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A DIRECT DECOMPOSITION OF THE MEASURE ALGEBRA OF A LOCALLY COMPACT ABELIAN GROUP

par N. Th. VAROPOULOS

0. Introduction and notations.

For any locally compact space X, let M(X) denote the Banach space of all complex bounded Radon measures on X. We shall in general follow N. Bourbaki [1] for measure theory.

For any two Radon measures on X μ and ν we shall write $\mu \perp \nu$ if they are mutually singular and $\mu \ll \nu$ if $|\mu|$ is absolutely continuous with respect to $|\nu|$. We shall say that $B \subseteq M(X)$, a subspace, is a (complex) band in M(X), if $\beta \in B \Longrightarrow \mathcal{R}\beta \in B$ and if $\{\mathcal{R}\beta; \beta \in B\}$ is the intersection of M(X) with a real band (cf. [1], ch. II). For $\{\beta_\alpha \in M(X)\}_{\alpha \in A}$ we denote by $\wr \beta_\alpha; \alpha \in A \wr \subseteq M(X)$ the (complex) band generated in M(X) by $\{\beta_\alpha\}_{\alpha \in A}$ i.e. the intersection of all (complex) bands containing $\{\beta_\alpha; \alpha \in A\}$. Also for B, B_1, B_2 bands in M(X) and $\mu \in M(X)$ we write :

$$B_1 \perp B_2 \Longleftrightarrow (\forall \beta_1 \in B_1, \quad \forall \beta_2 \in B_2 \Longrightarrow \beta_1 \perp \beta_2)$$

$$\mu \perp B \Longleftrightarrow \wr \mu \wr \perp B$$

$$B^\perp = \{m \in M(X); m \perp B\}$$

Let us also denote :

$$M^+ = M^+(X) = \{m \in M(X); m \geqslant 0\}$$

$$M_c = M_c(X) = \{m \in M(X); \quad \forall x \in X \quad \{x\} \text{ is } m\text{-null}\}$$

$$\Delta = \Delta(X) = M_c^\perp(X)$$

$$M_c^+ = M_c^+(X) = M^+(X) \cap M_c(X)$$

If Ω is a Borel subset let us denote :

$$B(\Omega) = \{m \in M(X);\quad X \setminus \Omega \quad \text{is } m\text{-null}\}$$

which is a band in $M(X)$.

Now we shall denote by G, in general, an additive locally compact abelian group, and follow freely well-established and standardised notations for it. E.g. we shall denate by $0 = 0_G$ its zero element; for P, $Q \subset G$ and $n \in \mathbf{Z}$ (the integers) we shall denote :

$$P + Q = \{x + y;\quad x \in P,\quad y \in Q\} \subset G$$

$$nP = \{sgn(n) \sum_{j=1}^{|n|} x_j;\quad x_j \in P \quad 1 \leqslant j \leqslant |n|\} \subset G$$

$$Gp(P) = Gp\{x; x \in P\}$$

Also we shall denote by $\hat{G}$ the dual group of G and for any $\mu \in M(G)$, $\hat{\mu}$ will denote the Fourier transform of μ. We let then (cf. [4], 5.6.9.) :

$$M_0(G) = \{m \in M(G);\quad \hat{m}(\chi) \xrightarrow[\chi \to \infty]{} 0,\quad \chi \in \hat{G}\} \subseteq M_c(G) \subseteq M(G).$$

Finally for any commutative Banach algebra R, we denote by $\tilde{R} = R + 1\mathbf{C}$ the Banach algebra we obtain by adjoining the identity to R and also :

$$R^2 = \left\{ \sum_{j=1}^{N} \lambda_j x_j y_j \mid N \geqslant 1;\ \lambda_j \in \mathbf{C} \text{ (the complex numbers)}; x_j, y_j \in R \right\}$$

Also we shall denote by $\mathfrak{M}(R)$ its maximal ideal space and by $\Sigma(R) \subseteq \mathfrak{M}(R)$ its Shilov boundary.

We shall not state here the main results obtained in this paper, which are concerned with a direct decomposition of the algebra $M(G)$, because they cannot be explained in a few words. We shall however state an application of our results.

THEOREM N. F. (Non Factorisation). — *For any non discrete, locally compact abelian group G :*

(i) $M_c/\overline{M_c^2}$ *is a non separable Banach space.*

(ii) $M_0/\overline{M_0^2}$ *is an infinite dimensional Banach space.*

(iii) *If further G is metrisable then* $M_0 \not\subset \overline{M_c^2}$.

The material of this paper is devided :

§ 1. We give some elementary algebraic and geometric results on independent subsets of a group G.

§ 2. We give some measure theoretic results on perfect, independent subsets of a locally compact group G.

§ 3. We obtain a direct decomposition of M(G), which is the main result of the paper.

§ 4. We give some application of § 3.

1. Algebraic and geometric results on independent sets.

DEFINITION 1.1.

A subset $\mathbf{P} \subset \mathrm{G}$ *of an abelian group will be called strongly independent if, for all* N, *positive integer, any family of* N *distinct points of* $\mathbf{P}$, $\{p_j \in \mathbf{P}\}_{j=1}^{\mathrm{N}}$ *and any family of* N *integers,* $\{n_j \in \mathbf{Z}\}_{j=1}^{\mathrm{N}}$, *such that* $\sum_{j=1}^{\mathrm{N}} n_j p_j = \mathrm{O}_{\mathrm{G}}$, *we must have* $\{n_j x; x \in \mathbf{P}\} = \mathrm{O}_{\mathrm{G}}$ *for* $1 \leqslant j \leqslant \mathrm{N}$.

For the rest of this paper, without further comments, we shall reserve the letter $\mathbf{P}$ for a strongly independent, perfect, metrisable subset of the locally compact abelian group G. We introduce here some more notations which will be kept fixed for the rest of the paper.

Let $m, k \in \mathbf{Z}$, $m \geqslant 0$, $k \geqslant 0$ and $g \in \mathrm{G}$, we denote then :

$$\Omega_m = \prod_{j=1}^{m} \mathbf{P}_j, \quad \mathbf{P}_j = \mathbf{P} \quad (1 \leqslant j \leqslant m) \quad \text{for} \quad m \geqslant 1,$$

and :

$$\omega_m : \Omega_m \to \mathrm{G} \quad \text{defined by} \quad \omega_m\,[(p_j)_{j=1}^{m}] = \sum_{j=1}^{m} p_j \in \mathrm{G}$$

ω_m then induces (cf [1], ch. V, § 6) :

$$\check{\omega}_m = \mathrm{M}(\Omega_m) \to \mathrm{M}(\mathrm{G}).$$

Let also :

$$\mathrm{R}_m^k = \bigcup_{1 \leqslant l_1 < l_2 < \dots < l_k \leqslant m} [\omega = (p_j)_{j=1}^{m} \in \Omega_m;\ p_{l_1} = p_{l_2} = \dots = p_{l_k}]$$

$$m \geqslant k \geqslant 2 \quad (\text{union over } l\text{'s});$$

$$D_m^k(g) = [\omega = (p_j)_{j=1}^m \in \Omega_m; \quad p_k = g]$$

for

$$m \geqslant k \geqslant 1\ ;$$

set also for convenience :

$$R_m^k = \varnothing \qquad \text{for} \qquad k > m \geqslant 0, \qquad \text{or} \qquad k = 1.$$

Let us also denote by $k(\mathbf{P}) = k \geqslant 2$ the smallest positive integer n such that $\{nx,\ x \in \mathbf{P}\} = O_G$, if such an integer exists; otherwise set $k = +\infty$. We shall call $k = k(\mathbf{P})$ the torsion of $\mathbf{P}$. If $k < +\infty$ we shall denote by $\mathbf{Z}\,(\text{mod}\ k)$ the integers modulo k, and for $n \in \mathbf{Z}$ let $n\,(\text{mod}\ k)$ be its class. If $k = +\infty$ for convenience we set

$$\mathbf{Z}\,(\text{mod}\ k) = \mathbf{Z} \quad \text{and for} \quad n \in \mathbf{Z}, \quad n\,(\text{mod}\ k) = n.$$

We now introduce:

DEFINITION 1.2.

