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FINITE DETERMINACY OF DICRITICAL SINGULARITIES IN $(\mathbb{C}^2, 0)$

by Gabriel CALSAMIGLIA-MENDLEWICZ (*)

Abstract. — For germs of singularities of holomorphic foliations in $(\mathbb{C}^2, 0)$ which are regular after one blowing-up we show that there exists a functional analytic invariant (the transverse structure to the exceptional divisor) and a finite number of numerical parameters that allow us to decide whether two such singularities are analytically equivalent. As a result we prove a formal-analytic rigidity theorem for this kind of singularities.

Résumé. — Nous montrons l'existence d'un invariant analytique fonctionnel (la structure transverse au diviseur exceptionnel) et d'un nombre fini de paramètres numériques associés aux germes de feuilletages holomorphes dans $(\mathbb{C}^2, 0)$ qui ne présentent pas de singularités après un éclatement. Ceux-ci permettent de décider si deux telles singularités sont analytiquement équivalentes. On dérive ensuite un théorème de rigidité formelle-analytique pour ce type de singularité.

1. Introduction

Given a holomorphic germ of 1-form ω in $(\mathbb{C}^2, 0)$ with an isolated zero at the origin we can define its associated singular foliation by holomorphic curves $\mathcal{F}_\omega$: the origin 0 is the singular set and its leaves are the integral curves of ω outside 0. Let $E : \widetilde{\mathbb{C}}^2 \longrightarrow (\mathbb{C}^2, 0)$ denote the quadratic blow up at the origin expressed in coordinates by $E(t, x) = (x, tx) = (X, Y)$, and E_0 its exceptional divisor corresponding to the set $\{x = 0\}$ in the chart (t, x). It is well known that $E^*(\omega)$ defines a regular foliation in the complement of E_0 which can be uniquely extended to a holomorphic foliation $\widetilde{\mathcal{F}_\omega}$ in a neighborhood of E_0 in $\widetilde{\mathbb{C}}^2$ with a finite set of isolated singularities on E_0. Let $\mathcal{D}_0$ denote the set of foliations $\mathcal{F}_\omega$ such that $\widetilde{\mathcal{F}_\omega}$ is a *regular* foliation. We

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are interested in describing the space of analytic equivalence classes of $\mathcal{D}_0$. The index theorem in ([3], p.592) forces E_0 to be generically transverse to $\widetilde{\mathcal{F}_\omega}$, so $\mathcal{F}_\omega$ has a dicritical singularity. However, there is a finite set of points $\Sigma_{\mathcal{F}_\omega} \subset E_0$ corresponding to the points $p \in E_0$ where the leaf of $\widetilde{\mathcal{F}_\omega}$ through p is tangent to the curve E_0 with contact order $r(p)+1$. A result by Klughertz [7] asserts that topologically there aren't any other invariants:

Theorem 1. — *Given $\mathcal{F}, \mathcal{F}' \in \mathcal{D}_0$, there exists a homeomorphism $\Psi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ sending leaves of $\mathcal{F}$ to leaves of $\mathcal{F}'$ if and only if there exists a bijection $\psi : \Sigma_{\mathcal{F}} \to \Sigma_{\mathcal{F}'}$ such that $r(\psi(p)) = r(p)$ for all $p \in \Sigma_{\mathcal{F}}$.*

In other words the partition of $\mathcal{D}_0$ into subsets whose elements are topologically equivalent can be described as $\mathcal{D}_0(n; r_1, \dots, r_n)$ where $n \in \mathbb{N}$ denotes the number of points of tangency, and $r_1, \dots, r_n \in \mathbb{N}^*$ their orders of tangency when $n \neq 0$. The case $n = 0$ is solved by Poincaré's linearization theorem: every $\mathcal{F} \in \mathcal{D}_0(0)$ is analytically equivalent to the radial foliation $\mathcal{F}_{Y\mathrm{d}X - X\mathrm{d}Y}$. Suzuki's example (see Section 2 below or [13]) shows that there are two elements in $\mathcal{D}_0(1; 1)$ which are not analytically equivalent. The obstruction is related to the following analytic invariant: given $\mathcal{F} \in \mathcal{D}_0(n; r_1, \dots, r_n)$, consider for each $p_i \in \Sigma_{\mathcal{F}}$ a local holomorphic first integral $F_i : U_i \to \mathbb{C}$ of $\widetilde{\mathcal{F}}$ in a neighborhood U_i of p_i in $\widetilde{\mathbb{C}}^2$.

Define $f_i = F_{i|U_i \cap E_0}$. The *group of invariance of $\mathcal{F}$ at p_i* is

$$H(\mathcal{F}, p_i) = \big\{ h \in \mathrm{Diff}(E_0, p_i) \mid f_i \circ h = f_i \big\}.$$

It is a cyclic group of germs of order $r(p_i) + 1$. We define the *transverse structure of $\mathcal{F}$* as

$$H(\mathcal{F}) = \bigcup_{p \in \Sigma_{\mathcal{F}}} H(\mathcal{F}, p).$$

Observe that if $\Psi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is a biholomorphism then $\Psi^*(\mathcal{F}) \in \mathcal{D}_0$ and the restriction $\psi \in \mathrm{Aut}(E_0)$ of Ψ to E_0 defines a bijection

$$h \longmapsto \psi^{-1} \circ h \circ \psi$$

from $H(\mathcal{F})$ to $H(\Psi^*(\mathcal{F}))$. We thus define the *projective class of the transverse structure* $H[\mathcal{F}]$ as the conjugacy class of $H(\mathcal{F})$ by holomorphic automorphisms of E_0, which are just Möbius transformations. The previous argument shows that $H[\mathcal{F}]$ depends only on the analytic class $[\mathcal{F}]$ of $\mathcal{F}$, and also that if $\Psi^*(\mathcal{F}) = \mathcal{F}'$, then up to linear conjugacy we can suppose $H(\mathcal{F}^{'}) = H(\mathcal{F})$. On the other hand, the fact that elements in $\mathcal{D}_0$ can be constructed using foliated surgery techniques (and Grauert's Theorem) allows us to realize any finite union of cyclic groups of germs of diffeomorphisms of (E_0, p) at points $p \in E_0$ of finite order as the transverse structure of an element of $\mathcal{D}_0$.

A natural question is to decide whether the projective class of the transverse structure determines the analytic class of the foliation completely. In the case of $\mathcal{D}_0(1;1)$ the answer is positive:

Theorem 2 (Cerveau). — *Given $\mathcal{F}, \mathcal{F}' \in \mathcal{D}_0(1;1)$, there exists a germ of biholomorphism $\Psi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ with $\Psi^*(\mathcal{F}') = \mathcal{F}$ if and only if $H[\mathcal{F}] = H[\mathcal{F}']$.*

In the remaining cases we provide examples showing that there are elements $\mathcal{F}, \mathcal{F}' \in \mathcal{D}_0$ with $H(\mathcal{F}) = H(\mathcal{F}')$ and $\# H(\mathcal{F}) > 2$ arbitrary which are not analytically equivalent. Our main result states that, apart from the projective class of the transverse structure, there are at most a finite number of analytic invariants of numerical nature in each topological class $\mathcal{D}_0(n; r_1, \dots, r_n)$:

Theorem 3. — *Let ω, ω' be two holomorphic 1-forms in $(\mathbb{C}^2, 0)$ defining foliations $\mathcal{F}, \mathcal{F}' \in D_0(n; r_1, \dots, r_n)$ respectively. Define*

$$N := r_1 + \dots + r_n, \quad \kappa := (N+1) + \max\{r_i\}(3N-2).$$

Suppose

(i) *$H(\mathcal{F}) = H(\mathcal{F}')$;*

(ii) *the jets of ω and ω' at 0 satisfy $j_0^\kappa(\omega) = j_0^\kappa(\omega')$.*

Then there exists a biholomorphism $\Psi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that

$$\Psi^*(\mathcal{F}') = \mathcal{F} \quad \textit{and} \quad \mathrm{d}\Psi(0,0) = \mathrm{Id}.$$

As a corollary we get a theorem of formal-analytic rigidity in $\mathcal{D}_0$:

Corollary 4. — *Two elements in $\mathcal{D}_0$ are formally equivalent if and only if they are analytically equivalent.*

Since the algebraic multiplicity of the elements in $\mathcal{D}_0(n; r_1, \dots, r_n)$ is $N+1$, we can state Theorem 3 in terms of the algebraic multiplicity instead of κ. This theorem can be reinterpreted in the following way: fix a 1-form ω such that $\mathcal{F}_\omega \in \mathcal{D}_0$. Consider the set

$$\mathcal{D}_0[\mathcal{F}_\omega] := \bigl\{\omega' \mid \mathcal{F}_{\omega'} \in \mathcal{D}_0 \text{ and } H(\mathcal{F}_\omega) = H(\mathcal{F}_{\omega'})\bigr\}. \tag{1.1}$$

The assertion is that each fiber of the map

$$j_0^\kappa : \mathcal{D}_0[\mathcal{F}_\omega] \longrightarrow \mathbb{C}^M \quad \text{where} \quad M := \binom{\kappa+1}{2} - \binom{N+1}{2}$$

defines a unique equivalence class in $\mathcal{D}_0$. Different fibers might define the same class or not. Nevertheless we have at most $\mathbb{C}^M$ different analytic classes with the same transverse structure.

