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THE LERAY MEASURE OF NODAL SETS FOR RANDOM EIGENFUNCTIONS ON THE TORUS

by Ferenc ORAVECZ, Zeév RUDNICK & Igor WIGMAN

Abstract. — We study nodal sets for typical eigenfunctions of the Laplacian on the standard torus in $d \geqslant 2$ dimensions. Making use of the multiplicities in the spectrum of the Laplacian, we put a Gaussian measure on the eigenspaces and use it to average over the eigenspace. We consider a sequence of eigenvalues with growing multiplicity $\mathcal{N} \to \infty$.

The quantity that we study is the Leray, or microcanonical, measure of the nodal set. We show that the expected value of the Leray measure of an eigenfunction is constant, equal to $1/\sqrt{2\pi}$. Our main result is that the variance of Leray measure is asymptotically $1/4\pi\mathcal{N}$, as $\mathcal{N} \to \infty$, at least in dimensions $d = 2$ and $d \geqslant 5$

Résumé. — Nous étudions les ensembles nodaux des fonctions propres du Laplacien sur le tore standard de dimension $d \geqslant 2$. En utilisant la multiplicité du spectre du Laplacien et en introduisant une mesure gaussienne sur l'espace propre, nous nous servons de cette dernière afin d'évaluer des moyennes dans l'espace. Nous considérons une suite de valeurs propres ayant une multiplicité croissante $\mathcal{N} \to \infty$.

La quantité que nous étudions est la mesure de Leray (mesure microcanonique). Nous montrons que la moyenne de la mesure de Leray pour une fonction propre est constante et qu'elle vaut $1/\sqrt{2\pi}$. Notre résultat principal précise que la variance de la mesure de Leray est asymptotiquement $1/4\pi\mathcal{N}$ lorsque $\mathcal{N} \to \infty$ pour $d = 2$ et $d \geqslant 5$.

1. Introduction

1.1. Background

The *nodal set* of a function is the set of points where the function vanishes. In this paper we study the nodal sets of eigenfunctions of the Laplacian $\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$ on the (standard) flat torus $\mathbb{R}^d/\mathbb{Z}^d$, $d \geqslant 2$.

Keywords: Nodal sets, Leray measure, eigenfunctions of the Laplacian, trigonometric polynomials.
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Of course we have the simple eigenfunctions such as $\cos(2\pi(mx+ny))$ or $\sin(2\pi mx)\sin(2\pi ny)$ with corresponding Laplace eigenvalue $4\pi^2(m^2+n^2)$, for which the nodal set have a very simple structure. However, on the standard torus such eigenfunctions are atypical, because the eigenvalues on the torus always have multiplicities. The dimension $\mathcal{N} = \mathcal{N}(E)$ of an eigenspace corresponding to eigenvalue $4\pi^2 E$ is the number of integer vectors $\lambda \in \mathbb{Z}^d$ so that $|\lambda|^2 = E$. In dimension $d \geqslant 5$ this grows as $E \to \infty$ roughly as $E^{\frac{d}{2}-1}$ but has more erratic behaviour for small d, particularly for $d = 2$.

We wish to study the nodal sets of *typical* eigenfunctions. For this we consider a *random* eigenfunction on the torus, that is a random linear combination

$$f(x) = \frac{1}{\sqrt{2\mathcal{N}}} \sum_{\lambda \in \mathbb{Z}^d : |\lambda|^2 = E} b_\lambda \cos 2\pi i \langle \lambda, x\rangle - c_\lambda \sin 2\pi i \langle \lambda, x\rangle$$

with $b_\lambda, c_\lambda \sim N(0,1)$ real Gaussians of zero mean and variance 1 which are independent save for the relations $b_{-\lambda} = b_\lambda$, $c_{-\lambda} = -c_\lambda$.

We denote by $\mathbb{E}(\bullet)$ the expected value of the quantity $\bullet$ in this ensemble. For instance, the expected amplitude of f is $\mathbb{E}(|f(x)|^2) = 1$.

1.2. Leray measure

The fundamental quantity that we study here is the Leray measure, or microcanonical measure, of the nodal set of a function f in our ensemble. This is defined as (see [10, Chapter III], [16, §3.3])

$$\mathcal{L}(f) := \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \operatorname{meas}\{x \in \mathbb{T} : |f(x)| < \epsilon\}. \tag{1.1}$$

and in fact we can define a measure on the nodal set by

$$\lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{x : |f(x)| < \epsilon} \phi(x) dx$$

which in statistical mechanics is the microcanonical ensemble. This measure also appears in number theory as the "singular integral" in the Hardy-Littlewood method and elsewhere, see e.g. [7, 4]. We may formally write

$$\mathcal{L}(f) = \int_{\mathbb{T}^d} \delta(f(x)) dx \ .$$

As is well known, the limit (1.1) exists when $\nabla f \neq 0$ on the nodal set, in which case

$$\mathcal{L}(f) = \int_{\{x : f(x) = 0\}} \frac{d\sigma(x)}{|\nabla f(x)|}$$

where $d\sigma$ is the Riemannian hypersurface measure on the nodal set (see § 4).

1.3. Results

The expected value of $\mathcal{L}(f)$ turns out to be constant (Theorem 4.1):

$$\mathbb{E}(\mathcal{L}) = \frac{1}{\sqrt{2\pi}} \ .$$

To compare, the expected *volume* (or hypersurface measure) of the nodal set of f in our ensemble is $\mathcal{I}_d\sqrt{E}$ for some constant $\mathcal{I}_d$ depending only on the dimension [18].

Our main result concerns the *variance* of $\mathcal{L}(f)$ as $\mathcal{N} \to \infty$:

THEOREM 1.1. — *In dimensions $d = 2$ and $d \geqslant 5$, as $\mathcal{N} \to \infty$,*

$$\mathrm{Var}(\mathcal{L}(f)) \sim \frac{1}{4\pi\mathcal{N}} \ .$$

We refer to [18] for estimates on the variance of the volume of the nodal sets.

Concerning remainder terms, in dimension $d = 2$ we show that $\mathrm{Var}(\mathcal{L}(f)) = 1/4\pi\mathcal{N} + O(1/\mathcal{N}^2)$. In dimension $d \geqslant 3$, we prove $\mathrm{Var}(\mathcal{L}(f)) = 1/4\pi\mathcal{N} + O(E^{\frac{d-3}{2}+\epsilon}/\mathcal{N}^2)$, for all $\epsilon > 0$. Thus whenever $\mathcal{N} > E^{\frac{d-3}{2}+\delta}$ for some $\delta > 0$ (which is always valid in dimension $d \geqslant 5$), then we get an asymptotic. In dimensions $d = 3, 4$ we are only able to show that the variance is bounded by $O(1/\mathcal{N})$, though we believe that the conclusion of Theorem 1.1 holds in those cases as well.

It is somewhat surprising that the result depends only on the dimension of the eigenspace and not on the way the frequencies λ are distributed. In dimension $d \geqslant 5$, the directions $\lambda/|\lambda|$ of the frequencies are uniformly distributed on the sphere S^{d-1} [17]. However, in two dimensions this need not be the case (though it holds for most values of E, see [8, 12, 9]). For instance there is an infinite sequence of eigenvalues where the dimension of the eigenspace goes to infinity but the set of directions $\lambda/|\lambda| \in S^1$ tends to an average of four equally spaced point masses [6].

1.4. Related work

The study of nodal lines of random waves goes back to Longuet-Higgins [13, 14] who computed various statistics of nodal lines for Gaussian random waves in connection with the analysis of ocean waves. Berry [2] suggested to model highly excited quantum states for classically chaotic systems by using various random wave models, and also computed fluctuations of various quantities in these models (see e.g. [3]). See also Zelditch [20]. The idea of

averaging over a single eigenspace in the presence of multiplicities appears in Bérard [1] who computed the expected surface measure of the nodal set for eigenfunctions of the Laplacian on spheres. Neuheisel [15] also worked on the sphere and studied the statistics of Leray measure. He gave an upper bound for the variance, which we believe is not sharp.

1.5. About the proof of Theorem 1.1

We compute the second moment $\mathbb{E}(\mathcal{L}^2)$ by means of Gaussian integration as an integral over the torus

$$\mathbb{E}(\mathcal{L}^2) = \frac{1}{2\pi}\int_{\mathbb{T}^d} \frac{dx}{\sqrt{1-u(x)^2}}$$

where

$$u(x) := \mathbb{E}(f(x+y)f(y)) = \frac{1}{\mathcal{N}}\sum_{|\lambda|^2=E} \cos 2\pi\langle\lambda, x\rangle$$

is the two-point function of our random process (which is translation invariant). This formula shows that one should single out points $x \in \mathbb{T}^d$ where $|u(x)|$ is close to 1 (clearly $|u(x)| \leqslant 1$). We will show (see section 6.3) that the total contribution to the integral near such (suitably defined) "singular" points is bounded by $O(\int_{\mathbb{T}^d} u(x)^4 dx)$.

Outside of these "singular" points, we may expand in a Taylor series $(1-u^2)^{-1/2} = 1 + \frac{1}{2}u^2 + O(u^4)$. The constant term 1 corresponds to the square of the expectation and thus we will get

$$\mathrm{Var}(\mathcal{L}) = \frac{1}{4\pi}\int_{\mathbb{T}^d} u(x)^2 dx + O\left(\int_{\mathbb{T}^d} u(x)^4 dx\right).$$

The second moment of u is immediately seen to equal $\int_{\mathbb{T}^d} u(x)^2 dx = 1/\mathcal{N}$, and it is easily seen that the fourth moment of u is at most $1/\mathcal{N}$. Thus we get an upper bound $\mathrm{Var}(\mathcal{L}) = O(1/\mathcal{N})$ (in any dimension $d \geqslant 2$). To obtain Theorem 1.1 one needs to show that the fourth moment of u is negligible relative to $1/\mathcal{N}$. In dimension $d = 2$ we have $\int_{\mathbb{T}^d} u(x)^4 dx \ll 1/\mathcal{N}^2$ by a geometric argument due to Zygmund [21]. In dimension $d \geqslant 3$, we can show that

$$\int_{\mathbb{T}^d} u(x)^4 dx \ll_\epsilon \frac{E^{\frac{d-3}{2}+\epsilon}}{\mathcal{N}^2}, \quad \forall \epsilon > 0 \tag{1.2}$$

which in dimension $d \geqslant 5$ suffices because $\mathcal{N} \approx E^{\frac{d}{2}-1}$ and so we get a bound of $1/\mathcal{N}E^{1/2-\epsilon}$.

Alternatively, note that $u(x)$ is itself an eigenfunction of the Laplacian and we want a bound on its L^4-norm relative to its L^2-norm. In dimension $d \geqslant 5$ a bound (valid for any Riemannian manifold) due to Sogge [19] suffices here. A stronger bound for the torus, due to Bourgain [5], will improve (1.2) for $d \geqslant 7$.

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2. Random eigenfunctions on the torus

2.1. The basic setup

We wish to consider eigenfunctions of the Laplacian on the standard flat torus:

$$\Delta\psi + 4\pi^2 E\psi = 0 \ .$$

These can be written as linear combinations of the basic exponentials $e^{2\pi i\langle\lambda,x\rangle}$, with $\lambda \in \mathbb{Z}^d$, $|\lambda|^2 = E$. The dimension $\mathcal{N}$ of the corresponding eigenspace is simply the number of ways of expressing E as a sum of d integer squares. For $d \geqslant 5$ this grows roughly as $E^{d/2-1}$ as $E \to \infty$. For $d \leqslant 4$ the dimension of the eigenspace need not grow with E. In the extreme case $d = 2$, $\mathcal{N}$ is given in terms of the prime decomposition of E as follows: If $E = 2^\alpha \prod_j p_j^{\beta_j} \prod_k q_k^{2\gamma_k}$ where $p_j \equiv 1 \mod 4$ and $q_k \equiv 3 \mod 4$ are odd primes, $\alpha, \beta_j, \gamma_k \geqslant$ are integers, then $\mathcal{N} = 4\prod_j(\beta_j + 1)$, and otherwise E is not a sum of two squares and $\mathcal{N} = 0$. On average (over integers which are sums of two squares) the dimension is $const \cdot \sqrt{\log E}$.

For some of our initial work, throughout sections § 4, 5 we will work in greater generality and instead of eigenspaces we will consider linear spaces $\mathcal{E} = \mathcal{E}(\Lambda)$ spanned by certain sets of exponentials $e^{2\pi i\langle\lambda,x\rangle}$ with $\lambda \in \Lambda \subset \mathbb{Z}^d$. We take into account the reflection symmetries of the torus by assuming that the frequency set Λ is invariant under the group of signed permutations $W_d = \{\pm 1\}^d \times S_d$, consisting of coordinate permutations and sign-change of any coordinate, e.g. $(\lambda_1, \lambda_2) \mapsto (-\lambda_1, \lambda_2)$ (for $d = 2$). We say that a

non-empty subset $\Lambda \subset \mathbb{Z}^d$ is "symmetric" if it is invariant under W_d, that is invariant under permutations of the coordinates and changing sign of each coordinate, and that $0 \notin \Lambda$.