We shall call reduced sum (on $\mathbf{P}$*, a strongly independent subset of* G *with torsion* k*) a formal expression* $\sum_{\alpha \in A} \dot{n}_\alpha p_\alpha$, *where* A *is a, possibly empty, finite index set, where*

$$\dot{n}_\alpha \in \mathbf{Z}\,(\text{mod}\ k) \quad \textit{and} \quad \dot{n}_\alpha \neq 0\,(\text{mod}\ k),$$

and where the points $(p_\alpha \in \mathbf{P})_{\alpha \in A}$ *are distinct.*

We shall then say that two reduced sums :

$$\mathfrak{M} = \sum_{\alpha \in A} \dot{m}_\alpha x_\alpha \qquad \text{and} \qquad \mathfrak{N} = \sum_{\beta \in B} \dot{n}_\beta y_\beta$$

are equivalent, and write $\mathfrak{M} \sim \mathfrak{N}$, if there exists a (1-1) correspondence, $\varphi : A \to B$, between A and B such that :

$$\dot{n}_{\phi(\alpha)} = \dot{m}_\alpha \qquad y_{\phi(\alpha)} = x_\alpha\ ; \qquad \alpha \in A.$$

We shall almost always abuse the above definition and its notations, in various obvious ways. So we shall say, for instance, that

$$\sum_{1 \leqslant j \leqslant M} m_j p_j \in G, \qquad m_j \in \mathbf{Z} \qquad (1 \leqslant j \leqslant M)$$

(the summation being taken, of course, for the group addition and the empty sum being interpreted as O_G) is a reduced sum, when we really mean that $\sum_{\alpha \in \{j \in \mathbf{Z}\,;\, 1 \leqslant j \leqslant M\}} [m_\alpha \,(\mathrm{mod}\, k)]\, p_\alpha$ is a reduced sum. Similarily we shall say that two reduced sums

$$\sum_{1 \leqslant j \leqslant M} m_j\, p_j \quad (m_j \in \mathbf{Z}) \qquad \text{and} \qquad \sum_{1 \leqslant j \leqslant N} n_j\, q_j \quad (n_j \in \mathbf{Z})$$

are equivalent when

$$\sum_{\alpha \in \{j \in \mathbf{Z}\,;\, 1 \leqslant j \leqslant M\}} [m_\alpha (\mathrm{mod}\, k)]\, p_\alpha \sim \sum_{\beta \in \{j \in \mathbf{Z}\,;\, 1 \leqslant j \leqslant M\}} [n_\beta (\mathrm{mod}\, k)]\, q_\beta,$$

observe that then

$$\sum_{1 \leqslant j \leqslant M} m_j\, p_j = \sum_{1 \leqslant j \leqslant N} n_j\, q_j \in \mathrm{G}.$$

We state now the fundamental:

Lemma 1.1.

Every $x \in \mathrm{Gp}(\mathbf{P})$ can be expressed as a reduced sum (on $\mathbf{P}$) in a unique way, up to equivalence.

Proof.

The only point to prove is that if:

$$\mathfrak{M} = \sum_{1 \leqslant j \leqslant M} m_j\, p_j \qquad \text{and} \qquad \mathfrak{N} = \sum_{1 \leqslant j \leqslant N} n_j\, q_j \qquad (m_j, n_j \in \mathbf{Z})$$

are two reduced sums such that:

$$\sum_{1 \leqslant j \leqslant M} m_j\, p_j = \sum_{1 \leqslant j \leqslant N} n_j\, q_j \in \mathrm{G} \qquad \text{then} \qquad \mathfrak{M} \sim \mathfrak{N}.$$

If $M = 0$ the above is simply a restatement of the definition of strong independence. Thus we proceed by induction on M and we observe that if $M \geqslant 1$ then $\sum_{1 \leqslant j \leqslant M-1} m_j\, p_j$ is also a reduced sum and:

$$\sum_{1 \leqslant j \leqslant M-1} m_j\, p_j = \sum_{1 \leqslant j \leqslant N} n_j\, q_j - m_M\, p_M. \tag{1.1}$$

Therefore there exists $\sum_{\alpha \in A} l_\alpha\, x_\alpha$ $(l_\alpha \in \mathbf{Z})$ a reduced sum such that

$$\{x_\alpha;\ \alpha \in A\} \subset \{q_j;\, p_M;\, 1 \leqslant j \leqslant N\}$$

and

$$\sum_{1 \leqslant j \leqslant M-1} m_j\, p_j = \sum_{\alpha \in A} l_\alpha\, x_\alpha. \tag{1.2}$$

Therefore if we use the inductive hypothesis on (1.2) and the fact that $p_M \notin \{p_j; 1 \leqslant j \leqslant M-1\}$ it follows that there exists $1 \leqslant j_0 \leqslant N$ such that $q_{j_0} = p_M$ and $n_{j_0} (\mathrm{mod}\, k) = m_M (\mathrm{mod}\, k)$; and that, combined with (1.1) and the inductive hypothesis, proves the inductive step.

LEMMA 1.2.

Let $\mathbf{P} \subset G$ *be a strongly independent subset of* G *with torsion* $k = k(\mathbf{P}) \geqslant 2$ *(possibly* $k = +\infty$*). And let* $m, n \in \mathbf{Z}$, $m \geqslant n \geqslant 0$ $m \geqslant 1$; *and let* $g \in G$. *Then if* $g \notin \mathrm{Gp}(\mathbf{P})$ we *have* $m\mathbf{P} \cap g + n\mathbf{P} = \emptyset$. *If on the other hand* $g \in \mathrm{Gp}(\mathbf{P})$ *and if* $g = \sum_{r \in \Gamma} \gamma_r g_r$ $(g_r \in \mathbf{P}; r \in \Gamma)$ *is the reduced sum expression of* g *then:*

(i). *If* $k > m > n$ *then:*

$$\omega_m^{-1}(m\mathbf{P} \cap n\mathbf{P}) = \emptyset.$$

(ii). *If* $m > n$, $m \geqslant k$ *then:*

$$\omega_m^{-1}(m\mathbf{P} \cap n\mathbf{P}) \subseteq R_m^k$$

(iii). *If* $k > m$ *and* $g \neq O_G$ *then:*

$$\omega_m^{-1}(m\mathbf{P} \cap g + n\mathbf{P}) \subset \bigcup_{r \in \Gamma} \bigcup_{1 \leqslant j \leqslant m} D_m^j(g_r)$$

(iv). *If* $m \geqslant k$ *and* $g \neq O_G$ *then:*

$$\omega_m^{-1}(m\mathbf{P} \cap g + n\mathbf{P}) \subset R_m^k \cup \bigcup_{r \in \Gamma} \bigcup_{1 \leqslant j \leqslant m} D_m^j(g_r)$$

(In the above inequalities, and in general, we assume that if

$$k = k(\mathbf{P}) = +\infty \quad \textit{then} \quad k > n \quad \textit{for all} \quad n \in \mathbf{Z}).$$

Proof.

