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Logarithmic Surfaces and Hyperbolicity

LOGARITHMIC SURFACES AND HYPERBOLICITY

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ABSTRACT. — In 1981 J. Noguchi proved that in a logarithmic algebraic manifold, having logarithmic irregularity strictly bigger than its dimension, any entire curve is algebraically degenerate.

In the present paper we are interested in the case of manifolds having logarithmic irregularity equal to its dimension. We restrict our attention to Brody curves, for which we resolve the problem completely in dimension 2: *in a logarithmic surface with logarithmic irregularity* 2 *and logarithmic Kodaira dimension* 2, *any Brody curve is algebraically degenerate.*

In the case of logarithmic Kodaira dimension 1, we still get the same result under a very mild condition on the Stein factorization map of the quasi-Albanese map of the log surface, but we show by giving a counter-example that the result is not true any more in general.

Finally we prove that a logarithmic surface having logarithmic irregularity 2 admits certain types of algebraically non degenerate entire curves if and only if its logarithmic Kodaira dimension is zero, and we also give a characterization of this case in terms of the quasi-Albanese map.

RÉSUMÉ. — J. Noguchi a démontré en 1981 que toute courbe entière est algébriquement dégénérée dans une variété algébrique logarithmique ayant une irrégularité logarithmique strictement plus grande que sa dimension.

Nous nous intéressons ici à des variétés dont l'irrégularité logarithmique est égale à la dimension. Nous nous restreignons au cas des courbes de Brody, pour lequel nous obtenons une solution complète du problème en dimension 2 : *toute courbe de Brody dans une surface logarithmique ayant une irrégularité logarithmique égale à* 2 *et de dimension de Kodaira logarithmique égale à* 2 *est algébriquement dégénérée.*

Nous obtenons encore le même résultat pour les variétés de dimension de Kodaira logarithmique égale à 1, sous une condition très faible portant sur la factorisation de Stein de l'application quasi-Albanese de la surface logarithmique. Nous démontrons également, par un contre-exemple, que le résultat ne tient plus sans cette condition.

Nous prouvons finalement qu'une surface logarithmique ayant une irrégularité logarithmique égale à 2 admet un certain type de courbes entières algébriques non dégénérées si et seulement si leur dimension de Kodaira logarithmique est égale à zéro ; nous donnons également une caractérisation de ce cas en termes de l'application quasi-Albanese.

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1. Introduction and results

We first introduce some notations (see Section 2 for the precise definitions). Let $\overline{X}$ be a projective manifold and $D \subset \overline{X}$ a normal crossing divisor. We call the pair $(\overline{X}, D)$ a log manifold and denote $X = \overline{X} \setminus D$. It is called a log surface if $\dim \overline{X} = 2$. Let $T^*_{\overline{X}}$ be its cotangent bundle and $\overline{T}^*_X$ its log cotangent bundle. We denote by $q_{\overline{X}} = \dim H^0(\overline{X}, T^*_{\overline{X}})$ its irregularity and $\bar{q}_X = \dim H^0(\overline{X}, \overline{T}^*_X)$ its log irregularity. We denote its log canonical bundle by $\overline{K}_X = \bigwedge^{\dim \overline{X}} \overline{T}^*_X = K_{\overline{X}}(D)$ and its log Kodaira dimension by $\bar{\kappa}_X = \kappa(\overline{X}, \overline{K}_X)$, the L-dimension of $\overline{K}_X$. We call $(\overline{X}, D)$ to be of log general type if $\bar{\kappa}_X = \dim \overline{X}$. Finally let $\alpha_X : X \to \mathcal{A}_X$ be the quasi-Albanese map. It is a holomorphic map which extends to a rational map $\bar{\alpha}_X : \overline{X} \cdots \to \bar{\mathcal{A}}_X$ (Iitaka '76 [16]), where $\bar{\mathcal{A}}_X$ is some compactification of $\mathcal{A}_X$ (see §2.2).

We know by the log-Bloch Theorem (Noguchi '81 [24]) that for any log manifold such that $\bar{q}_X > \dim \overline{X}$, any entire holomorphic curve $f : \mathbb{C} \to X$ is algebraically degenerate (this means $f(\mathbb{C})$ is contained in a proper algebraic subvariety of $\overline{X}$). More generally, by results of Noguchi-Winkelmann '02 [26] one has (defining the log structure and $\bar{q}_X$ on a Kähler manifold in a similar way as above:

THEOREM 1.1 (Noguchi, Noguchi-Winkelmann). — *Let $\overline{X}$ be a compact Kähler manifold and D be a hypersurface in $\overline{X}$. If $\bar{q}_X > \dim \overline{X}$, then any entire holomorphic curve $f : \mathbb{C} \to X$ is analytically degenerate (this means contained in a proper analytic subset of $\overline{X}$).*

In this paper we are interested in the case of log surfaces $(\overline{X}, D)$ with $\bar{q}_X = \dim \overline{X} = 2$.

The first part of this paper deals with the case of surfaces of log general type, that is $\bar{\kappa}_X = 2$. We restrict our attention to Brody curves, this means entire curves $f : \mathbb{C} \to X$ with bounded derivative f' in $\overline{X}$, for which we resolve the problem completely in the following main theorem of our paper.

THEOREM 1.2. — *Let $(\overline{X}, D)$ be a log surface with log irregularity $\bar{q}_X = 2$ and with log Kodaira dimension $\bar{\kappa}_X = 2$. Then every Brody curve $f : \mathbb{C} \to X$ is algebraically degenerate.*

The proof and some applications are given in Section 3.

The second part of this paper complements Theorem 1.2.

Let $(\overline{X}, D)$ be a log surface with log irregularity $\bar{q}_X = 2$. Let $\bar{\alpha}_X : \overline{X} \cdots \to \bar{\mathcal{A}}_X$ be the compactified quasi-Albanese map, I its finite set of

points of indeterminacy and $\bar{\alpha}_0 = \bar{\alpha}_X|_{\overline{X}\setminus I}$. In the case of dominant $\bar{\alpha}_X$, we consider the following condition:

(*) For all $z \in \mathcal{A}_X$ and $\overline{E}$ a connected component of the Zariski closure of $\bar{\alpha}_0^{-1}(z)$ with $\overline{E} \cap X \neq \emptyset$, any connected component of D intersecting $\overline{E}$ is contained in $\overline{E}$ (i.e. $\overline{E}$ is a connected component of $\overline{E} \cup D$).

We remark that condition (*) can be expressed intrinsically (see subsection 4.2) and is implied by the condition that all the fibers of $\alpha_X : X \to \mathcal{A}_X$ are compact. In particular, this condition is much weaker than the properness of α_X.

In the case $\bar{\kappa}_X = 1$ we have the following result:

THEOREM 1.3. — *Let $(\overline{X}, D)$ be a log surface with log irregularity $\bar{q}_X = 2$ and with log Kodaira dimension $\bar{\kappa}_X = 1$. Assume condition (*) in the case of dominant $\bar{\alpha}_X$. Then every entire curve $f : \mathbb{C} \to X$ is algebraically degenerate.*

The proof and some applications are given in Section 4.

As our counterexample in Proposition 4.7 shows, the additional condition (*) is necessary for the theorem to hold. Some reflections on our proof will reveal also that our condition (*) can in fact not be weakened further, at least when mild restrictions are imposed, for whose discussion and generalization to higher dimensions will be relegated to another paper.

Remark 1.4. — It is easily obtained from Hodge theory due to Deligne '71 [8] (see, for example, Catanese '84 [7]) that we have, for $(\overline{X}, D)$ a log surface

$$(1.1) \qquad \operatorname{rank}_{\mathbb{Z}} \operatorname{NS}(\overline{X}) \geqslant \operatorname{rank}_{\mathbb{Z}} \left\{ c_1(D_i) \right\}_{i=1}^{k} = k - \bar{q}_X + q_{\overline{X}}$$

where $D_1, \ldots, D_k$ are the irreducible components of D and $\operatorname{NS}(\overline{X})$ denotes the Neron-Severi group of $\overline{X}$. This may be deduced from the proof of Theorem 1.2 (i) of Noguchi-Winkelmann '02 [26], p. 605. But there does not seem to be an easy way to profit from this, unless one assumes some bound on the Neron-Severi group of $\overline{X}$.

Connected with this, we would like to mention again the work of Noguchi-Winkelmann '02 [26], which deals with the question of algebraic degeneracy in all dimensions and even with Kähler manifolds, especially with log tori or with log manifolds having small Neron-Severi groups, under the additional condition that all irreducible components D_i of D are ample. But as can be seen from the equation (1.1) above, their results never concern the case of log surfaces with log irregularity $\bar{q}_X \leqslant 2$.

We would also like to mention the preprint of Noguchi-Winkelmann-Yamnoi '05 [28] that we just received, which deals with the case of algebraic manifolds whose quasi-Albanese map is proper. More precisely, Noguchi-Winkelmann-Yamanoi '05 [28] deals with arbitrary holomorphic curves into arbitrary dimensional algebraic varieties of general type (this is the essential case), but only with proper Albanese map. Although we deal entirely with surfaces and mainly with Brody curves, our result do not require any condition in the case of log general type and our condition (*), which we use in the general case, is still much weaker than the condition of properness of the quasi-Albanese map as can, for example, be seen from the following simple example: Let X be the complement of a smooth ample divisor D in an abelian surface $\overline{X}$. Then $\overline{K}_X = D$ and so $\bar{\kappa}_X = 2$. Also $\bar{q}_X = q_{\overline{X}} = 2$ by equation (1.1). This means that the compactified quasi-Albanese map is the identity map and so the Albanese map is not proper.

We now give an indication of our methods of proof.

We first discuss the ideas of the proof for Theorem 1.2. We will reduce the proof of this theorem by a result of McQuillen and ElGoul and by log-Bloch's theorem to the claim that under the conditions of Theorem 1.2, $\alpha_X \circ f : \mathbb{C} \to \mathcal{A}_X$ is a translate of a complex one parameter subgroup of $\mathcal{A}_X$.

In the case $q_{\overline{X}} = 2$, the compactified quasi-Albanese map $\bar{\alpha}_X$ is a morphism and so the claim is trivial. If $q_{\overline{X}} < 2$, $\bar{\alpha}_X$ can have points of indeterminacy so that Brody curves are not preserved by $\bar{\alpha}_X$ in general. But from value distribution theory, the order of growth of a holomorphic curve is preserved under rational maps and Brody curves are of order at most 2. Using this, the key analysis in this proof consists of a detailed study of the geometry of the quasi-Albanese map (in particular at its points of indeterminacy) with respect to f to reduce the order of $\bar{\alpha}_X \circ f$ to 1 or less. Then $\bar{\alpha}_X \circ f$ is either constant or a leaf of a linear foliation on $\mathcal{A}_X$. We do this componentwise where in the case $q_{\overline{X}} = 1$, we use the fact from Noguchi-Winkelmann-Yamanoi '02 [27] that one can choose a metric on $\bar{\mathcal{A}}_X$ which lifts to the product metric on the universal cover $\mathbb{C} \times \mathbb{P}_1$ of $\bar{\mathcal{A}}_X$. In the case $q_{\overline{X}} = 0$ we take rational monomials of the components of $\bar{\alpha}_X$ motivated by arranging residues in a way that allows us to control the points of indeterminacy of the resulting map with respect to f.

We now discuss our proof for Theorem 1.3. We first prove the analogue in Proposition 4.3 of the structure theorem of Kawamata for open subsets of finite branched covers of semi-abelian varieties. We follow essentially the original ideas of Kawamata but with several new ingredients. For example,

in the case $\bar{\kappa}_X = 1$, one needs to observe that even though the quasi-Albanese map to the semi-abelian variety is not proper, the restriction to the generic fiber is. In the case $\bar{\kappa}_X = 0$, we need to observe that a complement of a (singular) curve in a semi-abelian variety is of log general type unless the curve is a translate of a algebraic subgroup. We reduce this observation by the addition theorem of Kawamata to the case where the semi-abelian variety is a simple abelian variety. For Theorem 1.3, the main observation is that condition (*) is equivalent to a condition on the Stein factorization of a desingularization of the quasi-Albanese map and that this allows us to use Proposition 4.3 to conclude that the base of the Iitaka fibration is hyperbolic.

We remark that we can give an elementary proof of the result of McQuillan and El Goul in the case of linear foliations on $\mathcal{A}_X$ by using techniques similar to those given in Bertheloot-Duval '01 [2].

The first named author would like to thank C.Peters and F.Catanese for valuable discussions some years ago on configurations of boundary curves of log surfaces in relation to log 1-forms. Finally we would like to thank the referee for many suggestions on how to clarify and improve the presentation of this paper.

2. Some Preliminaries

2.1. Log manifolds and residues of log 1-forms

Let $\overline{X}$ be a complex manifold with a normal crossing divisor D. This means that around any point x of $\overline{X}$, there exist local coordinates $z_1, \ldots, z_n$ centered at x such that D is defined by $z_1 z_2 \cdots z_\ell = 0$ in some neighborhood of x and for some $0 \leqslant \ell \leqslant n$. The pair $(\overline{X}, D)$ will be called a *log-manifold*. Let $X = \overline{X} \setminus D$.

