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THE SPACE OF BOUNDED ANALYTIC FUNCTIONS ON A REGION

by L. A. RUBEL and A. L. SHIELDS

Dedicated to the memory of David Lowdenslager

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CONTENTS

1. Introduction.

The origin of this work can be traced to two sources. The first is the work of J. Wolff [24] in which he produced a series of the form

$$F(z) = \sum \frac{a_n}{w_n - z}, \quad 0 < \Sigma \left| a_n \right| < \infty, \quad \left| w_n \right| < 1, \qquad (1.1)$$

$$n = 1, 2, 3, \ldots$$

such that

$$F(z) = 0 \qquad \text{all } \left| z \right| > 1. \qquad (1.2)$$

Related questions were investigated in some detail by Brown, Shields, and Zeller in [4], with the unit disc replaced by a general Jordan domain. In some sense, a substantial part of the present paper represents the mathematical content of that paper after such superstructure as series like (1.1) has been stripped away. To see the underlying idea, we may rewrite (1.2) as

$$F(z) = \int \frac{d\mu(w)}{w - z} = 0, \qquad \text{all} \qquad \left| z \right| > 1, \qquad (1.3)$$

where μ is the complex measure consisting of point masses $\{a_n\}$ at the points $\{w_n\}$ for $n = 1, 2, 3, \ldots$ But from (1.3), it follows easily that

$$\int f(w)\, d\mu(w) = 0 \qquad (1.4)$$

for each function f that is bounded and analytic in the unit disc, so that we are, in effect, concerned with measures of a special kind that annihilate all bounded analytic functions.

At this point, special restrictions on the domain G of the functions become irrelevant, except that to avoid trivialities, we suppose that G is a region and that not every bounded analytic function on G is a constant. (It is likely that many of our results and methods hold equally well for bounded analytic functions on Riemann surfaces, or for bounded analytic functions of several complex variables, but we do not pursue this line of thought.)

It is a natural step from here to pass to a detailed study of the duality

$$\langle f, \mu \rangle = \int f d\mu$$

between the bounded analytic functions f on the region G and the bounded complex measures μ that have all their mass in G. We conduct such a study, using as our main new tool the method of balayage, or sweeping of measures, borrowed from potential theory. We pay special attention to the weak topology α that the duality induces on the space $B_H(G)$ of all bounded analytic functions on G, and call the resulting topological vector space $\alpha(G)$. A generalized sequence $\{f_\gamma\}$, where γ lies in the partially ordered indexing set Γ, then converges to 0 provided that $\int f_\gamma d\mu$ converges to 0 for each such measure μ.

The other origin of our work is the extensive study that has been made, especially in case G is the open unit disc D, of the Banach algebra $H_\infty(G)$ of all bounded analytic functions on the region G, in the topology of uniform convergence. That study proceeds mostly by an analysis of the radial boundary values of the functions. One fact that emerges is that the ideal structure of $H_\infty(D)$ is extremely complicated. Another point is that these methods do not seem to carry over easily to more general regions. By imposing a weaker topology, the so-called strict topology β (introduced in this context by R. C. Buck [5] in the special case $G = D$) on the underlying space $B_H(G)$, some of these difficulties are removed, especially in the case $G = D$. A generalized sequence $\{f_\gamma\}$, $\gamma \in \Gamma$, converges to 0 in $\beta(G)$ provided that $\{f_\gamma k\}$ converges uniformly in G to 0 for each weight function k that is continuous on the closure of G and that vanishes on the boundary of G. For example, the resulting topological algebra $\beta(G)$ has the properties, in case $G = D$, that each closed ideal is principal, and that the closed maximal ideals correspond to points of the disc in a natural way.

The two spaces $\alpha(G)$ and $\beta(G)$ turn out to be closely related, although different, and this relationship can be exploited to advantage. For example, $\alpha(G)$ and $\beta(G)$ have the same closed subspaces, the same dual space, the same convergent sequences, the same compact sets, and the same bounded sets. In both $\alpha(G)$ and $\beta(G)$, a *sequence* $\{f_n\}$ is convergent if and only if it is boundedly convergent, that is, if and only if the $\{f_n\}$ are uniformly bounded and converge at each point of G. Of course, such a sequence must also converge uniformly on compact subsets of G. This fact, and others like it from the theory of normal families of analytic functions, are valuable tools.

At this point, it might be appropriate to address a few remarks to the more classically oriented analyst about our use of generalized sequences. As we have just remarked, the convergence of actual sequences in both $\alpha(G)$ and $\beta(G)$ is rather easy to deal with, whereas the convergence of generalized sequences is not so accessible. But because of such facts as Proposition 3.12, some consideration of generalized sequences is necessary. Nevertheless, because of Corollary 4.7 and Corollary 4.8, in studying the *linear* structure of $\alpha(G)$ and $\beta(G)$, and in particular the ideal structure of $\beta(G)$, only the convergence of actual sequences need be considered.

We proceed now from this preliminary description to a brief outline of the paper.

In § 2 we give the precise definitions of such entities as $B_H(G)$, $M'(G)$, etc., and prove some simple results about them. In § 3, we introduce the spaces $\alpha(G)$ and $\beta(G)$ and begin the study of their structure. In § 4, we discuss balayage. The first method of balayage, using an annular form of the Cauchy integral formula, sweeps a measure into the class of absolutely continuous measures with respect to planar Lebesgue measure. By this means, we prove that the space $M'(G)$ is separable and that $H_\infty(G)$ is its dual space. Further properties of $\alpha(G)$ and $\beta(G)$ follow from these considerations. Then we introduce the notion of a dominating subset S of G as one on which the supremum of each bounded analytic function is the same as its supremum on all of G, and prove that a set is dominating if and only if it is universal in the sense that any measure may be swept onto that set. This means that for any measure μ with all its mass in G, there is a measure ν with all its mass in S such that $\int f d\mu = \int f d\nu$ for each bounded analytic function f on G. This second method of balayage is effected by existence theorems from functional analysis. A third method is given rather explicitly in terms of a non-trivial measure μ, whose existence we assume, that has all its mass on a discrete subset of G and that has the property that $\int f d\mu = 0$ for each bounded analytic function f on G. We continue the section on balayage by characterizing sets of removable singularities for bounded analytic functions as those sets which are so thin that any measure can be swept a positive distance away from them. We conclude the section by proving, by our methods, a known result to the effect that any measure in the disc may be swept to the boundary of the disc as an absolutely continuous measure (with respect to Lebesgue linear measure), and conversely, that any such boundary measure may be swept inside the disc.

In the last section, we study the closed ideals in the topological algebra $\beta(G)$. In case G is the open unit disc, we prove that the principal ideal generated by a function f is dense if and only if f is an outer function, and is closed if and only if f is the product of an inner function and a function that is bounded away from 0. Here, we use « inner » and « outer » in the sense of Beurling. We prove that each clc ed ideal in $\beta(D)$ is the principal ideal generated by an inner function, nd use this result to characterize, for a rather large class of topological vector spaces of analytic functions, those closed subspaces that are invariant under multiplication by bounded analytic functions. We also prove, for a wide class of regions G, that the continuous multiplicative linear functionals on $\beta(G)$ correspond to point evaluations. We conclude the paper with a list of unsolved problems, some of which were mentioned in the body of the paper.

Some of the methods and results hold for classes of analytic functions satisfying growth restrictions other than boundedness, but we have not explored this question in any detail. The problem of polynominal approximation in spaces like $\alpha(G)$ and $\beta(G)$ was considered by the authors in [20] and has been further studied by D. Sarason (unpublished).

Some of the material of the present paper was presented to the American Mathematical Society, and an abstract of its principal results has appeared in [21].

2. Definitions and preliminaries.

2.1. By G we will always denote a connected open set in the complex plane. By G^- we denote the closure of G and by ∂G we denote the boundary of G. Both closure and boundary are taken with respect to the Riemann sphere, so that G^- and ∂G are compact.

2.2. We assume that there is a non-constant function that is bounded and analytic on G. Then the bounded analytic functions on G separate the points of G. For if f is a non-constant bounded analytic function and if $z_1 \neq z_2$, let $g(z) = f(z) - f(z_2)$ and let $h(z) = (z - z_1)^{-m} g(z)$, where m is the order of the zero of g at z_1, so that h vanishes at z_2 but not at z_1.

2.3. Given distinct points $z_1, ..., z_n$ in G and values $w_1, ..., w_n$, there is a bounded analytic function f that takes these values at these points. To show this, it is clearly sufficient to produce, for $j = 1, ..., n$, a bounded analytic function f_j that takes the value 1 at z_j and that vanishes at the other z_i. This is done as follows. Given $i \neq j$, choose φ_i so that $\varphi_i(z_i) = 0$, $\varphi_i(z_j) = 1$, and let f_j be the product of the φ_i.

2.4. $B_H(G)$ denotes the set of all bounded analytic functions on G, regarded as a complex vector space (or as an algebra over the complex numbers) with no topology. By $H_\infty(G)$, we denote the Banach algebra that has $B_H(G)$ as its underlying algebra, in the supremum norm

$$||f||_\infty = \sup \{|f(z)| : z \in G\}.$$

2.5. By M(G), we denote the set of all bounded complex-valued Borel measures μ that live in G (that is, the variation of μ over any set in the complement of G is 0). We consider M(G) as a Banach space in the norm

$$||\mu|| = \operatorname{var}(\mu) = |\mu|(G) = \int d\,|\mu|.$$

Now M(G) is paired with $B_H(G)$ via the inner product

$$\langle f, \mu \rangle = \int f d\mu.$$

There will exist measures $\mu \in M(G)$ such that $\langle f, \mu \rangle = 0$ for each $f \in B_H(G)$. We will discuss such measures in 2.8.

2.6. If f is continuous in G with $|f| \leqslant 1$ there, and if $\mu \in M(G)$, then

$$\left|\int f d\mu\right| \leqslant \int |f|\, d|\mu| \leqslant ||\mu||.$$

Equality holds in the first inequality if and only if $fd\mu$ is a constant multiple of a positive measure. Equality holds in the second inequality if and only if $|f| = 1$ almost everywhere with respect to $|\mu|$.

2.7. Proposition. — *If $f \in B_H(G)$ then the supremum norm of f is the same as the norm of f when it is regarded as a linear functional on the Banach space* M(G).

Proof. — By 2.6, the functional norm does not exceed the supremum norm. The reverse inequality follows by considering point measures.

2.8. We introduce an equivalence relation in $M(G) : \mu \sim \nu$ means that $\int f d\mu = \int f d\nu$ for all $f \in B_H(G)$. By N(G) we denote the set of measures equivalent to the zero measure. N(G) is a closed subspace of M(G) in the norm of 2.5.

2.9. Let $M'(G) = M(G)/N(G)$ be the space of equivalence classes $[\mu] = \mu + N(G)$ under the equivalence relation of 2.8, with the usual addition and scalar multiplication, under the quotient norm

$$||[\mu]|| = \inf \{||\mu + \nu|| : \nu \in N(G)\}.$$

A space related to M′(G) was studied in [10].

PROPOSITION 2.10. — *The infimum in* 2.9 *is attained if and only if there is a measure* μ' *equivalent to* μ *such that* $a\mu'$ *is a positive measure for some constant* a.