The dimension $\mathcal{N} = \dim \mathcal{E}$ is the number of the frequencies in Λ. Since Λ is symmetric and does not contain 0, $\mathcal{N}$ is even. We write $\Lambda/\pm$ to denote representatives of the equivalence class of Λ under $\lambda \mapsto -\lambda$.

Lemma 2.1. — *Any set Λ satisfying the symmetry conditions (i.e. invariant w.r.t. coordinate permutations and sign changes), spans $\mathbb{R}^d$.*

Proof. — Otherwise we have a nontrivial linear relation

$$\sum_{l=1}^{d} c_i \lambda_i = 0, \tag{2.1}$$

valid for all $\lambda \in \Lambda$. Since Λ is invariant under permutations, we may assume $\lambda_1 \neq 0$. Substituting λ and $\lambda' = (-\lambda_1', \lambda_2, \ldots, \lambda_d)$ and subtracting the equations we obtain $2\lambda_1 c_1 = 0$, which implies $c_1 = 0$. Repeating the argument for all c_i, we get a contradiction. $\square$

As a consequence of this lemma, we see that the set L_Λ of integer linear combinations of elements of $\Lambda \subseteq \mathbb{Z}^d$ is a sublattice of full rank, and hence its dual

$$L_\Lambda^* = \{v \in \mathbb{R}^d : \langle \lambda, v \rangle \in \mathbb{Z}, \forall \lambda \in \Lambda\}$$

is also a lattice in $\mathbb{R}^d$ (containing $\mathbb{Z}^d$).

2.2. A non-degeneracy condition

Assume that the set of frequencies Λ, which is assumed to be "symmetric", further satisfies the following "non-degeneracy" condition:

$$\exists \lambda \in \Lambda \text{ with } \lambda_1 \neq \pm\lambda_2 \text{ and } \lambda_1, \lambda_2 \neq 0\ . \tag{2.2}$$

By the symmetry of the set Λ, condition (2.2) is equivalent to requiring that for every $i \neq j$, there is $\lambda \in \Lambda$ with $\lambda_i \neq \pm\lambda_j$ and $\lambda_i, \lambda_j \neq 0$.

In the case of eigenfunctions of the Laplacian, where $\Lambda = \{\lambda \in \mathbb{Z}^d : |\lambda|^2 = E\}$, the non-degeneracy condition (2.2) holds as soon as $\mathcal{N} = \#\Lambda$ is sufficiently large, in fact if $\mathcal{N} > 3^d$. This is because any λ where there are no distinct indices $i \neq j$ with $\lambda_i, \lambda_j \neq 0$, $\lambda_i \neq \pm\lambda_j$ must be in the W_d-orbit of a vector of the form $\lambda(j, r) = (r, r, \ldots, r, 0, \ldots, 0)$ with the first j coordinates equal to $r > 0$ and the remaining $d - j$ coordinates equal to zero, and $E = jr^2$ (so r is determined uniquely by E and $0 \leqslant j \leqslant d$). The number of elements in the W_d-orbit of $\lambda(j, r)$ is $\binom{d}{j} 2^j$ and summing over all $0 \leqslant j \leqslant d$ gives at most 3^d possibilities.

2.3. Gaussian ensembles

For any symmetric set of frequencies $\Lambda \subset \mathbb{Z}^d$, we define an ensemble of Gaussian random functions $f \in \mathcal{E}$ by

$$f(x) = \frac{1}{\sqrt{2\mathcal{N}}} \sum_{\lambda \in \Lambda} b_\lambda \cos 2\pi i \langle \lambda, x \rangle - c_\lambda \sin 2\pi i \langle \lambda, x \rangle \tag{2.3}$$

with $b_\lambda, c_\lambda \sim N(0,1)$ real Gaussians of zero mean and variance 1 which are independent save for the relations $b_{-\lambda} = b_\lambda$, $c_{-\lambda} = -c_\lambda$. Thus we can rewrite

$$f(x) = \sqrt{\frac{2}{\mathcal{N}}} \sum_{\lambda \in \Lambda/\pm} b_\lambda \cos 2\pi i \langle \lambda, x \rangle - c_\lambda \sin 2\pi i \langle \lambda, x \rangle$$

where now only independent random variables appear.

Alternatively, we may identify $\mathcal{E} \cong \mathbb{R}^{\mathcal{N}}$ by taking coordinates $Z = (b_\lambda, c_\lambda)_{\lambda \in \Lambda/\pm}$ and putting the Gaussian probability measure

$$d\mu_{\mathcal{N}}(Z) = \frac{1}{(2\pi)^{\mathcal{N}/2}} \prod_{\lambda \in \Lambda/\pm} e^{-(b_\lambda^2 + c_\lambda^2)/2} db_\lambda dc_\lambda \ .$$

We define a set $\mathcal{B}$ by

$$\mathcal{B} = \{ w \in \mathbb{R}^d : \langle \lambda, w \rangle \in \mathbb{Z} \quad \forall \lambda \in \Lambda \text{ or } \langle \lambda, w \rangle \in \frac{1}{2} + \mathbb{Z} \quad \forall \lambda \in \Lambda \} \ .$$

Then clearly $\frac{1}{2} L_\Lambda^* \subseteq \mathcal{B} \subseteq L_\Lambda^*$ and so the projection of $\mathcal{B}$ on the torus $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ is finite. Note that if $x - y \in \mathcal{B}$, then for all $f \in \mathcal{E}$,

$$f(y) = \pm f(x), \qquad \text{and } \nabla f(y) = \pm \nabla f(y) \ .$$

For $a = (a_1, a_2) \in \mathbb{R}^2$, let

$$\mathcal{P}_{x,y}^a = \{ f \in \mathcal{E} : f(x) = a_1, f(y) = a_2 \} \ .$$

If $x - y \notin \mathcal{B}$ then this is an affine hyperplane of codimension two in $\mathcal{E}$. If $x - y \in \mathcal{B}$ then this is either empty or a hyperplane of codimension one in $\mathcal{E}$.

We define the two-point function of our ensemble as

$$u(x,y) = \mathbb{E}(f(x) f(y)) \ .$$

A simple computation shows that $u(x,y)$ depends only on the difference $x - y$, in fact $u(x,y) = u(x-y)$ where

$$u(z) = \frac{1}{\mathcal{N}} \sum_{\lambda \in \Lambda} \cos 2\pi \langle \lambda, z \rangle \ .$$

Lemma 2.2. — *$u(x) = \pm 1$ if and only if $x \in \mathcal{B}$.*

Proof. — If $x \in \mathcal{B}$ then $\cos 2\pi\langle\lambda, x\rangle$ are all equal, to either $+1$ or -1 and hence $u(x) = \pm 1$. On the other hand, since $|\cos 2\pi\langle\lambda, x\rangle| \leqslant 1$, if $u(x) = \pm 1$ then all the cosines $\cos 2\pi\langle\lambda, x\rangle$ have the same value, which is either $+1$ or -1, and this forces either $\langle\lambda, x\rangle \in \mathbb{Z}$ for all $\lambda \in \Lambda$, or $\langle\lambda, x\rangle \in \frac{1}{2} + \mathbb{Z}$ for all $\lambda \in \Lambda$, that is $x \in \mathcal{B}$. □

2.4. The singular set

We define the set of *singular* functions to be

$$Sing := \{f \in \mathcal{E} : \exists x \in \mathbb{T}^d,\ f(x) = 0 \text{ and } (\nabla f)(x) = \vec{0}\}.$$

Lemma 2.3. — *The set $Sing$ has codimension at least 1 in $\mathcal{E}$.*

Proof. — Define

$$\psi : \mathbb{T}^d \times \mathcal{E} \to \mathbb{R}^d \times \mathbb{R}$$
$$(x,\ f) \mapsto (\nabla f(x),\ f(x)),$$

Denoting $\pi_2 : \mathbb{T}^d \times \mathbb{R}^{\mathcal{N}} \to \mathbb{R}^{\mathcal{N}}$ the projection to the second factor, we have

$$Sing = \pi_2(\psi^{-1}(\{0\} \times \{0\})).$$

We prove that the Jacobian of ψ has maximal rank everywhere, and therefore $\psi^{-1}(\{0\} \times \{0\})$ is a smooth manifold of codimension $d+1$. It will then follow that $Sing \subset \mathbb{R}^{\mathcal{N}}$ has codimension $\geqslant 1$.

The $(d+1) \times (d+\mathcal{N})$ Jacobian matrix is

$$D\psi(x) = \begin{pmatrix} * & -2\pi\sqrt{\frac{2}{\mathcal{N}}}A(x) \\ * & \sqrt{\frac{2}{\mathcal{N}}}B(x) \end{pmatrix},$$

where $A(x)$ is a $d \times \mathcal{N}$ matrix defined by

$$A(x) = \Big((\sin 2\pi\langle\lambda, x\rangle\vec{\lambda},\ \cos 2\pi\langle\lambda, x\rangle\vec{\lambda})\Big)_{\lambda\in\Lambda/\pm},$$

and $B(x)$ is a $1 \times \mathcal{N}$ matrix defined by

$$B(x) = \Big((\cos 2\pi\langle\lambda, x\rangle,\ -\sin 2\pi\langle\lambda, x\rangle)\Big)_{\lambda\in\Lambda/\pm}.$$

Thus we want the $(d+1) \times \mathcal{N}$ matrix $\begin{pmatrix} A \\ B \end{pmatrix}$ to have rank $d+1$. However, ordering the vectors $\vec{\lambda}^{(j)} \in \Lambda/\pm$, it is a product of

$$\begin{pmatrix} \vec{0} & \vec{\lambda}^{(1)} & \vec{0} & \vec{\lambda}^{(2)} & \dots \\ 1 & 0 & 1 & 0 & \dots \end{pmatrix},$$

which is of rank $d+1$ by lemma 2.1 and

$$\begin{pmatrix} \cos 2\pi\langle\lambda^{(1)},x\rangle & -\sin 2\pi\langle\lambda^{(1)},x\rangle & 0 & 0 & \dots \\ \sin 2\pi\langle\lambda^{(1)},x\rangle & \cos 2\pi\langle\lambda^{(1)},x\rangle & 0 & 0 & \dots \\ 0 & 0 & \cos 2\pi\langle\lambda^{(2)},x\rangle & -\sin 2\pi\langle\lambda^{(2)},x\rangle & \dots \\ 0 & 0 & \sin 2\pi\langle\lambda^{(2)},x\rangle & \cos 2\pi\langle\lambda^{(2)},x\rangle & \dots \\ & & & & \ddots \end{pmatrix}$$

which is nonsingular. This immediately implies the result. □

The following is an immediate

Corollary 2.4. — *The set Sing has measure zero in $\mathcal{E}$.*

3. The Leray measure

We continue with our previous setting, that is $\Lambda \subset \mathbb{Z}^d$ is a symmetric, non-degenerate set of frequencies. We wish to define the Leray measure $\mathcal{L}(f)$ for $f \in \mathcal{E}$ by the limit

$$\mathcal{L}(f) = \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \operatorname{meas}\{x : |f(x)| < \epsilon\} .$$

It is well known that the limit exists for any nonsingular f (see [10, Chapter III], [16, §3.3]), and that in fact

$$\mathcal{L}(f) = \int_{\{x:f(x)=0\}} \frac{d\sigma(x)}{|\nabla f(x)|}$$

where $d\sigma(x)$ is the induced hypersurface measure.

We will need to know more refined information about the approach to the limit in the definition. For $\epsilon > 0$, set

$$\mathcal{L}_\epsilon(f) := \frac{1}{2\epsilon} \operatorname{meas}\{x : |f(x)| < \epsilon\} ,$$

so that $\mathcal{L}(f) = \lim_{\epsilon\to 0} \mathcal{L}_\epsilon(f)$.

For $\alpha > 0$, $\beta > 0$ let

$$\mathcal{E}(\alpha,\, \beta) = \{f \in \mathcal{E} :\ |f(x)| \leqslant \alpha \Rightarrow\ |\nabla f(x)| > \beta\} .$$

The sets $\mathcal{E}(\alpha,\, \beta)$ are open, and have the monotonicity property

$$\alpha_1 > \alpha_2 \Rightarrow \mathcal{E}(\alpha_1, \beta) \subseteq \mathcal{E}(\alpha_2, \beta)$$

and

$$\beta_1 > \beta_2 \Rightarrow \mathcal{E}(\alpha, \beta_1) \subseteq \mathcal{E}(\alpha, \beta_2) .$$

Moreover, for any sequence $\alpha_n, \beta_n \to 0$ we have

$$\mathcal{E} \setminus Sing = \bigcup_n \mathcal{E}(\alpha_n,\, \beta_n)\ .$$

Lemma 3.1. — *For $f \in \mathcal{E}(\alpha,\, \beta)$ and $0 < \epsilon < \alpha$, we have*

$$\mathcal{L}_\epsilon(f) < \frac{d^{3/2}}{\beta} 2\sqrt{E_{\max}}$$

where

$$E_{\max} = \max\{|\lambda|^2 : \lambda \in \Lambda\}\ .$$

We will first treat the one variable ($d = 1$) case and state it as a separate lemma (cf [11, Lemma 2]):

Lemma 3.2. — *Let $g(t)$ be a trigonometric polynomial of degree at most M so that there are $\alpha > 0$, $\beta > 0$ such that $|g'(t)| > \beta$ whenever $|g(t)| < \alpha$. Then for all $0 < \epsilon < \alpha$ we have*

$$\frac{1}{2\epsilon}\,\mathrm{meas}\{t : |g(t)| < \epsilon\} < \frac{2M}{\beta}\ .$$

Proof. — Decompose the open set $\{t : |g(t)| < \epsilon\}$ as a disjoint union of open intervals (a_k, b_k) (with $a_k < b_k$) and such that on each such interval, g' has constant sign, that is either $g' > \beta$ or $g' < -\beta$. We will show that the length $b_k - a_k$ of each such interval is at most $2\epsilon/\beta$ and that there are at most $2M$ such intervals.