Following Iitaka '82 [17], we define the logarithmic cotangent sheaf

$$\overline{T}^*_X = \bar{\Omega}X = \Omega(\overline{X}, \log D)$$

as the locally free subsheaf of the sheaf of meromorphic 1-forms on $\overline{X}$, whose restriction to X is $T^*_X = \Omega X$ (where we identify from now on vector bundles and their sheaves of sections) and whose localization at $x \in \overline{X}$ is of the form

$$(2.1) \qquad (\overline{T}^*_X)_x = \sum_{i=1}^{\ell} \mathcal{O}_{\overline{X},x} \frac{\mathrm{d}z_i}{z_i} + \sum_{j=\ell+1}^{n} \mathcal{O}_{\overline{X},x} \mathrm{d}z_j,$$

where the local coordinates $z_1, \ldots, z_n$ around x are chosen as before. Its dual, the logarithmic tangent sheaf $\bar{T}_X = T(\overline{X}, -\log D)$, is a locally free subsheaf of the holomorphic tangent bundle $T_{\overline{X}}$ over $\overline{X}$. Its restriction to X is identical to T_X, and its localization at $x \in \overline{X}$ is of the form

$$(2.2) \qquad (\bar{T}_X)_x = \sum_{i=1}^{\ell} \mathcal{O}_{\overline{X},x} z_i \frac{\partial}{\partial z_i} + \sum_{j=\ell+1}^{n} \mathcal{O}_{\overline{X},x} \frac{\partial}{\partial z_j}.$$

Let ω be a log 1-form defined around x, so that by (2.1) we have

$$(2.3) \qquad \omega_x = \sum_{i=1}^{\ell} (h_i)_x \frac{\mathrm{d}z_i}{z_i} + \sum_{j=\ell+1}^{n} (h_j)_x \mathrm{d}z_j.$$

Then we call, for $i = 1, \ldots, \ell$, the complex number $(h_i)_x(x) \in \mathbb{C}$ the *residue* of ω at x on the local irreducible branch of D given by $z_i = 0$. Since we do not assume simple normal crossing, we may have several such local irreducible components for D at x even if D is irreducible. But if $\overline{X}$ is compact, it is easy to see that for any (global) irreducible component D_j of D, the residue is constant on D_j (meaning it is the same for all points $x \in D_j$). In fact, whether $\overline{X}$ is compact or not, we have the exact sequence of sheaves

$$(2.4) \qquad 0 \to \Omega_{\overline{X}} \longrightarrow \bar{\Omega}_X \xrightarrow{\ \mathrm{Res}\ } \mathcal{O}_{\widehat{D}} \to 0,$$

where $\widehat{D}$ is the normalization of D.

2.2. Quasi-Albanese maps

We first recall the definition and some basic facts on semi-abelian varieties (see Iitaka '76 [16]).

A quasi-projective variety G is called a *semi-abelian variety* if it is a complex commutative Lie group which admits an exact sequence of groups

$$(2.5) \qquad 0 \to (\mathbb{C}^*)^{\ell} \longrightarrow G \xrightarrow{\ \pi\ } A \to 0,$$

where A is an abelian variety of dimension m. An important point in our analysis is that this exact sequence is not unique, but depends on a choice of ℓ generators for the kernel of π, which is an algebraic torus of dimension ℓ.

From the standard compactification $(\mathbb{C}^*)^{\ell} \subset (\mathbb{P}_1)^{\ell}$, which is equivariant with respect to the $(\mathbb{C}^*)^{\ell}$ action, we obtain a completion $\overline{G}$ of G as the $(\mathbb{P}_1)^{\ell}$ fiber bundle associated to the $(\mathbb{C}^*)^{\ell}$ principal bundle $G \to A$. This is a

smooth compactification of G with a simple normal crossing boundary divisor S. The projection map

$$\bar{\pi} : \bar{G} \longrightarrow A$$

has the structure of a $(\mathbb{P}_1)^\ell$-bundle. Here, $\bar{\pi}$ and $\bar{G}$ depend on the choice of the ℓ generators that identified our algebraic torus as $(\mathbb{C}^*)^\ell$.

We denote the natural action of G on $\bar{G}$ on the right as addition. It follows that the exponential map from the Lie algebra $\mathbb{C}^n$ to G is a group homomorphism and, hence, it is also the universal covering map of $G = \mathbb{C}^n/\Lambda$, where $\Lambda = \pi_1(G)$ is a discrete subgroup of $\mathbb{C}^n$ and $n = m + \ell$.

Following Iitaka '76 [16], we have the following explicit trivialization of the log tangent and cotangent bundles of $\bar{G}$: Let $z_1, \ldots, z_n$ be the standard coordinates of $\mathbb{C}^n$. Since $\mathrm{d}z_1, \ldots, \mathrm{d}z_n$ are invariant under the group action of translation on $\mathbb{C}^n$, they descend to forms on G. There they extend to logarithmic forms on $\bar{G}$ along S, which are elements of $H^0(\bar{G}, \bar{T}_G^*)$. These logarithmic 1-forms are everywhere linearly independent on $\bar{G}$. Thus, they globally trivialize the vector bundle $\bar{T}_G$. Finally, we note that these log 1-forms are invariant under the group action of G on $\bar{G}$, and, hence, the associated trivialization of $\bar{T}_G$ over $\bar{G}$ is also invariant.

Let now $(\bar{X}, D)$ be again a log surface and $\alpha_{\bar{X}} : \bar{X} \to \mathcal{A}_{\bar{X}}$ the Albanese map of $\bar{X}$ (it can be constant if $q_{\bar{X}} = 0$). Taking into account also the log 1-forms, Iitaka '76 [16] introduced the quasi-Albanese map $\alpha_X : X \to \mathcal{A}_X$, which is a holomorphic map to the semi-abelian variety $\mathcal{A}_X$, which comes equipped with the exact sequence

$$(2.6) \qquad 0 \to (\mathbb{C}^*)^\ell \to \mathcal{A}_X \xrightarrow{\pi} \mathcal{A}_{\bar{X}} \to 0$$

(Iitaka makes a noncanonical choice of ℓ generators for the algebraic torus for this construction). We have the commutative diagram

$$(2.7) \qquad \begin{array}{ccc} X & \xrightarrow{\alpha_X} & \mathcal{A}_X \\ & \alpha_{\bar{X}} \searrow & \downarrow \pi \\ & & \mathcal{A}_{\bar{X}} \end{array}$$

Iitaka also proved that α_X extends to a rational map $\bar{\alpha}_X : \overline{X} \cdots \to \bar{\mathcal{A}}_X$, and the diagram (2.7) extends to

(2.8)

$$
\begin{array}{ccc}
\overline{X} & \xrightarrow{\ \bar{\alpha}_X\ } & \bar{\mathcal{A}}_X \\
& {\scriptstyle \alpha_{\overline{X}}}\searrow & \ \downarrow{\scriptstyle \bar{\pi}} \\
& & \mathcal{A}_{\overline{X}}
\end{array}
$$

In general the $(\mathbb{P}_1)^\ell$-bundle $\bar{\mathcal{A}}_X \xrightarrow{\ \bar{\pi}\ } \mathcal{A}_{\overline{X}}$ is not trivial. But Noguchi-Winkelmann-Yamanoi '02 [27] observed that the transition functions of the $(\mathbb{P}_1)^\ell$-bundle $\bar{\pi} : \bar{\mathcal{A}}_X \to \mathcal{A}_{\overline{X}}$ (as the structure group $(\mathbb{C}^*)^\ell$ can always be reduced to the subgroup defined by $|z_i| = 1,\ i = 1, \ldots, \ell$) can be chosen to be isometries with respect to the product Fubini-Study metric on $(\mathbb{P}_1)^\ell$.

PROPOSITION 2.1. — *There exists a metric h on $\bar{\mathcal{A}}_X$ so that the universal cover map*

$$
\left(\mathbb{C}^m \times (\mathbb{P}_1)^\ell, \text{eucl.} \times \text{product FS} \right) \to (\bar{\mathcal{A}}_X, h)
$$

is a local isometry.

We first consider the case dim $\overline{X} = 2$ and $q_{\overline{X}} = 0$. This means we have a morphism $\alpha_X : X \to \mathbb{C}^* \times \mathbb{C}^*$ extending to a rational map $\bar{\alpha}_X : \overline{X} \cdots \to \mathbb{P}_1 \times \mathbb{P}_1$. Let $\omega \in H^0(\overline{X}, \overline{T}_X^*)$ be a log 1-form with residues $a_j \in \mathbb{Z}$ along the irreducible components D_j of D, $j = 1, \ldots, k$. We define a holomorphic function

(2.9) $$\Phi : X \longrightarrow \mathbb{C}^*, \quad \Phi(x) = \exp\left(\int_{x_0}^x \omega \right),$$

where x_0 is a fixed point in X. It is well defined since by the condition $q_{\overline{X}} = 0$, there are no non trivial periods in $\overline{X}$, and since the periods around the components of the divisor $D \subset \overline{X}$ of the integral have values which are entire multiples of 2π, and, hence, are eliminated by taking the exp-function. We claim that this function extends to a rational function $\overline{\Phi} : \overline{X} \cdots \to \mathbb{P}_1$: Let $P \in D_1$ (resp. $P \in D_1 \cap D_2$) and let z_1, z_2 be local coordinates around P such that $D_1 = \{z_1 = 0\}$ (resp. $D_1 = \{z_1 = 0\}$ and $D_2 = \{z_2 = 0\}$). Then it follows from equation (2.9) that there exists a holomorphic function $h : U(P) \to \mathbb{C}^*$ on a neighborhood $U(P)$ of P such that

(2.10) $\quad \overline{\Phi}(z_1, z_2) = z_1^{a_1} h(z_1, z_2) \quad \left(\text{resp. } \overline{\Phi}(z_1, z_2) = z_1^{a_1} z_2^{a_2} h(z_1, z_2)\right).$

In more detail, if $D_1 = \{z_1 = 0\}$ around P, then we have

$$\omega = h_1(z)\frac{\mathrm{d}z_1}{z_1} + h_2(z)\mathrm{d}z_2 = a_1\frac{\mathrm{d}z_1}{z_1} + \frac{h_1(z) - a_1}{z_1}\mathrm{d}z_1 + h_2(z)\mathrm{d}z_2.$$

By the first Riemann extension theorem, the function $(h_1(z) - a_1)/z_1$ extends to a holomorphic function, so we have $\omega = a_1\,\mathrm{d}z_1/z_1 + \omega_{\mathrm{hol}}$, where ω_{hol} is a holomorphic form. Now by (2.9) we get

$$\begin{aligned}
\Phi(z) &= \exp\Big(\int_{x_0}^{z}\omega\Big) = \exp\Big(\int_{x_0}^{z} a_1\frac{\mathrm{d}z_1}{z_1} + \omega_{\mathrm{hol}}\Big) \\
&= \exp\big(a_1(\log z_1 + 2\pi i\mathbb{Z})\big)\cdot\exp\big(\widetilde{h}(z)\big) \\
&= \big(\exp(\log z_1 + 2\pi i\mathbb{Z})\big)^{a_1}\cdot\exp\big(\widetilde{h}(z)\big) = z_1^{a_1}h(z).
\end{aligned}$$

The other equality follows in the same way.

From this we get by the second Riemann extension theorem, GAGA and by the local description of points of indeterminacy the following.

PROPOSITION 2.2. — *The holomorphic map* $\Phi : X \to \mathbb{C}^*$ *given by* (2.9) *extends to a rational function* $\overline{\Phi} : \overline{X} \to \mathbb{P}_1$. *It is a morphism outside the points of intersection of pairs of different irreducible components* $D_1, \ldots, D_k$ *of* D. *In particular points of indeterminacy never occur at self-intersection points of a component. More precisely, a point of* $D_{j_1}\cap D_{j_2} \subset \overline{X}$ *is in the set* I *of points of indeterminacy of* $\overline{\Phi}$ *if and only if* $a_{j_1}\cdot a_{j_2} < 0$, *and* $\overline{\Phi}(D_j \setminus I) \equiv 0$ (*resp.* ∞) *if and only if the residue* a_j *of* ω *along* D_j *is* > 0 (*resp.* < 0).

Using this we now describe the components of the map

$$\alpha_X = \big((\alpha_X)_1, (\alpha_X)_2\big) : X \longrightarrow (\mathbb{C}^*)^2$$

in more detail. The following basic facts follow from the exact sequence (2.4) and duality in Hodge theory, and they can be found for example in Noguchi-Winkelmann '02 [26]:

PROPOSITION 2.3. — *We can choose the basis* $\omega_1, \omega_2 \in H^0(\overline{X}, \overline{T}_X^*)$ *such that the residue* $a_{ij} \in \mathbb{Z}$ *for all* $i = 1, 2$ *and* $j = 1, \ldots, k$, *where* a_{ij} *is the residue of* ω_i *along the irreducible component* D_j *of* D. *The matrix of residues so obtained has rank 2:*

(2.11)

	D_1	D_2	$\cdots$	D_k
ω_1	a_{11}	a_{12}	$\cdots$	a_{1k}
ω_2	a_{21}	a_{22}	$\cdots$	a_{2k}

and we have, as in (2.9),

$$(2.12) \qquad (\alpha_X)_i(x) = \exp\Big(\int_{x_0}^x \omega_i\Big),$$

where x_0 is a fixed point in X.

Henceforth, in the case $q_{\overline{X}} = 0$, we assume that the components of α_X are given by such a choice of basis.

Next we consider the case where $\dim \overline{X} = 2$ and $q_{\overline{X}} = 1$. Let $x \in D$ be a point. By diagram (2.8) and the notation thereof, as $\mathcal{A}_{\overline{X}}$ is an elliptic curve, there is a small open neighborhood W of $\alpha_{\overline{X}}(x)$ such that $\overline{\pi}^{-1}(W) \simeq \mathbb{P}_1 \times W$. Let $\overline{V} \subset \alpha_{\overline{X}}^{-1}(W)$ be a small open ball centered at x in $\overline{X}$. Then on $V = \overline{V} \cap X$, α_X can be written as

$$\alpha_X = (\alpha_{\overline{X}}, \Phi)$$

where $\Phi : V \to \mathbb{C}^*$ is as in (2.9) (see Noguchi-Winkelmann '02 [26]). The same argument as we gave for Proposition 2.2 gives us:

PROPOSITION 2.4. — *The holomorphic map $\Phi : V \to \mathbb{C}^*$ given by (2.9) extends to a rational function $\overline{\Phi} : \overline{V} \to \mathbb{P}_1$. It is a morphism outside the points of intersection of pairs of different irreducible components $D_1, \ldots, D_k$ of D. In particular points of indeterminacy never occur at self-intersection points of a component. More precisely, a point of $D_{j_1} \cap D_{j_2} \subset \overline{X}$ is in the set I of points of indeterminacy of $\overline{\Phi}$ if and only if $a_{j_1} \cdot a_{j_2} < 0$, and $\overline{\Phi}(D_j \setminus I) \equiv 0$ (resp. ∞) if and only if the residue a_j of ω along D_j is > 0 (resp. < 0).*

2.3. Brody curves, maps of order 2 and limit sets of entire curves

Let $(\overline{X}, D)$ be a log manifold and $f : \mathbb{C} \to X$ be an entire curve. We recall that f is a Brody curve if the derivative of f with respect to some (and so any) hermitian metric on $\overline{X}$ is bounded.