A theorem of finite determinacy of a similar type was proven by Klughertz [7] in her doctoral thesis. Our appoach improves the order of the jet involved (in her statement the dependence is quadratic on N). On the other hand the methods used for the proof differ. We will *geometrically* construct a biholomorphism by choosing adequate generically transverse auxiliary foliations, whereas Klughertz used the *cohomological* methods developped in [8] to find a biholomorphism which is tangent to the identity up to a certain order.

Ortiz-Bobadilla, Rosales-Gonzalez and Voronin [10] have recently proved a formal-analytic rigidity theorem in $\mathcal{D}_0(n;1,\dots,1)$, after what formal normal forms are constructed for the analytic classes in $\mathcal{D}_0(n;1,\dots,1)$, and formal invariants are identified from this normal form. Again, the process lacks of a geometric interpretation, and we hope that our approach will shed some light on the problem of identifying the invariants and giving them a geometrical meaning, eventually enabling us to construct normal forms for every analytic class in $\mathcal{D}_0$.

Finally, we want to remark that a finite determinacy theorem for generalized cusps can be proven as a consequence of Theorem 3 and that the proof we present of the latter can be generalized to prove a theorem of finite determinacy for regular germs of holomorphic foliations defined in a neighborhood of the zero section of a Hopf component of negative auto-intersection having the same transverse structure (for a proof of these results see [1]). Similarly, Theorem 2 can be generalized to give the analytic classification of regular germs of foliations in a neighborhood of the zero section of a line bundle $L \to E$ over a rational or elliptic Riemann surface E having a single simple tangency with the curve E, provided that $c_1(L) < -1$. Both generalizations, and the method for the proof of these results, are inspired by the paper [2].

The content of the article is organized as follows: in Section 2 we provide examples of singularities in $\mathcal{D}_0$ that are not analytically equivalent; in Section 3 we prove Theorem 3 and in Section 4 we prove Corollary 4.

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2. Examples

Let us first establish some definitions that will be used throughout this article. For $\mathcal{F} \in \mathcal{D}_0$ we define $\Sigma_{\mathcal{F}}$ as the set of points $p \in E_0$ where the leaf $\widetilde{L}_p$ of $\widetilde{\mathcal{F}}$ is tangent to E_0 at p. The order of contact at p will be denoted by $r(p)+1$; remark that $r(p)$ is the order of the zero of the normal component to E_0 of the local 1-form defining $\widetilde{\mathcal{F}}$ at p. Define

$$\mathrm{Sep}(\widetilde{\mathcal{F}}) = \bigcup_{p \in \Sigma_{\mathcal{F}}} \widetilde{L}_p$$

as the set of *isolated separatrices*. The blow down of $\mathrm{Sep}(\widetilde{\mathcal{F}})$ by E will be denoted as $\mathrm{Sep}(\mathcal{F})$. The latter is a union of germs of generalized cusps. Observe that the foliation $\widetilde{\mathcal{F}}$ is the minimal resolution of $\mathcal{F}$. Nevertheless we will repeatedly use a different resolution $S_{\mathcal{F}}$ that we will call *extended resolution* which corresponds to the resolution of $\mathrm{Sep}(\mathcal{F})$ in the sense of (reducible) curves. It is the result of composing E with $S_{(p,r(p)+1)}$ at each point $p \in \Sigma_{\mathcal{F}}$, where $S_{(p,r(p)+1)}$ is defined inductively by the rules: $S_{(p,1)}$ is the blowing-up of the point p, and

$$S_{(p,i)} = S_{(p,i-1)} \circ S_{(\hat{p}_i,1)} \quad \text{where} \quad \widehat{p}_i = S^{-1}_{(p,i-1)}(p) \cap \overline{S^{-1}_{(p,i-1)}(E_0 \setminus p)}.$$

For each p we have $r(p)+1$ irreducible components $E_1^p, \dots, E^p_{r(p)+1}$ of the divisor $\mathcal{D}_{\mathcal{F}}$ associated to $S_{\mathcal{F}}$. The strict transform of each irreducible component of $\mathrm{Sep}(\mathcal{F})$ by $S_{\mathcal{F}}$ intersects transversely exactly one irreducible component of $\mathcal{D}_{\mathcal{F}}$. We call $\widehat{\mathcal{F}} = S^*_{\mathcal{F}}(\mathcal{F})$ the pull back of $\mathcal{F}$ by $S_{\mathcal{F}}$.

Next observe that for a point $p \in E_0$ we can find a neighborhood $U_p \subset \widetilde{\mathbb{C}}^2$ and a *local* biholomorphism $\Phi_p : (U_p, p) \to (\mathbb{C}^2, 0)$ which we call *normalizing chart of $\widetilde{\mathcal{F}}$ at p* such that $(u,v) = \Phi_p(t,x) = (\Phi_1(t,x), \Phi_2(t,x))$ with $\Phi_2(t,0) \equiv 0$ and such that $(\Phi_p^{-1})^*(\widetilde{\mathcal{F}}_{|U_p})$ can be described as the levels of the function $f_p(u,v) = v - u^{r(p)+1}$. It is important to remark that the change of coordinates is local. With this at hand it is obvious that for a point $p \in \Sigma_{\mathcal{F}}$ the group of invariance $H(\mathcal{F}, p)$ is cyclic of order $r(p)+1$; in fact $\Phi_1(t,0)$ conjugates it with the group of rotations of order $r(p)+1$. In particular if we choose any two elements $\mathcal{F}, \mathcal{F}' \in \mathcal{D}_0(n; r_1, \dots, r_n)$ with $\Sigma_{\mathcal{F}} = \{p_1, \dots, p_n\}$ and $\Sigma_{\mathcal{F}'} = \{p'_1, \dots, p'_n\}$, $r(p_i) = r(p'_i)$, we can find *germs* of biholomorphism $\psi_i : (E_0, p_i) \to (E_0, p'_i)$ conjugating $H(\mathcal{F}, p_i)$ with $H(\mathcal{F}', p'_i)$. However, in general, there does not exist an automorphism ψ of E_0 whose restriction to a neighborhood of p_i is ψ_i for $i = 1, \dots, n$, even in the case $n = 1$. In these cases $\mathcal{F}$ and $\mathcal{F}'$ cannot be analytically conjugated, for the existence of an equivalence would imply the existence of a ψ with the said properties. Suzuki's example is an instance of this phenomenon: define

$$\omega = (2Y^2 + X^3)\mathrm{d}X - 2XY\,\mathrm{d}Y,$$
$$\omega' = (Y^3 + Y^2 - XY)\mathrm{d}X - (2XY^2 + XY - X^2)\mathrm{d}Y.$$

They have first integrals

$$f(X,Y) = \frac{Y^2 - X^3}{X^2}, \quad f'(X,Y) = \frac{X}{Y}\mathrm{e}^{Y(Y+1)/X}$$

respectively, and define foliations $\mathcal{F}_\omega, \mathcal{F}_{\omega'} \in \mathcal{D}_0(1;1)$. In the (t,x) chart of $\widetilde{\mathbb{C}}^2$ we have $\Sigma_{\mathcal{F}_\omega} = \{(0,0)\}$ and $\Sigma_{\mathcal{F}_{\omega'}} = \{(1,0)\}$; $H(\mathcal{F}_\omega,(0,0)) = \langle h\rangle$ with $h(t) = -t$ and $H(\mathcal{F}_{\omega'},(1,0)) = \langle h'\rangle$. Consider the maps

$$H : (\mathbb{C},0) \longrightarrow \mathbb{C}^2, \quad t \longmapsto \big(t, h(t)\big),$$
$$H' : (\mathbb{C},1) \longrightarrow \mathbb{C}^2, \quad t \longmapsto \big(t, h'(t)\big)$$

and define $C = \operatorname{Im} H$, $C' = \operatorname{Im} H'$. C is algebraic in $\mathbb{C}^2$. Suppose there exists $\psi \in \operatorname{Aut}(E_0)$ such that $H'(t+1) = (\psi,\psi)\circ H(t)$. Recall that ψ is a rational function of t, so C' should also be algebraic. However C' is not algebraic (see [6] or [1]). Using this kind of argument it is possible to give necessary and sufficient conditions to decide which elements in $\mathcal{F} \in \mathcal{D}_0$ admit a meromorphic first integral (see [12]). In fact the conditions depend only on $H[\mathcal{F}]$.

