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A LINEAR EXTENSION OPERATOR FOR WHITNEY FIELDS ON CLOSED O-MINIMAL SETS

by Wiesław PAWŁUCKI (*)

Dedicated to my wife Jolanta

ABSTRACT. — A continuous linear extension operator, different from Whitney's, for C^p -Whitney fields (p finite) on a closed o-minimal subset of \mathbb{R}^n is constructed. The construction is based on special geometrical properties of o-minimal sets earlier studied by K. Kurdyka with the author.

RÉSUMÉ. — On construit un opérateur d'extension linéaire et continu pour les champs de Whitney de classe C^p (p fini) sur un sous-ensemble fermé o-minimal de \mathbb{R}^n . La construction, différente de celle de Whitney, est basée sur des propriétés géométriques spéciales des ensembles o-minimaux, étudiées avant par K. Kurdyka et l'auteur.

1. Introduction

In 1997 K. Kurdyka and the author gave in [6] the following o-minimal version of the Whitney extension theorem:

THEOREM 1.1 ([6]). — *Given any o-minimal structure on the ordered field of real numbers \mathbb{R} , a compact definable subset $E \subset \mathbb{R}^n$, a definable C^p -Whitney field F on E , where $p \in \mathbb{N} \setminus \{0\}$, then for any integer $q \geq p$, there exists a definable C^q -extension $f : \mathbb{R}^n \rightarrow \mathbb{R}$ of F which is C^q on $\mathbb{R}^n \setminus E$.*

However, the extension operator $F \mapsto f$ from [6] is not linear and it was not clear how the construction from [6] based on o-minimal geometry could be adapted to get an extension operator for *all* Whitney fields on

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any compact (or more generally closed) o-minimal subset E of \mathbb{R}^n . The present paper is devoted to this question. The main goal here is to prove the following

THEOREM 1.2. — *Let E be a closed o-minimal subset of \mathbb{R}^n and $p \in \mathbb{N}$. Let $\mathcal{E}^p(E)$ denote the Fréchet algebra of all C^p -Whitney fields on E .*

Then there exists a continuous linear extension operator $\mathcal{L} : \mathcal{E}^p(E) \longrightarrow C^p(\mathbb{R}^n)$ which has the following properties

- (1) \mathcal{L} is a finite composition of operators each of which either preserves definability or (only if $p > 0$) is an integration with respect to a parameter;
- (2) operators preserving definability in (1) are only of the following five types: substituting with a definable mapping; taking a linear combination with definable coefficients; differentiation; restriction to a definable subset and extending by zero;
- (3) there exists a constant $M > 0$ such that if ω is a modulus of continuity of a field F , then $M\omega$ is a modulus of continuity of $\mathcal{L}F$.

Since \mathcal{L} involves integration, it may not preserve definability in the initial o-minimal structure where E is definable. For example, if F is a (globally) subanalytic C^p -Whitney field, then $\mathcal{L}F$ can *a priori* involve the function $t \mapsto t \log t$, not subanalytic at 0. By a result of Lion and Rolin [7], we get in this case the following

COROLLARY 1.3. — *Let \mathcal{A} denote the algebra of real functions generated by (globally) subanalytic functions and their logarithms; i.e. \mathcal{A} consists of all functions of the form $P(h_1, \dots, h_m, \log h_1, \dots, \log h_m)$, where $h_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ ($i = 1, \dots, m$) are subanalytic, $m \in \mathbb{N} \setminus \{0\}$, $P \in \mathbb{R}[Y_1, \dots, Y_{2m}]$, and where we adopt the convention: $\log t = 0$, for $t \leq 0$. Let E be a closed subanalytic subset of \mathbb{R}^n and $p \in \mathbb{N}$.*

Then there exists a continuous linear extension operator $\mathcal{L} : \mathcal{E}^p(E) \longrightarrow C^p(\mathbb{R}^n)$ which has the following properties:

- (1) if F is a C^p -Whitney field on E all derivatives of which F^\times are (restrictions to E of) functions in \mathcal{A} , then $\mathcal{L}F \in \mathcal{A}$;
- (2) there exists a constant $M > 0$ such that if ω is a modulus of continuity of a field F , then $M\omega$ is a modulus of continuity of $\mathcal{L}F$.

The case $p = 0$ in Theorem 1.2, when integration is not used seems worth being stated separately

COROLLARY 1.4. — *Let E be a closed o-minimal subset of \mathbb{R}^n and let $\mathcal{C}(E)$ denote the Fréchet space of all real continuous functions on E*

Then there exists a continuous linear extension operator $\mathcal{L} : \mathcal{C}(E) \rightarrow \mathcal{C}(\mathbb{R}^n)$ preserving definability and such that there exists $M > 0$ such that, if ω is a modulus of continuity for $F \in \mathcal{C}(E)$, then $M\omega$ is a modulus of continuity for $\mathcal{L}F$.

By an o-minimal subset of an Euclidean space \mathbb{R}^n we mean a subset definable in any o-minimal structure on the ordered field of real numbers \mathbb{R} (see [2, 3] for the definition and fundamental properties).

We refer the reader to [13], [4], [8], [11] or/and [12] for basic facts on Whitney fields. It will be convenient for us to adopt the following definition of a Whitney field.

Let $p \in \mathbb{N} \setminus \{0\}$ and let A be a locally closed subset of \mathbb{R}^n ; i.e. contained and closed in some open subset $G \subset \mathbb{R}^n$. A \mathcal{C}^p -Whitney field on A is a polynomial

$$F(u, X) = \sum_{|\varkappa| \leq p} \frac{1}{\varkappa!} F^{\varkappa}(u) X^{\varkappa} \in \mathcal{C}(A)[X] = \mathcal{C}(A)[X_1, \dots, X_n],$$

which fulfills the following condition

(*) for each $c \in A$ and each $\alpha \in \mathbb{N}^n$ such that $|\alpha| \leq p$

$$D_X^\alpha F(a, 0) - D_X^\alpha F(b, a - b) = o(|a - b|^{p - |\alpha|}), \quad \text{when } A \ni a \rightarrow c, A \ni b \rightarrow c,$$

or equivalently (see [8], Chapter I, Theorem 2.2) - the condition

(**) for each $c \in A$

$$F(a, x - a) - F(b, x - b) = o(|x - a|^p + |x - b|^p),$$

uniformly with respect to $x \in \mathbb{R}^n$, when $A \ni a \rightarrow c, A \ni b \rightarrow c$.

We will denote by $\mathcal{E}^p(A)$ the real algebra of all \mathcal{C}^p -Whitney fields on A . It is a Fréchet algebra with the topology defined by the following system of seminorms

$$\|F\|_p^K = |F|_p^K + \sup_{\substack{a, b \in K \\ a \neq b \\ |\alpha| \leq p}} \frac{|D_X^\alpha F(a, 0) - D_X^\alpha F(b, a - b)|}{|a - b|^{p - |\alpha|}},$$

where K is a compact subset of A and $|\cdot|_p^K$ is a seminorm defined by

$$|F|_p^K = \sup_{\substack{a \in K \\ |\alpha| \leq p}} |F^\alpha(a)|.$$

Let $\mathcal{C}^p(G)$ denote the usual Fréchet algebra of real functions of class \mathcal{C}^p (\mathcal{C}^p -functions) on G . Then we have the following homomorphism of Fréchet

algebras

$$T : \mathcal{C}^p(G) \longrightarrow \mathcal{E}^p(A), \quad Tf(a, X) = T_a^p f(X) = \sum_{|\varkappa| \leq p} \frac{1}{\varkappa!} D^{\varkappa} f(a) X^{\varkappa},$$

and the Whitney extension theorem [13] says that there exists a linear continuous mapping

$$W : \mathcal{E}^p(A) \longrightarrow \mathcal{C}^p(G) \quad \text{such that} \quad T \circ W = id_{\mathcal{E}^p(A)},$$

called an *extension operator*.

A subset E of \mathbb{R}^n is said to be *1-regular* (with a constant $C \geq 1$) if any two points a, b of E can be joined in E by a rectifiable arc $\gamma : [0, 1] \longrightarrow E$ of length $|\gamma| \leq C|a - b|$.

If $F \in \mathcal{E}^p(A)$ and K is a compact 1-regular subset of A with a constant C , then

$$|F|_p^K \leq \|F\|_p^K \leq 2n^{\frac{p}{2}} C^p |F|_p^K \quad (\text{See [12], p.76, (2.5.1)}).$$

Consequently, if every compact subset L of A is contained in a 1-regular compact subset K of A , then the topology of $\mathcal{E}^p(A)$ is defined by the system of seminorms $|\cdot|_p^K$.