Following Noguchi-Ochiai '90 [25], we have the characteristic function $T_f(r, \omega) = \int_1^r \mathrm{d}t/t \int_{|z|<t} f^*\omega$ of a holomorphic map $f : \mathbb{C} \to Y$ with respect to a real continuous $(1,1)$-form ω on a Kähler manifold Y. If Y is compact and ω_H denotes the $(1,1)$-form associated to a hermitian metric H on Y, then it is easy to see (see [25], (5.2.19)) that

$$\rho_f := \varlimsup_{r \to \infty} \frac{\log T_f(r, \omega_H)}{\log r}$$

is independent of the hermitian metric H. The map f is said to be of *order at most* 2 if $\rho_f \leqslant 2$. Since the derivative of a Brody curve in the projective variety $\overline{X}$ is bounded with respect to hermitian metrics on $\overline{X}$, Brody curves are easily seen to be of order at most 2. By a classical theorem of Weierstrass, we get also that a curve $f : \mathbb{C} \to \mathbb{P}_N$ of order at most 2 which omits the coordinate hyperplanes can be written in the form $(1 : \exp(P_1(z)) : \cdots : \exp(P_n(z)))$ where the P_i's are polynomials of degree at most 2 in the variable $z \in \mathbb{C}$.

We now prove that the property of having order at most 2 is preserved under rational maps (see also [11] for a similar result).

LEMMA 2.5. — *Let $f : \mathbb{C} \to \mathbb{P}_N$ be a curve of order at most 2 and*

$$R : \mathbb{P}_N \cdots \to \mathbb{P}_M$$

be a rational map (not necessarily dominant) such that $f(\mathbb{C})$ is not contained in the set of indeterminacy of R. Then the curve $R \circ f : \mathbb{C} \to \mathbb{P}_M$ is of order at most 2.

Proof. — Let $f = (f_0 : \cdots : f_N)$ be a reduced representation and

$$R = (R_0 : \cdots : R_M)$$

be a (not necessarily reduced) representation by polynomials $R_0, \ldots, R_M$ of degree p. Then

$$\left(f_0^p : \cdots : f_N^p : R_0 \circ f : \cdots : R_M \circ f \right)$$

is a reduced representation of a curve $F : \mathbb{C} \to \mathbb{P}_{N+M+1}$, and without loss of generality $R_0 \circ f \not\equiv 0$. We have by [25], p. 183,

$$
\begin{aligned}
T_F(r, \omega_{\mathrm{FS}}) &= \int_0^{2\pi} \frac{1}{2} \log \left(|f_0^p|^2 + \cdots + |f_N^p|^2 + |R_0 \circ f|^2 \right. \\
&\qquad\qquad\qquad \left. + \cdots + |R_M \circ f|^2 \right) \mathrm{d}\theta + O(1) \\
&\leqslant \int_0^{2\pi} \frac{1}{2} \log \left(|C|(|f_0^p|^2 + \cdots + |f_N^p|^2) \right) \mathrm{d}\theta + O(1) \\
&= \int_0^{2\pi} \frac{p}{2} \log \left(|f_0|^2 + \cdots + |f_N|^2 \right) \mathrm{d}\theta + O(1) \\
&= p \cdot T_f(r, \omega_{\mathrm{FS}}) + O(1).
\end{aligned}
$$

By Noguchi-Ochiai '90 [25] (5.2.29) and (5.2.30) we get

$$T_{R \circ f}(r, \omega_{\text{FS}}) \leqslant \sum_{j=1}^{M} T_{R_j \circ f / R_0 \circ f}(r, \omega_{\text{FS}})$$

$$\leqslant M \cdot T_F(r, \omega_{\text{FS}}) + O(1) \leqslant M \cdot p \cdot T_f(r, \omega_{\text{FS}}) + O(1).$$

From this the lemma follows.   □

Finally, we need the definition and some simple observations on the limit set of an entire curve $f : \mathbb{C} \to \overline{X}$ as given by Nishino-Suzuki '80 [23] (Proposition 1 on p. 463 and Proposition 3 on p. 466). For $r > 0$, put $\Delta_r^c = \{z \in \mathbb{C} : |z| > r\}$. Let $\overline{f(\Delta_r^c)} \subset \overline{X}$ be the closure (with respect to the usual topology) of $f(\Delta_r^c)$ in $\overline{X}$, and $f(\infty) := \bigcap_{r>0} \overline{f(\Delta_r^c)}$. We remark that $f(\infty)$ is exactly the set of all points $p \in \overline{X}$ such that there exists a sequence $(z_v)_{v \in \mathbb{N}}$ with $|z_v| \to \infty$ and $f(z_v) \to p$.

PROPOSITION 2.6. — *Let $f : \mathbb{C} \to X$ be as above for a log surface $(\overline{X}, D)$.*

  a) *f extends to a holomorphic map $\widehat{f} : \mathbb{P}_1 \to \overline{X}$ if and only if $f(\infty)$ consists of exactly one point.*
  b) *If $f(\infty)$ is contained in a proper algebraic subvariety $C \subset \overline{X}$ and is not a point, then $f(\infty)$ is equal to a union of some of the irreducible components of C.*

LEMMA 2.7. — *For $a \in \mathbb{C}^*$, consider the entire curve $f : \mathbb{C} \to \mathbb{C}^* \times \mathbb{C}^*$, given by $z \mapsto (\exp(az), \exp(z^2))$. Then $f(\infty) \subset \mathbb{P}_1 \times \mathbb{P}_1$ contains $\{0, \infty\} \times \mathbb{P}_1$.*

Proof. — Since $f(\infty)$ is closed and since we can change the sequence $(z_\mu)_{\mu \in \mathbb{N}}$ to the sequence $(-z_\mu)_{\mu \in \mathbb{N}}$, it suffices to prove that $f(\infty)$ contains the points (∞, c) with $c \in \mathbb{C}^*$. Let $c \in \mathbb{C}^*$ fixed and $b \in \mathbb{C}$ be such that $c = \exp b$. We consider the four sequences $z_\mu = \pm\sqrt{b \pm 2\pi i \mu}$, $\mu \in \mathbb{N}$, where the four different sequences are obtained by the four different choices of signs. Then we always have $\exp(z_\mu^2) = c$, and for the arguments we have that $\arg(z_\mu)$ converges to one of the numbers $\frac{1}{4}\pi, \frac{3}{4}\pi, \frac{5}{4}\pi, \frac{7}{4}\pi$ according to the choice of signs. Then $|az_\mu| \to \infty$ and $\arg(az_\mu)$ converges to one of the numbers $\arg(a) + \frac{1}{4}\pi, \arg(a) + \frac{3}{4}\pi, \arg(a) + \frac{5}{4}\pi, \arg(a) + \frac{7}{4}\pi$. Since one of these four numbers is contained, modulo $2\pi i \mathbb{Z}$, in the open interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, there exists a choice of signs such that $\exp(az_\mu) \to \infty$.   □

LEMMA 2.8. — *Let $(\overline{X}, D)$ and $(\overline{Y}, E)$ be log surfaces, $\Psi : X \to Y$ a morphism and $\overline{\Psi} : \overline{X} \cdots \to \overline{Y}$ the rational extension of Ψ. Then*

$$\overline{\Psi}\big(f(\infty)\big) \supset \Psi \circ f(\infty),$$

where we understand $\overline{\Psi}(x)$ to be a curve for x a point of inderminancy of $\overline{\Psi}$.

Proof. — Let $Q \in \Psi \circ f(\infty)$. Then there is a sequence $(z_\mu)_{\mu \in \mathbb{N}}$ in $\mathbb{C}$ such that $\lim_{\mu \to \infty} |z_\mu| = \infty$ and $\lim_{\mu \to \infty} \Psi \circ f(z_\mu) = Q$. Let $\widehat{\overline{\Psi}} : \widehat{\overline{X}} \to \overline{Y}$ be a desingularization of $\overline{\Psi}$ via $\bar{r} : \widehat{\overline{X}} \to \overline{X}$ and $\widehat{f} : \mathbb{C} \to \widehat{\overline{X}}$ the lift of f to $\widehat{\overline{X}}$. As $\widehat{\overline{X}}$ is compact, after passing to a subsequence, $\widehat{f}(z_\mu)$ converges to a point $\widehat{P} \in \widehat{\overline{X}}$ and by continuity of $\bar{r}$, $P = \bar{r}(\widehat{P})$ lies in $f(\infty)$. Hence, the continuity of $\widehat{\overline{\Psi}}$ gives $Q = \widehat{\overline{\Psi}}(\widehat{P}) \in \overline{\Psi}(P) \subset \overline{\Psi}(f(\infty))$. $\qquad \square$

2.4. Foliations and the theorem of McQuillan-ElGoul

We first recall some basic notations and facts on foliations as given in Brunella '00 [4]: A foliation on the surface $\overline{X}$ can be defined by a collection of 1-forms $\omega_i \in \Omega^1_{\overline{X}}(U_i)$ with isolated singularities such that U_i, $i \in I$ is an open covering of $\overline{X}$, and that we have $\omega_i = f_{ij} \cdot \omega_j$ on $U_i \cap U_j$ with $f_{ij} \in \mathcal{O}^*_{\overline{X}}(U_i \cap U_j)$. These isolated singularities are called singularities of the foliation. The local integral curves of the forms ω_j glue together, up to reparametization, giving the so-called leaves of the foliation. Any meromorphic 1-form ω on $\overline{X}$ gives a foliation, namely choose an open covering $U_i, i \in I$ of $\overline{X}$ and meromorphic functions f_i on U_i such that $\omega_i := f_i \cdot \omega|_{U_i}$ are holomorphic forms on U_i with isolated singularities, only. Then $\omega_i, i \in I$ gives a foliation. By Brunella '00 [4] for algebraic $\overline{X}$ any foliation can be obtained like this, and, moreover, there is a one to one correspondance between foliations on the one hand and rational 1-forms modulo rational functions on the other hand.

We will use the following result of McQuillan '98 [22] and Brunella '99 [3], extended to the log context by El Goul '03 [13, Theorem 2.4.2] (in order to avoid misunderstandings we would like to point out that in this Theorem 2.4.2 of El Goul '03 [13] the foliation does not need to be tangent to the boundary divisor):

THEOREM 2.9. — *Let $(\overline{X}, D)$ be a log surface of log general type. Let $f : \mathbb{C} \to X$ be an entire curve. Suppose that there exists a foliation $\mathcal{F}$ on $\overline{X}$ such that f is (contained in) a leaf of $\mathcal{F}$. Then f is algebraically degenerate in $\overline{X}$.*

What we will need for the proof of Theorem 1.2 is only a corollary of a very special case of Theorem 2.9:

PROPOSITION 2.10. — *Let* $(\overline{X}, D)$ *be a log surface of log general type.* *Let* $\Psi : X \to \mathcal{A}_X$ *be a dominant morphism which extends to a rational map* $\overline{\Psi} : \overline{X} \cdots \to \bar{\mathcal{A}}_X$. *Let* $f : \mathbb{C} \to X$ *be an entire curve. Assume that the map* $\Psi \circ f : \mathbb{C} \to \mathcal{A}_X$ *is linearly degenerate with respect to the universal cover* $\mathbb{C}^2 \to \mathcal{A}_X$. *Then* f *is algebraically degenerate.*

Proof. — The map $\Psi \circ f : \mathbb{C} \to \mathcal{A}_X$ is linearly degenerate with respect to the universal cover $\mathbb{C}^2 \to \mathcal{A}_X$. Hence, $\Psi \circ f$ is a leaf of the linear foliation given by a 1-form with constant coefficients on $\mathbb{C}^2$, descending to a nowhere vanishing log 1-form ω on $\bar{\mathcal{A}}_X$ (see subsection 2.2). It corresponds to a meromorphic nowhere vanishing 1-form on $\bar{\mathcal{A}}_X$ without poles on $\mathcal{A}_X$. Now $\overline{\Psi} : \overline{X} - \to \bar{\mathcal{A}}_X$ is a dominant rational map, and so the meromorphic form ω pulls back via $\overline{\Psi}^*$ to a nonvanishing meromorphic 1-form $\overline{\Psi}^*\omega$ which is holomorphic over X. By construction we have that $f^*(\overline{\Psi}^*\omega) \equiv 0$. Let $S \subset \overline{X}$ be the divisor given by the meromorphic 1-form $\overline{\Psi}^*\omega$, namely the divisor given by the f_i, $i \in I$ with respect to an open covering U_i, $i \in I$ such that $f_i \cdot (\overline{\Psi}^*\omega)|_{U_i}$ is a holomorphic 1-form with isolated singularities only on U_i for all $i \in I$. If $f(\mathbb{C}) \subset S$ then f is algebraically degenerate. Otherwise f is contained in a leaf of the foliation given by the 1-form $\overline{\Psi}^*\omega$, namely by the $f_i \cdot (\overline{\Psi}^*\omega)|_{U_i}$. By Theorem 2.9 it follows that f is algebraically degenerate, which finishes the proof of Proposition 2.10. $\square$

2.5. The case of non-dominant quasi-Albanese map

The following is a direct consequence of the log-Bloch theorem and the universal property of the quasi-Albanese map.

PROPOSITION 2.11. — *Let* $(\overline{X}, D)$ *be a log surface with log irregularity* $\bar{q}_X = 2$. *Assume that the compactified quasi-Albanese map* $\bar{\alpha}_X : \overline{X} \cdots \to \bar{\mathcal{A}}_X$ *is not dominant. Then* $Y = \bar{\alpha}_X(X) \cap \mathcal{A}_X$ *is a hyperbolic algebraic curve. Hence every entire curve* $f : \mathbb{C} \to X$ *is algebraically degenerate.*

Proof. — The image $\overline{Y} := \bar{\alpha}_X(\overline{X}) \subset \bar{\mathcal{A}}_X$ of $\overline{X}$ under the compactified quasi-Albanese map $\bar{\alpha}_X : \overline{X} \to \bar{\mathcal{A}}_X$ is a proper algebraic subvariety. Since $\bar{q}_X = 2$, so in particular X admits nontrivial log 1-forms, $\overline{Y}$ cannot degenerate to a point, and so we have dim $\overline{Y} = 1$. By the log Bloch's theorem due to Noguchi '81 [24], the Zariski closure of the image of an entire curve in $\mathcal{A}_X$ is a translate of an algebraic subgroup of $\mathcal{A}_X$. But by the universal property of the quasi-Albanese map (see Iitaka '76 [16]), $Y = \overline{Y} \cap \mathcal{A}_X$ cannot be such a translate. Hence, Y is a hyperbolic algebraic curve. So the map $\bar{\alpha}_X \circ f$ is constant, and, hence, f is algebraically degenerate. $\square$

3. Proof of Theorem 1.2 and some applications

3.1. Reduction of the proof

We will reduce the proof of Theorem 1.2 by a result of McQuillen and ElGoul (which is Theorem 2.9 above) and by log-Bloch's theorem to the following.

