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ANISOTROPIC HÖLDER AND SOBOLEV SPACES FOR HYPERBOLIC DIFFEOMORPHISMS

by Viviane BALADI & Masato TSUJII (*)

ABSTRACT. — We study spectral properties of transfer operators for diffeomorphisms $T : X \to X$ on a Riemannian manifold X. Suppose that Ω is an isolated hyperbolic subset for T, with a compact isolating neighborhood $V \subset X$. We first introduce Banach spaces of distributions supported on V, which are anisotropic versions of the usual space of C^p functions $C^p(V)$ and of the generalized Sobolev spaces $W^{p,t}(V)$, respectively. We then show that the transfer operators associated to T and a smooth weight g extend boundedly to these spaces, and we give bounds on the essential spectral radii of such extensions in terms of hyperbolicity exponents.

RÉSUMÉ. — Nous étudions les propriétés spectrales des opérateurs de transfert associés aux difféomorphismes $T : X \to X$ sur une variété riemannienne X. Nous supposons qu'il existe un sous-ensemble hyperbolique Ω pour T, contenu dans un voisinage isolant compact V. Nous introduisons d'abord des espaces de Banach de distributions, supportées sur V, qui sont des versions anisotropes des espaces usuels de fonctions C^p, d'une part, et des espaces de Sobolev généralisés $W^{p,t}(V)$, d'autre part. Nous montrons ensuite que les opérateurs de transfert associés à T et à une fonction poids lisse g s'étendent continûment à ces espaces, et nous donnons des bornes pour les rayons spectraux essentiels de ces extensions, en fonction d'exposants d'hyperbolicité.

Keywords: Hyperbolic dynamics, transfer operator, Ruelle operator, spectrum, axiom A, Anosov, Perron-Frobenius, quasi-compact.
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1. Introduction

Let X be a d-dimensional C^∞ Riemannian manifold and let $T : X \to X$ be a diffeomorphism which is of class C^1 at least. For a given complex-valued continuous function g on X, we define the Ruelle transfer operator $\mathcal{L}_{T,g}$ by

$$\mathcal{L}_{T,g} : C^0(X) \longrightarrow C^0(X), \quad \mathcal{L}_{T,g}u(x) = g(x) \cdot u \circ T(x).$$

Such operators appear naturally in the study of fine statistical properties of dynamical systems and provide efficient methods, for instance, to estimate of decay of correlations. (We refer e.g. to [2].) Typical examples are the pull-back operator

$$T^*u := \mathcal{L}_{T,1}u = u \circ T, \tag{1.1}$$

and the Perron-Frobenius operator

$$\mathcal{P}u := \mathcal{L}_{T^{-1},|\det DT^{-1}|}u = |\det DT^{-1}| \cdot u \circ T^{-1}. \tag{1.2}$$

This paper is about spectral properties of the operator $\mathcal{L}_{T,g}$. We shall require a hyperbolicity assumption on the mapping T: Let $\Omega \subset X$ be a compact isolated invariant subset for T, with a compact isolating neighborhood V, that is, $\Omega = \bigcap_{m\in\mathbb{Z}} T^m(V)$. We assume that Ω is a hyperbolic subset, that is, there exists an invariant decomposition $T_\Omega M = E^u \oplus E^s$ of the tangent bundle over Ω, satisfying

$$\|DT^m|_{E^s}\| \leqslant C\lambda^m \quad \text{and} \quad \|DT^{-m}|_{E^u}\| \leqslant C\lambda^m,$$

for all $m \geqslant 0$ and $x \in \Omega$, with constants $C > 0$ and $0 < \lambda < 1$. Up to decomposing Ω, we may suppose that the dimensions of $E^u(x)$ and $E^s(x)$ are constant.

We define two local hyperbolicity exponents for each $x \in \Omega$ and each $m \geqslant 1$ by

$$\begin{aligned}\lambda_x(T^m) &= \sup_{v\in E^s(x)\setminus\{0\}} \frac{\|DT_x^m(v)\|}{\|v\|} \leqslant C\lambda^m \quad \text{and}\\ \nu_x(T^m) &= \inf_{v\in E^u(x)\setminus\{0\}} \frac{\|DT_x^m(v)\|}{\|v\|} \geqslant C^{-1}\lambda^{-m}.\end{aligned} \tag{1.3}$$

Let ω be the Riemannian volume form on X, and let $|\det DT|$ be the Jacobian of T, that is, the function given by $T^*\omega = |\det DT| \cdot \omega$. Put, for $m \geqslant 1$,

$$g^{(m)}(x) = \prod_{k=0}^{m-1} g\big(T^k(x)\big).$$

For real numbers $q \leqslant 0 \leqslant p$, and for $1 \leqslant t \leqslant \infty$, we set

$$(1.4) \quad R^{p,q,t}(T,g,\Omega,m) = \sup_{\Omega} \Big(\big| \det DT^m_x \big|^{-1/t} \cdot \big| g^{(m)}(x) \big| \times \max\big\{ \big(\lambda_x(T^m)\big)^p, \big(\nu_x(T^m)\big)^q \big\} \Big),$$

where we read $(\,.\,)^{1/\infty} = 1$ for $t = \infty$. As $\log R^{p,q,t}(T,g,\Omega,m)$ is subadditive with respect to m, we have

$$R^{p,q,t}(T,g,\Omega) := \lim_{m\to\infty} \sqrt[m]{R^{p,q,t}(T,g,\Omega,m)} = \inf_{m\geqslant 1} \sqrt[m]{R^{p,q,t}(T,g,\Omega,m)}.$$

In this paper, we introduce Banach spaces of distributions supported on V, show that the transfer operators $\mathcal{L}_{T,g}$ extend boundedly to those spaces and then give bounds for the essential spectral radii of these transfer operators, using the quantities $R^{p,q,t}(T,g,\Omega)$ introduced above. The main feature in our approach is that we work in Fourier coordinates. The definition and basic properties of the Banach spaces will be given in later sections. Here we state the main theorem as follows. Let $r_{\mathrm{ess}}(L|_{\mathcal{B}})$ be the essential spectral radius of a bounded linear operator $L : \mathcal{B} \to \mathcal{B}$. For non-integer $s > 0$, a mapping is of class C^s if all its partial derivatives of order $[s]$ are $(s-[s])$-Hölder.

THEOREM 1.1. — *Suppose that T is a C^r diffeomorphism for a real number $r > 1$, and let Ω be a hyperbolic invariant set with compact isolating neighborhood V, as described above. Then, for any real numbers $q < 0 < p$ with $p - q < r - 1$, there exist Banach spaces $C^{p,q}_*(T,V)$ and $W^{p,q,t}_*(T,V)$, for $1 < t < \infty$, of distributions supported on V, such that, for any C^{r-1} function $g : X \to \mathbb{C}$ supported on V,*

1) *(Hölder spaces) $\mathcal{L}_{T,g}$ extends boundedly to $\mathcal{L}_{T,g} : C^{p,q}_*(T,V) \circlearrowleft$ and*

$$r_{\mathrm{ess}}(\mathcal{L}_{T,g}|_{C^{p,q}_*(T,V)}) \leqslant R^{p,q,\infty}(T,g,\Omega).$$

2) *(Sobolev spaces) $\mathcal{L}_{T,g}$ extends boundedly to $\mathcal{L}_{T,g} : W^{p,q,t}_*(T,V) \circlearrowleft$ and*

$$r_{\mathrm{ess}}(\mathcal{L}_{T,g}|_{W^{p,q,t}_*(T,V)}) \leqslant R^{p,q,t}(T,g,\Omega), \quad \forall\, 1 < t < \infty.$$

The Banach spaces $C^{p,q}_*(T,V)$ and $W^{p,q,t}_*(T,V)$ contain the set $C^s(V)$ of C^s functions supported on V for any $s > p$.

Hyperbolic attractors and SRB measures

Let us see how to apply Theorem 1.1 to a hyperbolic attractor: Assume in addition to the above that Ω is an attracting hyperbolic set and take the isolating neighborhood V so that $T(V) \subset \mathrm{interior}(V)$. Consider the pull-back operator T^* defined by (1.1) and the Perron-Frobenius operator $\mathcal{P}$ defined by (1.2). Note that these operators are adjoint to each other:

$$\int_X T^* u \cdot v \mathrm{d}\omega = \int_X u \cdot \mathcal{P} v \mathrm{d}\omega. \tag{1.5}$$

Let $h : X \to [0,1]$ be a C^∞ function supported on V and satisfying $h \equiv 1$ on $T(V)$. Then the action of the operator $\mathcal{L}_{T^{-1},g}$ with

$$g(x) = \left| \det DT^{-1}(x) \right| \cdot h(x)$$

coincides with that of the Perron-Frobenius operator $\mathcal{P}$ on $C^{p,q}_*(T^{-1}, V)$ and $W^{p,q,t}_*(T^{-1}, V)$, $1 < t < \infty$. Therefore Theorem 1.1 easily implies:

Theorem 1.2. — *Let Ω be a hyperbolic attractor for a C^r diffeomorphism $T : X \to X$ with $r > 1$, and let V be a compact neighborhood of Ω such that $T(V) \subset \mathrm{interior}(V)$ and $\bigcap_{m \geqslant 0} T^m(V) = \Omega$. For real numbers $q < 0 < p$ with $p - q < r - 1$, the Perron-Frobenius operator $\mathcal{P}$ extends boundedly to*

$$\mathcal{P} : C^{p,q}_*(T^{-1}, V) \longrightarrow C^{p,q}_*(T^{-1}, V)$$

and also to

$$\mathcal{P} : W^{p,q,t}_*(T^{-1}, V) \longrightarrow W^{p,q,t}_*(T^{-1}, V),$$

and it holds

$$r_{\mathrm{ess}}(\mathcal{P}|_{C^{p,q}_*(T^{-1},V)}) \leqslant R^{-q,-p,1}(T, 1, \Omega),$$

and

$$r_{\mathrm{ess}}(\mathcal{P}|_{W^{p,q,t}_*(T^{-1},V)}) \leqslant R^{-q,-p,t/(t-1)}(T, 1, \Omega) \quad \text{for } 1 < t < \infty.$$

(The above bound is strictly less than 1 if $t > 1$ is close enough to 1.)

For real numbers $q < 0 < p$ with $p - q < r - 1$, the modified pull-back operator $T^*_h u := h \cdot (u \circ T)$ extends boundedly to

$$T^*_h : C^{p,q}_*(T, V) \longrightarrow C^{p,q}_*(T, V)$$

and also to

$$T^*_h : W^{p,q,t}_*(T, V) \longrightarrow W^{p,q,t}_*(T, V),$$

and it holds

$$r_{\mathrm{ess}}(T^*_h|_{C^{p,q}_*(T,V)}) \leqslant R^{p,q,\infty}(T, 1, V) < 1.$$

Also,

$$r_{\mathrm{ess}}(T^*_h|_{W^{p,q,t}_*(T,V)}) \leqslant R^{p,q,t}(T, 1, V), \quad \text{for } 1 < t < \infty.$$

(The above bound is strictly less than 1 if t is large enough.)

Once we have the estimates in Theorem 1.2, it is not difficult to see that the spectral radius of the modified pull-back operator T_h^* on $C_*^{p,q}(T,V)$, and on $W_*^{p,q,t}(T,V)$ for large enough t, are equal to one (h is a fixed point of T_h^*). If (T,Ω) is topologically mixing in addition, then 1 is the unique eigenvalue on the unit circle, it is a simple eigenvalue, and the fixed vector of the dual of T_h^* gives rise to the SRB measure μ on Ω: This corresponds to exponential decay of correlations for C^p observables and μ. (See Blank–Keller–Liverani [4, §3.2] for example.)

Remark 1.3. — From (1.5) and Theorem 1.2, if T is Anosov, the Perron-Frobenius operator $\mathcal{P}$ acts naturally on the (strong) dual spaces of $C_*^{p,q}(T,X)$ and $W_*^{p,q,t}(T,X)$, and we have for instance

$$r_{\mathrm{ess}}\big(\mathcal{P}|_{(C_*^{p,q}(T,X))^*}\big) \leqslant R^{p,q,\infty}(T,1,X) < 1,$$

for real numbers $q < 0 < p$ with $p - q < r - 1$.

Spectral stability

We point out that, in the setting of Theorem 1.1, there is $\epsilon > 0$ so that if $\widetilde{T}$ and $\widetilde{g}$, respectively, are ϵ-close to T and g, respectively, in the C^r, resp. C^{r-1}, topology, then the associated operator $\mathcal{L}_{\widetilde{T},\widetilde{g}}$ has same spectral properties than $\mathcal{L}_{T,g}$ on *the same Banach spaces.* Spectral stability can then be proved, as it has been done [4] or [7] for the norms defined there (see also the historical comments below).