As was shown by Glaeser [4] (see also [8], [12] or [11]) it is convenient to use a notion of a modulus of continuity in connection with Whitney fields. By a *modulus of continuity* we will understand any continuous, increasing and concave function $\omega : [0, +\infty) \longrightarrow [0, +\infty)$, vanishing at 0. By a modulus of continuity of a \mathcal{C}^p -Whitney field

$$F(u, X) = \sum_{|\varkappa| \leq p} \frac{1}{\varkappa!} F^{\varkappa}(u) X^{\varkappa}$$

on a subset A of \mathbb{R}^n we will understand such a modulus of continuity ω that

$$|D_X^\alpha(a, 0) - D_X^\alpha(b, a - b)| \leq \omega(|a - b|) |a - b|^{p - |\alpha|},$$

whenever $|\alpha| \leq p$ and $a, b \in A$. For a \mathcal{C}^p -function $f \in \mathcal{C}^p(G)$ on an open subset G , by its modulus of continuity we will understand a modulus of continuity of the \mathcal{C}^p -Whitney field Tf on G .

Every \mathcal{C}^p -Whitney field on a compact subset of \mathbb{R}^n admits a modulus of continuity. If a \mathcal{C}^p -Whitney field F on a subset A has a modulus of continuity ω , then it is easily seen that F extends by uniform continuity to a \mathcal{C}^p -Whitney field on \bar{A} with the same modulus of continuity. Whitney's extension operator [13] has the following property (see [4]):

There exists a constant M depending only on p and n such that, for every $F \in \mathcal{E}^p(A)$ admitting a modulus of continuity ω , $M\omega$ is a modulus of continuity for WF . (In fact a localization by a partition of unity is necessary.)

We have also the following

PROPOSITION 1.5. — *Let F be a C^p -Whitney field on a (locally) closed 1-regular with constant C subset A .*

- (1) *If ω is a modulus of continuity of F on A , then $|F^\alpha(a) - F^\alpha(b)| \leq \omega(|a - b|)$, whenever $|\alpha| = p$, $a, b \in A$.*
- (2) *If ω is a modulus of continuity such that $|F^\alpha(a) - F^\alpha(b)| \leq \omega(|a - b|)$, whenever $|\alpha| = p$, $a, b \in A$, then $n^{\frac{p}{2}}C^p\omega$ is a modulus of continuity of F on A .*

Proof. — (1) being trivial, for (2) see again [12], (2.5.1), p.76. □

Shortly, our construction of the extension operator \mathcal{L} is as follows. First we show how to extend C^p -Whitney fields from a linear subspace $\mathbb{R}^k \times 0$ of \mathbb{R}^n . Then we generalize the construction to the set of the form $\overline{\Omega} \times 0$, where Ω is open in \mathbb{R}^k for fields flat on $\partial\Omega \times 0$, simply by Hestenes Lemma. Using induction on dimension of A , this gives an extension operator for $A = \overline{\Gamma}$, where $\Gamma = \Omega \times 0$ assuming we have it already built for the *boundary* $\partial\Gamma = \overline{\Gamma} \setminus \Gamma$ of Γ which in this case is $\partial\Omega \times 0$. The next generalization is by taking $A = \overline{\Gamma}$, where Γ is a Λ_p -regular leaf of dimension k in the sense of [6], and again assuming the fields are flat on $\partial\Gamma$. Additionally, the extension can be chosen vanishing outside a *conical neighbourhood* of Γ ; i.e. the set $\{x \in \Omega \times \mathbb{R}^{n-k} : d(x, \Gamma) < \varepsilon d(x, \partial\Gamma)\}$, where Ω is the orthogonal projection of Γ to $\mathbb{R}^k \times 0$ and ε is a positive arbitrary constant. The next generalization is to the closure of a *finite tower* of Λ_p -regular leaves lying over a common *open Λ_p -regular cell* in \mathbb{R}^k . To finish the construction we will prove that every closed definable k -dimensional subset A admits a finite decomposition $A = M_0 \cup \dots \cup M_s$ such that each M_i is a finite tower of definable Λ_p -regular leaves in a suitable linear coordinate system and for any $i, j \in \{0, \dots, s\}$, where $i \neq j$, \overline{M}_i and \overline{M}_j are *simply separated relative to ∂M_i* ; i.e. $d(x, M_j) \geq Cd(x, \partial M_i)$, for each $x \in M_i$, with some positive constant C . (The proof of this Λ_p -regular Decomposition Theorem is based on [6] and [10].)

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2. Extension operator for a linear subspace

Observe that if Ω is an open subset of \mathbb{R}^k and $A = \Omega \times 0 \subset \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n$, then the algebra $\mathcal{E}^p(A)$ can be identified with the algebra of polynomials

$$F(u, W) = \sum_{|\alpha| \leq p} \frac{1}{\alpha!} F^\alpha(u) W^\alpha = \sum_{|\alpha| \leq p} \frac{1}{\alpha!} F^\alpha(u) W_1^{\alpha_1} \dots W_l^{\alpha_l},$$

where $l = n - k$ and $F^\alpha \in \mathcal{C}^{p-|\alpha|}(\Omega)$, for each $\alpha \in \mathbb{N}^l$ such that $|\alpha| \leq p$ (cf. [4], Chap.III, (8.4)).

Let us now consider the case $k = n - 1$ and $A = \mathbb{R}^k \times 0$. Then the extension operator will be produced using regularization of functions F^α by convolution. Strictly, we have the following

PROPOSITION 2.1. — *Let $\sigma \in \{0, \dots, p\}$, $g \in \mathcal{C}^{p-\sigma}(\mathbb{R}^k)$, $\varphi \in \mathcal{C}^p(\mathbb{R}^k)$. Assume that $\text{supp}\varphi$ is compact and put*

$$\varphi_w(v) = \frac{1}{w^k} \varphi\left(\frac{v}{w}\right)$$

$$\text{and } G(u, w) = \frac{1}{\sigma!} (g \star \varphi_w)(u) w^\sigma = \frac{1}{\sigma!} \int_{\mathbb{R}^k} g(u - v) \varphi_w(v) w^\sigma dv,$$

for $u \in \mathbb{R}^k$ and $w \in \mathbb{R}$, $w > 0$.

Then $G : \mathbb{R}^k \times (0, +\infty) \rightarrow \mathbb{R}$ is a \mathcal{C}^p -function and for every $(\alpha, \beta) \in \mathbb{N}^k \times \mathbb{N}$ such that $|\alpha| + \beta \leq p$

$$\lim_{w \rightarrow 0} D^{(\alpha, \beta)} G(u, w) = \begin{cases} 0, & \text{if } \beta < \sigma \\ D^\alpha g(u) \int \varphi, & \text{if } \beta = \sigma \\ \sum_{|\gamma| = \beta - \sigma} \omega_{\gamma\sigma} D^{\alpha + \gamma} g(u), & \text{if } \beta > \sigma \end{cases}$$

uniformly on compact subsets with respect to u , where $\omega_{\gamma\sigma}$ are some constants depending only on γ, σ and φ .

To prove Proposition 2.1 one needs the following

LEMMA 2.2. — For any $r \in \mathbb{R}$ and $\lambda \in \{0, \dots, p\}$

$$\frac{\partial^\lambda}{\partial w^\lambda} \left[w^r \varphi \left(\frac{v}{w} \right) \right] = w^{r-\lambda} \varphi_\lambda \left(\frac{v}{w} \right),$$

where φ_λ is a $\mathcal{C}^{p-\lambda}$ -function on \mathbb{R}^k with a compact support and $\int \varphi_\lambda = (k+r)(k+r-1)\dots(k+r-\lambda+1) \int \varphi$.

Proof of Lemma 2.2. —

$$\frac{\partial}{\partial w} \left[w^r \varphi \left(\frac{v}{w} \right) \right] = r w^{r-1} \varphi \left(\frac{v}{w} \right) + w^r \sum_{i=1}^k \frac{\partial \varphi}{\partial v_i} \left(\frac{v}{w} \right) \left(-\frac{v_i}{w^2} \right) = w^{r-1} \varphi_1 \left(\frac{v}{w} \right),$$

where $\varphi_1(v) = r\varphi(v) - \sum_{i=1}^k v_i \frac{\partial \varphi}{\partial v_i}(v)$. Moreover, integrating by parts,

$$\int \varphi_1 = r \int \varphi - \sum_{i=1}^k \int v_i \frac{\partial \varphi}{\partial v_i} = (r+k) \int \varphi$$

and Lemma 2.2 follows by induction. □

Proof of Proposition 2.1. — G is of class \mathcal{C}^p on $\mathbb{R}^k \times (0, +\infty)$, because

$$G(u, w) = \int g(v) \frac{1}{w^k} \varphi \left(\frac{u-v}{w} \right) w^\sigma dv.$$

(I) Assume first that $\beta \leq \sigma$ and $|\alpha| \leq p - \sigma$. Then

$$\begin{aligned} D^{(\alpha, \beta)} G(u, w) &= \frac{1}{\sigma!} \int D^\alpha g(u-v) w^{\sigma-\beta-k} \varphi_\beta \left(\frac{v}{w} \right) dv = \\ &= \frac{1}{\sigma!} w^{\sigma-\beta} \int D^\alpha g(u-v) \frac{1}{w^k} \varphi_\beta \left(\frac{v}{w} \right) dv \longrightarrow \frac{1}{\sigma!} 0^{\sigma-\beta} D^\alpha g(u) \int \varphi_\beta, \end{aligned}$$

when $w \rightarrow 0$, the convergence being uniform on compact subsets with respect to u . Consequently, the limit is 0, if $\beta < \sigma$ and $D^\alpha g \int \varphi$, if $\beta = \sigma$.