(i) [respectively: (ii)]. Let us make the contradictory hypothesis that there exists an element:

$$(p_j)_{1 \leqslant j \leqslant m} \in \omega_m^{-1}(m\mathbf{P} \cap n\mathbf{P})$$

[respectively: $(p_j)_{1 \leqslant j \leqslant m} \in \omega_m^{-1}(m\mathbf{P} \cap n\mathbf{P}) \setminus R_m^k$]

Then there exists $(q_j \in \mathbf{P};\ 1 \leqslant j \leqslant n)$ such that:

$$\sum_{1 \leqslant j \leqslant m} p_j = \sum_{1 \leqslant j \leqslant n} q_j \text{ (empty sums being interpreted as } O_G)$$

By the hypothesis then we see that there exists two reduced sums :

$$\mathcal{M} = \sum_{\alpha \in A} m_\alpha x_\alpha \quad \text{and} \quad \mathcal{N} = \sum_{\beta \in B} n_\beta y_\beta$$

$$(m_\alpha, n_\beta \in \mathbf{Z} ; \ \alpha \in A ; \ \beta \in B)$$

such that :

$$\sum_{\alpha \in A} m_\alpha x_\alpha = \sum_{1 \leqslant j \leqslant m} p_j = \sum_{1 \leqslant j \leqslant n} q_j = \sum_{\beta \in B} n_\beta y_\beta \tag{1.3}$$

$$\sum_{\alpha \in A} m_\alpha = m > n \geqslant \sum_{\beta \in B} n_\beta ; 1 \leqslant m_\alpha < k \ (\alpha \in A); \ 1 \leqslant n_\beta < k \ (\beta \in B) \tag{1.4}$$

Then (1.3) and Lemma 1.1 imply that $\mathcal{M} \sim \mathcal{N}$ which is not compatible with (1.4), and provides the required contradiction.

(iii) [respectively : (iv)]. Let :

$$(p_j)_{1 \leqslant j \leqslant m} \in \omega_m^{-1}(m\mathbf{P} \cap g + n\mathbf{P})$$

$$[\text{respectively} : \ (p_j)_{1 \leqslant j \leqslant m} \in \omega_m^{-1}(m\mathbf{P} \cap g + n\mathbf{P}) \setminus R_m^k]$$

what we have to prove is that:

$$\{p_j ; \ 1 \leqslant j \leqslant m\} \cap \{g_r ; \ r \in \Gamma\} \neq \varnothing \tag{1.5}$$

We suppose that (1.5) is not satisfied and proceeded to obtain a contradiction.

Now there exists ($q_j \in \mathbf{P}$; $1 \leqslant j \leqslant n$) such that :

$$\sum_{1 \leqslant j \leqslant m} p_j - \sum_{r \in \Gamma} \gamma_r g_r = \sum_{1 \leqslant j \leqslant n} q_j \in G \tag{1.6}$$

Also by the hypothesis there exists two reduced sums

$$\mathcal{M} = \sum_{\alpha \in A} m_\alpha x_\alpha \ \text{and} \ \mathcal{N} = \sum_{\beta \in B} n_\beta y_\beta \ (m_\alpha, n_\beta \in \mathbf{Z} ; \ \alpha \in A ; \ \beta \in B)$$

such that:

$$\sum_{\alpha \in A} m_\alpha x_\alpha = \sum_{1 \leqslant j \leqslant m} p_j ; \ \{x_\alpha ; \ \alpha \in A\} \subset \{p_j ; \ 1 \leqslant j \leqslant m\} ;$$

$$\sum_{\beta \in B} n_\beta y_\beta = \sum_{1 \leqslant j \leqslant n} q_j \tag{1.7}$$

$$\sum_{\alpha \in A} m_\alpha = m \geqslant n \geqslant \sum_{\beta \in B} n_\beta ; \ 1 \leqslant m_\alpha < k \ (\alpha \in A) ; \ 1 \leqslant n_\beta < k \ (\beta \in B) \tag{1.8}$$

Now since (1.5) is supposed to be false by the contradictory hypothesis, we see using (1.7) that:

$$\sum_{\alpha \in A} m_\alpha x_\alpha - \sum_{r \in \Gamma} \gamma_r g_r$$

is a reduced sum, and this fact, combined with Lemma 1.1 and (1.6), (1.7) and (1.8), implies that $\sum_{\alpha \in A} m_\alpha = \sum_{\beta \in B} n_\beta$ and that $\Gamma = \emptyset$, which contradicts the fact that $g \neq O_G$, and this complets the proof of the Lemma.

2. Measure theoretic results on independent sets.

In this paragraph again, as we have already said, **P** will denote a strongly independent, perfect, metrisable subset of the locally compact group, with torsion $k = k(\mathbf{P})$ (possibly $k = +\infty$). We have:

Lemma 2.1.

If $\mu, \nu \in M_c^+(G)$ *and are such that:*

(i) supp $\mu \subset m\mathbf{P}$.

(ii) supp $\nu \subset n\mathbf{P}$.

(iii) *All sets* $\{g + m'\mathbf{P} \,;\, g \in G,\ 0 \leqslant m' < m\}$ *are* μ*-null.*

(iv) *All sets* $\{g + n'\mathbf{P} \,;\, g \in G,\ 0 \leqslant n' < n\}$ *are* ν*-null.*

Then all sets $\{g + r\mathbf{P} \,;\, g \in G,\ 0 \leqslant r \leqslant m + n,\ (g, r) \neq (O_G, m + n)\}$ *are* $\mu * \nu$*-null.*

Proof.

Let $\bar{\mu} \in M^+(\Omega_m)$; $\bar{\nu} \in M^+(\Omega_n)$ such that $\check{\omega}_m(\bar{\mu}) = \mu$ and $\check{\omega}_n(\bar{\nu}) = \nu$ be fixed once and for all. Then we have, of course, $\check{\omega}_{m+n}(\bar{\mu} \otimes \bar{\nu}) = \mu * \nu$, and from (iii) and (iv) we deduced:

(iii)′ For all $g \in G$ and $1 \leqslant j \leqslant m$ we have $\bar{\mu}\,[D_m^j(g)] = 0$;

(iv)′ For all $g \in G$ and $1 \leqslant j \leqslant n$ we have $\bar{\nu}\,[D_n^j(g)] = 0$.

Let us also denote for $0 \leqslant r \leqslant m + n$ and $g \in G$:

$$\Delta_{r,g} = \omega_{m+n}^{-1}\,[(m + n)\,\mathbf{P} \cap g + r\mathbf{P}].$$

We see then that to prove the Lemma it suffices to prove that for all $1 \leqslant r \leqslant m + n$ and $g \in G$:

$$(g, r) \neq (O_G, m + n) \implies \bar{\mu} \otimes \bar{\nu}\,(\Delta_{r,g}) = 0. \qquad (2.1)$$

And applying Lemma 1.2 we see that to prove (2.1) it suffices to show:

(α) For all $g \in \mathrm{G}$ and $1 \leqslant j \leqslant m + n$ we have $\bar{\mu} \otimes \bar{\nu}\,[\mathrm{D}^{j}_{m+n}(g)] = 0$

(β) For all choice of $(l_j)_{j=1}^{k}$ such that $1 \leqslant l_1 < l_2 < \dots l_k \leqslant m + n$

we have

$$\bar{\mu} \otimes \bar{\nu}\,\{[\omega = (p_j)_{j=1}^{m+n} \in \Omega_{m+n}\,;\; p_{l_1} = p_{l_2} = \dots = p_{l_k}]\} = 0.$$

Condition (β) is vacuous unless $k \leqslant m + n$.

Proof of (α).

(α_1) If $1 \leqslant j \leqslant m$ the result follows from (iii)$'$;

(α_2) If $m + 1 \leqslant j \leqslant m + n$ the result follows from (iv)$'$.

Proof of (β).

(β_1) If $l_1 \leqslant m < l_k$ the result follows from an easy application of Fubini's theorem combined with (iii)$'$ and (iv)$'$.

(β_2) If $l_k \leqslant m$ [respectively: $m + 1 \leqslant l_1$] the result follows from condition (iii) [respectively: (iv)] and the fact that $\{k\,x\,;\; x \in \mathbf{P}\} = \mathrm{O}_{\mathrm{G}}$.

