsymbol{u} := u^i(x_{j,S}^i)$, we find a nonnegative $\mathcal{C}^1$-function $\psi_{j,S}^i$ such that

(A) $\operatorname{spt} \psi_{j,S}^i \subset S$,
(B) $\sup_S \psi_{j,S}^i < \operatorname{diam} S$,
(C) $\forall x \in \mathbf{R}^n:\ \operatorname{dist}(\nabla \psi_{j,S}^i(x), \operatorname{conv}\{\pm\Theta\gamma 2^{j+4} u^i(x_{j,S}^i)\}) < \tilde{\delta}$,
(D) $\forall x \in S \setminus G^i:\ \operatorname{dist}(\nabla \psi_{j,S}^i(x), \{\pm\Theta\gamma 2^{j+4} u^i(x_{j,S}^i)\}) < \tilde{\delta}$.

We set

$$\psi_j^i = \sum_{S \in \mathcal{T}_j^i} \psi_{j,S}^i, \qquad \psi^i = \sum_{j=1}^{c_B} \psi_j^i, \qquad \text{and} \qquad h = \sum_{i=1}^{n-1} \psi^i.$$

Using the condition (5.3) and (iv) we easily verify

(C') $\forall x \in \Omega:\ \operatorname{dist}(\nabla \psi_{j,S}^i(x), \operatorname{conv}\{\pm\Theta\gamma 2^{j+4} u^i(x)\}) < 2\tilde{\delta}$.

Let $i \in \{1, \dots, n-1\}$ and $j \in \{1, \dots, c_B\}$. If

$$x \in \Omega \setminus \bigcup\{\operatorname{interior} S;\ S \in \mathcal{T}_j^i\},$$

then using (A) we get

$$\psi_j^i(x) = 0 < \operatorname{dist}^2(x, \mathbf{R}^n \setminus \Omega).$$

If

$$x \in \bigcup\{\operatorname{interior} S;\ S \in \mathcal{T}_j^i\},$$

then by (ii) there is a uniquely determined $S \in \mathcal{T}_j^i$ with $x \in \operatorname{interior} S$. By (iv) and (B) we have

$$0 \leqslant \psi_j^i(x) = \psi_{j,S}^i(x) \leqslant \sup_S \psi_{j,S}^i < \operatorname{diam} S < \frac{1}{(n-1)c_B} \eta(x). \tag{5.4}$$

Now using (5.2) we get

$$\psi_j^i(x) < \frac{1}{(n-1)c_B} \eta(x) < \eta(x) \leqslant \operatorname{dist}^2(x, \mathbf{R}^n \setminus \Omega).$$

Thus we have $0 \leqslant \psi_j^i(x) \leqslant \mathrm{dist}^2(x, \mathbf{R}^n \setminus \Omega)$ on Ω. Now since $\psi_j^i = 0$ on $\mathbf{R}^n \setminus \Omega$ and $\mathcal{T}_j^i$ is locally finite in Ω, we see that ψ_j^i is differentiable on $\mathbf{R}^n$ and ψ_j^i is a $\mathcal{C}^1$-function on Ω. This implies that ψ^i and h are differentiable on $\mathbf{R}^n$ and are $\mathcal{C}^1$ on Ω. Clearly, $h = 0$ and $\nabla h = 0$ on $\mathbf{R}^n \setminus \Omega$. Thus we verified (a) and (c).

We denote the orthogonal projection of $\nabla\psi^i(x)$ to $\mathrm{lin}\{u^i(x)\}$ by $w^i(x)$. According to (C') we have

$$\begin{aligned} ||\nabla\psi^i(x) - w^i(x)|| &\leqslant \sum_{j=1}^{c_B} \mathrm{dist}(\nabla\psi_j^i(x), \mathrm{lin}\{u^i(x)\}) \\ &\leqslant 2c_B\tilde{\delta}, \qquad x \in \Omega, \qquad i = 1, \dots, n-1. \end{aligned} \tag{5.5}$$

