ANNALES

DE

L'INSTITUT FOURIER

Henrik ARATYN & Johan VAN DE LEUR

The symplectic Kadomtsev-Petviashvili hierarchy and rational solutions of Painlevé VI

Tome 55, nº 6 (2005), p. 1871-1903.

<http://aif.cedram.org/item?id=AIF_2005__55_6_1871_0>

Ann. Inst. Fourier, Grenoble
55, 6 (2005), 1871–1903

THE SYMPLECTIC KADOMTSEV-PETVIASHVILI HIERARCHY AND RATIONAL SOLUTIONS OF PAINLEVÉ VI

by Henrik ARATYN & Johan van de LEUR

1. Isomonodromic deformation problem, Painlevé VI equation and the Euler top equations.

Consider a Fuchsian system of linear differential equation with rational coefficients:

$$\frac{\partial}{\partial z}X(a,z) = -\sum_{i=1}^{3}\frac{A_i}{z-a_i}X(a,z), \quad \frac{\partial}{\partial a_i}X(a,z) = \frac{A_i}{z-a_i}X(a,z). \tag{1.1}$$

The three-dimensional Schlesinger equations

$$\frac{\partial}{\partial a_i}A_i = \sum_{j=1,\ j\neq i}^{3}\frac{[A_i,A_j]}{a_i-a_j}, \quad \frac{\partial}{\partial a_j}A_i = \frac{[A_i,A_j]}{a_j-a_i}, \quad i\neq j. \tag{1.2}$$

emerge as compatibility equations of the system (1.1) and describe monodromy preserving deformations for the linear differential equations in the complex plane.

Let us fix $a_1 = 0, a_2 = 1$ and $a_3 = x$ and work with 2×2 matrices A_0, A_1, A_x. The Schlesinger equations reduce to:

$$\dot{A}_0 = -\frac{1}{x}\left[A_0, A_x\right], \quad \dot{A}_1 = \frac{1}{1-x}\left[A_1, A_x\right],$$

Keywords: KP hierarchy, Grassmanian, Frobenius manifold, isomonodromic deformation, Painlevé VI.
Math. classification: 14M15, 17B65, 17B80, 22E67, 34M55, 37K10, 37K35.

$$\dot{A}_x = \frac{1}{x}\left[A_0, A_x\right] - \frac{1}{1-x}\left[A_1, A_x\right],$$

where $\dot{A} = dA/dx$. The matrix A_x can be eliminated by setting $A_x = -A_0 - A_1 - A_\infty$, where $A_\infty = -\sum_{i=1}^{3} A_i$ is an integral of the Schlesinger equations (1.2). The matrix A_∞ is a constant matrix with different eigenvalues, so it is diagonalizable.

We will now follow [16], [17], [18], see also [24] and describe a connection to the Painlevé VI equation.

Let $\pm\theta_0/2, \pm\theta_1/2, \pm\theta_x/2, \pm\theta_\infty/2$ be eigenvalues of A_0, A_1, A_x and A_∞ and so

$$\operatorname{tr}(A_0^2) = \frac{1}{2}\theta_0^2, \quad \operatorname{tr}(A_1^2) = \frac{1}{2}\theta_1^2, \quad \operatorname{tr}(A_x^2) = \frac{1}{2}\theta_x^2, \quad \operatorname{tr}(A_\infty^2) = \frac{1}{2}\theta_\infty^2.$$

We parametrize the traceless matrices A_0, A_1 as in [16], [17], [18], [24] :

$$A_i = \frac{1}{2}\begin{pmatrix} z_i & u_i(\theta_i - z_i) \\ (\theta_i + z_i)/u_i & -z_i \end{pmatrix}, \quad i = 0, 1. \tag{1.3}$$

Following [16], [17], [18], [24] we replace u_0 and u_1 by two new variables k and y:

$$k = xu_0(z_0 - \theta_0) - (1-x)u_1(z_1 - \theta_1), \quad ky = xu_0(z_0 - \theta_0) \tag{1.4}$$

as a result the above isomonodromic deformation problem leads to the Painlevé VI equation :

$$\ddot{y} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x}\right)\dot{y}^2 - \left(\frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x}\right)\dot{y} + \frac{y(y-1)(y-x)}{x^2(x-1)^2}\left[\alpha + \beta\frac{x}{y^2} + \gamma\frac{x-1}{(y-1)^2} + \delta\frac{x(x-1)}{(y-x)^2}\right] \tag{1.5}$$

characterized by the parameters $(\alpha, \beta, \gamma, \delta)$

$$\alpha = \frac{(1-\theta_\infty)^2}{2}, \quad \beta = -\frac{\theta_0^2}{2}, \quad \gamma = \frac{\theta_1^2}{2}, \quad \delta = \frac{1-\theta_x^2}{2}.$$

We will at this point reduce the number of parameters from four to two by setting $\rho = \theta_0 = \theta_1 = \theta_x$ and $\nu = \theta_\infty$. These constants ρ and ν parametrize $(\alpha, \beta, \gamma, \delta)$ as follows

$$\alpha = \frac{(1-\nu)^2}{2}, \quad \beta = -\frac{\rho^2}{2}, \quad \gamma = \frac{\rho^2}{2}, \quad \delta = \frac{1-\rho^2}{2}. \tag{1.6}$$

In this formulation it is convenient to define

$$\omega_1^2 = -\left(\frac{\rho^2}{2} + \operatorname{tr}(A_1 A_x)\right) = -\frac{\rho^2}{4} - \frac{\nu^2}{4} - \frac{1}{2}\nu z_0, \tag{1.7}$$

$$\omega_2^2 = -\left(\frac{\rho^2}{2} + \operatorname{tr}(A_0 A_1)\right) = -\frac{\rho^2}{4} + \frac{\nu^2}{4} + \frac{1}{2}\nu(z_0 + z_1), \tag{1.8}$$

$$\omega_3^2 = -\left(\frac{\rho^2}{2} + \operatorname{tr}(A_0 A_x)\right) = -\frac{\rho^2}{4} - \frac{\nu^2}{4} - \frac{1}{2}\nu z_1. \tag{1.9}$$

The functions $\omega_i, i = 1, 2, 3$ defined in (1.7)–(1.9) satisfy

$$\sum_{i=1}^{3} \omega_i^2 = -\frac{3\rho^2}{4} - \frac{\nu^2}{4} = -\mu^2, \tag{1.10}$$

which defines the scaling dimension μ.

One can also prove like in [13] that ω_i, $i = 1, 2, 3$, satisfy the time dependent Euler top equations:

$$\frac{d\omega_1}{dx} = \frac{\omega_2\omega_3}{x}, \qquad \frac{d\omega_2}{dx} = \frac{\omega_1\omega_3}{x(x-1)}, \qquad \frac{d\omega_3}{dx} = \frac{\omega_1\omega_2}{1-x}. \tag{1.11}$$

Next, introduce

$$\zeta = x(1-y)z_0 + (1-x)yz_1, \tag{1.12}$$

for which we have two equations [16], [17], [18], [24]:

$$\zeta = -x(1-x)\dot{y} + (1-\theta_\infty)y(1-y), \tag{1.13}$$

$$2\theta_\infty(z_0 + z_1) = 4\omega_2^2 + \rho^2 - \nu^2 = \frac{\rho^2(y-x)^2 - \zeta^2}{x(1-x)y(1-y)} + \rho^2 - \nu^2. \tag{1.14}$$

From which we can determine ω_2^2 in terms of y and $\dot{y}$ as

$$\omega_2^2 = \frac{\rho^2(y-x)^2 - \zeta^2}{4x(1-x)y(1-y)}.$$

From equations (1.12), (1.13) and (1.14) we can express z_0 or z_1 in terms of y and $\dot{y}$. This procedure yields:

$$\omega_1^2 = -\frac{\rho^2+\nu^2}{4} - \frac{\rho^2-\nu^2}{4}\frac{(1-x)y}{y-x} + \frac{\nu^2}{2}\frac{y(y-1)}{y-x} + \nu\frac{A}{y-x} - \frac{A_+A_-}{x(y-1)(y-x)}, \tag{1.15}$$

$$\omega_2^2 = \frac{A_+A_-}{x(1-x)y(y-1)}, \tag{1.16}$$

$$\omega_3^2 = -\frac{\rho^2+\nu^2}{4} - \frac{\rho^2-\nu^2}{4}\frac{x(1-y)}{y-x} - \frac{\nu^2}{2}\frac{y(y-1)}{y-x} - \nu\frac{A}{y-x} - \frac{A_+A_-}{(1-x)y(y-x)}, \tag{1.17}$$

where

$$A = \frac{1}{2}\left[\dot{y}x(x-1) - y(y-1)\right], \tag{1.18}$$

$$A_\pm = \frac{1}{2}\dot{y}x(x-1) - \frac{1}{2}(1-\theta_\infty)y(y-1) \pm \frac{1}{2}\rho(y-x) = A + \frac{1}{2}\nu y(y-1) \pm \frac{1}{2}\rho(y-x). \tag{1.19}$$

There are two natural ways to further reduce the system to a one parameter system characterized by a conformal scaling dimension μ only.

1) Set $\rho^2 = \nu^2$. Thus, from (1.10) $\rho^2 = \nu^2 = \mu^2$ with (*cf.* [13, 14, 15])

$$\alpha = \frac{(1 \mp \mu)^2}{2}, \quad \beta = -\frac{\mu^2}{2}, \quad \gamma = \frac{\mu^2}{2}, \quad \delta = \frac{1-\mu^2}{2}, \tag{1.20}$$

using that $\nu = \pm\mu$. For instance, for $\nu = 1/2$ we get $(\alpha, \beta, \gamma, \delta) = (1/8, -1/8, 1/8, 3/8)$, while for $\nu = -1/2$ we get $(\alpha, \beta, \gamma, \delta) = (9/8, -1/8, 1/8, 3/8)$. In this case $\omega_i, i = 1, 2, 3$ are defined through (1.7)–(1.9):

$$\omega_1^2 = -\frac{\mu^2}{2} - \frac{1}{2}\nu z_0, \quad \omega_2^2 = \frac{1}{2}\nu(z_0 + z_1), \quad \omega_3^2 = -\frac{\mu^2}{2} - \frac{1}{2}\nu z_1, \tag{1.21}$$

which now yields :

$$\omega_1^2 = -\frac{\mu^2}{2}\left(1 + \frac{y(1-y)}{y-x}\right) + \nu\frac{A}{y-x} - \frac{1}{x(y-1)(y-x)}A_+A_-, \tag{1.22}$$

$$\omega_2^2 = \frac{1}{x(1-x)y(y-1)}A_+A_-, \tag{1.23}$$

$$\omega_3^2 = -\frac{\mu^2}{2}\left(1 - \frac{y(1-y)}{y-x}\right) - \nu\frac{A}{y-x} - \frac{1}{(1-x)y(y-x)}A_+A_-, \tag{1.24}$$

where A is as in (1.18) and

$$\begin{aligned} A_\pm &= \frac{1}{2}\dot{y}x(x-1) - \frac{1}{2}(1-\nu)y(y-1) \pm \frac{1}{2}\mu(y-x) \\ &= A + \frac{1}{2}\nu y(y-1) \pm \frac{1}{2}\mu(y-x), \end{aligned} \tag{1.25}$$

with $\nu = \pm\mu$.

For $\nu = 1/2$ (and $\mu^2 = 1/4$) expressions (1.15)–(1.17) agree with results of [1].