Claim: *Let $(\overline{X}, D)$ be a log surface with $\bar{q}_X = 2$ and dominant $\bar{\alpha}_X$ and with log Kodaira dimension $\bar{\kappa}_X = 2$. Let $f : \mathbb{C} \to X$ be a Brody curve. Assume that f is not algebraically degenerate. Then $\alpha_X \circ f : \mathbb{C} \to \mathcal{A}_X$ is a translate of a complex one parameter subgroup of $\mathcal{A}_X$.*

For if $\bar{\alpha}_X$ is not dominant, then Theorem 1.2 follows from Proposition 2.11. If $\bar{\alpha}_X$ is dominant and f is not algebraically degenerate, then by the Claim $\alpha_X \circ f : \mathbb{C} \to \mathcal{A}_X$ is a translate of a complex 1-parameter subgroup of $\mathcal{A}_X$. Then Theorem 1.2 follows from Proposition 2.10.

So it suffices to prove the Claim for the various cases of $q_{\overline{X}}$ below.

3.2. The case $q_{\overline{X}} = 2$

The euclidean metric of the universal cover $\mathbb{C}^2$ of the Albanese torus $\mathcal{A}_{\overline{X}}$ descends to a metric h on it. We may choose a hermitean metric g on $\overline{X}$ such that $\alpha_{\overline{X}}^* h \leqslant g$ and we have

$$\mathbb{C} \xrightarrow{\ f\ } (\overline{X}, g) \xrightarrow{\ \alpha_{\overline{X}}\ } (\mathcal{A}_{\overline{X}}, h) \longleftarrow (\mathbb{C}^2, \text{eucl.}).$$

Now since f is a Brody curve, we have $|f'|_g \leqslant C$. By composing with the Albanese map, we therefore get

$$\left| (\alpha_{\overline{X}} \circ f)' \right|_h \leqslant C.$$

After lifting to $\mathbb{C}^2$, the components of $(\alpha_{\overline{X}} \circ f)'$ are bounded holomorphic functions. Hence, by Liouville's theorem, they are constant. So the map $\alpha_{\overline{X}} \circ f : \mathbb{C} \to \mathcal{A}_X$ is a translated complex 1-parameter subgroup of $\mathcal{A}_X$. $\quad\square$

3.3. The case $q_{\overline{X}} = 1$

We have the following diagram (see (2.8)):

(3.1)

$$
\begin{array}{ccc}
\mathbb{C} \xrightarrow{\;f\;} \overline{X} \dashrightarrow{\overline{\alpha}_X} & \overline{\mathcal{A}}_X \\
& \alpha_{\overline{X}} \searrow & \downarrow \pi \\
& & \mathcal{A}_{\overline{X}}
\end{array}
$$

As in the case $q_{\overline{X}} = 2$, we get that $\alpha_{\overline{X}} \circ f$ is linear with respect to the coordinates from the universal covering $\mathbb{C} \to \mathcal{A}_{\overline{X}}$. If $\alpha_{\overline{X}} \circ f$ is constant, then f is algebraically degenerate and we are done. So we assume from now on that this linear function is nonconstant.

Let $I \subset D \subset \overline{X}$ be the (finite) set of points of indeterminacy of $\overline{\alpha}_X$, $U \subset \mathcal{A}_{\overline{X}}$ a neighborhood of the finite set $\alpha_{\overline{X}}(I)$ consisting of small disks around each point of $\alpha_{\overline{X}}(I)$. Let $V = \alpha_{\overline{X}}^{-1}(U)$ and $W = f^{-1}(V)$.

Since $\overline{\alpha}_X$ is a morphism on the compact set $\overline{X} \setminus V$, $(\overline{\alpha}_X \circ f)'$ is bounded on $\mathbb{C} \setminus W$ with respect to any hermitian metric h on $\overline{\mathcal{A}}_X$.

Brody curves are of order $\leqslant 2$. Hence, by Lemma 2.5,

$$
\alpha_X \circ f : \mathbb{C} \longrightarrow \mathcal{A}_X \subset \overline{\mathcal{A}}_X
$$

is of order $\leqslant 2$.

Let $\mathbb{C} \times \mathbb{P}_1 \to \overline{\mathcal{A}}_X$ be the universal cover, and let

$$
\widetilde{(\alpha_X \circ f)} : \mathbb{C} \to \mathbb{C} \times \mathbb{C}^* \subset \mathbb{C} \times \mathbb{P}_1
$$

be a lift of the map $\alpha_X \circ f : \mathbb{C} \to \mathcal{A}_X \subset \overline{\mathcal{A}}_X$ to $\mathbb{C} \times \mathbb{P}_1$. If $\mathrm{pr}_1 : \mathbb{C} \times \mathbb{P}_1 \to \mathbb{C}$ (resp. $\mathrm{pr}_2 : \mathbb{C} \times \mathbb{P}_1 \to \mathbb{P}_1$) denote the projections to the first respectively the second factor, we get that $\mathrm{pr}_1 \circ \widetilde{(\alpha_X \circ f)}$ is a lift of $\alpha_{\overline{X}} \circ f$ through the universal cover $\mathbb{C} \to \mathcal{A}_{\overline{X}}$, which we know already to be a linear function. Define

$$
\Phi := \mathrm{pr}_2 \circ \widetilde{(\alpha_X \circ f)} : \mathbb{C} \longrightarrow \mathbb{C}^* \subset \mathbb{P}_1.
$$

If we prove that Φ is of the form $\Phi(z) = \exp(P(z))$ with a linear polynomial $P(z)$, then $\alpha_{\overline{X}} \circ f : \mathbb{C} \to \mathcal{A}_X$ is a translated complex 1-parameter subgroup of $\mathcal{A}_X$.

By Proposition 2.1, there exists a metric h on $\overline{\mathcal{A}}_X$ such that the universal covering map $(\mathbb{C} \times \mathbb{P}_1, \text{eucl.} \times \mathrm{FS}) \to (\overline{\mathcal{A}}_X, h)$ is a local isometry. By this, we get the existence of a constant C such that

(3.2) $$\left| (\Phi|_{\mathbb{C} \setminus W})' \right|_{FS} \leqslant C.$$

Furthermore, we get the following estimate for the characteristic function.

$$T_\Phi(r, \omega_{\text{FS}}) = \int_1^r \frac{dt}{t} \int_{|z|<t} \Phi^* \omega_{\text{FS}} \leqslant \int_1^r \frac{dt}{t} \int_{|z|<t} \widetilde{(\alpha_X \circ f)}^* \omega_{\text{eucl.} \times \text{FS}}$$

$$= \int_1^r \frac{dt}{t} \int_{|z|<t} (\alpha_X \circ f)^* \omega_h = T_{(\alpha_X \circ f)}(r, \omega_h).$$

Hence, from $\rho_{(\alpha_X \circ f)} \leqslant 2$, it follows $\rho_\Phi \leqslant 2$. So $\Phi(z) = \exp(P(z))$ with $\deg P \leqslant 2$. If $\deg P \leqslant 1$, the proof is finished. So we may assume $\deg P = 2$. We may assume that $P(z) = z^2 + c$ by a linear transformation $z \mapsto az + b$ in $\mathbb{C}$. Then, by a multiplicative transformation $w \mapsto \exp(c) \cdot w$, we may assume

$$P(z) = z^2.$$

Since $U \subset \mathcal{A}_{\overline{X}}$ is a small neighborhood of the finite set $\alpha_{\overline{X}}(I)$ and the map $\alpha_{\overline{X}} \circ f$ is *linear* with respect to the coordinates from the universal cover of $\mathcal{A}_{\overline{X}}$, we get that $\alpha_{\overline{X}} \circ f$ is also a universal covering map. Hence, up to a translation, $\alpha_{\overline{X}} \circ f$ is a group morphism with kernel $\Gamma \subset \mathbb{C}$. Then $W = (\alpha_{\overline{X}} \circ f)^{-1}(U)$ is the union of the translations by the lattice Γ of a finite number of small disks in $\mathbb{C}$.

Hence, there exists a sequence on the diagonal

$$(z_v = x_v + i x_v)_{v \to \infty} \subset \mathbb{C} \setminus W \quad \text{with} \quad x_v \to \infty.$$

We have

$$|\Phi'|_{FS}(z) = \frac{|2z| \exp(x^2 - y^2)}{1 + \exp\left(2(x^2 - y^2)\right)}$$

and, hence, since Φ' is bounded on $\mathbb{C} \setminus W$ by (3.2):

$$C \geqslant \left|\Phi'\right|_{FS}(z_v) = \frac{|2z_v| \exp(0)}{1 + \exp(0)} \longrightarrow \infty,$$

This is a contradiction since $|z_v| \to \infty$. So the assumption $\deg P = 2$ was wrong. $\qquad \square$

3.4. The case $q_{\overline{X}} = 0$

Let $(\overline{X}, D)$ be the log surface and assume $f : \mathbb{C} \to X$ is a Brody curve which is not algebraically degenerate with $f(\infty)$ its limit set. We recall (subsection 2.2) that for a rational function $\overline{\Phi} : \overline{X} \cdots \to \mathbb{P}_1$ which extends a holomorphic function $\Phi : X \to \mathbb{C}^*$, we always have $\Phi \circ f(z) = \exp(P(z))$, with $P(z)$ a polynomial of degree at most 2.

LEMMA 3.1. — *Assume there exists a rational function $\overline{\Phi} : \overline{X} \cdots \to \mathbb{P}_1$ which extends a holomorphic function $\Phi : X \to \mathbb{C}^*$ such that $\Phi \circ f(z) = \exp(P(z))$, with $P(z)$ a polynomial of degree (exactly) 2. Then we have $f(\infty) = \bigcup_{j=1}^{\ell} D_j$, where $D_1, \ldots, D_\ell$ is a subset of the set of the irreducible components $D_1, \ldots, D_k$ of D. Moreover, for any such rational function $\overline{\Phi}$, we have $\overline{\Phi}(D_j \setminus I_{\overline{\Phi}}) \equiv 0$ or $\equiv \infty$ for $j = 1, \ldots, \ell$, where $I_{\overline{\Phi}}$ denotes the set of points of indeterminacy of $\overline{\Phi}$.*

Proof. — Let $\overline{\Phi} : \overline{X} \cdots \to \mathbb{P}_1$ be a rational function which extends a holomorphic function $\Phi : X \to \mathbb{C}^*$ such that $\Phi \circ f(z) = \exp(P(z))$, with $P(z)$ a polynomial of degree (exactly) 2, and let $I_{\overline{\Phi}}$ be the set of points of indeterminacy of $\overline{\Phi}$. We first prove

$$(3.3) \qquad \overline{\Phi}(f(\infty) \setminus I_{\overline{\Phi}}) \subset \{0, \infty\} \subset \mathbb{P}_1.$$

We may assume by a linear coordinate change in $\mathbb{C}$ and a multiplicative transformation in $\mathbb{C}^* \subset \mathbb{P}_1$, that

$$P(z) = z^2.$$

Assume that (3.3) does not hold. Then there exists a point $p \in f(\infty) \setminus I_{\overline{\Phi}}$ such that $\overline{\Phi}(p) \in \mathbb{C}^*$. Let $U(p) \subset \overline{X}$ be a neighborhood such that its topological closure (with respect to the usual topology) $\overline{U}(p)$ does not contain any points of $I_{\overline{\Phi}}$. Note that $\overline{\Phi}$ is a holomorphic function in a neighborhood of $\overline{U}(p) \subset \overline{X}$. Since $p \in f(\infty)$, there exists a sequence $(z_v) = (x_v + iy_v)$, $v \in \mathbb{N}$, such that $|z_v| \to \infty$ and $f(z_v) \to p$. Without loss of generality, we may assume that $f(z_v) \in U(p)$ for all $v \in \mathbb{N}$. Then we have

$$(3.4) \qquad \exp(x_v^2 - y_v^2) = \left| \exp(x_v^2 - y_v^2) \cdot \exp(2ix_v^2 y_v^2) \right| = \left| \exp(z_v^2) \right|$$

$$= \left| \Phi \circ f(z_v) \right| \longrightarrow \left| \overline{\Phi}(p) \right| = C_1 > 0.$$

Since f is a Brody curve, its derivative is uniformly bounded on $\mathbb{C}$, and since $\overline{U}(p) \subset \overline{X}$ is compact, the derivative of $\overline{\Phi}|_{\overline{U}(p)}$ is bounded too. Hence, there exists a constant $C_2 > 0$ such that $|(\Phi \circ f)'|_{FS} \leqslant C_2$ on $f^{-1}(\overline{U}(p))$. So in particular

$$(3.5) \qquad C_2 \geqslant |(\Phi \circ f)'(z_v)|_{FS} = \frac{|2z_v| \cdot |\exp(z_v^2)|}{1 + |\exp(z_v^2)|^2}$$

$$= \frac{2|z_v| \cdot \exp(x_v^2 - y_v^2)}{1 + (\exp(x_v^2 - y_v^2))^2} \longrightarrow \infty$$

by (3.4) and as $|z_v| \to \infty$. This contradiction proves (3.3).

Now assume that there exists a rational function $\overline{\Phi} : \overline{X} \cdots \to \mathbb{P}_1$ which extends a holomorphic function $\Phi : X \to \mathbb{C}^*$ such that $\Phi \circ f(z) = \exp(P(z))$,

with $P(z)$ a polynomial of degree (exactly) 2. By (3.3) we have

$$\left(f(\infty) \setminus I_{\overline{\Phi}}\right) \subset \overline{\Phi}^{-1}\left(\{0,\infty\}\right) \subset D \subset \overline{X}.$$

Now the Lemma follows from Proposition 2.6. $\qquad\square$

For the rest of this subsection, we assume $\bar{q}_X = 2$ and $q_{\overline{X}} = 0$. Let $\overline{\Phi}_i = (\overline{\alpha}_X)_i : \overline{X} \cdots \to \mathbb{P}_1$, $i = 1, 2$, be the two components of the quasi-Albanese map (see subsection 2.1).