Organization of the paper. — After defining a version of our norms in $\mathbb{R}^d$ in Section 2, we proceed in the usual way: prove compact embeddings in Section 5 and a Lasota-Yorke type estimate in Section 6. In Section 8, we prove Theorem 1.1 by reducing to the model from Sections 2–6 starting from a C^r diffeomorphism on a manifold, and applying Hennion's [9] Theorem. For the Hölder spaces, our proof is elementary: it only uses integration by parts. For the Sobolev spaces, we require in addition a standard L^t estimate (Theorem 3.1) for (operator-valued) pseudodifferential operators with C^∞ symbols $P(\xi)$ depending only on ξ.

Comments. — To study spectral properties of Perron-Frobenius operators (and Ruelle transfer operators, more generally), it is primarily important to find appropriate spaces for them to act on. For C^r expanding dynamical systems ($r > 1$) and C^{r-1} weights, Ruelle [13], and later Fried [5] and Gundlach-Latushkin [8], showed that the Banach space of C^{r-1} functions worked nicely. For Anosov diffeomorphisms usual function spaces do not work. A remedy for this since the seventies is reduction to the expanding case, by taking (at least morally) a quotient along the stable foliation. This, however, limits the results severely since the stable foliations are in general only Hölder even if $r = \infty$. In the early nineties, Rugh [14], and then Fried [6], introduced some ideas which allow to bypass this reduction in the case of *analytic* Anosov diffeomorphisms. These results, together with the work of Kitaev [11] on the radius of convergence of dynamical Fredholm determinants for hyperbolic systems with finite differentiability, suggested that appropriate spaces of distributions could be constructed for C^r hyperbolic dynamics. The first major achievement in this direction was made in the work [4] by Blank, Keller and Liverani, in which they considered a Banach space of distributions and gave a bound on the essential spectral radius of the Perron-Frobenius operators acting on it. However their methods only allowed to exploit limited smoothness of the diffeomorphisms. In 2004, Gouëzel and Liverani [7, v1] improved the argument in [4] and introduced a new Banach space of distributions. More recently, in [7, v2], they removed technical assumptions in the first version: their results now are similar to ours. Also in 2004, the first-named author [3] gave a prototype of the use of Fourier coordinates that we exploit in the present paper, but under a strong assumption on the dynamical foliations. The present paper is also partly motivated by the argument in [1]. Finally, note that the spaces of distributions of Gouëzel and Liverani [7] are similar in spirit to the dual space of our Hölder spaces. The definition of the function spaces of Gouëzel and Liverani looks more geometric than ours. Our spaces are natural anisotropic versions of the usual Hölder and Sobolev spaces as we will see, and fit better in the standard theory of functional analysis.

2. Definition of the anisotropic norms

We recall a few facts on Sobolev and Hölder norms, which motivate our definition of anisotropic norms.

Fix an integer $d \geqslant 1$ and a C^∞ function $\chi : \mathbb{R} \to [0,1]$ with

$$\chi(s) = 1, \quad \text{for } s \leqslant 1, \qquad \chi(s) = 0, \quad \text{for } s \geqslant 2.$$

For $n \in \mathbb{Z}_+$, define $\chi_n : \mathbb{R}^d \to [0,1]$ as $\chi_n(\xi) = \chi(2^{-n}|\xi|)$ and, setting $\chi_{-1} \equiv 0$,

$$\psi_n : \mathbb{R}^d \longrightarrow [0,1], \quad \psi_n(\xi) = \chi_n(\xi) - \chi_{n-1}(\xi).$$

We have $1 = \sum_{n=0}^{\infty} \psi_n(\xi)$, and $\operatorname{supp}(\psi_n) \subset \{\xi \mid 2^{n-1} \leqslant |\xi| \leqslant 2^{n+1}\}$. Also $\psi_n(\xi) = \psi_1(2^{-n+1}\xi)$ for $n \geqslant 1$. Thus, for every multi-index α, there exists a constant C_α such that $\|\partial^\alpha \psi_n\|_{L^\infty} \leqslant C_\alpha 2^{-n|\alpha|}$ for all $n \geqslant 0$, and the inverse Fourier transform of ψ_n,

$$\widehat{\psi}_n(x) = (2\pi)^{-d} \int \mathrm{e}^{ix\xi} \psi_n(\xi) \mathrm{d}\xi,$$

decays rapidly, satisfies $\widehat{\psi}_n(x) = 2^{d(n-1)} \widehat{\psi}_1(2^{n-1}x)$ for $n \geqslant 1$ and all x, and is bounded uniformly in n with respect to the L^1-norm.

We decompose each C^∞ function $u : \mathbb{R}^d \to \mathbb{C}$ with compact support as

$$u = \sum_{n \geqslant 0} u_n$$

by defining for integer $n \in \mathbb{Z}_+$,

$$\begin{aligned} u_n(x) = \psi_n(D)u(x) &:= (2\pi)^{-d} \int \mathrm{e}^{i(x-y)\xi} \psi_n(\xi) u(y) \mathrm{d}y \mathrm{d}\xi \\ &= \widehat{\psi}_n * u(x). \end{aligned} \tag{2.1}$$

(Note that $\widehat{u}_n = \psi_n \cdot \widehat{u}$.)

Remark 2.1. — The operator $\psi_n(D)$ in (2.1) is the "pseudodifferential operator with symbol ψ_n." We refer to the books [10] and [15] for more about pseudodifferential operators, although our text is self-contained, except for Theorem 3.1.

From now on, we fix a compact subset $K \subset \mathbb{R}^d$ with non-empty interior. Let $C^\infty(K)$ be the space of complex-valued C^∞ functions on $\mathbb{R}^d$ supported on K. For a real number p and $1 < t < \infty$, we define on $C^\infty(K)$ the norms

$$\|u\|_{C^p_*} = \sup_{n \geqslant 0} 2^{pn} \|u_n\|_{L^\infty} \quad \text{and} \quad \|u\|_{W^{p,t}_*} = \Big\| \Big(\sum_{n \geqslant 0} 4^{pn} |u_n|^2 \Big)^{\frac{1}{2}} \Big\|_{L^t}.$$

It is known that the norm $\|u\|_{C^p_*}$ is equivalent to the C^p norm

$$\|u\|_{C^p} = \max \Big\{ \max_{|\alpha| \leqslant [p]} \sup_{x \in \mathbb{R}^d} |\partial^\alpha u(x)|, \\ \max_{|\alpha| = [p]} \sup_{x \in \mathbb{R}^d} \sup_{y \in \mathbb{R}^d / \{0\}} \frac{|\partial^\alpha u(x+y) - \partial^\alpha u(x)|}{|y|^{p-[p]}} \Big\}$$

provided that $p > 0$ is not an integer, and $\|u\|_{W^{p,t}_*}$ is equivalent to the generalized Sobolev norm

$$\|u\|_{W^{p,t}} = \big\| (1 + \Delta)^{\frac{1}{2}p} u \big\|_{L^t}$$

for any $p \in \mathbb{R}$ and $1 < t < \infty$. (See [16, Appendix A] for a brief account. Note that it is important to use $\ell^2(\mathbb{Z}_+)$ in the definition of $\|u\|_{W^{p,t}_*}$ [12].) The *little* Hölder space $C^p_*(K)$ is the completion of $C^\infty(K)$ with respect to the norm $\|.\|_{C_*^p}$. The generalized Sobolev space $W_*^{p,t}(K)$ for $1 < t < \infty$ is the completion of $C^\infty(K)$ with respect to the norm $\|.\|_{W_*^{p,t}}$.

Remark 2.2. — The *little* Hölder space $C_*^p(K)$ for non-integer $p > 0$ is the closure of $C^\infty(K)$ with respect to the C^p norm and is smaller than the Banach space of C^p functions. Thus our "Hölder" terminology is slightly incorrect and the notation $C_*^p(K)$ may deviate from the standard usage (cf. [16]).

We are going to introduce anisotropic versions of the norms and spaces above. Let $\mathbf{C}_+$ and $\mathbf{C}_-$ be closed cones in $\mathbb{R}^d$ with nonempty interiors such that $\mathbf{C}_+ \cap \mathbf{C}_- = \{0\}$. Let then $\varphi_+, \varphi_- : \mathbf{S}^{d-1} \to [0,1]$ be C^∞ functions on the unit sphere $\mathbf{S}^{d-1}$ in $\mathbb{R}^d$ satisfying

$$\varphi_+(\xi) = \begin{cases} 1, & \text{if } \xi \in \mathbf{S}^{d-1} \cap \mathbf{C}_+, \\ 0, & \text{if } \xi \in \mathbf{S}^{d-1} \cap \mathbf{C}_-, \end{cases} \qquad \varphi_-(\xi) = 1 - \varphi_+(\xi). \tag{2.2}$$

We shall work with combinations $\Theta = (\mathbf{C}_+, \mathbf{C}_-, \varphi_+, \varphi_-)$ as above. For another such combination $\Theta' = (\mathbf{C}'_+, \mathbf{C}'_-, \varphi'_+, \varphi'_-)$, we write $\Theta' < \Theta$ if

$$\mathrm{closure}(\mathbb{R}^d \setminus \mathbf{C}_+) \subset \mathrm{interior}(\mathbf{C}'_-) \cup \{0\}$$

(This implies $\mathbf{C}'_+ \subset \mathbf{C}_+$ and $\mathbf{C}'_- \supset \mathbf{C}_-$ in particular.) For $n \in \mathbb{Z}_+$ and $\sigma \in \{+,-\}$, we define

$$\psi_{\Theta,n,\sigma}(\xi) = \begin{cases} \psi_n(\xi)\,\varphi_\sigma(\xi/|\xi|), & \text{if } n > 0, \\ \frac{1}{2}\chi_n(\xi), & \text{if } n = 0. \end{cases}$$

Note that the $\psi_{\Theta,n,\sigma}(\xi)$ enjoy similar properties as those of the ψ_n, in particular the L^1-norm of the rapidly decaying function $\widehat{\psi}_{\Theta,n,\sigma}$ is bounded uniformly in n. For a C^∞ function $u : \mathbb{R}^d \to \mathbb{C}$ with compact support, an integer $n \in \mathbb{Z}_+$, $\sigma \in \{+,-\}$, and a combination $\Theta = (\mathbf{C}_+, \mathbf{C}_-, \varphi_+, \varphi_-)$, we define

$$u_{\Theta,n,\sigma} = \psi_{\Theta,n,\sigma}(D)u = \widehat{\psi}_{\Theta,n,\sigma} * u.$$

Since $1 = \sum_{n=0}^\infty \sum_{\sigma=\pm} \psi_{\Theta,n,\sigma}(\xi)$, we have $u = \sum_{n\geqslant 0}\sum_{\sigma=\pm} u_{\Theta,n,\sigma}$.

Let p and q be real numbers. For $u \in C^\infty(K)$, we define the anisotropic Hölder norm $\|u\|_{C_*^{\Theta,p,q}}$ by

$$\|u\|_{C_*^{\Theta,p,q}} = \max\left\{ \sup_{n\geqslant 0} 2^{pn} \|u_{\Theta,n,+}\|_{L^\infty}, \sup_{n\geqslant 0} 2^{qn} \|u_{\Theta,n,-}\|_{L^\infty} \right\}, \tag{2.3}$$

and the anisotropic Sobolev norm $\|u\|_{W_*^{\Theta,p,q,t}}$ for $1 < t < \infty$ by

$$\|u\|_{W_*^{\Theta,p,q,t}} = \left\| \Big(\sum_{n\geqslant 0} 4^{pn} |u_{\Theta,n,+}|^2 + 4^{qn} |u_{\Theta,n,-}|^2 \Big)^{\frac{1}{2}} \right\|_{L^t}. \tag{2.4}$$

Let $C_*^{\Theta,p,q}(K)$ be the completion of $C^\infty(K)$ for the norm $\|.\|_{C_*^{\Theta,p,q}}$. Likewise, for $1 < t < \infty$, let $W_*^{\Theta,p,q,t}(K)$ be the completion of $C^\infty(K)$ with respect to the norm $\|.\|_{W_*^{\Theta,p,q,t}}$. We will call these spaces $C_*^{\Theta,p,q}(K)$ and $W_*^{\Theta,p,q,t}(K)$ of distributions the anisotropic Hölder and Sobolev space respectively.(1) In Section 8, we will construct the Banach spaces in Theorem 1.1 by patching these Hölder and Sobolev spaces using local coordinates.