From equations (1.12) and (1.14) we find for $\rho^2 = \nu^2 = \mu^2$:

$$\begin{aligned} \mu^2\omega_2^2 x(1-x)y(1-y) - \mu^4(y-x)^2/4 + \big[x(1-y)(\omega_1^2 + \mu^2/2) \\ +(1-x)y(\omega_3^2 + \mu^2/2)\big]^2 = 0, \end{aligned} \tag{1.26}$$

which yields a solution of the Painlevé VI equation of the form :

$$y(x) = x\frac{\pm(x-1)\mu\omega_1\omega_2\omega_3 + x\omega_1^2\omega_2^2 + \omega_1^2\omega_3^2}{(x-1)^2\omega_2^2\omega_3^2 + x^2\omega_1^2\omega_2^2 + \omega_1^2\omega_3^2}. \tag{1.27}$$

2) In the second case we set $\rho = 0$ and therefore from (1.10) $\nu^2 = (2\mu)^2$ with (*cf.* [7,8, 25]) the result that

$$\alpha = \frac{(1 \pm 2\mu)^2}{2}, \quad \beta = 0, \quad \gamma = 0, \quad \delta = \frac{1}{2} \tag{1.28}$$

and (see (1.7)–(1.9))

$$\omega_1^2 = -\mu^2 - \frac{1}{2}\nu z_0, \quad \omega_2^2 = +\mu^2 + \frac{1}{2}\nu(z_0+z_1), \quad \omega_3^2 = -\mu^2 - \frac{1}{2}\nu z_1, \tag{1.29}$$

which now yields

$$\omega_1^2 = -\frac{(y-1)(y-x)}{x}\left[\frac{A}{(y-1)(y-x)} + \frac{\nu}{2}\right]^2, \tag{1.30}$$

$$\omega_2^2 = \frac{y(y-1)}{x(1-x)}\left[\frac{A}{y(y-1)} + \frac{\nu}{2}\right]^2, \tag{1.31}$$

$$\omega_3^2 = -\frac{y(y-x)}{(1-x)}\left[\frac{A}{y(y-x)} + \frac{\nu}{2}\right]^2. \tag{1.32}$$

From (1.14) we find for $\rho = 0$

$$\omega_2^2 = -\frac{\zeta^2}{4x(1-x)y(1-y)},$$

which together with definition (1.12) of ζ and (1.30) and (1.32) yields equation

$$4\mu^2\omega_2^2 x(1-x)y(1-y) + \left[x(1-y)(\omega_1^2+\mu^2) + (1-x)y(\omega_3^2+\mu^2)\right]^2 = 0. \tag{1.33}$$

As a general solution of (1.33) one obtains expressions

$$y(x) = -x\frac{x(\omega_1\omega_2 \mp \mu\omega_3)^2 + (\omega_1\omega_3 \pm \mu\omega_2)^2}{\left(\omega_3^2+\mu^2+x(\omega_2^2+\mu^2)\right)^2 + 4x\mu^2\omega_1^2}. \tag{1.34}$$

As an example we consider the case of $\mu = \pm 1$ with

$$\omega_1 = \frac{\sqrt{-b}(1-x)}{b-x}, \quad \omega_2 = -\frac{\sqrt{-b(b-1)}}{b-x}, \quad \omega_3 = \frac{\sqrt{b-1}x}{b-x}, \tag{1.35}$$

which satisfy the Euler top equations (1.11) and $\sum_{i=1}^3 \omega_i^2 = -1$, hence $\mu^2 = 1$. As one of two solutions to equation (1.33) we obtain

$$y(x) = -\frac{(b-1)x}{-b+x}, \tag{1.36}$$

which satisfies the Painlevé VI equation (1.5) with

$$(\alpha,\beta,\gamma,\delta) = \left((1-2\mu)^2/2, 0, 0, 1/2\right) = (1/2, 0, 0, 1/2),$$

corresponding to $\mu = 1$. Note, that introducing $a = (b-1)/b, a \neq 0$ we can rewrite (1.36) as

$$y(x) = \frac{ax}{1-(1-a)x},$$

which appeared in [25] as a one parameter family of rational solutions to Painlevé VI equation with $\mu = 1$.

As a second solution to equation (1.33) we obtain for (1.35)

$$y(x) := -\frac{x(b-1)(-b+x^2)^2}{(-b+x)(x^4-4bx^3+6bx^2-4bx+b^2)},$$

which satisfies the Painlevé VI equation (1.5) with

$$(\alpha,\beta,\gamma,\delta) = \big((1-2\mu)^2/2, 0, 0, 1/2\big) = (9/2, 0, 0, 1/2)\,.$$

corresponding to $\mu = -1$.

There is only one solution of equation (1.26):

$$y(x) = \frac{x^2-b}{2(-b+x)},$$

which yields a solution of the Painlevé VI equation (1.5) with

$$(\alpha,\beta,\gamma,\delta) = \big((1\pm\mu)^2/2, -\mu^2/2, \mu^2/2, (1-\mu^2)/2\big) = (2, -1/2, 1/2, 0)\,.$$

2. The Darboux-Egoroff equations.

The connection between the Painlevé VI equation and three-dimensional Frobenius manifolds is established through the Darboux-Egoroff equations for the rotation coefficients $\beta_{ij} = \beta_{ji}$:

$$\frac{\partial}{\partial u_k}\beta_{ij} = \beta_{ik}\beta_{kj}, \quad \text{distinct } i,j,k, \tag{2.1}$$

$$\sum_{k=1}^{n} \frac{\partial}{\partial u_k}\beta_{ij} = 0, \quad i \neq j. \tag{2.2}$$

In addition to these equations one also assumes the conformal condition:

$$\sum_{k=1}^{n} u_k \frac{\partial}{\partial u_k}\beta_{ij} = -\beta_{ij}. \tag{2.3}$$

The Darboux-Egoroff equations (2.1)-(2.2) appear as compatibility equations of a linear system :

$$\frac{\partial\Phi_{ij}(u,z)}{\partial u_k} = \beta_{ik}(u)\Phi_{kj}(u,z) \quad i \neq k, \tag{2.4}$$

$$\sum_{k=1}^{n} \frac{\partial\Phi_{ij}(u,z)}{\partial u_k} = z\Phi_{ij}(u,z). \tag{2.5}$$

Define the $n \times n$ matrices $\Phi = (\Phi_{ij})_{1\leqslant i,j\leqslant n}$, $B = (\beta_{ij})_{1\leqslant i,j\leqslant n}$ and $V_i = [B, E_{ii}]$, where $(E_{ij})_{k\ell} = \delta_{ik}\delta_{j\ell}$. Then the linear system (2.4)-(2.5) acquires the following form :

$$\frac{\partial\Phi(u,z)}{\partial u_i} = (zE_{ii} + V_i(u))\,\Phi(u,z), \quad i = 1,\ldots,n\,, \tag{2.6}$$

$$\sum_{k=1}^{n} \frac{\partial\Phi(u,z)}{\partial u_k} = z\Phi(u,z). \tag{2.7}$$

The conformal case $n = 3$ is very special. In that case

$$V = [B,U] = \left[\begin{pmatrix} 0 & \beta_{12} & \beta_{13} \\ \beta_{21} & 0 & \beta_{23} \\ \beta_{31} & \beta_{32} & 0 \end{pmatrix}, \begin{pmatrix} u_1 & 0 & 0 \\ 0 & u_2 & 0 \\ 0 & 0 & u_3 \end{pmatrix}\right] = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} \tag{2.8}$$

satisfies

$$\frac{\partial V}{\partial u_j} = [V_j, V]. \tag{2.9}$$

Note, that $\mathrm{Tr}(V^2)$ is an integration constant of equations (2.9), as it follows easily that $\partial\,\mathrm{Tr}(V^2)/\partial u_j = 0$ for all j.

For three-dimensional Frobenius manifolds, these equations exhibit isomonodromic dependence on canonical coordinates u and reduce to the class of the Painlevé VI equation (1.5) with $(\alpha, \beta, \gamma, \delta)$ parameters as in (1.20) or (1.28).

For vectorfields $I = \sum_{j=1}^{3} \partial/\partial u_j$ and $E = \sum_{j=1}^{3} u_j\partial/\partial u_j$ it follows from (2.9) that $I(V) = 0, E(V) = 0$ and accordingly V is a function of one variable x such that $I(x) = 0, E(x) = 0$. We choose

$$x = \frac{u_2 - u_1}{u_3 - u_1}. \tag{2.10}$$

Note that $\mathrm{tr}(V) = 0$ and $\det(V) = 0$ and V has eigenvalues $\mu, 0, -\mu$ where μ defines the integration constant $\mathrm{Tr}(V^2)$ of (2.9) through :

$$\mathrm{Tr}(V^2) = -2\left(\omega_1^2 + \omega_2^2 + \omega_3^2\right) = 2\mu^2.$$

Then $\omega_i, i = 1, 2, 3$ satisfy the Euler top equations (1.11) as a result of (2.9).

Note that $V(x)$, $V(u_1, u_2, u_3)$, *i.e.* V as function of x, respectively the u_i's, are connected as follows

$$V(x) = V(0, x, 1), \quad V(u_1, u_2, u_3) = V\left(\frac{u_2 - u_1}{u_3 - u_1}\right).$$

Since

$$\omega_1(u_1,u_2,u_3) = (u_3-u_2)\beta_{32}(u_1,u_2,u_3),$$
$$\omega_2(u_1,u_2,u_3) = (u_1-u_3)\beta_{13}(u_1,u_2,u_3),$$
$$\omega_3(u_1,u_2,u_3) = (u_2-u_1)\beta_{12}(u_1,u_2,u_3).$$

We find that

$$\omega_1(x) = (1-x)\beta_{23}(0,x,1), \quad \omega_2(x) = -\beta_{13}(0,x,1),$$
$$\omega_3(x) = x\beta_{12}(0,x,1).$$

In other words, it suffices to know the rotation coefficients $\beta_{ij}(0,x,1)$.

3. The tau-function.

We define the τ-function by equation:

$$\frac{\partial \log \tau}{\partial u_j} = \frac{1}{2}\operatorname{Tr}(V_j V) = \sum_{i=1}^{3} \beta_{ij}^2 (u_i - u_j) = \sum_{i,k=1}^{3} \epsilon_{ijk}^2 \frac{\omega_k^2}{(u_i-u_j)}, \tag{3.1}$$

in which we used $\beta_{ij} = \epsilon_{ijk}\omega_k/(u_j-u_i)$. This tau-function is related as

$$\tau_I = \frac{1}{\sqrt{\tau}}$$

to Dubrovin's isomonodromy tau-function τ_I [9].

The identity $I(\log\tau) = 0$, shows that τ is a function of two variables, which again can be identified with χ and h such that

$$h = u_2 - u_1. \tag{3.2}$$

It follows from (3.1) that

$$E(\log\tau(u)) = \frac{1}{2}\operatorname{Tr}\left(V^2\right) = \mu^2.$$

Making use of technical identities :

$$\frac{\partial x}{\partial u_1} = \frac{1}{h}(x-1)x, \quad \frac{\partial x}{\partial u_2} = \frac{1}{h}x, \quad \frac{\partial x}{\partial u_3} = -\frac{1}{h}x^2,$$

one easily derives

$$\frac{\partial}{\partial u_1} = \frac{x(x-1)}{h}\frac{\partial}{\partial x} - \frac{\partial}{\partial h}, \quad \frac{\partial}{\partial u_2} = \frac{x}{h}\frac{\partial}{\partial x} + \frac{\partial}{\partial h}, \quad \frac{\partial}{\partial u_3} = -\frac{x^2}{h}\frac{\partial}{\partial x},$$

from which

$$E = h\frac{\partial}{\partial h}$$

follows. Since $E(\log\tau) = h\partial\log\tau/\partial h = \mu^2$ we see that $\log\tau(x,h)$ decomposes as

$$\log\tau(x,h) = \mu^2\log h + \log\tau_0(x) \tag{3.3}$$

where τ_0 is a function of x only.