LEMMA 3.2. — Let $M = \left(\begin{smallmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{smallmatrix}\right) \in M(2 \times 2, \mathbb{Z})$ be a nonsingular matrix (not necessarily invertible over $\mathbb{Z}$). Then the map

$$(\overline{\Psi}_1, \overline{\Psi}_2) = (\overline{\Phi}_1^{m_{11}} \overline{\Phi}_2^{m_{12}}, \overline{\Phi}_1^{m_{21}} \overline{\Phi}_2^{m_{22}}) : \overline{X} \cdots \longrightarrow (\mathbb{P}_1)^2$$

is a dominant rational map extending the dominant morphism

$$(3.6) \qquad\qquad (\Psi_1, \Psi_2) : X \longrightarrow (\mathbb{C}^*)^2,$$

$$x \longmapsto \left(\exp\left(\int_{x_0}^x m_{11}\omega_1 + m_{12}\omega_2\right), \exp\left(\int_{x_0}^x m_{21}\omega_1 + m_{22}\omega_2\right)\right).$$

(We remark that the map (Ψ_1, Ψ_2) is not the quasi-Albanese map in general, but it factors through the quasi-Albanese map by a finite étale map.)

Proof. — It suffices to prove that the morphism $(\Psi_1, \Psi_2) : X \to (\mathbb{C}^*)^2$ is dominant, the rest is clear from the properties of the quasi-Albanese map. Let ω_1, ω_2 be the two linearly independent log forms corresponding to Φ_1, Φ_2. By construction $(\Psi_1, \Psi_2) : X \to (\mathbb{C}^*)^2$ is of the form

$$(3.7) \quad (\Psi_1, \Psi_2)(x)$$

$$= \left(\exp\left(\int_{x_0}^x m_{11}\omega_1 + m_{12}\omega_2\right), \exp\left(\int_{x_0}^x m_{21}\omega_1 + m_{22}\omega_2\right)\right)$$

where x_0 is a fixed point in X. It suffices to prove that the lift of this map to the universal cover, $X \longrightarrow \mathbb{C}^2$,

$$x \longmapsto \left(m_{11}\left(\int_{x_0}^x \omega_1\right) + m_{12}\left(\int_{x_0}^x \omega_2\right), m_{21}\left(\int_{x_0}^x \omega_1\right) + m_{22}\left(\int_{x_0}^x \omega_2\right)\right)$$

has rank 2 in some points. But this is true since M is non singular and since the lift to the universal cover of the quasi-Albanese map, this means

$$X \longrightarrow \mathbb{C}^2, \quad x \longmapsto \left(\int_{x_0}^x \omega_1, \int_{x_0}^x \omega_2\right)$$

has this property. $\qquad\square$

By Lemma 2.5, we have $\Psi_i \circ f = \exp(Q_i(z))$ with $\deg Q_i \leqslant 2$. Then if one of the Q_i has degree 0, it follows immediately that f is algebraically degenerate (since $\Psi_i \circ f$ is constant then). Our aim is now to choose the nonsingular matrix M such that $\deg Q_i \leqslant 1$ for $i = 1, 2$. Then if one of the Q_i has degree 0, one is done. If both polynomials Q_i have degree 1, then $(\Psi_1, \Psi_2) \circ f : \mathbb{C} \to \mathbb{C}^* \times \mathbb{C}^*$ is a translate of a complex 1-parameter subgroup of $\mathbb{C}^* \times \mathbb{C}^*$. But we have the commutative diagram

$$
\begin{array}{ccc}
\mathbb{C}^2 & \xrightarrow{\;M\;} & \mathbb{C}^2 \\
\downarrow & & \downarrow \\
\mathcal{A}_X & \longrightarrow & (\mathbb{C}^*)^2
\end{array}
$$

where M is a linear isomorphism given by the matrix M and the vertical arrows are the universal covering maps. So $\alpha_{\overline{X}} \circ f : \mathbb{C} \to \mathcal{A}_X$ is also a translated complex 1-parameter subgroup of $\mathcal{A}_X$.

By Lemma 2.5 we have $\Phi_i \circ f = \exp(P_i(z))$ with $\deg P_i \leqslant 2$ for the two components of the quasi-Albanese map. If $\deg P_i \leqslant 1$ for $i = 1, 2$, we put $M = I_2$, the identity matrix, and the proof is finished. Otherwise we may assume without loss of generality that $\deg P_2 = 2$. Then by Lemma 3.1, we have $f(\infty) = \bigcup_{j=1}^{\ell} D_j$. Consider the submatrix of the residues of ω_1, ω_2 with respect to the first ℓ divisors forming $f(\infty)$

(3.8)

$$
\begin{array}{ccccc}
 & D_1 & D_2 & \ldots & D_\ell \\
\omega_1 & a_{11} & a_{12} & \ldots & a_{1\ell} \\
\omega_2 & a_{21} & a_{22} & \ldots & a_{2\ell}
\end{array}
$$

If this matrix has rank 2, there exists a nonsingular matrix M with coefficients in $\mathbb{Z}$ such that after passing from the log forms ω_1, ω_2 corresponding to the map (Φ_1, Φ_2), to the forms $m_{11}\omega_1 + m_{12}\omega_2, m_{21}\omega_1 + m_{22}\omega_2$ corresponding to (Ψ_1, Ψ_2), the matrix of residues (3.8) has at least one residue $= 0$ in every line.

If the matrix of residues (3.8) of (Φ_1, Φ_2) has rank 1, we can choose the nonsingular matrix M with $m_{11} \neq 0$, $m_{21} = 0$ and $m_{22} = 1$ such that after passing from the log forms ω_1, ω_2 corresponding to the map (Φ_1, Φ_2), to the forms $m_{11}\omega_1 + m_{12}\omega_2, m_{21}\omega_1 + m_{22}\omega_2$ corresponding to (Ψ_1, Ψ_2) the matrix of residues (3.8) has all residues $= 0$ in the first line. Here, there is the difficult case where the resulting matrix does not have any zero in the second line.

If the matrix of residues (3.8) of (Φ_1, Φ_2) has rank 0, we just take $M = I_2$ the identity matrix.

In all cases, except the difficult one, there exist some residues $= 0$ in every line of the residue matrix for (Ψ_1, Ψ_2). So by Lemma 3.1 and equation (2.10), we have $\deg Q_i \leqslant 1$ for $i = 1, 2$, and we are done.

So we are left with the only (difficult) case that the matrix of residues of (Ψ_1, Ψ_2) with respect to the log forms $\widetilde{\omega}_1 := m_{11}\omega_1 + m_{12}\omega_2$ and ω_2 looks as follows (with $a_{2j} \neq 0$ for all $j = 1, \ldots, \ell$)

(3.9)

$$
\begin{array}{ccccc}
 & D_1 & D_2 & \ldots & D_\ell \\
\widetilde{\omega}_1 & 0 & 0 & \ldots & 0 \\
\omega_2 & a_{21} & a_{22} & \ldots & a_{2\ell}
\end{array}
$$

and that $\Psi_i \circ f = \exp(Q_i(z))$ with $\deg Q_1 \leqslant 1$ (which follows again by Lemma 3.1 and equation (2.10)) and $\deg Q_2 = 2$. In this case, we have to use the explicit geometry of the entire curve f with respect to the map (Ψ_1, Ψ_2) in a similar way as we did in the proof for the case $q_{\overline{X}} = 1$ above.

LEMMA 3.3. — a) The rational function $\overline{\Psi}_1 : \overline{X} \cdots \to \mathbb{P}_1$ is a morphism in a neighborhood of $f(\infty) = \bigcup_{j=1}^{\ell} D_j \subset \overline{X}$.

b) There exists at most one irreducible component of $f(\infty)$, say D_1, such that $\overline{\Psi}_1(D_1) = \mathbb{P}_1$. If it does not exist, then for all points $x \in D_i$ we have $\overline{\Psi}_1(x) \in \mathbb{C}^*$. If it exists, there exists exactly one point $x_0 \in f(\infty)$ and exactly one point $x_\infty \in f(\infty)$, both lying in D_1, such that $\overline{\Psi}_1(x_0) = 0$ and $\overline{\Psi}_1(x_\infty) = \infty$ (in particular, for all points $x \in f(\infty) \setminus \{x_0, x_\infty\}$, we have $\overline{\Psi}_1(x) \in \mathbb{C}^*$). Furthermore, x_0 respectively x_∞ are intersection points of D_1 with components D_j with $j \geqslant \ell + 1$ (meaning components not belonging to $f(\infty)$) with residues $a_{1j} > 0$ respectively $a_{1j} < 0$.

Proof. — a) This follows immediately from Proposition 2.2 since by (3.9), all the residues of $\overline{\Psi}_1$ on all components of $f(\infty)$ are $= 0$.

b) Since $\overline{\Psi}_1$ is a morphism around $f(\infty)$, the image of every irreducible component of $f(\infty)$ is a (closed) subvariety in $\mathbb{P}_1$, and hence a single point or all of $\mathbb{P}_1$. Since the residue of $\overline{\Psi}_1$ on each component of $f(\infty)$ is 0, it follows by (2.10) that the image of each component, if it is a single point, has to lie in $\mathbb{C}^*$. By the same token, we also get that any point $x \in f(\infty)$ which is mapped to 0 respectively to ∞ by $\overline{\Psi}_1$ has to be an intersection point with a component of D having residue > 0 respectively < 0, and which cannot be one of the components of $f(\infty)$. Since D is a normal crossing divisor, no three irreducible components meet in one point, and so no point mapping to 0 or to ∞ can lie on the intersection of different irreducible components of $f(\infty)$. We will now prove that there can exist at most one point $x = x_0 \in f(\infty)$ with $\overline{\Psi}_1(x) = 0$ and at most one point

$x = x_\infty \in f(\infty)$ with $\bar{\Psi}_1(x) = \infty$. From this it follows immediately that there can be at most one component of $f(\infty)$ mapping onto $\mathbb{P}_1$, which then has to contain both x_0 and x_∞.

Let $\{(x_0)_1, \dots, (x_0)_m\} = \{x \in f(\infty) : \bar{\Psi}_1(x) = 0\}$. Take a neighborhood $U(f(\infty))$ such that $\bar{\Psi}_1$ is still a morphism in a neighborhood of the closure $\bar{U}(f(\infty))$ of $U(f(\infty))$ (this is possible since there is only a finite number of points of indeterminacy of $\bar{\Psi}_1$). Take small neighborhoods $U_p = U_p((x_0)_p)$, $p = 1, \dots, m$, such that their closures are still contained in $U(f(\infty))$.

Assume $m \geqslant 2$. For $p = 1, 2$, take sequences $(z_v^{(p)})$, $v \in \mathbb{N}$, such that $|z_v^{(p)}| \to \infty$ and $f(z_v^{(p)}) \to (x_0)_p$. Without loss of generality, we may assume that $f(z_v^{(p)}) \in U_p$ for all $v \in \mathbb{N}$. Let $[z_v^{(1)}, z_v^{(2)}] \subset \mathbb{C}$ be the (linear) segment between $z_v^{(1)}$ and $z_v^{(2)}$ in $\mathbb{C}$. Then there exists a point

$$w_v \in \left([z_v^{(1)}, z_v^{(2)}] \cap f^{-1}\big(U(f(\infty))\big)\right) \setminus \bigcup_{p=1}^{m} f^{-1}(U_p).$$

The image of the segment $[z_v^{(1)}, z_v^{(2)}]$ under f joins the two points $f(z_v^{(1)}) \in U_1$ and $f(z_v^{(2)}) \in U_2$. Since the two neighborhoods are disjoint, it has to cross the boundaries of both of them. Let w_v be the inverse image of such a crossing point with one of these boundaries.

By (3.9) we have $\bar{\Psi}_1 \circ f(z) = \exp(az + b)$ with $a, b \in \mathbb{C}$. Since $f(z_v^{(p)}) \to (x_0)_p$ we get $\bar{\Psi}_1 \circ f(z_v^{(p)}) \to \bar{\Psi}_1((x_0)_p) = 0$, $p = 1, 2$. So $\exp(az_v^{(p)} + b) \to 0$, meaning $\mathrm{Re}(az_v^{(p)} + b) \to -\infty$, $p = 1, 2$. Now $w_v \in [z_v^{(1)}, z_v^{(2)}]$, so there exist λ_v with $0 \leqslant \lambda_v \leqslant 1$ such that $w_v = \lambda_v z_v^{(1)} + (1 - \lambda_v) z_v^{(2)}$. Then we have

$$(3.10) \quad \mathrm{Re}(aw_v + b) = \mathrm{Re}\big(\lambda_v(az_v^{(1)} + b) + (1 - \lambda_v)(az_v^{(2)} + b)\big)$$

$$= \lambda_v \mathrm{Re}\big(az_v^{(1)} + b\big) + (1 - \lambda_v)\mathrm{Re}\big(az_v^{(2)} + b\big) \longrightarrow -\infty$$

since $\lambda_v, 1 - \lambda_v \geqslant 0$. In particular $|w_v| \to \infty$.

Consider the sequence $f(w_v)$, $v \in \mathbb{N}$. After passing to a subsequence, we get $f(w_v) \to P \in \bar{X}$. By construction, we have $P \in f(\infty)$, and by (3.10) we get

$$(3.11) \qquad \bar{\Psi}_1(P) = \lim_{v \to \infty} \bar{\Psi}_1\big(f(w_v)\big) = \lim_{v \to \infty} \exp(aw_v + b) = 0.$$

But $w_v \in f^{-1}(U(f(\infty))) \setminus \bigcup_{p=1}^{m} f^{-1}(U_p((x_0)_p))$, so $P \neq (x_0)_v$, $v = 1, \dots, m$. This is a contradiction, since

$$\{(x_0)_1, \dots, (x_0)_m\} = \{x \in f(\infty) : \bar{\Psi}_1(x) = 0\} \ni P.$$

So our assumption $m \geqslant 2$ was wrong, and there exists at most one point $x_0 \in f(\infty)$ such that $\bar{\Psi}_1(x_0) = 0$.