3. Preliminaries

In studying the anisotropic Hölder and Sobolev norms, it is convenient to work in different "coordinates" that we introduce next. Let

$$\Gamma = \big\{(n,\sigma) \mid n \in \mathbb{Z}_+,\ \sigma \in \{+,-\}\big\}$$

and put

$$\mathbb{C}^\Gamma = \big\{\big(f_{n,\sigma}\big)_{(n,\sigma)\in\Gamma} \mid f_{n,\sigma} \in \mathbb{C}\big\}.$$

For real numbers p and q, and for $\mathbf{f} = (f_{n,\sigma})_{(n,\sigma)\in\Gamma}$ and $\mathbf{g} = (g_{n,\sigma})_{(n,\sigma)\in\Gamma}$ in $\mathbb{C}^\Gamma$, we define a norm associated to a scalar product

$$|\mathbf{f}|_{\mathcal{W}^{p,q}} = \sqrt{(\mathbf{f},\mathbf{f})_{\mathcal{W}^{p,q}}}, \quad (\mathbf{f},\mathbf{g})_{\mathcal{W}^{p,q}} = \sum_{n=0}^{\infty} \big(4^{pn} f_{n,+} \cdot \overline{g_{n,+}} + 4^{qn} f_{n,-} \cdot \overline{g_{n,-}}\big),$$

and a norm $|\mathbf{f}|_{\mathcal{C}^{p,q}} = \max\big\{\sup_{n\geqslant 0} 2^{pn}|f_{n,+}|, \sup_{n\geqslant 0} 2^{qn}|f_{n,-}|\big\}$. We then set

$$\mathcal{W}^{p,q} = \big\{\mathbf{f} \in \mathbb{C}^\Gamma\ ;\ |\mathbf{f}|_{\mathcal{W}^{p,q}} < \infty\big\} \quad \text{and} \quad \mathcal{C}^{p,q} = \big\{\mathbf{f} \in \mathbb{C}^\Gamma\ ;\ |\mathbf{f}|_{\mathcal{C}^{p,q}} < \infty\big\}.$$

Recall that $K \subset \mathbb{R}^d$ is a fixed compact set. The operation

$$\mathcal{Q}_\Theta u = \big(\psi_{\Theta,n,\sigma}(D)u\big)_{(n,\sigma)\in\Gamma}$$

gives the correspondences

$$\mathcal{Q}_\Theta : C_*^{\Theta,p,q}(K) \longrightarrow L^\infty(\mathbb{R}^d, \mathcal{C}^{p,q}), \quad \mathcal{Q}_\Theta : W_*^{\Theta,p,q,t}(K) \longrightarrow L^t(\mathbb{R}^d, \mathcal{W}^{p,q}).$$

If we define norms

$$\big\|\mathbf{u}\big\|_{p,q,\infty} = \big\||\mathbf{u}|_{\mathcal{C}^{p,q}}(x)\big\|_{L^\infty} \quad \text{for } \mathbf{u} \in L^\infty(\mathbb{R}^d, \mathcal{C}^{p,q})$$

and

$$\big\|\mathbf{u}\big\|_{p,q,t} = \big\||\mathbf{u}|_{\mathcal{W}^{p,q}}(x)\big\|_{L^t} \quad \text{for } \mathbf{u} \in L^t(\mathbb{R}^d, \mathcal{W}^{p,q}),$$

(1) Note that we relate the spaces to isometric images of L^t spaces in Appendix A.

respectively, the anisotropic Hölder norm and the Sobolev norms coincide with their respective pull-backs by $\mathcal{Q}_\Theta$:

$$\|u\|_{C_*^{\Theta,p,q}} = \|\mathcal{Q}_\Theta u\|_{p,q,\infty} \quad \text{and} \quad \|u\|_{W_*^{\Theta,p,q,t}} = \|\mathcal{Q}_\Theta u\|_{p,q,t}. \tag{3.1}$$

The pseudodifferential operator $\psi(D)$ with symbol $\psi \in C_0^\infty(\mathbb{R}^d)$ extends to a continuous operator $\psi(D) : L^t(\mathbb{R}^d) \to L^t(\mathbb{R}^d)$ for $1 \leqslant t \leqslant \infty$ whose operator norm is bounded by $\|\widehat{\psi}\|_{L^1}$, because

$$\big\|\psi(D)u\big\|_{L^t} = \|\widehat{\psi} * u\|_{L^t} \leqslant \|\widehat{\psi}\|_{L^1} \cdot \|u\|_{L^t} \tag{3.2}$$

by Young's inequality. We will use the following more general result on operator-valued pseudodifferential operators:

Theorem 3.1 (see [16, Thm. 0.11.F]). — *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and let $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be the space of bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$ equipped with the operator norm. If $P(.) \in C^\infty(\mathbb{R}^d, \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2))$ satisfies*

$$\big\|D_\xi^\alpha P(\xi)\big\|_{\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)} \leqslant C_\alpha\big(1+|\xi|^2\big)^{-\frac{1}{2}|\alpha|} \tag{3.3}$$

for each multi-index α, then for each $1 < t < \infty$ the operator

$$P(D) : L^t(\mathbb{R}^d, \mathcal{H}_1) \longrightarrow L^t(\mathbb{R}^d, \mathcal{H}_2)$$

is bounded.

Remark 3.2. — The operator-valued pseudodifferential operator $P(D)$ is defined by

$$P(D)u(x) = (2\pi)^{-d} \int \mathrm{e}^{i(x-y)\xi} P(\xi) u(y) \mathrm{d}\xi \mathrm{d}y.$$

Remark 3.3. — The proof of Theorem 3.1 does not need much knowledge on the theory of pseudodifferential operators and, in fact, is rather simple. Since the case $t = 2$ is proved by using Parseval's identity, one only has to check that the arguments in Sections 0.2 and 0.11 of [16] extend straightforwardly to the operator-valued case.

By "integration by parts on w", we will mean application, for $f \in C^2(\mathbb{R}^d)$ and $g \in C_0^1(\mathbb{R}^d)$, with $\sum_{j=1}^d (\partial_j f)^2 \neq 0$ on $\mathrm{supp}(g)$, of the formula

$$\begin{aligned}\int \mathrm{e}^{if(w)} g(w) \mathrm{d}w &= -\sum_{k=1}^d \int i\big(\partial_k f(w)\big)\mathrm{e}^{if(w)}\, \frac{i(\partial_k f(w)) \cdot g(w)}{\sum_{j=1}^d (\partial_j f(w))^2} \mathrm{d}w \\ &= i \int \mathrm{e}^{if(w)} \sum_{k=1}^d \partial_k \left(\frac{\partial_k f(w) \cdot g(w)}{\sum_{j=1}^d (\partial_j f(w))^2} \right) \mathrm{d}w,\end{aligned}$$

where $w = (w_k)_{k=1}^d \in \mathbb{R}^d$, and ∂_k denotes partial differentiation with respect to w_k.

If $f \in C^{1+\delta}(\mathbb{R}^d)$ and $g \in C_0^\delta(\mathbb{R}^d)$, for $\delta \in (0,1)$, and $\sum_{j=1}^d (\partial_j f)^2 \neq 0$ on $\mathrm{supp}(g)$, we shall consider the following "regularised integration by parts".[(2)] Set, for $k = 1, \dots, d$

$$h_k := \frac{i(\partial_k f(w)) \cdot g(w)}{\sum_{j=1}^d (\partial_j f(w))^2}.$$

Each h_k belongs to $C_0^\delta(\mathbb{R}^d)$. Let $h_{k,\epsilon}$, for small $\epsilon > 0$, be the convolution of h_k with $\epsilon^{-d} v(x/\epsilon)$, where the C^∞ function $v : \mathbb{R}^d \to [0,1]$ is supported in the unit ball and satisfies $\int v(x)\mathrm{d}x = 1$. There is C, independent of f and g, so that for each small $\epsilon > 0$ and all k,

$$\|\partial_k h_{k,\epsilon}\|_{L^\infty} \leqslant C \|h_k\|_{C^\delta} \epsilon^{\delta - 1}, \quad \|h_k - h_{k,\epsilon}\|_{L^\infty} \leqslant C \|h_k\|_{C^\delta} \epsilon^\delta.$$

Finally, for every real number $\Lambda \geqslant 1$

$$\begin{aligned} \int \mathrm{e}^{i\Lambda f(w)} g(w) \mathrm{d}w &= -\sum_{k=1}^d \int i\, \partial_k f(w) \mathrm{e}^{i\Lambda f(w)} \cdot h_k(w) \mathrm{d}w \\ &= \int \frac{\mathrm{e}^{i\Lambda f(w)}}{\Lambda} \sum_{k=1}^d \partial_k h_{k,\epsilon}(w) \mathrm{d}w \\ &\quad - \sum_{k=1}^d \int i\, \partial_k f(w) \mathrm{e}^{i\Lambda f(w)} \big(h_k(w) - h_{k,\epsilon}(w)\big) \mathrm{d}w. \end{aligned} \tag{3.4}$$

4. A pseudolocal property

Although the pseudodifferential operators $\psi_{\Theta,n,\sigma}(D)$ are not local operators, i.e., $u_{\Theta,n,\sigma} = \psi_{\Theta,n,\sigma}(D)u$ does not necessarily vanish outside of the support of u, we have the following rapid decay property, which will be used in Sections 5 and 7:

Lemma 4.1. — *For all positive real numbers b, c, ϵ and each $1 < t \leqslant \infty$, there exists a constant $C = C(b, c, \epsilon, t) > 0$ such that*

$$\big|u_{\Theta,n,\sigma}(x)\big| \leqslant \frac{C \sum_{\tau=\pm} \sum_{\ell \geqslant 0} 2^{-c \max\{n,\ell\}} \|u_{\Theta,\ell,\tau}\|_{L^t}}{d(x, \mathrm{supp}(u))^b},$$

for all $n \geqslant 1$, all $u \in C^\infty(K)$, and all $x \in \mathbb{R}^d$ satisfying $d(x, \mathrm{supp}(u)) > \epsilon$.

(2) We thank S. Gouëzel for suggesting this.

Note that the numerator of the right hand side above is bounded by $C\|u\|_{C_*^{\Theta,p,q}}$ in the case $t=\infty$, and by $C\|u\|_{W_*^{\Theta,p,q,t}}$ in the case $1<t<\infty$ provided $c>-q$.

Proof. — Choose a C^∞ function $\rho:\mathbb{R}^d\to[0,\infty)$ supported in the disk of radius $\frac{1}{4}\epsilon$ centered at the origin and so that $\int\rho(x)\mathrm{d}x=1$. Fix $u\in C^\infty(K)$. Let $U(\epsilon)$ be the ϵ-neighborhood of $\mathrm{supp}(u)$. Put

$$\chi_0(x)=\int \mathbf{1}_{U(\epsilon/4)}(y)\cdot\rho(x-y)\mathrm{d}y,$$

where $\mathbf{1}_Z$ denotes the indicator function of a subset $Z\subset\mathbb{R}^d$. Then χ_0 is supported in $U(\frac{1}{2}\epsilon)$, with $0\leqslant\chi_0(x)\leqslant 1$ for any $x\in\mathbb{R}^d$, and $\chi_0(x)=1$ for $x\in\mathrm{supp}(u)$. Since $\|\chi_0\|_{C_*^c}$ is bounded by a constant depending only on c and ϵ, we have

$$\big\|\psi_j(D)\chi_0\big\|_{L^\infty}\leqslant C(c,\epsilon)2^{-cj}. \tag{4.1}$$

Furthermore, integrating several times by parts on ξ in

$$\psi_j(D)\,\chi_0(y)=(2\pi)^{-d}\int \mathrm{e}^{i(y-w)\xi}\,\psi_j(\xi)\,\chi_0(w)\mathrm{d}\xi\mathrm{d}w,$$

we can see that for any $y\in\mathbb{R}^d$ satisfying $d(y,\mathrm{supp}(\chi_0))\geqslant\frac{1}{4}\epsilon$

$$\big|\psi_j(D)\chi_0(y)\big|\leqslant C(b,c,\epsilon)\cdot 2^{-cj}d(y,\mathrm{supp}(\chi_0))^{-b}. \tag{4.2}$$

We assume $d(x,\mathrm{supp}(u))>\epsilon$ henceforth and estimate

$$\psi_{\Theta,n,\sigma}(D)u(x)=\psi_{\Theta,n,\sigma}(D)(\chi_0 u)(x)=\sum_{(\ell,\tau)\in\Gamma}\widehat{\psi}_{\Theta,n,\sigma}*(\chi_0 u_{\Theta,\ell,\tau})(x).$$

By the Hölder inequality, we have

(4.3)
$$\begin{aligned}\big|\widehat{\psi}_{\Theta,n,\sigma}*(\chi_0 u_{\Theta,\ell,\tau})(x)\big|&\leqslant\big\|\mathbf{1}_{U(\epsilon/2)}(.)\cdot\widehat{\psi}_{\Theta,n,\sigma}(x-\cdot)\big\|_{L^{t'}}\cdot\big\|\chi_0 u_{\Theta,\ell,\tau}\big\|_{L^t}\\&\leqslant C(b,c,\epsilon,t')\cdot 2^{-cn}d\big(x,\mathrm{supp}(u)\big)^{-b}\cdot\big\|u_{\Theta,\ell,\tau}\big\|_{L^t}\end{aligned}$$

for any n and ℓ, where t' is the conjugate exponent of t, i.e. $t^{-1}+(t')^{-1}=1$.

