ANNALES

DE

L'INSTITUT FOURIER

Bo TAN, Zhi-Xiong WEN, Jun WU & Zhi-Ying WEN

Substitutions with Cofinal Fixed Points

Tome 56, nº 7 (2006), p. 2551-2563.

<http://aif.cedram.org/item?id=AIF_2006__56_7_2551_0>

Ann. Inst. Fourier, Grenoble
56, 7 (2006) 2551-2563

SUBSTITUTIONS WITH COFINAL FIXED POINTS

by Bo TAN, Zhi-Xiong WEN, Jun WU & Zhi-Ying WEN

ABSTRACT. — Let φ be a substitution over a 2-letter alphabet, say $\{a, b\}$. If $\varphi(a)$ and $\varphi(b)$ begin with a and b respectively, φ has two fixed points beginning with a and b respectively.

We characterize substitutions with two cofinal fixed points (i.e., which differ only by prefixes). The proof is a combinatorial one, based on the study of repetitions of words in the fixed points.

RÉSUMÉ. — Soit φ une substitution en un alphabet $\{a, b\}$ de deux lettres. Si $\varphi(a)$ et $\varphi(b)$ commencent par a et b respectivement, alors φ possède deux points fixes débutants par a et b respectivement.

Nous caractériserons les substitutions avec deux points fixes co-finaux (c'est-à-dire, qui diffèrent que par leur préfixe). La démonstration est combinatoire, elle se base sur une étude de répétitions de mots dans les points fixes.

1. Introduction and Main Result

Combinatorial properties of finite words and infinite sequences are of increasing importance in various fields of mathematics, physics as well as computer science, bioinformatics $\cdots$ (the reader is referred to the series of books by Lothaire [5, 6, 7], and the references therein for more information).

Substitutions are very simple combinatorial objects, and play an important rôle in the study of sequences. Roughly speaking, substitutions are rules to replace letters by words, and by iteration, they produce sequences. The combinatorial properties of these sequences are of great interest in various subject (see, for instance, the book by Allouche and Shallit [1], the book by Fogg [9] and the references therein).

Note that a substitution can be naturally extended to a morphism of the corresponding free group; If this extension is an automorphism, the substitution is said to be invertible. These substitutions have been intensively

Keywords: Cofinal sequences, substitution.
Math. classification: 68R15, 11B85.

studied by different authors: Wen and Wen [12] showed that the monoid of invertible substitutions over a 2-letter alphabet is finitely generated, also they gave a set of the three generators. Ei and Ito [4] provided a geometrical characterization of these invertible substitutions. A survey on this subject can be found in Chapter 9 in [9]. The situation in a bigger alphabet is much more complicated; For example, Wen and Zhang [14] showed that the monoid of invertible substitutions over a 3-letter alphabet is not finitely generated. The structure of invertible substitutions over a 3-letter alphabet is studied in Tan, Wen and Zhang [11].

The sequences generated by invertible substitutions have very interesting properties. In the 2-letter case, these sequences are shown to be the so-called Sturmian sequences (with some trivial exceptions). For a sequence s, the complexity p is defined to be the function which associates with each positive integer n the number of different words of length n occurring in s. Sturmian sequences are exactly the ones of minimal complexity (that is, $p(n) = n + 1$) amongst all non periodic sequences. There are many equivalent definitions of Sturmian sequence. For example, Sturmian sequences can be geometrically generated by the intersections of a straight line with a grid in the plane. Sturmian sequence are extensively studied and we refer the reader to Chapter 2 in [6] or Chapter 6 in [9] for more details and information. Up to now, there are few results on the sequences generated by invertible substitutions over a bigger alphabet. A special class of sequences over three letters which can be generated by invertible substitutions were studied in Arnoux and Rauzy [2].

The aim of the present paper is to characterize the substitutions over a 2-letter alphabet with two cofinal fixed points. This characterization is motivated by some interesting properties of the invertible substitutions.

Let us introduce some definitions and notations first.

Let $S = \{a, b\}$ be an alphabet consisting of two letters. Let S^* and $\tilde{S}$ be respectively the free monoid (with the empty word, denoted by ε, as identity element) and the free group generated by S. Let $S^+ = S^* \backslash \{\varepsilon\}$. Let $S^{\mathbb{N}^*}$ be the set of (one-sided) sequences over S, indexed by $\mathbb{N}^*$.

A morphism $\varphi : S^* \to S^*$ is called a substitution. In this work, we deal only with non-erasing substitutions, which means that both $\varphi(a)$ and $\varphi(b)$ are different from ε. Since a substitution φ is determined by the images of the generators, we can identify a substitution φ with the ordered pair $\big(\varphi(a), \varphi(b)\big)$. Thus (ab, a) denotes the well known Fibonacci substitution which maps a to ab and b to a.