(b) Suppose that $x \in \Omega \setminus G^{i_0}$, $i_0 \in \{1, \dots, n-1\}$. Then there exists the biggest $j_0 \in \{1, \dots, c_B\}$ such that $x \in \bigcup \mathcal{T}_{j_0}^{i_0}$. Using (C) and (D) we infer

$$\begin{aligned} ||\nabla\psi^{i_0}(x)|| &\geqslant ||\nabla\psi_{j_0}^{i_0}(x)|| - \sum_{j=1}^{j_0-1} ||\nabla\psi_j^{i_0}(x)|| \\ &\geqslant (\Theta\gamma 2^{j_0+4} - \tilde{\delta}) - \sum_{j=1}^{j_0-1} (\Theta\gamma 2^{j+4} + \tilde{\delta}) \\ &\geqslant 32\Theta\gamma - c_B\tilde{\delta} > 8\Theta\gamma. \end{aligned} \tag{5.6}$$

Using the inequalities (5.5) and (5.6) we infer

$$\begin{aligned} ||w^{i_0}(x)|| &\geqslant ||\nabla\psi^{i_0}(x)|| - ||\nabla\psi^{i_0}(x) - w^{i_0}(x)|| \\ &\geqslant ||\nabla\psi^{i_0}(x)|| - 2c_B\tilde{\delta} \geqslant 8\Theta\gamma - \Theta\gamma = 7\Theta\gamma. \end{aligned} \tag{5.7}$$

Finally, using orthogonality of $w^i(x)$'s, (5.5), and (5.7), we get

$$\begin{aligned} ||\nabla h(x)|| = \left|\left|\sum_{i=1}^{n-1} \nabla\psi^i(x)\right|\right| &\geqslant \left|\left|\sum_{i=1}^{n-1} w^i(x)\right|\right| - \sum_{i=1}^{n-1} ||\nabla\psi^i(x) - w^i(x)|| \\ &\geqslant ||w^{i_0}(x)|| - (n-1)2c_B\tilde{\delta} > 7\Theta\gamma - \Theta\gamma > \Theta\gamma. \end{aligned}$$

Thus we have $\{x \in \Omega;\ ||\nabla h(x)|| \leqslant \Theta\gamma\} \subset \bigcap_{i=1}^{n-1} G^i$. Consequently,

$$\Phi_\infty(\{x \in \Omega;\ ||\nabla h(x)|| \leqslant \Theta\gamma\}) < \varepsilon.$$

(d) We estimate

$$\begin{aligned}|\langle \nabla h(x), v(x)\rangle| &= \left|\sum_{i=1}^{n-1}\langle w^i(x), v(x)\rangle + \sum_{i=1}^{n-1}\langle \nabla\psi^i(x) - w^i(x), v(x)\rangle\right| \\ &= \left|\sum_{i=1}^{n-1} 0 + \sum_{i=1}^{n-1}\langle \nabla\psi^i(x) - w^i(x), v(x)\rangle\right| \\ &\leqslant \sum_{i=1}^{n-1} ||\nabla\psi^i(x) - w^i(x)|| \leqslant 2c_B\tilde{\delta}(n-1) < \delta\end{aligned}$$

by (5.5) and the choice of $\tilde{\delta}$.

(e) By (5.4) we have

$$0 \leqslant h(x) \leqslant \sum_{i=1}^{n-1}\sum_{j=1}^{c_B} |\psi_j^i(x)| \leqslant (n-1)c_B\frac{1}{(n-1)c_B}\eta(x) = \eta(x), \qquad x \in \Omega.$$

(f) Using (C') and the inequality $2nc_B\tilde{\delta} < \Theta\gamma$, we obtain

$$\begin{aligned}||\nabla h(x)|| &\leqslant \sum_{i=1}^{n-1}\sum_{j=1}^{c_B} ||\nabla\psi_j^i(x)|| \leqslant \sum_{i=1}^{n-1}\sum_{j=1}^{c_B}(\Theta\gamma 2^{j+4} + 2\tilde{\delta}) \\ &= (n-1)c_B(\Theta\gamma 2^{c_B+4} + 2\tilde{\delta}) \leqslant \gamma, \qquad x \in \Omega.\end{aligned}$$

□

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Manuscrit reçu le 2 février 2007,
accepté le 5 avril 2007.

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