The proof for x_∞ is exactly the same, just change 0 and ∞ in the proof above, and in (3.10), change $-\infty$ to $+\infty$. □

Now we can finish up the proof of the claim: If there does not exist an irreducible component D_1 of $f(\infty)$ such that $\overline{\Psi}_1(D_1) = \mathbb{P}_1$ (case 1), we leave (Ψ_1, Ψ_2) as it is. If it does exist (case 2), then we may assume $x_0 \in D_1 \cap D_{l+1}$. The matrix of residues (3.9) extends to

(3.12)

	D_1	D_2	$\ldots$	D_ℓ	$D_{\ell+1}$
$\widetilde{\omega}_1$	0	0	$\ldots$	0	$a_{1,\ell+1}$
ω_2	a_{21}	a_{22}	$\ldots$	$a_{2\ell}$	$a_{2,\ell+1}$

with $a_{1,\ell+1} > 0$. We put

$$\widetilde{\omega}_2 := a_{1,\ell+1}\omega_2 - a_{2,\ell+1}\omega_1$$

Then the new matrix of residues becomes

(3.13)

	D_1	D_2	$\ldots$	D_ℓ	$D_{\ell+1}$
$\widetilde{\omega}_1$	0	0	$\ldots$	0	$a_{1,\ell+1}$
$\widetilde{\omega}_2$	a_{21}	a_{22}	$\ldots$	$a_{2\ell}$	0

By Proposition 2.2, we get that $x_0 \in D_1 \cap D_{\ell+1}$ cannot be a point of indeterminacy of $\overline{\Psi}_2$, since the corresponding product of residues is zero. In both cases we get, by a linear coordinate change in $\mathbb{C}$ and a multiplicative transformation in $\mathbb{C}^* \subset \mathbb{P}_1$, that

$$(\Psi_1, \Psi_2) \circ f(z) = \big(\exp(az), \exp(z^2)\big), \quad a \in \mathbb{C}^*.$$

By Lemma 2.7 and Lemma 2.8, we get

(3.14) $\{0\} \times \mathbb{P}_1 \subset (\Psi_1, \Psi_2) \circ f(\infty) \subset (\overline{\Psi}_1, \overline{\Psi}_2)(f(\infty)).$

We will reach a contradiction to (3.14) in both cases. In case 1, we have $\overline{\Psi}_1(f(\infty)) \subset \mathbb{C}^*$, so

$$(\overline{\Psi}_1, \overline{\Psi}_2)\big(f(\infty)\big) \cap \big(\{0\} \times \mathbb{P}_1\big) = \emptyset,$$

contradicting (3.14). In case 2, we get that $(\overline{\Psi}_1|_{f(\infty)})^{-1}(0) = \{x_0\}$ and x_0 is not a point of indeterminacy of $\overline{\Psi}_2$. So we have

$$\big(\overline{\Psi}_1, \overline{\Psi}_2\big)(f(\infty)) \cap \big(\{0\} \times \mathbb{P}_1\big) = \{0\} \times \big\{\overline{\Psi}_2(x_0)\big\}$$

which is strictly contained in $\{0\} \times \mathbb{P}_1$, contradicting (3.14).

This concludes the proof of the claim, and, by the reduction in subsection 3.1, the proof of Theorem 1.2. □

3.5. Some applications

Theorem 1.2 gives us the following corollaries on hyperbolicity.

COROLLARY 3.4. — *Let* $(\overline{X}, D)$ *a log surface with log irregularity* $\bar{q}_X = 2$ *and log Kodaira dimension* $\bar{\kappa}_X > 0$. *Suppose that* X *does not contain non-hyperbolic algebraic curves and that* D *is hyperbolically stratified* (*this means that every irreducible component minus all the others is a hyperbolic curve*). *Then* X *is complete hyperbolic and hyperbolically imbedded in* $\overline{X}$.

Proof. — One has $\bar{\kappa}_X \neq 1$, otherwise the Iitaka fibration of X would contain non-hyperbolic fibers and so X would have non-hyperbolic algebraic curves. Hence this follows from Theorem 1.2 and from the main result of Green '77 [14]. □

Another consequence of Theorem 1.2 is the "best possible" result for algebraic degeneracy in the three component case of complements of plane curves (see Dethloff-Schumacher-Wong '95 [11] and [12], Bertheloot-Duval '01 [2]):

COROLLARY 3.5. — *Let* $D \subset \mathbb{P}_2$ *be a normal crossing curve of degree at least 4 consisting of three components. Then every Brody curve* $f : \mathbb{C} \to \mathbb{P}_2 \setminus D$ *is algebraically degenerate.*

Proof. — $\overline{K}_{(\mathbb{P}_2 \setminus D)} = \mathcal{O}((\deg D) - 3)$ is very ample and, hence, big. Let $D_i = \{f_i = 0\}$, where f_i is a homogeneous equation for D_i, for $i = 1, 2, 3$. Then

$$\omega_j = \mathrm{d} \log \Big(\frac{f_j^{\deg f_3}}{f_3^{\deg f_j}} \Big), \quad j = 1, 2,$$

are linearly independent log forms, and it is easy to see that there are no others. Hence, $\bar{q}_{(\mathbb{P}_2 \setminus D)} = 2$. □

4. Proof of Theorem 1.3 and applications

4.1. Kawamata's Theorem

Recall that for a fibered variety, i.e., a dominant algebraic morphism between irreducible, reduced quasiprojective varieties with irreducible general fibers, we have the following addition theorem of Kawamata '77 ([18]) in dimension two:

THEOREM 4.1. — *Let $f : V \to B$ be a fibered variety for a nonsingular algebraic surface V and a nonsingular algebraic curve B. Let F be a general fiber of f. Then*

$$\bar{\kappa}_V \geqslant \bar{\kappa}_F + \bar{\kappa}_B.$$

Recall also the following result of Kawamata '81 [19, Theorems 26, 27]:

THEOREM 4.2. — *Let X be a normal algebraic variety, $\mathcal{A}$ a semi-abelian variety and let $f : X \to \mathcal{A}$ be a finite morphism. Then $\bar{\kappa}_X \geqslant 0$ and there exist a connected complex algebraic subgroup $\mathcal{B} \subset \mathcal{A}$, étale covers $\widetilde{X}$ of X and $\widetilde{\mathcal{B}}$ of $\mathcal{B}$, and a normal algebraic variety $\widetilde{Y}$ such that*

   (i) *$\widetilde{Y}$ is finite over $\mathcal{A}/\mathcal{B}$;*
   (ii) *$\widetilde{X}$ is a fiber bundle over $\widetilde{Y}$ with fiber $\widetilde{\mathcal{B}}$ and translations by $\widetilde{\mathcal{B}}$ as structure group;*
   (iii) *$\bar{\kappa}_{\widetilde{Y}} = \dim \widetilde{Y} = \bar{\kappa}_X$.*

If further $\bar{\kappa}_X = 0$ and f is surjective, then f is an étale morphism.

For the proof of Theorem 1.3 we will need the following extension of this theorem in the case of surfaces.

PROPOSITION 4.3. — *Let X be a normal algebraic surface, $\mathcal{A}$ a semi-abelian surface and let $f : X \to \mathcal{A}$ be a finite morphism. Let $X_0 \subset X$ be an open algebraic subvariety.*

1) In the case $\bar{\kappa}_{X_0} = 1$, let $\Phi : X^* \to Y^*$ be the logarithmic Iitaka fibration of X_0, $\Psi : X^* \to X_0$ the birational morphism relating X_0 to X^*, and for $y \in Y^*$, let $X_y^* = \Phi^{-1}(y)$, $X_y = \Psi(X_y^*) \subset X_0 \subset X$ and $B_y = f(X_y)$. Then for generic $y \in Y^*$, $X_y \subset X$ is a closed subvariety and B_y is a translate of a complex one parameter algebraic subgroup $\mathcal{B}$ of $\mathcal{A}$. Moreover, there are étale covers $\widetilde{X}$ of X and $\widetilde{\mathcal{B}}$ of $\mathcal{B}$, and a smooth algebraic curve $\widetilde{Y}$ such that

   (i) *$\widetilde{Y}$ is finite over $\mathcal{A}/\mathcal{B}$.*
   (ii) *$\widetilde{X}$ is a fiber bundle over $\widetilde{Y}$ with fiber $\widetilde{\mathcal{B}}$ and translations by $\widetilde{\mathcal{B}}$ as structure group. In particular, X and $\widetilde{X}$ are smooth.*
   (iii) *$\dim \widetilde{Y} = \bar{\kappa}_{X_0} = 1$.*

In particular, for generic $y \in Y^*$, $X_y \subset X$ is equal to the image of a suitable fiber of the fiber bundle $\widetilde{X}$ over $\widetilde{Y}$, and the image of a generic fiber of this fiber bundle is of the form X_y for $y \in Y^*$.

2) If $\bar{\kappa}_{X_0} = 0$, then f is an étale morphism and $X \setminus X_0$ is finite.

Proof. — Since the proof uses among others the same ideas as that given by Kawamata in [19] (see Theorems 13, 23, 26 and 27 of Kawamata's paper [19]), we just indicate the differences and what has to be added:

1) In the case $\bar{\kappa}_{X_0} = 1$, the key point is to observe that despite the composed map $X^* \overset{\Psi}{\to} X_0 \hookrightarrow X$ being not proper in general, for generic $y \in Y^*$, $X_y \subset X$ is a closed subvariety. In order to see this, let $\overline{X}_y \subset \overline{X}$ be the (Zariski) closure of X_y in $\overline{X}$ (it is the curve $\overline{\Psi}(\overline{X}_y^*)$). Then $\overline{X}_y \cap X$ is the closure of X_y in X, $\overline{B}_y = f(\overline{X}_y \cap X)$ is the closure of $B_y \subset \mathcal{A}$ and we have for the logarithmic Kodaira dimensions (for generic $y \in Y^*$):

$$0 = \bar{\kappa}_{X_y^*} = \bar{\kappa}_{X_y} \geqslant \bar{\kappa}_{\overline{X}_y \cap X} \geqslant \bar{\kappa}_{\overline{B}_y} \geqslant 0,$$

where the last inequality follows since we can restrict the constant 1-forms of $\mathcal{A}$ to $\overline{B}_y$, and all the others follow by lifting back log multi-canonical forms through logarithmic (!) morphisms. So we get in particular for the inclusion of algebraic curves $X_y \subset \overline{X}_y \cap X$ that $0 = \bar{\kappa}_{X_y} = \bar{\kappa}_{\overline{X}_y \cap X}$ (remark that $X_y \subset X$ is algebraic since $X_y \subset X_0$ is algebraic and $X_0 \subset X$ is an open algebraic subvariety). But an algebraic curve having log Kodaira dimension zero is either isomorphic to an elliptic curve or to $\mathbb{C}^* \subset \mathbb{P}_1$, and if we delete any other point of any of these algebraic curves, they become hyperbolic, so their log Kodaira dimension becomes one. This means $X_y = \overline{X}_y \cap X$ is a closed subvariety of X. The rest of the proof is exactly analogous to the proof of Kawamata's Theorem 27. We observe here that the normal curve $\widetilde{Y}$, having singularities only in codimension 2, is automatically smooth.

2) As f is a finite morphism between algebraic surfaces, it is surjective. So the first part of Theorem 4.2 gives $0 = \bar{\kappa}_{X_0} \geqslant \bar{\kappa}_X \geqslant 0$. Hence the first part of our claim follows from the last part of Theorem 4.2. Since X is finite and étale over $\mathcal{A}$, X is also a semi-abelien variety since it is a quotient of affine space by a discrete subgroup and the kernel in X of the map to the abelian variety quotient of $\mathcal{A}$ is a finite cover of that of $\mathcal{A}$. Let $\overline{X}$ be a compactification of X as given in subsection 2.2. If $X \setminus X_0$ is not finite, then there exist a curve C in $\overline{X}$ intersecting X but not X_0. We will reach a contradiction.

Assume that X is not a simple abelian surface. We follow the proof of Theorem 26 in [19] to obain a contradiction. As X is semi-abelian, it is a fiber bundle over a flat curve B, i.e. B has trivial log-canonical bundle so that B is either $\mathbb{C}^*$ or an elliptic curve. Moreover, the fiber bundle has flat curves as fibers. Now the log-Kodaira dimension becomes positive if we remove a finite nonempty set from a flat curve, whose log-Kodaira dimension is 0. If C intersects the generic fiber of this fiber bundle, then we have a contradiction with the addition theorem above. Hence, C is vertical

and so $X \setminus C$ is a fiber space over a curve of positive Kodaira dimension. This again contradicts the addition theorem.

Hence we may assume that X is a simple abelian surface. Now it is well known that C is then an ample divisor in X so that its L-dimension is 2. This latter fact can be seen directly as follows.

If $C^2 > 0$, then Riemann-Roch gives that

$$h^0(mC) - h^1(mC) + h^2(mC) = O(m^2)$$

while Serre-duality gives $h^2(mC) = h^0(-mC) = 0$. Hence $h^0(mC) = O(m^2)$ and the L-dimension of C is 2. If $C^2 \leqslant 0$, then the algebraic subgroup $G = \{t \in X \mid C + t = C\}$ must contain differences of points in C and so is at least 1-dimensional. As X is simple, $G = X$ which is a contradiction.

To compute the log-Kodaira dimension of $X \setminus C$, we blow up X successively until the total reduced transform of C becomes normal crossing, where we let E_i be the exceptional divisor at the i-th step. Then we know that at each step, the resulting canonical divisor is $\sum_i E_i$ and $C = \widetilde{C} + \sum_i m_i E_i$ as divisors ($m_i > 0$), where we have suppressed the pullback symbols and $\widetilde{C}$ is the strict transform of $C \subset \overline{X}$. Hence, for the final blown up variety $\widehat{X}$ of X, the log-canonical divisor of $(\widehat{X}, \widetilde{C})$ is $K = \widetilde{C} + \sum_i E_i$. But this means that a multiple of K is C plus an effective divisor and therefore the log-Kodaira dimension of $\widehat{X} \setminus \widetilde{C}$ is 2. This is a contradiction since then $X_0 \subset \widehat{X} \setminus \widetilde{C}$ must have the same log-Kodaira dimension. $\qquad\square$

4.2. Stein factorization and condition (*)

Let $(\overline{X}, D)$ be a log surface with log irregularity $\bar{q}_X = 2$. Let $\bar{\alpha}_X : \overline{X} \cdots \to \bar{\mathcal{A}}_X$ be the compactified Albanese map, I its finite set of points of indeterminacy and $\bar{\alpha}_0 = \bar{\alpha}_X|_{\overline{X} \setminus I}$. We assume in this subsection that $\bar{\alpha}_X$ is dominant. Recall the following condition:

(*) For all $z \in \mathcal{A}_X$ and $\overline{E}$ a connected component of the Zariski closure of $\bar{\alpha}_0^{-1}(z)$ with $\overline{E} \cap X \neq \emptyset$, any connected component of D intersecting $\overline{E}$ is contained in $\overline{E}$ (i.e. $\overline{E}$ is a connected component of $\overline{E} \cup D$).