Suppose that on (a_k, b_k), $g' > \beta$; then g is increasing, and $g(a_k) = -\epsilon$, $g(b_k) = +\epsilon$. Then the length of the interval is

$$\begin{aligned} b_k - a_k &= \int_{a_k}^{b_k} \frac{g'(t)}{g'(t)} dt < \frac{1}{\beta} \int_{a_k}^{b_k} g'(t) dt \\ &= \frac{g(b_k) - g(a_k)}{\beta} = \frac{2\epsilon}{\beta}\ . \end{aligned}$$

Likewise, if $g' < -\beta$ on (a_k, b_k) then $g(a_k) = +\epsilon$, $g(b_k) = -\epsilon$, and

$$\begin{aligned} b_k - a_k &= \int_{a_k}^{b_k} \frac{-g'(t)}{-g'(t)} dt < \frac{1}{\beta} \int_{a_k}^{b_k} -g'(t) dt \\ &= \frac{g(a_k) - g(b_k)}{\beta} = \frac{2\epsilon}{\beta} \end{aligned}$$

as required.

In both cases, each interval has an endpoint where $g(t) = +\epsilon$, and hence the number of such intervals is bounded by the number of solutions of $g(t) = +\epsilon$ which is at most $2M$ since g is a trigonometric polynomial of degree at most M. □

We now prove Lemma 3.1 by reduction to the case $d = 1$:

Proof. — Decompose the set $\{x : |f(x)| < \epsilon\}$ as a union $\cup_{j=1}^{d} W_j$ where

$$W_j = \{y : |f(y)| < \epsilon, \quad |\frac{\partial f}{\partial x_j}(y)| \geqslant |\frac{\partial f}{\partial x_k}(y)| \quad \forall k \neq j\}$$

and it suffices to show that

$$\mathrm{meas}(W_j) < 2\epsilon \frac{\sqrt{d}}{\beta} 4\sqrt{E_{\max}} \ .$$

For simplicity we fix $j = 1$. On W_1, we have

$$|\frac{\partial f}{\partial x_1}(y)| > \frac{\beta}{\sqrt{d}}$$

since $|f(y)| < \epsilon < \alpha$ implies (recall $f \in \mathcal{E}(\alpha, \beta)$)

$$\beta^2 < |\nabla f(y)|^2 = \sum_{k=1}^{d} |\frac{\partial f}{\partial x_k}(y)|^2 \leqslant d|\frac{\partial f}{\partial x_1}(y)|^2 \ .$$

For $y \in \mathbb{T}^{d-1}$ set

$$I(y) = \{t \in \mathbb{T}^1 : (t, y) \in W_1\}$$

which is a subset of $\mathbb{T}^1$. Then slice-integration gives

$$\mathrm{meas}(W_1) = \int_{\mathbb{T}^{d-1}} \mathrm{meas}(I(y)) dy$$

and so it suffices to show

$$\mathrm{meas}(I(y)) < 2\epsilon \frac{\sqrt{d}}{\beta} 4\sqrt{E_{\max}} \ .$$

Now on $I(y)$, the one-variable trigonometric polynomial $g(t) := f(t, y)$ satisfies $|g(t)| = |f(t, y)| < \epsilon$, and

$$|g'(t)| = |\frac{\partial f}{\partial x_1}(t, y)| > \frac{\beta}{\sqrt{d}} \ .$$

Moreover $g(t)$ is of degree at most $\sqrt{E_{\max}}$ because

$$f(t, y) = \sum_{\lambda \in \Lambda} a_\lambda e(\lambda_1 t + \sum_{j=2}^{d} \lambda_j y_j)$$

and for all frequencies in the sum we have $\lambda_1^2 \leqslant |\lambda|^2 \leqslant E_{\max}$. Thus by Lemma 3.2 we find that $\mathrm{meas}(I(y)) < 2\epsilon \frac{\sqrt{d}}{\beta} 2\sqrt{E_{\max}}$ as required. $\square$

4. The expected value of $\mathcal{L}$

In this section, we give a formula for the expected value of $\mathcal{L}(f)$:

THEOREM 4.1. — *Suppose that Λ is symmetric and satisfies the nondegeneracy condition* (2.2). *Then the Leray measure $\mathcal{L}(f)$ is integrable (with respect to the Gaussian measure), and*

$$\mathbb{E}(\mathcal{L}) = \frac{1}{\sqrt{2\pi}}. \tag{4.1}$$

4.1. A formal treatment

To compute the expectation of $\mathcal{L}(f)$, we formally write it as

$$\mathcal{L}(f) = \int_{\mathbb{T}^d} \delta(f(x))dx$$

and hence formally

$$\mathbb{E}(\mathcal{L}(f)) = \mathbb{E}(\int_{\mathbb{T}^d} \delta(f(x))dx) = \int_{\mathbb{T}^d} \mathbb{E}(\delta(f(x))dx$$

Now for each fixed $x \in \mathbb{T}^d$, the random variable $f(x)$ is a sum of Gaussians hence is itself a Gaussian whose mean is zero and variance is computed to be unity. Hence the expected value $\mathbb{E}(\delta(f(x))$ should be

$$\mathbb{E}(\delta(f(x)) = \int_{-\infty}^{\infty} \delta(a) e^{-a^2/2} \frac{da}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}}$$

which gives the result $\mathbb{E}(\mathcal{L}) = 1/\sqrt{2\pi}$. Justifying this simple manipulation in a rigorous fashion turns out to be rather tedious will be done below, with some parts relegated to an appendix.

4.2. A rigorous proof

The Leray measure $\mathcal{L}(f)$ is defined outside of the singular set, which has measure zero in $\mathcal{E}$, in fact forms a closed subset of codimension $\geqslant 1$ (Lemma 2.3). We compute the expectation of the nodal measure $\mathcal{L}$ as follows: We consider the increasing sequence of open subsets $\mathcal{E}(\frac{1}{n}, \frac{1}{n})$, $n = 1, 2, \ldots$, whose union is the set of nonsingular elements $\mathcal{E} \backslash Sing$. We choose subsets $H_n \subset \mathcal{E}(\frac{1}{n}, \frac{1}{n})$ which are (finite) unions of *disjoint* open balls, so that

$$\bigcup_n H_n = \mathcal{E} \backslash Sing$$

and in fact the H_n exhaust almost all nonsingular f's, in the sense that $\mu(H_n) \to 1$. (This is possible by Vitali's covering theorem). We will show that the limit

$$\mathbb{E}(\mathcal{L}) = \lim_n \int_{H_n} \mathcal{L}(f) d\mu(f)$$

exists and equals $1/\sqrt{2\pi}$.

By definition,

$$\int_{H_n} \mathcal{L}(f) d\mu(f) = \int_{H_n} \lim_{\epsilon \to 0} \mathcal{L}_\epsilon(f) d\mu(f)$$

where

$$\mathcal{L}_\epsilon(f) := \frac{1}{2\epsilon} \int_{\mathbb{T}^d} \chi(\frac{f(x)}{\epsilon}) dx \ .$$

By Lemma 3.1, on H_n, $\mathcal{L}_\epsilon(f) \leqslant c_n$ is bounded uniformly for all $\epsilon < \frac{1}{n}$. Thus by the dominated convergence theorem we can exchange limits:

$$\int_{H_n} \lim_{\epsilon \to 0} \mathcal{L}_\epsilon(f) d\mu(f) = \lim_{\epsilon \to 0} \int_{H_n} \mathcal{L}_\epsilon(f) d\mu(f) \ .$$

On the integral, we use Fubini's theorem to change the order of integration

$$\int_{H_n} \mathcal{L}_\epsilon(f) d\mu(f) = \int_{\mathbb{T}^d} \left(\frac{1}{2\epsilon} \int_{H_n} \chi(\frac{f(x)}{\epsilon}) d\mu(f) \right) dx \ .$$

For the inner integral, we note that for each x, $f(x)$ is a Gaussian random variable of mean zero and variance $\mathbb{E}(f(x)^2) = 1$ and hence setting

$$\mathcal{P}_x^a = \{ f \in \mathcal{E} : f(x) = a \}$$

which is an affine hyperplane of $\mathcal{E}$ of codimension one, we have

$$\frac{1}{2\epsilon} \int_{H_n} \chi(\frac{f(x)}{\epsilon}) d\mu(f) = \frac{1}{2\epsilon} \int_{|a|<\epsilon} \mu_x^a(\mathcal{P}_x^a \cap H_n) e^{-a^2/2} \frac{da}{\sqrt{2\pi}}$$

where μ_x^a is the induced Gaussian probability measure on the hyperplane $\mathcal{P}_x^a$. Thus

$$\int_{H_n} \mathcal{L}(f) d\mu(f) = \lim_{\epsilon \to 0} \int_{\mathbb{T}^d} \frac{1}{2\epsilon} \int_{|a|<\epsilon} \mu_x^a(\mathcal{P}_x^a \cap H_n) e^{-a^2/2} \frac{da}{\sqrt{2\pi}} dx \ .$$

Now the function

$$\mu_x^a(\mathcal{P}_x^a \cap H_n)$$

is bounded by $\mu_x^a(\mathcal{P}_x^a) = 1$ and is *continuous* in both a and in x because we chose H_n to be a disjoint union of balls, and the volume of the intersection of a hyperplane with this kind of nice set is a continuous function of the hyperplane (since this is true for a ball). Hence we may move the limit

$\epsilon \to 0$ inside the integral over $\mathbb{T}^d$, and find, by the fundamental theorem of calculus, that

$$\lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_{|a|<\epsilon} \mu_x^a(\mathcal{P}_x^a \cap H_n) e^{-a^2/2} \frac{da}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \mu_x^0(\mathcal{P}_x^0 \cap H_n) \ .$$

Thus we find that

$$\int_{H_n} \mathcal{L}(f) d\mu(f) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}^d} \mu_x^0(\mathcal{P}_x^0 \cap H_n) dx \ .$$

Now the functions

$$g_n(x) := \mu_x^0(\mathcal{P}_x^0 \cap H_n)$$

are *continuous* in x, and are bounded: $g_n(x) \leqslant \mu_x^0(\mathcal{P}_x^0) = 1$ and moreover for each x their limit is

$$\lim_{n\to\infty} g_n(x) = \mu_x^0(\mathcal{P}_x^0) = 1$$

because by Proposition A.1 the singular set has measure zero in $\mathcal{P}_x^0$ for each x and the H_n exhaust all the nonsingular elements up to measure zero.

Thus we may in taking the limit $n \to \infty$ move the limit under the integral to get

$$\begin{aligned} \lim_n \int_{H_n} \mathcal{L}(f) d\mu(f) &= \lim_n \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}^d} g_n(x) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}^d} \lim_n g_n(x) dx = \frac{1}{\sqrt{2\pi}} \end{aligned}$$

as required. □

5. A formula for the variance of $\mathcal{L}$

In this section we give a formula for the variance of $\mathcal{L}(f)$ in terms of the two-point function

$$u(x,y) = \mathbb{E}(f(x)f(y)) = \frac{1}{\mathcal{N}} \sum_{\lambda\in\Lambda} \cos 2\pi\langle \lambda, z\rangle \ .$$

The main result of this section is

Theorem 5.1. — *Let $d \geqslant 2$. For any symmetric set of frequencies $\Lambda \subset \mathbb{Z}^d$ satisfying the non-degeneracy condition* (2.2)*, the second moment of $\mathcal{L}$ is given by*

$$\mathbb{E}(\mathcal{L}^2) = \frac{1}{2\pi} \int_{\mathbb{T}^d} \frac{dz}{\sqrt{1-u(z)^2}} \ .$$

Thus the variance of $\mathcal{L}$ is

$$\mathrm{Var}(\mathcal{L}) = \frac{1}{2\pi}\int_{\mathbb{T}^d} \frac{dz}{\sqrt{1-u(z)^2}} - \frac{1}{2\pi}\ .$$

5.1. A formal derivation

It is simple to formally derive Theorem 5.1: Writing $\mathcal{L}(f) = \int_{\mathbb{T}^d} \delta(f(x))dx$ we have

$$\begin{aligned}\mathbb{E}(\mathcal{L}^2) &= \mathbb{E}\left(\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \delta(f(x))\delta(f(y))dxdy\right)\\ &= \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \mathbb{E}\Big(\delta(f(x))\delta(f(y))\Big)dxdy\ .\end{aligned}$$

Now replace the vector $(f(x), f(y))$ by a Gaussian vector $a = (a_1, a_2)$ with covariance matrix

$$\begin{pmatrix} \mathbb{E}(f(x)^2) & \mathbb{E}(f(x)f(y)) \\ \mathbb{E}(f(y)f(x)) & \mathbb{E}(f(y)^2)\end{pmatrix} = \begin{pmatrix} 1 & u(x-y) \\ u(y-x) & 1\end{pmatrix} = \Sigma(x-y) \tag{5.1}$$

whose determinant is $\det\Sigma(x-y) = 1-u(x-y)^2$. Thus

$$\begin{aligned}\mathbb{E}(\delta(f(x))\delta(f(y))) &= \iint_{\mathbb{R}^2} \delta(a_1)\delta(a_2)\frac{e^{-\frac{1}{2}a\Sigma^{-1}a^T}}{\sqrt{\det\Sigma}}\frac{da_1da_2}{2\pi}\\ &= \frac{1}{2\pi}\frac{1}{\sqrt{1-u(x-y)^2}}\ .\end{aligned}$$

This gives

$$\mathbb{E}(\mathcal{L}^2) = \frac{1}{2\pi}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{1}{\sqrt{1-u(x-y)^2}}dxdy = \frac{1}{2\pi}\int_{\mathbb{T}^d}\frac{1}{\sqrt{1-u(z)^2}}dz$$

as claimed. The rigorous proof of this formula takes up the rest of the section.