Proof. — If such a pair a, μ' exists, then we may choose a with $|a| = 1$. Now let f be the constant function with value a. We have for any $\nu \in N(G)$,

$$||[\mu]|| \leqslant ||\mu'|| = \int f d\mu' = \int f d(\mu + \nu) \leqslant ||f|| \, ||\mu + \nu|| = ||\mu + \nu||.$$

Hence $||\mu'|| = ||[\mu]||$.

In the other direction, we use Proposition 2.7, a later result (Theorem 4.5), and the Hahn-Banach Theorem, to show that to each $\mu \in M(G)$ there corresponds an $f \in H_\infty(G)$ with $||f|| = 1$, such that $\int f d\mu = ||[\mu]||$. Suppose now that μ' attains the infimum. Then we have

$$||\mu'|| = ||[\mu]|| = \int f d\mu' \leqslant ||f|| \, ||\mu'|| = ||\mu'||.$$

Hence by 2.6, $fd\mu'$ is a positive measure and $|f(z)| = 1$ almost everywhere with respect to $|\mu'|$. In particular, $|f(z)| = 1$ for at least one $z \in G$, and by the maximum principle, this implies that f is a constant.

2.11. Let $K = K(G)$ denote the set of all non-negative continuous functions k defined on G^- such that k vanishes on ∂G. It can be easily shown that given $\mu_1, \ldots, \mu_n \in M(G)$, there is some $k \in K$ for which

$$\int \frac{1}{k} d|\mu_i| < \infty, \qquad i = 1, \ldots, n.$$

The proof involves exhausting G by an increasing sequence of open subsets G_j, with $G_j^- \subseteq G_{j+1}$ for $j = 1, 2, \ldots$ Note that $\lim |\mu_i|(G_{j+1} - G_j^-) = 0$ as $j \to \infty$ for each $i = 1, \ldots, n$. We omit the details.

3. Topologies.

In this section we introduce two natural locally convex Hausdorff topologies on $B_H(G)$ and study some of their properties.

3.1. DEFINITION. — *Let* $\alpha(G)$ *denote* $B_H(G)$ *under the weak topology arising from the duality between* $B_H(G)$ *and* $M'(G)$ *with the inner product*

$$\langle f, [\mu]\rangle = \langle f, \mu\rangle = \int f d\mu.$$

The basic neighborhoods of 0 *are of the form*

$$N(\mu_1, \ldots, \mu_n : \varepsilon) = \{f \in B_H(G) : \left|\int f d\mu_i\right| < \varepsilon, i = 1, \ldots, n\}.$$

This is the weakest topology on $B_H(G)$ in which all the elements of $M'(G)$, regarded as linear functionals on $B_H(G)$, are continuous. A net (generalized sequence) $\{f_\gamma\}$, $\gamma \in \Gamma$, where Γ is the partially ordered index set, converges to 0 in this topology if and only if the net $\{\int f_\gamma d\mu\}$ converges to 0 for each $\mu \in M(G)$. In particular, an α-convergent net is pointwise convergent. The converse is false in general, but is valid when the functions are uniformly bounded (Theorem 3.7). See [16], Chapter 2, for a discussion of topological concepts in terms of nets.

3.2. DEFINITION. — $\beta(G)$ *denotes* $B_H(G)$ *under the topology given by the seminorms*

$$\|f\|_k = \sup \{|f(z)\, k(z)| : z \in G\},$$

where k ranges over the class $K(G)$.

The basic neighborhoods of 0 are the sets $\{f : \|f\|_k < \varepsilon\}$, where $k \in K$ and $\varepsilon > 0$. A net $\{f_\gamma\}$ converges to 0 in this topology if and only if the associated net $\{f_\gamma k\}$ converges to 0 uniformly on G for each fixed $k \in K(G)$. In particular, a β-convergent net must converge uniformly on compact subsets of G. (The converse is true for bounded nets, by Theorem 3.7) This topology was discussed briefly by Buck [5] in the case $G = D = \{z : |z| < 1\}$, who called it the strict topology.

3.3. PROPOSITION. — *Every* α-*open set is also* β-*open* (*that is, the* β-*topology is at least as strong as the* α-*topology*), *and every* β-*open set is open in* $H_\infty(G)$.

Proof. — For the first part, it suffices to show that each basic α-neighborhood of 0 contains a basic β-neighborhood of 0. Let the α-neighborhood be determined by $\varepsilon > 0$ and measures $\mu_1, \ldots, \mu_n$ in $M(G)$. We may assume that $\|\mu_i\| \leqslant 1$ for $i = 1, \ldots, n$. By 2.11, there is a $k \in K(G)$ for which $\int (1/k)\, d\,|\mu_i| < 1$, $i = 1, \ldots, n$. Let

$$E = E_k = \{f \in B_H(G) : \|f\|_k < \varepsilon\}.$$

Then for $f \in E$,

$$\left|\int f d\mu_i\right| \leqslant \int |kf|\,(1/k)\, d\,|\mu_i| < \varepsilon.$$

Thus, E is contained in the α-neighborhood.

Now consider a basic β-neighborhood $E = E_k$ as above. Let $d = \max\{k(z) : z \in G\}$. Since for $f \in E$, $|f(z)| < \varepsilon/k(z)$, $z \in G$, we see that the H_∞-ball $\{f : \|f\|_\infty < \varepsilon/d\}$ is contained in the β-neighborhood.

3.4. THEOREM. — *$\alpha(G)$ and $\beta(G)$ have the same dual space, namely* $M'(G)$.

Proof. — As noted in 3.1, the elements of $M'(G)$ are continuous linear functionals on $\alpha(G)$. Since β is at least as strong as α, they are also continuous on $\beta(G)$. It therefore suffices to show that every β-continuous linear functional is given by integration with respect to some element of $M(G)$.

Let $\lambda : \beta(G) \to \mathbf{C}$ be a β-continuous linear functional. By the general theory of locally convex topological vector spaces, there is some k in $K(G)$ such that λ is continuous with respect to the seminorm $\|\ \ \|_k$.

The collection $\{kf\}$, $f \in B_H$, is a vector subspace of the Banach space $C_0(G^-)$ of all continuous functions on G^- that vanish on ∂G, with the supremum norm. Further, λ is a bounded linear functional on this subspace. By the Hahn-Banach Theorem, λ can be extended to be a bounded linear functional on $C_0(G^-)$. It follows from the Riesz representation theorem that λ can be represented by some measure μ in $M(G)$:

$$\lambda(f) = \int f d\mu \qquad f \in B_H(G),$$

and the proof is complete.

3.5. COROLLARY. — *$\alpha(G)$ and $\beta(G)$ have the same closed linear subspaces and the same closed convex sets.*

This follows directly from the preceding theorem by well-known results in functional analysis. See [9], Chapter V, § 2.14, for details.

Problem. — We have seen that the α-topology is the weakest topology on $B_H(G)$ with respect to which $M'(G)$ is the dual space. Is the β-topology the strongest topology having this property? In other words, is the β-topology the Mackey topology on $B_H(G)$ with respect to the duality $\langle B_H(G), M'(G)\rangle$?

3.6. PROPOSITION. — *$\alpha(G)$ and $\beta(G)$ have the same bounded sets, namely the norm-bounded sets.*

Proof. — Recall that a set S in a topological vector space is said to be bounded if to each neighborhood U of 0, there corresponds a positive number ε such that $\varepsilon S \subseteq U$. If a set is bounded in a given topology then it is bounded in any weaker topology. It follows that the norm-bounded sets are also α-bounded and β-bounded.

In the other direction, it is enough to show that if S is α-bounded, then S is norm-bounded. Let us regard the elements of S as linear functionals on the Banach space M(G), and apply the uniform boundedness principle. These functionals are pointwise bounded on M(G) since for any measure μ in M(G) we may consider the α-neighborhood U of 0 given by

$$U = \{f \in B_H(G) : \left|\int f d\mu\right| < 1\}.$$

Since $\varepsilon S \subseteq U$ for some $\varepsilon > 0$, we have $\left|\int f d\mu\right| < 1/\varepsilon$ for all $f \in S$.

By the uniform boundedness principle, the set S, regarded as a set of linear functionals on M(G), is bounded in norm. But by Theorem 2.7, the H_∞ norm is the same as the linear functional norm, and the proof is complete.

3.7. THEOREM. — *Let $\{f_\gamma\}$, $\gamma \in \Gamma$, be a uniformly bounded net of functions $f_\gamma \in B_H(G)$ - say $|f_\gamma(z)| \leqslant m$ for all $z \in G$ and all $\gamma \in \Gamma$. Then the following statements are equivalent:*

i) *$\{f_\gamma\}$ is β-convergent to* 0,

ii) *$\{f_\gamma\}$ is α-convergent to* 0,

iii) *$\{f_\gamma\}$ converges pointwise* to 0,

iv) *$\{f_\gamma\}$ converges to* 0 *uniformly on compact subsets of* G.

Proof. — It is clear that i) implies ii) since the β-topology is at least as strong as the α-topology. On considering point measures, it is clear that ii) implies iii). That iii) implies iv) is a familiar fact from the theory of normal families of analytic functions. Expressed a little differently (see [8], Chapter V) the bounded sets in the space H(G) of all analytic functions on G, in the topology of uniform convergence on compact subsets of G, are precisely those sets of functions that are uniformly bounded on each compact subset of G. In particular, a uniformly bounded set of analytic functions is a bounded set in H(G). But H(G) has the property that the bounded sets have compact closures. Hence every subset of $\{f_\gamma\}$ has at least one cluster point. But by iii), the net $\{f_\gamma\}$ has at most one cluster point, namely 0, and iv) follows.

It remains to be shown that iv) implies i). Let there be given a function $k \in K(G)$ and a number $\varepsilon > 0$. There is no loss of generality in supposing that $k(z) \leqslant m$ in G. There is a compact set $C \subseteq G$ such that $k(z) < \varepsilon/m$ for $z \in G - C$. Since $\{f_\gamma\}$ converges uniformly on C, there is an index γ_0 such that $|f_\gamma| < \varepsilon/m$ on C whenever $\gamma > \gamma_0$. Hence

$$k(z)\,|f_\gamma(z)| < \varepsilon \qquad\qquad z \in G, \gamma > \gamma_0,$$

which completes the proof.

3.8. Corollary. — *The α and β topologies agree on bounded sets.*

This follows from the fact that two topologies with the same convergent nets must agree.

3.9. Corollary. — *The α and β topologies, restricted to any bounded set, are metric.*

Proof. — The topology of uniform convergence on compact subsets of G is determined by the countable family of seminorms $\|\ \|_n$ given by

$$\|f\|_n = \sup\{|f(z)| : z \in K_n\}$$

where $\{K_n\}$ is an increasing sequence of compact subsets of G such that any compact subset of G is contained in one of the K_n. The topology determined by these seminorms has the metric ρ given by

$$\rho(f, g) = \sum_{n=0}^{\infty} 2^{-n} \frac{\|f - g\|_n}{1 + \|f - g\|_n}.$$

We remark that the fact that the topology is metric is equivalent, via [9], p. 426, Theorem 1, to the fact (see Theorem 4.5) that $M'(G)$ is separable.

3.10. COROLLARY. — *The α and β topologies have the same compact sets, namely the bounded and closed sets.*

Proof. — If S is a bounded set in $\alpha(G)$ or $\beta(G)$, then S is a normal family by Theorem 3.6. Hence each sequence of elements of S contains a subsequence that converges uniformly on compact subsets of G. By Theorem 3.7, this implies convergence in $\alpha(G)$ and $\beta(G)$. Because S is metric, this shows that S is compact. Conversely, in any topological vector space, a compact set is always closed and bounded.