It follows from equations (2.9) and (3.1) that

$$\frac{\partial^2\log\tau}{\partial u_i\partial u_j} = -\beta_{ij}^2, \quad i\neq j.$$

which translates to the following parametrization of ω_i's in terms of a single isomonodromic tau function :

$$\begin{aligned}
\omega_2^2 &= x(x-1)\Big(\frac{d^2}{dx^2}\ln(\tau_0)(x)\Big) + (2x-1)\Big(\frac{d}{dx}\ln(\tau_0)(x)\Big)\\
&= \frac{d}{dx}\Big[x(x-1)\frac{d}{dx}\ln(\tau_0)(x)\Big],\\
\omega_3^2 &= -x^2(x-1)\Big(\frac{d^2}{dx^2}\ln(\tau_0)(x)\Big) - x^2\Big(\frac{d}{dx}\ln(\tau_0)(x)\Big) - \mu^2\\
&= -x^2\frac{d}{dx}\Big[(x-1)\frac{d}{dx}\ln(\tau_0)(x)\Big] - \mu^2,\\
\omega_1^2 &= x(x-1)^2\Big(\frac{d^2}{dx^2}\ln(\tau_0)(x)\Big) + (x-1)^2\Big(\frac{d}{dx}\ln(\tau_0)(x)\Big)\\
&= (x-1)^2\frac{d}{dx}\Big[x\frac{d}{dx}\ln(\tau_0)(x)\Big].
\end{aligned} \tag{3.4}$$

One verifies that indeed $\omega_1^2+\omega_2^2+\omega_3^2 = -\mu^2$. Moreover,

$$\frac{d\ln\tau_0}{dx} = \frac{\omega_1^2}{x(1-x)} + \frac{\omega_2^2}{x}.$$

4. The CKP hierarchy.

The symplectic Kadomtsev-Petviashvili or CKP hierarchy [5] can be obtained as a reduction of the KP hierarchy,

$$\frac{\partial}{\partial t_n}L = [(L^n)_+, L], \tag{4.1}$$

$$\text{for } \; L = L(t,\partial) = \partial + \ell^{(-1)}(t)\partial^{-1} + \ell^{(-2)}(t)\partial^{-2} + \cdots,$$

where $x = t_1$ and $\partial = \frac{\partial}{\partial x}$, by assuming the extra condition

$$L^* = -L. \tag{4.2}$$

By taking the adjoint, *i.e.*, * of (4.1), one sees that $\frac{\partial L}{\partial t_n} = 0$ for n even. Date, Jimbo, Kashiwara and Miwa [5], [19] construct such L's from certain special KP wave functions $\psi(t,z) = P(t,z)e^{\sum_i t_i z^i}$ (recall $L(t,\partial) = P(t,\partial)\partial P(t,\partial)^{-1}$), where one then puts all even times t_n equal to 0. Recall that a KP wave function satisfies

$$L\psi(t,z) = z\psi(t,z), \quad \frac{\partial \psi(t,z)}{\partial t_n} = (L^n)_+\psi(t,z), \tag{4.3}$$

and

$$\operatorname{Res}_z \psi(t,z)\psi^*(s,z) = 0. \tag{4.4}$$

The special wave functions which lead to an L that has condition (4.2) satisfy

$$\psi^*(t,z) = \psi(\tilde{t},-z), \quad \text{where} \quad \tilde{t}_i = (-)^{i+1}t_i. \tag{4.5}$$

We call such a ψ a CKP wave function. Note that this implies that $L(t,\partial)^* = -L(\tilde{t},\partial)$ and that

$$\operatorname{Res}_z \psi(t,z)\psi(\tilde{s},-z) = 0.$$

One can put all even times equal to 0, but we will not do that here.

The CKP wave functions correspond to very special points in the Sato Grassmannian, which consists of all linear spaces

$$W \subset H_+ \oplus H_- = \mathbb{C}[z] \oplus z^{-1}\mathbb{C}[[z^{-1}]],$$

such that the projection on H_+ has finite index. Namely, W corresponds to a CKP wave function if the index is 0 and for any $f(z), g(z) \in W$ one has $\operatorname{Res}_z f(z)g(-z) = 0$. The corresponding CKP tau functions satisfy $\tau(\tilde{t}) = \tau(t)$.

We will now generalize this to the multi-component case and show that a CKP reduction of the multi-component KP hierarchy gives the Darboux-Egoroff system. The n component KP hierarchy [4], [20] consists of the equations in $t_j^{(i)}$, $1 \leqslant i \leqslant n$, $j = 1, 2, \dots$

$$\frac{\partial}{\partial t_j^{(i)}} L = [(L^j C_i)_+, L], \quad \frac{\partial}{\partial t_j^{(i)}} C_k = [(L^j C_i)_+, C_k], \tag{4.6}$$

for the commuting $n \times n$-matrix pseudo-differential operators, L, C_i, $i = 1, 2, \dots n$, with $\sum_i C_i = I$ of the form

$$\begin{aligned} L &= \partial + L^{(-1)}\partial^{-1} + L^{(-2)}\partial^{-2} + \cdots, \\ C_i &= E_{ii} + C_i^{(-1)}\partial^{-1} + C_i^{(-2)}\partial^{-2} + \cdots, \quad 1 \leqslant i \leqslant n, \end{aligned} \tag{4.7}$$

where $x = t_1^{(1)} + t_1^{(2)} + \cdots + t_1^{(n)}$. The corresponding wave function has the form

$$\Psi(t,z) = P(t,z)\exp\left(\sum_{i=1}^{n}\sum_{j=1}^{\infty} t_j^{(i)} z^j E_{ii}\right),$$

where $P(t,z) = I + P^{(-1)}(t)z^{-1} + \cdots$, and satisfies

$$L\Psi(t,z) = z\Psi(t,z), \quad C_i\Psi(t,z) = \Psi(t,z)E_{ii},$$

$$\frac{\partial\Psi(t,z)}{\partial t_j^{(i)}} = (L^jC_i)_+\Psi(t,z) \tag{4.8}$$

and

$$\operatorname{Res}_z \Psi(t,z)\Psi^*(s,z)^T = 0.$$

From this we deduce that

$$L = P(t,\partial)\partial P(t,\partial)^{-1} \text{ and } C_i = P(t,\partial)E_{ii}P(t,\partial)^{-1}.$$

Using this, the simplest equations in (4.8) are

$$\frac{\partial\Psi(t,z)}{\partial t_1^{(i)}} = (zE_{ii} + V_i(t))\Psi(t,z), \tag{4.9}$$

where $V_i(t) = [B(t), E_{ii}]$ and $B(t) = P^{(-1)}(t)$. In terms of the matrix coefficients β_{ij} of B we obtain (2.1) for $u_i = t_1^{(i)}$.

The Sato Grassmannian becomes vector valued, *i.e.*,

$$H_+ \oplus H_- = (\mathbb{C}[z])^n \oplus z^{-1}(\mathbb{C}[[z^{-1}]])^n.$$

The same restriction as in the 1-component case (4.5), viz.,

$$\Psi(t,z) = \Psi^*(\tilde{t},-z), \quad \text{where} \quad \tilde{t}_n^{(i)} = (-)^{n+1}t_n^{(i)}.$$

leads to $L^*(\tilde{t}) = -L(t)$, $C_i^*(\tilde{t}) = C_i(t)$ and

$$\operatorname{Res}_z \Psi(t,z)\Psi(\tilde{s},-z)^T = 0, \tag{4.10}$$

which we call the multi-component CKP hierarchy. But more importantly, it also gives the restriction

$$\beta_{ij}(t) = \beta_{ji}(\tilde{t}). \tag{4.11}$$

Such CKP wave functions correspond to points W in the Grassmannian for which

$$\operatorname{Res}_z f(z)^T g(-z) = \operatorname{Res}_z \sum_{i=1}^{n} f_i(z)g_i(-z) = 0$$

for any $f(z) = (f_1(z), f_2(z), \ldots, f_n(z))^T$, $g(z) = (g_1(z), g_2(z), \ldots, g_n(z))^T \in W$.

If we finally assume that $L = \partial$, then Ψ, W also satisfy

$$\frac{\partial \Psi(t,z)}{\partial x} = \sum_{i=1}^{n} \frac{\partial \Psi(t,z)}{\partial t_1^{(i)}} = z\Psi(t,z), \quad zW \subset W \tag{4.12}$$

and thus β_{ij} satisfies (2.2) for $u_i = t_1^{(i)}$. Now differentiating (4.10) n times to x for $n = 0, 1, 2, \ldots$ and applying (4.12) leads to

$$\Psi(t,z)\Psi(\tilde{t},-z)^T = I.$$

These special points in the Grassmannian can all be constructed as follows [21]. Let $G(z)$ be an element in $GL_n(\mathbb{C}[z,z^{-1}])$ that satisfies

$$G(z)G(-z)^T = I, \tag{4.13}$$

then $W = G(z)H_+$. Clearly, any two $f(z), g(z) \in W$ can be written as $f(z) = G(z)a(z)$, $g(z) = G(z)b(z)$ with $a(z), b(z) \in H_+$, then $zf(z) = zG(z)a(z) = G(z)za(z) \in W$, since $za(z) \in H_+$. Moreover,

$$\begin{aligned}\operatorname{Res}_z f(z)^T g(-z) &= \operatorname{Res}_z a(z)^T G(z)^T G(-z) b(-z)\\ &= \operatorname{Res}_z a(z)^T b(-z) = 0.\end{aligned}$$

We now take very special elements in this twisted loop group, *i.e.*, elements that correspond to certain points of the Grassmannian that have a basis of homogeneous elements in z. Choose integers $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_n$ such that $\mu_{n+1-j} = -\mu_j$. Then take $G(z)$ of the form

$$G(z) = N(z)S^{-1} = Nz^{-\mu}S^{-1}, \quad \text{where} \quad \mu = \operatorname{diag}(\mu_1, \mu_2, \ldots, \mu_n)$$

and $N = (n_{ij})_{1 \leqslant i,j \leqslant n}$ a constant matrix that satisfies

$$N^T N = \sum_{j=1}^{n} (-1)^{\mu_j} E_{j,n+1-j} \tag{4.14}$$

and

$$\begin{aligned}S = \delta_{n,2m+1} E_{m+1,m+1} + \sum_{j=1}^{m} \frac{1}{\sqrt{2}} (E_{jj} + iE_{n+1-j,j}\\ + E_{j,n+1-j} - iE_{n+1-j,n+1-j}),\end{aligned}$$

for $n = 2m$ or $n = 2m+1$. Then [2]

$$\sum_{i=1}^{n} \sum_{j=1}^{\infty} j t_j^{(i)} \frac{\partial \Psi(t,z)}{\partial t_j^{(i)}} = z \frac{\partial \Psi(t,z)}{\partial z},$$

from which one deduces that

$$\sum_{i=1}^{n} \sum_{j=1}^{\infty} j t_j^{(i)} \frac{\partial \beta_{ij}}{\partial t_j^{(i)}} = -\beta_{ij}. \tag{4.15}$$

We next put $t_j^{(i)} = 0$ for all i and all $j > 1$ and $u_i = t_1^{(i)}$, then we obtain the situation of Section 2.