A substitution φ can be naturally extended to be a morphism of the free group $\tilde{S}$ which we also denote by φ. If φ is an automorphism of $\tilde{S}$, we say that φ is an invertible substitution.

A substitution φ also acts on $S^{\mathbb{N}^*}$:

$$\varphi(x_1x_2\dots x_n\dots) = \varphi(x_1)\varphi(x_2)\dots\varphi(x_n)\dots.$$

If ξ is a sequence such that $\varphi(\xi) = \xi$, we say that ξ is a fixed point of φ.

Two sequences ξ_1 and ξ_2 are said to be cofinal if there exist two finite words w_1 and w_2, and a sequence ξ, such that $\xi_1 = w_1\xi$ and $\xi_2 = w_2\xi$.

The following result by Wen, Wen, and Wu [13] motivated the present study.

THEOREM 1.1. — *(1) Let φ be a primitive invertible substitution with two fixed points ξ_1 and ξ_2. Then φ is in the monoid $\langle\tau,\ \pi\rangle$ generated by the substitutions $\tau = (ba, a)$ and $\pi = (b, a)$ and the total number of π's and τ's in any decomposition of φ is even. Furthermore, there exists an infinite sequence ξ such that $\xi_1 = ab\xi$ and $\xi_2 = ba\xi$.*

(2) Let φ be a primitive substitution with two fixed points ξ_1 and ξ_2. If $\xi_1 = ab\xi$ and $\xi_2 = ba\xi$ for some sequence ξ, then φ is invertible.

Here are our results.

THEOREM 1.2. — *A substitution φ has two cofinal fixed points if and only if φ is of one of the following types:*

- **(i)** $\varphi(a) = ab^k$, $\varphi(b) = b^m$ *for some* $k \geqslant 1$, $m \geqslant 2$;
- **(ii)** $\varphi(a) = a^k$, $\varphi(b) = ba^m$ *for some* $k \geqslant 2$, $m \geqslant 1$;
- **(iii)** $\varphi(a) = (ab)^k a$, $\varphi(b) = (ba)^m b$ *for some* $k \geqslant 1$, $m \geqslant 1$;
- **(iv)** $\varphi(a) = aw(b^k w)^{k-1}$, $\varphi(b) = b^k w$ *for some word* w, $k \geqslant 1$;
- **(v)** $\varphi(a) = a^k w$, $\varphi(b) = bw(a^k w)^{k-1}$ *for some word* w, $k \geqslant 1$;
- **(vi)** *φ is an invertible substitution, and $\varphi \in \langle\pi,\ \tau\rangle$ satisfying the total number of π's and τ's in any decomposition of φ is even.*

Remark that the substitutions of types (i) and (ii) are non-primitive, and the substitutions of type (iii) are periodic. Also remark that types (i) and (ii) are symmetric from each other, as well as types (iv) and (v).

The cofinal sequences ultimately coincide. Let us mention that other coincidence conditions on a substitution σ, defined in terms of prefixes of $\sigma^n(a)$ and $\sigma^n(b)$, have been considered by other authors (see for example [3]). It will be interesting to compare them with our work.

2. Preliminaries

If $w \in S^*$ is a word, we denote by $|w|$ its length (i.e., the total number of letters appearing in w) and by $|w|_a$ (resp. $|w|_b$) the number of times the letter a (resp. b) occurs in w. Denote by $L(w)$ the vector $(|w|_a, |w|_b)$.

A word v is a factor of a word w, and then we write $v \prec w$, if there exist $u, u' \in S^*$ such that $w = uvu'$. We say that v is a prefix (resp. suffix) of a word w, and then we write $v \triangleleft w$ (resp. $v \triangleright w$), if there exists $u \in S^*$ such that $w = vu$ (resp. $w = uv$). The notion of factors and prefix extends in a natural way to infinite sequences.

A word $u \in S^+$ is said to be primitive if it is not a power of some word (i.e., $u = v^k$ for some $v \in S^+$ implies $k = 1$).

Let $w = x_1 x_2 \cdots x_n \in S^*$ $(x_i \in S)$, we denote by w^{-1} the inverse word of w, that is $w^{-1} = x_n^{-1} \cdots x_2^{-1} x_1^{-1}$. Let $w = uv$, then $wv^{-1} := u$ by convention.

A sequence $\eta = \eta_1 \eta_2 \cdots$ is said to be ultimately periodic, if there exist k and n in $\mathbb{N}$ such that, for any $j \geqslant k$, $\eta_j = \eta_{j+n}$.