And that completes the proof of the Lemma.

At this stage it will be necessary to introduce some more notations:

A mapping $\sigma : \Omega_m \to \Omega_m$ will be called a symmetry operation of Ω_m, if there exists $s = s\,(\sigma) \in \mathfrak{S}_m$ the symmetric group of m elements, such that:

$$\sigma\,[(p_j)_{j=1}^{m}] = (q_j)_{j=1}^{m} \in \Omega_m \quad \text{with } q_j = p_{j^s} \quad (m \geqslant 1)$$

$j \to j^s$ being the action of the permutation $s \in \mathfrak{S}_m$.

We shall denote the set of symmetry operations of Ω_m by Σ_m, in (1-1) correspondence with $\mathfrak{S}_m$. Each $\sigma \in \Sigma_m$ induces $\check{\sigma} : \mathrm{M}(\Omega_m) \to \mathrm{M}(\Omega_m)$ a symmetry operation of $\mathrm{M}\,(\Omega_m)$.

A (complex) band $\mathrm{B} \subseteq \mathrm{M}\,(\Omega_m)$ will be called symmetric if

$$\check{\sigma}\,(\mathrm{B}) \subseteq \mathrm{B} \quad (\sigma \in \Sigma_m)\,;$$

we denote by $\mathfrak{S}_m$ the set of all symmetric bands of $\mathrm{M}\,(\Omega_m)$. For $\mathrm{B} \subseteq \mathrm{M}\,(\Omega_m)$ a band we denote by:

$$\mathrm{B}^{\Sigma} = \bigcap_{\mathrm{B} \subset \mathrm{S} \in \mathfrak{S}_m} \mathrm{S} = \{ \check{\sigma}(\beta) ; \quad \beta \in \mathrm{B} ; \quad \sigma \in \Sigma_m \}$$

LEMMA 2.2.

If $\mathrm{B} \subseteq \mathrm{M}(\Omega_m)$ *is a band and* $m \geqslant 2$ *then:*

(i) *If* $x, y \in \mathrm{M}(m\mathrm{P}) \subset \mathrm{M}(\mathrm{G})$, $x \ll y$ *and* $y \in \check{\omega}_m(\mathrm{B})$ *then* $x \in \check{\omega}_m(\mathrm{B})$; *in particular* $\mathcal{R}\, y \in \check{\omega}_m(\mathrm{B})$.

(ii) $\check{\omega}_m^{-1}\{\check{\omega}_m[\mathrm{B}(\mathrm{R}_m^2)\} \cap \mathrm{M}^+(\Omega_m) \subset \mathrm{B}(\mathrm{R}_m^2)$

$\check{\omega}_m^{-1}\{\check{\omega}_m[\mathrm{B}(\Omega_m \setminus \mathrm{R}_m^2)]\} \cap \mathrm{M}^+(\Omega_m) \subset \mathrm{B}(\Omega_m \setminus \mathrm{R}_m^2)$.

(iii) *If* $\alpha, \beta \in \mathrm{B}(\Omega_m \setminus \mathrm{R}_m^2) \cap \mathrm{M}^+(\Omega_m)$ *and* $\check{\omega}_m(\alpha) \ll \check{\omega}_m(\beta)$, *then* $\alpha \in \{\beta\}^{\Sigma} = \{\check{\sigma}(\beta) ; \sigma \in \Sigma_m\}$.

(iv) *If* $\{\gamma_\phi \in \mathrm{B}(\Omega_m \setminus \mathrm{R}_m)\}_{\phi \in \Phi}$ *is a family of measures such that for all* $\varphi \in \Phi$ $\check{\omega}_m(\gamma_\phi) \geqslant 0$ *then there exists a family*

$$\{\delta_\phi \in \mathrm{B}(\Omega_m \setminus \mathrm{R}_m^2) \cap \{\gamma_\phi\}^{\Sigma} \cap \mathrm{M}^+(\Omega_m)\}_{\phi \in \Phi}$$

such that for all $\varphi \in \Phi$ $\check{\omega}_m(\delta_\phi) = \check{\omega}_m(\gamma_\phi)$, *and such that if for* $\varphi, \psi \in \Phi$ $\omega_m(\gamma_\phi) \geqslant \check{\omega}_m(\gamma_\psi)$ *then* $\delta_\phi \geqslant \delta_\psi$.

(v) *If* B *is symmetric and* $\mathrm{B} \subseteq \mathrm{B}(\Omega_m \setminus \mathrm{R}_m^2)$ *then* $\check{\omega}_m(\mathrm{B})$ *is a band of* $\mathrm{M}(m\mathrm{P}) \subseteq \mathrm{M}(\mathrm{G})$.

Proof.

(i) It is an immediate consequence of the fact that $\mathrm{B} \subset \mathrm{M}(\Omega_m)$ is a band (cf. [1], ch. V, § 6, n° 3).

(ii) It is an immediate consequence of $\omega_m^{-1}[\omega_m(\mathrm{R}_m^2)] = \mathrm{R}_m^2$ which follows from Lemma 1.1.

(iii) and (iv) We consider $\bar{\omega}_m$ the restriction of ω_m to $\Omega_m \setminus \mathrm{R}_m^2$:

$$\bar{\omega}_m : \Omega_m \setminus \mathrm{R}_m^2 \to m\mathbf{P}.$$

Then $\Omega_m \setminus \mathrm{R}_m^2$ is « un espace polonais » (cf. [2], § 6, No. 1, Prop. 2 and § 2, No. 9, Prop. 16).

Also applying Lemma 1.1 we see that the conditions of the « Borel cross section theorem » (cf. [2], § 6, No. 8) are verified for the equivalence relation on $\Omega_m \setminus \mathrm{R}_m^2 : x \sim y \Leftrightarrow \bar{\omega}_m(x) = \bar{\omega}_m(y)$. From that we see that we can split $\Omega_m \setminus \mathrm{R}_m^2 = \bigcup_{r \in \mathfrak{S}_m} \mathrm{A}_r$ ($\mathrm{A}_r \subset \Omega_m \setminus \mathrm{R}_m^2$ Borel subset ; $r \in \mathfrak{S}_m$) into m ! Borel subsets such that:

(α) $r \neq s \Longleftrightarrow A_r \cap A_s = \emptyset$.

(β) If $\sigma \in \Sigma_m$ and $s = s(\sigma) \in \mathfrak{S}_m$ is the associated permutation then $\sigma(A_r) = A_{rs}$ (rs being the group product in $\mathfrak{S}_m$).

(γ) For each $s \in \mathfrak{S}_m$ there exists $b_s : \omega_m(\Omega_m \setminus R^2_m) \to A_s$ a Borel function with $\omega_m \circ b_s = 1$ and $b_s \circ (\omega_m \mid_{A_s}) = 1$ (1 being the identity mapping of a space) (Cf. [2], § 6, No. 7 and § 2, No. 10, Prop. 17).

Now let $\mu \in M(\Omega_m \setminus R^2_m)$ be arbitrary; with the above decomposition of the space $\Omega_m \setminus R^2_m$ we associate the orthogonal (Riesz-Lebesgue) decomposition of μ:

$$\mu = \sum_{s \in \mathfrak{S}_m} \mu_s \qquad \text{with } \mu_s \in B(A_s)$$

Observe then that if $\sigma \in \Sigma_m$ and $s = s(\sigma) \in \mathfrak{S}_m$ is the corresponding permutation we have for any $r \in \mathfrak{S}_m$ (using the identification between the spaces A_t ($t \in \mathfrak{S}_m$) induced by the equivalence relation $\sim$) :

$$[\check{\sigma}(\mu)]_r = \mu_{rs^{-1}}.$$

We also denote in general by :

$$\mu^\Sigma = \sum_{\sigma \in \Sigma_m} \check{\sigma}(\mu).$$

Using these notations and observations we see that if $\alpha, \beta \in M^+(\Omega_m)$ are as in (iii) we have for all $r \in \mathfrak{S}_m$:

$$\check{\omega}_m[(\beta^\Sigma)_r] = \sum_{s \in \mathfrak{S}_m} \check{\omega}_m(\beta_s) = \check{\omega}_m(\beta) \gg \check{\omega}_m(\alpha) \gg \check{\omega}_m(\alpha_r)$$

From that using the Borel isomorphisme between A_r and $\omega_m(\Omega_m \setminus R^2_m)$, induced by $\omega_m \mid_{A_r}$ and b_r, as in (γ) we see that: $\alpha_r \ll (\beta^\Sigma)_r$ and therefore also $\alpha \ll \beta^\Sigma \in \{\beta\}^\Sigma$ and that proves (iii).