Suppose that $\ell\geqslant n+3$. Then we have

$$\psi_{\Theta,n,\sigma}(D)\big(\big(\psi_j(D)\chi_0\big)\cdot u_{\Theta,\ell,\tau}\big)=0\quad\text{for } j<\ell-2,$$

because $\mathrm{supp}(\psi_{\Theta,n,\sigma})$ does not meet $\mathrm{supp}(\psi_j)+\mathrm{supp}(\psi_{\Theta,\ell,\tau})$ which supports the Fourier transform of $(\psi_j(D)\chi_0)\cdot u_{\Theta,\ell,\tau}$. Thus

$$\psi_{\Theta,n,\sigma}(D)(\chi_0 u_{\Theta,\ell,\tau})=\sum_{j\geqslant\ell-2}\widehat{\psi}_{\Theta,n,\sigma}*\big(\big(\psi_j(D)\chi_0\big)\cdot u_{\Theta,\ell,\tau}\big).$$

For each $j \geqslant \ell - 2$ with $\ell \geqslant n+3$, we can see from (4.1)–(4.2) that

$$\begin{aligned}
&\left|\widehat{\psi}_{\Theta,n,\sigma} * \big((\psi_j(D)\chi_0) \cdot u_{\Theta,\ell,\tau}\big)(x)\right| \\
&\qquad\leqslant \|\widehat{\psi}_{\Theta,n,\sigma}\|_{L^\infty} \cdot \|\mathbf{1}_{\mathbb{R}^d\setminus U(\delta)} \cdot \psi_j(D)\chi_0\|_{L^{t'}} \cdot \|u_{\Theta,\ell,\tau}\|_{L^t} \\
&\qquad\qquad + \left\|\mathbf{1}_{U(\delta)} \cdot \widehat{\psi}_{\Theta,n,\sigma}(x-\cdot)\right\|_{L^{t'}} \cdot \|\psi_j(D)\chi_0\|_{L^\infty} \cdot \|u_{\Theta,\ell,\tau}\|_{L^t} \\
&\qquad\leqslant C(b,c,\epsilon,t) \cdot 2^{-cj} d\big(x, \mathrm{supp}(u)\big)^{-b} \cdot \|u_{\Theta,\ell,\tau}\|_{L^t},
\end{aligned}$$

where $\delta = \frac{1}{2}\epsilon + \frac{1}{4}d(x, \mathrm{supp}(u))$. (We decomposed the domain of integration in the convolution into $U(\delta)$ and its complement.) Hence, if $\ell \geqslant n+3$, we have

$$\left|\psi_{\Theta,n,\sigma}(D)(\chi_0 u_{\Theta,\ell,\tau})(x)\right| \leqslant C(b,c,\epsilon,t) \cdot 2^{-c\ell} d(x, \mathrm{supp}(u))^{-b} \cdot \|u_{\Theta,\ell,\tau}\|_{L^t}.$$

With this and (4.3) we conclude the proof of the lemma. □

5. Compact embeddings

Recall that $K \subset \mathbb{R}^d$ is a compact subset with non-empty interior. If $p' \leqslant p$ and $q' \leqslant q$, we have the obvious continuous inclusions

$$\begin{aligned}
&C_*^{\Theta,p,q}(K) \subset C_*^{\Theta,p',q'}(K), \\
&W_*^{\Theta,p,q,t}(K) \subset W_*^{\Theta,p',q',t}(K) \quad \text{for } 1 < t < \infty.
\end{aligned} \tag{5.1}$$

Here we prove:

Proposition 5.1. — *If $p' < p$ and $q' < q$, the inclusions* (5.1) *are compact.*

Proof. — Take any sequence $u^{(k)}$, $k \geqslant 1$, in $C^\infty(K)$ such that

$$\|u^{(k)}\|_{C_*^{\Theta,p,q}} < E \quad \big(\text{resp. } \|u^{(k)}\|_{W_*^{\Theta,p,q,t}} < E\big)$$

for some positive constant $E > 0$. We show that there exists a subsequence $\{k(j)\}$ such that $\{u^{(k(j))}\}$ is a Cauchy sequence in the norm $\|.\|_{C_*^{\Theta,p',q'}}$ (resp. $\|.\|_{W_*^{\Theta,p',q',t}}$). For each $(n,\sigma) \in \Gamma$, the Fourier transform $\hat{u}^{(k)}_{\Theta,n,\sigma}$ of $u^{(k)}_{\Theta,n,\sigma}$ is a C^∞ function supported on $\{\xi \mid 2^{n-1} \leqslant |\xi| \leqslant 2^{n+1}\}$, and its first order derivatives are bounded uniformly for $k \geqslant 1$ and $\xi \in \mathbb{R}^d$ since $(1+|x|)u^{(k)}_{\Theta,n,\sigma}(x)$ are uniformly bounded in L^1norm from Lemma 4.1. Hence, by Ascoli-Arzelá's Theorem and by the diagonal argument, we can choose a subsequence $\{k(j)\}$ such that the sequences $\{\hat{u}^{(k(j))}_{\Theta,n,\sigma}\}_{j=0}^\infty$ are all Cauchy sequences with respect to the L^∞-norm and so is the sequence

$\{u^{(k(j))}_{\Theta,n,\sigma}\}_{j=0}^{\infty}$. This is the subsequence with the required property. Indeed, for given $\epsilon > 0$, we can choose an integer $N > 0$ so that

$$\sum_{n>N} \big(2^{(q'-q)n} + 2^{(p'-p)n}\big) E < \frac{1}{2}\epsilon,$$

and then we have

$$\big\|u^{(k(j))} - u^{(k(j'))}\big\|_{C_*^{\Theta,p',q'}} \leqslant \frac{1}{2}\epsilon + \sum_{n\leqslant N} \big(2^{p'n}\big\|u^{(k(j))}_{\Theta,n,+} - u^{(k(j'))}_{\Theta,n,+}\big\|_{L^\infty} + 2^{q'n}\big\|u^{(k(j))}_{\Theta,n,-} - u^{(k(j'))}_{\Theta,n,-}\big\|_{L^\infty}\big),$$

(resp. the same inequality with the norms $\|.\|_{C_*^{\Theta,p',q'}}$ and $\|.\|_{L^\infty}$ replaced by $\|.\|_{W_*^{\Theta,p',q',t}}$ and $\|.\|_{L^t}$). The right hand side is $< \epsilon$ for large enough j, j'. □

6. A Lasota-Yorke type inequality

Let $r > 1$. Let $K, K' \subset \mathbb{R}^d$ be compact subsets with non-empty interiors, and take a compact neighborhood W of K. Let $T : W \to K'$ be a C^r diffeomorphism onto its image. Let $g : \mathbb{R}^d \to \mathbb{C}$ be a C^{r-1} function such that $\operatorname{supp}(g) \subset K$. In this section we study the transfer operator on $\mathbb{R}^d$:

$$L : C^{r-1}(K') \longrightarrow C^{r-1}(K), \quad Lu(x) = g(x) \cdot u \circ T(x).$$

For two fixed combinations

$$\Theta = (\mathbf{C}_+, \mathbf{C}_-, \varphi_+, \varphi_-) \quad \text{and} \quad \Theta' = (\mathbf{C}'_+, \mathbf{C}'_-, \varphi'_+, \varphi'_-)$$

as in Section 2, we make the following *cone-hyperbolicity* assumption on T:

$$\text{(6.1)} \qquad DT_x^{\mathrm{tr}}\big(\mathbb{R}^d \setminus \operatorname{interior}(\mathbf{C}_+)\big) \subset \operatorname{interior}(\mathbf{C}'_-) \cup \{0\} \quad \text{for all } x \in W,$$

where DT_x^{tr} denotes the transpose of the derivative of T at x. We put

$$\|T\|_+ = \sup_{x\in\operatorname{supp}(g)} \ \sup_{0\neq DT_x^{\mathrm{tr}}(\xi)\notin \mathbf{C}'_-} \frac{\|DT_x^{\mathrm{tr}}(\xi)\|}{\|\xi\|},$$

$$\|T\|_- = \inf_{x\in\operatorname{supp}(g)} \ \inf_{0\neq \xi\notin \mathbf{C}_+} \frac{\|DT_x^{\mathrm{tr}}(\xi)\|}{\|\xi\|}.$$

Note that $\|T\|_+$ is a largest (i.e. weakest) contraction and that $\|T\|_-$ is a smallest (i.e. weakest) expansion.

THEOREM 6.1. — *Fix Θ and Θ' and assume* (6.1). *For any $q < 0 < p$ such that $p - q < r - 1$, the operator L extends to continuous operators*

$$L : C_*^{\Theta,p,q}(K') \longrightarrow C_*^{\Theta',p,q}(K), \quad L : W_*^{\Theta,p,q,t}(K') \longrightarrow W_*^{\Theta',p,q,t}(K)$$

for $1 < t < \infty$. Furthermore, for any $0 \leqslant p' < p$ and $q' < q$ such that $p - q' < r - 1$, we have the following Lasota-Yorke type inequalities:

- Hölder case. — *There exist a constant C, that does not depend on T or g, and a constant $C(T, g)$, that may depend on T and g, such that for any $u \in C_*^{\Theta,p,q}(K)$*

$$\|Lu\|_{C_*^{\Theta',p,q}} \leqslant C\|g\|_\infty \cdot \max\{\|T\|_+^p, \|T\|_-^q\} \cdot \|u\|_{C_*^{\Theta,p,q}} + C(T, g) \cdot \|u\|_{C_*^{\Theta,p',q'}}.$$

- Sobolev case. — *For each $1 < t < \infty$, there exist a constant $C(t)$, that does not depend on T or g, and a constant $C(T, g, t)$, that may depend on T and g, such that for any $u \in W_*^{\Theta,p,q,t}(K)$*

$$\|Lu\|_{W_*^{\Theta',p,q,t}} \leqslant C(t)\|g\|_\infty \frac{\max\{\|T\|_+^p, \|T\|_-^q\}}{\inf |\det DT|^{1/t}} \|u\|_{W_*^{\Theta,p,q,t}} + C(T, g, t) \cdot \|u\|_{W_*^{\Theta,p',q',t}}.$$

For the proof of Theorem 6.1, we need more notation. By (6.1) there exists a closed cone $\widetilde{\mathbf{C}}_+$ such that $\widetilde{\mathbf{C}}_+ \subset \text{interior}(\mathbf{C}_+) \cup \{0\}$ and that

$$DT_x^{\text{tr}}\big(\mathbb{R}^d \setminus \text{interior}(\widetilde{\mathbf{C}}_+)\big) \subset \text{interior}(\mathbf{C}'_-) \cup \{0\} \quad \text{for all } x \in \text{supp}(g). \tag{6.2}$$

Fix also a closed cone $\widetilde{\mathbf{C}}_- \subset \text{interior}\,(\mathbf{C}_-) \cup \{0\}$ and let $\tilde\varphi_+$, $\tilde\varphi_- : \mathbf{S}^{d-1} \to [0, 1]$ be C^∞ functions satisfying

$$\widetilde\varphi_+(\xi) = \begin{cases} 1, & \text{if } \xi \notin \mathbf{S}^{d-1} \cap \mathbf{C}_-, \\ 0, & \text{if } \xi \in \mathbf{S}^{d-1} \cap \widetilde{\mathbf{C}}_-. \end{cases} \qquad \widetilde\varphi_-(\xi) = \begin{cases} 0, & \text{if } \xi \in \mathbf{S}^{d-1} \cap \widetilde{\mathbf{C}}_+, \\ 1, & \text{if } \xi \notin \mathbf{S}^{d-1} \cap \mathbf{C}_+. \end{cases}$$

Recall the function χ we fixed in the beginning. Put, for $\ell \geqslant 1$,

$$\widetilde\psi_\ell(\xi) = \chi\big(2^{-\ell-1}|\xi|\big) - \chi\big(2^{-\ell+2}|\xi|\big),$$

(note that $\widetilde\psi_\ell = \psi_{\ell-1} + \psi_\ell + \psi_{\ell+1}$ if $\ell \geqslant 2$) and then define, for $(\ell, \tau) \in \Gamma$,

$$\widetilde\psi_{\Theta,\ell,\tau}(\xi) = \begin{cases} \widetilde\psi_\ell(\xi)\,\widetilde\varphi_\tau(\xi/|\xi|), & \text{if } \ell \geqslant 1, \\ \chi\big(\frac{1}{2}|\xi|\big), & \text{if } \ell = 0. \end{cases}$$

Note that $\widetilde\psi_{\Theta,\ell,\tau}(\xi) = 1$ if $\xi \in \text{supp}(\psi_{\Theta,\ell,\tau})$.