3.11. COROLLARY. — *The spaces $\alpha(G)$ and $\beta(G)$ have the same convergent sequences, namely the bounded sequences that converge pointwise.*

Proof. — A bounded and pointwise convergent sequence must be β-convergent, by Theorem 3.7. Also, every β-convergent sequence is α-convergent. Consideration of point measures shows that an α-convergent sequence must be pointwise convergent. It remains to be shown that an α-convergent sequence $\{f_n\}$ is bounded, and this follows from the uniform boundedness principle. For if we regard the functions f_n as linear functionals on $M(G)$, then they are pointwise bounded since $\lim \int f_n d\mu$ exists by hypothesis. They are therefore bounded in the linear functional norm. By Theorem 2.7, this norm is equal to the supremum norm, and the corollary is proved.

3.12. PROPOSITION. — *Neither $\alpha(G)$ nor $\beta(G)$ satisfies the first axiom of countability and consequently neither $\alpha(G)$ nor $\beta(G)$ is a metric space.*

Proof. — The idea of this proof is from a private communication by R. C. Buck to one of the authors. Let f be a function in $B_H(G)$ with $||f|| = 1$, f not a constant. Let

$$S = \{f^n + nf^m : m, n = 1, 2, 3, ...\}.$$

Now the sequence $\{f^n\}$ converges to 0 in $\alpha(G)$ and $\beta(G)$ by Corollary 3.11, since it is uniformly bounded and converges pointwise to 0. We prove that $0 \in S^-$ but that no sequence of elements of S can converge to 0. Let U be a neighborhood of 0 in either $\alpha(G)$ or $\beta(G)$, and let V be a neighborhood of 0 in the corresponding space, such that $V + V \subseteq U$. Choose n so large that $f^n \in V$, and then choose m so large that $nf^m \in V$. It follows that $f^n + nf^m \in U$, so that $0 \in S^-$. On the other hand, suppose that $\{s_k\}$, $k = 1, 2, 3, \ldots$, were a sequence of elements of S that converged to 0, say

$$s_k = f^{n_k} + n_k f^{m_k}.$$

Since the s_k must be uniformly bounded, it follows that the n_k must be bounded, so that some integer n occurs infinitely often in the sequence $\{n_k\}$. Passing to a subsequence, we have

$$s_k = f^n + nf^{m_k}.$$

If $m_k \to \infty$ then $s_k \to f^n \neq 0$, while if for some m, $m_k = m$ for infinitely many k, then $f^n + nf^m$ is not 0, but is a limit point of $\{s_k\}$. In any event, then, we have a contradiction.

For the definitions of the terms used in the next theorem, see [16], § 12.2, § 19.2.

3.13. PROPOSITION. — *The space* $\alpha(G)$ *is neither barrelled nor bornological. The same is true for* $\beta(G)$.

Proof. — The closed unit ball $U = \{f : ||f||_\infty \leqslant 1\}$ in $H_\infty(G)$ is a barrel in $\alpha(G)$ and $\beta(G)$. That is, U is convex, circled, and absorbing. Also, U is both α-closed and β-closed, since, by Theorem 3.7, the α-convergent or β-convergent nets of functions in U are just the pointwise convergent nets, and U is obviously closed in the topology of pointwise convergence. But U is not a neighborhood of 0 in either $\alpha(G)$ or $\beta(G)$. To see this, choose a function $f \in B_H(G)$, $||f|| = 1$, f non-constant. Then the sequence $\{2f^n\}$ converges to 0 in $\alpha(G)$ and in $\beta(G)$ by Corollary 3.11, yet $2f^n$ is never in U. Therefore neither $\alpha(G)$ nor $\beta(G)$ is barrelled.

Since U absorbs all bounded sets but is not a neighborhood of 0, neither space is bornological.

This completes our list of the similarities between $\alpha(G)$ and $\beta(G)$. Despite the similarities, they are different spaces.

3.14. THEOREM. — *The* α *and* β *topologies are not the same.*

Proof. — Choose a compact subset C of G that has a non-empty interior, and choose a function $k \in K(G)$ such that $k > 0$ on C. Let

$$E = \{ f \in B_H(G) : ||f||_k < 1 \}.$$

The functions in E are uniformly bounded on C. But no α-neighbourhood of 0 can have this property. Indeed, given any finite collection of measures, there is an $f \in B_H(G)$, $f \neq 0$, that is orthogonal to all of them. This follows from the fact (see 2.3) that $B_H(G)$ is an infinite-dimensional vector space. No matter what the number $\varepsilon > 0$, the functions $\{ nf \}$, $n = 1, 2, 3, \ldots$, all lie in the α-neighbourhood of 0 determined by these measures and this number ε. But the functions $\{ nf \}$ are not uniformly bounded on C since f cannot vanish on all of C.

The next result provides an alternate proof that the α and β topologies are different.

3.15. Proposition. — *The space* $\beta(G)$ *is topologically complete while the space* $\alpha(G)$ *is not.*

Proof. — To prove that $\beta(G)$ is complete, let $\{ f_\gamma \}$, $\gamma \in \Gamma$, be a Cauchy net in $\beta(G)$, so that if $k \in K(G)$ is given, the net $\{ f_\gamma k \}$ is a Cauchy net in the uniform topology and is consequently uniformly convergent. It follows that on each compact subset of G, $\{ f_\gamma \}$ is uniformly convergent, so that the limit function f is analytic in G. Also, for each $k \in K(G)$, $\sup \{ |f(z)\, k(z)| : z \in G \} < \infty$, and it follows that f is bounded.

To prove that $\alpha(G)$ is not complete, choose a bounded, continuous, and non-analytic function f in G. Let Γ_0 be the family of all finite subsets of $M(G)$, partially ordered by inclusion. Given $\gamma \in \Gamma_0$, $\gamma = \{ \mu_1, \ldots \mu_n \}$, choose a function $f_\gamma \in B_H(G)$ such that

$$\int f_\gamma \, d\mu_i = \int f \, d\mu_i, \qquad i = 1, \ldots n.$$

Such a choice is possible because $B_H(G)$ is infinite-dimensional. This net $\{ f_\gamma \}$ is a Cauchy net in $\alpha(G)$, but does not converge to an element of $\alpha(G)$. Thus $\alpha(G)$ is not complete. Another way of seeing this is by applying the well-known fact that the dual of an infinite-dimensional Banach space is never complete in the weak-star topology, and using Corollary 4.6.

3.16. Proposition. — $\beta(G)$ *is a topological algebra. That is, multiplication is jointly continuous in* $\beta(G)$.

Proof. — Let $\{ f_\gamma \}$, $\{ g_\gamma \}$, $\gamma \in \Gamma$ be two nets that converge to 0 in $\beta(G)$, and suppose that $k \in K(G)$ is given. Then

$$|f_\gamma g_\gamma k| = |f_\gamma k^{1/2}| \, |g_\gamma k^{1/2}|,$$

and since $k^{1/2} \in K(G)$, it follows that $\{ f_\gamma g_\gamma k \}$ converges uniformly to 0 in G. Hence $\{ f_\gamma g_\gamma \}$ converges to 0 in $\beta(G)$. Now suppose that $\{ f_\gamma \}$, $\{ g_\gamma \}$ are two nets such that $\{ f_\gamma \}$ converges to f in $\beta(G)$ and $\{ g_\gamma \}$ converges to g in $\beta(G)$. By the above, $\{ (f_\gamma - f)(g_\gamma - g) \}$ tends to 0 in $\beta(G)$. Hence

$$(f_\gamma g_\gamma - fg) + f(g - g_\gamma) + g(f - f_\gamma) \to 0$$

in $\beta(G)$. But the second and third summands tend to 0 in $\beta(G)$ and so the first must also. The result is proved.

3.17. *Remarks.* — We conclude this section with some additional facts about the structure of $\alpha(G)$ and $\beta(G)$, omitting the proofs.

PROPOSITION. — *There exists a sequence* $\{ f_n \}$ *of invertible elements of* $B_H(G)$ *such that* $\{ f_n \}$ *converges boundedly to* 1 *but such that* $\{ 1/f_n \}$ *does not converge boundedly.*

COROLLARY. — *The algebra* $\beta(G)$ *is not multiplicatively convex in the sense of Michael* [18].

PROPOSITION. — *The weak topology of the unit ball in* M′(G) *is not metric.*

4. Balayage.

In this section, we show by balayage or " sweeping " that any measure in M(G) can be replaced by an equivalent measure that has special properties.

4.1. THEOREM. — *Given a measure* μ *in* M(G), *there exists a measure* ν *in* M(G), *such that*

$$\nu \sim \mu, \quad ||\nu|| \leqslant ||\mu||$$

and such that ν *is absolutely continuous with respect to planar Lebesgue measure.*

DEFINITION. — *Let* λ *denote planar Lebesgue measure.*

DEFINITION. — *Let* $L^1(G)$ *denote the subspace of* $M(G)$ *consisting of all measures* μ *that are absolutely continuous with respect to* λ, *with the norm* $\|\mu\| = \int d|\mu|$ *as before.*

Proof of theorem. — Following a suggestion of J. L. Doob, we first sweep a point measure, and then sweep the general measure by an integration process. Our original proof was longer. It is easy to modify the proof to make ν even better behaved. For example, we could assure that $d\nu = \varphi d\lambda$, where φ is an infinitely differentiable function of the two real coordinate variables.

In 4.24 we give a proof via functional analysis of a slightly weakened version of Theorem 4.1.

i) Let ε_w be the unit point mass at the point w, $w \in G$, let $d = d(w, \partial G)$ denote the distance from w to the boundary of G, let $d' = \min(d, 1)$, and finally let $a = d'/3$, $b = 2d'/3$. Then the closed annulus $A_w = \{z : a \leqslant |z - w| \leqslant b\}$ is a subset of G and varies continuously with w in the sense that a and b are continuous functions of w. We now define a measure ν_w by

$$\nu_w(E) = \frac{1}{b-a}\int_{t=a}^{t=b}\left(\frac{1}{2\pi i}\int_{|\zeta - w| = t} \chi_E(\zeta)\frac{d\zeta}{\zeta - w}\right) dt,$$

where E is any Borel subset of G and χ_E is its characteristic function. Then $f(w) = \int f d\nu_w$ for each $f \in B_H(G)$, so that $\nu_w \sim \varepsilon_w$. This assertion is just the Cauchy integral formula averaged over an annulus. Also, $\|\nu_w\| = 1$, as a simple estimation shows. It is clear that $\nu_w \in L^1(G)$.

ii) For any measure $\mu \in M(G)$, let ν be defined by

$$\nu(E) = \int \nu_w(E)\, d\mu(w) = \int \left(\int \chi_E(z)\, d\nu_w(z)\right) d\mu(w),$$

where E ranges over the Borel subsets of G. For $f \in B_H(G)$, we have

$$\int f(z)\, d\nu(z) = \int \left(\int f(z)\, d\nu_w(z)\right) d\mu(w) = \int f(w)\, d\mu(w),$$

so that $\nu \sim \mu$. Also, if φ is any continuous function on G^- that vanishes on ∂G then

$$\left|\int \varphi d\nu\right| \leqslant \int \left(\int |\varphi(z)|\, d|\nu_w|(z)\right) d|\mu|(w) \leqslant \|\varphi\|_\infty \|\mu\|,$$

where $\|\varphi\|_\infty = \sup\{|\varphi(z)| : z \in G\}$. Hence $\|\nu\| \leqslant \|\mu\|$.

iii) Finally, let E be a subset of G such that $\lambda(E) = 0$. Then $\nu_w(E) = 0$ for each $w \in G$, and consequently $\nu(E) = 0$. Hence ν is absolutely continuous with respect to λ, and the proof is complete.