Also just above, if $\{\gamma_\varphi \in M(\Omega_m)\}_{\varphi \in \Phi}$ is a family as in (iv) we have for any fixed $r \in \mathfrak{S}_m$ and all $\varphi \in \Phi$:

$$\check{\omega}_m[(\gamma_\varphi^\Sigma)_r] = \sum_{s \in \mathfrak{S}_m} \check{\omega}_m[(\gamma_\varphi)_s] = \check{\omega}_m(\gamma_\varphi) \geqslant 0$$

and thus using the Borel isomorphisme $\omega_m \mid_{A_r} \leftrightarrow b_r$ we see that this implies that $(\gamma_\varphi^\Sigma)_r \geqslant 0$ ($\varphi \in \Phi$). It suffices then to set $\delta_\varphi = (\gamma_\varphi^\Sigma)_r$ ($\varphi \in \Phi$) to obtain (iv).

(v). It is an immediate consequences of (i), (ii) and (iv), and of the definition of the band (cf. [1], ch. II).

3. The direct decomposition of M (G).

We introduce some more notations. Let us denote by :

$$T_1 = M_c(\mathbf{P}) = \{m \in M_c(G);\ \operatorname{supp} m \subset \mathbf{P}\}$$

and by :

$$T_n = T_1 \hat{\otimes} T_1 \dots \hat{\otimes} T_1$$

the tensor product of T_1 with itself n times [5]. Also for any $\theta \in T_1'$, the dual space of T_1, we can identify $\theta^n = \theta \otimes \theta \otimes \dots \otimes \theta$ (n times) with an element of $(T_n)'$ the dual space of T_n. We then denote by :

$$S_n = T_n \Big/ \bigcap_{\theta \in T_1'} \operatorname{Ker} \theta^n$$

which is also a Banach space.

Finally for any collection $(B_\alpha)_{\alpha \in A}$ of Banach spaces we shall denote by :

$$B = \bigoplus_{\alpha \in A} B_\alpha = \{b = (b_\alpha)_{\alpha \in A} \in \prod_{\alpha \in A} B_\alpha\ ;\ \sum_{\alpha \in A} ||b_\alpha||_{B_\alpha} < +\infty\}$$

which for the norm

$$||(b_\alpha)_{\alpha \in A}|| = \sum_{\alpha \in A} ||b_\alpha||_{B_\alpha}$$

becomes a Banach space; the direct Banach sum of the $(B_\alpha)_{\alpha \in A}$.

We then observe that

$$T = \bigoplus_{n \geqslant 1} T_n \qquad \text{and} \qquad S = \bigoplus_{n \geqslant 1} S_n$$

can be given a natural Banach algebra structure for which the natural projection :

$$p : T \to S$$

becomes a Banach algebra surjective homomorphisme [: for $t_m \in T_m$ and $t_n \in T_n$ we define $t_m \cdot t_n = t_m \otimes t_n \in T_{m+n}$ and the extend by bilinearity and continuity. We then observe that Ker p is an ideal of T and so we can define a multiplication in S.]

Now the natural identification

$$T_1 = M_c(\mathbf{P}) \to M(G)$$

induces a mapp

$$T_n \to M(G)$$

and also a mapp :

$$\tau : T \to M(G)$$

which is easily seen to be a Banach algebra homomorphisme. Finally if we tensor τ with $i : \Delta(G) \to M(G)$ the natural injection we obtain :

$$\pi = i \hat{\otimes} \tau : \Delta \hat{\otimes} T \to M(G)$$

also a Banach algebra homomorphisme. Observe now that we can identify canonically and isometricaly $\Delta(G) \hat{\otimes} T$ as a Banach space with the direct Banach sum (cf. [5] exposés n^{os} 1 and 4) :

$$\Delta \hat{\otimes} T = \bigoplus_{g \in G;\, n \geqslant 1} \delta_g \mathbf{C} \otimes T_n$$

and let us denote :

$$\pi_n^g = \pi \big|_{\delta_g \mathbf{C} \otimes T_n} \quad \text{and} \quad \pi_n = \pi_n^{0_G} \quad (g \in G, n \geqslant 1)$$

We now state :

LEMMA 3.1.

(i) *For any* $g \in G$ *and* $n \geqslant 1$; $\mathrm{I}m\, \pi_n^g$ *is a (complex) band in* $M(G)$.

(ii) $\Pi = \mathrm{I}m\, \pi \subset M_c(G)$; *and* Π *is a band of* $M(G)$.

(iii) *If* $g_j \in G$, $n_j \in \mathbf{Z}$ $n_j \geqslant 1$ *and* $x_j \in \mathrm{I}m\pi_{n_j}^{g_j}$ *for* $j = 1, 2$; *then :*

$$(g_1, n_1) \neq (g_2, n_2) \implies x_1 \perp x_2$$

(iv) $I = M_c \cap \Pi^\perp = \{m \in M_c(G); \forall\, y \in \Pi, y \perp m\}$ *is an ideal of* $M(G)$

(v) $\operatorname{Ker} \tau = \operatorname{Ker} p \subset T$.

To prove the Lemma (and in general) as we have already said, we shall preserve all the notations already introduced in § 1 and § 2. Before starting with the proof we make some :

Remarks.

(3.i) We can identify T_m with a complex symmetric band of $M(\Omega_m)$ by the natural isometric injection:

$$\varphi_m : T_m \to M(\Omega_m).$$

To see that we just have to observe that for all $x \in T_m$ there exists a family $(\mu_j \in M_c^+(\mathbf{P}))_{j=1}^m$ such that

$$x \in \hat{\bigotimes}_{1 \leqslant j \leqslant m} L_1(\mathbf{P}; \mu_j) = L_1(\Omega_m ; \bigotimes_{1 \leqslant j \leqslant m} \mu_j)$$

(cf. [5] exposés n^{os} 4, 5, 6) and to remark that the natural injection of $L_1(\Omega_m; \bigotimes_{1 \leqslant j \leqslant m} \mu_j)$ into $M(\Omega_m)$ is isometric. Observe also that then $\pi_m = \check{\omega}_m \circ \varphi_m$ $(m \geqslant 1)$.

(3.ii) For all $g \in G$, $m \geqslant l \geqslant 1$ and $t_m \in T_m \subset M(\Omega_m)$ (for the above identification) the sets R_m^l and $D_m^l(g)$ are t_m-null subsets of Ω_m. This is a simple consequence of Fubini's theorem applied to the product space Ω_m, and of the definition of $M_c(P)$.

(3.iii) Observe that for all $g \in G$ and $n \geqslant 1$

$$I m\, \pi_n^g = \delta_g \star I m\, \pi_n.$$

Proof of Lemma 3.1.

(i) By remark (3.iii) we may assume that $g = O_G$, and using then remark (3.i) we see that our result is a consequence of Lemma 2.2 (v).