Next, fix a closed cone $\widetilde{\mathbf{C}}'_- \subset \text{interior}(\mathbf{C}'_-) \cup \{0\}$ and take integers $h^-_{\min}$ and $h^+_{\max}$ such that for all $x \in W$

$$\big\|DT_x^{\text{tr}}(\xi)\big\| < 2^{h^+_{\max}-4}\|\xi\| \quad \text{if } DT_x^{\text{tr}}(\xi) \notin \widetilde{\mathbf{C}}'_-,$$

$$2^{h^-_{\min}+4}\|\xi\| < \big\|DT_x^{\text{tr}}(\xi)\big\| \quad \text{if } \xi \notin \widetilde{\mathbf{C}}_+.$$

By modifying the cones $\widetilde{\mathbf{C}}_+$ and $\widetilde{\mathbf{C}}'_-$ if necessary, we may and do assume

$$2^{h^-_{\min}} > 2^{-5}\|T\|_-, \qquad 2^{h^+_{\max}} < 2^{+5}\|T\|_+.$$

We write $(\ell,\tau) \hookrightarrow (n,\sigma)$ if either

- $(\tau,\sigma) = (+,+)$ and $n \leqslant \ell + h^+_{\max}$, or
- $(\tau,\sigma) = (-,-)$ and $\ell + h^-_{\min} \leqslant n$, or
- $(\tau,\sigma) = (+,-)$ and $(n \geqslant h^-_{\min}$ or $\ell \geqslant -h^+_{\max})$.

We write $(\ell,\tau) \not\hookrightarrow (n,\sigma)$ otherwise.

By the definition of $\not\hookrightarrow$ and by (6.2), there exists an integer $N(T) > 0$ such that, if $(\ell,\tau) \not\hookrightarrow (n,\sigma)$ and $\max\{n,\ell\} \geqslant N(T)$, we have

$$d\big(\mathrm{supp}(\psi_{\Theta',n,\sigma}), DT^{\mathrm{tr}}_x\big(\mathrm{supp}(\widetilde{\psi}_{\Theta,\ell,\tau})\big)\big) \geqslant 2^{\max\{n,\ell\}-N(T)} \tag{6.3}$$

for $x \in \mathrm{supp}(g)$. Indeed, the case $(\tau,\sigma) = (-,+)$ follows from (6.2). Taking $N(T) \geqslant \max\{3, h^-_{\min} + 3\}$, the case $(-,-)$ follows from the definition of $h^-_{\min}$. The case $(+,+)$ follows from the definition of $h^+_{\max}$ if $\xi \in DT^{\mathrm{tr}}_x(\mathrm{supp}(\widetilde{\psi}_{\Theta,\ell,\tau}))$ is such that $DT^{\mathrm{tr}}_x(\xi) \notin \widetilde{\mathbf{C}}'_-$, and, taking

$$N(T) \geqslant \max\{3, -h^+_{\max} + 3\},$$

from the fact that $\widetilde{\mathbf{C}}'_- \subset$ interior $(\mathbf{C}'_-)$ if $DT^{\mathrm{tr}}_x(\xi) \in \widetilde{\mathbf{C}}'_-$. Finally, the case $(+,-)$ does not occur if we take $N(T) \geqslant \max\{h^-_{\min}, -h^+_{\max}\}$.

Proof of Theorem 6.1. — For $v := Lu$, we have

$$v_{\Theta',n,\sigma} = \sum_{(\ell,\tau)\in\Gamma} \psi_{\Theta',n,\sigma}(D)L(u_{\Theta,\ell,\tau}).$$

We define $\mathbf{S}$ as the formal matrix of operators

$$S^{\ell,\tau}_{n,\sigma}u = \begin{cases} \psi_{\Theta',n,\sigma}(D)L\,u, & \text{if } (\ell,\tau) \hookrightarrow (n,\sigma), \\ \psi_{\Theta',n,\sigma}(D)L\,\tilde{\psi}_{\Theta,\ell,\tau}(D)u & \text{if } (\ell,\tau) \not\hookrightarrow (n,\sigma), \end{cases}$$

for $((\ell,\tau),(n,\sigma)) \in \Gamma\times\Gamma$. That is, we set

$$\mathbf{S}\big((u_{\Theta,\ell,\tau})_{(\ell,\tau)\in\Gamma}\big) = \Big(\sum_{(\ell,\tau)\in\Gamma} S^{\ell,\tau}_{n,\sigma}u_{\Theta,\ell,\tau}\Big)_{(n,\sigma)\in\Gamma}.$$

Since $\widetilde{\psi}_{\Theta,\ell,\tau}(D)u_{\Theta,\ell,\tau} = u_{\Theta,\ell,\tau}$, we have the commutative relation $\mathbf{S}\circ\mathcal{Q}_\Theta = \mathcal{Q}_{\Theta'}\circ L$. For the proof of Theorem 6.1, it is enough to show

$$\|\mathbf{S}\mathbf{u}\|_{p,q,\infty} < C2^{\max\{ph^+_{\max},qh^-_{\min}\}}\|g\|_{L^\infty}\|\mathbf{u}\|_{p,q,\infty} + C(T,g)\|\mathbf{u}\|_{p',q',\infty}$$

and that, for $1 < t < \infty$,

$$\|\mathbf{S}\mathbf{u}\|_{p,q,t} < \frac{C(t)2^{\max\{ph^+_{\max},qh^-_{\min}\}}\|g\|_{L^\infty}}{\inf|\det DT|^{1/t}}\|\mathbf{u}\|_{p,q,t} + C(T,g,t)\|\mathbf{u}\|_{p',q',t}.$$

To prove the above inequalities, we split the matrix of operators $\mathbf{S}$ into two parts:

$$\mathbf{S}_0 = (\widetilde{S}^{\ell,\tau}_{n,\sigma}), \quad \widetilde{S}^{\ell,\tau}_{n,\sigma} = \begin{cases} S^{\ell,\tau}_{n,\sigma}, & \text{if } (\ell,\tau) \hookrightarrow (n,\sigma), \\ 0 & \text{if } (\ell,\tau) \not\hookrightarrow (n,\sigma), \end{cases}$$

and $\mathbf{S}_1 = \mathbf{S} - \mathbf{S}_0$. We first consider $\mathbf{S}_0$. This is the composition $\Phi(D) \circ \Psi \circ \mathbf{L}$ of

- the operator $\mathbf{L}$ defined by $\mathbf{L}(\mathbf{u})(x) = g(x) \cdot \mathbf{u} \circ T(x)$,
- the operator Ψ defined by

$$\Psi\big((f_{\ell,\tau})_{(\ell,\tau)\in\Gamma}\big)_{(n,\sigma)} = \sum_{(\ell,\tau)\hookrightarrow(n,\sigma)} f_{\ell,\tau},$$

  where $\sum_{(\ell,\tau)\hookrightarrow(n,\sigma)}$ is the sum over $(\ell,\tau) \in \Gamma$ such that $(\ell,\tau) \hookrightarrow (n,\sigma)$,
- the pseudodifferential operator $\Phi(D)$ with symbol $\Phi : \mathbb{R}^d \to \mathcal{L}(\mathbb{C}^\Gamma, \mathbb{C}^\Gamma)$,

$$\Phi(\xi)\big((f_{\ell,\tau})_{(\ell,\tau)\in\Gamma}\big)_{(n,\sigma)} = \psi_{\Theta',n,\sigma}(\xi) f_{n,\sigma}.$$

Clearly $\|\mathbf{L}\|_{p,q,\infty} \leqslant \|g\|_\infty$ and $\|\mathbf{L}\|_{p,q,t} \leqslant \|g\|_\infty \sup|\det DT|^{-1/t}$ for $1 < t < \infty$. Also we can prove

$$\|\Psi\|_{p,q,t} \leqslant C 2^{\max\{p h^+_{\max}, q h^-_{\min}\}} \quad \text{for } 1 < t \leqslant \infty \tag{6.4}$$

as follows. Set $c(+) = p$, $c(-) = q$ and observe that there is C so that

$$\begin{aligned} \sum_{(\ell,\tau):(\ell,\tau)\hookrightarrow(n,\sigma)} 2^{c(\sigma)n - c(\tau)\ell} &\leqslant C 2^{\max\{p h^+_{\max}, q h^-_{\min}\}}, \quad \forall (n,\sigma), \\ \sum_{(n,\sigma):(\ell,\tau)\hookrightarrow(n,\sigma)} 2^{c(\sigma)n - c(\tau)\ell} &\leqslant C 2^{\max\{p h^+_{\max}, q h^-_{\min}\}}, \quad \forall (\ell,\tau). \end{aligned} \tag{6.5}$$

For $\mathbf{f}(x) = (f_{n,\sigma}(x))_{(n,\sigma)\in\Gamma}$, we have, at each point $x \in \mathbb{R}^d$,

$$\big|\Psi(\mathbf{f})\big|_{C^{p,q}}(x) \leqslant C 2^{\max\{p h^+_{\max}, q h^-_{\min}\}} |\mathbf{f}|_{C^{p,q}}(x) \tag{6.6}$$

and also,

$$\big|\Psi(\mathbf{f})\big|_{\mathcal{W}^{p,q}}(x) \leqslant C 2^{\max\{p h^+_{\max}, q h^-_{\min}\}} |\mathbf{f}|_{\mathcal{W}^{p,q}}(x). \tag{6.7}$$

The latter inequality is obtained by using Cauchy-Schwartz and (6.5), as follows:

$$\begin{aligned}
\big|\Psi(\mathbf{f})\big|_{\mathcal{W}^{p,q}}(x)^2 &= \sum_{(n,\sigma)\in\Gamma}\Big(\sum_{(\ell,\tau):(\ell,\tau)\hookrightarrow(n,\sigma)} 2^{c(\sigma)n-c(\tau)\ell}2^{c(\tau)\ell}f_{\ell,\tau}\Big)^2\\
&\leqslant \sum_{(n,\sigma)\in\Gamma}\Big(\sum_{(\ell,\tau):(\ell,\tau)\hookrightarrow(n,\sigma)} 2^{c(\sigma)n-c(\tau)\ell}\Big)\Big(\sum_{(\ell,\tau)\hookrightarrow(n,\sigma)} 2^{c(\sigma)n-c(\tau)\ell}\cdot 2^{2c(\tau)\ell}\big|f_{\ell,\tau}(x)\big|^2\Big)\\
&\leqslant C2^{\max\{ph^+_{\max},qh^-_{\min}\}}\sum_{(n,\sigma)\in\Gamma}\sum_{(\ell,\tau):(\ell,\tau)\hookrightarrow(n,\sigma)} 2^{c(\sigma)n-c(\tau)\ell}\cdot 2^{2c(\tau)\ell}\big|f_{\ell,\tau}(x)\big|^2\\
&\leqslant C2^{\max\{ph^+_{\max},qh^-_{\min}\}}\sum_{(\ell,\tau)\in\Gamma}\sum_{(n,\sigma):(\ell,\tau)\hookrightarrow(n,\sigma)} 2^{c(\sigma)n-c(\tau)\ell}\cdot 2^{2c(\tau)\ell}|f_{\ell,\tau}(x)|^2\\
&\leqslant C2^{2\max\{ph^+_{\max},qh^-_{\min}\}}\sum_{(\ell,\tau)\in\Gamma} 2^{2c(\tau)\ell}\big|f_{\ell,\tau}(x)\big|^2.
\end{aligned}$$