5.2. Integrability of the kernel

Lemma 5.2. — *Let $\Lambda \subset \mathbb{R}^d$ be invariant under permutations and coordinate sign changes. Then*

$$\sum_{\lambda\in\Lambda}\langle\lambda,\xi\rangle^2 = \frac{1}{d}\sum_{\lambda\in\Lambda}||\lambda||^2\cdot||\xi||^2$$

Proof. — We write the quadratic form in the LHS as

$$Q(\xi) = \sum_{\lambda\in\Lambda} \langle \lambda, \xi\rangle^2 = \sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j$$

where

$$a_{ij} = \sum_{\lambda\in\Lambda} \lambda_i\lambda_j \ .$$

If $i \neq j$ use the symmetry under the sign change of the i-th coordinate to change variables and deduce that $a_{ij} = 0$. For $i = j$ we find

$$a_{ii} = \sum_{\lambda\in\Lambda} \lambda_i^2$$

and the latter sum is independent of i since Λ is symmetric under permutations; hence we may average the RHS over i to find

$$a_{ii} = \frac{1}{d}\sum_{i=1}^{d}\sum_{\lambda\in\Lambda} \lambda_i^2 = \frac{1}{d}\sum_{\lambda\in\Lambda} ||\lambda||^2$$

as required. □

Lemma 5.3. — *For $d > 1$, the kernel $1/\sqrt{\det\Sigma(z)} = 1/\sqrt{1-u(z)^2}$ is integrable on $\mathbb{T}^d$.*

Proof. — We need to check near the zeros of $\det\Sigma(z)$, that is at points where $u(z) = \pm 1$. By Lemma 2.2 this implies that z lies in the finite set $\mathcal{B}/\mathbb{Z}^d$. At such points z_0, all the cosines $\cos 2\pi\langle\lambda, z_0\rangle$ have the same value, which is either $+1$ or -1, and expanding in a small neighbourhood we have

$$\cos 2\pi\langle\lambda, z\rangle \sim \pm(1 - \frac{1}{2}\langle\lambda, z - z_0\rangle^2) \ .$$

Thus

$$\det\Sigma(z) = 1 - u(z)^2 \sim \frac{1}{2\mathcal{N}}\sum_{\lambda\in\Lambda}\langle\lambda, z - z_0\rangle^2 \ .$$

By Lemma 5.2, we thus have

$$\det\Sigma(z) \sim \frac{1}{2d}\left(\frac{1}{\mathcal{N}}\sum_{\lambda\in\Lambda}|\lambda|^2\right)|z - z_0|^2$$

and therefore

$$\frac{1}{\sqrt{\det\Sigma(z)}} \sim const.\frac{1}{|z - z_0|}$$

near z_0, which is integrable if (and only if) $d > 1$. □

5.3. Proof of Theorem 5.1

We have

$$\int_{H_n} \mathcal{L}(f)^2 d\mu(f) = \int_{H_n} \lim_{\epsilon_1,\epsilon_2\to 0} \mathcal{L}_{\epsilon_1}(f)\mathcal{L}_{\epsilon_2}(f) d\mu(f) \ .$$

By Lemma 3.1 and the dominated convergence theorem, we may take the limit outside the integral sign and get

$$\int_{H_n} \mathcal{L}(f)^2 d\mu(f) = \lim_{\epsilon_1,\epsilon_2\to 0} \int_{H_n} \frac{1}{4\epsilon_1\epsilon_2} \iint_{\mathbb{T}^d\times\mathbb{T}^d} \chi(\frac{f(x)}{\epsilon_1})\chi(\frac{f(y)}{\epsilon_2}) dxdyd\mu(f)$$

which by Fubini's theorem and the change of variable $y = x + z$, equals

$$\lim_{\epsilon_1,\epsilon_2\to 0} \iint_{\mathbb{T}^d\times\mathbb{T}^d} \left(\frac{1}{4\epsilon_1\epsilon_2} \int_{H_n} \chi(\frac{f(x)}{\epsilon_1})\chi(\frac{f(x+z)}{\epsilon_2}) d\mu(f) \right) dxdz \ .$$

5.3.1. Excising the singular set

Fix $\epsilon_1, \epsilon_2 > 0$ and let $S(\epsilon_1, \epsilon_2) \subset \mathbb{T}^d$ be a subset of measure at most $(\epsilon_1\epsilon_2)^2$ surrounding the finitely many points of $\mathcal{B}/\mathbb{Z}^d$. Then using $\chi \leqslant 1$ we have

$$\int_{\mathbb{T}^d} \int_{S(\epsilon_1,\epsilon_2)} \left(\frac{1}{4\epsilon_1\epsilon_2} \int_{H_n} \chi(\frac{f(x)}{\epsilon_1})\chi(\frac{f(x+z)}{\epsilon_2}) d\mu(f) \right) dxdz \leqslant \frac{\operatorname{meas} S(\epsilon_1,\epsilon_2)}{4\epsilon_1\epsilon_2} < \epsilon_1\epsilon_2$$

and hence the in the limit $\epsilon_1, \epsilon_2 \to 0$ this gives zero contribution. Thus

$$\int_{H_n} \mathcal{L}(f)^2 d\mu(f) = \lim_{\epsilon_1,\epsilon_2\to 0} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d\setminus S(\epsilon_1,\epsilon_2)} \left(\frac{1}{4\epsilon_1\epsilon_2} \int_{H_n} \chi(\frac{f(x)}{\epsilon_1})\chi(\frac{f(x+z)}{\epsilon_2}) d\mu(f) \right) dxdy \ .$$

5.3.2. Gaussian integration

For fixed $\epsilon_1, \epsilon_2 > 0$ we evaluate the inner integral as in the formal derivation of § 5.1 by replacing the vector $(f(x), f(y))$ by a Gaussian vector $(a_1, a_2) \in \mathbb{R}^2$ with covariance matrix $\Sigma(z)$ given in (5.1). For $x - y \notin \mathcal{B}/\mathbb{Z}^d$ and $a = (a_1, a_2) \in \mathbb{R}^2$, set

$$\mathcal{P}^a_{x,y} = \{f \in \mathcal{E} : f(x) = a_1, f(y) = a_2\} \ ,$$

which is an affine subspace of codimension two. Let $\mu^a_{x,y}$ be the induced Gaussian probability measure on $\mathcal{P}^a_{x,y}$. Then for $z = x - y \notin \mathcal{B}/\mathbb{Z}^d$,

$$\frac{1}{4\epsilon_1\epsilon_2}\int_{H_n}\chi\left(\frac{f(x)}{\epsilon_1}\right)\chi\left(\frac{f(x+z)}{\epsilon_2}\right)d\mu(f) =$$

$$\frac{1}{\sqrt{\det\Sigma(z)}}\frac{1}{4\epsilon_1\epsilon_2}\iint_{|a_1|<\epsilon_1,|a_2|<\epsilon_2} e^{-\frac{1}{2}a\Sigma^{-1}(z)a^T}\mu^a_{x,x+z}(\mathcal{P}^a_{x,x+z}\cap H_n)\frac{da_1da_2}{2\pi}.$$

Thus we find

$$\int_{H_n}\mathcal{L}(f)^2d\mu(f) = \lim_{\epsilon_1,\epsilon_2\to 0}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d\setminus S(\epsilon_1,\epsilon_2)} K_n(x,x+z;\epsilon_1,\epsilon_2)dxdz$$

where

$$K_n(x,x+z;\epsilon_1,\epsilon_2) =$$

$$\frac{1}{\sqrt{\det\Sigma(z)}}\frac{1}{4\epsilon_1\epsilon_2}\iint_{|a_1|<\epsilon_1,|a_2|<\epsilon_2} e^{-\frac{1}{2}a\Sigma^{-1}(z)a^T}\mu^a_{x,x+z}(\mathcal{P}^a_{x,x+z}\cap H_n)\frac{da_1da_2}{2\pi}.$$

5.3.3. Excising more points

Fix $\delta > 0$, and for ϵ_1, ϵ_2 sufficiently small fix a set $D \subset \mathbb{T}^d$ so that

(1) $S(\epsilon_1,\epsilon_2) \subset D$

(2) D contains the measure zero set of $z = x - y$ for which Proposition B.1 fails to hold.

(3) $\displaystyle\int_D \frac{dz}{\sqrt{\det\Sigma(z)}} < \delta.$

Then we can bound

$$K_n(x,x+z;\epsilon_1,\epsilon_2) \leqslant \frac{1}{2\pi\sqrt{\det\Sigma(z)}}$$

by using $e^{-\frac{1}{2}a\Sigma^{-1}(z)a^T} \leqslant 1$ and $\mu^a_{x,x+z}(\mathcal{P}^a_{x,x+z}\cap H_n) \leqslant \mu^a_{x,x+z}(\mathcal{P}^a_{x,x+z}) = 1$. Thus

$$\int_{H_n}\mathcal{L}(f)^2d\mu(f) = \lim_{\epsilon_1,\epsilon_2\to 0}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d\setminus D} K_n(x,x+z;\epsilon_1,\epsilon_2)dxdz + O(\delta)\ ,$$

with the implied constant in $O(\delta)$ independent of n.

5.3.4. A switch of limit and integration

Since K_n is dominated by $1/\sqrt{\det\Sigma(z)}$, which is integrable by Lemma 5.3, we may use the dominated convergence theorem to switch the limit $\epsilon_1, \epsilon_2 \to 0$ and the integral to get

$$\int_{H_n} \mathcal{L}(f)^2 d\mu(f) = \int_{\mathbb{T}^d} \int_{\mathbb{T}^d \setminus D} \lim_{\epsilon_1,\epsilon_2 \to 0} K_n(x, x+z; \epsilon_1, \epsilon_2)\, dxdz + O(\delta)\ .$$

where the implied constant is independent of n.