(II) Now assume that $\beta \leq \sigma$ and $|\alpha| > p - \sigma$. Then $\alpha = \gamma + \delta$, where $|\gamma| = p - \sigma$ and $\delta \neq 0$.

$$\begin{aligned} D^{(\gamma, \beta)} G(u, w) &= \frac{1}{\sigma!} \int D^\gamma g(u-v) w^{\sigma-\beta-k} \varphi_\beta \left(\frac{v}{w} \right) dv = \\ &= \frac{1}{\sigma!} \int D^\gamma g(v) w^{\sigma-\beta-k} \varphi_\beta \left(\frac{u-v}{w} \right) dv. \\ D^{(\alpha, \beta)} G(u, w) &= \frac{1}{\sigma!} \int D^\gamma g(v) w^{\sigma-\beta-k} w^{-|\delta|} D^\delta \varphi_\beta \left(\frac{u-v}{w} \right) dv = \\ &= \frac{1}{\sigma!} w^{\sigma-\beta-|\delta|} \int D^\gamma g(u-wv) D^\delta \varphi_\beta(v) dv. \end{aligned}$$

Notice that $\sigma - \beta - |\delta| = p - |\alpha| - \beta \geq 0$ and $\int D^\delta \varphi_\beta(v) dv = 0$, since φ_β has a compact support. Consequently, $D^{(\alpha, \beta)} G(u, w) \rightarrow 0$, when $w \rightarrow 0$.

(III) Finally, let $\beta > \sigma$. Then $|\alpha| \leq p - \beta < p - \sigma$ and $\beta = \sigma + \rho$, where $\rho > 0$. By the case (I),

$$D^{(\alpha, \sigma)}G(u, w) = \frac{1}{\sigma!} \int D^\alpha g(u - vw)\varphi_\sigma(v)dv.$$

$D^\alpha g$ being of class $p - \sigma - |\alpha| \geq \rho$, one obtains

$$\begin{aligned} D^{(\alpha, \beta)}G(u, w) &= D^{(0, \rho)}(D^{(\alpha, \sigma)}G)(u, w) \\ &= \frac{1}{\sigma!} \sum_{|\mu|=\rho} \int D^{\alpha+\mu}g(u - vw)(-v)^\mu \varphi_\sigma(v)dv, \end{aligned}$$

which tends to $\sum_{|\mu|=\rho} \omega_{\mu\sigma} D^{\alpha+\mu}g(u)$ uniformly on compact subsets with respect to u , when $w \rightarrow 0$, where $\omega_{\mu\sigma} = \frac{1}{\sigma!} \int (-v)^\mu \varphi_\sigma(v)dv$. □

PROPOSITION 2.3. — *Let $\varphi \in \mathcal{C}^p(\mathbb{R}^k)$ be with compact support and such that $\int \varphi = 1$. Then the formula*

$$\begin{aligned} L(gW^\sigma)(u, w) &= \left(\frac{w}{|w|}\right)^\sigma \left[\frac{1}{\sigma!} (g \star \varphi_{|w|})(u) |w|^\sigma \right. \\ &\quad \left. - \sum_{0 < |\gamma| \leq p - \sigma} \frac{1}{(\sigma + |\gamma|)!} \omega_{\gamma\sigma} L(D^\gamma g W^{\sigma+|\gamma|})(u, |w|) \right], \end{aligned}$$

for $\sigma \in \{1, \dots, p\}$, $g \in \mathcal{C}^{p-\sigma}(\mathbb{R}^k)$, $u \in \mathbb{R}^k$ and $w \in \mathbb{R} \setminus \{0\}$, completed by putting

$$L(gW^\sigma)(u, 0) = 0, \quad \text{and} \quad L(gW^0) = L(g) = g, \quad \text{for } g \in \mathcal{C}^p(\mathbb{R}^k),$$

defines (inductively) a continuous linear extension operator $L = L_p : \mathcal{E}^p(\mathbb{R}^k \times 0) \rightarrow \mathcal{C}^p(\mathbb{R}^{k+1})$.

Moreover, there exists a constant $M > 0$ (depending only on k, p and φ) such that if ω is a modulus of continuity of a field $F \in \mathcal{E}^p(\mathbb{R}^k \times 0)$, then $M\omega$ is a modulus of continuity of the \mathcal{C}^p -function LF .

Proof. — This follows immediately from Proposition 2.1. □

Now we generalize our extension operator to any linear subspace of \mathbb{R}^n .

PROPOSITION 2.4. — *Let $\mathbb{R}^k \times 0 \subset \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l$, where $l > 1$. Then the formula*

$$L_p(gW_1^{\alpha_1} \dots W_l^{\alpha_l}) = L_p(L_{p-\alpha_l}(gW_1^{\alpha_1} \dots W_{l-1}^{\alpha_{l-1}})W_l^{\alpha_l}),$$

where $\alpha = (\alpha_1, \dots, \alpha_l) \in \mathbb{N}^l, |\alpha| \leq p$ and $g \in \mathcal{C}^{p-|\alpha|}(\mathbb{R}^k)$, defines by induction on l a linear continuous extension operator $L = L_p : \mathcal{E}^p(\mathbb{R}^k \times 0) \rightarrow \mathcal{C}^p(\mathbb{R}^n)$.

Moreover, there is a constant $M > 0$ such that if ω is a modulus of continuity for $F \in \mathcal{E}^p(\mathbb{R}^k \times 0)$, then $M\omega$ is a modulus of continuity for LF .

Proof. — This follows easily by induction from Proposition 2.3. □

3. A generalization to the ideal of \mathcal{C}^p -Whitney fields on $\bar{\Omega} \times 0$ p -flat on $\partial\Omega \times 0$ (Ω - an open Λ_p -regular cell in $\mathbb{R}^k = \mathbb{R}^k \times 0 \subset \mathbb{R}^k \times \mathbb{R}^l$)

If A is any locally closed subset of \mathbb{R}^n and B any closed subset of A , $\mathcal{E}^p(A, B)$ will denote the ideal of all \mathcal{C}^p -Whitney fields F on A p -flat on B ; i.e. $F^\alpha(u) = 0$, when $|\alpha| \leq p$ and $u \in B$. It is closed in $\mathcal{E}^p(A)$.

Let first Ω be any open subset of \mathbb{R}^k . By the Hestenes Lemma (see [12], Lemma 4.3, p.80)

$$\mathcal{E}^p(\bar{\Omega} \times 0, \partial\Omega \times 0) = \{F = \sum_{|\alpha| \leq p} \frac{1}{\alpha!} F^\alpha W^\alpha : F^\alpha \in \mathcal{C}^{p-|\alpha|}(\Omega),$$

$$\lim_{u \rightarrow a} D^\beta F^\alpha(u) = 0, \text{ if } a \in \partial\Omega, |\beta| \leq p - |\alpha|\},$$

and putting

$$\tilde{F}^\alpha(u) = \begin{cases} F^\alpha(u), & \text{if } u \in \Omega \\ 0, & \text{if } u \in \mathbb{R}^k \setminus \Omega \end{cases} \quad \text{and} \quad \tilde{F} = \sum_{\alpha} \frac{1}{\alpha!} \tilde{F}^\alpha W^\alpha,$$

one obtains a linear continuous extension operator

$$\mathcal{E}^p(\bar{\Omega} \times 0, \partial\Omega \times 0) \ni F \rightarrow \tilde{F} \in \mathcal{E}^p(\mathbb{R}^k \times 0, (\mathbb{R}^k \setminus \Omega) \times 0)$$

preserving modulus of continuity.

Now we will consider the case when Ω is an open Λ_p -regular cell in \mathbb{R}^k (cf. [6]). We will first recall the notion of Λ_p -regular mapping. Let $\psi : D \rightarrow \mathbb{R}^m$ be a mapping on an open subset $D \subset \mathbb{R}^n$. We say that ψ is Λ_p -regular (on D) if it is of class \mathcal{C}^p and there is a constant $C \geq 0$ such that

$$|D^{\varkappa} \psi(x)| \leq C/d(x, \partial D)^{|\varkappa|-1}, \quad \text{whenever } 1 \leq |\varkappa| \leq p \quad \text{and } x \in D.$$

Remark 3.1. — Let ψ be Λ_p -regular on D . Then

- (1) it is Λ_p -regular on every open $D' \subset D$;

- (2) if $A \subset \Omega$ is a 1-regular subset, then the restriction $\psi|_A$ is Lipschitz and thus it has a continuous extension $\bar{\psi}|_{\bar{A}}$.