(ii) and (iii). By remark (3.iii) again, in the proof of (iii) we may assume that $g_1 = O_G$ and $n_1 \geqslant n_2$ (it suffices to translate the two spaces, and interchange them between themselves if need be). Then from Lemma 1.2 since $(O_G, n_1) \neq (g_2, n_2)$, we have:

$$\omega_{n_1}^{-1}(n_1 \mathbf{P} \cap g_2 + n_2 \mathbf{P}) \subseteq R_{n_1}^2 \cup \bigcup_{r \in \Gamma} \bigcup_{1 \leqslant j \leqslant n_1} D_{n_1}^j(g_r);$$

$g_r \in G$, $\operatorname{card} \Gamma < +\infty$ and from that, and remark (3.ii) it follows then that for any $x \in I m\, \pi_{n_1}$ the set $g_2 + n_2 \mathbf{P}$ is x-null and since:

$$y \in I m\, \pi_{n_2}^{g_2} \Longrightarrow \operatorname{supp} y \subset g_2 + n_2 \mathbf{P}$$

we have $x \perp y$ and that completes the proof of (iii). Now to prove (ii) it suffices to set $n_1 = n > 0$ and $n_2 = 0$ in the above argument and obtain:

$$x \in I m\, \pi_n^g \quad \text{and} \quad \delta \in \Delta \Longrightarrow x \perp \delta \tag{3.1}$$

(iv) Since by remark (3.iii) Π and thus also I are translation invariant it suffices that we prove that I is an ideal in $M_c(G)$ and for that it suffices that we prove :

$$\mu, \nu \in M_c^+(G), \quad \mu \perp \Pi \Longrightarrow \mu \star \nu \perp \Pi \tag{3.2}$$

We claim that in fact it suffices to prove (3.2) making the extra assumption

(A) There exists $m, n \in \mathbf{Z}$; $m \geqslant 1$ $n \geqslant 1$ such that :

(A_1) supp $\mu \subset m\mathbf{P}$;

(A_2) supp $\nu \subset n\mathbf{P}$;

(A_3) All the sets $\{g + m'\mathbf{P};\ g \in G,\ 0 \leqslant m' < m\}$ are μ-null;

(A_4) All the sets $\{g + n'\mathbf{P};\ g \in G,\ 0 \leqslant n' < n\}$ are ν-null.

Indeed the family

$$\mathcal{R}(\mathbf{P}) = \{g + r\ \mathbf{P};\ g \in G,\ r \geqslant 0\}$$

generates a Raicov system of sets (cf [3] and [8], § 6) thus :

$$I(\mathbf{P}) = \{x \in M(G);\quad \forall R \in \mathcal{R}(\mathbf{P}) \text{ is } x\text{-null}\}$$

is an ideal of M(G). Therefore we may assume that μ and ν as in (3.2) are orthogonal to $I(\mathbf{P})$. It is an easy matter then to verify, taking into account the translation invariance of Π also Lemma 1.2, that any μ and ν as in (3.2) and orthogonal to $I(\mathbf{P})$ can be decomposed into denumerable orthogonal sums $\mu = \sum_{j=1}^{\infty} \mu_j$ and $\nu = \sum_{j=1}^{\infty} \nu_j$ of components which after appropriate translation satisfy (A). (For some m, n depending on the component of course).

Now with the assumption (A) on μ and ν holding for some $m, n \in \mathbf{Z}$ $m \geqslant 1$, $n \geqslant 1$; we see at once :

(α) $\mu \star \nu \perp I\,m\,\pi_r^g$ if $g \in G$, $r > m + n$ (cf. proof of (iii) above).

(β) $\mu \star \nu \perp I\,m\,\pi_r^g$ if $g \in G$, $r < m + n$ by Lemma 2.1 and (A).

(γ) $\mu \star \nu \perp I\,m\,\pi_{m+n}^g$ if $g \in G$, $g \neq O_G$ either by Lemma 2.1 and (A) or by the proof of (iii) above. Thus it only remains for us to verify :

$$(\delta)\ \mu \star \nu \perp I\,m\,\pi_{m+n}.$$

We proceed to prove (δ). Towards that for the projections :

$$\check{\omega}_m :\ M(\Omega_m) \to M(m\,\mathbf{P}),$$

$$\check{\omega}_n : \mathrm{M}(\Omega_n) \to \mathrm{M}(n\,\mathbf{P}),$$
$$\check{\omega}_{m+n} : \mathrm{M}(\Omega_{m+n}) \to \mathrm{M}\,[(m+n)\,\mathbf{P}]$$

We choose some $\bar{\mu} \in \mathrm{M}^+(\Omega_m)$ and $\bar{\nu} \in \mathrm{M}^+(\Omega_n)$ such that: $\check{\omega}_m(\bar{\mu}) = \mu$; $\check{\omega}_n(\bar{\nu}) = \nu$ therefore also $\check{\omega}_{m+n}(\bar{\mu} \otimes \bar{\nu}) = \mu \star \nu$ and $\bar{\mu} \perp \mathrm{T}_m$. Now to prove ($\delta$) we must show that for all $t \in \mathrm{T}_{m+n}$ we have $\mu \star \nu \perp \pi_{m+n}(t)$; and to see that last fact it suffice to prove:

$$\theta \in \mathrm{M}^+(\Omega_{m+n});\quad \check{\omega}_{m+n}(\theta) \ll \mu \star \nu \Longrightarrow \theta \perp \mathrm{T}_{m+n} \tag{3.3}$$

But since $\bar{\mu} \perp \mathrm{T}_m$ we have:

$$\bar{\mu} \otimes \bar{\nu} \perp \mathrm{T}_{m+n} = \mathrm{T}_m \hat{\otimes} \mathrm{T}_n \subset \mathrm{M}(\Omega_{m+n}) \tag{3.4}$$

and since $\check{\omega}_{m+n}(\bar{\mu} \otimes \bar{\nu}) = \mu \star \nu$ we see from Lemma 2.2 that:

$$\theta \in \mathrm{M}^+(\Omega_{m+n})\,;\ \check{\omega}_{m+n}(\theta) \ll \mu \star \nu \Rightarrow \theta \in \{\bar{\mu} \otimes \bar{\nu}\}^{\Sigma} + \mathrm{B}(\mathrm{R}^2_{m+n}).$$

But $\mathrm{B}(\mathrm{R}^2_{m+n}) \perp \mathrm{T}_{m+n}$; and since $\mathrm{T}^{\Sigma}_{m+n} = \mathrm{T}_{m+n}$ we see that:

$$\bar{\mu} \otimes \bar{\nu} \perp \mathrm{T}_{m+n} \Rightarrow \{\bar{\mu} \otimes \bar{\nu}\}^{\Sigma} \perp \mathrm{T}_{m+n}$$

thus by (3.4) we have:

$$\{\bar{\mu} \otimes \bar{\nu}\}^{\Sigma} + \mathrm{B}(\mathrm{R}^2_{m+n}) \perp \mathrm{T}_{m+n}$$

and that combined with (3.5) proves (3.3) and complets the proof.

(v) Taking (iii) into account we see that to prove (v) it suffices to prove that for all $n \in \mathbf{Z}$ $n \geqslant 1$

$$\mathrm{Ker}\ \pi_n = \bigcap_{\theta \in \mathrm{T}_1'} \mathrm{Ker}\ \theta^n \subset \mathrm{T}_n$$

We prove this in two stages:

$$(\alpha) \qquad \mathrm{Ker}\ \pi_n = \bigcap_{f \in \mathbf{C}(\mathbf{P}) \subset \mathrm{T}_1'} \mathrm{Ker}\, f^n \qquad (n \geqslant 1)$$

$$(\beta) \qquad \bigcap_{f \in \mathbf{C}(\mathbf{P}) \subset \mathrm{T}_1'} \mathrm{Ker}\, f^n = \bigcap_{\theta \in \mathrm{T}_1'} \mathrm{Ker}\ \theta^n \qquad (n \geqslant 1)$$

To prove (α) and (β) we fix $n \in \mathbf{Z}$ $n \geqslant 1$ once and for all.