When $\#H(\mathcal{F}) = 2$ the correspondence $[\mathcal{F}] \mapsto H[\mathcal{F}]$ is injective (see Theorem 2). In the remaining cases this is no longer true. Mattei [8] showed that *locally* (in the sense of unfoldings) there exists a vector space of dimension $\frac{1}{2}N(N-1)$ of analytic classes once we have fixed a transverse structure. In next paragraph we construct some explicit families of counterexamples which give a clear idea of the kind of obstructions that appear.

The first family is related to the fact that $H(\mathcal{F})$ does not determine the analytic class of $\operatorname{Sep}(\mathcal{F})$. Fix $n \geqslant 2$. In the (t,x) chart of $\widetilde{\mathbb{C}}^2$ choose points $p_i = (t_i,0)$ and $r_i \in \mathbb{N}^*$ for $i = 1,\dots,n$. Define

$$P(t) = \int (t-t_1)^{r_1}\cdots(t-t_n)^{r_n}\,\mathrm{d}t$$

with $P(0) = 0$. In the same chart consider foliations

$$\widetilde{\mathcal{F}} = \big\{P(t) + x = \mathrm{C^{te}}\big\}, \quad \widetilde{\mathcal{F}}' = \big\{P(t) + x\big(1 + (t-t_1)\big) = \mathrm{C^{te}}\big\}. \tag{2.1}$$

They extend to regular holomorphic foliations in $\widetilde{\mathbb{C}}^2$, and we call $\mathcal{F} = E^{-1*}(\widetilde{\mathcal{F}})$ and $\mathcal{F}' = E^{-1*}(\widetilde{\mathcal{F}}')$ the singular foliations they define in $(\mathbb{C}^2,0)$ after implosion. They admit meromorphic first integrals in (X,Y) with common denominator $X^{r_1+\cdots+r_n+1}$. From (2.1) we deduce that $\mathcal{F},\mathcal{F}' \in$

$\mathcal{D}_0(n; r_1, \ldots, r_n)$ with $\Sigma_{\mathcal{F}} = \Sigma_{\mathcal{F}'} = \{p_1, \ldots, p_n\}$, and by the definition of the transverse structure $H(\mathcal{F}) = H(\mathcal{F}')$.

LEMMA 2.1. — *One has $[\mathcal{F}] \neq [\mathcal{F}']$ for a generic choice of p_i's.*

Proof. — If $n > 2$, generically in the choice of p_i's we have that

$$\operatorname{Aut}\big(H(\mathcal{F})\big) := \big\{\varphi \in \operatorname{Aut}(E_0) \mid \varphi H(\mathcal{F})\varphi^{-1} = H(\mathcal{F})\big\} = \{\mathrm{Id}\}. \tag{2.2}$$

Suppose $\Psi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is a biholomorphism such that $\Psi^*(\mathcal{F}') = \mathcal{F}$. From $H(\mathcal{F}) = H(\mathcal{F}')$ and (2.2) we deduce that there exists $\lambda \in \mathbb{C}^*$ such that $\mathrm{d}\Psi(0,0) = \lambda\,\mathrm{Id}$. We also have that $\Psi(\mathrm{Sep}(\mathcal{F})) = \mathrm{Sep}(\mathcal{F}')$, but by the choices made in (2.1), it is easily seen by studying the action of Ψ on the divisor of the extended resolution of $\mathcal{F}$ and $\mathcal{F}'$ and the position of the points of intersection of $\mathrm{Sep}(\mathcal{F})$ with it that there is no possible value for λ. In the case $n = 2$, $\operatorname{Aut}(H(\mathcal{F}))$ consists of two elements, but generically the case where $\widehat{\Psi}_0 \neq \mathrm{Id}$ is excluded by a similar argument. □

Motivated by this proof we establish the following definition:

DEFINITION 2.2. — *Given $\mathcal{F} \in \mathcal{D}_0(n; r_1, \ldots, r_n)$ and two families P, Q of n points in $\mathcal{D}_{\mathcal{F}}$ we say that $P \sim Q$ if and only if there exists a biholomorphism $\Psi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ whose lifting $\widehat{\Psi}$ to a neighborhood of $\mathcal{D}_{\mathcal{F}}$ satisfies $\widehat{\Psi}_{|E_0} = \mathrm{Id}$ and $\widehat{\Psi}(P) = Q$. Define $Q_{\mathcal{F}} \subset \mathcal{D}_{\mathcal{F}}$ as the set of n singularities of $\widehat{\mathcal{F}}$ which are* not *corners of $\mathcal{D}_F$ and $q(\mathcal{F}) := [Q_{\mathcal{F}}]$ its class by the equivalence relation $\sim$.*

Observe that, although $q(\mathcal{F})$ is not an *analytic* invariant of $\mathcal{F}$, it is invariant by the subgroup of biholomorphisms which fix every point in E_0. By using coordinates it is easily seen that the space of classes of points of type $Q_{\mathcal{F}}$ is isomorphic to a subset of $\mathbb{C}P^{n-1}$. On the other hand, $q(\mathcal{F})$ depends only on $\mathrm{Sep}(\mathcal{F})$.

The second family of examples shows that even fixing $H(\mathcal{F})$ and $\mathrm{Sep}(\mathcal{F})$ there are analytically different elements in $\mathcal{D}_0$. Fix $r \geqslant 3$ and consider

$$\mathcal{F} = \big\{f := (X^{r+1} + Y^r)/X^r = \mathrm{C^{te}}\big\},$$
$$\mathcal{F}' = \big\{f(X,Y) \cdot (1 + X) = \mathrm{C^{te}}\big\}$$

contained in $\mathcal{D}_0(1; r-1)$. After one blowing up we have

$$\widetilde{\mathcal{F}} = \big\{\tilde{f} := x + t^r = \mathrm{C^{te}}\big\},$$
$$\widetilde{\mathcal{F}}' = \{\tilde{f}' := (x + t^r)(1 + x) = \mathrm{C^{te}}\}.$$

Clearly $H(\mathcal{F}) = H(\mathcal{F})$, $\mathrm{Sep}(\mathcal{F}) = \mathrm{Sep}(\mathcal{F}')$ and $\operatorname{Aut}(H(\mathcal{F}))$ is the set of nonzero homotetias in the t variable. Suppose there exists a biholomorphism $\Psi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that $\Psi^*(\mathcal{F}) = \mathcal{F}'$. From the previous facts we get that after blowing up $\mathbb{C}^2$ at 0, Ψ lifts to

$$\widetilde{\Psi}(t,x) = \big(\lambda t + x\phi_1(t,x), x\big(\mu + \phi_2(t,x)\big)\big)$$

for some holomorphic functions ϕ_1, ϕ_2 defined in a neighborhood of $E_0 \setminus \infty$ and $\lambda, \mu \in \mathbb{C}^*$. Since Ψ conjugates the foliations we have

$$\tilde{f}\big(\widetilde{\Psi}(t,x)\big) = \lambda^r \cdot \tilde{f}'(t,x).$$

From this last equation we get

$$x\phi_2(t,x) = \lambda^r(x+t^r)(1+x) - \big(\lambda t + x\phi_1(t,x)\big)^r - \mu x. \tag{2.3}$$

Now from the fact that $\widetilde{\Psi}$ is the lifting of Ψ we have that the left hand side of equation (2.3) must be a series of the form $\sum_{i\geqslant 1} A_i(t)x^i$ where A_i are polynomials in t and $\deg A_i \leqslant i$. Thus, to eliminate the $t^r x$ term we need $\phi_1(t,0) = \lambda t$, but this will produce a nonzero term in the $t^r x^2$-monomial which cannot be cancelled with any other term of the right hand side of equation (2.3), producing a contradiction. Hence $[\mathcal{F}] \neq [\mathcal{F}']$.