5. The case n=3.

We will now give an example of the previous construction, viz., the case that $n = 3$ and $-\mu_1 = \mu_3 = \mu \in \mathbb{N}$ and $\mu_2 = 0$. Hence, the point of the Grassmannian is given by

$$N(z)H_+ = N \begin{pmatrix} z^{-\mu} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{\mu} \end{pmatrix} H_+.$$

More precise, let $n_i = (n_{1i}, n_{2i}, n_{3i})^T$ and $e_1 = (1,0,0)^T$, $e_2 = (0,1,0)^T$ and $e_3 = (0,0,1)^T$, then this point of the Grassmannian has as basis

$$n_1 z^{-\mu},\ n_1 z^{1-\mu},\ \dots,\ n_1 z^{-1}, n_1,\ n_2,\ n_1 z,\ n_2 z,\ \dots,\ , n_1 z^{\mu-1},\ n_2 z^{\mu-1},$$
$$e_1 z^{\mu},\ e_2 z^{\mu},\ e_3 z^{\mu},\ e_1 z^{\mu+1},\ e_2 z^{\mu+1},\ \cdots.$$

Using this one can calculate in a similar way as in [22] (using the boson-fermion correspondence or vertex operator constructions) the wave function:

$$\Psi(t,z) = P(t,z) \exp\Big(\sum_{i=1}^{n} \sum_{j=1}^{\infty} t_j^{(i)} z^j E_{ii} \Big),$$

where

$$P_{jj}(t,z) = \frac{\hat{\tau}(t_\ell^{(k)} - \delta_{kj}(\ell z^\ell)^{-1})}{\hat{\tau}(t)},$$
$$P_{ij}(t,z) = z^{-1} \frac{\hat{\tau}_{ij}(t_\ell^{(k)} - \delta_{kj}(\ell z^\ell)^{-1})}{\hat{\tau}(t)} \quad \text{for } i \neq j$$

and where

$$\hat{\tau}(t) = \det \sum_{k=1}^{3} \sum_{i=0}^{\mu-1} \Big(\sum_{j=1}^{2\mu} n_{k1} S_{\mu+i-j+1}(t^{(k)}) E_{3i+k,j} + \sum_{j=1}^{\mu} n_{k2} S_{i-j+1}(t^{(k)}) E_{3i+k,2\mu+j} \Big).$$

The functions $S_i(x)$ are the elementary Schur polynomials, defined by:

$$\sum_{j\in\mathbb{Z}} S_j(x) z^j = e^{\sum_{k=1}^{\infty} x_k z^k}.$$

The tau function $\hat{\tau}_{ij}(t)$ is up to a multiplicative factor -1 equal to the above determinant where we replace the j-th row by

$$\begin{pmatrix} n_{i1} S_{\mu-1}(t^{(i)}) & \cdots & n_{i1} S_1(t^{(i)}) & n_{i1} & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Then

$$\beta_{ij}(t) = \frac{\hat\tau_{ij}(t)}{\hat\tau(t)}. \tag{5.1}$$

As we have seen in Section 2 it suffices to calculate $\beta_{ij}(t)$ only for $t_1^{(2)} = x$, $t_1^{(3)} = 1$ However we will not do that yet, we will take $t_j^{(1)} = s_j$, for $j = 1, 2, 3, \ldots$, $t_1^{(3)} = 1$ and all other $t_i^{(j)} = 0$ and write $\beta_{ij}(s)$ for the resulting β_{ij}. In fact we will make this substitution in $\hat\tau(t)$ and $\hat\tau_{ij}(t)$. This might lead to $\hat\tau(s) = \hat\tau_{ij}(s) = 0$ in such a way that $\beta_{ij}(s) = \frac{\hat\tau_{ij}(s)}{\hat\tau(s)} \neq 0$. However, as we shall see later, this will not happen.

Since, we can multiply the columns of the matrices of $\hat\tau_{ij}(s)$ and $\hat\tau(s)$ by a constant we can change the vectors $n_1 = (n_{11}, n_{21}, n_{31})^T$ and $n_2 = (n_{12}, n_{22}, n_{32})^T$. This will multiply $\hat\tau(s)$ by a scalar, but also $\hat\tau_{ij}(s)$ by the same scalar, hence $\beta_{ij}(s)$ remains the same. In a similar way $\beta_{ij}(t)$ does not change if we permute the rows of $\hat\tau(s)$ and $\hat\tau_{ij}(s)$ in the same way. We thus choose

$$n_1 = (\alpha, 1, a)^T, \quad n_2 = (-a, 0, \alpha)^T, \quad \text{with } \alpha, a \neq 0 \text{ and } \alpha^2 + a^2 = -1.$$

Then our new $\hat\tau(s)$ becomes:

$$\hat\tau(s) = \det \sum_{i=1}^{\mu} \Bigg(\alpha E_{i,\mu+i} - a E_{i,2\mu+i} + \sum_{j=1}^{2\mu} S_{\mu+i-j}(s) E_{\mu+i,j} + \frac{a}{(\mu+i-j)!} E_{2\mu+i,j} + \sum_{j=1}^{\mu} \frac{\alpha}{(i-j)!} E_{2\mu+i,2\mu+j} \Bigg),$$

where we assume that $k! = \infty$ for $k < 0$. And $\hat\tau_{12}(s)$, $\hat\tau_{13}(s)$ and $\hat\tau_{32}(s)$ is -1 times the same determinant, but now with the $\mu+1$-th, $2\mu+1$-th, $\mu+1$-th row, respectively, replaced by

$$(0\ \cdots\ 0\ \alpha \mid 0\ \cdots\ 0 \mid 0\ \cdots\ 0), \quad (0\ \cdots\ 0\ \alpha \mid 0\ \cdots\ 0 \mid 0\ \cdots\ 0)$$
$$\left(\tfrac{a}{(\mu-1)!}\ \tfrac{a}{(\mu-2)!}\ \cdots\ \tfrac{a}{0!} \mid 0\ \cdots\ 0 \mid 0\ \cdots\ 0 \right), \quad \text{respectively.}$$

Next subtract a multiple of the $2\mu+j$-th column from the $\mu+j$-th column, then one sees that

$$\hat\tau(s) = \det \sum_{i=1}^{\mu} \Bigg(E_{\mu+i,\mu+i} + \sum_{j=1}^{\mu} S_{\mu+i-j}(s) E_{ij} - a^2 S_{i-j}(s) E_{i,\mu+j} + \sum_{j=1}^{2\mu} \frac{1}{(\mu+i-j)!} E_{\mu+i,j} \Bigg), \tag{5.2}$$

and $\hat\tau_{12}(s)$, $\hat\tau_{13}(s)$, $\hat\tau_{32}(s)$, respectively is the same determinant with the 1-th, $\mu+1$-th, 1-th row replaced by,

$$(0\ \cdots\ 0\ -\alpha \mid 0\ \cdots\ 0)\,,\quad \left(0\ \cdots\ 0\ -\tfrac{\alpha}{a} \mid 0\ \cdots\ 0\right),$$
$$\left(-\tfrac{a}{(\mu-1)!}\ -\tfrac{a}{(\mu-2)!}\ -\cdots\ -\tfrac{a}{0!} \mid 0\ \cdots\ 0\right),\quad \text{respectively.}$$

Now multiply the matrix in (5.2) from the left with the matrix

$$\sum_{1\leqslant j\leqslant i\leqslant 2\mu} \frac{(-1)^{i-j}}{(i-j)!}E_{ij},$$

Then $\hat\tau(s)$ does not change and now becomes equal to

$$\hat\tau(s) = \det \sum_{i=1}^{\mu} E_{\mu+i,\mu+i} + \sum_{j=1}^{2\mu}\left((T^{\mu}_{2\mu-j}(s)\right)^{(\mu-i)} E_{ij}, \tag{5.3}$$

with

$$T^{\mu}_k(s) = \sum_{j=0}^{k-\mu}\frac{(-1)^j}{j!}S_{k-j}(s) - a^2 \sum_{j=k-\mu+1}^{k}\frac{(-1)^j}{j!}S_{k-j}(s) \ \text{ and}$$
$$(T^{\mu}_k(s))^{(p)} = \frac{\partial^p T^{\mu}_k(s)}{\partial s_1^p}. \tag{5.4}$$

Multiplying the determinant of the other $\hat\tau_{ij}(s)$ by the same matrix, one obtains that $\hat\tau_{12}(s)$, $\hat\tau_{13}(s)$, $\hat\tau_{32}(s)$, respectively is the same determinant with the 1-th, $\mu+1$-th, 1-th row replaced by,

$$\alpha\left(\tfrac{(-1)^{\mu}}{(\mu-1)!}\ \tfrac{(-1)^{\mu-1}}{(\mu-2)!}\ \cdots\ \tfrac{-1}{0!} \mid 0\ \cdots\ 0\right),$$

$$\tfrac{\alpha}{a}\left(\tfrac{(-1)^{\mu}}{(\mu-1)!}\ \tfrac{(-1)^{\mu-1}}{(\mu-2)!}\ \cdots\ \tfrac{-1}{0!} \mid 0\ \cdots\ 0\right),$$

$$-a\,(n0\ \cdots\ 0\ 1 \mid 0\ \cdots\ 0)\,,\quad \text{respectively.}$$

Now permuting the first μ rows of the matrix gives that

$$\hat\tau(s) = (-)^{\frac{\mu(\mu-1)}{2}}\mathrm{W}\left(T^{\mu}_{2\mu-1}(s), T^{\mu}_{2\mu-2}(s), \cdots, T^{\mu}_{\mu}(s)\right), \tag{5.5}$$
$$\hat\tau_{12}(s) = -(-)^{\frac{\mu(\mu-1)}{2}}\alpha\mathrm{W}\Big(T^{\mu}_{2\mu-1}(s) + \frac{T^{\mu}_{2\mu-2}(s)}{\mu-1}, T^{\mu}_{2\mu-2}(s) + \frac{T^{\mu}_{2\mu-3}(s)}{\mu-2}, \cdots, T^{\mu}_{\mu+1}(s) + T^{\mu}_{\mu}(s)\Big),$$

$$\hat{\tau}_{13}(s) = -(-)^{\frac{\mu(\mu-1)}{2}} \frac{\alpha}{a} \mathrm{W}\Big(T^{\mu}_{2\mu-1}(s) + \frac{T^{\mu}_{2\mu-2}(s)}{\mu-1}, T^{\mu}_{2\mu-2}(s) + \frac{T^{\mu}_{2\mu-3}(s)}{\mu-2}, \cdots, T^{\mu}_{\mu+1}(s) + T^{\mu}_{\mu}(s), T^{\mu}_{\mu-1}(s)\Big),$$

$$\hat{\tau}_{32}(s) = (-)^{\frac{\mu(\mu-1)}{2}} a \mathrm{W}\Big(T^{\mu}_{2\mu-1}(s), T^{\mu}_{2\mu-2}(s), \cdots, T^{\mu}_{\mu+1}(s)\Big),$$

where W stands for the Wronskian determinant:

$$W(f_1(s), f_2(s), \ldots, f_n(s)) = \det\left(\frac{\partial^{i-1} f_j(s)}{\partial s_1^{i-1}}\right)_{1\leqslant i,j\leqslant n}.$$

Thus, by (5.1) we have an expression for $\beta_{ij}(s)$ and hence can calculate the $\omega_i(s)$'s. Now put all $s_j = 0$ for $j > 1$ and write x for s_1, then

(5.6)

$$\omega_1(x) = -a(1-x)\frac{\mathrm{W}\left(T^{\mu}_{2\mu-1}(x), T^{\mu}_{2\mu-2}(x), \cdots, T^{\mu}_{\mu+1}(x)\right)}{\mathrm{W}\left(T^{\mu}_{2\mu-1}(x), T^{\mu}_{2\mu-2}(x), \cdots, T^{\mu}_{\mu}(x)\right)},$$

$$\omega_2(x) = -\frac{\alpha}{a}\frac{\mathrm{W}\left(T^{\mu}_{2\mu-1}(x) + \frac{T^{\mu}_{2\mu-2}(x)}{\mu-1}, T^{\mu}_{2\mu-2}(x) + \frac{T^{\mu}_{2\mu-3}(x)}{\mu-2}, \cdots, T^{\mu}_{\mu+1}(x) + T^{\mu}_{\mu}(x), T^{\mu}_{\mu-1}(x)\right)}{\mathrm{W}\left(T^{\mu}_{2\mu-1}(x), T^{\mu}_{2\mu-2}(x), \cdots, T^{\mu}_{\mu}(x)\right)},$$

$$\omega_3(x) = -\alpha x\frac{\mathrm{W}\left(T^{\mu}_{2\mu-1}(x) + \frac{T^{\mu}_{2\mu-2}(x)}{\mu-1}, T^{\mu}_{2\mu-2}(x) + \frac{T^{\mu}_{2\mu-3}(x)}{\mu-2}, \cdots, T^{\mu}_{\mu+1}(x) + T^{\mu}_{\mu}(x)\right)}{\mathrm{W}\left(T^{\mu}_{2\mu-1}(x), T^{\mu}_{2\mu-2}(x), \cdots, T^{\mu}_{\mu}(x)\right)}$$

satisfy the Euler top equations (1.11). We will show later that $\sum_{i=1}^{3} \omega_i(x) = -\mu^2$.