Let $\varphi : S^* \to S^*$ be a substitution, the substitution matrix of φ is defined to be $M_\varphi = (L(\varphi(a))^t, L(\varphi(b))^t)$. Then we have $M_{\varphi \circ \phi} = M_\varphi M_\phi$ for any substitutions φ and ϕ. If M_φ is primitive (i.e., there exists $k \geqslant 1$ such that $M^k > 0$), we call φ a primitive substitution.

We denote by $\mathrm{Aut}(\tilde{S})$ the group of automorphisms of $\tilde{S}$. It is known ([8]) that $\mathrm{Aut}(\tilde{S})$ is generated by the following three special automorphisms

$$\sigma = (ab, a), \quad \pi = (b, a), \quad \delta = (a, b^{-1}).$$

A substitution φ which also is an automorphism of $\tilde{S}$ is called an invertible substitution. The monoid of invertible substitutions, denoted by $\mathrm{IS}(S)$, is generated by the three substitutions $\pi = (b, a), \sigma = (ab, a)$, and $\tau = (ba, a)$ (see [12]):

$$\mathrm{IS}(S) = \langle \pi, \sigma, \tau \rangle$$

(where the notation $\langle \sigma_1, \cdots, \sigma_n \rangle$ denotes the monoid of substitutions generated by $\sigma_1, \cdots, \sigma_n$).

Let us remark that the writing of an invertible substitution as a product of π, σ, and τ is not unique. For example: $(aba,\ a) = \sigma \circ \pi \circ \tau = \tau \circ \pi \circ \sigma$. On the other hand, by considering the determinant of the corresponding substitution matrices, the parity of the total number of π, σ and τ in the decomposition of an invertible substitution is independent of the decomposition.

The following elementary lemmas will be used in the proof of the main theorem. See for example [5, 6] for more details.

LEMMA 2.1. — *Let $u, v \in S^+$. If $uv = vu$, then there exist $w \in S^+$ and $m, n \in \mathbb{N}$ such that*

$$u = w^m, \quad v = w^n.$$

LEMMA 2.2. — *Let $u, v, w \in S^+$. If $|u| \geqslant |v|$, and $wu = uv^k$ for some $k \in \mathbb{N}$, then there exist $n \in \mathbb{N}$ and $v_1 \triangleright v$ such that $u = v_1 v^n$.*

3. Proof for the non-primitive and periodic cases

In this section, we prove Theorem 1.2 in the non-primitive and periodic cases. We consider the non-primitive case first.

If the substitution is not primitive, then either $\varphi(a) = a^m$ or $\varphi(b) = b^m$ for some $m \geqslant 1$. We will only consider the case $\varphi(a) = a^m$.

Since φ has two fixed points, we have $m \geqslant 2$, and thus one of the fixed points is a^∞. Since the two fixed points are cofinal, the other one (beginning with b) will be ua^∞ for some word $u \in S^*$.

We write $\varphi(b) = bv$ with $v \in S^*$ and claim that $|v|_b = 0$.

Indeed, assume that $|v|_b \geqslant 1$, then $|\varphi(b)|_b \geqslant 2$, thus $|\varphi^n(b)|_b \geqslant 2^n$, and the fixed point beginning with b will contain infinitely many of b, this is a contradiction.

Hence, $|v|_b = 0$, and $\varphi(b) = ba^k$. The non-primitive cases follow.

Now we turn to the periodic cases. Suppose that one fixed point is ultimately periodic. Since two fixed points are cofinal, the other is also ultimately periodic.

Séébold [10] enumerated the substitutions with periodic fixed point as follows.

PROPOSITION 3.1. — *Let φ be a primitive substitution over S. φ has a ultimately periodic fixed point if and only if φ is of one of the following types:*

1. $\varphi(a) = v^p$, $\varphi(b) = v^q$, $p, q \geqslant 1$, $v \in S^*$;
2. $\varphi(a) = (ab)^p a$, $\varphi(b) = (ba)^q b$, $p, q \geqslant 1$.

By Proposition 3.1, φ will be of one of the above types. But φ's of type 1 have only one fixed point.

And it is trivial to check that φ's of type 2 have two cofinal fixed points.

4. Proof for other cases

In this section, we prove Theorem 1.2 in other cases.

4.1. Sufficiency

The proof of sufficiency is just direct checking. And by Theorem 1.1 and the symmetry of types (iv) and (v), we need only consider the substitutions of type

(iv) $\varphi(a) = aw(b^k w)^{k-1}$, $\varphi(b) = b^k w$ for some word w, $k \geqslant 1$.

Setting $x = w(b^k w)^{k-1}$, we have

$$\xi_1 = ax\varphi(x)\varphi^2(x)\varphi^3(x)\cdots ;$$
$$\xi_2 = b^k x\varphi(x)\varphi^2(x)\varphi^3(x)\cdots ,$$

thus the fixed points are cofinal.