We are thus interested in determining other analytic invariants. This is quite a difficult problem even for the reducible curves $\operatorname{Sep}(\mathcal{F})$ associated to $\mathcal{F} \in \mathcal{D}_0$, for which, except in some cases (see [15], [14]), a complete list of analytic invariants is unknown.

In the examples above a short calculation shows that, denoting by ω and ω' the forms defining the foliations $\mathcal{F}$ and $\mathcal{F}'$ we have that

$$j^{N+1}(\omega) = j^{N+1}(\omega') \quad \text{but} \quad j^{N+2}(\omega) \neq j^{N+2}(\omega').$$

In the case of germs of *curves* we know that there is finite determinacy (see [6]): if two equations of such germs coincide up to a sufficiently high order, they are analytically equivalent. Our approach is to prove a theorem of finite determinacy in $\mathcal{D}_0(n; r_1, \dots, r_n)$ (see Theorem 3) whose proof is given in the following section.

3. Proof of Theorem 3

Consider $\mathcal{F} = \mathcal{F}_\omega \in \mathcal{D}_0(n; r_1, \dots, r_n)$ and suppose, without loss of generality, that $\Sigma_{\mathcal{F}} = \{p_1, \dots, p_n\}$ is contained in the (t,x) chart of $\widetilde{\mathbb{C}}^2$. Denote by $p_\infty = (\infty, 0)$ the point at infinity in this chart. Let C_i be the separatrix whose strict transform $\widetilde{C}_i$ passes through p_i. Take irreducible Weierstrass polynomials in Y, $f_i(X,Y)$ such that $C_i = \{f_i = 0\}$ for $i = 1, \dots, n, \infty$, and a unit $\phi \in \mathcal{O}^*_{(\mathbb{C}^2,0)}$. Define

$$N = r_1 + \cdots + r_n, \quad F := \prod_{i=1}^{n} f_i$$

and the meromorphic function in $(\mathbb{C}^2,0)$

$$g = \frac{f_\infty^{N+n+1}}{F}\cdot\phi.$$

It defines a germ of holomorphic foliation $\mathcal{G} = \{g = \mathrm{C}^{\mathrm{te}}\}$ with a dicritical singularity at 0 whose blowing up

$$\widetilde{\mathcal{G}} := E^*(\mathcal{G}) = \{\tilde{g} := g\circ E = \mathrm{C}^{\mathrm{te}}\}$$

has the following properties:

LEMMA 3.1. — (i) *$E_0, \widetilde{C}_1, \ldots, \widetilde{C}_n, \widetilde{C}_\infty$ are invariant by $\widetilde{\mathcal{G}}$;*

(ii) $\mathrm{Sing}(\widetilde{\mathcal{G}}) = \{p_1, \ldots, p_n, p_\infty\}$;

(iii) *$\widetilde{\mathcal{G}}$ is dicritic at $p_1, \ldots, p_n$ and has a saddle with local holomorphic first integral and index $-1/(N+n+1)$ at p_∞;*

(iv) *the holonomy of $\widetilde{\mathcal{G}}$ at p_i along E_0 is trivial.*

Moreover, for a generic choice of unit ϕ, we have that the set $\mathrm{tang}(\widetilde{\mathcal{F}}, \widetilde{\mathcal{G}})$ *of tangencies between $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ satisfies*

$$\mathrm{tang}(\widetilde{\mathcal{F}}, \widetilde{\mathcal{G}}) = \widetilde{C}_\infty + \sum_{i=1}^{n}(\widetilde{C}_i + \widetilde{T}_i) \tag{3.1}$$

where $\widetilde{T}_i$ is a regular irreducible analytic set tangent to E_0 at p_i with order $r_i = r(p_i)$ of contact, when $\widetilde{C}_i$ and E_0 are tangent with contact r_i+1 at p_i. In this case we will say that $(\mathcal{F},\mathcal{G})$ (or $(\widetilde{\mathcal{F}},\widetilde{\mathcal{G}})$) *are companion foliations (see Figure* 3.1 *for diagrams).*

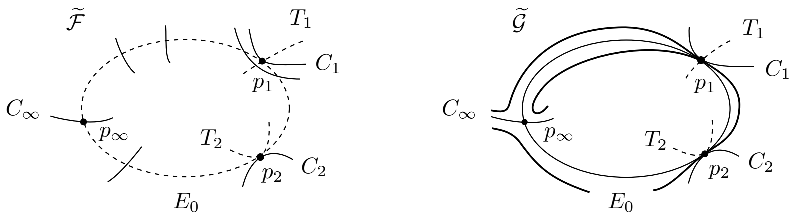

Figure 3.1. Companion foliations $(\widetilde{\mathcal{F}},\widetilde{\mathcal{G}})$ with $r(p_1)=1$, $r(p_2)>1$.

Proof. — For the proofs of (i)–(iv) it suffices to say that

$$\mathrm{div}(\tilde{g}) = (N+n+1)\widetilde{C}_\infty + E_0 - \sum_{i=1}^{n}\widetilde{C}_i.$$

For the proof of (3.1) observe that $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ are transverse at all $p\in E_0\setminus\{p_1,\ldots,p_n,p_\infty\}$. At p_∞, $\widetilde{C}_\infty$ is a common separatrix of $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ and since p_∞ is a saddle for $\widetilde{\mathcal{G}}$ there are no other components of $\operatorname{tang}(\widetilde{\mathcal{F}},\widetilde{\mathcal{G}})$ there.

For $i=1,\dots,n$, consider a normalizing chart for $\widetilde{\mathcal{F}}$ around p_i: (t,x), where $p_i=(0,0)$. In this chart $\widetilde{\mathcal{F}}$ is expressed as the levels of $f(t,x)=x-t^{r_i+1}$ and $\widetilde{\mathcal{G}}$ as the levels of

$$g(t,x)=\frac{x}{x-t^{r_i+1}}\cdot\nu(t,x)$$

where ν is a unit depending on ϕ and the normalizing chart. Hence the components of tangency are described in this chart by the expression

$$\begin{aligned}(3.2)\qquad 0&=\mathrm{d}f\wedge\big((x-t^{r_i+1})^2\mathrm{d}g\big)\\ &=(x-t^{r_i+1})[x\partial_t\nu+(r_i+1)t^{r_i}(\nu+x\partial_x\nu)]\mathrm{d}x\wedge\mathrm{d}t.\end{aligned}$$

For generic values of $j^1(\nu)$ (which depend on generic values of $j^1(\phi)$) the set $\{x\partial_t\nu+(r_i+1)t^{r_i}(\nu+x\partial_x\nu)=0\}$ is regular at $(0,0)$ and has contact r_i with the set $\{x=0\}$. Since the inverse of the normalizing chart sends $\{x=0\}$ to E_0 and preserves orders of tangency we get, after (3.2), two components of $\operatorname{tang}(\widetilde{\mathcal{F}},\widetilde{\mathcal{G}})$ at p_i, $\widetilde{C}_i$ and $\widetilde{T}_i$, with the required properties. □