Taking the supremum and L^t norm of both sides of (6.6), (6.7), respectively, we obtain (6.4). The operator $\Phi(D)$ is bounded with respect to the norm $\|.\|_{p,q,t}$ for $1<t\leqslant\infty$: If $t=\infty$, this follows from (3.2) since, as mentioned in Section 2, $\widehat{\psi}_{\Theta',n,\sigma}$ is bounded uniformly for $(n,\sigma)\in\Gamma$ in L^1-norm, and the case $1<t<\infty$ follows from Theorem 3.1. Thus we conclude

$$\|\mathbf{S}_0\|_{p,q,\infty}\leqslant C\|g\|_\infty\cdot 2^{\max\{ph^+_{\max},qh^-_{\min}\}},$$

and

$$\|\mathbf{S}_0\|_{p,q,t}\leqslant \frac{C(t)\|g\|_\infty\cdot 2^{\max\{ph^+_{\max},qh^-_{\min}\}}}{\inf|\det DT|^{1/t}}\quad\text{for } 1<t<\infty.$$

Next we consider $\mathbf{S}_1$. It only remains to show the following two estimates:

$$\|\mathbf{S}_1\mathbf{u}\|_{p,q,\infty}<C(T,g)\|\mathbf{u}\|_{p',q',\infty}\ \text{ and }\ \|\mathbf{S}_1\mathbf{u}\|_{p,q,t}<C(T,g,t)\|\mathbf{u}\|_{p',q',t},$$

for all $1<t<\infty$. For this, it is enough to prove that for $1<t\leqslant\infty$,

$$\|S^{\ell,\tau}_{n,\sigma}u\|_{L^t}\leqslant C(T,g)2^{-(r-1)\max\{n,\ell\}}\|u\|_{L^t}\quad\text{if }\ (\ell,\tau)\not\hookrightarrow(n,\sigma). \tag{6.8}$$

Indeed, setting $c'(+)=p'$, and $c'(-)=q'$, (6.8) implies that

$$\begin{aligned}
\|\mathbf{S}_1\mathbf{u}\|_{p,q,\infty}&\leqslant \sup_{(n,\sigma)\in\Gamma}\sum_{(\ell,\tau)\not\hookrightarrow(n,\sigma)} 2^{c(\sigma)n}\|S^{\ell,\tau}_{n,\sigma}u_{\Theta,\ell,\tau}\|_{L^\infty}\\
&\leqslant C(T,g)\sup_{(n,\sigma)\in\Gamma}\Big(\sum_{(\ell,\tau)\not\hookrightarrow(n,\sigma)} 2^{c(\sigma)n-c'(\tau)\ell-(r-1)\max\{n,\ell\}}\Big)\|\mathbf{u}\|_{p',q',\infty},
\end{aligned}$$

and, for $1 < t < \infty$, (in the first inequality below, the triangle inequality is used twice, pointwise and for L^t)

$$\begin{aligned}\|\mathbf{S}_1\mathbf{u}\|_{p,q,t} &\leqslant \sum_{(n,\sigma)\in\Gamma}\sum_{(\ell,\tau)\not\hookrightarrow(n,\sigma)} 2^{c(\sigma)n}\|S^{\ell,\tau}_{n,\sigma}u_{\Theta,\ell,\tau}\|_{L^t}\\ &\leqslant C(T,g)\Big(\sum_{(n,\sigma)\in\Gamma}\sum_{(\ell,\tau)\not\hookrightarrow(n,\sigma)} 2^{c(\sigma)n-c'(\tau)\ell-(r-1)\max\{n,\ell\}}\Big)\|\mathbf{u}\|_{p',q',t}.\end{aligned}$$

The sums in (.) above are finite from the assumption $p - q' < r - 1$.

We prove (6.8). Since (6.8) is obvious when $\max\{n,\ell\} < N(T)$, we will assume $\max\{n,\ell\} \geqslant N(T)$. Rewrite the operator $S^{\ell,\tau}_{n,\sigma}$ in the case $(\ell,\tau)\not\hookrightarrow(n,\sigma)$ as

$$(S^{\ell,\tau}_{n,\sigma}u)(x) = (2\pi)^{-2d}\int V^{\ell,\tau}_{n,\sigma}(x,y)\cdot u\circ T(y)\,\big|\det DT(y)\big|\,\mathrm{d}y,$$

where, extending T to a bilipschitz diffeomorphism of $\mathbb{R}^d$ we write

$$V^{\ell,\tau}_{n,\sigma}(x,y) = \int \mathrm{e}^{i(x-w)\xi+i(T(w)-T(y))\eta}g(w) \times \psi_{\Theta',n,\sigma}(\xi)\widetilde{\psi}_{\Theta,\ell,\tau}(\eta)\,\mathrm{d}w\,\mathrm{d}\xi\, d\eta. \tag{6.9}$$

Since $\|u\circ T\cdot|\det DT|\|_{L^t} \leqslant C(T)\|u\|_{L^t}$, the inequality (6.8) follows if we show that there exists $C(T,g)$ such that for all $(\ell,\tau)\not\hookrightarrow(n,\sigma)$ and all $1 < t \leqslant \infty$ the operator norm of the integral operator

$$H^{\ell,\tau}_{n,\sigma} : v \longmapsto \int V^{\ell,\tau}_{n,\sigma}(\cdot,y)v(y)dy$$

acting on $L^t(\mathbb{R}^d)$ is bounded by $C(T,g)\cdot 2^{-(r-1)\max\{n,\ell\}}$.

Define the positive-valued integrable function $b : \mathbb{R}^d \to \mathbb{R}$ by

$$b(x) = 1 \ \text{ if } |x| \leqslant 1, \quad b(x) = |x|^{-d-1} \ \text{ if } |x| > 1. \tag{6.10}$$

The required estimate on $H^{\ell,\tau}_{n,\sigma}$ follows if we show

$$\big|V^{\ell,\tau}_{n,\sigma}(x,y)\big| \leqslant C(T,g)2^{-(r-1)\max\{n,\ell\}}\cdot 2^{d\min\{n,\ell\}}\, b\big(2^{\min\{n,\ell\}}(x-y)\big) \tag{6.11}$$

for some $C(T,g) > 0$ and all $(\ell,\tau)\not\hookrightarrow(n,\sigma)$. Indeed, as the right hand side of (6.11) is written as a function of $x-y$, say $B(x-y)$, we have, by Young's inequality,

$$\begin{aligned}\|H^{\ell,\tau}_{n,\sigma}v\|_{L^t} &\leqslant \big\|B * |v|\big\|_{L^t} \leqslant \|B\|_{L^1}\cdot\|v\|_{L^t}\\ &\leqslant C(T,g)2^{-(r-1)\max\{n,\ell\}}\cdot\|b\|_{L^1}\cdot\|v\|_{L^t}.\end{aligned}$$

Below we prove the estimate (6.11). Integrating (6.9) by parts $[r]-1$ times on w (in particular, if $1<r<2$ we do nothing), we obtain

$$(6.12)\qquad V^{\ell,\tau}_{n,\sigma}(x,y)=\int \mathrm{e}^{i(x-w)\xi+i(T(w)-T(y))\eta}F(\xi,\eta,w)\\ \times \psi_{\Theta',n,\sigma}(\xi)\widetilde{\psi}_{\Theta,\ell,\tau}(\eta)\,\mathrm{d}w\mathrm{d}\xi\mathrm{d}\eta,$$

where $F(\xi,\eta,w)$ is a $C^{r-[r]}$ function in w which is C^∞ in the variables ξ and η. Using (6.3), we can see that for all α, β

$$(6.13)\qquad \big\|\partial_\xi^\alpha\partial_\eta^\beta F(\xi,\eta,.)\big\|_{C^{r-[r]}}\leqslant C_{\alpha,\beta}(T,g)2^{-n|\alpha|-\ell|\beta|-([r]-1)\max\{n,\ell\}}.$$

Assume first that r is an integer (then, $r=[r]\geqslant 2$). Put

$$G_{n,\ell}(\xi,\eta,w)=F(\xi,\eta,w)\,\psi_{\Theta',n,\sigma}(\xi)\,\widetilde{\psi}_{\Theta,\ell,\tau}(\eta).$$

Consider the scaling

$$\widetilde{G}_{n,\ell}(\xi,\eta,w)=G_{n,\ell}(2^{n-1}\xi,2^{\ell-1}\eta,w).$$

Then, denoting by $\mathcal{F}$ the inverse Fourier transform with respect to the variable (ξ,η), we have

$$\mathcal{F}G_{n,\ell}(u,v,w)=2^{(n-1)d+(\ell-1)d}\mathcal{F}\widetilde{G}_{n,\ell}(2^{n-1}u,2^{\ell-1}v,w).$$

The estimate (6.13) implies that

$$\big\|\partial_\xi^\alpha\partial_\eta^\beta \widetilde{G}_{n,\ell}(\xi,\eta,.)\big\|_{C^{r-[r]}}\leqslant C_{\alpha,\beta}(T,g)2^{-([r]-1)\max\{n,\ell\}}.$$

Integrating by parts with respect to ξ and η, this gives that the functions

$$W_{n,\ell}(u,v,w)=2^{([r]-1)\max\{n,\ell\}}\mathcal{F}\widetilde{G}_{n,\ell}(u,v,w)$$

as functions of u and v are smooth rapidly decaying functions, uniformly bounded (as rapidly decaying functions) with respect to w, n, and ℓ. From this, it is easy to see that

$$\big|W_{n,\ell}(u,v,w)\big|\leqslant C(T,g)b(u)b(v).$$

By definition,

$$(6.14)\qquad \big|V^{\ell,\tau}_{n,\sigma}(x,y)\big|\leqslant\int\big|\mathcal{F}G_{n,\ell}(x-w,T(w)-T(y),w)\big|\mathrm{d}w\\ \leqslant 2^{-([r]-1)\max\{n,\ell\}+(n-1)d+(\ell-1)d}\\ \times\int\big|W_{n,\ell}(2^{n-1}(x-w),2^{\ell-1}(T(w)-T(y)),w)\big|\mathrm{d}w.$$

With this, it is not difficult to conclude (6.11) for integer $r\geqslant 2$, by using either the change of variables

$$u=2^{n-1}(x-w)\quad\text{or}\quad v=2^{\ell-1}\big(T(w)-T(y)\big).$$

Indeed, the case when $\|x-y\| \leqslant 2^{-\min\{n,\ell\}}$ is straightforward. If $\|x-y\| > 2^{-\min\{n,\ell\}}$, letting $q_0 \leqslant \min\{\ell,n\}$ be the integer so that $\|x-y\| \in [2^{-q_0}, 2^{-q_0+1})$, use that at least one of the following conditions holds:

$$\begin{aligned}\|u\| &= 2^{n-1}\|x-w\| \geqslant 2^{C(T)+n-q_0} \geqslant 2^{C(T)+\min\{n,\ell\}-q_0},\\ \|v\| &= 2^{\ell-1}\big\|T(w)-T(y)\big\| > 2^{C(T)}\|w-y\| > 2^{C(T)+\ell-q_0}\\ &\geqslant 2^{C(T)+\min\{n,\ell\}-q_0}.\end{aligned}$$

If $r>1$ is not an integer, we start from (6.12) and rewrite $V^{\ell,\tau}_{n,\sigma}(x,y)$ as

$$\int \mathrm{e}^{i\Lambda(x-w)(\xi/\Lambda)+i\Lambda(T(w)-T(y))(\eta/\Lambda)} F(\xi,\eta,w) \times \psi_{\Theta',n,\sigma}(\xi)\,\widetilde{\psi}_{\Theta,\ell,\tau}(\eta)\,\mathrm{d}w\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{6.15}$$

for $\Lambda = 2^{\max\{\ell,n\}}$. Recalling (3.4), we apply to (6.15) one regularised integration by parts for $\delta = r-[r]$ (noting that T is $C^{1+\delta}$). We get two terms $F_{1,\epsilon}(\xi,\eta,w)$ and $F_{2,\epsilon}(\xi,\eta,w)$. Choosing $\epsilon=\Lambda^{-1}$, we may apply the above procedure to each of them. □

7. Partitions of unity

Let $r>0$ and recall $K\subset\mathbb{R}^d$ is compact with nonempty interior. A C^r partition of unity on K is by definition a finite family of C^r functions $g_i:\mathbb{R}^d\to[0,1]$, $1\leqslant i\leqslant I$, such that $\sum_i g_i(x)=1$ for $x\in K$ and $\sum_i g_i(x)\leqslant 1$ for $x\in\mathbb{R}^d$. The intersection multiplicity of a partition of unity is $\nu := \sup_x \#\{i \mid x\in \operatorname{supp}(g_i)\}$. For $u\in C^\infty(K)$, we set $u_i := g_i u$ so that $u=\sum_i u_i$. In this section, we compare the norms of u and those of the u_i's. (This will be useful to refine partitions in the proof of Theorem 1.1 in the next section.)