Note, see (5.5), that $\hat{\tau}(s)$ and $\hat{\tau}_{ij}(s)$ are Wronskians of functions wich satisfy

$$\frac{\partial f(s)}{\partial s_p} = \frac{\partial^p f(s)}{\partial s_1^p}, \quad p = 2, 3, 4, \ldots$$

Hence they are 1-component KP tau-functions. In the next sections we will show that these Wronskians can be obtained in the (1-component) 2-vector 1-constrained CKP hierarchy.

6. The 2-vector 1-constrained CKP hierarchy.

The Lax operator L of the (1-component) 2-vector 1-constrained CKP hierarchy can be written as (see [3])

(6.1) $$L = \partial + \Phi_1(t)\partial^{-1}\Phi_1^*(t) + \Phi_2(t)\partial^{-1}\Phi_2^*(t),$$

where $\Phi_j(t)$ is an eigenfunction and $\Phi_j^*(t) = \Phi_j(\tilde{t})$ an adjoint eigenfunction, satisfying

$$\frac{\partial \Phi_j(t)}{\partial t_n} = (L^n)_+ \Phi_j(t), \quad \frac{\partial \Phi_j^*(t)}{\partial t_n} = -((L^*)^n)_+ \Phi_j^*(t). \tag{6.2}$$

Recall that the Sato KP Grassmannian consists of all linear spaces

$$W \subset H_+ \oplus H_- = \mathbb{C}[z] \oplus z^{-1}\mathbb{C}[[z^{-1}]],$$

such that the projection on H_+ has finite index. We introduce a natural filtration on Grassmannian

$$\cdots \subset H_{k+1} \subset H_k \subset H_{k-1} \subset H_{k-2} \subset \cdots,$$

consisting of the linear subspaces

$$H_k = \left\{ \sum_{j=k}^{N} a_j z^j | a_j \in \mathbb{C} \right\}.$$

On the space H we have a bilinear form, viz. if $f(z) = \sum_j a_j z^j$ and $g(z) = \sum_j b_j z^j$ are in H, then we define

$$(f(z), g(z)) = \operatorname{Res}_z f(z)g(z) = \sum_j a_j b_{-j-1}. \tag{6.3}$$

Then the polynomial Sato Grassmannian $\operatorname{Gr}(H)$ consists of all linear subspaces of $W \subset H$ such that

$$H_k \subset W \subset H_\ell \quad \text{for certain } k > \ell. \tag{6.4}$$

The space $\operatorname{Gr}(H)$ has a subdivision into different components:

$$\operatorname{Gr}^{(j)}(H) = \left\{ W \in \operatorname{Gr}(H) | H_k \subset W,\ j = k - \dim(W/H_k) \right\}.$$

Clearly, the subspace H_k belongs to $\operatorname{Gr}^{(k)}(H)$. The polynomial CKP Sato Grassmannian consists of linear subspaces of $\operatorname{Gr}^{(0)}(H)$ such that for any $f(z), g(z) \in W$ one has $(f(z), g(-z)) = 0$. To describe the spaces corresponding to the 2-vector 1-constrained CKP hierarchy, such W must also satisfy the following condition [11], [12], [3]. There exists a subspace

$$W' \subset W \quad \text{of codimension 2 such that } zW' \subset W. \tag{6.5}$$

We assume that there is no larger subspace W' with $zW' \subset W$. Let $\psi_W(t, z)$ be the wave function corresponding to such W, then the $\Phi_j(t)$ can be constructed as follows. Let

$$zW + W = W \oplus \mathbb{C}zf_1(z) \oplus \mathbb{C}zf_2(z)$$

with $f_i(z) \in W$. Choose two independent elements $h_i(z) \in \mathbb{C}f_1(z) \oplus \mathbb{C}f_2(z)$ such that

$$(h_1(z), zh_2(-z) = (h_2(z), zh_1(-z)) = 0,$$

then up to a scalar constant c_j one has

$$\Phi_j(t) = c_j \left(\psi_W(t, z), zh_j(-z)\right).$$

7. Bäcklund-Darboux transformations.

In the next section we will define subspaces W that are related to the tau-functions $\hat{\tau}(s)$ of Section 5. Since Bäcklund-Darboux transformations will play an important role in our construction, we will describe the elementary ones first. For $W \in \mathrm{Gr}(H)$, let $W^\perp$ be the orthocomplement of W in H w.r.t. the bilinear form (6.3). Then, $W^\perp$ also belongs to $\mathrm{Gr}(H)$.

For each $W \in \mathrm{Gr}(H)$ we denote the wave function corresponding to W by ψ_W. The dual wave function of ψ_W, which we denote by ψ_W^* can be characterized as follows [26], [10]:

Proposition 7.1. — *Let W and $\tilde{W}$ be two subspaces in* $\mathrm{Gr}(H)$. *Then $\tilde{W}$ is the space W^* corresponding to the dual wave function, if and only if $\tilde{W} = W^\perp$ with $W^\perp$ the orthocomplement of W w.r.t. the bilinear form (6.3) on H. Moreover*

$$(\psi_W(t,z), \psi_W^*(s,z)) = 0.$$

Let $W \in \mathrm{Gr}^{(k)}(H)$ then

$$\psi_W(t,z) = P_W(t,\partial)e^{\sum_{j=1}^\infty t_j z^j}, \quad \psi_W^*(t,z) = P_W^{*-1}(t,\partial)e^{-\sum_{j=1}^\infty t_j z^j},$$

where $P_W(t,\partial)$ is an k^{th} order pseudo-differential operator. The corresponding KP Lax operator L_W is equal to

$$L_W(t,\partial) = P_W(t,\partial)\partial P_W^{-1}(t,\partial). \tag{7.1}$$

From now on we will use the notation ψ_W and L_W whenever we want to emphasize its dependence on a point W of the Sato Grassmannian $\mathrm{Gr}(H)$.

Eigenfunctions $\Phi(t)$ and adjoint eigenfunctions $\Psi(t)$ of the KP Lax operator, satisfy (6.2) and can be expressed in wave and adjoint wave functions, viz. there exist functions $f, g \in H$ such that

$$\Phi(t) = (\psi_W(t,z), f(z)), \quad \Psi(t) = (\psi_W^*(t,z), g(z)). \tag{7.2}$$

Such (adjoint) eigenfunctions induce elementary Bäcklund-Darboux transformations [10]. Assume that we have the following data $W \in \mathrm{Gr}^{(k)}(H)$, $W^\perp$, $\psi_W(t,z)$ and $\psi_W^*(t,z)$, then the (adjoint) eigenfunctions (7.2) induce new KP wave functions:

$$\begin{aligned} \psi_{W'}(t,z) &= \left(\Phi(t)\partial\Phi(t)^{-1}\right)\psi_W(t,z), \\ \psi_{W'}^*(t,z) &= \left(\Phi(t)\partial\Phi(t)^{-1}\right)^{*-1}\psi_W^*(t,z), \\ \psi_{W''}(t,z) &= \left(-\Psi(t)\partial\Psi(t)^{-1}\right)^{*-1}\psi_W(t,z), \\ \psi_{W''}^*(t,z) &= \left(-\Psi(t)\partial\Psi(t)^{-1}\right)\psi_W^*(t,z), \end{aligned} \tag{7.3}$$

and new tau-functions

$$\tau_{W'}(t) = \Phi(t)\tau_w(t), \quad \tau_{W''}(t) = \Psi(t)\tau_W(t), \tag{7.4}$$

where

$$\begin{aligned} W' &= \{w \in W | (w(z), f(z)) = 0\} \in \mathrm{Gr}^{(k+1)}(H), \quad W'^{\perp} = W^{\perp} + \mathbb{C}f, \\ W'' &= W + \mathbb{C}g \in \mathrm{Gr}^{(k-1)}(H), \quad W''^{\perp} = \{w \in W^{\perp} | (w(z), g(z)) = 0\}. \end{aligned} \tag{7.5}$$

Now applying n consecutive elementary Bäcklund-Darboux transformations such that one obtains

$$W' = \left\{w \in W | (w(z), f_i(z)) = 0, i = 1, 2, \ldots, n\right\}$$

from W, then (see [10])

$$\tau_{W'}(t) = W(\Phi_1(t), \Phi_2(t), \ldots, \Phi_n(t))\tau_W(t),$$

where one has to take derivatives w.r.t. x and where

$$\Phi_j(t) = (\psi_W(t, z), f_j(z))$$

and

$$\psi_{W'}(t, z) = \frac{1}{\tau_{W'}(t)} W\left(\Phi_1(t), \Phi_2(t), \ldots, \Phi_n(t), \psi_W(t, z)\right). \tag{7.6}$$

8. Subspaces W_μ.

In this section we will construct Subspaces W_μ in the 2-vector 1-constrained CKP hierarchy related to the solutions of Section 5 of the time-dependent Euler top equations. Let $a \in \mathbb{C}$ with $a \neq 0, \pm i$ be the parameter of Section 5. Define $b = -a^2$, then $b \neq 0, 1$ and introduce

$$r_0(z) = be^z. \tag{8.1}$$

Unfortunately $r_0(z)$ is not an element of H. However, since we always assume that $H_k \subset W$ for $k >> 0$, we will write e^z and will mean in fact $\sum_{j=0}^{N} \frac{z^j}{j!}$ with $N > 2k >> 0$. Having this in mind, we define for $i = 1, 2, \cdots$ the elements.

$$r_i(z) = z^{-i}\left(be^z + (1-b)\sum_{j=0}^{i-1} \frac{z^j}{j!}\right). \tag{8.2}$$

Note that

$$r_{i+1}(z) = z^{-1}\left(r_i(z) + \frac{1-b}{i!}\right) \tag{8.3}$$

and a straightforward calculation shows:

Lemma 8.1. — *For* $i, j > 0$

$$(r_i(z), r_j(-z)) = 0.$$

Now define for $\mu = 1, 2, \cdots$, the point $W_\mu \in \mathrm{Gr}(H)$

$$W_\mu = \text{linear span}\{r_1(z), r_2(z), \ldots, r_\mu(z)\} \oplus H_\mu. \tag{8.4}$$

From now on we will assume that μ can also be 0, then $W_0 = H_0$. From the definition (8.2) of the functions $r_i(z)$ it is clear that

$$(f(z), r_i(-z)) = (f(z), g(-z)) = 0 \quad \text{for all } f(z), g(z) \in H_\mu, \quad 0 \leqslant i \leqslant \mu.$$

From Lemma 8.1 it is then clear that W_μ satisfies the CKP condition, to be more precise

Proposition 8.1. — $W_\mu \in \mathrm{Gr}^{(0)}(H)$ *satisfies the CKP condition and*

$$W_\mu = \big\{f(z) \in H_{-\mu} | (f(z), r_i(-z)) = 0, \textit{for } 1 \leqslant i \leqslant \mu\big\}.$$

Next define the subspace $U_\mu \subset W_\mu$ of codimension 2 for $\mu \geqslant 2$, of codimension 1 if $\mu = 1$ and of codimension 0 if $\mu = 0$ by

$$U_\mu = \big\{f(z) \in W_\mu | (f(z), 1) = (f(z), r_0(-z)) = 0\big\}. \tag{8.5}$$

Now let $g(z) \in U_\mu$, then $zg(z) \in H_{-\mu+1}$ and $(zg(z), r_j(-z)) = 0$ for all $1 \leqslant j \leqslant \mu$.