Let $(\mathcal{F},\mathcal{G})$ and $(\mathcal{F}',\mathcal{G}')$ be two pairs of companion foliations where $\mathcal{G}=\{g=\mathrm{C^{te}}\}$ and $\mathcal{G}'=\{g'=\mathrm{C^{te}}\}$ associated to $\mathcal{F},\mathcal{F}'\in\mathcal{D}_0(n;r_1,\dots,r_n)$ with $\Sigma_{\mathcal{F}}=\Sigma_{\mathcal{F}'}$. For a point $p\in E_0\setminus\{\Sigma_{\mathcal{F}},p_\infty\}$, consider holomorphic first integrals $F_p,F'_p:(U_p,p)\to(\mathbb{C},0)$ of $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}'$ respectively defined in a small neighborhood U_p of p. By transversality and Lemma 3.1, for each $q\in U_p$ there is a unique point $\Psi_p(q)\in U_p$ such that

$$(3.3)\qquad F'_p\big(\Psi_p(q)\big)=F_p(q),\quad g'\big(\Psi_p(q)\big)=g(q).$$

By holomorphicity of all the foliations under consideration, $\Psi_p:(U_p,p)\to(U_p,p)$ defines a germ of biholomorphism. If $p,p'\in E_0\setminus\{\Sigma_{\mathcal{F}},p_\infty\}$ and $U_p\cap U_{p'}\neq\emptyset$ then $\Psi_p\equiv\Psi_{p'}$ on $U_p\cap U_{p'}$. Thus we have a biholomorphism $\Psi:U\to U$ from a neighborhood U of $E_0\setminus\{\Sigma_{\mathcal{F}},\infty\}$ in $\widetilde{\mathbb{C}}^2$ to itself. The properties of $\mathcal{G}$ and $\mathcal{G}'$ at p_∞ and its relations with $\mathcal{F}$ and $\mathcal{F}'$ allow us to extend Ψ to a neigborhood of p_∞ as a biholomorphism by using a theorem of Mattei and Moussu (see [9], p. 482). The following lemma provides necessary and sufficient conditions for Ψ to extend to neighborhoods of all points of $\Sigma_{\mathcal{F}}$:

Lemma 3.2. — *Ψ extends to a neighborhood of E_0 as a biholomorphism if and only if the following conditions are fulfilled:*

(a) $H(\mathcal{F})=H(\mathcal{F}')$;
(b) $q(\mathcal{F})=q(\mathcal{F}')$;

(c) *there exist homeomorphisms* $\psi_i : T_i \longrightarrow T_i'$ *between the irreducible components of* $tang(\mathcal{F}, \mathcal{G})$ *and* $tang(\mathcal{F}', \mathcal{G}')$ *not invariant by any of the foliations passing through* $p_i \in \Sigma_{\mathcal{F}}$ *such that*

$$F_i'\big(\psi_i(q)\big) = F_i(q), \quad g'\big(\psi_i(q)\big) = g(q) \tag{3.4}$$

for all $q \in T_i$ *and local first integrals* F_i *and* F_i' *of* $\widetilde{\mathcal{F}}$ *and* $\widetilde{\mathcal{F}}'$ *respectively around* p_i *whose restriction to* E_0 *coincide.*

Proof. — Suppose first that Ψ extends to a neighborhood of E_0 as a biholomorphism. Then it is the identity on E_0 and we have already seen that this implies (a) and (b). For the proof of (c) it suffices to observe that $\Psi(T_i) = T_i'$, take $\psi_i \equiv \Psi_{|T_i}$ and apply the equations (3.3) defining Ψ in the neighborhood of p_i.

For the converse, we have to consider the extended resolution $S_{\mathcal{F}}$ of $\mathcal{F}$ and $\mathcal{F}'$. Call $\widehat{\mathcal{F}} = S_{\mathcal{F}}^*(\mathcal{F})$, $\widehat{\mathcal{F}}' = S_{\mathcal{F}}^*(\mathcal{F}')$, $\widehat{\mathcal{G}} = S_{\mathcal{F}}^*(\mathcal{G})$, $w\widehat{\mathcal{G}}' = S_{\mathcal{F}}^*(\mathcal{G}')$ and $D_{\mathcal{F}}$ the exceptional divisor associated to $S_{\mathcal{F}}$. $D_{\mathcal{F}}$ consists of $N + n + 1$ irreducible components: one of them is transverse to $\widehat{\mathcal{F}}$ and $\hat{\mathcal{F}}'$, and we still call it E_0 (by abuse of language). From each p_i there is a chain of $r_i + 1$ irreducible components that we will call $E_1^i, \dots, E_{r_i+1}^i$ where E_j^i and E_{j+1}^i intersect transversally at a corner for $j = 1, \dots, r_i$ and $p_i = E_0 \cap E_{r_i+1}^i$.

Call $\widehat{C}_i$ and $\widehat{T}_i$ the strict transforms of C_i and T_i respectively by $S_{\mathcal{F}}$ for $i = 1, \dots, n, \infty$. The relevant properties of $\widehat{\mathcal{F}}$ are:

1) $E_1^i, \dots, E_{r_i+1}^i, \widehat{C}_i$ and $\widehat{C}_\infty$ are invariant by $\widehat{\mathcal{F}}$ for $i = 1, \dots, n$.
2) $\mathrm{Sing}(\widehat{\mathcal{F}}) = \{\text{corners}, Q_1, \dots, Q_n\}$ where $Q_i \in E_{r_i+1}^i$ is not a corner. All singularities are reduced saddles.
3) For $j = 1, \dots, r_i + 1$ the holonomy of $\widehat{\mathcal{F}}$ along E_j^i of the singularities in E_j^i have degree j.

For the companion foliation $\widehat{\mathcal{G}}$ the relevant properties are:

4) $\widehat{\mathcal{G}}$ is regular and transversal to $E_{r_i+1}^i$ in all its points. $E_0, E_1^i, \dots, E_r^i, \widehat{C}_i$ and $\widehat{C}_\infty$ are invariant by $\widehat{\mathcal{G}}$ for $i = 1, \dots, n$.
5) $\mathrm{Sing}(\widehat{\mathcal{G}}) = \{Q_1^i, Q_2^i, p_\infty, \text{corners not contained in } \ E_{r_i+1} : i = 1, \dots, n\}$ where $Q_1^i \in E_1^i$ and $Q_2^i \in E_{r_i}^i$ are not corners. All singularities are reduced saddles with holomorphic first integral.
6) From Lemma 3.1, $\widehat{T}_i$ is a regular disc transverse to $E_{r_i}^i$

All these properties are also satisfied by $\widehat{\mathcal{F}}'$ and $\widehat{\mathcal{G}}'$.

Condition (b) implies that after a diagonal linear change of coordinates in the original foliations we can suppose $\mathrm{Sing}(\widehat{\mathcal{F}}) = \mathrm{Sing}(\widehat{\mathcal{F}}') = \{Q_1, \dots, Q_n\}$. Condition (a) implies that the holonomy maps of Q_i along $E_{r_i+1}^i$, and the index of Q_i are the same for $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{F}}'$. As before we can use the theorem of Mattei and Moussu to extend $\widehat{\Psi}$, the lifting of Ψ, to the separatrix $\widehat{C}_i$ for $i = 1, \dots, n$.

Now fix $i \in \{1, \dots, n\}$. Condition (a) implies the existence of the first integrals F_i and F'_i with the properties stated in (c). Let L, L' be leaves of $\widehat{\mathcal{F}}$ and $\widehat{\mathcal{F}}'$ respectively with $L \cap E_0 = L' \cap E_0 \neq \{p_i\}$ sufficiently close to p_i. The construction requires that $\Psi(L) = L'$. Observe that $L \cap \widehat{T}_i = \{W_1, \dots, W_{r_i}\}$, by properties 3) and 6). Observe that these points correspond to the critical points of the restriction $\widehat{g}_{|L}$ of $\widehat{g} := S^*_{\mathcal{F}}(g)$ to L. Take a disc $D \subset L$ containing $L \cap \widehat{T}_i$, and such that the image of ∂D by Ψ has already been defined. Let $D' \subset L'$ be the disc containing $L' \cap \widehat{T}'_i = \{W'_1, \dots, W'_{r_i}\}$ such that $\partial D' = \Psi(\partial D)$. If L is sufficiently close to $E^i_{r_i}$ then all the multiplicities $v(W_j)$ of the critical points of $\widehat{g}_{|D}$ coincide and are equal to some $v > 1$. Applying Hurwitz's formula applied to the pasting of two copies of $\widehat{g}_{|D}$ we get

$$2 = 2 \cdot (r_i + 1) - 2\Big(\sum_{q \in D} (v(q) - 1)\Big)$$

which means that $v = 2$ and that $v(q) = 1$ for all $q \in D \setminus \{W_1, \dots, W_{r_i}\}$. The second equation in (3.4) means that by renaming the points we can suppose $\widehat{g}'(W'_j) = \widehat{g}(W_j) =: w_j \in \mathbb{C}$ for $j = 1, \dots, r_i$. Observe that $\widehat{g}^{-1}(w_j)$ contains r_i points. Define $V = \{w_1, \dots, w_{r_i}\}$. Suppose D is big enough to contain $\widehat{g}^{-1}(V) \cap L =: W$. Define $W' = \widehat{g}'^{-1}(V) \cap L'$. Thus $\widehat{g}_{|D \setminus W} : D \setminus W \longrightarrow \widehat{g}(D) \setminus V$ is a $(r_i + 1) : 1$ holomorphic covering. We can copy the construction with $\widehat{g}'$ and observe that the image of both coverings is the same. Thus, using a covering argument, we can find a unique topological extension $\widehat{\Psi}_D$ making the following diagram commutative

$$\begin{array}{ccc} D \setminus W & \xrightarrow{\widehat{\Psi}_D} & D' \setminus W' \\ \hat{g}\big\downarrow & & \big\downarrow\widehat{g}' \\ \widehat{g}(D) \setminus V & \xrightarrow{=} & \hat{g}'(D') \setminus V \end{array}$$

which is holomorphic and extends holomorphically to W because we can interpret $\widehat{\Psi}_D$ as a holomorphic map between subsets of discs. This can be done for each leaf L not containing p_i.