Lemma 7.1. — *Let $q\leqslant 0\leqslant p$, and let p' and q' be real numbers with $p'<p$ and $q'<q$. For every C^r partition of unity $\{g_i\}$ whose intersection multiplicity is ν, there are constants $C(\{g_i\})$ and $C(\{g_i\},t)$ (that may depend on the g_i's) so that for any $u\in C^\infty(K)$*

$$\|u\|_{C^{\Theta,p,q}_*} \leqslant \nu\cdot \max_{1\leqslant i\leqslant I}\|u_i\|_{C^{\Theta,p,q}_*} + C\big(\{g_i\}\big)\sum_{1\leqslant i\leqslant I}\|u_i\|_{C^{\Theta,p',q'}_*}$$

and, for all $1<t<\infty$

$$\|u\|_{W^{\Theta,p,q,t}_*} \leqslant \nu\Big[\sum_{1\leqslant i\leqslant I}\|u_i\|^t_{W^{\Theta,p,q,t}_*}\Big]^{1/t} + C\big(\{g_i\},t\big)\sum_{1\leqslant i\leqslant I}\|u_i\|_{W^{\Theta,p',q',t}_*}.$$

Proof. — Let $U(i,\epsilon)$ be the ϵ-neighborhood of the support of g_i. Take $\epsilon > 0$ so small that the intersection multiplicity of the sets $U(i,\epsilon)$ is ν. Decompose $\mathcal{Q}_\Theta u_i$ (recall Section 3) into

$$\mathbf{u}_i^{\mathrm{body}} = \mathbf{1}_{U(i,\epsilon)} \cdot \mathcal{Q}_\Theta u_i \quad \text{and} \quad \mathbf{u}_i^{\mathrm{tail}} = \mathcal{Q}_\Theta u_i - \mathbf{u}_i^{\mathrm{body}}.$$

On the one hand, Lemma 4.1 implies

$$\|\mathbf{u}_i^{\mathrm{tail}}\|_{p,q,\infty} \leqslant C\|u_i\|_{C_*^{\Theta,p',q'}} \quad \text{and} \quad \|\mathbf{u}_i^{\mathrm{tail}}\|_{p,q,t} \leqslant C(t)\|u_i\|_{W_*^{\Theta,p',q',t}}.$$

On the other hand, since the intersection multiplicity is ν, we have

$$\Big\|\sum_i \mathbf{u}_i^{\mathrm{body}}\Big\|_{p,q,\infty} \leqslant \nu \max_i \big\|\mathbf{u}_i^{\mathrm{body}}\big\|_{p,q,\infty},$$

and, using the Hölder inequality,

$$\Big\|\sum_i \mathbf{u}_i^{\mathrm{body}}\Big\|_{p,q,t} \leqslant \nu^{1/t'}\Big[\sum_i \big\|\mathbf{u}_i^{\mathrm{body}}\big\|_{p,q,t}^t\Big]^{1/t}.$$

Therefore we obtain the estimates in the lemma by using (3.1). □

The next proposition gives bounds in the opposite direction.

Proposition 7.2. — *Let $q \leqslant 0 \leqslant p$, and let p' and q' be real numbers with $p' < p$, $q' < q$ and $p - q' < r$. If $\Theta' < \Theta$, there are constants C_0 and $C_0(t)$ so that for every C^r partition of unity $\{g_i\}$ there are constants $C(\{g_i\})$ and $C(\{g_i\}, t)$ (which may depend on the g_i's) so that for all $u \in C^\infty(K)$*

$$\max_{1\leqslant i\leqslant I} \|u_i\|_{C_*^{\Theta',p,q}} \leqslant C_0\|u\|_{C_*^{\Theta,p,q}} + C\big(\{g_i\}\big)\|u\|_{C_*^{\Theta,p',q'}},$$

and, for $1 < t < \infty$,

$$\Big[\sum_{1\leqslant i\leqslant I} \|u_i\|_{W_*^{\Theta',p,q,t}}^t\Big]^{1/t} \leqslant C_0(t)\|u\|_{W_*^{\Theta,p,q,t}} + C\big(\{g_i\}, t\big)\|u\|_{W_*^{\Theta,p',q',t}}.$$

Proof. — We revisit the proof of Theorem 6.1, setting $T = \mathrm{id}$. (Note that assumption (6.1) holds since we are assuming $\Theta' < \Theta$.) Recall $\Phi(D)$ and Ψ there, and let $\mathbf{S}^{(i)}$, $\mathbf{S}_0^{(i)}$, $\mathbf{S}_1^{(i)}$ and $\mathbf{L}^{(i)}$ be the operators defined in the same way as $\mathbf{S}$, $\mathbf{S}_0$, $\mathbf{S}_1$ and $\mathbf{L}$ respectively with g replaced by g_i. Obviously $|\mathbf{L}_i(\mathbf{f})|_{C^{p,q}}(x) \leqslant |g_i(x)| \cdot |\mathbf{f}|_{C^{p,q}}(x)$ and $|\mathbf{L}_i(\mathbf{f})|_{\mathcal{W}^{p,q}}(x) \leqslant |g_i(x)| \cdot |\mathbf{f}|_{\mathcal{W}^{p,q}}(x)$ at each point x. These and (6.6)–(6.7) imply

$$\max_i \big\|\Psi \circ \mathbf{L}^{(i)}(\mathbf{u})\big\|_{p,q,\infty} \leqslant C_1\|\mathbf{u}\|_{p,q}$$

and

$$\Big[\sum_i \big\|\Psi \circ \mathbf{L}^{(i)}(\mathbf{u})\big\|_{p,q,t}^t\Big]^{1/t} \leqslant C_1\|\mathbf{u}\|_{p,q,t}, \quad \forall 1 < t < \infty$$

for $\mathbf{u} = (u_{\Theta,n,\sigma})_{(n,\sigma)\in\Gamma}$. By boundedness of $\Phi(D)$, the same estimates hold with $\Psi \circ \mathbf{L}^{(i)}$ replaced by $\mathbf{S}_0^{(i)} = \Phi(D) \circ \Psi \circ \mathbf{L}^{(i)}$. The conclusion of the proposition then follows from those estimates and the estimates on the operators $\mathbf{S}_1^{(i)}$ parallel to that on $\mathbf{S}_1$ in the proof of Theorem 6.1. □

8. Transfer operators for hyperbolic diffeomorphisms

In this section we prove Theorem 1.1 by reducing to the model of Sections 2–7.

Proof of Theorem 1.1. — We first define the spaces $C_*^{p,q}(T,V)$ and $W_*^{p,q,t}(T,V)$, by using local charts to patch the anisotropic Hölder and Sobolev spaces from Section 2. Fix a finite system of C^∞ local charts $\{(V_j,\kappa_j)\}_{j=1}^J$ that cover the compact isolating neighborhood V of Ω, and a finite system of pairs of closed cones[3] $\{(\mathbf{C}_{j,+},\mathbf{C}_{j,-})\}_{j=1}^J$ in $\mathbb{R}^d$ with the properties that for all $1 \leqslant j,k \leqslant J$:

(a) The closure of $\kappa_j(V_j)$ is a compact subset of $\mathbb{R}^d$.

(b) The cones $\mathbf{C}_{j,\pm}$ are transversal to each other: $\mathbf{C}_{j,+} \cap \mathbf{C}_{j,-} = \{0\}$.

(c) If $x \in V_j \cap \Omega$, the cones $(D\kappa_j)^*(\mathbf{C}_{j,+})$ and $(D\kappa_j)^*(\mathbf{C}_{j,-})$ in the cotangent space contain the normal subspaces of $E^u(x)$ and $E^s(x)$, respectively.

(d) If $T^{-1}(V_k) \cap V_j \neq \emptyset$, setting $U_{jk} = \kappa_j(T^{-1}(V_k) \cap V_j)$, the map in charts
$$T_{jk} := \kappa_k \circ T \circ \kappa_j^{-1} : U_{jk} \to \mathbb{R}^d$$
enjoys the cone-hyperbolicity condition:

$$DT_{jk,x}^{\mathrm{tr}}\big(\mathbb{R}^d \setminus \mathrm{interior}(\mathbf{C}_{k,+})\big) \subset \mathrm{interior}(\mathbf{C}_{j,-}) \cup \{0\}, \quad \forall x \in U_{jk}. \tag{8.1}$$

Choose C^∞ functions $\varphi_{j,+},\varphi_{j,-} : \mathbf{S}^{d-1} \to [0,1]$ for $1 \leqslant j \leqslant J$ which satisfy condition (2.2) with $\mathbf{C}_\pm = \mathbf{C}_{j,\pm}$, giving combinations
$$\Theta_j = (\mathbf{C}_{j,+},\mathbf{C}_{j,-},\varphi_{j,+},\varphi_{j,-})$$
as in Section 2. Choose finally a C^∞ partition of the unity $\{\phi_j\}$ on V subordinate to the covering $\{V_j\}_{j=1}^J$, that is, the support of each $\phi_j : X \to [0,1]$ is contained in V_j and we have $\sum_{j=1}^J \phi_j \equiv 1$ on V.

We define the Banach spaces $C_*^{p,q}(T,V)$ and $W_*^{p,q,t}(T,V)$ for $1 < t < \infty$, respectively, to be the completion of $C^\infty(V)$ for the norm
$$\|u\|_{C_*^{p,q}(T,V)} := \max_{1\leqslant j\leqslant J} \big\|(\phi_j \cdot u) \circ \kappa_j^{-1}\big\|_{C_*^{\Theta_j,p,q}}$$

[3] We regard $\mathbf{C}_{j,\pm}$ as constant cone fields in the *cotangent* bundle $T^*\mathbb{R}^d$.

and

$$\|u\|_{W_*^{p,q,t}(T,V)} := \max_{1\leqslant j\leqslant J}\big\|(\phi_j\cdot u)\circ\kappa_j^{-1}\big\|_{W_*^{\Theta_j,p,q,t}}. \tag{8.2}$$

By this definition, we have that $C_*^{p,q}(T,V)$ and $W_*^{p,q,t}(T,V)$ contain $C^s(V)$ for $s>p$ and $W^{p,t}(V)$, respectively, as dense subsets. Take and fix real numbers $0\leqslant p'<p$ and $q'<q$ such that $p-q'<r-1$. By Lemma 5.1 and a finite diagonal argument over $\{1,\dots,J\}$, we can see that the inclusions $C_*^{p,q}(T,V)\subset C_*^{p',q'}(T,V)$ and $W_*^{p,q,t}(T,V)\subset W_*^{p',q',t}(T,V)$ are compact.