4.2. Earlier, in (2.8), we considered the subspace N(G) of M(G) consisting of all measures orthogonal to all bounded analytic functions. We now consider the corresponding subspace of $L^1(G)$. Let $N_\lambda(G)$ consist of all measures μ in $L^1(G)$ such that $\mu \sim 0$. The inclusion map is a map of $L^1(G)$ into M(G), and it induces a map of the quotient space $L^1(G)/N_\lambda(G)$ into $M'(G)$. By Theorem 4.1, this map is onto and isometric.

4.3. COROLLARY. — *The space* $L^1(G)/N_\lambda(G)$ *is isometrically isomorphic to* $M'(G)$ *under the natural correspondence.*

4.4. COROLLARY. — *The space* $M'(G)$ *is separable.* (See the remark following Corollary 3.9 for another proof of this result.)

Thus, $M'(G)$ is considerably " nicer " than the rather pathological space M(G).

4.5. THEOREM. — *The space* $H_\infty(G)$ *is the conjugate space of the separable Banach space* $M'(G)$.

Proof. — It is enough to prove that $H^\infty(G)$ is the conjugate space of $L^1(G)/N_\lambda(G)$, by Corollary 4.3. Let $L^\infty(G)$ be the space of equivalence classes of complex-valued functions on G that are bounded almost everywhere (with respect to λ) in the essential supremum norm. We make the usual abuse of notation by not distinguishing between bounded functions and the equivalence classes of $L^\infty(G)$ that they belong to. Since $B_H(G)$ may be regarded as a linear subspace of the space $L^\infty(G)$, it will be sufficient to show that $B_H(G)$ is weak-star closed in $L^\infty(G)$. By a result of Banach ([2], Chapitre VII, Théorème 5), it is sufficient to show that $B_H(G)$ is sequentially closed, that is, that each sequence in $B_H(G)$ that converges in the weak-star topology of $L^\infty(G)$ to a limit in $L^\infty(G)$ actually has its limit in $B_H(G)$.

Let $\{f_n\}$ be a sequence of functions in $B_H(G)$, and suppose that $f \in L^\infty(G)$ and that $f_n \to f$ in the weak-star topology of $L^\infty(G)$. Then, by the definition of weak-star convergence, $\int f_n d\mu$ converges (to $\int f d\mu$) for each $\mu \in L^1(G)$. Consequently, $\int f_n d\mu$ converges for each $\mu \in M(G)$, by Theorem 4.1. As in the proof of Corollary 3.11, the f_n are uniformly bounded and converge uniformly on compact subsets of G to a function $f' \in B_H(G)$. Since $\int f' d\mu = \int f d\mu$ for each $\mu \in L^1(G)$, it follows that $f = f'$ (actually that $f = f'$ almost everywhere with respect to λ), and the proof is complete.

4.6. COROLLARY. — *The* α*-topology on* $B_H(G)$ *is precisely the weak-star topology on* $H_\infty(G)$ *as the dual of* $M'(G)$.

The proof is immediate from the preceding theorem and the definitions. Since $M'(G)$ is a separable Banach space, we have the following corollaries.

4.7. COROLLARY. — *A linear subspace of* $\alpha(G)$ *is closed if and only if it is sequentially closed.*

See [2], Chapitre VIII, Théorème 5.

4.8. COROLLARY. — *A linear subspace of* $\beta(G)$ *is closed if and only if it is sequentially closed.*

We make much use of this fact in the sequel. It follows from the previous corollary and the facts that $\alpha(G)$ and $\beta(G)$ have the same closed linear subspaces (Corollary 3.5) and the same convergent sequences (Corollary 3.11).

4.9. COROLLARY. — *The space* $\alpha(G)$ *is separable. In fact, for each subset* S *of* $\alpha(G)$, *there is a countable subset* $A \subseteq S$ *such that each element of* S *is the limit of a sequence of elements of* A. *The same assertion holds for* $\beta(G)$.

See [2], Chapitre VIII, Théorème 4, for the assertion about $\alpha(G)$. The fact $\beta(G)$ that has the same convergent sequences as $\alpha(G)$ implies the assertion for $\beta(G)$.

4.10. DEFINITION. — *A subset* S *of* G *is called dominating, provided that*

$$\sup\{|f(z)| : z \in S\} = \sup\{|f(z)| : z \in G\}$$

for each $f \in B_H(G)$.

4.11. DEFINITION. — *Let* S *be a subset of* G. *We denote by* $M(S)$ *the set of all measures* $\mu \in M(G)$ *that live in* S.

$M(S)$ is a closed subspace of $M(G)$ in the variation norm.

4.12. DEFINITION. — *A subset* S *of* G *is called universal, provided that for each* μ *in* $M(G)$ *there exists a measure* ν *in* $M(S)$ *such that* $\nu \sim \mu$.

We say that μ can be swept onto S in this case. Theorem 4.14 shows, among other things, that this sweeping can be accomplished with an arbitrarily small increase in norm.

4.13. DEFINITION. — *A subset* S *of* G *is called strongly universal provided that for each* μ *in* M(G) *and each* $\varepsilon > 0$ *there exists a measure* ν *in* M(S) *such that* $\nu \sim \mu$ *and* $||\nu|| \leqslant (1 + \varepsilon) ||\mu||$.

A result equivalent to the next one was proved in [4] for the case where G is a Jordan domain. It can be used in conjunction with Proposition 4.15 to give an alternative proof that M′ (G) is separable.

4.14. THEOREM. — *Let* S *be a subset of* G. *Then the following assertions are equivalent.*

i) S *is strongly universal*

ii) S *is universal*

iii) S *is dominating.*

Proof. — It is trivial that i) implies ii). To prove that ii) implies iii), suppose the contrary. Then there would exist a point $\zeta \in G - S$ and a function $f \in B_H(G)$ such that

$$f(\zeta) = 1, \quad |f(z)| \leqslant r < 1 \text{ for } z \in S.$$

Let $\nu \in M(S)$ be equivalent to the point measure ε_ζ at ζ. Then

$$1 = (f(\zeta))^n = \int f^n \, d\varepsilon_\zeta = \int f^n \, d\nu$$

so that

$$1 \leqslant r^n ||\nu|| \quad \text{for } n = 1, 2, 3, \ldots,$$

which is impossible.

To show that iii) implies i), we may suppose without loss of generality that S is a countable set, say $S = \{ s_n \}$, $n = 1, 2, 3, \ldots$, since a dense subset of a dominating set is dominating, and any superset of a strongly universal set is strongly universal. In this case, M (S) may be identified with the space l^1 of absolutely summable sequences. Consider now the operation T of restricting a bounded analytic function f to S. Since S is dominating, T is an isometric mapping of H_∞ (G) onto a subspace E of the space l^∞ of all bounded sequences. Without attempting to further describe E, we assert that E is weak-star closed in l^∞ as the dual space of l^1. This follows just as in the proof of Theorem 4.5. It is enough to show that E contains all limits of sequences of elements of E. But a weak-star convergent sequence in l^∞ is bounded. If the sequence is $\{Tf_n\}$, it follows that the f_n are uniformly bounded on G. Hence $\{ f_n \}$ is a normal family, and passing to a subsequence if necessary, the f_n converge uniformly on compact subsets of G to a bounded analytic function f. But $\lim Tf_n = Tf$ at each point of S, and the assertion is proved.

Now let $N = E^\perp$ be the subspace of l^1 orthogonal to E. Since E is weak-star closed, it is the annihilator of N, that is, $E = N^\perp$. Hence E is the dual space of the quotient Banach space l^1/N. In this case, its norm as a linear functional on E is equal to its quotient norm.

To complete the proof, let us choose $\mu \in M(G)$. We may regard μ as a linear functional on $H_\infty(G)$, and consequently on E. As such, it is continuous in the weak-star topology of E. To prove this, we need only show that the null space of μ, namely $E_\mu = \{ Tf \in E : \int f d\mu = 0 \}$, is a weak-star closed subspace of E (see [13], Corollary to Theorem 2.62). But a subspace is weak-star closed if it contains all its sequential limits. Let $\{ Tf_n \}$ be a weak-star convergent sequence in E_μ, with limit Tf. In l^∞, a weak-star convergent sequence is bounded. It follows that $\{ f_n \}$ is a bounded sequence since T is an isometry, and on passing to a subsequence if necessary, we see that $\{ f_n \}$ converges boundedly and pointwise to a function in $B_H(G)$ that must be the function f, since it agrees with f on the dominating set S. Applying Corollary 3.11, we see that $0 = \lim \int f_n d\mu = \int f d\mu$ so that $f \in E_\mu$, and we have proved that E_μ is closed.

Applying the remarks of the next to the last paragraph, we see that μ can be identified with an element σ of l^1/N, and the norm of μ as a linear functional is equal to the quotient norm of σ. In other words, given $\varepsilon > 0$, we can find a measure $\nu \in M(S)$ such that $\nu \sim \mu$ and $\| \nu \| \leqslant (1 + \varepsilon) \| \mu \|^*$, where $\| \mu \|^*$ denotes the norm of μ as a linear functional on $H_\infty(G)$. But

$$\| \mu \|^* = \sup \{ | \int f d\mu | : f \in H_\infty(G), \| f \|_\infty \leqslant 1 \},$$

and by Theorem 4.5, this is the norm of $[\mu]$ in the space $M'(G)$, which completes the proof, since $\| [\mu] \| \leqslant \| \mu \|$.

4.15. PROPOSITION. — *There exists a countable dominating subset of* G *that has no limit point in* G.

Proof. — G can be written as the union of a sequence $\{F_n\}$ of open sets F_n having compact closures, with

$$F_n \subseteq F_n^- \subseteq F_{n+1} \qquad n = 1, 2, 3, \dots .$$

Let $C_n = F_n^- - F_{n-1}$, with the understanding that F_0 is the empty set. Then the C_n are compact sets. By the maximum principle, for all $f \in B_H(G)$, we have

$$m_n = m_n(f) = \max\{|f(z)| : z \in C_n\} = \max\{|f(z)| : z \in F_n\}$$

$$m_n \to \|f\|_\infty \qquad \text{as } n \to \infty.$$

Let U denote the unit ball of H_∞, and let $\{\varepsilon_n\}$ be a sequence of positive numbers decreasing to 0. By the uniform equicontinuity of the functions in U, there exists for each n a positive number δ_n such that if $z, w \in C_n$ and $|z - w| < \delta_n$ then $|f(z) - f(w)| < \varepsilon_n$ for all $f \in U$.

For each n, we choose a finite subset $E_n \subseteq C_n$ such that each point of C_n has distance less that $\delta_n/2$ from some point of E_n. Then

$$\max\{|f(z)| : z \in C_n\} \leqslant \max\{|f(z)| : z \in E_n\} + \varepsilon_n.$$

Thus, the union of the sets E_n is a dominating set with no limit points inside G.

4.16. Proposition. — *Every dominating subset of* G *contains a countable dominating subset of* G *that has no limit point in* G.