This follows from the following observation:

$$(zg(z), r_j(-z)) = (g(z), zr_j(-z)) = \left(g(z), -r_{j-1}(-z) - \frac{1-b}{(j-1)!}\right) = 0,$$

for $j = 1, 2, \ldots, \mu$, since $g(z)$ is perpendicular to 1, $r_i(-z)$ for $0 \leqslant i \leqslant \mu$. Hence, W_μ has a subspace W' of codimension 2 such that $zW' \subset W_\mu$, hence

Proposition 8.2. — W_μ *with* $\mu > 1$ *also belongs to the 2-vector 1-constrained KP hierarchy.*

Note that W_1 belongs to the 1-vector 1-constrained KP. From Proposition 8.1 and Section 7 it is clear that W_μ can be obtained from $H_{-\mu} \in \mathrm{Gr}^{(\mu)}(H)$ by μ consecutive elementary Bäcklund-Darboux transformations. Now $\tau_{H_{-\mu}} = 1$ and $\psi_{H_{-\mu}}(t, z) = z^{-\mu}\psi_0(t, z)$ where $\psi_0(t, z) = e^{\sum_{i=0}^\infty t_i z^i}$. Let $\tau_\mu(t) = \tau_{W_\mu}(t)$ and $\psi_\mu(t, z) = \psi_{W_\mu}(t, z)$, then

$$\tau_\mu(t) = W\left(R_1^\mu(t), R_2^\mu(t), \ldots, R_\mu^\mu(t)\right), \tag{8.6}$$

where

$$R_i^\mu(t) := (z^{-\mu}\psi_0(t,z), r_i(-z)) = (\psi_0(t,z), z^{-\mu} r_i(-z)) \tag{8.7}$$
$$= \sum_{k=0}^{i-1} \frac{(-1)^{k-i}}{k!} S_{\mu+i-k-1}(t) + b \sum_{k=i}^{\mu+i-1} \frac{(-1)^{k-i}}{k!} S_{\mu+i-k-1}(t).$$

Here $S_k(t)$ are the elementary Schur functions. The corresponding wave function is given by

$$\psi_\mu(t,z) = \frac{1}{\tau_\mu(t)} W\left((R_1^\mu(t), R_2^\mu(t), \ldots, R_\mu^\mu(t), z^{-\mu}\psi_0(t,z)\right). \tag{8.8}$$

Note that

$$R_i^\mu(t) = (-)^i T^\mu_{\mu+i-1}(t), \quad \text{with } b = -a^2,$$

Hence

$$\tau_\mu(s) = (-)^{\frac{\mu(\mu+1)}{2}} \hat\tau(s).$$

In order to describe the other tau-functions of Section 5, we want to find the right expression for the Lax operator $L = L_\mu = L_{W_\mu}$. For this we study W_μ and zW_μ. Recall from (8.4) that

$$W_\mu = \text{linear span}\{r_1(z), r_2(z), \ldots, r_\mu(z)\} \oplus H_\mu,$$

and

$$W_\mu^\perp = \text{linear span}\{r_1(-z), r_2(-z), \ldots, r_\mu(-z)\} \oplus H_\mu,$$

hence

$$zW_\mu = \text{linear span}\{zr_1(z), zr_2(z), \ldots, zr_\mu(z)\} \oplus H_{\mu+1},$$
$$(zW_\mu)^\perp = \text{linear span}\{z^{-1}r_1(-z), z^{-1}r_2(-z), \ldots, z^{-1}r_\mu(-z)\} \oplus H_{\mu-1}.$$

From now on we assume in this section that $\mu > 1$. In that case it is straightforward to check that

$$zW_\mu + W_\mu = W_\mu \oplus \mathbb{C}zr_1(z) \oplus \mathbb{C}zr_2(z).$$

Putting

$$h_1(z) = r_1(z) - r_2(z) \quad \text{and} \quad h_2(z) = r_2(z), \tag{8.9}$$

one easily verifies that

$$(h_1(z), zh_2(-z)) = (h_2(z), zh_1(-z)) = (h_1(-z), zh_2(z)) \tag{8.10}$$
$$= (h_2(-z), zh_1(z)) = 0.$$

Using the construction of the Lax operator as in Section 6 we see that

$$L_\mu = \partial + \sum_{i=1}^{2} c_i(\psi_\mu(t,z), zh_i(-z))\partial^{-1}(\psi_\mu^*(t,z), zh_i(z)) \tag{8.11}$$
$$= \partial + \sum_{i=1}^{2} c_i(\psi_\mu(t,z), zh_i(-z))\partial^{-1}(\psi_\mu(\tilde t,z), zh_i(-z)).$$

We want to determine the c_i's, for this we let L_μ act on ψ_μ, this gives

$$(8.12)\quad \begin{aligned} z\psi_\mu(t,z) &= \frac{\partial \psi_\mu(t,z)}{\partial x} + \sum_{i=1}^{2} c_i(\psi_\mu(t,z), zh_i(-z))\partial^{-1} \\ &\qquad \cdot (\psi_\mu^*(t,z), zh_i(z))\psi_\mu(t,z) \\ &= \frac{\partial \psi_\mu(t,z)}{\partial x} + \sum_{i=1}^{2} c_i(\psi_\mu(t,z), zh_i(-z))(\psi_\mu(\tilde{t},z), zh_i(-z)) \\ &\qquad \cdot \psi_{W_\mu+\mathbb{C}zh_i(z)}(t,z). \end{aligned}$$

Now take the bilinear form with the elements $h_j(-z)$. Since (8.10) holds, and $h_1(-z)$ (resp. $h_2(-z)$) is perpendicular to W_μ and $W_\mu+\mathbb{C}zh_2(z)$ (resp. $W_\mu+\mathbb{C}zh_1(z)$) we obtain

$$\begin{aligned} (\psi_\mu(t,z), zh_i(-z)) = c_i(\psi_\mu(t,z), zh_i(-z))(\psi_\mu(\tilde{t},z), zh_i(-z)) \\ \cdot(\psi_{W_\mu+\mathbb{C}zh_i(z)}(t,z), h_i(-z)). \end{aligned}$$

Hence

$$(8.13)\qquad c_i = \left((\psi_\mu(\tilde{t},z), zh_i(-z))\left(\psi_{W_\mu+\mathbb{C}zh_i(z)}(t,z), h_i(-z)\right)\right)^{-1}.$$

We are now going to determine these c_i's. Note that

$$(8.14)\qquad zh_1(z) = r_0(z) - r_1(z) \quad\text{and}\quad zh_2(z) = 1 - b + r_1(z).$$

Using this we see that

$$\begin{aligned} W_\mu + \mathbb{C}zh_1(z) &= \text{linear span}\{r_0(z), r_1(z), \dots r_\mu(z)\} + H_\mu, \\ W_\mu + \mathbb{C}zh_2(z) &= \text{linear span}\{1, r_1(z), r_2(z), \dots r_\mu(z)\} + H_\mu. \end{aligned}$$

The fact that

$$(r_0(z), r_1(-z)) = -b,\ (r_0(z), r_i(-z)) = 0$$

and

$$(1, r_j(-z)) = -\frac{1}{(j-1)!} \quad\text{for}\quad i>1,\ j\geqslant 1,$$

gives the following, more convenient description of $W_\mu + \mathbb{C}zh_1(z)$ and $W_\mu + \mathbb{C}zh_2(z)$:

$$(8.15)\quad \begin{aligned} W_\mu + \mathbb{C}zh_1(z) &= \{f(z)\in H_{-\mu} | (f(z), r_i(-z)) = 0 \\ &\qquad\qquad \text{for } i = 2,3,\dots,\mu\}, \\ W_\mu + \mathbb{C}zh_2(z) &= \{f(z)\in H_{-\mu} | \Big(f(z), r_{i+1}(-z) - \frac{r_i(-z)}{i}\Big) = 0 \\ &\qquad\qquad \text{for } i = 1,2,\dots,\mu-1\}. \end{aligned}$$

Thus,

$$\tau_{W_\mu+\mathbb{C}zh_1(z)}(t) = \det\big(\big(\psi_0(t,z), z^{i-\mu-1}r_{j+1}(-z)\big)\big)_{1\leqslant i,j\leqslant \mu-1} = W(R_2^\mu(t), R_3^\mu(t), \dots, R_\mu^\mu(t)),$$

$$\tau_{W_\mu+\mathbb{C}zh_2(z)}(t) = \det\Big(\psi_0(t,z), z^{i-\mu-1}\Big(r_{j+1}(-z) - \frac{r_j(-z)}{j}\Big)\Big)_{1\leqslant i,\ j\leqslant \mu-1} = W\Big(R_2^\mu(t)-R_1^\mu(t), R_3^\mu(t)-\frac{R_2^\mu(t)}{2}, \dots, R_\mu^\mu(t)-\frac{R_{\mu-1}^\mu(t)}{\mu-1}\Big). \tag{8.16}$$

and the corresponding wave functions are equal to:

$$\psi_{W_\mu+\mathbb{C}zh_1(z)}(t,z) = \frac{1}{\tau_{W_\mu+\mathbb{C}zh_1(z)}(t)} W\big(W(R_2^\mu(t), R_3^\mu(t), \dots, R_\mu^\mu(t), z^{-\mu}\psi_0(t,z)\big). \tag{8.17}$$

and

$$\psi_{W_\mu+\mathbb{C}zh_2(z)}(t,z) = \frac{1}{\tau_{W_\mu+\mathbb{C}zh_2(z)}(t)} W\Big(R_2^\mu(t)-R_1^\mu(t), R_3^\mu(t)-\frac{R_2^\mu(t)}{2}, \dots, R_\mu^\mu(t)-\frac{R_{\mu-1}^\mu(t)}{\mu-1}, z^{-\mu}\psi_0(t,z)\Big). \tag{8.18}$$

From this we deduce that

$$\begin{aligned}\big(\psi_{W_\mu+\mathbb{C}zh_1(z)}(t,z), h_1(-z)\big) &= \big(\psi_{W_\mu+\mathbb{C}zh_1(z)}(t,z), r_1(-z)\big) = (-)^{\mu-1}\frac{\tau_\mu(t)}{\tau_{W_\mu+\mathbb{C}zh_1(z)}(t)},\\ \big(\psi_{W_\mu+\mathbb{C}zh_2(z)}(t,z), h_2(-z)\big) &= \big(\psi_{W_\mu+\mathbb{C}zh_2(z)}(t,z), r_2(-z)\big) = (-)^{\mu-1}\frac{\tau_\mu(t)}{\tau_{W_\mu+\mathbb{C}zh_2(z)}(t)}.\end{aligned} \tag{8.19}$$