5.3.5. Taking the limit $\epsilon_1, \epsilon_2 \to 0$

The function

$$(x, z, a) \mapsto e^{-\frac{1}{2} a \Sigma^{-1}(z) a^T} \mu^a_{x,x+z}(\mathcal{P}^a_{x,x+z} \cap H_n)$$

is continuous on $\mathbb{T}^d \times \mathbb{T}^d \setminus D \times \mathbb{R}^2$ by construction of H_n to have continuous intersection with hyperplanes of fixed dimension. Thus for $z \notin D$, we may use the fundamental theorem of calculus to get

$$\lim_{\epsilon_1,\epsilon_2 \to 0} \frac{1}{4\epsilon_1\epsilon_2} \iint_{|a_1|<\epsilon_1, |a_2|<\epsilon_2} e^{-\frac{1}{2} a \Sigma^{-1}(z) a^T} \mu^a_{x,x+z}(\mathcal{P}^a_{x,x+z} \cap H_n) da_1 da_2 = \mu^0_{x,x+z}(\mathcal{P}^0_{x,x+z} \cap H_n)\ .$$

Therefore for $z \notin D$,

$$\lim_{\epsilon_1,\epsilon_2 \to 0} K_n(x, x+z; \epsilon_1, \epsilon_2) = \frac{\mu^0_{x,x+z}(\mathcal{P}^0_{x,x+z} \cap H_n)}{2\pi\sqrt{\det \Sigma(z)}}\ .$$

This gives

$$\int_{H_n} \mathcal{L}(f)^2 d\mu(f) = \int_{\mathbb{T}^d} \int_{\mathbb{T}^d \setminus D} \frac{\mu^0_{x,x+z}(\mathcal{P}^0_{x,x+z} \cap H_n)}{2\pi\sqrt{\det \Sigma(z)}} dxdz + O(\delta)\ .$$

5.3.6. The limit $n \to \infty$

Taking now the limit $n \to \infty$, and using continuity of $\mu^0_{x,x+z}(\mathcal{P}^0_{x,x+z} \cap H_n)$ on $\mathbb{T}^d \times \mathbb{T}^d \setminus D$ (which is is due to the construction of H_n) and using Proposition B.1 to guarantee that for $z \notin D$, the intersection of $\mathcal{P}^0_{x,x+z}$ with the singular set has measure zero in $\mathcal{P}^0_{x,x+z}$, we find

$$\lim_{n\to\infty} \mu^0_{x,x+z}(\mathcal{P}^0_{x,x+z} \cap H_n) = \mu^0_{x,x+z}(\mathcal{P}^0_{x,x+z}) = 1$$

and thus

$$\lim_{n\to\infty} \int_{H_n} \mathcal{L}(f)^2 d\mu(f) = \int_{\mathbb{T}^d \setminus D} \frac{dz}{2\pi\sqrt{\det \Sigma(z)}} + O(\delta)\ .$$

Since $\delta > 0$ is arbitrary and $1/\sqrt{\det \Sigma(z)}$ is integrable on $\mathbb{T}^d$, we finally conclude that

$$\mathbb{E}(\mathcal{L}^2) = \lim_{n\to\infty} \int_{H_n} \mathcal{L}(f)^2 d\mu(f) = \frac{1}{2\pi} \int_{\mathbb{T}^d} \frac{dz}{\sqrt{\det \Sigma(z)}} \ .$$

This concludes the proof of Theorem 5.1. □

6. The asymptotics of the variance

In the previous section we showed that the second moment of the Leray measure for the ensemble of trigonometric polynomials associated to any symmetric set of frequencies is given by

$$\mathbb{E}(\mathcal{L}^2) = \frac{1}{2\pi} \int\limits_{\mathbb{T}^d} \frac{dx}{\sqrt{1-u^2(x)}} \tag{6.1}$$

where $u(x) = \frac{1}{\mathcal{N}} \sum\limits_{\lambda\in\Lambda} \cos 2\pi\langle\lambda,\, x\rangle$ is the two-point function of the process.

From now on, we specialize to the case that

$$\Lambda = \{\lambda \in \mathbb{Z}^d : |\lambda|^2 = E\} \ .$$

In this section we show:

Proposition 6.1. — *The second moment of $\mathcal{L}(f)$ is given by*

$$\mathbb{E}(\mathcal{L}^2) = \frac{1}{2\pi} + \frac{1}{4\pi\mathcal{N}} + O\bigg(\int_{\mathbb{T}^d} u(x)^4 dx\bigg) \ .$$

In section § 7 we will see that for $d = 2$ and $d \geqslant 5$, the fourth moment of u is negligible relative to $1/\mathcal{N}$ and hence we will obtain

$$\mathrm{Var}(\mathcal{L}) \sim \frac{1}{4\pi\mathcal{N}}$$

as $\mathcal{N} \to \infty$, which is Theorem 1.1.

We now set about the proof of Proposition 6.1.

6.1. Singular points

Definition 6.2. — *A point $x \in \mathbb{T}^d$ is a positive singular point if there is a set of frequencies $\Lambda_x \subset \Lambda$ with density $\frac{|\Lambda_x|}{|\Lambda|} > 1 - \frac{1}{4d}$ for which $\cos 2\pi\langle\lambda, x\rangle > 3/4$ for all $\lambda \in \Lambda_x$. Similarly we define a negative singular point to be a point x where there is a set $\tilde{\Lambda}_x \subset \Lambda$ of density $> 1 - \frac{1}{4d}$ for which $\cos 2\pi\langle\lambda, x\rangle < -3/4$ for all $\lambda \in \tilde{\Lambda}_x$.*

An example is the origin, where $\cos 2\pi\langle\lambda, 0\rangle = 1$.

Let $M \approx \sqrt{E}$ be a large integer[(1)]. We decompose the unit cube (the torus) as a disjoint union (with boundary overlaps) of M^d closed cubes $I_{\vec{k}}$ of side length $1/M$ centered at $\vec{k}/M$, $\vec{k} \in \mathbb{Z}^d$.

Definition 6.3. — *A cube $I_{\vec{k}}$ is a positive (resp. negative) singular cube if it contains a positive (resp. negative) singular point.*

Lemma 6.4. — *For a positive (respectively, negative) singular cube I, there is a subset of frequencies $\Lambda_I \subset \Lambda$ with with density $\frac{|\Lambda_I|}{|\Lambda|} > 1 - \frac{1}{4d}$ for which $\cos 2\pi\langle\lambda, y\rangle > 1/2$ (respectively, $\cos 2\pi\langle\lambda, y\rangle < -1/2$) for all $y \in I$ and all $\lambda \in \Lambda_I$.*

Proof. — Let $x \in \Lambda$ be a positive singular point, and let $\Lambda_I = \Lambda_x$ be the set of frequencies for which $\cos 2\pi\langle\lambda, x\rangle > 3/4$. It suffices to show that if $|y - x| \ll 1/M$ then $\cos 2\pi\langle\lambda, y\rangle > 1/2$ for all $\lambda \in \Lambda_x$.

By the mean value theorem and Cauchy-Schwartz,

$$\begin{aligned} |\cos 2\pi\langle\lambda, y\rangle - \cos 2\pi\langle\lambda, x\rangle| &= |\langle -2\pi \sin 2\pi\langle\lambda, \xi\rangle\lambda, x - y\rangle| \\ &\leqslant 2\pi|\lambda||x - y| \ll \frac{\sqrt{E}}{M} \end{aligned}$$

and hence if $M \gg \sqrt{E}$ (all implied constants are absolute, depending only on the dimension d) and $\cos 2\pi\langle\lambda, x\rangle > 3/4$ then

$$\cos 2\pi\langle\lambda, y\rangle \geqslant \cos 2\pi\langle\lambda, x\rangle - |\cos 2\pi\langle\lambda, y\rangle - \cos 2\pi\langle\lambda, x\rangle| > \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

as required. The case of negative singular cubes is analogous. □

As Lemma 6.4 shows, singular cubes cannot be both positive and negative.

Let B be the union of all singular cubes. Since the volume of each cube is $1/M^d$, the number of such cubes is $M^d \operatorname{meas}(B)$.

Lemma 6.5. — *i) If $x \notin B$ then $|u(x)| < 1 - \frac{1}{16d}$.*
ii) If $x \in B$ then $|u(x)| > \frac{1}{2} - \frac{3}{8d} \geqslant \frac{1}{16}$.
iii) $\operatorname{meas}(B) \leqslant 16^4 \int_{\mathbb{T}^d} u(x)^4 dx$.

Proof. — i) If $x \notin B$, then x is neither a positive nor a negative singular point, hence there are subsets $\Lambda', \Lambda'' \subset \Lambda$ each of density $> \frac{1}{4d}$ for which

(1) It suffices to take $M = \lfloor 16\pi\sqrt{d}\sqrt{E}\rfloor$.

$\cos 2\pi\langle\lambda, x\rangle \leqslant \frac{3}{4}$ for all $\lambda \in \Lambda'$ and $\cos 2\pi\langle\lambda, x\rangle \geqslant -\frac{3}{4}$ for all $\lambda \in \Lambda''$. Hence

$$\begin{aligned} u(x) &= \frac{1}{\mathcal{N}} \sum_{\lambda\in\Lambda'} \cos 2\pi\langle\lambda, x\rangle + \frac{1}{\mathcal{N}} \sum_{\lambda\notin\Lambda'} \cos 2\pi\langle\lambda, x\rangle \\ &\leqslant \frac{3}{4}\frac{|\Lambda'|}{\mathcal{N}} + \frac{\mathcal{N} - |\Lambda'|}{\mathcal{N}} \\ &= 1 - \frac{1}{4}\frac{|\Lambda'|}{\mathcal{N}} < 1 - \frac{1}{16d}\,. \end{aligned}$$

Likewise, using Λ'' instead of Λ', we also have $u(x) > -1 + \frac{1}{16d}$ and hence $|u(x)| < 1 - \frac{1}{16d}$.

ii) Suppose $x \in B$ lies in a positive singular cube. Then by Lemma 6.4 there is $\Lambda' \subset \Lambda$, with $\frac{|\Lambda'|}{|\Lambda|} > 1 - \frac{1}{4d}$, such that $\cos 2\pi\langle\lambda, x\rangle > \frac{1}{2}$ for all $\lambda \in \Lambda'$.

Hence

$$\begin{aligned} u(x) &= \frac{1}{\mathcal{N}} \sum_{\lambda\in\Lambda'} \cos 2\pi\langle\lambda, x\rangle + \frac{1}{\mathcal{N}} \sum_{\lambda\notin\Lambda'} \cos 2\pi\langle\lambda, x\rangle \\ &> \frac{1}{\mathcal{N}} \sum_{\lambda\in\Lambda'} \frac{1}{2} + \frac{1}{\mathcal{N}} \sum_{\lambda\notin\Lambda'} (-1) \\ &= \frac{1}{2}\frac{|\Lambda'|}{\mathcal{N}} - \frac{\mathcal{N} - |\Lambda'|}{\mathcal{N}} = \frac{3}{2}\frac{|\Lambda'|}{\mathcal{N}} - 1 \\ &> \frac{3}{2}(1 - \frac{1}{4d}) - 1 = \frac{1}{2} - \frac{3}{8d} \geqslant \frac{1}{16}\,. \end{aligned}$$

Thus $u(x) > \frac{1}{2} - \frac{3}{8d} \geqslant \frac{1}{16}$. Likewise if x lies in a negative singular cube we will find that $u(x) < -\frac{1}{16}$ and hence for all $x \in B$ we have $|u(x)| > \frac{1}{16}$.

iii) follows from (ii) by a Chebyshev type inequality. □

We separately compute the contributions I_B, I_{B^c}, of the singular set B and its complement B^c to (6.1).

6.2. The contribution of B^c

This will be the main term. For $x \notin B$, since $|u(x)|$ is bounded away from 1, we may use the Taylor expansion

$$\frac{1}{\sqrt{1 - u(x)^2}} = 1 + \frac{1}{2}u(x)^2 + O(u(x)^4)$$

(the implied constant independent of Λ!) to find

$$\begin{aligned} I_{B^c} &= \frac{1}{2\pi}\int_{B^c} \frac{dx}{\sqrt{1 - u(x)^2}} = \frac{1}{2\pi}\int_{B^c} \left(1 + \frac{1}{2}u(x)^2 + O(u(x)^4)\right) dx \\ &= \frac{1}{2\pi} + \frac{1}{4\pi}\int_{\mathbb{T}^d} u(x)^2 dx + O(\mathrm{meas}(B)) + O(\int_{\mathbb{T}^d} u(x)^4 dx) \end{aligned}$$

on using $|u(x)| \leqslant 1$; then since $\int_{\mathbb{T}^d} u(x)^2 dx = \frac{1}{\mathcal{N}}$ and $\mathrm{meas}(B) \ll \int_{\mathbb{T}^d} u^4$ by Lemma 6.5(iii), we find

$$I_{B^c} = \frac{1}{2\pi} + \frac{1}{4\pi\mathcal{N}} + O\Big(\int_{\mathbb{T}^d} u^4\Big) . \tag{6.2}$$

6.3. The contribution of the singular set B

To estimate I_B, we will show that each integral over a single singular cube contributes $O(1/M^{d-1}\sqrt{E})$. Since the number of singular cubes is $M^d\,\mathrm{meas}(B)$, we will find that the total contribution of I_B is bounded by

$$I_B \ll \mathrm{meas}(B)\frac{M}{\sqrt{E}} \approx \mathrm{meas}(B) \ll \int_{\mathbb{T}^d} u^4 \tag{6.3}$$

because we assume that $M \approx \sqrt{E}$. Together with (6.2), this will prove Proposition 6.1.

6.4. A bound for the Hessian of u on a cube

The Hessian of u is $H = (\frac{\partial^2 u}{\partial x_i \partial x_j})$. We will need to know:

Lemma 6.6. — *The Hessian of u at any point in a positive singular cube is negative definite and satisfies*

$$\xi^T H \xi \leqslant -\frac{\pi^2 E}{2d}||\xi||^2 .$$

Likewise for a negative singular cube the Hessian is positive definite and satisfies $\xi^T H \xi \geqslant \frac{\pi^2 E}{2d}||\xi||^2$.