Proof. — Our proof is a continuation of the preceding proof. Let S be a dominating set in G and let n be a positive integer. For each point z of E_n, we choose a point w of S that is in C_n and whose distance from z is less than $\delta_n/2$ if this is possible. Let S′ denote the totality of the points so chosen for $n = 1, 2, 3, \dots$. Then S′ is at most countable, and has no limit points in G.

To prove that S′ is dominating, we fix a function $f \in U$ and choose a sequence $\{\zeta_k\}$ of points in S such that $|f(\zeta_k)| \to \|f\|_\infty$ as $k \to \infty$. We may choose the ζ_k so that they approach the boundary of G. Each point ζ_k is in a set C_n for at least one index n. Let $n(k)$ be the first such index. Then $n(k) \to \infty$ as $k \to \infty$ since ζ_k approaches the boundary of G. Hence, if $\varepsilon > 0$ is given, we can choose k so that

$$|f(\zeta_k)| > \|f\|_\infty - \varepsilon, \qquad \zeta_k \in C_n, \qquad \varepsilon_n < \varepsilon,$$

where $n = n(k)$. The set E_n contains a point z whose distance from ζ_k

is less than $\delta_n/2$. Hence S′ must contain a point w whose distance from z is at most $\delta_n/2$. Thus $|w - \zeta_k| < \delta_n$ and consequently

$$|f(w)| > |f(\zeta_k)| - \varepsilon > ||f||_\infty - 2\varepsilon$$

and the proof is complete.

4.17. THEOREM. — *Let* $S = \{z_n\}$ *be a countable subset of* G *with no limit points in* G. *Then* S *is dominating if and only if there is a measure* $\mu \in M(S)$, $\mu \neq 0$, *such that* $\mu \sim 0$.

Proof. — Suppose first that S is dominating. Then by Theorem 4.14, S is strongly universal. If S were to carry no non-zero measure equivalent to 0, it would follow that $||\mu|| = ||[\mu]||$ for every measure $\mu \in M(S)$, contrary to Theorem 2.10. In the other direction, let μ be a measure in M (S) with $\mu \sim 0$ but $\mu \neq 0$. There are complex numbers $\{a_n\}$, $n = 1, 2, 3, \ldots$, such that

$$0 < \Sigma |a_n| < \infty$$

and

$$\Sigma a_n f(z_n) = \int f d\mu = 0$$

for each $f \in B_H(G)$. Consider the function

$$A(z) = \sum \frac{a_n}{z_n - z} = \int \frac{d\mu(w)}{w - z} \qquad z \in G - S$$

This function is analytic in $G - S$ and has simple poles at each point z_n for which $a_n \neq 0$, since S has no limit point in G. In particular, A (z) is not identically 0, and therefore, since G is connected, the zeros of A (z) in G are isolated. We claim that

$$A(z) f(z) = \int \frac{f(w)}{w - z} d\mu(w) \quad z \in G - S \tag{4.17.1}$$

for each $f \in B_H(G)$. This is equivalent to the assertion

$$\int \frac{f(w) - f(z)}{w - z} d\mu(w) = 0 \quad z \in G - S.$$

But this follows from the hypothesis, since for each $z \in G - S$, the integrand is a bounded analytic function of w. More precisely, the integrand is the function g defined by $g(w) = (f(w) - f(z))/(w - z)$ if $w \neq z$, and $g(z) = f'(z)$.

Now suppose that S is not dominating. Then there exists a function $f \in B_H(G)$ and a point $z_0 \notin S$ such that

$$f(z_0) = 1, \quad |f(z_n)| \leqslant r < 1 \quad \text{for } n = 1, 2, 3, \dots .$$

We may further assume that $A(z_0) \neq 0$, since we could otherwise move z_0 slightly, and renormalize f. Applying formula (4.17.1) to f^n, we have

$$A(z_0) = A(z_0)(f(z_0))^n = \int \frac{(f(w))^n}{w - z_0} d\mu(w),$$

which is impossible for large n. This completes the proof.

4.18. We remark that formula (4.17.1) is a discrete Cauchy integral formula for bounded analytic functions. In effect, it gives an explicit formula for sweeping the unit point measure at the point z, $z \notin S$, onto the set S. It leads to a general balayage formula as follows. If $\rho \in M(G)$, then the measure $\sigma \in M(S)$ given by the following expression is equivalent to ρ:

$$d\sigma(w) = \left(\int \frac{d\rho(z)}{A(z)(w - z)} \right) d\mu(w).$$

We leave the details to the reader.

We now indicate a connection between balayage and the problem of characterizing sets of removable singularities for bounded analytic functions.

4.19. DEFINITION. — *Given a connected open set* G' *and a compact subset* E *of* G', *let* $G = G' - E$. *We say that* E *is a set of removable singularities for bounded analytic functions in* G *provided that each* $f \in B_H(G)$ *has a bounded analytic extension* $F \in B_H(G')$.

It is well known (see [1], Chapter IV, Section 4C) that the above property of E is independent of the set G', so that it makes sense simply to speak of E as being a set of removable singularities for bounded analytic functions. In some sense, such sets E are thin sets. We shall prove a result to the effect that E is a set of removable singularities for bounded analytic functions if and only if E is so thin that each measure in M (G) can be swept a positive distance away from E. First we require some preliminaries.

We shall use the following notation:

$$E_\varepsilon = \{ z \in G : \text{distance}(z, E) > \varepsilon \}, \ \varepsilon > 0.$$

4.20. DEFINITION. — *Given* G′, E, *and* G *as in the preceding definition, we say that a measure* $\mu \in M(G)$ *is holomorphically free of* E, *if, for some* $\varepsilon > 0$, *there is a measure* $\nu \sim \mu$ *that lives in* E_ε .

4.21. LEMMA. — *If every measure* μ *in* M(G) *is holomorphically free of* E, *then for some* $\varepsilon > 0$, E_ε *is a dominating subset of* G.

Proof. — Let us write E_n in place of $E_{1/n}$. Proceeding by contraction, if the lemma were false, we could find a sequence $\{ f_n \}$ of functions in $B_H(G)$ and a sequence $\{ z_n \}$ of points in G such that

$$\sup \{|f_n(z)| : z \in E_n\} \leqslant 4^{-n}$$

$$|f_n(z_n)| > \frac{3}{4}, \; ||f_n|| = 1.$$

We define a measure $\mu \in M(G)$ by

$$\mu = \Sigma\, 3^{-k} \varepsilon_k,$$

where ε_k is the unit point mass at z_k. By hypothesis, there is a measure ν that lives in E_N for some positive integer N, with $\nu \sim \mu$. For $n \geqslant N$, we have $E_N \subseteq E_n$, and therefore

$$|\textstyle\int f_n\, d\nu| \leqslant 4^{-n}\, ||\nu||. \tag{4.21.1}$$

On the other hand,

$$|\textstyle\int f_n\, d\mu| = |\Sigma\, 3^{-k} f_n(z_k)| > \frac{3}{4} 3^{-n} - \\ (\sum_{k<n} + \sum_{k>n})\, 3^{-k}\, |f_n(z_k)| \geqslant \frac{3}{4} 3^{-n} - \frac{1}{2} 4^{-n} - \frac{1}{2}\, 3^{-n},$$

which contradicts (4.21.1) for large n.

4.22. THEOREM. — *Let* G′ *be a bounded open set, let* E *be a compact subset of* G′, *and let* G = G′ — E. *Then* E *is a set of removable singularities for bounded analytic functions in* G *if and only if each measure* μ *in* M(G) *is holomorphically free of* E.

Proof. — By the preceding lemma and by Theorem 4.14, it is sufficient to show that E is removable if and only if E_ε is dominating for some $\varepsilon > 0$. Suppose first that E is not removable, and let $\varepsilon > 0$ be given. There exists a non-constant bounded analytic function f on the complement C of E with respect to the Riemann sphere (see [1], Chapter IV, Section 4C). By the maximum principle, we see that

$$\sup \{|f(z)| : z \in G\} = \sup \{|f(z)| : z \in C\}.$$

But

$$\sup \{|f(z)| : z \in E_\varepsilon\} < \sup \{|f(z)| : z \in C\}$$

since $(E_\varepsilon)^-$ is a compact subset of C. Hence E_ε is not a dominating set.

Conversely, suppose that E is removable, and choose ε with $0 < \varepsilon < \text{distance}(E, \partial G')$. Then E_ε is a dominating set. For, given $f \in B_H(G)$, there is a bounded analytic continuation, which we still denote by f, into all of G'. But $G' - E_\varepsilon$ is a compact subset of G', and thus

$$\sup \{|f(z)| : z \in G'\} = \sup \{|f(z)| : z \in E_\varepsilon\}$$

as was to be proved.

The next result deals with the sweeping of measures on the unit disc D onto the boundary of D, and conversely with the sweeping of measures on the boundary of D into the interior.

4.23. THEOREM. — *The spaces* $M'(D)$ *and* $L^1(-\pi, \pi)/N^1$ *are isometrically isomorphic, where*

$$N^1 = \{h \in L^1(-\pi, \pi) : \int_{-\pi}^{\pi} f(e^{i\theta})\, h(\theta)\, d\theta = 0, \quad \text{all} \quad f \in B_H(D)\}.$$

This means that, given a measure $\mu \in M(D)$, and a positive number ε, there exists a function $h \in L^1(-\pi, \pi)$ such that

$$\int_D f(z)\, d\mu(z) = \int_{-\pi}^{\pi} f(e^{i\theta})\, h(\theta)\, d\theta \text{ for each } f \in B_H(D), \tag{4.23.1}$$

and such that $\|h\|_1 \leqslant (1 + \varepsilon) \|\mu\|$, and conversely, that given any function $h \in L^1(-\pi, \pi)$, and a positive number ε, there is a measure μ in $M(D)$ for which (4.23.1) holds, and such that $\|\mu\| \leqslant (1 + \varepsilon) \|h\|_1$.

Remark. — A result equivalent to this theorem was proved in [4], Theorem 6. The fact that each $\mu \in M(D)$ may be swept into $L^1(-\pi, \pi)$ follows from Theorem 3.4 and [7], Theorem 2. It was proved in [19], Theorem A that each equivalence class of L^1/N^1 contains a unique function h of minimal norm.

Proof of the theorem. — Suppose we are given a function $h \in L^1(-\pi, \pi)$. We define

$$L(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\, h(\theta)\, d\theta, \qquad f \in B_H(D). \tag{4.23.2}$$

To prove that there exists a measure $\mu \in M(D)$ such that

$$L(f) = \int_D f(z)\, d\mu(z), \qquad f \in B_H(D), \tag{4.23.3}$$

it is enough to prove that L is a continuous linear functional on $\alpha(D)$. The linearity is obvious. To prove that L is continuous, it is enough to prove that $L^{-1}(0) = \{f \in B_H(D) : L(f) = 0\}$ is a closed subset of $\alpha(D)$. But since $L^{-1}(0)$ is a linear subspace of $\alpha(D)$, it is enough, by Corollary 4.7, to prove that it is sequentially closed. Let us suppose then that f_n converges to f in $\alpha(D)$, and that for each n, $L(f_n) = 0$. We must prove that $L(f) = 0$. By Corollary 3.11, we know that the f_n are uniformly bounded, say $||f_n||_\infty \leqslant 1$, and pointwise convergent to f in D. Let us consider the f_n as elements of the unit ball in $H_\infty(D)$ as the dual of $L^1(-\pi, \pi)/N^1$. By the Alaoglu theorem, we may pass to a subsequence so that now $\{f_n\}$ converges, say to F, in this space. This means that

$$\int f_n(e^{i\theta})\, H(\theta)\, d\theta \to \int F(e^{i\theta})\, H(\theta)\, d\theta$$

for each $H \in L^1(-\pi, \pi)$. By choosing, for each $z_0 \in D$, $H(\theta) = (2\pi)^{-1} e^{i\theta}/(e^{i\theta} - z_0)$, we see by the Cauchy integral formula that $F = f$. Since $\int F(e^{i\theta})\, h(\theta)\, d\theta = 0$, we are done.