We shall say (after [6]) that S is an *open Λ_p -regular (definable in a given o -minimal structure) cell* in \mathbb{R}^n iff

- (1) S is an open interval in \mathbb{R} , when $n = 1$;
- (2) $S = \{(x', x_n) : x' \in T, \psi_1(x') < x_n < \psi_2(x')\}$, where T is an open Λ_p -regular (definable) cell in \mathbb{R}^{n-1} and each ψ_i ($i = 1, 2$) is a function on T being either real Λ_p -regular (definable) function on T , or identically equal to $-\infty$, or identically equal to $+\infty$, and $\psi_1(x') < \psi_2(x')$, for all $x' \in T$, when $n > 1$.

Remark 3.2. — Such a cell S is 1-regular and if ψ_i is finite it is Lipschitz on T , thus it admits a continuous extension $\bar{\psi}_i$ to \bar{T} .

For any open (definable) Λ_p -regular cell in \mathbb{R}^n , one defines, by induction on n , a sequence $\rho_j : \bar{S} \rightarrow \mathbb{R} \cup \{+\infty\}$ ($j = 1, \dots, 2n$) of the *functions associated with the cell S* :

- (1) When $n = 1$ and $S = (a_1, a_2)$, we put

$$\rho_1(x) = \begin{cases} x - a_1, & \text{if } a_1 \in \mathbb{R} \\ +\infty, & \text{if } a_1 = -\infty \end{cases} \quad \text{and} \quad \begin{cases} \rho_2(x) = a_2 - x, & \text{if } a_2 \in \mathbb{R} \\ +\infty, & \text{if } a_2 = +\infty. \end{cases}$$

- (2) When $n > 1$ and $S = \{(x', x_n) : x' \in T, \psi_1(x') < x_n < \psi_2(x')\}$, let σ_j ($j = 1, \dots, 2n - 2$) be the functions associated with T . We put, for any $x = (x', x_n) \in \bar{S}$, $\rho_j(x) = \sigma_j(x')$ for $j = 1, \dots, 2n - 2$ and

$$\rho_{2n-1}(x) = \begin{cases} x_n - \bar{\psi}_1(x'), & \text{if } \psi_1 : T \rightarrow \mathbb{R} \\ +\infty, & \text{if } \psi_1 \equiv -\infty \end{cases} \quad \text{and}$$

$$\rho_{2n}(x) = \begin{cases} \bar{\psi}_2(x') - x_n, & \text{if } \psi_2 : T \rightarrow \mathbb{R} \\ +\infty, & \text{if } \psi_2 \equiv +\infty. \end{cases}$$

Remark 3.3 ([6], Lemma 3). — There exists a constant $\Theta > 0$ such that

$$\Theta \min_j \rho_j(x) \leq d(x, \partial S) \leq \min_j \rho_j(x), \quad \text{for } x \in \bar{S}.$$

(We adopt the convention: $d(x, \emptyset) = +\infty$.)

Remark 3.4 ([6], Lemma 4). — The functions ρ_j which are finite are Λ_p -regular on S , Lipschitz on \bar{S} and definable, if S is so.

LEMMA 3.5 (cf. [6], Lemma 5). — Let $\varphi_\nu : \Omega \rightarrow \mathbb{R}$ ($\nu = 1, \dots, m$) be Λ_p -regular functions on an open subset $\Omega \subset \mathbb{R}^k$. Assume that $r(u) :=$

$(\sum_{\nu=1}^m \varphi_\nu^2(u))^{\frac{1}{2}} \neq 0$ for each $u \in \Omega$. Then there exists a constant $\tilde{C} > 0$ such that for each $u \in \Omega$

$$\left| D^\alpha \left(\frac{1}{r} \right) (u) \right| \leq \frac{\tilde{C}}{r(u) \min(r(u), d(u, \partial\Omega))^{|\alpha|}}, \text{ where } 0 \leq |\alpha| \leq p;$$

consequently
$$\left| D^\alpha \left(\frac{1}{r} \right) (u) \right| \leq \frac{\tilde{C}}{\min(r(u), d(u, \partial\Omega))^{|\alpha|+1}}.$$

Proof. — Induction on $|\alpha|$. □

PROPOSITION 3.6 (cf. [6], Lemmas 6-7). — Let Ω be an open subset of \mathbb{R}^k , let $f \in C^p(\Omega \times \mathbb{R}^l)$ and $r \in C^p(\Omega)$, and let $t : \Omega \rightarrow (0, +\infty)$ be any positive function such that $t(u) \leq d(u, \partial\Omega)$ for any $u \in \Omega$. Let $\varepsilon > 0$ and put

$$\Delta_\varepsilon := \{(u, w) \in \Omega \times \mathbb{R}^l : |w| < \varepsilon t(u)\}.$$

Assume that there exists a constant $\tilde{C} > 0$ such that $|D^\alpha(\frac{1}{r})| \leq \frac{\tilde{C}}{t^{|\alpha|+1}}$, when $\alpha \in \mathbb{N}^k$, and for each $c \in \partial\Omega$, $D^\varkappa f(u, w) = o(t(u)^{p-|\varkappa|})$, when $\Delta_\varepsilon \ni (u, w) \rightarrow (c, 0)$ and $\varkappa \in \mathbb{N}^k \times \mathbb{N}^l$, $|\varkappa| \leq p$.

Let $\xi : \mathbb{R} \rightarrow \mathbb{R}$ be any C^p -function. Fix $i \in \{1, \dots, l\}$ and put

$$g(u, w) := \xi\left(\frac{w_i}{r(u)}\right) f(u, w), \text{ for } (u, w) \in \Omega \times \mathbb{R}^l.$$

Then for each $c \in \partial\Omega$, $D^\varkappa g(u, w) = o(t(u)^{p-|\varkappa|})$, when $\Delta_\varepsilon \ni (u, w) \rightarrow (c, 0)$ and $\varkappa \in \mathbb{N}^k \times \mathbb{N}^l$, $|\varkappa| \leq p$.

Proof. — Put $h(u, w) = \xi\left(\frac{w_i}{r(u)}\right)$. By the Leibniz formula

$D^\varkappa g = \sum_{\lambda \leq \varkappa} \binom{\varkappa}{\lambda} D^\lambda h D^{\varkappa-\lambda} f$, so it suffices to check that there exists a constant $C'_\varepsilon > 0$ such that $|D^\lambda h(u, w)| \leq C'_\varepsilon t(u)^{-|\lambda|}$, when $(u, w) \in \Delta_\varepsilon$ and $|\lambda| \leq p$. First, it is easy to see this for $h_0(u, w) := \frac{w_i}{r(u)}$ using Lemma 3.5.

Then for $h = \xi \circ h_0$ we have

$$\frac{\partial h}{\partial x_j} = (\xi' \circ h_0) \frac{\partial h_0}{\partial x_j}, \text{ where } (x_1, \dots, x_n) = (u_1, \dots, u_k, w_1, \dots, w_l)$$

and $D^\lambda \left(\frac{\partial h}{\partial x_j} \right) = \sum_{\mu \leq \lambda} \binom{\lambda}{\mu} D^\mu (\xi' \circ h_0) D^{\lambda-\mu} \left(\frac{\partial h_0}{\partial x_j} \right)$, if $|\lambda| \leq p - 1$, so we conclude by induction. □

Remark 3.7. — Suppose that f is a \mathcal{C}^p -function on the whole space $\mathbb{R}^k \times \mathbb{R}^l$ and such that for each $c \in \partial\Omega$, $D^\varkappa f(u, 0) = o(t(u)^{p-|\varkappa|})$, when $\Omega \ni u \rightarrow c$ and $\varkappa \in \mathbb{N}^k \times \mathbb{N}^l, |\varkappa| \leq p$.

Then for each $c \in \partial\Omega$, $D^\varkappa f(u, w) = o(t(u)^{p-|\varkappa|})$, when $\Delta_\varepsilon \ni (u, w) \rightarrow (c, 0)$ and $\varkappa \in \mathbb{N}^k \times \mathbb{N}^l, |\varkappa| \leq p$. This follows immediately from the Taylor formula

$$D^\varkappa f(u, w) = \sum_{|\lambda| \leq p-|\varkappa|} \frac{1}{\lambda!} D^{\varkappa+(0,\lambda)} f(u, 0) w^\lambda + o(|w|^{p-|\varkappa|}),$$

when $u \rightarrow c, w \rightarrow 0$.