The holomorphicity in the transverse direction to $\widehat{\mathcal{F}}$ comes from (3.4). Thus after blowing E^i_j down for $j = 1, \dots, r_i + 1$, Ψ extends biholomorphically to $U_i \setminus p_i$ where U_i is a neighborhood of p_i in $\widetilde{\mathbb{C}}^2$, and by Hartogs' theorem (see [11], p. 341) we extend it to p_i. $\square$

The following lemma finishes the proof of Theorem 3:

LEMMA 3.3. — *The hypotheses in Theorem* 3 *imply the existence of functions* g, g' *satisfying* (a), (b) *and* (c) *of Lemma* 3.2.

Proof: Suppose $\mathcal{F} = \mathcal{F}_\omega$ and $\mathcal{F}' = \mathcal{F}_{\omega'} \in \mathcal{D}_0(n; r_1, \ldots, r_n)$ with $H(\mathcal{F}) = H(\mathcal{F}')$. Thus (a) is already satisfied. Recall $N = r_1 + \cdots + r_n$ and that the algebraic multiplicity of ω and ω' is $N + 1$. After a linear change of coordinates we can suppose $\Sigma_{\mathcal{F}} = \{p_1, \ldots, p_n\}$ with coordinates $p_i = (t_i, 0)$ in the chart (t, x) of $\widetilde{\mathbb{C}}^2$ such that $|t_i| \neq |t_j|$ for $i \neq j$. A direct calculation shows that $j^{N+2}(\omega) = j^{N+2}(\omega')$ implies (b). For the proof of (c), consider a companion foliation $\mathcal{G} = \{g = \mathrm{C^{te}}\}$ for $\mathcal{F}$. Recall that $C_i = \{f_i = 0\}$ (resp. $C'_i = \{f'_i = 0\}$) is the Weiertrass polynomial of the separatrix of $\mathcal{F}$ (resp. $\mathcal{F}'$) whose strict transform by E passes through p_i for $i = 1, \ldots, n, \infty$. Let $F = \prod_{i=1}^n f_i$, $F' = \prod_{i=1}^n f'_i$. We need to find a unit $u \in \mathcal{O}^*_{(\mathbb{C}^2,0)}$ such that

$$\mathcal{G}' = \left\{ g' := \frac{f'^{N+n+1}_\infty}{F'} \cdot u = \mathrm{C^{te}} \right\}$$

is a companion foliation for $\mathcal{F}'$ and

- (*) $\operatorname{tang}(\mathcal{F}, \mathcal{G}) = \operatorname{tang}(\mathcal{F}', \mathcal{G}')$;
- (**) g and g' satisfy (3.4) on each component of type T_i of $\operatorname{tang}(\mathcal{F}, \mathcal{G})$, where $\psi_i : T_i \to T_i$ is defined by the first equation of (3.4).

We know $C_i = \{f_i = 0\}$ is invariant by $\mathcal{F}_\omega$ so

$$\mathrm{d}f_i \wedge \omega = f_i \cdot H_i \mathrm{d}X \wedge \mathrm{d}Y$$

for some holomorphic function $H_i \in \mathcal{O}_{(\mathbb{C}^2,0)}$. The divisor $\operatorname{tang}(\mathcal{F}, \mathcal{G})$ is defined by

$$\begin{aligned} 0 &= (F^2 \mathrm{d}g) \wedge \omega \\ &= f_\infty^{N+n+1} F \Big(H_\infty - \sum_{i=1}^n H_i \Big) \mathrm{d}X \wedge \mathrm{d}Y =: f_\infty^{N+n+1} F H \mathrm{d}X \wedge \mathrm{d}Y. \end{aligned} \tag{3.5}$$

As we saw in Lemma 3.1 the divisor $T = \{H = 0\}$ has n irreducible components $T_1, \ldots, T_n$ with multiplicity one. Each T_i is a generalized cusp of type $(r_i, r_i + 1)$ (when $r_i = 1$ it is just a regular disc). We will decompose the problem of constructing u in two steps by finding functions $\phi \in \mathcal{O}^*_{(\mathbb{C}^2,0)}$ and $\varphi \in \mathcal{O}_{(\mathbb{C}^2,0)}$ such that $u = \phi + H \cdot \varphi$. The idea is that conditions $(*)$ and $(**)$ define the values of ϕ and φ on the analytic subset of dimension one T. We then need to find holomorphic functions defined in the whole neighborhood of the origin in $\mathbb{C}^2$ taking the same values on T. For this purpose we need to construct the ψ_i's appearing in (3.4) first:

Using Lemma 3.4 (i) consider F_i, F'_i local holomorphic first integrals of $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{F}}'$ respectively around p_i such that $F_{i|V_i} \equiv F'_{i|V_i}$ in a neighborhood $V_i \subset E_0$ of p_i. Consider the following diagram:

$$\begin{tikzcd}
\widetilde{T}_i \setminus p_i \arrow[d, "\pi_i"'] & & & \widetilde{T}_i \setminus p_i \arrow[d, "\pi'_i"] \\
(\widetilde{T}_i \setminus p_i)/F_i & \cong & (V_i \setminus p_i)/F_i = (V_i \setminus p_i)/F'_i & \cong & (\widetilde{T}_i \setminus p_i)/F'_i
\end{tikzcd}$$

where π_i and π'_i are $r_i : 1$ holomorphic coverings corresponding to the projections onto the local leave spaces. Hence we can construct a homeomorphism $\psi_i : \widetilde{T}_i \setminus p_i \to \widetilde{T}_i \setminus p_i$ such that $\pi'_i \circ \psi_i = \pi_i$. In fact there are r_i different possibilities for constructing ψ_i. After blowing down we can consider $\psi_i : T_i \to T_i$ as a homeomorphism.

Define the function $\phi_i := \phi_{|T_i} : (T_i, p_i) \to (\mathbb{C}, 1)$ using condition (**):

$$\phi_i(q) = \Big(\frac{f_\infty \circ \psi_i^{-1}}{f'_\infty}\Big)^{N+n+1} \Big(\prod_{i=1}^{n} \frac{f'_i}{f_i \circ \psi_i^{-1}}\Big)(q) \tag{3.6}$$

for $q \in T_i$. From now on we will suppose $j^s(\omega) = j^s(\omega')$ and find bounds for s to insure that the construction can be done.

Claim 1. — If $s \geqslant (N+1) + \max\{r_i\}(N-1)$ there exists $\phi \in \mathcal{O}^*_{(\mathbb{C}^2,0)}$ such that $\phi_{|T_i} = \phi_i$ for $i = 1, \ldots, n$. Moreover,

$$\mathrm{d}\phi(X,Y) = X^{(s-(N+1))/\max\{r_i\}}\nu(X,Y)$$

for some holomorphic germ of 1-form ν.

Proof of Claim 1. — We need to analyse the relations between the jets of the 1-form $\omega = \sum_{j \geqslant N+1}(P_j \mathrm{d}X + Q_j \mathrm{d}Y)$ where $P_j(X,Y)$, $Q_j(X,Y)$ are homogeneous polynomials of degree j and the form defining $\widetilde{\mathcal{F}}_\omega$ in the (t,x) chart of $\widetilde{\mathbb{C}}^2$:

$$\widetilde{\omega}(t,x) := \frac{E^*\omega(t,x)}{x^{N+2}} = \sum_{j \geqslant 0} x^j \left[Q_{j+N+1}(1,t)\mathrm{d}t + R_{j+N+2}(t)\mathrm{d}x\right]$$

where $R_{j+N+2}(t) := P_{j+N+2}(1,t) + tQ_{j+N+2}(1,t)$.