Suppose now that we are given a measure $\mu \in M(D)$. We define L by (4.23.3) and must prove that it has a representation (4.32.2). Analogously to the above considerations, it is enough to prove that if $f_n \to f$ in the weak-star topology on $H_\infty(D)$ as the dual of $L^1(-\pi, \pi)/N^1$, and if $L(f_n) = 0$, then $L(f) = 0$. But, by the uniform boundedness principle, if the f_n converge in this topology, then they must be bounded in the essential supremum norm as functions on $(-\pi, \pi)$. Therefore, they must be uniformly bounded as functions on D. We see also that they must converge pointwise in D by choosing the functions H as above. Hence $f_n \to f$ in $\alpha(D)$ and the result follows.

4.24. *Another approach to Theorem 4.1.* We give a non-constructive proof here, based on functional analysis methods, that given a measure $\mu \in M(G)$ and a positive number ε, there exists a measure $\nu \in L^1(G)$ with $\nu \sim \mu$ and $||\nu|| \leqslant ||\mu|| + \varepsilon$. We recall (see 4.2) that $N_\lambda(G)$ is the annihilator in $L^1(G)$ of $B_H(G)$ and that (see the proof of Theorem 4.5) $B_H(G)$ is a weak-star closed subspace of $L^\infty(G)$. Recall also that we write $[\mu]$ for the equivalence class of $M'(G)$ that μ belongs to. In case μ belongs to $L^1(G)$, let us write $[[\mu]] = \mu + N_\lambda(G)$ for the equivalence class of $L^1(G)/N_\lambda(G)$ that μ belongs to.

Then we have, from the theory of Banach spaces, that

a) $B_H(G) = N_\lambda(G)^\perp$

b) $(L^1(G)/N_\lambda(G))^*$ is isometrically isomorphic to $H^\infty(G)$

c) for $\nu \in L^1(G)$, we have

$$\|[[\nu]]\| = \sup \{|\textstyle\int f d\nu| : f \in H^\infty(G), \|f\| = 1\}.$$

The isomorphism in b) is the following: to the function $f \in H_\infty$ we let correspond the linear functional L_f given by $L_f([[\nu]]) = \int f d\nu$.

Now let the measure $\mu \in M(G)$ be given. We define the linear functional $L_\mu : B_H(G) \to \mathbf{C}$ by $L_\mu(f) = \int f d\mu$. Then L_μ is a continuous linear functional on $B_H(G)$ as a subspace of $L^\infty(G)$ in the weak-star topology. This is the same topology as the weak-star topology on $B_H(G)$ as the dual of $L^1(G)/N_\lambda(G)$. That L_μ is continuous is proved by proving that the null space of L_μ is sequentially closed, and this is done by an argument with normal families. This is similar to other proofs we have given and we omit the details. As a consequence, we see that there exists an element $[[\rho]]$ of $L^1(G)/N_\lambda(G)$ such that $L_\mu(f) = \int f d\rho$ for all $f \in B_H(G)$. There exists an element $\nu \in [[\rho]]$ such that

$$\|\nu\| \leqslant \|[[\rho]]\| + \varepsilon.$$

We now assert that $\|[\mu]\| = \|[[\rho]]\|$. If this is proved, we have the desired result since then

$$\|\nu\| \leqslant \|[[\rho]]\| + \varepsilon = \|[\mu]\| + \varepsilon \leqslant \|\mu\| + \varepsilon,$$

and we have already shown that $\nu \sim \mu$. Now if we take $f \in H^\infty(G)$ with $\|f\| \leqslant 1$ then we have $|\int f d\mu| \leqslant \|[\mu]\|$ so that $|\int f d\rho| \leqslant \|[\mu]\|$. By c), this implies that $\|[[\rho]]\| \leqslant \|[\mu]\|$. For the opposite inequality, choose a sequence $\{\rho_n\}$ of elements of $[[\rho]]$ such that $\|\rho_n\| \to \|[[\rho]]\|$. Since $\|\rho_n\| \geqslant \|[\mu]\|$, we are done.

We remark, finally, that by Proposition 2.10, we have proved, by these methods, the full strength of Theorem 4.1 except in the case when some constant multiple of a measure equivalent to μ is a positive measure. It does not seem that these methods will handle the exceptional case.

5. Closed ideals in $\beta(G)$.

Our study of the closed ideals in $\beta(G)$ is most successful in the case of the unit disc, $G = D$. We assume some familiarity with the theory of bounded analytic functions in D, in particular the fact (see, for example [14], Chapter 5) that every bounded analytic function in the unit disc has a unique representation as the product of an inner function and a bounded outer function. The radial boundary values of a bounded analytic function f in D, which exist almost everywhere, will be denoted by $f(e^{i\theta})$. Inner functions f are characterized by the property that multiplication by them is an isometry on $H_\infty(D)$, or equivalently that $|f(e^{i\theta})| = 1$ almost everywhere. An inner function f has the representation $f = BS$ where B is a Blaschke product over the zeros of f in D.

$$B(z) = z^p \prod \frac{-\bar{z}_n}{|z_n|} \frac{z - z_n}{1 - z\bar{z}_n}$$

and where S is a singular function

$$S(z) = \exp\left[-\int \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(\theta)\right],$$

where μ is a singular non-negative measure with respect to Lebesgue measure. An outer function Ω has the representation

$$\Omega(z) = \exp\left[-\int \frac{e^{i\theta} + z}{e^{i\theta} - z} h(\theta)\, d\theta\right]$$

where $h \in L^1(-\pi, \pi)$. The outer function Ω belongs to $H_\infty(D)$ if and only if ess inf $\{h(\theta) : -\pi < \theta \leqslant \pi\} > -\infty$.

Notation. By $(f) = f\, B_H(G)$, we denote the principal ideal generated by f in $B_H(G)$.

R. C. Buck has conjectured [5] that if $f \in \beta(D)$, then (f) is dense in $\beta(D)$ if and only if f has no zeros. We show that this conjecture has to be modified — the ideal is dense if and only if f has no inner factor. In other words, the topological units in $\beta(D)$ are just the outer functions. Roughly speaking, it might be said that the ideal structure of $\beta(D)$ is as simple as it is because the β topology is fairly strong on the one hand, yet weak enough so that the dual of $\beta(D)$ is the same as the dual of $\alpha(D)$, namely $M'(D)$.

5.1. THEOREM. — *The principal ideal* (f) *is dense in* $\beta(D)$ *if and only if* f *is an outer function.*

Proof. — First, suppose that f is an outer function. It will be enough to show that the constant function 1 belongs to $(f)^-$. We may assume that $\|f\| = 1$. Then

$$f(z) = \exp\left[\int \frac{z+e^{i\theta}}{z-e^{i\theta}}\, h(\theta)\, d\theta\right] \tag{5.1.1}$$

where $h(\theta) = \log |f(e^{i\theta})|$ is a non-positive integrable function. Let

$$h_n(\theta) = \min(h(\theta), n),$$

$$g_n(\theta) = h(\theta) - h_n(\theta),$$

$$F_n(z) = \exp - \int \frac{z+e^{i\theta}}{z-e^{i\theta}}\, h_n(\theta)\, d\theta,$$

so that

$$f(z)\, F_n(z) = \exp \int \frac{z+e^{i\theta}}{z-e^{i\theta}}\, g_n(\theta)\, d\theta. \tag{5.1.2}$$

Now $g_n(\theta) \geqslant 0$, and g_n converges to 0 in the L^1 metric. Hence

$$\|fF_n\|_\infty \leqslant 1,$$

and $f(z)\, F_n(z) \to 1$ as $n \to \infty$ for each z in D, and we have proved that $1 \in (f_n)^-$.

For the converse, assume that f has a non-trivial inner factor φ. Then $(f) \subseteq (\varphi) \neq \beta(D)$, and the result follows from the next result, which we understand has also been obtained by Paul Hessler.

5.2. PROPOSITION. — *If* φ *is an inner function, then* (φ) *is closed in* $\beta(D)$.

Proof. — By Corollary 4.7, since an ideal is *a fortiori* a linear subspace, it is enough to show that (φ) is sequentially closed. Let us then assume that $\varphi f_n \to g$ in $\beta(D)$. By Corollary 3.11 the functions $\{\varphi f_n\}$ are uniformly bounded. But $\|\varphi f_n\| = \|f_n\|$ since multiplication by an inner function is an isometry, and it follows that the functions f_n are uniformly bounded. By passing to a subsequence, we may assume that $\{f_n\}$ converges, say $f_n \to f$, in $\beta(D)$. Hence $g = \varphi f$, and it follows that $g \in (\varphi)$, which was to be proved.

5.3. PROPOSITION. — *If $f \in \beta(D)$ and if φ is the inner factor of f, then (f) is dense in (φ).*

Proof. — Let g denote the outer factor of f, so that $f = \varphi g$. From the proof of Theorem 5.1, there exists a sequence $\{g_n\}$ of bounded analytic functions such that $g_n g \to 1$ in $\beta(D)$.

A function $f \in \beta(G)$ is called a unit if $fg = 1$ for some $g \in \beta(G)$; f is a unit if and only if $|f|$ is bounded away from 0 in G.

5.4. THEOREM. — *The principal ideal (f) is closed if and only if the outer factor of f is a unit.*

Proof. — Let g be the outer factor of f and let φ be the inner factor of f. If g is a unit, then (f) is closed, by Proposition 5.2. If g is not a unit, then (f) does not contain φ, for if $\varphi = fh = \varphi gh$, then $gh = 1$. But $\varphi \in (f)^-$ by the preceding theorem, and the proof is complete.

5.5. THEOREM. — *Every closed ideal in $\beta(D)$ is the principal ideal generated by an inner function.*

Remark. — Since, by Corollary 3.5, the closed ideals of $\beta(D)$ correspond to the closed subspaces of $\alpha(D)$ that are invariant under multiplication by bounded analytic functions, the result follows from a result of Srinavasan (see [12], p. 25) and Theorem 4.23. We prove the result here by using Beurling's characterization of the closed invariant subspaces of H_2 (see [3], or [14], Chapter 7), as being of the form φH_2, where φ is an inner function. It would be good to find a direct intrinsic proof. By solving a simple extremal problem, the result can be shown to follow from the assertion that if f and g are two inner functions with no non-trivial common factors, then there exist sequences $\{F_n\}$ and $\{G_n\}$ of bounded analytic functions such that $fF_n + gG_n$ converges boundedly to 1. It has been remarked to the authors that this fact follows from the Corona theorem of Carleson [6], but because of its depth and the difficulty of its proof, the Corona theorem is hardly an appropriate tool for this problem.