Let now Ω be an open Λ_p -regular cell in \mathbb{R}^k and ρ_j ($j = 1, \dots, 2k$) - the functions associated with Ω . We define an extension operator

$$\mathcal{L} : \mathcal{E}^p(\bar{\Omega} \times 0, \partial\Omega \times 0) \longrightarrow \mathcal{C}^p(\mathbb{R}^n), \quad \text{where } \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^l,$$

by the following formula

$$\mathcal{L}F(u, w) = \begin{cases} \prod_{i=1}^l \prod_{j=1}^{2k} \xi\left(Q \frac{w_i}{\rho_j(u)}\right) (L\tilde{F})(u, w), & \text{if } u \in \Omega \\ 0, & \text{if } u \in \mathbb{R}^k \setminus \Omega, \end{cases}$$

where Q is any real number $> \sqrt{l}\Theta^{-1}$, Θ is a constant from Remark 3.3 and $\xi : \mathbb{R} \rightarrow \mathbb{R}$ is a (definable, if we wish) \mathcal{C}^p -function equal to 1 in a neighborhood of 0, and equal to 0 outside the open interval $(-1, 1)$.

To check that $\mathcal{L}F \in \mathcal{C}^p(\mathbb{R}^n)$ we use repeatedly Proposition 3.6 with $r = \rho_j \neq +\infty$ and $t(u) = d(u, \partial\Omega)$ (at the beginning we take $f = L\tilde{F}$ as in Remark 3.7) and the Hestenes Lemma. The factors involving $\rho_j \equiv +\infty$ being obviously 1 can be omitted in the above formula.

Observe that if ε is any constant from $(0, 1)$, we can choose Q in such a way that $\mathcal{L}F$ is p -flat outside the set

$$\begin{aligned} \Delta_\varepsilon(\Omega \times 0) &:= \{x \in \mathbb{R}^n : d(x, \bar{\Omega} \times 0) < \varepsilon d(x, \partial\Omega \times 0)\} \\ &= \{(u, w) \in \Omega \times \mathbb{R}^l : |w| < \frac{\varepsilon}{\sqrt{1-\varepsilon^2}} d(u, \partial\Omega)\}. \end{aligned}$$

Remark 3.8. — If r and t are as in Proposition 3.6 and $F \in \mathcal{E}^p(\bar{\Omega} \times 0, \partial\Omega \times 0)$ is such that, for each $c \in \partial\Omega$, $F^\varkappa(u, 0) = o(t(u)^{p-|\varkappa|})$, when $\Omega \ni u \rightarrow c$ and $|\varkappa| \leq p$, the above formula for an extension of F can be modified by putting

$$\mathcal{L}'F(u, w) = \begin{cases} \prod_{i=1}^l \xi\left(\sqrt{l} \frac{w_i}{r(u)}\right) \mathcal{L}F(u, w), & \text{if } u \in \Omega \\ 0, & \text{if } u \in \mathbb{R}^k \setminus \Omega. \end{cases}$$

Then $\mathcal{L}'F$ is p -flat, outside the neighborhood $\{(u, w) \in \Omega \times \mathbb{R}^l : |w| < r(u)\}$ of $\Omega \times 0$ and outside $\Delta_\varepsilon(\Omega \times 0)$.

In order that $\mathcal{L}F$ (or $\mathcal{L}'F$) and F have the same (up to a multiplicative constant) modulus of continuity we will prove the following

PROPOSITION 3.9. — *Under the assumptions of Proposition 3.6 assume additionally that Ω is 1-regular, $r \in \mathcal{C}^{p+1}(\Omega)$ such that*

$$|D^\alpha \left(\frac{1}{r}\right)| \leq \frac{\tilde{c}}{t^{|\alpha|+1}}, \quad \text{when } \alpha \in \mathbb{N}^k, |\alpha| \leq p + 1$$

and t is Lipschitz. Then there exists a constant $M > 0$ such that if ω is a modulus of continuity for f on Δ_ε satisfying

$$|D^\varkappa f(u, w)| \leq \omega(t(u))t(u)^{p-|\varkappa|},$$

when $(u, w) \in \Delta_\varepsilon$ and $|\varkappa| \leq p$, then $M\omega$ is a modulus of continuity for g on Δ_ε satisfying

$$|D^\varkappa g(u, w)| \leq M\omega(t(u))t(u)^{p-|\varkappa|},$$

when $(u, w) \in \Delta_\varepsilon$ and $|\varkappa| \leq p$.

Proof. — In view of the proof of Proposition 3.6, it suffices to check that, for a constant $M > 0$, $M\omega$ is a modulus of continuity for g on Δ_ε . First observe that Δ_ε is 1-regular, because Ω is so and the function t is Lipschitz. There exists a constant $C \geq 1$ such that $|t(u_1) - t(u_2)| \leq C|u_1 - u_2|$, for any $u_1, u_2 \in \Omega$.

Fix any $\varkappa \in \mathbb{N}^{k+l}$ such that $|\varkappa| = p$, any $\lambda \leq \varkappa$ and any two points $x_i = (u_i, w_i) \in \Delta_\varepsilon$ ($i = 1, 2$). We have to estimate

$$|D^\lambda h(x_1)D^{\varkappa-\lambda} f(x_1) - D^\lambda h(x_2)D^{\varkappa-\lambda} f(x_2)|.$$

Case I: $t(u_i) \leq 2C|x_1 - x_2|$ ($i = 1, 2$).

$$\begin{aligned} \text{Then } |D^\lambda h(x_i)D^{\varkappa-\lambda} f(x_i)| &\leq C'_\varepsilon t(u_i)^{-|\lambda|} \omega(t(u_i))t(u_i)^{p-|\varkappa-\lambda|} \\ &\leq C'_\varepsilon \omega(2C|x_1 - x_2|) \leq 2CC'_\varepsilon \omega(|x_1 - x_2|). \end{aligned}$$

Case II: $t(u_1) > 2C|x_1 - x_2|$.

$$\text{Then } |u_1 - u_2| \leq C|x_1 - x_2| < \frac{1}{2}t(u_1) \leq \frac{1}{2}d(u_1, \Omega); \text{ thus } [x_1, x_2] \subset \Omega \times \mathbb{R}^l.$$

$$\text{We have } |D^\lambda h(x_1)[D^{\varkappa-\lambda} f(x_1) - D^{\varkappa-\lambda} f(x_2)]| \leq |D^\lambda h(x_1)| \times$$

$$\left[\sum_{1 \leq |\mu| \leq p-|\varkappa-\lambda|} \frac{1}{\mu!} |D^{\varkappa-\lambda+\mu} f(x_1)| |x_1 - x_2|^{|\mu|} + \omega(|x_1 - x_2|) |x_1 - x_2|^{p-|\varkappa-\lambda|} \right] \leq$$

$$M_1 \omega(t(u_1))t(u_1)^{-1} |x_1 - x_2| + M_2 \omega(|x_1 - x_2|) \leq M' \omega(|x_1 - x_2|),$$

where M_1, M_2 and M' are positive constants and we use: $\omega(s)t \leq \omega(t)s$ if $t \leq s$.

On the other hand $[[D^\lambda h(x_1) - D^\lambda h(x_2)]D^{\varkappa-\lambda} f(x_2)] \leq$

$$\sup_{x \in [x_1, x_2]} \sum_{j=1}^{k+l} |D^{\lambda+(j)} h(x)| |x_1 - x_2| |D^{\varkappa-\lambda} f(x_2)|.$$

For any $x = (u, w) \in [x_1, x_2]$, $2|t(u_1) - t(u)| \leq 2C|u_1 - u| \leq 2C|x_1 - x_2| < t(u_1)$ and $2|w_1 - w| \leq 2C|x_1 - x_2| < t(u_1)$; thus $\frac{1}{2}t(u_1) < t(u) < \frac{3}{2}t(u_1)$ and $|w| \leq |w_1| + |w_1 - w| < \varepsilon t(u_1) + t(u) \leq (2\varepsilon + 1)t(u)$.

Consequently $x \in \Delta_{2\varepsilon+1}$ and

$$|D^{\lambda+(j)} h(x)| \leq C'_{2\varepsilon+1} t(u)^{-|\lambda|-1} \leq 2^{|\lambda|+1} C'_{2\varepsilon+1} t(u_1)^{-|\lambda|-1}$$

and

$$|D^{\varkappa-\lambda} f(x_2)| \leq \omega(t(u_2))t(u_2)^{|\lambda|} \leq \left(\frac{3}{2}\right)^{|\lambda|+1} \omega(t(u_1))t(u_1)^{|\lambda|}.$$

The needed inequality follows. □

Remark 3.10. — Suppose that f is a \mathcal{C}^p -function on the whole space $\mathbb{R}^k \times \mathbb{R}^l$ and ω is its modulus of continuity such that

$$|D^{\varkappa} f(u, 0)| \leq \omega(t(u))t(u)^{p-|\varkappa|},$$

when $u \in \Omega$ and $\varkappa \in \mathbb{N}^{k+l}$, $|\varkappa| \leq p$.