(α) Let $x \in \bigcap_{f \in \mathbf{C}(\mathbf{P}) \subset \mathrm{T}_1'} \mathrm{Ker}\, f^n$ and set for all $\chi \in \widehat{\mathrm{G}}$

$$f_\chi = \chi|_{\mathbf{P}} \in \mathbf{C}(\mathbf{P})$$

Then we have:

$$0 = \langle x, f^n_\chi \rangle = \langle x, \chi \circ \omega_n \rangle = \langle \pi_n(x), \chi \rangle = [\pi_n(x)]^\wedge(\chi) \tag{3.6}$$

and χ being arbitrary we deduce that $\pi_n(x) = 0$ and $x \in \operatorname{Ker} \pi_n$. Conversely let $x \in \operatorname{Ker} \pi_n \subset T_n$. Then for all $f \in \mathbf{C}(\mathbf{P})$ there exists ψ_f a bounded function on $n\mathbf{P} \subset G$ (ψ_f can in fact be assumed Borelian, but this is not essential) such that:

$$f^n\big|_{\Omega_n \setminus R_n^2} \equiv \psi_f \circ \omega_n\big|_{\Omega_n \setminus R_n^2}$$

and since by remark (3.iii) R_n^2 is an x-null set we have for all $f \in \mathbf{C}(\mathbf{P})$:

$$\langle x, f^n \rangle = \langle x, \psi_f \circ \omega_n \rangle = \langle \pi_n(x), \psi_f \rangle = 0$$

therefore also $x \in \bigcap_{f \in \mathbf{C}(\mathbf{P}) \subset T_1'} \operatorname{Ker} f^n$. And that complets the proof of (α).

(β) We shall prove that:

$$\bigcap_{f \in \mathbf{C}(\mathbf{P}) \subset T_1'} \operatorname{Ker} f^n \subseteq \bigcap_{\theta \in T_1'} \operatorname{Ker} \theta^n$$

the inclusion the other way is obvious. Towards that let us fix

$$x \in \bigcap_{f \in \mathbf{C}(\mathbf{P}) \subset T_1'} \operatorname{Ker} f^n \quad \text{and prove that} \quad x \in \bigcap_{\theta \in T_1'} \operatorname{Ker} \theta^n.$$

Now it is well-known that for any $\mu \in M(\mathbf{P})$ the unit ball of $\mathbf{C}(\mathbf{P})$ ($\subseteq L^\infty(\mathbf{P}; \mu)$) is dence in the unit ball of $L^\infty(\mathbf{P}; \mu)$ for the weak topology $\sigma[L^\infty(\mathbf{P}; \mu); L_1(\mathbf{P}; \mu)]$. From that it follows by decomposing $M_c(\mathbf{P}) = \bigoplus L_1(\mathbf{P}; \mu_\alpha)$ into orthogonal bands, that the unit ball of $\mathbf{C}(\mathbf{P})$ ($\subseteq [M_c(\mathbf{P})]'$) is dence in $[M_c(\mathbf{P})]_1'$ the unit ball of $[M_c(\mathbf{P})]' = T_1'$, for the weak topology $\sigma(T_1'; T_1)$.

So for any $\theta \in T_1'$ there exists a net $\{f_\nu \in \mathbf{C}(\mathbf{P}) \subset T_1\}_{\nu \in N}$ such that: $\|f_\nu^n\|_{\mathbf{C}(\mathbf{P})} \leqslant \|\theta\|_{T_1'}$; $f_\nu \underset{\nu \in N}{\rightarrow} \theta$ for the topology $\sigma(T_1'; T_1)$ for that net it follows that: $f_\nu^n \underset{\nu \in N}{\rightarrow} \theta^n$ for the weak topology $\sigma(T_n'; T_n)$ (e.g. use the explicit expression of elements of T_n; *cf* [5], exposés n^{os} 5 et 6). Thus since $\langle x, f_\nu^n \rangle = 0$ ($\nu \in N$) we obtain $\langle x, \theta^n \rangle = 0$ and θ being arbitrary we see that we have in fact proved the required result that

$$x \in \bigcap_{\theta \in T_1'} \operatorname{Ker} \theta^n.$$

with this, the proof of Lemma 3.1 is complete.

Now using Lemma (3.1) (v) we see that τ induces an injection $j: S \rightarrow M(G)$ which if tensored with $i \cdot \Delta \rightarrow M(G)$ gives:

$$k = i \hat{\otimes} j : \Delta \hat{\otimes} S \rightarrow M(G).$$

And Lemma 3.1 implies then that k identifies topologically $\Delta \hat{\otimes} S$ with $\Pi = \operatorname{Im} \pi = \operatorname{Im} k$. We are now able to state:

THEOREM D (DECOMPOSITION).

To every **P**, *perfect, metrisable strongly independent subset of* G, *there corresponds a canonical topological and algebraic identification of the Banach algebra* $\Delta(G) \hat{\otimes} S$ *with a closed subalgebra* $\Pi \subset M(G)$.

Then Π *is a band of* $M(G)$, *and* $I = \Pi^{\perp} \cap M_c(G)$ *is an ideal of* $M(G)$, *and we have the direct and orthogonal* (Riesz-Lebesgue) *decompositions:*

$$D(\mathbf{P}): \; M_c(G) = \Pi \oplus I \,; \; M(G) = L \oplus I \,; \; L = \Delta(G) \oplus \Pi$$

The closed subalgebra $L \subseteq M(G)$ *can then be identified, topologically and algebraically in a canonical fashion with the Banach algebra* $\Delta(G) \hat{\otimes} \tilde{S}$.

Remark (3 iv).

The identification of L and $\Delta \hat{\otimes} \tilde{S}$ is obtained by:

$$L = \Delta \oplus \Pi \cong \Delta \oplus (\Delta \hat{\otimes} S) \cong \Delta \hat{\otimes} (S \oplus 1\, \mathbf{C}) = \Delta \hat{\otimes} \tilde{S}.$$

4. Applications.

For our applications we shall need to couple Theorem D with the following previous result of ours [8].

If G *is a non discrete locally compact abelian group then:*

(i) *There exists* $\mathbf{P} \subset G$ *a perfect, metrisable strongly independent subset.*

(ii) *If in addition* G *is metrisable we may assume that* **P** *is as in* (i) *and such that:*

$$M_0(\mathbf{P}) = \{m \in M_0(G) \,; \quad \operatorname{supp} m \subset \mathbf{P}\} \neq \{0\}.$$

Remark.

(4.i) In [8] we prove Theorem R (ii); (i) follows from that by considering a metrisable non discrete subgroup $H \subset G$ (cf. [4], 2.4, 2.5.2).

(4.ii) If $\mathbf{P}$ is as in (i) then $M_c(\mathbf{P})$ is a nom separable Banach space. This is seen using simple arguments of general topology and Radon measure theory (cf. [8] Lemma 7.1 and Remark (7.iii)).

(4.iii) If $\mathbf{P}$ is as in (ii) then $M_0(\mathbf{P})$ is an infinite dimensional Banach space, since for any $\mu \in M_0(\mathbf{P})$ $(\subseteq M_c(\mathbf{P}))$

$$M_0(\mathbf{P}) \supseteq L_1(\mathbf{P}, \mu).$$

Application I.

Proof of Theorem N.F.

To see parts (i) and (iii), and the special case of part (ii) when G is metrisable, of the Theorem N.F., it suffices to combine Theorem D, Theorem R, Remarks (4.ii) and (4.iii) and the simple observation that $\overline{(\Delta \hat{\otimes} S)^2} \subset \Delta \hat{\otimes} S$ is a direct summand such that:

$$\Delta \hat{\otimes} S = \overline{(\Delta \hat{\otimes} S)^2} \oplus [\Delta \hat{\otimes} M_c(\mathbf{P})].$$

(We use also the fact that $M_0(G)$ is a translation invariant band.)