For $m\geqslant 1$ and j, k so that

$$V_{m,jk} := T^{-m}(V_k)\cap V_j\cap\Big(\bigcap_{i=0}^{m}T^{-i}(V)\Big)\neq\emptyset,$$

we may consider the map in charts

$$T_{jk}^m=\kappa_k\circ T^m\circ\kappa_j^{-1}:\kappa_j(V_{m,jk})\longrightarrow\mathbb{R}^d.$$

Note that (8.1) implies that

$$(DT_{jk,x}^m)^{\mathrm{tr}}\big(\mathbb{R}^d\setminus\mathrm{interior}(\mathbf{C}_{k,+})\big)\subset\mathrm{interior}(\mathbf{C}_{j,-})\cup\{0\}, \tag{8.3}$$

for all $x\in\kappa_j(V_{m,jk})$. For $1<t\leqslant\infty$, we set

$$\Lambda_{m,t}=\max_j\max_k\sup_{x\in\kappa_j(V_{m,jk})}\frac{\big|g^{(m)}\circ\kappa_j^{-1}(x)\big|\cdot\max\big\{(\|T_{jk}^m\|_+(x))^p,(\|T_{jk}^m\|_-(x))^q\big\}}{|\det DT_x^m|^{1/t}}$$

where

$$\|T_{jk}^m\|_+(x)=\sup\left\{\frac{\|(DT_{jk}^m)_x^{\mathrm{tr}}(\xi)\|}{\|\xi\|}\ ;\ 0\neq(DT_{jk}^m)_x^{\mathrm{tr}}(\xi)\notin\mathbf{C}_{j,-}\right\},$$

and

$$\|T_{jk}^m\|_-(x)=\inf\left\{\frac{\|(DT_{jk}^m)_x^{\mathrm{tr}}(\xi)\|}{\|\xi\|}\ ;\ 0\neq\xi\notin\mathbf{C}_{k,+}\right\}.$$

Then a standard argument on hyperbolic sets gives a constant $C(t)>1$ that does not depend on $m>0$ such that

$$C(t)^{-1}R^{p,q,t}(T,g,\Omega,m)\leqslant\Lambda_{m,t}\leqslant C(t)R^{p,q,t}(T,g,\Omega,m). \tag{8.4}$$

The definition of $\Lambda_{m,t}$ involves first taking a maximum and a product, and then taking the supremum over x. We shall apply Theorem 6.1 in a moment: the upper bound there corresponds to taking a supremum first. Since different points in $\kappa_j(V_{m,jk})$ may have very different itineraries, it is necessary to refine our partition of unity, depending on m. This will not cause problems since we can take arbitrarily fine finite C^∞ partitions of unity on $\mathbb{R}^d$, with intersection multiplicities bounded uniformly by a constant depending only on d. Using such a partition of unity, we decompose the function $u_{jk} = (\phi_k(\phi_j \circ T^{-m}) \cdot u) \circ \kappa_k^{-1}$ into $u_{jk,i}$ for $1 \leqslant i \leqslant I_{jk}$. Take combinations $\Theta'_k < \Theta_k$ (close to Θ_k) so that the iterated cone-hyperbolicity condition (8.3) holds with Θ_k replaced by Θ'_k. For each m, by taking a sufficiently fine partition of unity, we can apply Theorem 6.1 to obtain, for $1 \leqslant i \leqslant I_{jk}$,

$$\|g^{(m)} \circ \kappa_j^{-1} \cdot u_{jk,i} \circ T^m_{jk}\|_{C_*^{\Theta_j,p,q}} \leqslant 2\Lambda_{m,\infty} \cdot \|u_{jk,i}\|_{C_*^{\Theta'_k,p,q}} + C\|u_{jk,i}\|_{C_*^{\Theta'_k,p',q'}}.$$

Then, using Lemma 7.1 and Proposition 7.2, we get

$$\|g^{(m)} \circ \kappa_j^{-1} \cdot u_{jk} \circ T^m_{jk}\|_{C_*^{\Theta_j,p,q}} \leqslant C_1 \cdot \Lambda_{m,\infty} \cdot \|u_{jk}\|_{C_*^{\Theta_k,p,q}} + C_1(m) \cdot \|u_{jk}\|_{C_*^{\Theta_k,p',q'}},$$

where C_1 is a constant that does not depend on m. Thus, using Proposition 7.2 again, we obtain the following Lasota-Yorke type inequality:

$$\|\mathcal{L}^m_{T,g} u\|_{C_*^{p,q}(T,V)} \leqslant C_2 \cdot J \cdot \Lambda_{m,\infty} \cdot \|u\|_{C_*^{p,q}(T,V)} + C_2(m)\|u\|_{C_*^{p',q'}(T,V)}, \quad m \geqslant 1.$$

Likewise, we obtain for $1 < t < \infty$

$$\|\mathcal{L}^m_{T,g} u\|_{W_*^{p,q,t}(T,V)} \leqslant C_2(t) \cdot J \cdot \Lambda_{m,t} \cdot \|u\|_{W_*^{p,q,t}(T,V)} + C_2(m,t)\|u\|_{W_*^{p',q',t}(T,V)}.$$

Finally Hennion's Theorem [9] gives the claimed upper bounds

$$\liminf_{m\to\infty} \big(C(t)\Lambda_{m,t}\big)^{1/m} = R^{p,q,t}(T,g,\Omega)$$

for the essential spectral radius of $\mathcal{L}_{T,g}$. □

Remark 8.1. — The proof above applies to (hyperbolic) mixed transfer operators [11].

Remark 8.2. — Though it is not explicit in our notation, the definition of the spaces $C_*^{p,q}(T,V)$ and $W_*^{p,q,t}(T,V)$ depends on the system of charts $\{(V_j,\kappa_j)\}_{j=1}^J$, the set of combinations $\{(\mathbf{C}_{j,+},\mathbf{C}_{j,-},\varphi_{j,+},\varphi_{j,-})\}_{j=1}^J$, and the partition of unity $\{\phi_j\}_{j=1}^J$. Choosing a different system of local charts, a different set of combinations, or a different partition of unity, does not *a priori* give rise to equivalent norms, though Theorem 6.1 gives relations. This is a little unpleasant, but does not cause problems.

Appendix A. Relating the anisotropic Banach spaces with $L^t(K)$

Recalling $\varphi_\pm$ from Section 2, define for real numbers p and q the symbols $\Psi_{\Theta,p,+}(\xi) = (1+|\xi|^2)^{\frac{1}{2}p}\varphi_+\big(\xi/|\xi|\big)$ and $\Psi_{\Theta,q,-}(\xi) = (1+|\xi|^2)^{\frac{1}{2}q}\varphi_-\big(\xi/|\xi|\big)$, and, recalling the compact set $K \subset \mathbb{R}^d$ with nonempty interior, define norms for $u \in C^\infty(K)$ and $1 < t < \infty$ by

$$\text{(A.1)} \qquad \begin{cases} \|u\|^{\dagger\dagger}_{W_*^{\Theta,p,q,t}} &= \|\Psi_{\Theta,p,+}(D)u + \Psi_{\Theta,q,-}(D)u\|_{L^t}, \\ \|u\|^{\dagger}_{W_*^{\Theta,p,q,t}} &= \|\Psi_{\Theta,p,+}(D)u\|_{L^t} + \|\Psi_{\Theta,q,-}(D)u\|_{L^t}. \end{cases}$$

Let $\|.\|_{W_*^{\Theta,p,q,t}}$ be the norm defined in Section 2. We shall prove below (using Theorem 3.1) that for each $1 < t < \infty$ there is a constant $C > 0$ so that

$$\text{(A.2)} \quad C^{-1}\|u\|_{W_*^{\Theta,p,q,t}} \leqslant \|u\|^{\dagger}_{W_*^{\Theta,p,q,t}} \leqslant C\|u\|_{W_*^{\Theta,p,q,t}} \quad \text{for } u \in C^\infty(K).$$

Let $W_{\dagger}^{\Theta,p,q,t}(K)$ be the completion of $C^\infty(K)$ with respect to $\|.\|^{\dagger}_{W_*^{\Theta,p,q,t}}$. Then $W_{\dagger}^{\Theta,p,q,t}(K) = W_*^{\Theta,p,q,t}(K)$ by (A.2).

Let $W_{\dagger\dagger}^{\Theta,p,q,t}(K)$ be the completion of $C^\infty(K)$ with respect to $\|.\|^{\dagger\dagger}_{W_*^{\Theta,p,q,t}}$. Then

$$\text{(A.3)} \quad W_{\dagger\dagger}^{\Theta,p,q,t}(K) = \big\{u \in \mathcal{S}'(\mathbb{R}^d) \mid \operatorname{supp}(u) \subset K, \\ \|u\|^{\dagger\dagger}_{W_*^{\Theta,p,q,t}} < \infty\big\}, \quad \forall 1 < t < \infty.$$

Clearly,

$$\text{(A.4)} \qquad \|u\|^{\dagger\dagger}_{W_*^{\Theta,p,q,t}} \leqslant \|u\|^{\dagger}_{W_*^{\Theta,p,q,t}} \quad \text{for } u \in C^\infty(K).$$

Though we do not know whether the norm $\|.\|^{\dagger\dagger}_{W_*^{\Theta,p,q,t}}$ is equivalent to $\|.\|_{W_*^{\Theta,p,q,t}}$, we show below (using Theorem 3.1) that, for each $1 < t < \infty$, if $\Theta' > \Theta$ and $p \geqslant q$,

$$\text{(A.5)} \qquad \|u\|^{\dagger}_{W_*^{\Theta,p,q,t}} \leqslant C\|u\|^{\dagger\dagger}_{W_*^{\Theta',p,q,t}} \quad \text{for } u \in C^\infty(K)$$

for some constant $C > 0$. From (A.2), (A.4) and (A.5), it follows that, if $\Theta' > \Theta$ then for all $1 < t < \infty$ and $p \geqslant q$, then $W_*^{\Theta',p,q,t}(K) \subset W_*^{\Theta,p,q,t}(K)$, and, also,

$$W_{\dagger\dagger}^{\Theta',p,q,t}(K) \subset W_*^{\Theta,p,q,t}(K) \subset W_{\dagger\dagger}^{\Theta,p,q,t}(K).$$

Let $W_{\dagger\dagger}^{\Theta,p,q,t}(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$ be the isometric image of the restriction to $L^t(\mathbb{R}^d)$ of the inverse of the bijective operator

$$\Psi_{\Theta,p,+}(D) + \Psi_{\Theta,q,-}(D) : \mathcal{S}'(\mathbb{R}^d) \longrightarrow \mathcal{S}'(\mathbb{R}^d).$$

For $1 < t < \infty$, (A.3) implies that $W^{\Theta,p,q,t}_{\dagger\dagger}(K)$ is just (isometrically)

$$\big\{u \in W^{\Theta,p,q,t}_{\dagger\dagger}(\mathbb{R}^d) \mid \operatorname{supp}(u) \subset K\big\}.$$

The space $W^{\Theta,p,q,t}_{\dagger}(K)$ may be described in a similar (although not as neat) way using the injective (non surjective) operator

$$\big(\Psi_{\Theta,p,+}(D), \Psi_{\Theta,q,-}(D)\big) : \mathcal{S}'(\mathbb{R}^d) \longrightarrow \mathcal{S}'(\mathbb{R}^d) \oplus \mathcal{S}'(\mathbb{R}^d).$$

To finish, we prove (A.2) and (A.5). For the proof of (A.2), it is enough to show

$$\text{(A.6)} \qquad C^{-1}\|\Psi_{\Theta,p,+}(D)u\|_{L^t} \leqslant \Big\|\Big(\sum_{n\geqslant 0}(2^{pn}\psi_{\Theta,n,+}(D)u)^2\Big)^{\frac12}\Big\|_{L^t} \leqslant C\|\Psi_{\Theta,p,+}(D)u\|_{L^t}$$

for some constant $C > 0$ and the corresponding claim for $\|\Psi_{\Theta,q,-}(D)u\|_{L^t}$. Since the norm $\|.\|_{W^{0,t}_*}$ is equivalent to the norm $\|.\|_{L^t}$ as noted in Section 2, and since $\psi_n(\xi)\Psi_{\Theta,p,+}(\xi) = a^p(\xi)\psi_{\Theta,n,+}(\xi)$ for $n \geqslant 1$, with $a^p(\xi) = (1 + |\xi|^2)^{\frac12 p}$, we have

$$\begin{aligned} &C^{-1}\|\Psi_{\Theta,p,+}(D)u\|_{L^t} \\ &\quad\leqslant \Big\|\Big(\sum_{n\geqslant 0}(a^p(D)\psi_{\Theta,n,+}(D)u)^2\Big)^{\frac12}\Big\|_{L^t} = \big\|\Psi_{\Theta,p,+}(D)u\big\|_{W^{0,t}_*} \\ &\quad\leqslant C\|\Psi_{\Theta,p,+}(D)u\|_{L^t}. \end{aligned}$$

It is easy to see that there exist $P, Q \in C^\infty(\mathbb{R}^d, \mathcal{L}(\ell^2, \ell^2))$ satisfying (3.3) such that the corresponding pseudodifferential operators $P(D)$ and $Q(D)$ on $L(\mathbb{R}^d, \ell^2)$ transform $(2^{pn}\psi_{\Theta,n,+}(D)u)_{n\geqslant 0}$ to $(a^p(D)\psi_{\Theta,n,+}(D)u)_{n\geqslant 0}$ and the reverse. Thus Theorem 3.1 gives (A.6). The corresponding claim for $\|\Psi_{\Theta,q,-}(D)u\|_{L^t}$ is shown in a parallel manner.

In order to prove (A.5), it is enough to show

$$\max\big\{\|\Psi_{\Theta,p,+}(D)u\|_{L^t}, \|\Psi_{\Theta,q,-}(D)u\|_{L^t}\big\} \leqslant C\big\|\Psi_{\Theta',q,-}(D)u + \Psi_{\Theta',p,+}(D)u\big\|_{L^t}$$

for some constant C. Since $\Psi_{\Theta,p,+}(\xi)/(\Psi_{\Theta',p,+}(\xi) + \Psi_{\Theta',q,-}(\xi))$ and $\Psi_{\Theta,q,-}(\xi)/(\Psi_{\Theta',p,+}(\xi) + \Psi_{\Theta',q,-}(\xi))$ both satisfy (3.3), this follows from Theorem 3.1.

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