4.2. Necessity

We show the necessity by a series of lemmas and propositions.

Lemma 4.1. — *Let φ be a primitive substitution, u, v words in S^+ such that $a \triangleleft u$, $b \triangleleft v$, $|u| \neq |v|$, and the last letters of u and v are distinct. If moreover $\varphi(u) = uw$, $\varphi(v) = vw$ for some $w \in S^*$, then $|\varphi(a)| \geqslant 3$, $|\varphi(b)| \geqslant 3$.*

Proof. — Obviously $a \triangleleft \varphi(a), b \triangleleft \varphi(b)$.

Since φ is a primitive substitution, $|\varphi(a)|_b \geqslant 1$ and $|\varphi(b)|_a \geqslant 1$. The conditions $\varphi(u) = uw$ and $\varphi(v) = vw$ together imply $\varphi(uv^{-1}) = uv^{-1}$; and $|u| \neq |v|$ implies $L(uv^{-1}) \neq (0,0)$, then 1 is an eigenvalue of M_φ, i.e., $\det(M_\varphi - I) = 0$. So $|\varphi(a)|_a \geqslant 2$ and $|\varphi(b)|_b \geqslant 2$, and thus $|\varphi(a)| \geqslant 3$, $|\varphi(b)| \geqslant 3$. □

Proposition 4.2. — *Let φ be a primitive substitution, u, v words in S^+ such that $a \triangleleft u$, $b \triangleleft v$, $|u| \neq |v|$, and the last letters of u and v are distinct. If moreover $\varphi(u) = uw$, $\varphi(v) = vw$ for some $w \in S^*$ and the fixed points of φ are not ultimately periodic, then φ is of one of the following types:*

(i) *$\varphi(a) = ax(b^k w)^{k-1}$, $\varphi(b) = b^k x$ for some word x, $k \geqslant 2$, and then $u = a,\ v = b^k$.*

(ii) *$\varphi(a) = a^k x$, $\varphi(b) = bx(a^k x)^{k-1}$ for some word x, $k \geqslant 2$, and then $u = a^k,\ v = b$.*

Proof. — First we have $a \triangleleft \varphi(a),\ b \triangleleft \varphi(b)$, and by Lemma 4.1, $|\varphi(a)| \geqslant 3$, $|\varphi(b)| \geqslant 3$.

Due to the symmetry between a and b, we may assume that $|\varphi(a)| \geqslant |\varphi(b)|$.

We consider several cases, depending on the length of the word u and on its last letter.

Case 1: $\mathbf{a \triangleright u}$ and $|\mathbf{u}| \geqslant 3$.

Since the last letters of u and v are distinct, we have $b \triangleright v$.

Write $u = au_1a$ with $u_1 \in S^+$. Since $\varphi(u) = uw$, we have $\varphi(u) = \varphi(au_1a) = au_1aw$. Since $|\varphi(a)| \geqslant 3$, $|\varphi(b)| \geqslant 3$, we have $|\varphi(au_1)| = |\varphi(a)| + |\varphi(u_1)| > |u|$. Therefore $\varphi(a)$ is a suffix of w. Similarly, $\varphi(b)$ is a suffix of w. Now since $|\varphi(a)| \geqslant |\varphi(b)|$, $\varphi(b)$ is a suffix of $\varphi(a)$.

Claim. — There exist $n \in \mathbb{N}$ and $w_1 \triangleright \varphi(b)$, $w_1 \neq \varphi(b)$ such that $\varphi(a) = w_1\varphi(b)^n$.

One has $v = bv_1b$ for some $v_1 \in S^*$.

<u>Subcase 1:</u> a is not a factor of v_1, that is, $v = b^k$ for some $k \geqslant 2$.

Since $\varphi(v) = \varphi(b^k) = \varphi(b)^k = b^kw$, we have $|\varphi(b)^k| > |w| > |\varphi(a)|$. This joint to the fact that $\varphi(a)$ is a suffix of w gives Claim 1.

<u>Subcase 2:</u> a is a factor of v_1. Thus $v = bv_2ab^k$ for some $v_2 \in S^*$ and $k \in \mathbb{N}$.

(i) If $|\varphi(b)^k| \geqslant |\varphi(a)|$. As in Subcase 1, Claim 1 holds.

(ii) If $|\varphi(b)^k| < |\varphi(a)|$. There exists $x \in S^*$ such that $x\varphi(a) = \varphi(a)\varphi(b)^k$. Due to Lemma 2.2, Claim 1 holds.

The proof of Claim 1 is completed.

By Claim 1, we can write $\varphi(b) = w_2w_1$, $\varphi(a) = w_1(w_2w_1)^n$ for some $w_1, w_2 \in S^*$.