Then there exists a constant $M'' > 0$ such that

$$|D^{\varkappa} f(u, w)| \leq M'' \omega(t(u))t(u)^{p-|\varkappa|},$$

when $(u, w) \in \Delta_\varepsilon$, and $\varkappa \in \mathbb{N}^{k+l}$, $|\varkappa| \leq p$.

Indeed, this follows immediately from

$$|D^{\varkappa} f(u, w) - \sum_{|\lambda| \leq p-|\varkappa|} \frac{1}{\lambda!} D^{\varkappa+(0,\lambda)} f(u, 0)w^\lambda| \leq \omega(|w|)|w|^{p-|\varkappa|}.$$

Remark 3.11. — If Ω is an open Λ_{p+1} -regular cell in \mathbb{R}^k and ξ is a \mathcal{C}^{p+1} -function, then there exists a positive constant M , such that, for any $F \in \mathcal{E}^p(\bar{\Omega} \times 0, \partial\Omega \times 0)$ (respectively, fulfilling additional conditions: $|F^{\varkappa}(u, 0)| \leq \omega(r(u))r(u)^{p-|\varkappa|}$, when $u \in \Omega$, $\varkappa \in \mathbb{N}^{k+l}$, $|\varkappa| \leq p$) if ω is a modulus of continuity for F , then $M\omega$ is a modulus of continuity for $\mathcal{L}F$ (respectively, for $\mathcal{L}'F$).

4. A generalization to the ideal of C^p -Whitney fields on the closure of a Λ_p -regular leaf p -flat on its boundary

Now we will transpose the extension operator \mathcal{L} to the closure of any Λ_p -regular leaf. A subset $E \subset \mathbb{R}^n$ is called a (definable) Λ_p -regular leaf of dimension k in \mathbb{R}^n if it is the graph $E = \{(u, \varphi(u)) : u \in \Omega\}$ of a (definable) Λ_p -regular mapping $\varphi : \Omega \rightarrow \mathbb{R}^l$ defined on an open (definable) Λ_p -regular cell Ω in \mathbb{R}^k . A reduction of this case to the previous one will be by the following Lipschitz automorphism

$$\bar{\Omega} \times \mathbb{R}^l \ni (u, w) \mapsto (u, w + \bar{\varphi}(u)) \in \bar{\Omega} \times \mathbb{R}^l$$

and the following

PROPOSITION 4.1 (cf. [6], Proposition 3). — Let $\varphi : \Omega \rightarrow \mathbb{R}^l$ be a Λ_p -regular mapping defined on an open subset $\Omega \subset \mathbb{R}^k$. Let $t : \Omega \rightarrow (0, +\infty)$ be any function such that $t(u) \leq d(u, \partial\Omega)$, for each $u \in \Omega$. Let E be any closed subset of $\Omega \times \mathbb{R}^l$ and

$$F(u, w; U, W) = \sum_{|\alpha|+|\beta| \leq p} \frac{1}{\alpha! \beta!} F^{(\alpha, \beta)}(u, w) U^\alpha W^\beta \quad \begin{cases} U = (U_1, \dots, U_k), \\ W = (W_1, \dots, W_l) \end{cases}$$

a C^p -Whitney field on E such that, for any $c \in \partial\Omega$

$$F^{(\alpha, \beta)}(u, w) = o(t(u)^{p-|\alpha|-|\beta|}), \text{ when } u \rightarrow c \text{ and } |\alpha| + |\beta| \leq p.$$

Let $F_\varphi(u, v; U, V)$ be a polynomial in (U, V) of degree $\leq p$ such that

$$F_\varphi(u, v; U, V) = \sum_{|\alpha|+|\beta| \leq p} \frac{1}{\alpha! \beta!} F^{(\alpha, \beta)}(u, v + \varphi(u)) U^\alpha (V + \sum_{1 \leq \varkappa \leq p} \frac{1}{\varkappa!} D^\varkappa \varphi(u) U^\varkappa)^\beta \text{ mod}(U, V)^{p+1}$$

defined for $(u, v) \in E_\varphi$, where $E_\varphi = \{(u, v) \in \Omega \times \mathbb{R}^l : (u, v + \varphi(u)) \in E\}$.

Then F_φ is a C^p -Whitney field on E_φ such that, for any $c \in \partial\Omega$

$$F_\varphi^{(\alpha, \beta)}(u, v) = o(t(u)^{p-|\alpha|-|\beta|}), \text{ when } u \rightarrow c \text{ and } |\alpha| + |\beta| \leq p.$$

Proof. — It is easy to check that F_φ fulfills the condition (**) from Introduction, thus it is a C^p -Whitney field on E_φ . Besides

$$F_\varphi(u, v; U, V) = \sum_{|\alpha|+|\beta| \leq p} \frac{1}{\alpha! \beta!} F^{(\alpha, \beta)}(u, v + \varphi(u)) U^\alpha \times \sum_{\gamma + \sum_{\varkappa} \delta_{\varkappa} = \beta} \frac{\beta!}{\gamma! \prod \delta_{\varkappa}!} V^\gamma \prod_{\varkappa} \left[\frac{1}{\varkappa!^{|\delta_{\varkappa}|}} U^{|\delta_{\varkappa}| \varkappa} (D^\varkappa \varphi(u))^{\delta_{\varkappa}} \right] \text{ mod}(U, V)^{p+1},$$

thus

$$F_\varphi^{(\sigma, \gamma)}(u, v) = \sum_{\alpha + \sum_{\varkappa} |\delta_{\varkappa}| \varkappa = \sigma} [\cdot] F^{(\alpha, \gamma + \sum_{\varkappa} \delta_{\varkappa})}(u, v + \varphi(u)) \prod_{\varkappa} (D^{\varkappa} \varphi(u))^{\delta_{\varkappa}},$$

where $[\cdot]$ denotes constants. To conclude notice that

$$\begin{aligned} & F^{(\alpha, \gamma + \sum_{\varkappa} \delta_{\varkappa})}(u, v + \varphi(u)) \prod_{\varkappa} (D^{\varkappa} \varphi(u))^{\delta_{\varkappa}} = \\ & o(1) t(u)^{p - |\alpha| - |\gamma| - \sum_{\varkappa} |\delta_{\varkappa}|} C \prod_{\varkappa} d(u, \partial\Omega)^{-|\delta_{\varkappa}| \varkappa + |\delta_{\varkappa}|} = \\ & o(t(u)^{p - |\sigma| - |\gamma|}). \end{aligned}$$

□

Remark 4.2. — If $E = \{(u, \varphi(u)) : u \in \Omega\}$ (resp. $E = \Omega \times \mathbb{R}^l$), then F_φ extends to a C^p -Whitney field on $\overline{E}_\varphi = \overline{\Omega} \times 0$ (resp. $\overline{E}_\varphi = \overline{\Omega} \times \mathbb{R}^l$) p -flat on $\partial E_\varphi = \partial\Omega \times 0$ (resp. $\partial E_\varphi = \partial\Omega \times \mathbb{R}^l$).

Proof. — The both cases follow from the Hestenes Lemma. □

PROPOSITION 4.3. — Under the assumptions of Proposition 4.1, assume additionally that the mapping φ is Λ_{p+1} -regular, E and Ω are both 1-regular and \overline{E} and $\partial\Omega \times \mathbb{R}^l$ are simply separated (*). Then there exists a constant $M > 0$ such that, for each $F \in \mathcal{E}^p(\overline{E}, \partial E)$, if ω is a modulus of continuity of F , then $M\omega$ is a modulus of continuity of F_φ .

Moreover, if $|F^{\varkappa}(u, w)| \leq \omega(t(u)) t(u)^{p - |\varkappa|}$, when $(u, w) \in E$ and $|\varkappa| \leq p$, then $|F_\varphi^{\varkappa}(u, v)| \leq M\omega(t(u)) t(u)^{p - |\varkappa|}$, when $(u, v) \in E_\varphi$ and $|\varkappa| \leq p$.