Now to prove part (ii) of Theorem N.F. for a general non discrete locally compact abelian group we consider $H \subset G$ a compact subgroup such that G/H is metrisable and non discrete (cf. [9], § 1, p. 450). Then the natural projection $p : G \to G/H$ induces (cf.]1[, ch. V, § 6) a Banach algebra homomorphisme $\check{p} : M(G) \to M(G/H)$ such that

$$\check{p}(M_0(G)) = M_0(G/H)$$

(that last point is immediate since H is compact (cf. [1], ch. VII).) From that we see at once that since $M_0(G/H) / \overline{[M_0(G/H)]^2}$ is infinite dimensional so is $M_0(G) / \overline{(M_0(G))^2}$ which completes the proof.

Before giving our next application we make:

Remark (4.iv) It is trivial to verify that if R_1 and R_2 are two commutative Banach algebras with identity then we can identify canonically $\mathcal{M}(R_1 \hat{\otimes} R_2) = \mathcal{M}(R_1) \times \mathcal{M}(R_2)$; for that identification, it is seen at once that $\Sigma(R_1) \times \Sigma(R_2) \subset \Sigma(R_1 \hat{\otimes} R_2)$.

(That last inclusion in fact is never strict, and we have always $\Sigma(R_1) \times \Sigma(R_2) = \Sigma(R_1 \hat{\otimes} R_2)$; but that last point is not quite trivial and will not be needed).

Application II.

(i) For any $\mathbf{P} \subset G$ using the decomposition $D(\mathbf{P})$ we can identify canonically $\mathcal{M}(\Delta \hat{\otimes} S)$ with a closed subset of $\mathcal{M}[M(G)]$.

(ii) Using Remark (4.iv) we can identify canonically

$$\mathcal{M}(\Delta \hat{\otimes} \tilde{S}) = \Gamma \times \mathcal{M}(\tilde{S})$$

where Γ is the Bohr compactification of $\hat{G}$.

(iii) We leave it to the reader to verify that every $\theta \in [M_c(\mathbf{P})]_1'$ (for notation cf. Proof of Lemma 3.1 (v)) induces canonically a multiplicative linear form on $\tilde{S}$. (θ induces canonically a multiplicative linear form θ^∞ on $T = \bigoplus_{n \geqslant 1} T_n$ by setting $\theta^\infty = \bigoplus_{n \geqslant 1} \theta^n$, we have to verify that $\operatorname{Ker} \theta^\infty \supseteq \operatorname{Ker} p$ which is immediate). The above correspondence defines a topological canonical identification between $\mathcal{M}(\tilde{S})$ and $[M_c(\mathbf{P})]_1'$ (The unit ball $[M_c(\mathbf{P})]_1'$ is topologised with the weak topology $\sigma(T_1', T_1)$).

(iv) We have $\mathcal{M}(\tilde{S}) = \Sigma(\tilde{S})$ and thus, by Remark (4.iv),

$$\mathcal{M}(\Delta \hat{\otimes} \tilde{S}) = \Sigma(\Delta \hat{\otimes} \tilde{S}).$$

We do not give detailed verification of the above statements (and in particular no proof of (iv)) because they were proved directly in the particular case $k(\mathbf{P}) = +\infty$ (and G an I-Group) by A. B. Simon [6], [7]. So we are confident that the reader after consulting [7] will have no difficulty to fill in the gaps for himself.

There are a number of other applications that can be obtained by specialising **P**, we shall examine them in a future publication. At this stage we content ourselves (preserving all our previous notations) to state, and give only a few indications of the proof a particularly simple one:

Application III.

Let G be a compact metrisable abelian group and **P** be a Kronecker or a K_p ([4] 5.1.2) subset then:

(i) $M_0(G) \subseteq I$.

(ii) The decomposition D (**P**) induces canonically a direct decomposition:

$$M/M_0 = L \oplus (I/M_0).$$

(iii) If G is a non discrete locally compact abelian group the natural involution $\mu \to \mu^* = \overline{\mu(-x)}$ of M (G) induces an involution in M/M_0 for which it becomes a non symmetric algebra.

Indication of Proof.

(i) $\to$ (ii) $\to$ (iii) almost trivially.

Proof of (i) : Taking into account Remark (3.iii) and Lemma 3.1 and also the fact that $M_0(G)$ is a translation invariant band, we see that suffices to show that $\operatorname{Im} \pi_n \cap M_0(G) = \{0\}$ for $n \geqslant 1$ (For $n = 1$ this fact is well-known cf. [4], 5.6.10).

Now let $x \in T_n$ be such that that $\pi_n(x) = \mu \in M_0(G)$ and let us assume that **P** is a Kronecker set. Then if $f \in \mathbf{C}(\mathbf{P})$ and $|f| \equiv 1$ approximating uniformly f on **P** by a net of characters $(\chi_\nu \in \hat{G})_{\nu \in N}$ such that $\chi_\nu \underset{\nu \in N}{\to} \infty$, we see that (cf. equation (3.6)) $\langle x, f^n \rangle = 0$. From that it can be deduced that $\mu = \pi_n(x) = 0$ (cf. Proof of Lemma 3.1 (v). We use the fact that $\{f \in \mathbf{C}(\mathbf{P}) ; |f| \equiv 1\}$ is dence in $[M_c(\mathbf{P})]_1'$ for the topology $\sigma(T_1' ; T_1)$).

One major disadvantage of the decomposition D (**P**) is that if $k(\mathbf{P}) > 2$ it is not symmetric (not stable by the involution $\mu \to \mu^* = \overline{\mu(-x)}$ of the algebra M (G) i.e. $I^* \neq I$ and $\Pi^* \neq \Pi$ (if $k(P) = 2$ then it is symmetric since $\mathbf{P} = -\mathbf{P}$). This can be amended at once, if both **P** and $-$**P** are considered at the same time. More explicitly, let the decompositions associated to **P** and $-$**P** be:

$$D(\mathbf{P}) : M_c(G) = \Pi \oplus I \,;\; M(G) = L \oplus I \,;\; L \cong \Delta \hat{\oplus} \tilde{S}$$

$$D(-\mathbf{P}) : M_c(G) = \Pi^- \oplus I^- ;\; M(G) = L^- \oplus I^- ;\; L^- \cong \Delta \hat{\oplus} \tilde{S}^-$$

then we have:

$$\Pi^* = \Pi^- \; ; \; I^* = I^- \; ; \; L^* = L^-$$

and we have:

THEOREM D_8 (SYMMETRIC DECOMPOSITION).

The subalgebra $\mathbf{K} = L \cdot L^- \subset M(G)$ *is a closed symmetric subalgebra and if* $k(\mathbf{P}) > 2$ *it can be identified topologically and algebraically, in a canonical fashion with* $\Delta \hat{\otimes} \tilde{S} \hat{\otimes} \tilde{S}^-$. *Also we have a direct and orthogonal (Riesz-Lebesgue) decomposition:*

$$\mathbf{D}_8(\mathbf{P}) : \; M(G) = \mathbf{K} \oplus \mathbf{J}$$

where $\mathbf{J}$ *is an ideal (for that last fact when* $k(\mathbf{P}) = +\infty$ *and* G *an* I-*group,* cf. [6]).

The proof of Theorem D_8 is very similar to that of Theorem D, and does not involve any new ideas; the details however are much more complicated and tedious to expose, since furthermore the main application of $D_8(\mathbf{P})$ (for the important special cases of I-groups) has been obtained directly in [7]; writting down the proof of Theorem D_8 would serve no great purpose, and anyway, is a task beyond the literary capacity of the author.

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