Since $a \triangleleft \varphi(a)$ and $b \triangleleft \varphi(b)$, we have that neither w_1 nor w_2 is the empty word. Then $a \triangleleft w_1$, $b \triangleleft w_2$.

One has $u = au_1a$ with $u_1 \in S^*$ (remember that $|u| \geqslant 3$). Let $\alpha \in \{a,\ b\}$ be the last letter of u_1.

Claim. — $|\varphi(au_1\alpha^{-1})| < |u|$, $|\varphi(au_1)| > |u|$.

The inequality $|\varphi(au_1)| > |u|$ has been shown before Claim 1.

Now suppose that Claim 2 does not hold; Then $|\varphi(au_1\alpha^{-1})| \geqslant |u|$. Since $\varphi(u) = uw$, we have $|\varphi(\alpha a)| \leqslant |w|$, and so $\varphi(\alpha a)$ is a suffix of w.

Since $\varphi(b)$ is a suffix of $\varphi(a)$, $w_2w_1w_1(w_2w_1)^n$ is a suffix of w.

<u>Subcase 1:</u> a is not a factor of v_1, that is, $v = b^k$, for some $k \geqslant 2$.

Since $\varphi(v) = \varphi(b^k) = \varphi(b)^k = (w_2w_1)^k$, we have $|\varphi(b)^k| > |w|$. Then $w \triangleright \varphi(b)^k$ and $w_2w_1w_1(w_2w_1)^n \triangleright (w_2w_1)^k$. Therefore $w_2w_1w_1 = w_1w_2w_1$, and $w_1w_2 = w_2w_1$. Then, due to Lemma 2.1 we have

$$w_1 = w_3^p, \quad w_2 = w_3^q,$$

for some $w_3 \in S^+$ and $p,\ q \in \mathbb{N}$. This implies the fixed point is periodic. A contradiction.

Subcase 2. If a is a factor of v_1, thus $v = bv_2ab^k$ for some $v_2 \in S^*$ and $k \in \mathbb{N}$.

(i) If $|\varphi(b)^k| \geqslant |\varphi(\alpha a)|$. By arguing as in Subcase 1, we get a contradiction.

(ii) If $|\varphi(b)^k| < |\varphi(\alpha a)|$. We have $|\varphi(a)\varphi(b)^k| \geqslant |\varphi(\alpha a)|$, and $\varphi(\alpha a) \triangleright \varphi(a)\varphi(b)^k$. This also implies $w_1w_2 = w_2w_1$. In the same way, this is a contradiction.

Claim 2 follows.

From Claim 2, we have $|\varphi(au_1\alpha^{-1})| < |u| = |au_1\alpha^{-1}|+2$, this contradicts Lemma 4.1. Thus Case 1 is impossible.

Case 2: $\mathbf{a \triangleright u}$ and $\mathbf{|u| = 2}$.

In this case, $u = aa$.

Subcase 1: $|v| = 1$.

Then $v = b$, $\varphi(b) = bw$, and $\varphi(aa) = aaw$; This contradicts $|\varphi(a)| \geqslant |\varphi(b)|$.

Subcase 2: $|v| = 2$.

This contradicts $|u| \neq |v|$.

Subcase 3: $|v| \geqslant 3$.

We can write $v = bv_1b$.

Claim. — $|v_1|_a \leqslant 1$.

If $|v_1|_a \geqslant 2$, take any word v_2 satisfying $|v_2|_a = |v_1|_a - 2$, $|v_2|_b = |v_1|_b$. Since $|\varphi(b)| \geqslant 3$ (due to Lemma 4.1) and $\varphi(aa) = aaw$, we have

$$\begin{aligned}|\varphi(v_2)| &= |\varphi(v)| - 2|\varphi(b)| - 2|\varphi(a)| = |bv_1bw| - 2|\varphi(b)| - 2|\varphi(a)| \\ &\leqslant |v_2| + 4 + |w| - 6 - |\varphi(aa)| \leqslant |v_2| + 4 + |w| - 6 - 2 - |w| \\ &= |v_2| - 4,\end{aligned}$$

which contradicts Lemma 4.1. Claim 3 follows.

According to Claim 3, we have $|v_1|_a = 0$ or 1.

Suppose first that $|v_1|_a = 0$.