Proof. — The Hessian H_λ of $\cos 2\pi\langle\lambda, x\rangle$ is given by

$$(H_\lambda)_{i,j} = -(2\pi)^2 \cos 2\pi\langle\lambda, x\rangle\lambda_i\lambda_j = -(2\pi)^2 \cos 2\pi\langle\lambda, x\rangle(\lambda\lambda^T)_{i,j}$$

(if we think of λ as a column vector) for which

$$\xi^T H_\lambda \xi = -\cos 2\pi\langle\lambda, x\rangle\langle\lambda, 2\pi\xi\rangle^2 .$$

Let $\Lambda' \subset \Lambda$ be a set of frequencies of density $> 1 - \frac{1}{4d}$ so that for all x in the singular cube, and all $\lambda \in \Lambda'$, we have $\cos 2\pi\langle\lambda, x\rangle > 1/2$. Then for $\lambda \in \Lambda'$ (the weak inequality is introduced to cover the case that $\langle\lambda, \xi\rangle = 0$)

$$\xi^T H_\lambda \xi \leqslant -\frac{1}{2}\langle\lambda, 2\pi\xi\rangle^2 .$$

For the remaining $\lambda \notin \Lambda'$, we use $-\cos 2\pi\langle\lambda, x\rangle \leqslant 1$ to get $\xi^T H_\lambda \xi \leqslant \langle\lambda, 2\pi\xi\rangle^2$. Hence the Hessian H of u at x satisfies

$$\begin{aligned}\xi^T H\xi &= \frac{1}{\mathcal{N}}\sum_{\lambda\in\Lambda}\xi^T H_\lambda\xi \\ &\leqslant -\frac{1}{2}\frac{1}{\mathcal{N}}\sum_{\lambda\in\Lambda'}\langle\lambda, 2\pi\xi\rangle^2 + \frac{1}{\mathcal{N}}\sum_{\lambda\notin\Lambda'}\langle\lambda, 2\pi\xi\rangle^2 \\ &= -\frac{1}{2}\frac{1}{\mathcal{N}}\sum_{\text{all }\lambda}\langle\lambda, 2\pi\xi\rangle^2 + \frac{3}{2}\frac{1}{\mathcal{N}}\sum_{\lambda\notin\Lambda'}\langle\lambda, 2\pi\xi\rangle^2\end{aligned}$$

for all ξ. By Lemma 5.2 we have

$$-\frac{1}{2}\frac{1}{\mathcal{N}}\sum_{\text{all }\lambda}\langle\lambda, 2\pi\xi\rangle^2 = -\frac{2\pi^2 E}{d}||\xi||^2 .$$

For the sum over $\lambda \notin \Lambda'$, use Cauchy-Schwartz to write

$$\langle\lambda, 2\pi\xi\rangle^2 \leqslant 4\pi^2 E||\xi||^2$$

and the sum over these $\lambda \notin \Lambda'$ is hence bounded by

$$\frac{3}{2}\frac{\mathcal{N} - |\Lambda'|}{\mathcal{N}}4\pi^2 E||\xi||^2 \leqslant \frac{3\pi^2}{2d}E||\xi||^2$$

(since $\frac{|\Lambda'|}{\mathcal{N}} \geqslant 1 - \frac{1}{4d}$). Thus we find

$$\xi^T H\xi \leqslant -\frac{\pi^2}{2d}E||\xi||^2$$

as required. □

6.5. The contribution of a singular cube

To find the contribution to the integral of each singular cube I_k, assume the cube contains a positive singular point.

Pick a point $x_0 \in I_k$ for which $u(x_0)$ is maximal in I_k. Now use the Taylor expansion around x_0 with remainder

$$u(x) = u(x_0) + \nabla u(x_0)\cdot(x - x_0) + R_2(x)$$

where the remainder $R_2(x)$ can be given in terms of the Hessian H of u as

$$R_2(x) = \frac{1}{2}(x - x_0)^T H(z)(x - x_0)$$

where z is some point on the line segment between x_0 and x. Since the cube is convex, z also belongs to the singular cube. Thus by Lemma 6.6, we have

$$R_2(x) \leqslant -\frac{\pi^2 E}{4d}||x - x_0||^2 .$$

The directional derivative at x_0 of u in the direction of any other point in the cube is nonpositive (since the function is decreasing as we go from x_0 to nearby points in the cube) and hence

$$\nabla u(x_0) \cdot (x - x_0) \leqslant 0$$

for all points x in the cube, as this quantity is a positive multiple of the directional derivative of u at x_0 in the direction of the line joining x_0 to x. Thus

$$\begin{aligned} u(x) &= u(x_0) + \nabla u(x_0) \cdot (x - x_0) + \frac{1}{2}(x - x_0)^T H(z)(x - x_0) \\ &\leqslant 1 + 0 - \frac{\pi^2 E}{4d}||x - x_0||^2 . \end{aligned}$$

Therefore

$$1 - u^2 \gg E||x - x_0||^2$$

amd hence the integral over a positive singular cube is bounded by

$$\int_{||x-x_0|| \ll 1/M} \frac{dx}{\sqrt{E||x - x_0||^2}} \ll \frac{1}{\sqrt{E}} \int_0^{1/M} \frac{r^{d-1} dr}{r} \approx \frac{1}{\sqrt{E} M^{d-1}} .$$

The case of a negative singular cube is analogous; instead of using a maximum of u in the cube we take x_0 to be a minimum of u in the cube and show that $u(x) \geqslant -1 + \frac{\pi^2 E}{4d}||x - x_0||^2$.

Thus we have proved (6.3) and hence are done with the proof of Proposition 6.1.

7. Bounding the fourth moment of the two-point function

In this section we bound the fourth moment of the two-point function

$$u(x) = \frac{1}{\mathcal{N}} \sum_{\lambda \in \Lambda} e^{2\pi i \langle \lambda, x \rangle} .$$

Note that

$$\int_{\mathbb{T}^d} u(x)^4 dx = \frac{1}{\mathcal{N}^4} \#\{\lambda_1, \lambda_2, \lambda_3, \lambda_4 : \lambda_1 + \lambda_2 = \lambda_3 + \lambda_4\} .$$

The number of solutions of the equation

$$\lambda_1 + \lambda_2 = \lambda_3 + \lambda_4, \qquad \lambda_i \in \Lambda \tag{7.1}$$

is at most $\mathcal{N}^3$ since fixing three of the variables determines the fourth one. Thus

$$\int u^4 dx \leqslant \frac{1}{\mathcal{N}} \ . \tag{7.2}$$

This bound used no special property of the set of frequencies Λ. For the set $\Lambda_E = \{\lambda : |\lambda|^2 = E\}$ we can do much better.

Proposition 7.1. — *i) In dimension $d = 2$, we have*

$$\int u^4 \ll \frac{1}{\mathcal{N}^2} \ .$$

ii) In dimension $d \geqslant 3$,

$$\int_{\mathbb{T}^d} u(x)^4 dx \ll_\epsilon \frac{E^{\frac{d-3}{2}+\epsilon}}{\mathcal{N}^2}$$

for all $\epsilon > 0$.

To prove the proposition, we need to bound the number of solutions of (7.1). A simple geometric argument pointed out by Zygmund [21] shows that in dimension $d = 2$, the only solutions of (7.1) are "diagonal" solutions, that is $\lambda_1 = \lambda_3$, or $\lambda_1 + \lambda_2 = 0 = \lambda_3 + \lambda_4$ etcetera. This gives the required bound in two dimensions.

For higher dimensions, we want to show that the number of solutions of (7.1) is $\ll \mathcal{N}^2 E^{\frac{d-3}{2}+\epsilon}$. Fix λ_3, λ_4. If $\lambda_3 + \lambda_4 = 0$ then $\lambda_1 + \lambda_2 = 0$ and there are $\mathcal{N}^2$ such pairs. So we may ignore them and assume that $\nu := \lambda_3 + \lambda_4 \neq 0$ and then we wish to show that there are at most $E^{\frac{d-3}{2}+\epsilon}$ choices of of λ_1, λ_2 with $\lambda_1 + \lambda_2 = \nu$ given. Since $\lambda_2 = \nu - \lambda_1$ is determined by λ_1, we thus need to show:

Lemma 7.2. — *Let $d \geqslant 3$ and $0 \neq \nu \in \mathbb{Z}^d$. Then the number of $\lambda \in \mathbb{Z}^d$ with*

$$|\lambda|^2 = E = |\nu - \lambda|^2 \tag{7.3}$$

is at most $c(\epsilon) E^{\frac{d-3}{2}+\epsilon}$ for all $\epsilon > 0$ with $c(\epsilon) > 0$ independent of ν.

Proof. — To see this, rewrite the equations as

$$|\lambda|^2 = E, \quad 2\langle \lambda, \nu \rangle = |\nu|^2$$

or

$$\sum_{j=1}^d x_j^2 = E, \quad 2\sum_{j=1}^d \nu_j x_j = |\nu|^2 \ .$$

Fix the last $d-3$ coordinates $x_4, \dots, x_d$ (there are at most $E^{\frac{d-3}{2}}$ such choices) and lets count the number of solutions of the resulting system of equations

$$x_1^2 + x_2^2 + x_3^2 = R, \quad \nu_1 x_1 + \nu_2 x_2 + \nu_3 x_3 = S \tag{7.4}$$

where $R \leqslant E$ and $|\nu_i|, |S| \ll E$. The number of solutions of (7.3) is thus bounded by $E^{\frac{d-3}{2}}$ times the number of solutions of equations such as (7.4). So it suffices to show that the number of solutions of (7.4) is at most $c(\epsilon)E^\epsilon$ uniformly in ν.

Solving the linear equation for x_3 and substituting in the quadratic equation gives an inhomogeneous quadratic equation

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1 + ex_2 + f = 0$$

where all coefficients are integers which are at most polynomial in E and the homogeneous quadratic part is positive definite. Then one may complete the square and change variables to get an equation

$$x^2 + Dy^2 = k$$

where $D > 0$, and D, k are polynomial in E. Thus the number of solutions of (7.4) is bounded by the number $r_D(k)$ of representations of an integer k by the quadratic form $x^2 + Dy^2$.

Now we claim that $r_D(k)$ is at most

$$r_D(k) \leqslant 6\tau(k) \tag{7.5}$$

where $\tau(k)$ is the number of divisors of k. Since $\tau(k) \ll k^\epsilon$, $\forall \epsilon > 0$, this will imply that the number of solutions to (7.4) is at most $c(\epsilon)E^\epsilon$ uniformly in ν and conclude the proof of the lemma.

The uniform estimate (7.5) follows from factorization into prime ideals in the ring of integers of the imaginary quadratic extension $\mathbb{Q}(\sqrt{-D})$: Indeed, $r_D(k)$ is at most the number $\rho(k)$ of ideals of norm k, times the number of units of the field, which is at most 6. Now the Dirichlet series $\zeta_D(s) := \sum_{k \geqslant 1} \rho(k)/k^s$ is the Dedekind zeta function of the field $\mathbb{Q}(\sqrt{-D})$, and by class-field theory there is a factorization $\zeta_D(s) = \zeta(s)L(s,\chi)$ where $\zeta(s)$ is the Riemann zeta function, and $L(s,\chi)$ is the Dirichlet L-function associated to the quadratic character χ attached to $\mathbb{Q}(\sqrt{-D})$. Thus $\rho(k) = \sum_{m|k} \chi(m)$ and therefore $\rho(k)$ is bounded by the number $\tau(k)$ of divisors of k. Thus $r_D(k) \leqslant 6\tau(k)$. □

Remark 7.3. — For higher dimensions, one can improve on the trivial bound (7.2) by noting that $u(x)$ is itself an eigenfunction of the Laplacian

with eigenvalue $4\pi^2 E$, and then appealing to the general results of Sogge [19] on L^p-norms of eigenfunctions. We recall these: Let

$$M_{d,p}(E) = \sup_{\Delta f + 4\pi^2 E f = 0} \frac{||f||_p}{||f||_2} .$$

Then for $p \leqslant 4$ we have (using $|u| \leqslant 1$) that $\int u^4 \leqslant ||u||_p^p$ and hence

$$\int_{\mathbb{T}^d} u^4(x)dx \ll_p \left(\frac{M_{d,p}(E)}{\sqrt{\mathcal{N}}}\right)^p .$$

Sogge showed that for eigenfunctions of the Laplacian on any smooth compact Riemannian manifold, and for for $p = p_d := 2(d+1)/(d-1)$, one has $M_{d,p}(E) \ll E^{1/2p_d}$. Since $p_d \leqslant 4$ for $d \geqslant 3$, we have

$$\int_{\mathbb{T}^d} u^4(x)dx \ll \frac{E^{1/2}}{\mathcal{N}^{p_d/2}} .$$

In dimension $d \geqslant 5$ we have $\mathcal{N} \approx E^{\frac{d}{2}-1}$ and hence we find

$$\int_{\mathbb{T}^d} u^4(x)dx \ll \frac{1}{\mathcal{N}^{1+\alpha(d)}}, \quad \alpha(d) = \frac{2}{d-1} - \frac{1}{d-2}$$

which improves on (7.2) whenever $d > 3$ since $\alpha(d) > 0$ for $d > 3$.

For the torus in dimension $d \geqslant 4$, Bourgain [5] showed that for $p \geqslant \frac{2(d+1)}{d-3}$,

$$M_{d,p}(E) \ll E^{\frac{d-2}{4} - \frac{d}{2p} + \epsilon}, \forall \epsilon > 0$$

which improves on Proposition 7.1 in dimension $d \geqslant 7$ (when we may take $p = 4$).