Lemma 3.4. — *If $\mathcal{F}_\omega, \mathcal{F}_{\omega'} \in \mathcal{D}_0(n; r_1, \ldots, r_n)$ satisfy $j^s_0(\omega) = j^s_0(\omega')$, then:*

(i) $\tilde{\omega'}(t,x) = \widetilde{\omega}(t,x) + x^{s-(N+1)}\omega_2(t,x)$ *for some holomorphic* 1*-form* ω_2. *We define* $K(s) := s - (N+1) \in \mathbb{N}$.

(ii) *If moreover,* $H(\widetilde{\mathcal{F}}_\omega, p) = H(\widetilde{\mathcal{F}}_{\omega'}, p)$ *for* $p \in E_0$*, given a local holomorphic first integral* f *of* $\widetilde{\mathcal{F}}_\omega$ *there exists a local holomorphic first integral* f' *of* $\widetilde{\mathcal{F}}_{\omega'}$ *in a neighborhood of* p *such that*

$$f'(t,x) = f(t,x) + x^{K(s)+1} h(t,x) \tag{3.7}$$

for some holomorphic function h *defined in a neighborhood of* p*.*

(iii) *Equation* (3.7) *implies that* $j^{K(s)+r_i}(f_i) = j^{K(s)+r_i}(f'_i)$ *for the Weierstrass polynomials* f_i, f'_i *of* C_i *and* C'_i *respectively.*

Proof. — To prove (i), observe that in (3.7), the terms in the j-th member of the sum depend on the $(j+1)+N+1$ jet of ω. For the proof of (ii) we can construct f' by extending the function $f(t,0)$ along the leaves of $\widetilde{\mathcal{F}}_{\omega'}$ in a neighborhood of $p = (t_0, 0)$. We can assume $\partial f/\partial x(p) \neq 0$. We write $h(t,x) = f(t,x) - f'(t,x) = \sum_{i\geqslant 0} h_i(t)x^i$ with h_i holomorphic functions of t. From item (i) we get

$$x^{K(s)}(\omega_2 \wedge \mathrm{d}f') + \widetilde{\omega} \wedge \mathrm{d}h = \widetilde{\omega}' \wedge \mathrm{d}f' \equiv 0$$

Since $\widetilde{\omega} = A\mathrm{d}t + B\mathrm{d}x$ is regular at p, and $\mathrm{d}h = \sum_{i\geqslant 1} \frac{\partial h_i}{\partial t} x^i \mathrm{d}t + i x^{i-1} h_i(t)\mathrm{d}x$ we get, by comparing jets:

$$0 \equiv \sum_{i=0}^{K(s)} x^{i-1} i A h_i - \sum_{i=0}^{K(s)-1} B x^i \frac{\partial h_i}{\partial t}$$

By hypothesis $h_0 \equiv 0$. Inductively we get $h_1(t) \equiv \ldots \equiv h_{K(s)}(t) \equiv 0$.

For the proof of (iii) we take Puiseux parametrizations $w \mapsto (w^{r_i+1}, Q_i(w))$ and $w \mapsto (w^{r_i+1}, Q'_i(w))$ of C_i and C'_i respectively. By (3.7) we get, by comparing terms after the blowing up, $j^{K(s)+r_i}(Q_i) = j^{K(s)+r_i}(Q'_i)$ which implies the assertion in (iii). $\square$

Now consider a Puiseux parametrization

$$\tau_i : \mathbb{D} \longrightarrow T_i, \quad w \longmapsto \big(w^{r_i}, \widehat{P}_i(w)\big)$$

from a small disc $\mathbb{D}$ to T_i. Hence there exists a homeomorphism $b_i : \mathbb{D} \to \mathbb{D}$ such that

$$\psi_i(w^{r_i}, \widehat{P}_i(w)) = \big(\big(b_i(w)\big)^{r_i}, \widehat{P}_i\big(b_i(w)\big)\big)$$

In fact b_i is holomorphic outside 0, which implies that it is also holomorphic there. Suppose $b_i(w) = \sum_{j\geqslant 1} b^i_j w^j$ and that we have chosen ψ_i by imposing $b^0_1 = 1$. Using (3.7) and the fact that $h \circ E^{-1} \circ \tau_i(w) = a_i w^{r_i} + \cdots$ with $a_i \neq 0$ we see inductively that $b^i_2 = b^i_3 = \cdots = b^i_{K(s)} = 0$. This together with (3.6) and Lemma 3.4 (iii) implies that

$$\phi_i\big(w^{r_i}, \widehat{P}_i(w)\big) = 1 + w^{K(s)} \tilde{\phi}_i(w) \tag{3.8}$$

for some holomorphic function $\tilde{\phi}_i$. Since $\widehat{P}_i(w) = t_i w^{r_i} \dots$ and $|t_i| \neq |t_j|$ if $i \neq j$ we can apply the following interpolation result due to Cartan (see [5], p. 102), which has been adapted to our situation:

For each $i \in \{1, \dots, n\}$ consider a germ of analytic irreducible set

$$T_i = \Big\{(X, Y) \in \mathbb{C}^2 \mid Y^{r_i} + \sum_{j=0}^{r_i - 1} \alpha_j(X) Y^j = 0\Big\}$$

with its Puiseux parametrization $x \mapsto (x^{r_i}, \widehat{P}_i(x))$. Suppose that $\widehat{P}_i(x) = A_i x^{r_i} + \cdots$ for $A_i \in \mathbb{C}$ with $A_i^{r_k} \neq A_j^{r_k}$ if $i \neq j$ and $i, j, k \in \{1, \dots, n\}$.

Lemma 3.5. — *Let $\nu_i : T_i \longrightarrow \mathbb{C}$ be a continuous function such that $\nu_i(0,0) = a \in \mathbb{C}$ (independently of i) and $\nu_i(x^{r_i}, \widehat{P}_i(x)) = a + c_i x^{\ell_i} + \cdots$ is holomorphic with $c_i \neq 0$. If $\ell := \min \ell_i / r_i \geqslant (\sum_{i=1}^n r_i) - 1 =: N - 1$, then there exists a holomorphic function $\nu \in \mathcal{O}_{(\mathbb{C}^2, 0)}$ such that*

(i) $\nu_{|T_i} = \nu_i$;

(ii) $\mathrm{d}\nu(X, Y) = X^{\ell - N} \eta(X, Y)$ *for a holomorphic* 1*-form* η.

In other words, ν extends holomorphically to a neighborhood of $(0,0)$ in $\mathbb{C}^2$ the functions ν_i defined on the analytic subsets T_i (of dimension 1).

Proof of Lemma 3.5. — For each $i \in \{1, \dots, n\}$ choose a branch $(.)^{1/r_i}$ of the r_i-th root and a primitive r_i-th root of unity ζ_i. We index with j the set of all branches of the union T of all the T_i's: let $r_0 := 0$ and for $j \in \{1, \dots, N\}$ define $P_j(X) := \widehat{P}_i(\zeta_i^{[j]} X^{1/r_i})$ where $[j] = j - (r_0 + \cdots + r_{i-1})$ if $j \in \{r_0 + \cdots + r_{i-1} + 1, \dots, r_0 + \cdots + r_i\}$. We claim that the expression

$$\nu(X, Y) := \sum_{i=1}^{N} \Big(\prod_{j \neq i} \frac{Y - P_j(X)}{P_i(X) - P_j(X)} \Big) \nu_i\big(X, P_i(X)\big) \tag{3.9}$$

defines an univaluated, continuous function in $(\mathbb{C}^2, 0)$ with $\nu(p) = \nu_i(p)$ for $p \in T_i$. Outside $\{X = 0\}$ this is a consequence of the symmetry of the expression, for when we follow a loop around the origin in the X-plane, we exchange the order of the members of the sum, leaving its value unchanged.