We have $v = b^k$ for some $k \geqslant 3$ and $\varphi(b^k) = b^kw$. Since $|\varphi(b)|_a \geqslant 1$, b^k is a prefix of $\varphi(b)$. As in the proof of Case 1, one shows that $\varphi(a)$ is a suffix of w, $\varphi(b) = w_2w_1$, $\varphi(a) = w_1(w_2w_1)^n$ for some $w_1,\ w_2 \in S^+$ with

$a^2 \lhd w_1,\ b^k \lhd w_2$. So we write $w_1 = a^2 w_3,\ w_2 = b^k w_4$. We have

$$\varphi(u) = \varphi(aa) = a^2 w_3 (b^k w_4 a^2 w_3)^n a^2 w_3 (b^k w_4 a^2 w_3)^n = aaw,$$

and

$$\varphi(v) = \varphi(b^k) = b^k w_4 a^2 w_3 (b^k w_4 a^2 w_3)^{k-1} = b^k w.$$

Hence $w_3 b^k w_4 a^2 \lhd w$ and $w_4 a^2 w_3 b^k \lhd w$, but $|w_4 a^2 w_3 b^k| = |w_3 b^k w_4 a^2|$, thus $w_4 a^2 w_3 b^k = w_3 b^k w_4 a^2$, which is impossible (since the last letters differ).

Now suppose that $|v_1|_a = 1$.

We have $v = b^k a b^l$ for some $k,\ l \in \mathbb{N}$. Let $\varphi(a) = a^m w_5$, where $m \geqslant 2$ and $w_5 \in S^+$ with $b \lhd w_5$. Then

$$\begin{aligned}\varphi(v) &= \varphi(b^k a b^l) = b^k a b^l w \\ &= b^k a b^l a^{-1} a^{-1} \varphi(aa) = b^k a b^l a^{m-2} w_5 a^m w_5.\end{aligned}$$

Again, by arguing as in the proof of Case 1, one shows that $\varphi(a)$ is a suffix of w, $\varphi(b) = w_2 w_1$, and $\varphi(a) = w_1 (w_2 w_1)^n$ for some $w_1,\ w_2 \in S^+$ satisfying $a^m \lhd w_1,\ b^k \lhd w_2$. Thus $|\varphi(b)|_a \geqslant |w_1|_a \geqslant 2$, and this implies $b^k a b^l$ is a prefix of $\varphi(b)$. Since $\varphi(b^k a b^l) = b^k a b^l w$, $\varphi(b^{k-1} a b^l)$ is a suffix of w. This, together with the facts that $\varphi(a)$ is a suffix of w and $\varphi(u) = \varphi(aa) = aaw$, gives $w_1 w_2 = w_2 w_1$. Then the fixed point is periodic, and this is a contradiction.

This study of Subcases 1, 2, and 3 shows that Case 2 is impossible.

Case 3: $\mathbf{a \rhd u}$ and $\mathbf{|u| = 1}$.

In this case, $u = a$ and $\varphi(a) = aw$. Since $|u| \neq |v|$, we have $|v| \geqslant 2$. Write $v = b v_1 b$.

<u>Subcase 1:</u> a is a factor of v.

For any word v_2 satisfying $|v_2|_a = |v|_a - 1$, $|v_2|_b = |v|_b$, we have

$$\begin{aligned}|\varphi(v_2)| &= |\varphi(v)| - |\varphi(a)| = |v| + |w| - 1 - |w| \\ &= |v_2| - 2,\end{aligned}$$

which contradicts Lemma 4.1.

<u>Subcase 2:</u> a is not a factor of v.

Since $|u| \neq |v|$, we have $v = b^k$ for some $k \geqslant 2$. By $\varphi(u) = \varphi(a) = aw$, and $\varphi(v) = \varphi(b^k) = b^k w$, we have

$$\varphi(a) = ax(b^k x)^{k-1}, \quad \varphi(b) = b^k x,$$

for some word x, $k \geqslant 2$, and $u = a,\ v = b^k$.

This is just the case (i) in Proposition 4.2.

Case 4: $b \triangleright u$.

Write $u = au_1b$, $v = bv_1a$. By arguing as in the proof of Case 1, one shows that $\varphi(a)$ is a suffix of w, $\varphi(b) = w_2w_1$, $\varphi(a) = w_1(w_2w_1)^n$ for some $w_1,\ w_2 \in S^+$.

Subcase 1: If $|v| \geqslant 3$.

As in the proof of Case 1, if α is the last letter of v_1, then $|\varphi(bv_1\alpha^{-1})| < |v|$, $|\varphi(bv_1)| > |v|$. This contradicts Lemma 4.1.

Subcase 2: If $|v| = 2,\ |u| = 2$.

This contradicts $|u| \neq |v|$.

Subcase 3: If $|v| = 2,\ |u| \geqslant 3$.

In this case, $v = ba$, and $u = au_1b$ for some word u_1. Since $\varphi(v) = \varphi(ba) = baw$, $\varphi(u) = \varphi(au_1b) = au_1bw$, we have $|\varphi(u_1)| = |u_1|$. This contradicts Lemma 4.1.