Appendix A. The intersection of the singular set with codimension one hyperplanes

We consider the hyperplane

$$\mathcal{P}_x^a = \{f \in \mathcal{E} : \ f(x) = a\}$$

and show that the set of singular functions in $\mathcal{P}_x^a$ has measure zero. Assume that the set of frequencies Λ, which is assumed to be "symmetric", further satisfies the non-degeneracy condition (2.2), that is:

$$\exists \lambda \in \Lambda \text{ with } \lambda_1 \neq \pm\lambda_2 \text{ and } \lambda_1, \lambda_2 \neq 0 . \tag{A.1}$$

By the symmetry of the set Λ, condition (A.1) is equivalent to requiring that for every $i \neq j$, there is $\lambda \in \Lambda$ with $\lambda_i \neq \pm\lambda_j$ and $\lambda_i, \lambda_j \neq 0$.

Proposition A.1. — *Assume that Λ is symmetric and satisfies the nondegeneracy condition* (A.1). *Then for all $x \in \mathbb{T}^d$, and all a, the intersection $\mathcal{P}_x^a \cap Sing$ has measure zero in $\mathcal{P}_x^a$.*

In order to prove Proposition A.1, we will need some lemmas.

Let $L_\Lambda \subset \mathbb{Z}^d$ be the lattice spanned by Λ. By Lemma 2.1, it is a sublattice of full rank, hence its dual L_Λ^* is still a lattice in $\mathcal{E}$. In § 2.3 we defined the set $\mathcal{B}$ by

$$\mathcal{B} = \{w \in \mathbb{R}^d : \langle \lambda, w\rangle \in \mathbb{Z} \quad \forall \lambda \in \Lambda \text{ or } \langle \lambda, w\rangle \in \frac{1}{2} + \mathbb{Z} \quad \forall \lambda \in \Lambda\} .$$

Let

$$\mathcal{B}_x := \{y \in \mathbb{T}^d : x - y \in \mathcal{B}\} .$$

Note that if $y \in \mathcal{B}_x$ then for all $f \in \mathcal{E}$, $f(x) = \pm f(y)$ and $\nabla f(x) = \pm \nabla f(y)$.

Lemma A.2. — *Suppose that Λ is symmetric and satisfies the nondegeneracy condition* (A.1). *If $w \notin \mathcal{B}$ then there are no nonzero solutions $(\vec{c}, b', b'') \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$, satisfying*

$$\langle \vec{c}, \lambda\rangle = b'' \sin 2\pi\langle w, \lambda\rangle \tag{A.2}$$

$$b' = b'' \cos 2\pi\langle w, \lambda\rangle \tag{A.3}$$

for all $\lambda \in \Lambda$.

Proof. — If $b'' = 0$ then $b' = 0$ and since Λ spans $\mathbb{R}^d$ by Lemma 2.1, we find $\vec{c} = 0$. Otherwise, from (A.3) we find that $\forall \lambda \in \Lambda$

$$\sin 2\pi\langle w, \lambda\rangle = \pm\sqrt{1 - (\frac{b'}{b''})^2} \tag{A.4}$$

(necessarily $|b'| \leqslant |b''|$). Set

$$\gamma = \sqrt{(b'')^2 - (b')^2} .$$

We will show that $\vec{c} = \vec{0}$, which implies that $\sin 2\pi\langle w, \lambda\rangle = 0$ for all $\lambda \in \Lambda$, and thus $\cos 2\pi\langle w, \lambda\rangle = \pm 1$; by (A.3), $\cos 2\pi\langle w, \lambda\rangle$ is constant and so is either $+1$ for all $\lambda \in \Lambda$ or equals -1 for all $\lambda \in \Lambda$, hence we will find that $w \in \mathcal{B}$, contradicting our assumption.

Fix $j = 1, \ldots, d$ and we wish to see $c_j = 0$; by symmetry we may take $j = 1$. Find $\lambda \in \Lambda$ satisfying condition (A.1). Next, replacing λ by $-\lambda$ if necessary, we may assume that

$$\langle \vec{c}, \lambda\rangle = +\gamma$$

that is

$$\lambda_1 c_1 + \sum_{i \neq 1} c_i \lambda_i = +\gamma . \tag{A.5}$$

Let $\hat{\lambda} = (-\lambda_1, \lambda_2, \dots) \in \Lambda$ be the result of changing the sign of the first coordinate of λ. Then $\langle \hat{\lambda}, \vec{c} \rangle = \pm\gamma$, that is

$$-\lambda_1 c_1 + \sum_{i \neq 1} c_i \lambda_i = \pm\gamma \ . \tag{A.6}$$

If the sign is $+$, we compare (A.6) with (A.5) to deduce that

$$c_1 \lambda_1 = 0$$

and since $\lambda_1 \neq 0$ we find that $c_1 = 0$.

Otherwise, if the sign in (A.6) is $-$, we compare with (A.5) to find

$$c_1 \lambda_1 = +\gamma \ . \tag{A.7}$$

Repeating the above argument with λ replaced by $(\lambda_2, \lambda_1, \dots) \in \Lambda$ (that is we switch the first and second coordinates), we find that either $c_1 = 0$ or else

$$c_1 \lambda_2 = +\gamma \tag{A.8}$$

and together with (A.7) we find that

$$c_1 \lambda_2 = +\gamma = c_1 \lambda_1 \ .$$

Since $\lambda_2 \neq \lambda_1$ we find again that $c_1 = 0$. □

Lemma A.3. — *Suppose that Λ is symmetric and satisfies the nondegeneracy condition* (A.1). *Then for every $x \in \mathbb{T}^d$, the map Ψ_x given by*

$$\begin{aligned} \Psi_x : (\mathbb{T}^d \backslash \mathcal{B}_x) \times \mathcal{E} &\to \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \\ (y, f) &\mapsto (\nabla f(y), f(y), f(x)) \end{aligned} \tag{A.9}$$

is a submersion.

Proof. — We wish to show that the derivative $D_{y,f}\Psi_x : \mathbb{R}^d \times \mathbb{R}^{\mathcal{N}} \to \mathbb{R}^{d+2}$ at the point (y, f) has rank $d+2$. For this it suffices to show that the $(d+2) \times \mathcal{N}$ matrix $\frac{\partial \Psi_x}{\partial f}$ has rank $d+2$. Now

$$\frac{\partial \Psi_x}{\partial f} = \bigoplus_{\lambda \in \Lambda/\pm} \sqrt{\frac{2}{\mathcal{N}}} \begin{pmatrix} -2\pi \sin 2\pi\langle \lambda, y \rangle \vec{\lambda} & -2\pi \cos 2\pi\langle \lambda, y \rangle \vec{\lambda} \\ \cos 2\pi\langle \lambda, y \rangle & -\sin 2\pi\langle \lambda, y \rangle \\ \cos 2\pi\langle \lambda, x \rangle & -\sin 2\pi\langle \lambda, x \rangle \end{pmatrix} .$$

Post-multiplying it by the (block-diagonal) invertible matrix

$$\bigoplus_{\lambda \in \Lambda/\pm} \sqrt{\frac{\mathcal{N}}{2}} \begin{pmatrix} -\sin 2\pi\langle \lambda, y \rangle & \cos 2\pi\langle \lambda, y \rangle \\ -\cos 2\pi\langle \lambda, y \rangle & -\sin 2\pi\langle \lambda, y \rangle \end{pmatrix}$$

gives the $(d+2)\times\mathcal{N}$ matrix

$$\bigoplus_{\lambda\in\Lambda/\pm}\begin{pmatrix} 2\pi\vec{\lambda} & \vec{0} \\ 0 & 1 \\ \sin 2\pi\langle\lambda, x-y\rangle & \cos 2\pi\langle\lambda, x-y\rangle \end{pmatrix} .$$

Thus we want to show that the rank of this matrix is $d+2$.

For this it suffices to show that the rows are linearly independent, that is there is no non-trivial solution $(\vec{c}, b', b'')\in\mathbb{R}^{d+2}$ to the system

$$\begin{aligned} \langle\vec{c},\lambda\rangle &= b''\sin 2\pi\langle x-y,\lambda\rangle \\ b' &= b''\cos 2\pi\langle x-y,\lambda\rangle \end{aligned}$$

which by Lemma A.2 this has no solutions if $x-y\notin\mathcal{B}$, that is if $y\notin\mathcal{B}_x$. □

Proof of Proposition A.1. — We will partition $\mathcal{P}_x^a\cap Sing$ into two sets: The set $Sing_x^{\text{in}}$ of those f for which all singular points of the nodal set of f lie in $\mathcal{B}_x$ (here necessarily $a=0$), and the set $Sing_x^{\text{out}}$ of those f for which there is a singular point of the nodal set outside $\mathcal{B}_x$. We will show that each has measure zero.

We first show that $Sing_x^{\text{in}}$ has measure zero. We will in fact see that it is a linear subspace of codimension d in $\mathcal{P}_x^0$. Note that if $y\in\mathcal{B}_x$ then $f(y)=\pm f(x)$ and $\nabla f(y)=\pm\nabla f(x)$ and so

$$Sing_x^{\text{in}} = \{f\in\mathcal{E} : f(x)=0, \quad \nabla f(x)=\vec{0}\} .$$

Thus $Sing_x^{\text{in}}$ are the solutions to the linear system of equations

$$f(x)=0, \quad \nabla f(x)=0 .$$

The $(d+1)\times|\Lambda|$ matrix of this system is

$$\bigoplus_{\lambda\in\Lambda/\pm}\begin{pmatrix} -2\pi\sin 2\pi\langle\lambda,x\rangle\vec{\lambda} & -2\pi\cos 2\pi\langle\lambda,x\rangle\vec{\lambda} \\ \cos 2\pi\langle\lambda,x\rangle & -\sin 2\pi\langle\lambda,x\rangle \end{pmatrix}$$

which as we have seen in the proof of Lemma 2.3 has rank $d+1$, and thus $Sing_x^{\text{in}}\subset\mathcal{P}_x^0$ has codimension d in $\mathcal{P}_x^0$.

We now turn to $Sing_x^{\text{out}}$. Let $\pi_{\mathcal{E}}:\mathbb{T}^d\times\mathcal{E}\to\mathcal{E}$ be the projection on the second factor; then by the definition (A.9) of Ψ_x,

$$\pi_{\mathcal{E}}(\Psi_x^{-1}(\vec{0},0,a)) = Sing_x^{\text{out}} .$$

Lemma A.3 shows, in particular, that $(\vec{0},0,a)$ is a regular value of Ψ_x, so that $\Psi_x^{-1}(\vec{0},0,a)$ is a submanifold of $\mathbb{T}^d\times\mathcal{E}$ of codimension $d+2$, that is $\Psi_x^{-1}(\vec{0},0,a)\subset\mathbb{T}^d\times\mathcal{P}_x^a$ has dimension $|\Lambda|-2$. Therefore $Sing_x^{\text{out}} = \pi_{\mathcal{E}}(\Psi_x^{-1}(\vec{0},0,a))\subset\mathcal{P}_x^a$ has dimension at most $|\Lambda|-2$ in the $(|\Lambda|-1)$-dimensional space $\mathcal{P}_x^a$ and hence has measure zero. □

Appendix B. The intersection of the singular set with codimension two hyperplanes

For $a = (a_1, a_2) \in \mathbb{R}^2$, let

$$\mathcal{P}^a_{x,y} = \{f \in \mathcal{E} : f(x) = a_1, f(y) = a_2\} .$$

If $x - y \notin \mathcal{B}$ then this is an affine hyperplane of codimension two in $\mathcal{E}$. If $x - y \in \mathcal{B}$ then this is either empty or a hyperplane of codimension one in $\mathcal{E}$.

Proposition B.1. — *For $d \geqslant 2$, for any symmetric set of frequencies Λ satisfying the non-degeneracy condition* (A.1)*, there is a set of measure zero $S = S_\Lambda \subset \mathbb{T}^d$ so that for $x - y \neq S$, the intersection $\mathcal{P}^a_{x,y} \cap Sing$ has measure zero in $\mathcal{P}^a_{x,y}$.*

The proof of Proposition B.1 follows along the lines of Proposition A.1, proving that the codimension is $\geqslant 1$. We will need a lemma about the nonexistence of solutions to certain systems of equations:

Lemma B.2. — *Let $d \geqslant 2$. Then for any symmetric set of frequencies Λ satisfying the non-degeneracy condition* (A.1)*, there is a set $S \subset \mathbb{T}^d$ of measure zero so that if $x - y \notin S$ then there do not exist $z \in \mathbb{T}^d$, numbers $b_1, b_2 \neq 0$ and b_3 and $\vec{c} \in \mathbb{R}^d$, which satisfy*

$$b_3 + i\langle \vec{c},\, \lambda\rangle = b_1 e^{2\pi i\langle \lambda,\, x-z\rangle} + b_2 e^{2\pi i\langle \lambda,\, y-z\rangle} \tag{B.1}$$

for every $\lambda \in \Lambda$.