Condition (*) is a natural condition in the sense that all the data can be read directly from two linearly independent log 1-forms, ω_1, ω_2 without referring to α_X or $\bar{\alpha}_X$. Indeed, α_X is not dominant if and only if $\omega_1 \wedge \omega_2 \equiv 0$ and a curve C in $\overline{X}$ such that neither ω_1 nor ω_2 is identically ∞ along C

is exceptional with respect to $\bar{\alpha}_X$ if and only if $i^*\omega_1 \equiv 0 \equiv i^*\omega_2$ where $i : C \to \overline{X}$ is the inclusion map.

PROPOSITION 4.4. — *Condition* (*) *is equivalent to the following condition: If* $\bar{\widehat{\alpha}}_X : \widehat{\overline{X}} \to \bar{\mathcal{A}}_X$ *is a desingularization of the rational map* $\bar{\alpha}_X$ *which is biholomorphic on* X, *then for the Stein factorization* $\widehat{\overline{X}} \xrightarrow{\bar{\beta}} \overline{Y} \xrightarrow{\bar{\gamma}} \bar{\mathcal{A}}_X$ *of the morphism* $\bar{\widehat{\alpha}}_X$ *we have that* $\bar{\beta}(X) \subset \overline{Y}$ *is an open algebraic subvariety.*

Proof. — We first prove the following lemma.

LEMMA 4.5. — *Let* $\bar{\widehat{\alpha}}_X : \widehat{\overline{X}} \to \bar{\mathcal{A}}_X$ *be a desingularization of the rational map* $\bar{\alpha}_X$ *via* $r : \widehat{\overline{X}} \to \overline{X}$ *which is biholomorphic on* X *and* $\widehat{\overline{X}} \xrightarrow{\bar{\beta}} \overline{Y} \xrightarrow{\bar{\gamma}} \bar{\mathcal{A}}_X$ *the Stein factorization of the morphism* $\bar{\widehat{\alpha}}_X$. *We identify* X *and* $r^{-1}(X)$ *and let* $\widehat{D}$ *be the reduced total transform of* D. *Then condition* (*) *is equivalent to the following condition.*

(**) *For all* $z \in \mathcal{A}_X$ *and* $\widehat{\overline{E}}$ *a connected component of* $\bar{\widehat{\alpha}}_0^{-1}(z)$ *with* $\widehat{\overline{E}} \cap X \neq \emptyset$, *any connected component of* $\widehat{D}$ *intersecting* $\widehat{\overline{E}}$ *is contained in* $\widehat{\overline{E}}$ *(i.e.* $\widehat{\overline{E}}$ *is a connected component of* $\widehat{\overline{E}} \cup \widehat{D}$).

Proof of the lemma. — (*) implies (**). Assume that (**) does not hold. Then there exist $z \in \mathcal{A}_X$ and a connected component $\widehat{\overline{E}}$ of $\bar{\widehat{\alpha}}^{-1}(z)$ with nonempty intersection with X. Also there is a connected component of $\widehat{D}$ having nonempty intersection with but not contained in $\widehat{\overline{E}}$. Let $\sum_k \widehat{D}_k$ be its decomposition into irreducible components. We may assume that $\widehat{D}_1 \not\subset \widehat{\overline{E}}$ so that $\bar{\widehat{\alpha}}_X|_{\widehat{D}_1} \not\equiv z$ and that there is a point $\widehat{x} \in \widehat{D}_1 \cap \widehat{\overline{E}}$.

Now $r(\widehat{\overline{E}})$ is a connected subset of a connected component $\overline{E}$ of the Zariski closure of $\bar{\alpha}_0^{-1}(z)$. We distinguish two cases.

Case 1 : $r(\widehat{D}_1) \subset D$ *is a curve.* — Then $r(\widehat{D}_1)$ is an irreducible component of D intersecting $\overline{E}$ at $r(\widehat{x})$, but $r(\widehat{D}_1) \not\subset \overline{E}$ because $\bar{\alpha}_X|_{r(\widehat{D}_1)\setminus I} \not\equiv z$. This contradicts (*).

Case 2 : $r(\widehat{D}_1) \subset D$ *is a point.* — In this case $r(\widehat{x}) \in I$ is a point where $\bar{\alpha}_X$ has a point of indeterminacy. By Propositions 2.2 and 2.4, $r(\widehat{x}) \in \overline{E}$ is an intersection point of two irreducible components D_i and D_j of D such that the form ω defining Φ (only around $r(\widehat{x})$ in the case of $q_{\overline{X}} = 1$) has strictly opposite signs along D_i and D_j, and $\bar{\alpha}_X$ maps $D_i \setminus I$ and $D_j \setminus I$ to $\bar{\mathcal{A}}_X \setminus \mathcal{A}_X$ and so, in particular, $\bar{\alpha}_X|_{D_i\setminus I} \not\equiv z$ and so (*) cannot hold.

(**) implies (*). Assume that (*) does not hold. Then there exist z in $\mathcal{A}_X$, a connected component $\overline{E}$ of the Zariski closure of $\bar{\alpha}_0^{-1}(z)$ with $\overline{E} \cap X \neq \emptyset$ and a connected component D_0 of D such that $D_0 \cap \overline{E} \neq \emptyset$

and $D_0 \not\subset \overline{E}$. Let $\widehat{D}_0$ and $\widehat{\overline{E}}$ be the proper transforms of D_0 and $\overline{E}$ under $r : \widehat{\overline{X}} \to \overline{X}$ respectively. Then any connected component of $\widehat{\overline{E}}$ intersects $\widehat{D}_0$ or is connected to $\widehat{D}_0$ by the exceptional divisor F of r. Let $\widehat{E}_0$ be such a component. Since $D_0 \not\subset \overline{E}$, we have $\widehat{D}_0 \not\subset \widehat{\overline{\alpha}}_X^{-1}(z) \supset \widehat{E}_0$. So there is a component of $F \cup \widehat{D}_0$ intersecting with but not contained in $\widehat{E}_0$. This contradicts (**). $\qquad\square$

We now continue with the proof of the proposition.

Assume (**), we prove that $\bar{\beta}(X) \subset \overline{Y}$ is an open algebraic subvariety by giving an explicit description for it: By definition of the Stein factorization, points of $\overline{Y}$ corresponds to connected components of the fibers of the morphism $\widehat{\bar{\alpha}}_X : \widehat{\overline{X}} \to \bar{\mathcal{A}}_X$, and $\bar{\beta} : \widehat{\overline{X}} \to \overline{Y}$ is the canonical surjective map which contracts these connected components to points. Since $\bar{\beta}$ is a proper birational morphism, $\bar{\beta}(\widehat{D}) \subset \overline{Y}$ is algebraic. Since $\bar{\alpha}_X$ is dominant, $\overline{Y}$ is a surface and so $\bar{\beta}(\widehat{D})$ can be decomposed into a pure 1-dimensional subvariety Γ and a finite set $G = \{y_1, \ldots, y_s\}$ in $\overline{Y}$. Let $y \in \bar{\beta}(X)$, then $y = \bar{\beta}(\widehat{x})$ for some $\widehat{x} \in X$. Let $z = \widehat{\bar{\alpha}}_X(\widehat{x})$. By the definition of $\bar{\beta}$, $\bar{\beta}^{-1}(y)$ is a connected component $\widehat{\overline{E}}$ of $\widehat{\bar{\alpha}}^{-1}(z)$. By (**), all connected components of $\widehat{D}$ intersecting $\widehat{\overline{E}}$ map to the point y, and the others map to a closed algebraic set not containing y. This means that $y \notin \Gamma$ and so $\bar{\beta}(X)$ is just $\overline{Y} \setminus \Gamma$ minus a finite number of points in G, which is an algebraic set.

Conversely, assume that (**) does not hold. Then there exist $z \in \mathcal{A}_X$, a connected component $\widehat{\overline{E}}$ of $\widehat{\bar{\alpha}}^{-1}(z)$ with $\widehat{\overline{E}} \cap X \neq \emptyset$ and an irreducible component $\widehat{D}_0$ of $\widehat{D}$ with $\widehat{x} \in \widehat{D}_0 \cap \widehat{\overline{E}}$ and $\widehat{D}_0 \not\subset \overline{E}$. So $\Gamma = \bar{\beta}(\widehat{D}_0) \subset \overline{Y}$ is a 1-dimensional subvariety containing the point $y_0 = \bar{\beta}(\widehat{x}) = \bar{\beta}(\widehat{\overline{E}})$. So $\bar{\beta}$ has a 1-dimensional fiber over y_0 and so all other fibers over a neighborhood U of y_0 are single points. Hence $U \cap \Gamma \cap \bar{\beta}(X) = \{y_0\}$ and $\bar{\beta}(X) \subset \overline{Y}$ cannot be open. $\qquad\square$

4.3. End of the proof of Theorem 1.3

For this proof we adopt the notations of Theorem 4.2, meaning we allow arbitrary boundary divisors. This is not a problem since we do not need the condition that $f : \mathbb{C} \to X$ is Brody in this theorem and since we can always lift an entire curve f through a birational map which is biholomorphic on X and the property of f to be algebraically degenerate is of course invariant under such a birational map.

By the same token we also may assume that $\bar{\alpha}_X : \overline{X} \to \bar{\mathcal{A}}_X$ is (already) a morphism. Hence, let

$$\overline{X} \xrightarrow{\bar{\beta}} \overline{Y} \xrightarrow{\bar{\gamma}} \bar{\mathcal{A}}_X$$

be a Stein factorization of the morphism $\bar{\alpha}_X$ and $\beta = \bar{\beta}|_X$. Then $\overline{Y}$ is a normal variety which compactifies $Y = (\bar{\gamma})^{-1}(\mathcal{A}_X)$, and f is algebraically degenerate (in X) if and only if the map $\beta \circ f$ is algebraically degenerate (in Y).

We apply Proposition 4.3 to the finite morphism $\gamma = \bar{\gamma}|_Y : Y \to \mathcal{A}_X$ and to $Y_0 := \beta(X) \subset Y$, which is an open subvariety by Proposition 4.4. In order to simplify notations, we assume that X is the total space of the Iitaka fibration of Y_0 (there is no problem with this since the entire curve $f : \mathbb{C} \to X$ lifts through a birational morphism over X). So let $\Phi : X \to Z^*$ be the log Iitaka fibration. Then Proposition 4.3 says that there exist a semi-abelian curve $\mathcal{B} \subset \mathcal{A}_X$, étale covers $\widetilde{Y}$ and $\widetilde{\mathcal{B}}$ of Y and $\mathcal{B}$, respectively, and a smooth curve $\widetilde{Z}$ such that

1) $\widetilde{Z}$ is finite over $\mathcal{A}_X/\mathcal{B}$;
2) $\widetilde{Y}$ is a fiber bundle over $\widetilde{Z}$ with fiber $\widetilde{\mathcal{B}}$ and translations by $\widetilde{\mathcal{B}}$ as structure group.

Moreover, if for $z \in Z^*$ we put $X_z = \Phi^{-1}(z)$, $Y_z = \beta(X_z) \subset Y_0 \subset Y$ and $B_z = \gamma(Y_z)$, then for generic $z \in Z^*$, $Y_z \subset Y$ is a closed subvariety, which is equal to the image of a suitable fiber of the fiber bundle $p : \widetilde{Y} \to \widetilde{Z}$, and, vice versa, the image of a generic fiber of this fiber bundle is of the form Y_z for generic $z \in Z^*$.

Let $\widetilde{Y}_0 \subset \widetilde{Y}$ be the inverse image of Y_0 in $\widetilde{Y}$ under the étale cover $\widetilde{Y} \to Y$, and let $\widetilde{Z}_0 \subset \widetilde{Z}$ be the image of $\widetilde{Y}_0$ under the fiber bundle projection $p : \widetilde{Y} \to \widetilde{Z}$. Then $\widetilde{Z}_0 \subset \widetilde{Z}$ is an open algebraic subcurve: In fact, since $\widetilde{Y}_0 \subset \widetilde{Y}$ is an open algebraic subsurface, its complement can only contain a finite number of curves or isolated points, which can only contain a finite number of fibers of the fiber bundle $\widetilde{Y} \to \widetilde{Z}$, so the complement of $\widetilde{Z}_0$ in $\widetilde{Z}$ contains at most a finite number of points. Let still $\widetilde{W}_0 \subset \widetilde{Y}$ be the inverse image of $\widetilde{Z}_0$ under the fiber bundle projection $p : \widetilde{Y} \to \widetilde{Z}$.

We claim that $\widetilde{W}_0 \setminus \widetilde{Y}_0$ is of codimension at least 2 in $\widetilde{W}_0$: In fact, by Proposition 4.3 the generic fiber of the fiber bundle $\widetilde{Y} \to \widetilde{Z}$ projects to a fiber $Y_z \subset Y_0$, so is contained in $\widetilde{Y}_0$. We know that since $\widetilde{Y}_0 \subset \widetilde{Y}$ is an open algebraic subsurface, its complement can only contain a finite number of (closed) curves or isolated points. But since such a closed curve does not hit the generic fiber of the fiber bundle (so it is contained in the union of a finite number of fibers since the base $\widetilde{Z}$ is of dimension 1), it is itself equal

to a fiber, but then it is also in the complement of $\widetilde{W}_0$. This proves the claim.

We now claim that $\widetilde{W}_0$ is of log Kodaira dimension $\bar{\kappa}_{\widetilde{W}_0} \geqslant 1$: Since $p : \widetilde{Y} \to \widetilde{Z}$ is a fiber bundle over the smooth curve $\widetilde{Z}$ and $\widetilde{Y}$ is étale over Y, both Y and $\widetilde{Y}$ are smooth. Let $X^+ = \bar{\beta}^{-1}(Y_0) \subset \overline{X}$. Then X^+ has the same log Kodaira dimension as Y_0 as they are properly birational. But $X^+ \setminus X$ lies only on the fibers of $\bar{\beta}$ above Y_0 and so any pluricanonical logarithmic form for $(\overline{X}, D)$ must actually be a pluricanonical form over X^+ by the Riemann extension theorem applied to Y_0. Hence $1 = \bar{\kappa}_X = \bar{\kappa}_{X^+} = \bar{\kappa}_{Y_0} \leqslant \bar{\kappa}_{\widetilde{Y}_0} = \bar{\kappa}_{\widetilde{W}_0}$ where the last equality follows again by the Riemann extension theorem.