This study of Subcases 1, 2, and 3 shows that Case 4 is impossible. □

The above proposition corresponds to cases (iv) and (v) with $k \geqslant 2$ in Theorem 1.2.

Proposition 4.3. — *Let $u,\ v \in S^+$. Assume $a \triangleleft u,\ b \triangleleft v$, $|u| = |v| \geqslant 3$, and the last letters of u and v are distinct. Then there are* NO *primitive substitutions φ whose fixed points are non-ultimately periodic and such that $\varphi(u) = uw$, $\varphi(v) = vw$ for some $w \in S^*$.*

Proof. — Suppose that φ is a primitive substitution such that $\varphi(u) = uw$, $\varphi(v) = vw$ for some $w \in S^*$.

We have $a \triangleleft \varphi(a)$, $b \triangleleft \varphi(b)$ and $|\varphi(a)| \geqslant 2$, $|\varphi(b)| \geqslant 2$. Without loss of generality, we assume $|\varphi(a)| \geqslant |\varphi(b)|$.

We consider several cases.

Case 1: $|\varphi(a)| \geqslant 3,\ |\varphi(b)| \geqslant 3$.

As in the proof of Proposition 4.2, this is impossible.

Case 2: $|\varphi(a)| = 2$.

Since $|\varphi(a)| \geqslant |\varphi(b)|$, and $|\varphi(b)| \geqslant 2$, we have $|\varphi(a)| = |\varphi(b)| = 2$. Since φ is a primitive substitution and $a \triangleleft \varphi(a)$, $b \triangleleft \varphi(b)$, it follows that $\varphi(a) = ab$, $\varphi(b) = ba$, and then the fixed points are the Thue-Morse sequences. But $\varphi(uv^{-1}) \neq uv^{-1}$. Thus $\varphi(u) = uw$, $\varphi(v) = vw$ does not hold. So, Case 2 is impossible.

Case 3: $|\varphi(a)| \geqslant 3$, $|\varphi(b)| = 2$ and $|u| = |v| = 3$.

Subcase 1: $a \triangleright u$.

We have $u = a\alpha a$, $v = b\beta b$ with $\alpha,\ \beta \in \{a,\ b\}$. Thus

$$\begin{aligned}|\varphi(u)| &= |\varphi(a\alpha a)| \geqslant 2|\varphi(a)| + |\varphi(b)| \\ &> 2|\varphi(b)| + |\varphi(a)| \geqslant |\varphi(b\beta b)| \\ &= |\varphi(v)|,\end{aligned}$$

which contradicts $|\varphi(u)| = |u| + |w| = |v| + |w| = |\varphi(v)|$. Thus, Subcase 1 is impossible.

Subcase 2: $b \triangleright u$.

We have $u = a\alpha b$, $v = b\beta a$ with $\alpha,\ \beta \in \{a,\ b\}$.

(i) If $\alpha \neq \beta$. As in Case 1, this is impossible.

(ii) If $\alpha = \beta = b$. By $\varphi(v) = vw$ and $|\varphi(b)| = 2$, we have $\varphi(b) = bb$, which contradicts that φ is a primitive substitution.

(iii) If $\alpha = \beta = a$. Then $u = aab$, $v = baa$, aab is a prefix of $\varphi(a)$, and $\varphi(b) = ba$. As in the proof of Claim 1, one shows that $\varphi(a) = w_1(\varphi(b))^n$ for some $n \in \mathbb{N}$ and for some proper suffix w_1 of $\varphi(b)$ (proper meaning $w_1 \neq \varepsilon$ and $w_1 \neq \varphi(b)$). Since $\varphi(b) = ba$, it follows that $\varphi(a) = a(ba)^n$ for some $n \in \mathbb{N}$, which contradicts that aab is a prefix of $\varphi(a)$.

Thus Subcase 2 is impossible.

The study of Subcases 1 and 2 shows that Case 3 is impossible.

Case 4: $|\varphi(a)| \geqslant 3$, $|\varphi(b)| = 2$ and $|u| = |v| \geqslant 4$.

As in the proof of Proposition 4.2, this is impossible. □

Proof of the Necessity. — Let φ be a primitive substitution with two non-ultimately periodic fixed points ξ_1 and ξ_2 which are of the form $\xi_1 = u\xi$, $\xi_2 = v\xi$, where u and v are words.

Obviously, at least one of u, v is not the empty word. Also we can assume that the last letters of u and v are distinct. Then

$$\begin{aligned}u\xi = \varphi(u\xi) &= \varphi(uv^{-1}v\xi) \\ &= \varphi(uv^{-1})v\xi.\end{aligned}$$

We claim that $u = \varphi(uv^{-1})v$. Otherwise $u \neq \varphi(uv^{-1})v$, so there exists $u_1 \in S^+$ such that $\xi = u_1\xi$. Thus ξ is periodic, and ξ_1 and ξ_2 are ultimately periodic. This causes a contradiction.