Define $K := \mathcal{M}(X)$ the field of meromorphic functions in X, and $\overline{K}$ its algebraic closure. Obviously $P_j \in \overline{K}$ and $P_i \neq P_j$ if $i \neq j$. Hence for any $a \in \mathbb{C}$ the polynomial in $\overline{K}[Y]$ of degree $N - 1$ defined by

$$\Big(\sum_{i=1}^{N} \Big(\prod_{j \neq i} \frac{Y - P_j(X)}{P_i(X) - P_j(X)} \Big) a \Big) - a \tag{3.10}$$

has N different roots, and is therefore the zero polynomial. From $A_i^{r_k} \neq A_j^{r_k}$ if $i \neq j$ we obtain $|P_i(X) - P_j(X)| = |X| \cdot |h_{ij}(X)|$ for a continuous function h_{ij} such that $h_{ij}(0) \neq 0$. From the hypotheses on ν_i we have

$|\nu_i(X,P_i(X))-a| \leqslant |X|^{N-1}h_i(|X|)$ for real continuous functions h_i defined in the neighborhood of 0. Therefore,

$$\big|\nu(X,Y)-a)\big| \leqslant \Big(\sum_{i=1}^{N}\Big(\prod_{j\neq i}\frac{|Y-P_j(X)|}{|h_{ij}(X)|}\Big)h_i\big(|X|\big)\Big)\frac{|X|^{N-1}}{|X|^{N-1}} \longrightarrow 0 \tag{3.11}$$

when $(X,Y)\to(0,0)$. A similar argument can be used to prove continuity on the remaining points of $\{X=0\}$. Hence ν is continuous on $(\mathbb{C}^2,0)$ and trivially holomorphic on $(\mathbb{C}^2,0)\setminus\{X=0\}$, hence holomorphic on $(\mathbb{C}^2,0)$.

For the proof of (ii) write $\nu(X,Y)=\sum_{j=0}^{N-1}a_j(X)Y^j$ where $a_0(X)-a$ and $a_j(X)$ are holomorphic functions of X. By (3.11) they are zero up to order $\ell-(N-1)$ for $j=1,\ldots,N-1$, and taking derivatives we have that $X^{\ell-N}$ divides all terms of $\mathrm{d}\nu(X,Y)$. □

This finishes the proof of Claim 1. Let us continue with the proof of Lemma 3.3. Condition (*) is satisfied if $\varphi_i := \varphi_{|T_i}$ satisfies

$$\begin{aligned} 0 &= ({F'}^2\mathrm{d}g')\wedge\omega'_{|T_i} \\ &= F'\cdot {f'_\infty}^{N+n+1}(\phi H'\mathrm{d}X\wedge\mathrm{d}Y+\mathrm{d}\phi\wedge\omega'+\varphi_i\mathrm{d}H\wedge\omega')_{|T_i} \end{aligned} \tag{3.12}$$

where H' is obtained by a process similar to equation (3.5) but using the f_i''s. In fact, from Lemma 3.4 (iii) we have

$$j^{K(s)+N}(H)=j^{K(s)+N}(H'). \tag{3.13}$$

Equation (3.12) is equivalent to

$$\phi(q)H'(q)\mathrm{d}X\wedge\mathrm{d}Y+\mathrm{d}\phi\wedge\omega'(q)+\varphi_i(q)\big(\mathrm{d}H\wedge\omega'(q)\big)=0 \tag{3.14}$$

for each point $q\in T_i$. The expression in (3.14) defines the values of φ_i.

Claim 2. — *If $s\geqslant(N+1)+\max\{r_i\}(3N-2)=:\kappa$ there exists $\varphi\in\mathcal{O}_{(\mathbb{C}^2,0)}$ such that $\varphi_{|T_i}=\varphi_i$ for $i=1,\ldots,n$.*

Proof of Claim 2. — To use Lemma 3.5 for interpolation, we need to analyse the order of $\varphi_i(w^{r_i},\widehat{P}(w))$. This can be done using (3.14):

- $H'(w^{r_i},\widehat{P}_i(w))$ has order $r_i(K(s)+N)$, from (3.13) and the fact that $\widehat{P}_i$ has order r_i;
- $\mathrm{d}H(w^{r_i},\widehat{P}_i(w))$ has order r_iN-1, since the tangent cone of H is $\prod_{i=1}^n(Y-t_iX)^{r_i}$;
- $\omega'(w^{r_i},P_i'(w))$ has order $r_i(N+2)$, since
$$j^{N+1}(\omega')=\prod_{i=1}^n(Y-t_iX)^{r_i}(Y\mathrm{d}X-X\mathrm{d}Y);$$
- $\mathrm{d}\phi(w^{r_i},P_i'(w))$ has order $r_i(K(s)/\max r_i-N)$, by the the second part of Claim 1.

Hence, if

$$\begin{cases} r_i(K(s)+N)-(r_iN-1)-r_i(N+2)\geqslant r_i(N-1), \\ r_i\Big(\dfrac{K(s)}{\max r_i}-N\Big)-(r_iN-1)\geqslant r_i(N-1) \quad \text{for} \quad i=1,\dots,n \end{cases} \tag{3.15}$$

then we have $\varphi_i(w^{r_i},P_i'(w))=w^{r_i(N-1)}\widetilde{\varphi}_i(w)$ for some holomorphic function $\widetilde{\varphi}_i$. The hypothesis $s\geqslant\kappa$ guarantees that the hypothesis of Claim 1 and the inequalities in (3.15) are satisfied. Applying Lemma 3.5 we obtain the desired function φ. □

4. Proof of Corollary 4

Proof of Corollary 4. — Take $\mathcal{F}_\omega$ and $\mathcal{F}_{\omega'}$ in $\mathcal{D}_0$, and suppose there exists a formal equivalence $\widehat{\phi}$ in $(\mathbb{C}^2,0)$ and a formal power series $\widehat{h}$ such that $\widehat{\phi}^*\omega'=\widehat{h}\omega$. After a linear change of coordinates we can suppose $\Sigma_{\mathcal{F}_\omega}=\Sigma_{\mathcal{F}_{\omega'}}$, with the same orders of tangency and $\hat{\phi}$ tangent to the identity. Given $\ell\in\mathbb{N}$ there exist 1-forms ω_ℓ e ω'_ℓ such that $\mathcal{F}_{\omega_\ell}=\mathcal{F}_\omega$, $\mathcal{F}_{\omega'_\ell}$ is analytically equivalent to $\mathcal{F}_{\omega'}$, $j^\ell(\omega_\ell)=j^\ell(\omega'_\ell)$, $H(\omega)=H(\omega_\ell)$ and $H(\omega')=H(\omega'_\ell)$. Let us prove $H(\omega')=H(\omega)$. Let $h_p\in H(\widetilde{\mathcal{F}}_\omega,p)$ (resp. $h'_p\in H(\widetilde{\mathcal{F}}_{\omega'},p)$) be a generator, where $p\in\Sigma_\omega$. Given $s\in\mathbb{N}$, there exists a big enough $\ell(s)\in\mathbb{N}$ such that $j^{\ell(s)}(\omega_{l(s)})$ determines $j^s(h_p)$ uniquely where the jet of h_p is taken in a global coordinate t of E_0. Therefore we have $j^s(h_p)=j^s(h'_p)$ for each $s\in\mathbb{N}$ and $h_p=h'_p$. Now apply Theorem 3 to ω_ℓ and ω'_ℓ for a big ℓ and we get a biholomorphism from $\mathcal{F}_\omega$ to $\mathcal{F}_{\omega'}$. □

5. Addendum

In section 3, when $n=1$ we need to define the companion foliation in a different manner to be able to construct the biholomorphism. The problem is that in this case the set of tangencies between $\mathcal{F}$ and $\mathcal{G}$ is not a regular curve after applying a blow-up. To avoid this we consider the function defining the companion foliation $\mathcal{G}$ to be a product $g=\frac{f_\infty^{r+2}}{f_1\cdot f_2}$ where $f_1=0$ is the isolated separatrix and $f_2=0$ is a (regular) separatrix tangent to some other direction p_2. This produces a radial singularity at the point p_2 in E_0 for the companion foliation $\mathcal{G}$. The construction of the biholomorphism between two elements $\mathcal{F},\mathcal{F}'\in\mathcal{D}_0(1;r)$ by using the values of their companion foliations extends without any extra conditions to a neighborhood of p_2, as can be seen by blowing this point once and using the same argument as at p_∞.

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