For the other eigenfunctions we find, using (8.14):

$$\begin{aligned}\big(\psi_\mu(\tilde t,z), zh_1(-z)\big) &= (-)^{\mu+1}\frac{\tau_{W'}(\tilde t)}{\tau_\mu(\tilde t)},\\ \big(\psi_\mu(\tilde t,z), zh_2(-z)\big) &= (b-1)\frac{\tau_{W''}(\tilde t)}{\tau_\mu(\tilde t)},\end{aligned} \tag{8.20}$$

where

$$\begin{aligned}W' &= \big\{f(z)\in H_{-\mu} | (f(z), r_i(-z)) = 0 \text{ for } i=0,1,\dots,\mu\big\},\\ W'' &= \big\{f(z)\in H_{-\mu} | (f(z),1) = 0 \ \text{ and } (f(z), r_i(-z)) = 0 \ \text{ for } \ i = 1,2,\dots,\mu\big\}\end{aligned} \tag{8.21}$$

and

$$\tau_{W'}(t) = W\left(R_0^\mu(t), R_1^\mu(t), R_2^\mu(t), \dots, R_\mu^\mu(t)\right), \tag{8.22}$$
$$\tau_{W''}(t) = W\left(R_1^\mu(t), R_2^\mu(t), \dots, R_\mu^\mu(t), S_{\mu-1}(t)\right).$$

Now combining (8.13), (8.20) and (8.19), we find that

$$c_1 = \frac{\tau_\mu(\tilde{t})\tau_{W_\mu+\mathbb{C}zh_1(z)}(t)}{\tau_{W'}(\tilde{t})\tau_\mu(t)} = \frac{\tau_{W_\mu+\mathbb{C}zh_1(z)}(t)}{\tau_{W'}(\tilde{t})}, \tag{8.23}$$

$$c_2 = (-)^{\mu-1}(b-1)^{-1}\frac{\tau_\mu(\tilde{t})\tau_{W_\mu+\mathbb{C}zh_2(z)}(t)}{\tau_{W''}(\tilde{t})\tau_\mu(t)}$$
$$= (-)^{\mu-1}(b-1)^{-1}\frac{\tau_{W_\mu+\mathbb{C}zh_2(z)}(t)}{\tau_{W''}(\tilde{t})},$$

since $\tau_\mu(\tilde{t}) = \tau_\mu(t)$. Since these c_i's are just constants, it suffices to substitute $t=0$, *i.e.* $t_j = 0$ for all $j = 1,2,3,\dots$, in (8.23), this gives

$$c_1 = \frac{\tau_{W_\mu+\mathbb{C}zh_1(z)}(0)}{\tau_{W'}(0)}, \quad c_2 = (-)^{\mu-1}(b-1)^{-1}\frac{\tau_{W_\mu+\mathbb{C}zh_2(z)}(0)}{\tau_{W''}(0)}. \tag{8.24}$$

We now calculate these tau-functions for $t=0$:

$$\tau_{W_\mu+\mathbb{C}zh_1(z)}(0) = \det\left(\left(z^{i-\mu-1}, r_{j+1}(-z)\right)\right)_{1\leqslant i,j\leqslant \mu-1} \tag{8.25}$$
$$= \det\left(\frac{(-)^{\mu-i}b}{(\mu+j-i+1)!}\right)_{1\leqslant i,j\leqslant\mu-1}$$
$$= (-)^{\frac{\mu(\mu-1)}{2}}b^{\mu-1}\det\left(\frac{1}{(\mu+j-i+1)!}\right)_{1\leqslant i,j\leqslant\mu-1}$$
$$= (-)^{\frac{\mu(\mu-1)}{2}}b^{\mu-1}S^{(\mu-1)}_{\mu+1,\mu+1,\dots,\mu+1}(1,0,0,\dots),$$

where (see [23])

$$S^{(k)}_{\lambda_1,\lambda_2,\dots,\lambda_k}(t_1,t_2,t_3,\dots) = \det\left(S_{\lambda_i+j-i}(t_1,t_2,t_3,\dots)\right)_{1\leqslant i,j\leqslant k},$$

the Schur function corresponding to the partition $\lambda_1,\lambda_2,\dots,\lambda_k$. Here $S_\ell(t_1,t_2,t_3,\dots)$ is the elementary Schur function. In a similar way one shows that

$$\tau_{W'}(0) = \det\left(\left(z^{i-\mu-1}, r_{j-1}(-z)\right)\right)_{1\leqslant i,j\leqslant\mu+1} \tag{8.26}$$
$$= (-)^{\frac{\mu(\mu-1)}{2}+1}b^\mu S^{(\mu+1)}_{\mu-1,\mu-1,\dots,\mu-1}(1,0,0,\dots)$$

and

$$\tau_{W''}(0) = \det\begin{pmatrix} (z^{-\mu}, r_1(-z)) & \cdots & (z^{-\mu}, r_\mu(-z)) & (z^{-\mu},1) \\ (z^{1-\mu}, r_1(-z)) & \cdots & (z^{1-\mu}, r_\mu(-z)) & (z^{1-\mu},1) \\ \vdots & & \vdots & \vdots \\ (r_1(-z),1) & \cdots & (r_\mu(-z),1) & (1,1) \end{pmatrix} \tag{8.27}$$
$$= (-)^{\frac{\mu(\mu-1)}{2}}b^{\mu-1}S^{(\mu)}_{\mu,\mu,\dots,\mu,\mu-1}(1,0,0,\dots).$$

And finally the most complicated one:

$$\begin{aligned}
\tau_{W_\mu+\mathbb{C}zh_2(z)}(0) &= \det\left(\left(z^{i-\mu-1}, \left(r_{j+1}(-z) - \frac{r_j(-z)}{j}\right)\right)\right)_{1\leqslant i,j\leqslant \mu-1} \\
&= \det\left(\frac{(-)^{\mu-i}b}{(\mu+j-i+1)!} - \frac{(-)^{\mu-i}b}{(\mu+j-i)!j}\right)_{1\leqslant i,j\leqslant \mu-1} \\
&= (-)^{\frac{(\mu-1)(\mu+2)}{2}} b^{\mu-1} \det\left(\frac{\mu-i+1}{(\mu+j-i+1)!j}\right)_{1\leqslant i,j\leqslant \mu-1} \\
&= (-)^{\frac{(\mu-1)(\mu+2)}{2}} \mu b^{\mu-1} \det\left(\frac{1}{(\mu+j-i+1)!}\right)_{1\leqslant i,j\leqslant \mu-1} \\
&= (-)^{\frac{(\mu-1)(\mu+2)}{2}} \mu b^{\mu-1} S^{(\mu-1)}_{\mu+1,\mu+1,\ldots,\mu+1}(1,0,0,\ldots).
\end{aligned} \tag{8.28}$$

We conclude from all this that

$$\begin{aligned}
c_1 &= -b^{-1}\frac{S^{(\mu-1)}_{\mu+1,\mu+1,\ldots,\mu+1}(1,0,0,\ldots)}{S^{(\mu+1)}_{\mu-1,\mu-1,\ldots,\mu-1}(1,0,0,\ldots)}, \\
c_2 &= (b-1)^{-1}\mu\frac{S^{(\mu-1)}_{\mu+1,\mu+1,\ldots,\mu+1}(1,0,0,\ldots)}{S^{(\mu)}_{\mu,\mu,\ldots,\mu,\mu-1}(1,0,0,\ldots)}.
\end{aligned} \tag{8.29}$$

Now using the fact that

$$S^{(\mu-1)}_{\mu+1,\mu+1,\ldots,\mu+1}(1,0,0,\ldots) = S^{(\mu+1)}_{\mu-1,\mu-1,\ldots,\mu-1}(1,0,0,\ldots) = \mu\frac{\prod_{i=1}^{\mu-1}(i!)^2}{\prod_{i=1}^{2\mu-1} i!},$$

$$S^{(\mu)}_{\mu,\mu,\ldots,\mu,\mu-1}(1,0,0,\ldots) = \mu^2\frac{\prod_{i=1}^{\mu-1}(i!)^2}{\prod_{i=1}^{2\mu-1} i!},$$

we obtain

$$c_1 = -\frac{1}{b} \quad \text{and} \quad c_2 = \frac{1}{b-1}. \tag{8.30}$$

So finally

$$\begin{aligned}
L_\mu &= \partial - b^{-1}(\psi_\mu(t,z), zh_1(-z))\partial^{-1}(\psi_\mu(\tilde{t},z), zh_1(-z)) \\
&\quad + (b-1)^{-1}(\psi_\mu(t,z), zh_2(-z))\partial^{-1}(\psi_\mu(\tilde{t},z), zh_2(-z)) \\
&= \partial + \left(\psi_\mu(t,z), (-)^{\mu+1}\sqrt{-b}e^{-z}\right)\partial^{-1}\left(\psi_\mu(\tilde{t},z), (-)^{\mu+1}\sqrt{-b}e^{-z}\right) \\
&\quad + \left(\psi_\mu(t,z), \sqrt{b-1}\right)\partial^{-1}\left(\psi_\mu(\tilde{t},z), \sqrt{b-1}\right).
\end{aligned} \tag{8.31}$$

We have added the term $(-)^{\mu+1}$ here, in order to get rid of this term later on in this section. Note that (see (8.20))

$$\begin{aligned}(\psi_\mu(t,z),(-)^{\mu+1}\sqrt{-b}e^{-z}) &= \frac{1}{\sqrt{-b}}\frac{\tau_{W'}(t)}{\tau_\mu(t)}\\ &= \frac{1}{\sqrt{-b}}\frac{W\left(R_0^\mu(t),R_1^\mu(t),R_2^\mu(t),\dots,R_\mu^\mu(t)\right)}{W\left(R_1^\mu(t),R_2^\mu(t),\dots,R_\mu^\mu(t)\right)},\\ \left(\psi_\mu(t,z),\sqrt{b-1}\right) &= \sqrt{b-1}\frac{\tau_{W''}(t)}{\tau_\mu(t)}\\ &= \sqrt{b-1}\frac{W\left(R_1^\mu(t),R_2^\mu(t),\dots,R_\mu^\mu(t),S_{\mu-1}(t)\right)}{W\left(R_1^\mu(t),R_2^\mu(t),\dots,R_\mu^\mu(t)\right)}.\end{aligned}$$

Using (8.23), (8.16) and (8.30) we find that also

$$\begin{aligned}(\psi_\mu(\tilde t,z),(-)^{\mu+1}\sqrt{-b}e^{-z}) &= \sqrt{-b}\frac{\tau_{W_\mu+\mathbb{C}zh_1(z)}(t)}{\tau_\mu(t)}\\ &= \sqrt{-b}\frac{W\left(R_2^\mu(t),R_3^\mu(t),\dots,R_\mu^\mu(t)\right)}{W\left(R_1^\mu(t),R_2^\mu(t),\dots,R_\mu^\mu(t)\right)},\end{aligned}$$

$$\begin{aligned}&\left(\psi_\mu(\tilde t,z),\sqrt{b-1}\right)\\ &= (-)^{\mu-1}\sqrt{b-1}\frac{\tau_{W_\mu+\mathbb{C}zh_2(z)}(t)}{\tau_\mu(t)}\\ &= (-)^{\mu-1}\sqrt{b-1}\frac{W\left(R_2^\mu(t)-R_1^\mu(t),R_3^\mu(t)-\frac{R_2^\mu(t)}{2},\dots,R_\mu^\mu(t)-\frac{R_{\mu-1}^\mu(t)}{\mu-1}\right)}{W\left(R_1^\mu(t),R_2^\mu(t),\dots,R_\mu^\mu(t)\right)}.\end{aligned}$$