Proof. — Observe that E_φ is 1-regular. Let $\sigma \in \mathbb{N}^k, \gamma \in \mathbb{N}^l$ be such that $|\sigma| + |\gamma| = p$ and let $(u_i, v_i) \in E_\varphi, (i = 1, 2)$. We have to estimate

$$\begin{aligned} & |F_\varphi^{(\sigma, \gamma)}(u_1, v_1) - F_\varphi^{(\sigma, \gamma)}(u_2, v_2)| \leq \\ & \sum_{\alpha + \sum_{\varkappa} |\delta_{\varkappa}| \varkappa = \sigma} [\cdot] |F^{(\alpha, \gamma + \sum_{\varkappa} \delta_{\varkappa})}(u_1, v_1 + \varphi(u_1)) \prod_{\varkappa} (D^{\varkappa} \varphi(u_1))^{\delta_{\varkappa}} - \\ & F^{(\alpha, \gamma + \sum_{\varkappa} \delta_{\varkappa})}(u_2, v_2 + \varphi(u_2)) \prod_{\varkappa} (D^{\varkappa} \varphi(u_2))^{\delta_{\varkappa}}|. \end{aligned}$$

Fix $\lambda = (\alpha, \gamma + \sum_{\varkappa} \delta_{\varkappa})$ and put $x_i = (u_i, v_i + \varphi(u_i))$ and

$$\theta(u) = \prod_{\varkappa} (D^{\varkappa} \varphi(u))^{\delta_{\varkappa}}.$$

(*) See the beginning of Section 5 for the definition of simple separation.

Case I: $|x_1 - x_2| \geq \frac{1}{2}d(u_i, \partial\Omega)$ for $i = 1, 2$.

$$\begin{aligned} |F^\lambda(x_i)\theta(u_i)| &\leq \omega(d(x_i, \partial E))d(x_i, \partial E)^{p-|\lambda|}|\theta(u_i)| \leq \\ &\omega(Cd(u_i, \partial\Omega))[Cd(u_i, \partial\Omega)]^{p-|\lambda|}|\theta(u_i)| \leq \\ \omega(2C|x_1 - x_2|)[Cd(u_i, \partial\Omega)]^{p-|\lambda|} \prod_{z \in \mathcal{X}} d(u_i, \partial\Omega)^{-|\delta_{**}||z|+|\delta_{**}|} &\leq M\omega(|x_1 - x_2|). \end{aligned}$$

Case II: $|x_1 - x_2| \leq \frac{1}{2}d(u_1, \partial\Omega)$.

$$\begin{aligned} &|F^\lambda(x_1)\theta(u_1) - F^\lambda(x_2)\theta(u_2)| \leq \\ &|F^\lambda(x_1) - F^\lambda(x_2)||\theta(u_2)| + |F^\lambda(x_1)||\theta(u_1) - \theta(u_2)| \leq \\ &\left[\sum_{1 \leq |\mu| \leq p-|\lambda|} \frac{1}{\mu!} |F^{\lambda+\mu}(x_1)||x_2 - x_1|^{|\mu|} + \omega(|x_1 - x_2|)|x_1 - x_2|^{p-|\lambda|} \right] |\theta(u_2)| + \\ &|F^\lambda(x_1)| \sup_{z \in [u_1, u_2]} \sum_{j=1}^k |D^{(j)}\theta(z)||u_1 - u_2| \leq \\ &\left[\sum_{1 \leq |\mu| \leq p-|\lambda|} \frac{1}{\mu!} \omega(d(x_1, \partial E))d(x_1, \partial E)^{p-|\lambda|-|\mu|} |x_1 - x_2|d(u_1, \partial\Omega)^{|\mu|-1} + \right. \\ &\quad \left. \omega(|x_1 - x_2|)d(u_1, \partial\Omega)^{p-|\lambda|} \right] |\theta(u_2)| + \\ &\omega(d(x_1, \partial\Omega))|x_1 - x_2| \sup_{z \in [u_1, u_2]} \sum_{j=1}^k |D^{(j)}\theta(z)| \\ &\left[C_1\omega(d(u_1, \partial\Omega))|x_1 - x_2|d(u_1, \partial\Omega)^{p-|\lambda|-1} + \right. \\ &\quad \left. \omega(|x_1 - x_2|)d(u_1, \partial\Omega)^{p-|\lambda|} \right] |\theta(u_2)| + \\ &C_2\omega(d(u_1, \partial\Omega))|x_1 - x_2| \sup_{z \in [u_1, u_2]} \prod_{z \in \mathcal{X}} d(z, \partial\Omega)^{-|\delta_{**}||z|+|\delta_{**}|-1}. \end{aligned}$$

Now it suffices to observe that $\omega(d(u_1, \partial\Omega))|x_1 - x_2| \leq \omega(|x_1 - x_2|)d(u_1, \partial\Omega)$ and $d(z, \partial\Omega) \geq d(u_1, \partial\Omega) - |z - u_1| \geq d(u_1, \partial\Omega) - |x_1 - x_2| \geq \frac{1}{2}d(u_1, \partial\Omega)$, if $z \in [u_1, u_2]$. □

Assume now that $E = \{(u, \varphi(u)) : u \in \Omega\}$ is a Λ_p -regular leaf of dimension k in \mathbb{R}^n . We define an extension operator $\mathcal{L} : \mathcal{E}^p(\bar{E}, \partial E) \rightarrow \mathcal{C}^p(\mathbb{R}^n)$ by the formula

$$\mathcal{L}F = \begin{cases} (\mathcal{L}F_\varphi)_{-\varphi}, & \text{on } \Omega \times \mathbb{R}^l \\ 0, & \text{on } (\mathbb{R}^k \setminus \Omega) \times \mathbb{R}^l. \end{cases}$$

For any constant $\varepsilon > 0$, we can specify this operator in such a way that for each $F \in \mathcal{E}^p(\overline{E}, \partial E)$, $\mathcal{L}F$ is flat outside the neighborhood $\Delta_\varepsilon(E) := \{x \in \mathbb{R}^n : d(x, E) < \varepsilon d(x, \partial E)\}$.

5. A generalization to a finite tower of Λ_p -regular leaves

Here we will generalize the extension operator \mathcal{L} to the ideal $\mathcal{E}^p(\overline{E}, \partial E)$, where E is a finite disjoint union $E = E_1 \cup \dots \cup E_s$ of graphs of Λ_p -regular mappings $\varphi_\sigma : \Omega \rightarrow \mathbb{R}^l$ ($\sigma = 1, \dots, s$) defined on a common open Λ_p -regular cell $\Omega \subset \mathbb{R}^k$. Put $r_\sigma(u) := |\varphi_\sigma(u) - \varphi_s(u)|$ for $\sigma = 1, \dots, s - 1$ and $u \in \Omega$.

We first define $\mathcal{L}F$ for any $F \in \mathcal{E}^p(\overline{E}, \overline{E}_1 \cup \dots \cup \overline{E}_{s-1} \cup \partial E_s)$.

Then we put

$$\mathcal{L}F = \begin{cases} \left[\prod_{\sigma=1}^{s-1} \prod_{i=1}^l \xi \left(\sqrt{l} \frac{w_i}{r_\sigma(u)} \right) \mathcal{L}((F|_{\overline{E}_s})_{\varphi_s}) \right]_{-\varphi_s}, & \text{on } \Omega \times \mathbb{R}^l \\ 0, & \text{on } (\mathbb{R}^k \setminus \Omega) \times \mathbb{R}^l, \end{cases}$$

which gives an extension operator according to Proposition 3.6 (used repeatedly with $t(u) := \min(\{r_\sigma(u)\}, d(u, \partial\Omega))$), Remark 3.8 and Proposition 4.1.

Let now consider a general case where F is any element of $\mathcal{E}^p(\overline{E}, \partial E)$. Proceeding by induction, assume that $\mathcal{L}(F|_{\overline{E}_1 \cup \dots \cup \overline{E}_{s-1}})$ has already been defined. Then $H := F - T\mathcal{L}(F|_{\overline{E}_1 \cup \dots \cup \overline{E}_{s-1}})|_{\overline{E}} \in \mathcal{E}^p(\overline{E}, \overline{E}_1 \cup \dots \cup \overline{E}_{s-1} \cup \partial E_s)$ and we put

$$\mathcal{L}F = \mathcal{L}H + \mathcal{L}(F|_{\overline{E}_1 \cup \dots \cup \overline{E}_{s-1}}).$$

For any $\varepsilon > 0$, we can specify this operator in such a way that $\mathcal{L}F$ is p -flat outside the set $\Delta_\varepsilon(E) := \{x \in \mathbb{R}^n : d(x, E) < \varepsilon d(x, \partial E)\}$.

6. Extension operator for a closed definable subset of \mathbb{R}^n

DEFINITION 6.1 (cf. [10]). — Let $A, B, Z \subset \mathbb{R}^n$. We say that A and B are simply Z -separated if one of the following equivalent conditions holds

- (1) $\exists M > 0 \forall x \in A, \quad d(x, B) \geq Md(x, Z)$;
- (2) $\exists C > 0 \forall x \in \mathbb{R}^n, \quad d(x, A) + d(x, B) \geq Cd(x, Z)$. (If (1) holds, one can take $C = M/(M + 1)$.)

We say that A and B are simply separated if they are simply $A \cap B$ -separated.