Proof. — Let I be a closed ideal in $\beta(D)$. If $f \in I$ and if φ is the inner factor of f, then $\varphi \in I$ by Proposition 5.3. It is sufficient to show that I contains one inner function φ_0 that divides all the other inner functions in I.

Let J denote the closure of I in the Hilbert space H_2 of all functions f analytic in D such that $\|f\|_2 < \infty$, where

$$\|f\|_2 = \lim_{r\to 1-} \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 \, d\theta \right\}^{1/2}.$$

Then J is a closed subspace of H_2 that is invariant under multiplication by z, and so by Beurling's theorem, J is generated by some one inner function φ_0; that is, $J = \varphi_0 H_2$. Consequently, φ_0 divides all the inner functions in I, and it is therefore enough to show that φ_0 itself belongs to I.

Since φ_0 is in the H_2 closure of I, there exists a sequence $\{f_n\}$ of functions $f_n \in I$ such that $f_n \to \varphi_0$ in the H_2-metric. We write $f_n = \varphi_n g_n$, where φ_n is inner and g_n is outer. By passing to a subsequence, we may assume that $\{\varphi_n\}$ converges in $\beta(D)$, say $\varphi_n \to \varphi$. In particular, $\varphi \in I$. Since $\|g_n\|_2 = \|f_n\|_2$, we may use the weak compactness of the unit ball in Hilbert space, and by passing to a subsequence, may assume that $\{g_n\}$ converges weakly in H_2, say $g_n \to g$. In particular, $g_n(z) \to g(z)$ for each $z \in D$, since evaluation at points of D is, by the Cauchy integral formula, a continuous linear functional on H_2. It follows that $\varphi g = \varphi_0$. From the next lemma, with $g_0 = 1$, it follows that φ is inner. But from the equation $\varphi g = \varphi_0$, with φ inner and $g \in H_2$, it follows that $g \in H_\infty$, since $H_2 \subseteq H_1$ and the inner-outer factorization of functions in H_1 is unique. Hence $\varphi_0 \in IH_\infty = I$, and we are done.

5.6. Lemma. — *Suppose that $\{f_n\}$, $n = 1, 2, 3, \ldots$, is a sequence of functions in H_2, that $f_n \to f_0$ in H_2, $f_0 \neq 0$, that $f_n = \varphi_n g_n$ is the inner-outer factorization of f_n, $n = 1, 2, 3, \ldots$, and that $f_0 = \varphi_0 g_0$ is the inner-outer factorization of f_0. Suppose further that $\varphi_n \to \varphi$ in $\beta(D)$ and that $g_n \to g$ weakly in H_2. Then φ is an inner function and $g_n \to g$ in the H_2 metric.*

We remark that g need not be an outer function.

Proof. — It is clear that $|\varphi(z)| \leqslant 1$ for each $z \in D$, so that $|\varphi(e^{i\theta})| \leqslant 1$ almost everywhere. To prove that φ is inner, it is enough to prove that $|\varphi(e^{i\theta})| \geqslant 1$ almost everywhere. If, on the contrary, $|\varphi(e^{i\theta})| < 1$ on a set of positive measure, then for some set of positive measure and for some $\varepsilon > 0$, we would have $|\varphi(e^{i\theta})| < 1 - \varepsilon$ on that set. It would follow that for each $h \in H_2$, $h \neq 0$, $\|\varphi h\|_2 < \|h\|_2$. From the fact that $g_n \to g$ weakly, we see that $\|g\|_2 \leqslant \liminf \|g_n\|_2$. Hence

$$||g||_2 \leqslant \lim ||g_n||_2 = \lim ||f_n||_2 = ||\varphi_0 g_0|| = ||g_0||,$$
$$||g_0|| = ||\varphi_0 g_0|| = ||\varphi g|| \leqslant ||g||.$$

Hence, we have equality throughout, so that $||\varphi g|| = ||g||$, and φ is consequently an inner function. Also, $||g|| = \lim ||g_n||$, and this, plus weak convergence of $\{g_n\}$ to g, implies that $\{g_n\}$ converges to g in norm, since then

$$(g - g_n, g - g_n) \to 0.$$

Theorem 5.5 can be used to extend Beurling's theorem on invariant subspaces of H_2 to other spaces of analytic functions, and in particular to the spaces H_p, $1 \leqslant p < \infty$. These spaces H_p were treated in this connection by Helson in [12]. We restrict outselves to spaces of functions of bounded characteristic, that is, functions that are quotients of two bounded analytic functions. Another characterization of such functions f is given by the criterion.

$$\int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| \, d\theta \leqslant m < \infty \qquad \text{for all } r < 1.$$

Every function of bounded characteristic is the product of an outer function and the quotient of two inner functions that have no non-trivial common inner factor. A space of functions is said to be invariant if it is taken into itself by multiplication by each bounded analytic function.

5.7. THEOREM. — *Let* E *be a topological vector space whose elements are functions analytic in the unit disc* D, *that statisfies the following five conditions :*

i) *every function in* E *has bounded characteristic;*

ii) *if $f \in$ E and $f = \varphi_1 g/\varphi_2$, where g is an outer function and φ_1 and φ_2 are inner functions with no non-trivial common inner factor, then $g/\varphi_2 \in$ E.*

iii) *the bounded analytic functions in* D *form a dense subset of* E.

iv) *for each function f that is bounded and analytic in* D, *let T_f be the transformation defined by $T_f g = fg$. Then T_f maps* E *into* E *and T_f is continuous.*

v) *if* S *is a closed subspace of* E *and if $\{f_n\}$ is a uniformly bounded sequence of analytic functions in* S *that converges pointwise in* D *to a function f, then $f \in$ S.*

Then each closed invariant subspace V *of* E *is generated by an inner function* φ, *that is,* $V = \varphi E$.

Proof. — We show first that if f is an outer function in E, then the constant function 1 is in the E-closure of $f B_H(D)$. The proof is almost identical with the proof of Theorem 5.1. The function f has the representation (5.1.1), where $h(\theta)$ is an integrable function, not necessarily bounded above. We define h_n and F_n as before. Then each F_n belongs to $B_H(D)$, and fF_n has the representation (5.1.2). Hence $\|fF_n\|_\infty \leqslant 1$ and $f(z)F_n(z) \to 1$ as $n \to \infty$ for each $z \in D$, and therefore by condition (v) of the present theorem, the function 1 belongs to the smallest closed subspace of E that contains $f B_H(D)$.

Now let V be a closed invariant subspace of E, and let $f = \varphi_1 g/\varphi_2$ be the inner-outer representation of f. Then $\varphi_1 \in V$. Indeed, as we have just seen, there is a sequence $\{f_n\}$ of bounded analytic functions f_n such that $gf_n \to 1$ in E. Using (iv), we see that $f(\varphi_2 f_n) = \varphi_1(gf_n) \to \varphi_1$ in E, and hence $\varphi_1 \in V$.

We now let $V' = V \cap B_H(D)$, and claim that V' is a closed ideal in $B_H(D)$. First, V' is clearly an ideal. Next, to prove that V' is closed, it is enough to prove that it is sequentially closed. Let $\{f_n\}$ be a sequence of functions in V' such that $f_n \to f$, say, in $\beta(D)$. Using (v) and the fact that $f \in B_H(D)$, we see that $f \in V$.

Now by Theorem 5.5, $V' = \varphi_0 B_H(D)$ for some inner function φ_0. We now prove that $V = \varphi_0 E$. First, to see that $V \subseteq \varphi_0 E$, let f be any function in V, and write $f = \varphi_1 g/\varphi_2$ as its inner-outer representation. From the first part of our proof, we see that $\varphi_1 \in V$, hence $\varphi_1 \in V'$, and consequently $\varphi_1 = \varphi_0 \varphi'$ for some inner function φ', and hence $f = \varphi_0 \varphi' g/\varphi_2$. But from ii), we have that $g/\varphi_2 \in E$, and from iv), since $\varphi' \in B_H(D)$, we have that $g\varphi'/\varphi_2 \in E$ and so $f \in \varphi_0 E$. It remains to show that $\varphi_0 E \subseteq V$, that is, that $\varphi_0 f \in V$ for each $f \in E$. By iii), there is a net $\{f_\gamma\}$, $\gamma \in \Gamma$, of functions $f_\gamma \in B_H(D)$ such that $\{f_\gamma\}$ converges to f in E. But by iv), $\varphi_0 f_\gamma \to \varphi_0 f$ in E and hence $\varphi_0 f \in V$, since $\varphi_0 f_\gamma \in V'$ for each $\gamma \in \Gamma$, and the result is proved.

5.8. *Remarks.*

a) The second half of condition (iv), namely the continuity of the multiplication operator, follows automatically from the closed graph theorem if E is complete metric space in which evaluation at any given point of D is a continuous linear functional on E.

b) In many spaces E that actually arise, bounded pointwise convergence implies weak convergence, and this gives condition (v).

c) It is perhaps true that conditions (i)-(v) imply that if φ is an inner function, then φE is always a closed subspace, but we do not have a proof of this.

5.9. Applications.

We now show briefly that the familiar spaces H_p, $1 \leqslant p < \infty$, satisfy the hypotheses of Theorem 5.7, so that each closed invariant subspace of H_p is generated by an inner function. The case $p = 2$ is, of course, the theorem of Beurling we used in the proof of Theorem 5.5, which was used in turn in the proof of Theorem 5.7. The case $p = 1$ was treated by de Leeuw and Rudin [17]. The general case was treated by Helson [12].

i) $\int |f(re^{i\theta})|^p \, d\theta \leqslant m < \infty$ for $r < 1$ implies that

$$\int \log^+ |f(re^{i\theta})| \, d\theta \leqslant m' < \infty \text{ for } r < 1,$$

so that each $f \in H_p$ has bounded characteristic.

ii) Passing to the boundary of the disc, we have

$$\int \left| \frac{g(e^{i\theta})}{\varphi_2(e^{i\theta})} \right|^p d\theta = \int \left| \frac{g(e^{i\theta})}{\varphi_2(e^{i\theta})} \varphi_1(e^{i\theta}) \right|^p d\theta$$

since $|\varphi_1(e^{i\theta})| = 1$ almost everywhere.

iii) The polynomials are bounded analytic functions in D, and by Fejér's theorem, the Cesàro means of the partial sums of the Taylor's series of a function in H_p must converge to the function in the metric of H_p.

iv) It is obvious that T_f maps H_p into H_p, and the continuity is also clear since

$$\|T_f g - T_f g'\|_p = \|f(g - g')\|_p \leqslant \|f\|_\infty \|g - g'\|_p.$$

v) Let us suppose, by the way of contradiction, that we have a sequence $\{f_n\}$ of bounded analytic functions in S, that $f_n \to f$ in $\beta(D)$, but that $f \notin S$. By the Hahn-Banach theorem, there exists a function

$$g \in L^q(-\pi, \pi) \text{ such that } \int f_n(e^{i\theta}) \, g(e^{i\theta}) d\theta = 0$$

but such that $\int fg\,d\theta \neq 0$. But by Hölder's inequality, if we define ν by $d\nu(\theta) = g(\theta)\,d\theta$, then ν is a measure on $(-\pi, \pi)$ that is absolutely continuous with respect to Lebesgue measure. By Theorem 4.23, this measure ν can be swept inside the disc, in the sense that there is a measure $\mu \in M(D)$ such that

$$\int f(z)\,d\mu(z) = \int f(e^{i\theta})\,d\nu(\theta).$$

But then we have $0 = \int f_n\,d\mu \rightarrow \int f\,d\mu \neq 0$, and the assertion is proved by contradiction.