Proof. — We choose $\lambda \in \Lambda$ satisfying condition (A.1), that is $\lambda_1, \lambda_2 \neq 0$ and $\lambda_1 \neq \pm\lambda_2$. Taking the norm-square of (B.1), we have

$$b_3^2 + \langle \vec{c}, \lambda\rangle^2 = b_1^2 + b_2^2 + 2b_1 b_2 \cos 2\pi\langle \lambda, x - y\rangle .$$

Now repeat this with λ replaced by

$$\lambda^\epsilon := (\epsilon_1\lambda_1, \epsilon_2\lambda_2, \ldots, \epsilon_d\lambda_d)$$

and sum the resulting equalities over all $\epsilon \in \{\pm 1\}^d$, each weighted by

$$\chi_{1,2}(\epsilon) = \epsilon_1\epsilon_2 .$$

This gives

$$\sum_{\epsilon\in\{\pm1\}^d} \chi_{1,2}(\epsilon)\left(b_3^2 + \langle \vec{c}, \lambda^\epsilon\rangle^2\right) = \sum_{\epsilon\in\{\pm1\}^d} \chi_{1,2}(\epsilon)\left(b_1^2 + b_2^2 + 2b_1 b_2 \cos 2\pi\langle \lambda^\epsilon, x - y\rangle\right) .$$

Now use

$$\sum_{\epsilon\in\{\pm1\}^d} \chi_{1,2}(\epsilon) = 0$$

to get

$$\sum_{\epsilon\in\{\pm1\}^d} \chi_{1,2}(\epsilon)\langle \vec{c}, \lambda^\epsilon\rangle^2 = 2b_1b_2 \sum_{\epsilon\in\{\pm1\}^d} \chi_{1,2}(\epsilon)\cos 2\pi\langle \lambda^\epsilon, x-y\rangle \ .$$

Expand

$$\langle \vec{c}, \lambda^\epsilon\rangle^2 = \sum_{j,k=1}^{d} \lambda_j\lambda_k c_j c_k \epsilon_j\epsilon_k$$

and use

$$\sum_{\epsilon\in\{\pm1\}^d} \chi_{1,2}(\epsilon)\epsilon_j\epsilon_k = \begin{cases} 2^d, & (j,k) = (1,2) \text{ or } (2,1) \\ 0 & \text{otherwise} \end{cases}$$

to get

$$\sum_{\epsilon\in\{\pm1\}^d} \chi_{1,2}(\epsilon)\langle \vec{c}, \lambda^\epsilon\rangle^2 = 2^{d+1}c_1c_2\lambda_1\lambda_2 \ .$$

Thus we find

$$2^{d+1}c_1c_2\lambda_1\lambda_2 = 2b_1b_2 \sum_{\epsilon\in\{\pm1\}^d} \chi_{1,2}(\epsilon)\cos 2\pi\langle \lambda^\epsilon, x-y\rangle \ . \tag{B.2}$$

We repeat the argument with λ replaced by

$$\tilde{\lambda} = (\lambda_2, \lambda_1, \lambda_3, \ldots, \lambda_d)$$

that is we have permuted the first and second coordinates of λ. Then we get

$$2^{d+1}c_1c_2\lambda_2\lambda_1 = 2b_1b_2 \sum_{\epsilon\in\{\pm1\}^d} \chi_{1,2}(\epsilon)\cos 2\pi\langle \tilde{\lambda}^\epsilon, x-y\rangle \ . \tag{B.3}$$

Comparing (B.2) with (B.3) and dividing by $2b_1b_2$ (which is nonzero by assumption), we get

$$\sum_{\epsilon\in\{\pm1\}^d} \chi_{1,2}(\epsilon)\cos 2\pi\langle \lambda^\epsilon, x-y\rangle = \sum_{\epsilon\in\{\pm1\}^d} \chi_{1,2}(\epsilon)\cos 2\pi\langle \tilde{\lambda}^\epsilon, x-y\rangle \ . \tag{B.4}$$

Writing

$$\cos 2\pi\langle \lambda^\epsilon, x-y\rangle = \frac{\exp 2\pi i\langle \lambda^\epsilon, x-y\rangle + \exp 2\pi i\langle \lambda^{-\epsilon}, x-y\rangle}{2}$$

and noting that $\chi_{1,2}(-\epsilon) = \chi_{1,2}(\epsilon) = \epsilon_1\epsilon_2$, we may rewrite (B.4) as

$$\sum_{\epsilon\in\{\pm1\}^d} \epsilon_1\epsilon_2 \exp 2\pi i\langle \lambda^\epsilon, x-y\rangle = \sum_{\epsilon\in\{\pm1\}^d} \epsilon_1\epsilon_2 \exp 2\pi i\langle \tilde{\lambda}^\epsilon, x-y\rangle \ . \tag{B.5}$$

If we use the identity

$$\sum_{\epsilon_3,\dots,\epsilon_d=\pm 1} \exp 2\pi i \sum_{j=3}^{d} \epsilon_j \lambda_j (x_j - y_j) = 2^{d-2} \prod_{j=3}^{d} \cos 2\pi \lambda_j (x_j - y_j)$$

and some simple trigonometric identities, then (B.5) becomes

$$\begin{aligned} &2^{d-1} \sin 2\pi\lambda_1(x_1-y_1) \sin 2\pi\lambda_2(x_2-y_2) \prod_{j=3}^{d} \cos 2\pi\lambda_j(x_j-y_j) \\ &\qquad = 2^{d-1} \sin 2\pi\lambda_2(x_1-y_1) \sin 2\pi\lambda_1(x_2-y_2) \prod_{j=3}^{d} \cos 2\pi\lambda_j(x_j-y_j)\ . \end{aligned}$$

This forces either

$$\sin 2\pi\lambda_1(x_1-y_1) \sin 2\pi\lambda_2(x_2-y_2) = \sin 2\pi\lambda_2(x_1-y_1) \sin 2\pi\lambda_1(x_2-y_2)\ ,$$

which is a measure zero condition on $x-y$ since we assume that $\lambda_1, \lambda_2 \neq 0$ and $\lambda_1 \neq \pm\lambda_2$, or else $d \geqslant 3$ and there is some $j \neq 1, 2$ with $\lambda_j \neq 0$ for which $\cos 2\pi\lambda_j(x_j - y_j) = 0$, which is again a measure zero condition on $x - y$. □

As before, we denote by $\mathcal{B}_x = x + \mathcal{B}$. For $x, y \in \mathbb{T}^d$, $x - y \notin \mathcal{B}$, consider the map

$$\begin{aligned} \Psi_{x,y} : \mathbb{T}^d \backslash (\mathcal{B}_x \cup \mathcal{B}_y) \times \mathcal{E} &\to \mathbb{R}^{d+3} \\ (z, f) &\mapsto (\nabla f(z), f(z), f(x), f(y)) \end{aligned} \tag{B.6}$$

Lemma B.3. — *Suppose that Λ is symmetric and satisfies the non-degeneracy condition* (A.1). *Then there is a set $S = S_\Lambda \subset \mathbb{T}^d$ of measure zero so that if $x - y \notin S$, then $\Psi_{x,y}$ is a submersion.*

Proof. — We wish to show that the derivative $D_{z,f}\Psi_{x,y} : \mathbb{R}^d \times \mathbb{R}^{\mathcal{N}} \to \mathbb{R}^{d+3}$ at the point (z, f) has rank $d+3$. For this it suffices to show that the $(d+3) \times \mathcal{N}$ matrix $\frac{\partial \Psi_{x,y}}{\partial f}$ has rank $d+3$. Now

$$\frac{\partial \Psi_{x,y}}{\partial f} = \bigoplus_{\lambda \in \Lambda/\pm} \sqrt{\frac{2}{\mathcal{N}}} \begin{pmatrix} -2\pi \sin 2\pi\langle\lambda, z\rangle \vec{\lambda} & -2\pi \cos 2\pi\langle\lambda, z\rangle \vec{\lambda} \\ \cos 2\pi\langle\lambda, z\rangle & -\sin 2\pi\langle\lambda, z\rangle \\ \cos 2\pi\langle\lambda, x\rangle & -\sin 2\pi\langle\lambda, x\rangle \\ \cos 2\pi\langle\lambda, y\rangle & -\sin 2\pi\langle\lambda, y\rangle \end{pmatrix}.$$

Post-multiplying it by the (block-diagonal) invertible matrix

$$\bigoplus_{\lambda \in \Lambda/\pm} \sqrt{\frac{\mathcal{N}}{2}} \begin{pmatrix} -\sin 2\pi\langle\lambda, z\rangle & \cos 2\pi\langle\lambda, z\rangle \\ -\cos 2\pi\langle\lambda, z\rangle & -\sin 2\pi\langle\lambda, z\rangle \end{pmatrix}$$

gives the $(d+3)\times\mathcal{N}$ matrix

$$\bigoplus_{\lambda\in\Lambda/\pm}\begin{pmatrix} 2\pi\vec{\lambda} & \vec{0} \\ 0 & 1 \\ \sin 2\pi\langle\lambda, x-z\rangle & \cos 2\pi\langle\lambda, x-z\rangle \\ \sin 2\pi\langle\lambda, y-z\rangle & \cos 2\pi\langle\lambda, y-z\rangle \end{pmatrix}.$$

Thus we want to show that the rank of this matrix is $d+3$, that is that the rows are linearly independent, i.e. that is there is no non-trivial solution $(\vec{c}, b_1, b_2, b_3)\in\mathbb{R}^{d+3}$ so that

$$\langle\vec{c},\,\lambda\rangle = b_1\sin 2\pi\langle\lambda,\,x-z\rangle + b_2\sin 2\pi\langle\lambda,\,y-z\rangle$$
$$b_3 = b_1\cos 2\pi\langle\lambda,\,x-z\rangle + b_2\cos 2\pi\langle\lambda,\,y-z\rangle,$$

for all $\lambda\in\Lambda$. We may write the system in a complex form as

$$b_3 + i\langle\vec{c},\,\lambda\rangle = b_1 e^{2\pi i\langle\lambda,\,x-z\rangle} + b_2 e^{2\pi i\langle\lambda,\,y-z\rangle}.$$

If either of b_1, b_2 is zero, we are in the same situation as in Lemma A.2 and so we deduce that either $x-z\in\mathcal{B}$ or $y-z\in\mathcal{B}*$, contradicting our assumption that $z\notin\mathcal{B}_x\cup\mathcal{B}_y$. If both $b_1, b_2\neq 0$, then Lemma B.2 implies the result of Proposition B.1. □

Proof of Proposition B.1: Given the measure zero set S of Lemma B.2, and $x, y\in\mathbb{T}^d$ with $x-y\notin S$, we write the set of singular elements in $\mathcal{P}^a_{x,y}$ as a union of two subsets each of which we will show to have measure zero:

$$\mathcal{P}^a_{x,y}\cap Sing = Sing^{\text{in}}_{x,y}\cup Sing^{\text{out}}_{x,y}$$

where:

i) $Sing^{\text{in}}_{x,y}$ consists of those $f\in\mathcal{P}^a_{x,y}$ for which all singular points of the nodal set (that is z so that $f(z)=0$, $\nabla f(z)=0$) lie in $\mathcal{B}_x\cup\mathcal{B}_y$. If $z\in\mathcal{B}_x$ then $f(x)=\pm f(z)$ and $\nabla f(x)=\pm\nabla f(z)$ so either $f(x)=0$, $\nabla f(x)=0$ or the same with y replacing x. If both $a_1, a_2\neq 0$ then $Sing^{\text{in}}_{x,y}=\emptyset$, and in any case we will see that $Sing^{\text{in}}_{x,y}$ has measure zero in $\mathcal{P}^a_{x,y}$: Indeed, as we saw in Lemma 2.3, for every $x\in\mathbb{T}^d$, the linear space

$$\{f\in\mathcal{E} : f(x)=0, \nabla f(x)=0\}$$

has codimension $d+1$ in $\mathcal{E}$. Since $\mathcal{P}^a_{x,y}$ has codimension 2 in $\mathcal{E}$, we find that $Sing^{\text{in}}_{x,y}$ is a union of two affine hyperplanes of codimension at least $d-1\geqslant 1$ in $\mathcal{P}^a_{x,y}$ (recall $d\geqslant 2$), and therefore has measure zero in $\mathcal{P}^a_{x,y}$.

ii) $Sing^{\text{out}}_{x,y}$ consists of $f\in\mathcal{P}^a_{x,y}$ for which there is a singular point z of the nodal set outside of $\mathcal{B}_x\cup\mathcal{B}_y$. Thus in the notation of (B.6),

$$Sing^{\text{out}}_{x,y} = \pi_{\mathcal{E}}\circ\Psi^{-1}_{x,y}(\vec{0}, 0, a)$$

where $\pi_{\mathcal{E}} : \mathbb{T}^d \times \mathcal{E} \to \mathcal{E}$ is the projection onto the second factor. Since $x-y \notin S$, we may use Lemma B.3 to deduce that $\Psi_{x,y}^{-1}(\vec{0}, 0, a)$ is a submanifold of $\mathbb{T}^d \times \mathcal{E}$ of codimension $d+3$, hence its projection $\pi_{\mathcal{E}} \circ \Psi_{x,y}^{-1}(\vec{0}, 0, a)$ has codimension at least 3 in $\mathcal{E}$ and hence codimension at least one in $\mathcal{P}_{x,y}^a$. Thus $\mathcal{P}_{x,y}^a \cap Sing$ has measure zero in $\mathcal{P}_{x,y}$, in fact has codimension at least one. $\square$

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