Next we claim that $\bar{\kappa}_{\widetilde{Z}_0} = 1$: Let $s_1, s_2 \in H^0(\widetilde{Y}, m\overline{K}_{\widetilde{W}_0})$ be two linearly independent log multi-canonical sections. Consider the exact sequence of locally free sheaves

$$0 \to p^* \overline{K}_{\widetilde{Z}_0} \longrightarrow \overline{\Omega}_{\widetilde{W}_0} \longrightarrow \overline{K}_{\widetilde{Y}/\widetilde{Z}} \to 0,$$

from which it follows that $\overline{K}_{\widetilde{W}_0}$ is isomorphic to $p^* \overline{K}_{\widetilde{Z}_0} \otimes \overline{K}_{\widetilde{Y}/\widetilde{Z}}$. But $\overline{K}_{\widetilde{Y}/\widetilde{Z}}$ is dual to

$$\ker \left(\bar{\mathrm{d}}p : \overline{T}_{\widetilde{W}_0} \to \pi^* \overline{T}_{\widetilde{Z}_0} \right)$$

and this latter has a global nowhere vanishing section v (that generates the Lie group action) and, hence, is trivial. So $\overline{K}_{\widetilde{W}_0}$ is isomorphic to $\overline{K}_{\widetilde{Z}_0}$. Hence s_1, s_2 give two linearly independent sections of $m\overline{K}_{\widetilde{Z}_0}$.

Theorem 1.3 is now immediate: In fact, let $f : \mathbb{C} \to X$ be an entire curve. If $p : \widetilde{Y} \to \widetilde{Z}$ denotes the fiber bundle projection, then the entire curve $\beta \circ f : \mathbb{C} \to Y_0$ lifts by the étale cover $\widetilde{Y} \to Y$ to a curve having image in $\widetilde{Y}_0 \subset \widetilde{W}_0$, projecting by p to an entire curve having image in the curve $\widetilde{Z}_0$. But since $\bar{\kappa}_{\widetilde{Z}_0} = 1$, the curve $\widetilde{Z}_0$ is hyperbolic, so this entire curve is constant. Hence, the image of the entire curve $\beta \circ f$ lies in the image in Y of a fiber of the fiber bundle $\widetilde{Y} \to \widetilde{Z}$, and so the entire curve $f : \mathbb{C} \to X$ is algebraically degenerate. $\qquad\square$

Remark. — The counter-example in Proposition 4.7 shows that, without condition (*), it can happen that entire curves $f : \mathbb{C} \to X$ can be Zariski-dense. In fact, condition (*) does not hold in this counter-example: With the notations as in Proposition 4.7, $\bar{\alpha}_X = \alpha_{\overline{X}} : \overline{X} \to E \times E$ is the blowup morphism of the two points Q_1, Q_2 and the exceptional fibers $\alpha_{\overline{X}}^{-1}(Q_1)$ and $\alpha_{\overline{X}}^{-1}(Q_2)$ intersect D but do not contain it.

4.4. Some applications

The following result generalizes Kawamata's theorem for normal surfaces finite over a semi-abelian surface to surfaces with log irregularity 2 by relating it to entire holomorphic curves.

THEOREM 4.6. — *Let $(\overline{X}, D)$ be a log surface with log irregularity $\bar{q}_X = 2$. In the case of dominant $\bar{\alpha}_X$, assume condition (*). Then the following are equivalent:*

1) *There is an entire curve $f : \mathbb{C} \to X$ such that $f^*w \equiv 0$ for some $w \in H^0(\overline{T}^*_X)$ and f is not algebraically degenerate.*
2) $\bar{\kappa}_X = 0$.
3) $\bar{\alpha}_X$ *is birational and $\mathcal{A}_X \setminus \alpha_X(X)$ is finite.*

Proof. — We use Theorem 1.3 and the theorem of McQuillan-ElGoul to deduce 2) from 1). To deduce 3) from 2), we use the addition theorem of Kawamata for open surfaces and Proposition 4.3. Finally, we deduce 1) from 3) by elementary methods.

1) ⇒ 2). Suppose that there exists an algebraically nondegenerate entire curve in X lying in the foliation defined by a log 1-form. By Theorem 2.9 and Proposition 2.11 we may assume that $\bar{\alpha}_X$ is dominant and $\bar{\kappa}_X \leqslant 1$. This means in particular that there are linearly independent log 1-forms whose wedge is not identically 0. But this wedge gives a log 2-form and hence $\bar{\kappa}_X \neq -\infty$. But $\bar{\kappa}_X \neq 1$ by Theorem 1.3. Hence $\bar{\kappa}_X = 0$.

2) ⇒ 3). Assume that α_X (and hence $\bar{\bar{\alpha}}_X$) is not dominant. Then $\bar{\bar{\alpha}}_X$ factors through a morphism $\bar{\alpha} : \widehat{\overline{X}} \to \overline{Y} \subset \bar{\mathcal{A}}_X$ where $Y = \overline{Y} \cap \mathcal{A}_X$ is a hyperbolic curve by Proposition 2.11 and so $\bar{\kappa}_Y = 1$. Let $\overline{F}$ be a general fiber of $\bar{\alpha}$, then $\overline{F}$ is a smooth projective curve having transversal intersection with $\widehat{D}$, the reduced total transform of D. Hence, by the smoothness of $\bar{\alpha}$ along $\overline{F}$, we have the following exact sequence

$$0 \to \overline{K}_F \longrightarrow \Omega(\widehat{\overline{X}}, \widehat{D})|_{\overline{F}} \longrightarrow \mathcal{O}_F \to 0.$$

Hence $\overline{K}_X|_F = \wedge^2\Omega(\widehat{\overline{X}}, \widehat{D})|_{\overline{F}} = \overline{K}_F$. Since $\bar{\kappa}_X \geqslant 0$, there is a nontrivial section of some tensor power of the log canonical sheaf $\overline{K}_X$ over $\widehat{\overline{X}}$. Since this section remains nontrivial over $\overline{F}$, we see that $\bar{\kappa}_F \geqslant 0$. By Theorem 4.1, we have $0 = \bar{\kappa}_X \geqslant \bar{\kappa}_F + \bar{\kappa}_B \geqslant 1$. This contradiction shows that α_X is dominant (This fact also follows directly from Theorem 28 of [19]). Keeping the same notation as Proposition 4.4 with the normal surface $Y = (\bar{\gamma})^{-1}(\mathcal{A}_X)$, then $\gamma = \bar{\gamma}|_Y : Y \to \mathcal{A}_X$ is a surjective finite map and we have $\bar{\kappa}_Y \geqslant 0$ by Theorem 4.2. As $\beta = \bar{\beta}|_X$ is a morphism

to Y, we have $0 = \bar{\kappa}_X \geqslant \bar{\kappa}_Y$. Hence, $\bar{\kappa}_Y = 0$ and Y is étale over $\mathcal{A}_X$ by Theorem 4.2. This means that Y is a semi-abelian variety and so by the universal property of the Albanese map α, $\gamma : Y \to \mathcal{A}_X$ is an isomorphism. By condition (*) and Proposition 4.4, we have that $Y_0 = \beta(X)$ is an open subvariety of the normal surface Y. As $\beta = \bar{\beta}|_X$ is a morphism to Y_0, we have $0 = \bar{\kappa}_X \geqslant \bar{\kappa}_{Y_0} \geqslant \bar{\kappa}_Y = 0$; thus forcing equality. Hence $Y \setminus Y_0$ is finite by Theorem 4.3 and the result follows.

3) $\Rightarrow$ 1). Keeping the same notation as in Proposition 4.4, $Y_0 = \alpha_X(X)$ is an open subset of $\mathcal{A}_X$ with finite complement. Let $X^+ = \bar{\hat{\alpha}}^{-1}(Y_0)$. As $\bar{\hat{\alpha}}$ is a birational morphism on X^+ that restricts to α_X on X, we see that $X^+ \setminus X$ is an exceptional divisor of $\bar{\hat{\alpha}}$ in X^+ whose image is contained in the image of the exceptional divisor E of α_X. Consider the finite sets $S_1 := \alpha_X(E) \subset \mathcal{A}_X$, $S_2 := \mathcal{A}_X \setminus Y_0$ and $S' = S_1 \cup S_2$. By a translation if necessary, we may assume that the universal covering map $h : \mathbb{C}^2 \to \mathcal{A}_X$ is a morphism of additive groups and that $\Gamma = \ker h$ does not intersect $S = h^{-1}(S')$. Consider the linear map $f_0 : \mathbb{C} \to \mathbb{C}^2$, $z \mapsto (z, az)$ for $a \in \mathbb{C}$. Then $f_0^* w \equiv 0$ for the linear 1-form $w = \mathrm{d}z_2 - a\,\mathrm{d}z_1$ where (z_1, z_2) is the standard coordinate for $\mathbb{C}^2$. Now if $f_1 = h \circ f_0$ is algebraically degenerate, then its image lies in an elliptic curve or a rational curve in $\bar{\mathcal{A}}_X$ and the latter must intersect the boundary of $\mathcal{A}_X$ at at least and hence exactly two points. In either case f_1 is an étale covering over its image, which is either $\mathbb{C}^*$ or an elliptic curve, and so $f_0^{-1}(\Gamma) = \ker f_1 \neq \{(0,0)\}$. Therefore, since log 1-forms on $\mathcal{A}_X$ correspond bijectively with linear 1-forms on $\mathbb{C}^2$, it suffices to produce an a such that $f_0^{-1}(\Gamma' \cup S) = \emptyset$, $\Gamma' = \Gamma \setminus \{(0,0)\}$, for then $f_1(\mathbb{C})$ does not intersect S' and $\alpha_X^{-1} \circ f_1$ would be the required holomorphic curve on X as α_X^{-1} is a well defined holomorphic map outside S'. For this, we only need to choose $a \in \mathbb{C}$ outside the countable set $\{v/u \mid (u,v) \in \Gamma' \cup S\}$. $\square$

The additional condition (*) on α_X is essential for Theorems 1.3 and 4.6, as the following counterexamples shows. We remark here however that it can be weakened still for Theorem 4.6, as can be seen from the proofs we give. We also remark that one can prove that 2) and 3) of Theorem 4.6 are equivalent even without condition (*).

PROPOSITION 4.7 (Counterexample for $\bar{\kappa}_X = 1$). — *Let E be an elliptic curve and $p : E \times E \to E$ the projection to the first factor. Let $P_1, P_2 \in E$ be two distinct points, and $Q_i \in p^{-1}(P_i)$, $i = 1, 2$ two points. Let $b : \bar{X} \to E \times E$ be the blow up of $E \times E$ in the points Q_1, Q_2. Let D be the union of the proper transforms of $p^{-1}(P_i)$, $i = 1, 2$ in $\bar{X}$, and $X := \bar{X} \setminus D$. Then $\bar{q}_X = 2$ and $\bar{\kappa}_X = 1$, but X admits entire curves $f : \mathbb{C} \to X$ which are*

neither algebraically degenerate nor linearly degenerate (i.e. the condition given in 1) of Theorem 4.6).

Proof. — First it is easy to see that $\bar{q}_X \geqslant 2$, since $q_{E \times E} = 2$ and linearly independent 1-forms on $E \times E$ lift to linearly independent 1-forms on $\overline{X}$.

The fact that there exist entire curves $f : \mathbb{C} \to X$ which are not algebraically degenerate is an easy application of the main result of Buzzard-Lu '00 [6]: The map $p \circ b : X \to E$ is a surjective holomorphic map defining X into an elliptic fibered surface over the curve E without any branching. So by Theorem 1.7 of [6], X is dominable by $\mathbb{C}^2$ and there are (a lot of) entire curves $f : \mathbb{C} \to X$ which are neither algebraically degenerate nor linearly degenerate (i.e. the condition given in 1) of Theorem 4.6). By Theorem 1.1 this still implies in particular that $\bar{q}_X \leqslant 2$, so $\bar{q}_X = 2$.

It is easy to see that $\bar{\kappa}_X \leqslant 1$: Since $K_{E \times E} = 0$ and by the standard formula for the relation of the canonical divisors under blow ups of points we get

$$\overline{K}_X = (p \circ b)^{-1}(P_1) + (p \circ b)^{-1}(P_2).$$

Hence, for a generic point $P \in E$, the fiber $F := (p \circ b)^{-1}(P)$ has the property that the restriction to F of the line bundles $(\overline{K}_X)^{\otimes m}$ are trivial, hence, their sections cannot seperate points. Hence, $\bar{\kappa}_X \leqslant 1$.

It remains to prove that $\bar{\kappa}_X \geqslant 1$. By the Riemann-Roch theorem for the curve E and the divisor $P_1 + P_2 \subset E$ of degree 2 (see for example Hartshorne '77 [15], p. 295), we get that for the log curve $E_0 := E \setminus \{P_1, P_2\} \subset E$, we have

$$h^0(E, \overline{K}_{E_0}) = h^0(E, P_1 + P_2) = \deg(P_1 + P_2) = 2.$$

Hence, there exist two linearly independent log 1-forms η_1, η_2 on $E_0 \subset E$. The key point is now that we can use them to construct two linearly independent sections of the line bundle $\overline{K}_X$ on $\overline{X}$ (since this will prove $\bar{\kappa}_X \geqslant 1$).

Locally, let x be the base coordinate and y the fiber coordinate of the projection $p : E \times E \to E$, both linear with respect to the linear structure of E. We blow up the origin of (x, y) and let $X = \overline{X} \setminus D$, where D is the proper transform of $x = 0$ by b. Then in a neighborhood of the point of interection of the exceptional curve of b and the proper transform of $y = 0$ by b, X is parametrized by x, z where $y = zx$. Any log 1-form of $E \setminus \{0\}$ is of the form $f(x)\mathrm{d}x/x$ near 0. Now

$$\mathrm{d}y = z\,\mathrm{d}x + x\,\mathrm{d}z$$

is a 1-form on $\overline{X}$ and, hence,

$$(4.1) \qquad\qquad f(x)\frac{\mathrm{d}x}{x} \wedge \mathrm{d}y = f(x)\mathrm{d}x \wedge \mathrm{d}z$$

is a log 2-form on $(\overline{X}, \{x = 0\})$, which by (4.1) doesn't have any poles on X and, hence, is a log 2-form on $(\overline{X}, D)$. By this local argument we thus see that the forms $\eta_1 \wedge \mathrm{d}y, \eta_2 \wedge \mathrm{d}y$ are two linearly independent global sections of the line bundle $\overline{K}_X$ on $\overline{X}$. $\qquad\square$

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