PROPOSITION 6.2. — Let $E_i \supset E'_i$ ($i = 1, \dots, s$) be closed subsets of \mathbb{R}^n and let $C > 0$ be a constant such that, for any $i, j \in \{1, \dots, s\}, i \neq j$ and any $x \in \mathbb{R}^n$

$$d(x, E_i) + d(x, E_j) \geq Cd(x, E'_i).$$

Let $\varepsilon \in (0, C/2]$. Put $\Gamma_\varepsilon(E_i, E'_i) := \{x \in \mathbb{R}^n : d(x, E_i) < \varepsilon d(x, E'_i)\}$. Suppose that, for each $i = 1, \dots, s$

$$\mathcal{L}_i : \mathcal{E}^p(E_i, E'_i) \longrightarrow \mathcal{C}^p(\mathbb{R}^n)$$

is an extension operator such that $\mathcal{L}_i F$ is p -flat outside $\Gamma_\varepsilon(E_i, E'_i)$, for any $F \in \mathcal{E}^p(E_i, E'_i)$.

Then the formula

$$\mathcal{L}F = \sum_{i=1}^s \mathcal{L}_i(F|E_i)$$

defines an extension operator $\mathcal{L} : \mathcal{E}^p(\bigcup_i E_i, \bigcup_i E'_i) \longrightarrow \mathcal{C}^p(\mathbb{R}^n)$. Moreover, if each \mathcal{L}_i preserves (up to a multiplicative constant) a modulus of continuity, then \mathcal{L} has the same property.

Proof. — It suffices to check that $\Gamma_\varepsilon(E_i, E'_i) \cap \Gamma_\varepsilon(E_j, E'_j) = \emptyset$, if $i \neq j$. If there were $x \in \Gamma_\varepsilon(E_i, E'_i) \cap \Gamma_\varepsilon(E_j, E'_j)$, then

$$2\varepsilon[d(x, E'_i) + d(x, E'_j)] > 2[d(x, E_i) + d(x, E_j)] \geq C[d(x, E'_i) + d(x, E'_j)],$$

a contradiction. □

A proof of the following theorem will be given in the next section.

Λ_p -REGULAR DECOMPOSITION THEOREM 6.3. — Let E be a closed subset of \mathbb{R}^n definable in some fixed o-minimal structure on the ordered field of the real numbers \mathbb{R} . Let $k = \dim E$. Let Z be any definable subset of E of dimension $< k$.

Then there exists a finite decomposition

$$E = M_1 \cup \dots \cup M_s \cup A$$

such that each M_i is a finite tower of Λ_p -regular k -dimensional definable leaves in an appropriate linear coordinate system, A is a closed definable subset of $\dim < k$ containing Z and, for any $i, j \in \{1, \dots, s\}$ ($i \neq j$), \overline{M}_i and \overline{M}_j are simply ∂M_i -separated and, for any i , \overline{M}_i and A are simply ∂M_i -separated.

In order to define an extension operator for any closed definable subset $E \subset \mathbb{R}^n$ we will use induction on $\dim E$. By the induction hypothesis we have an extension operator

$$\mathcal{L}_0 : \mathcal{E}^p(\cup_{i=1}^s \partial M_i \cup A) \longrightarrow \mathcal{C}^p(\mathbb{R}^n),$$

and by Section 5 combined with Proposition 6.2 we have an extension operator

$$\mathcal{L}_1 : \mathcal{E}^p(E, \cup_{i=1}^s \partial M_i \cup A) \longrightarrow \mathcal{C}^p(\mathbb{R}^n).$$

Now an extension operator for E is defined by the formula

$$\mathcal{L}F = \mathcal{L}_1[F - T\mathcal{L}_0(F|_{\cup_i \partial M_i \cup A})|_E] + \mathcal{L}_0(F|_{\cup_i \partial M_i \cup A}).$$

7. Proof of Λ_p -regular Decomposition Theorem

Let $P \subset \mathbb{R}^n$ be any definable subset and V - a linear subspace of \mathbb{R}^n of dimension $n - k$. Following [10], we will say that P is *perfectly situated relative to V* if, for a/any linear complement W of V in \mathbb{R}^n , P can be represented as a disjoint union

$$P = \bigcup \{ \hat{\varphi} : \varphi \in \mathcal{F} \}$$

of a finite family \mathcal{F} of definable \mathcal{C}^1 -mappings $\varphi : \Delta_\varphi \longrightarrow V$ defined on connected \mathcal{C}^1 -submanifolds $\Delta_\varphi \subset W$ and with bounded derivatives ($\hat{\varphi}$ stands here for the graph $\{u + \varphi(u) : u \in \Delta_\varphi\}$ of φ).

We will use the following

THEOREM 7.1 (cf. [10], Theorem 0). — *Let $\Sigma = \{ \sigma \subset \{1, \dots, n\} : \text{card } \sigma = n - k \} = \{ \sigma_1, \dots, \sigma_q \}$, where $q = \binom{n}{k}$.*

Let $V_i = \bigoplus_{\nu \in \sigma_i} \mathbb{R}e_\nu$ ($i = 1, \dots, q$), where e_1, \dots, e_n is the canonical basis in \mathbb{R}^n .

Any definable closed subset $E \subset \mathbb{R}^n$ of dimension k is a union $E = \bigcup_{i=1}^q E_i$ of definable closed subsets E_i such that, for each i , E_i is perfectly situated relative to V_i and, for each $j \neq i$, E_i and E_j are simply separated and $\dim(E_i \cap E_j) < k$.

From the last theorem and easy properties of simply separated sets (see [10], Proposition 2; (1) and (3)), it follows that it suffices to prove Λ_p -regular Decomposition Theorem for each E_i and $Z_i = (Z \cap E_i) \cup (\bigcup_{j \neq i} E_i \cap E_j)$ separately, therefore - up to a permutation of variables - it suffices to prove it assuming that E is perfectly situated relative to $0 \times \mathbb{R}^l$, where $l = n - k$. The proof in this case is based on the following two propositions.

PROPOSITION 7.2 ([6], Proposition 2). — *If $\varphi : \Omega \rightarrow \mathbb{R}^l$ is a definable Λ_1 -regular mapping defined on an open $\Omega \subset \mathbb{R}^k$, then there exists a closed definable subset Z of Ω such that $\dim Z < k$ and $\varphi|_{\Omega \setminus Z}$ is Λ_p -regular mapping on $\Omega \setminus Z$.*

PROPOSITION 7.3 ([6], Proposition 4). — *For any definable open subset $\Omega \subset \mathbb{R}^k$, there exists a finite family \mathcal{S} of disjoint subsets of Ω such that $\dim(\Omega \setminus \bigcup \mathcal{S}) < k$ and each $S \in \mathcal{S}$ is an open definable Λ_p -regular cell in an appropriate linear system of coordinates in \mathbb{R}^k .*

Proof of Proposition 7.3. — See [6], Proposition 4, where the set is assumed bounded, but this assumption is not essential. Alternatively, first one can apply [10]; Theorem 1, (B_k) to get the case $p = 1$ of Proposition 7.3, which is the theorem of Kurdyka [5] and Parusiński [9], and then by induction on k one gets the case of any $p \geq 1$, applying Proposition 7.2. \square

To finish the proof of the theorem, first represent E as union of graphs with bounded derivatives:

$$E = \bigcup \{ \hat{\varphi} : \varphi \in \mathcal{F} \},$$

as in the beginning of the section. Adding to Z all the graphs with $\dim \Delta_\varphi < k$, one can assume that

$$E = Z \cup \bigcup \{ \hat{\varphi} : \varphi \in \mathcal{F}_* \},$$

where $\mathcal{F}_* = \{ \varphi \in \mathcal{F} : \Delta_\varphi \text{ non-empty open in } \mathbb{R}^k \}$. By Proposition 7.2, for each $\varphi \in \mathcal{F}_*$ there exists a closed definable subset K_φ of Δ_φ of $\dim < k$ such that $\varphi|_{\Delta_\varphi \setminus K_\varphi}$ is Λ_p -regular. Let

$$\Theta := \overline{\pi(Z)} \cup \bigcup \{ \partial \Delta_\varphi \cup K_\varphi : \varphi \in \mathcal{F}_* \},$$

where $\pi : \mathbb{R}^k \times \mathbb{R}^l \rightarrow \mathbb{R}^k$ is the canonical projection. Take a family \mathcal{S} as in Proposition 7.3 for the open subset

$$\Omega := \bigcup \{ \Delta_\varphi : \varphi \in \mathcal{F}_* \} \setminus \Theta.$$

Now it suffices to define, for each $S \in \mathcal{S}$

$$M_S := E \cup \pi^{-1}(S) \quad \text{and} \quad A := E \setminus \bigcup \{ M_S : S \in \mathcal{S} \}.$$

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