It is a consequence of Helmer's Theorem [11] that in the algebra of analytic functions in the complex plane, every finitely generated ideal is closed in the topology of uniform convergence on compact subsets. This contrasts with the algebra $\beta(D)$, as the next result shows.

5.10. Proposition. — *There is a finitely generated ideal in* $\beta(D)$ *that is not closed.*

Proof. — We outline the proof, which is along familiar lines. By Theorem 5.5, it is enough to construct a finitely generated ideal in $B_H(D)$ that is not principal. We choose two sequences $\{z_n\}$ and $\{w_n\}$ of points of D having no points in common, such that $\Sigma\,(1 - |z_n|) < \infty$ and such that $|z_n - w_n|$ converges to 0 extremely rapidly as n tends to ∞. Let B_1 be the Blaschke product formed with the zeros $\{z_n\}$ and let B_2 be the Blaschke product formed with the zeros $\{w_n\}$. Suppose that the ideal in $B_H(D)$ generated by B_1 and B_2 were principal, with generator f. Then f has no zeros since B_1 and B_2 have no common zeros. Thus $1/f$ is an analytic function of bounded characteristic, and it follows that for some positive constant c,

$$|f(z)| \geqslant \exp\left\{\frac{-c}{1-|z|}\right\} \quad \text{all } z \in D.$$

We would also have $f = g_1 B_1 + g_2 B_2$ for some pair g_1, g_2 of bounded analytic functions. In particular, $|f(z_n)| = |B_2(z_n)|\,|g_2(z_n)|$, which is impossible, because g_2 is bounded while B_2 is extremely small at the points $\{z_n\}$.

5.11. Proposition. — *There is a maximal ideal in* $\beta(D)$ *that is not closed.*

Proof. — If a maximal ideal is closed, then by Theorem 5.5, it is generated by an inner function φ. But the multiples of φ cannot form a maximal ideal unless φ is a single Blaschke factor. To see this, observe that if φ contains a Blaschke product of at least two factors, or a Blaschke product and a singular function, then the inner function formed by deleting one factor will generate a larger ideal. On the other hand, if φ has no zeros, then there is a square root of φ that is an inner function that generates a larger proper ideal. Now let I be the ideal of all functions $f \in B_H(D)$ such that $f(x) \to 0$ as $x \to 1-$. There is a maximal ideal J that contains I, and by the above remarks, it follows that J cannot be closed, since the functions in I have no common zeros.

5.12. In a general region G, we make the following definitions, where as before, (f) denotes the principal ideal in $B_H(G)$ generated by f.

a) The function $f \in \beta(G)$ is called an exterior function if (f) is dense in $\beta(G)$.

b) The function $f \in \beta(G)$ is called an interior function if (f) is closed in $\beta(G)$.

Note that if f is both interior and exterior, then f is a unit, since then $fB_H = B_H$.

Problem. — For which regions G does each $f \in \beta(G)$ have an unique factorization, modulo units, as the product of an interior function and an exterior function ?

Problem. — For which regions G is each closed ideal in $\beta(G)$ principal ?

We call a transformation $T : H_\infty(G) \to H_\infty(G)$ an approximate isometry provided that there exist two positive numbers a and b such that

$$a\|f\| \leqslant \|Tf\| \leqslant b\|f\| \quad \text{for each } f \in H_\infty(G).$$

Problem. — For which regions G do the interior functions coincide with those functions φ such that multiplication by φ is an approximate isometry on $H_\infty(G)$?

5.13. In the disc, there is an intrinsic way of finding the inner factor of a function $f \in B_H(D)$. Let us suppose for simplicity that $f(0) \neq 0$. For each interior factor φ of f, let

$$\rho(\varphi) = \sup \{|u(0)\,\varphi(0)| : u \text{ is a unit, } ||u\varphi||_\infty = 1\}.$$

Among all such φ with $||\varphi|| = 1$, there is a function φ_0 that minimizes ρ, and it can be shown that f/φ_0 is an exterior function. We omit the details.

5.14. We now consider maximal ideals in $\beta(G)$. From the Gelfand theory of Banach algebras, we know that there is an one-to-one correspondance between maximal ideals in $H_\infty(G)$ and multiplicative linear functionals on $H_\infty(G)$; the ideal associated with a given multiplicative linear functional is just the null space of the functional. Since $B_H(G)$ is the underlying algebra both of $H_\infty(G)$ and of $\beta(G)$, it follows that a maximal ideal is closed in $\beta(G)$ if and only if the associated multiplicative linear functional is β-continuous see [13], Corollary to Theorem 2.6.2).

5.15. Let X be the class of all continuous multiplicative linear functionals on $\beta(G)$, and suppose that G is bounded, so that $e \in B_H(G)$, where $e : G \to \mathbf{C}$ is the identity map, $e(z) = z$ for $z \in G$. If $\chi \in X$, we let χ^* be the complex number $\chi^* = \chi(e)$.

5.16. Let G' denote the set consisting of the points of G and of all points that are removable singularities for all functions in $B_H(G)$. We may think of the functions in $B_H(G)$ as being defined on G'.

For $\zeta \in G'$, let χ_ζ denote the functional of evaluation at ζ, $\chi_\zeta(f) = f(\zeta)$. Then $\chi_\zeta \in X$.

The question arises : for which regions G do we obtain all the elements of X in this manner ? Rudin [22] has shown by an ingenious construction that not all regions have this property. We shall show, however, that there is a wide class of regions that do have this property.

5.17. THEOREM. — *If G is a bounded region and if ∂G is the union of non-degenerate continua and isolated points, then the only continuous multiplicative linear functionals on $\beta(G)$ are the point evaluations at points of G'.*

Proof. — From the easily proved fact that if G is an open set in the plane, and if p is an isolated point of ∂G then $G \cup \{p\}$ is an open set, we know that if we adjoin the isolated boundary points of G to G, we obtain a region whose boundary is now the union of non-degenerate continua. This new set has no removable singularities in its boundary (any point $z_0 \in \partial G$ is a non-removable singularity for the bounded analytic function $(z - z_0)^{1/2}(z - z_1)^{1/2}$, where z_1 is a point on the same component of ∂G as z_0) and so this new set is precisely G'. Now let $\chi \in X$ be given. We first show that $\chi^* = \chi(e)$ belongs to G'. Indeed, χ^* cannot belong to the exterior of G', since then the function f given by $f(z) = (z - \chi^*)^{-1}$ would belong to $B_H(G)$, which would lead to the contradiction

$$1 = \chi(1) = \chi(z - \chi^*)\chi(f) = 0 \times \chi(f) = 0.$$

Also, χ^* cannot belong to the boundary of G, for in this case there would be another point $z_0 \neq \chi^*$ in the same component of ∂G as χ^*. Let

$$g_n(z) = (z - z_0)\left(\frac{z - \chi^*}{z - z_0}\right)^{1/n},$$

for some branch of the n-th root. Then $g_n \in \beta(G')$ and $\chi(g_n) = 0$ since $(\chi(g_n))^n = \chi(g_n^n) = 0$. The functions g_n are uniformly bounded, and hence some subsequence converges in $\beta(G)$ to a function that must have the form $c(z - z_0)$, where c is a constant of modulus 1. Hence $c(\chi^* - z_0) = \chi(c(z - z_0)) = 0$, which is a contradiction.

We now know that $\chi^* \in G'$, and we must show that $\chi(f) = f(\chi^*)$ for all $f \in B_H(G)$. Let us choose $f \in B_H(G)$, and let $g \in B_H(G)$ be defined by

$$g(z) = \frac{f(z) - f(\chi^*)}{z - \chi^*} \quad \text{for } z \neq \chi^*,$$

with $g(\chi^*) = f'(\chi^*)$. We then have that

$\chi(f) - f(\chi^*) = \chi(z - \chi^*)\,\chi(g) = 0$, and the proof is done.

5.18. In conclusion, we list some unsolved problems, some of which have already been mentioned in the text.

a) Is the β-topology on $B_H(G)$ the same as the Mackey topology in the pairing $< B_H(G), M'(G) >$?

b) Suppose that τ is a Hausdorff topology on $B_H(G)$ that makes it into a locally convex space in which the convergent sequences are precisely the uniformly bounded and pointwise convergent sequences. Must τ be intermediate between the α and β topologies ?

c) If we think of $B_H(G)$ as a space of multiplication operators on $L^2(G)$ (the space of equivalence classes of complex-valued functions f

on G with $||f||^2 = \int |f|^2 \, d\lambda$, where λ is Lebesgue planar measure), then it follows from Theorem 4.1 that the weak operator topology on $B_H(G)$ is identical with the α topology. Is the strong operator topology identical with the β topology ?

d) Is multiplication in $\alpha(G)$ jointly continuous ?

e) For which regions G does each $f \in B_H(G)$ have an unique factorization, modulo units, as a product of an interior factor and an exterior factor ?

f) For which regions G is every closed ideal in $\beta(G)$ principal ?

g) Give a concrete description of the continuous multiplicative linear functionals on $\beta(G)$ for the general region G. For which regions G are they just the point evaluations at points of G and at removable singularities ?

h) Does there exist a natural way of assigning a topology to $B_H(G)$ so that the continuous multiplicative linear functionals are just those described in the preceeding problem ?

i) Let D denote the open unit disc, and let $\{a_n\}$, $n = 0, 1, 2, \ldots$, be a given sequence of complex numbers. Suppose that if f is any function in $B_H(D)$, with power series $f(z) = \Sigma \, b_n z^n$, then the limit

$$\lim_{r \to 1-} \Sigma \; a_n \, b_n \, r^n$$

exists. Is $\{a_n\}$ necessarily one side of the sequence of Fourier coefficients of some integrable function ? That is does $\{a_n\}$ determine a continuous linear function on $\beta(D)$?

(This question is referred to in [4], p. 181. In essence, it was raised by A. E. Taylor in [23], p. 33.)

Since the preparation of this paper for publication, some of the questions raised have been answered. Collins has given a negative answer to problem a), Waelbroeck has given a negative answer to problem d), and Shields and Wells have given a positive answer to problem i).

Added in proof. — Some of the questions posed have been answered in the meantime. For question a), J. B. Conway has shown that the β topology is not the Mackey topology, in his paper : Subspaces of $C(S)_\beta$, the space $(1^\infty, \beta)$, and (H^∞, β), *Bull. Amer. Math. Soc.* 72 (1966), 79-81.

For question d), L. Waelbroeck has shown in a private communication that multiplication in $\alpha(G)$ is not jointly continuous. Finally, the conjecture in problem i) is correct as has been shown (unpublished) by Piranian, Shields, and Wells.

We give some additional bibliographical information. In reference 10, Havin shows that $H^\infty(G)$ is a conjugate Banach space, which is precisely our Theorem 4.5. Another paper of V. P. Havin that should be mentioned, in connection with balayage, is the article, Analytical representation of linear functionals on spaces of harmonic and analytic functions, continuous in a closed domain, *Doklady Akad. Nauk SSSR,* Vol. 151 (1963), 505-508. Finally, we mention the paper by T. P. Srinivasan and Ju-Kwei Wang, On closed ideals of analytic functions, *Bull. Amer. Math. Soc.* 16 (1965), 49-52 where a result close to our Theorem 5.5 is proved.

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