Hence $u = \varphi(uv^{-1})v$, so $\varphi(uv^{-1}) = uv^{-1}$.

Since $|\varphi(a)| \geqslant 2$, $|\varphi(b)| \geqslant 2$, then $u \neq \varepsilon$, $v \neq \varepsilon$, and there exists $w \in S^+$ such that $\varphi(u) = uw$, $\varphi(v) = vw$.

Without loss of generality, we assume $a \triangleleft u$, $b \triangleleft v$.

Case 1: $|u| = |v| = 1$.

Then $u = a,\ v = b$. In this case, $\varphi(a) = aw,\ \varphi(b) = bw$.

This corresponds to the cases (iv) and (v) with $k = 1$ in the statement of the theorem.

Case 2: $|u| = |v| = 2$.

(i) If $u = aa,\ v = bb$. Since $\varphi(aa) = aaw,\ \varphi(bb) = bbw$, there exist words w_1 and w_2 such that $w_1 aa w_1 = w = w_2 bb w_2$. This is impossible.

(ii) If $u = ab, v = ba$. By Theorem 1.1, φ is an invertible substitution, and $\varphi \in \langle \pi,\ \tau \rangle$ is such that the total number of π's and τ's in any decomposition of φ is even. This is just the case (vi) in the theorem.

The above argument, together with Propositions 4.2 and 4.3, yields the necessity. □

Acknowledgement

The authors are very happy to express their gratitude to Prof. Jacques Peyrière for his careful reading and polishing of the manuscript.

The authors are supported by NSFC no. 10501053, 10571140, 10571104 and 10571138 respectively.

BIBLIOGRAPHY

[1] J. Allouche & J. Shallit, *Automatic sequences: Theory and Applications*, Cambridge University Press, Cambrige, 2002.

[2] P. Arnoux & G. Rauzy, "Représentation géométrique de suites de complexité $2n + 1$", *Bull. Soc. Math.* **119** (1991), no. 2, p. 199-215, France.

[3] P. Arnoux & S.Ito, "Pisot substitutions and Rauzy fractals", *Bull. Belg. Math. Soc. Simon Stevin* **8** (2001), no. 2, p. 181-207.

[4] H. Ei & S. Ito, "Decomposition theorem on invertible substitutions", *Osaka J. Math.* **35** (1998), no. 4, p. 821-834.

[5] M. Lothaire, *Combinatorics on words*, second ed., Cambridge University Press, Cambridge, 1997.

[6] ———, *Algebraic combinatorics on words*, Cambridge University Press, Cambridge, 2002.

[7] ———, *Applied combinatorics on words*, Cambridge University Press, Cambridge, 2005.

[8] J. Nielsen, "Die Isomorphismengruppen der freien Gruppen", *Math. Ann* **91** (1924), p. 169-209, Available at `http://mathlib.sub.uni-goettingen.de/JFM/digit.php?an=JFM+50.0078.04`.

[9] N. Pytheas Fogg, *Substitutions in Dynamics, Arithmetics and Combinatorics*, Lecture Notes in Mathematics, vol. 1794, Springer, Berlin, 2002.

[10] P. Séébold, "An effective solution to the D0L periodicity problem in the binary case", *EATCS Bull.* **36** (1988), p. 137-151.

[11] B. Tan, Z.-X. Wen & Y. P. Zhang, "The structure of invertible substitutions on a three-letter alphabet", *Adv. in Appl. Math.* **32** (2004), no. 4, p. 736-753.

[12] Z.-X. Wen & Z.-Y. Wen, "Local isomorphism of the invertible substitutions", *C. R. Acad. Sci. Paris Sér. I Math.* **318** (1994), no. 4, p. 299-304.

[13] Z.-X. Wen, Z.-Y. Wen & J. Wu, "On invertible substitutions with two fixed points", *C. R. Math. Acad. Sci. Paris* **334** (2002), no. 9, p. 727-731.

[14] Z.-X. Wen & Y. P. Zhang, "Some remarks on invertible substitutions on three letter alphabet", *Chinese Sci. Bull.* **44** (1999), no. 19, p. 1755-1760.

Bo TAN, Zhi-Xiong WEN & Jun WU
Huazhong University of Science and Technology
Department of Mathematics
Wuhan, 430074 (P.R. China)
bo_tan@163.com
zhxwen@whu.edu.cn
wujunyu@public.wh.hb.cn

Zhi-Ying WEN
Tsinghua University
Department of Mathematics
Beijing, 100084 (P.R. China)
wenzy@tsinghua.edu.cn