We thus obtain in this way that

$$\beta_{12}(s) = \left(\psi_\mu(\tilde s,z),\sqrt{b-1}\right) \quad \text{with } \alpha = \sqrt{b-1}$$

and

$$\beta_{32}(s) = (\psi_\mu(\tilde s,z),(-)^{\mu+1}\sqrt{-b}e^{-z}) \quad \text{with } a = (-)^{\mu+1}\sqrt{-b}.$$

To obtain $\beta_{13}(s)$, we calculate the so-called squared eigenfunction potential

$$\partial^{-1}\left(\psi_\mu(\tilde t,z),\sqrt{b-1}\right)(\psi_\mu(t,z),(-)^{\mu+1}\sqrt{-b}e^{-z})$$

of $\left(\psi_\mu(\tilde t,z),\sqrt{b-1}\right)$ and $(\psi_\mu(t,z),(-)^{\mu+1}\sqrt{-b}e^{-z})$. Let

$$w(z) = b_1r_1(-z)+\sum_{i=1}^{\mu-1}b_{i+1}\left(r_{i+1}(-z)-\frac{r_i(-z)}{i}\right)+\sum_{j>\mu}b_jz^j,$$

be an arbitrary element of $W_\mu^\perp$, then

$$
\begin{aligned}
&\partial^{-1}\left(\psi_\mu(\tilde{t},z),\sqrt{b{-}1}\right)(\psi_\mu(t,z),(-)^{\mu+1}\sqrt{-b}e^{-z})\\
&=\partial^{-1}\left(\psi_\mu(\tilde{t},z),\sqrt{b{-}1}\right)(\psi_\mu(t,z),(-)^{\mu+1}\sqrt{-b}(e^{-z}+w(z)))\\
&=\left(\psi_\mu(\tilde{t},z),\sqrt{b{-}1}\right)\left(\psi_\mu(\tilde{t},z),\sqrt{b{-}1}\right)^{-1}\partial^{-1}\left(\psi_\mu(\tilde{t},z),\sqrt{b{-}1}\right)\\
&\qquad\qquad\cdot\left(\psi_\mu(t,z),(-)^{\mu+1}\sqrt{-b}(e^{-z}+w(z))\right)\\
&=\left(\psi_\mu(\tilde{t},z),\sqrt{b{-}1}\right)\left(\psi_{W_\mu+\mathbb{C}zh_2(z)}(t,z),(-)^{\mu+1}\sqrt{-b}(e^{-z}+w(z))\right)\\
&=\left(\psi_\mu(\tilde{t},z),\sqrt{b{-}1}\right)\left(\psi_{W_\mu+\mathbb{C}zh_2(z)}(t,z),(-)^{\mu+1}\sqrt{-b}\left(\frac{r_0(-z)}{b}+b_1r_1(-z)\right)\right).
\end{aligned}
$$

Using (8.23) and (8.30) we find that

$$
\partial^{-1}\left(\psi_\mu(\tilde{t},z),\sqrt{b{-}1}\right)(\psi_\mu(t,z),(-)^{\mu+1}\sqrt{-\,b}e^{-z})=-\sqrt{\tfrac{b-1}{-b}}\,\tfrac{\tau_{W'''}(t)}{\tau_\mu(t)},\tag{8.32}
$$

where

$$
\begin{aligned}
\tau_{W'''}(t)=W\Bigg(&R_2^\mu(t)-\frac{R_1^\mu(t)}{1},R_3^\mu(t)-\frac{R_2^\mu(t)}{2},\ldots,R_\mu^\mu(t)\\
&-\frac{R_{\mu-1}^\mu(t)}{\mu-1},R_0^\mu(t)+bb_1R_1^\mu(t)\Bigg).
\end{aligned}\tag{8.33}
$$

Now comparing (5.1), (5.5), (8.32) and (8.33) we see that

$$b_1=0.$$

For this $b_1=0$, the tau-function $\tau_{W'''}$ corresponds to the following point in the Grassmannian:

$$
\begin{aligned}
W'''=\Big\{&f(z)\in H_{-\mu}|(f(z),r_0(-z))=0\\
&\text{and}\ \left(f(z),r_i(-z)-\frac{r_{i-1}(-z)}{i-1}\right)=0\ \text{ for }\ i=2,3\ldots,\mu\Big\}.
\end{aligned}\tag{8.34}
$$

Hence, using the fact that $\alpha=\sqrt{b-1}$ and $a=(-)^{\mu+1}\sqrt{-b}$ one finds that

$$\beta_{13}(s)=\partial^{-1}\left(\psi_\mu(\tilde{s},z),\sqrt{b-1}\right)(\psi_\mu(s,z),(-)^{\mu+1}\sqrt{-b}e^{-z}).$$

We now calculate the squared eigenfunction potential in a different way. Let

$$w(z)=\sum_{i=1}^{\mu}a_ir_i(-z)+\sum_{j>\mu}a_jz^j,$$

be an arbitrary element of $W_\mu^\perp$, a straightforward calculation shows that

$$\begin{aligned}\partial^{-1}&(\psi_\mu(\tilde t,z),(-)^{\mu+1}\sqrt{-b}e^{-z})\left(\psi_\mu(t,z),\sqrt{b-1}\right)\\ &=\partial^{-1}(\psi_\mu(\tilde t,z),(-)^{\mu+1}\sqrt{-b}e^{-z})\left(\psi_\mu(t,z),\sqrt{b-1}(1+w(z))\right)\\ &=(\psi_\mu(\tilde t,z),(-)^{\mu+1}\sqrt{-b}e^{-z})\left(\psi_{W_\mu+\mathbb{C}zh_1(z)}(t,z),\sqrt{b-1}(1+a_1r_1(-z))\right).\end{aligned}$$

Using (8.23) and (8.30) we find that

$$\begin{aligned}&\partial^{-1}(\psi_\mu(\tilde t,z),(-)^{\mu+1}\sqrt{-b}e^{-z})\left(\psi_\mu(t,z),\sqrt{b-1}\right)\\ &\qquad=\sqrt{\frac{b-1}{-b}}\frac{\tau_{W'}(\tilde t)\tau_{W''''}(t)}{\tau_\mu(\tilde t)\tau_{W_\mu+\mathbb{C}zh_1(z)}(t)}=\sqrt{-b(b-1)}\frac{\tau_{W''''}(t)}{\tau_\mu(t)},\end{aligned}\tag{8.35}$$

where

$$\tau_{W''''}(t)=W\left(R_2^\mu(t),R_3^\mu(t),\ldots,R_\mu^\mu(t),S_{\mu-1}(t)+a_1R_1^\mu(t)\right).\tag{8.36}$$

This is the tau-function corresponding to the following point of the Grassmannian

$$\begin{aligned}W''''=\{f(z)\in H_{-\mu}|&(f(z),1+a_1r_1(-z))=0\\ &\text{and}\ \ (f(z),r_i(-z))=0\quad\text{for}\ \ i=2,3\ldots,\mu\}.\end{aligned}\tag{8.37}$$

Hence

$$\begin{aligned}&\partial^{-1}(\psi_\mu(\tilde t,z),(-)^{\mu+1}\sqrt{-b}e^{-z})\left(\psi_\mu(t,z),\sqrt{b-1}\right)\\ &\quad=\sqrt{-b(b-1)}\left(\frac{W\left(R_2^\mu(t),R_3^\mu(t),\ldots,R_\mu^\mu(t),S_{\mu-1}(t)\right)}{W\left(R_1^\mu(t),R_2^\mu(t),\ldots,R_\mu^\mu(t)\right)}-(-)^\mu a_1\right).\end{aligned}\tag{8.38}$$

It is not clear yet what the value of a_1 one should take.

We now put all $t_i=0$ for $i>1$, and write $f(x)$ for $f(x,0,0,\ldots)$. Comparing (8.32) and (8.35), we see that

$$b\tau_{W''''}(x)=\tau_{W'''}(x).\tag{8.39}$$

To calculate a_1 we substitute $x=0$. We find that

$$\begin{aligned}\tau_\mu(0)=\tau_{W_\mu}(0)&=\det\left((z^{i-\mu-1},r_j(-z))\right)_{1\leqslant i,j\leqslant\mu}\\ &=\det\left(\frac{(-)^{\mu-i}b}{(\mu+j-i)!}\right)_{1\leqslant i,j\leqslant\mu}\\ &=(-)^{\frac{\mu(\mu-1)}{2}}b^\mu\det\left(\frac{1}{(\mu+j-i)!}\right)_{1\leqslant i,j\leqslant\mu}\\ &=(-)^{\frac{\mu(\mu-1)}{2}}b^\mu\mu S^{(\mu)}_{\mu,\mu,\ldots,\mu}(1,0,0,\ldots)\\ &=(-)^{\frac{\mu(\mu-1)}{2}}b^\mu\mu\frac{\prod_{i=1}^{\mu-1}(i!)^2}{\prod_{i=1}^{2\mu-1}i!}.\end{aligned}\tag{8.40}$$

In a similar way we find that $\tau_{W'''}(0) = \tau_\mu(0)$ and that

$$\tau_{W''''}(0) = (-)^{\frac{\mu(\mu-1)}{2}}(b^{\mu-1} - (-)^\mu a_1 b^\mu)\mu \frac{\prod_{i=1}^{\mu-1}(i!)^2}{\prod_{i=1}^{2\mu-1} i!}. \tag{8.41}$$

Comparing all the results (8.39)–(8.41) we conclude that

$$a_1 = 0.$$

Since we know that $\sum_{i=1}^3 \omega_i^2(x)$ is equal to a constant, it suffices to calculate this value for $x = 0$. We find that

$$\begin{aligned}
\omega_1(0) = \beta_{32}(0) &= \frac{1}{\sqrt{-b}} \frac{\tau_{W'}(0)}{\tau_\mu(0)} \\
&= \frac{1}{\sqrt{-b}} \frac{(-)^{\frac{\mu(\mu-1)}{2}+1} b^\mu S^{(\mu+1)}_{\mu-1,\mu-1,\dots,\mu-1}(1,0,0,\dots)}{(-)^{\frac{\mu(\mu-1)}{2}} b^\mu S^{(\mu)}_{\mu,\mu,\dots,\mu}(1,0,0,\dots)} \\
&= -\frac{1}{\sqrt{-b}}\mu, \\
\omega_2(0) = -\beta_{13}(0) &= -\sqrt{-b(b-1)}\frac{\tau_{W''''}(0)}{\tau_\mu(0)} \\
&= -\sqrt{-b(b-1)}\frac{(-)^{\frac{\mu(\mu-1)}{2}} b^{\mu-1} S^{(\mu-1)}_{\mu+1,\mu+1,\dots,\mu+1}(1,0,0,\dots)}{(-)^{\frac{\mu(\mu-1)}{2}} b^{\mu-1} S^{(\mu)}_{\mu,\mu,\dots,\mu}(1,0,0,\dots)} \\
&= -\sqrt{-b(b-1)}\frac{\mu}{b} = \sqrt{\frac{b-1}{-b}}\mu,
\end{aligned}$$

$$\omega_3(0) = 0\beta_{12}(0) = 0.$$

Hence,

$$\sum_{i=1}^3 \omega_i^2(0) = \left(\frac{1}{\sqrt{-b}}\mu\right)^2 + \left(\sqrt{\frac{b-1}{-b}}\mu\right)^2 = -\mu^2$$

and

$$\sum_{i=1}^3 \omega_i^2(x) = -\mu^2. \tag{8.42}$$

We next calculate $R_i^\mu(x)$:

$$\begin{aligned}
R_i^\mu(x) &= \sum_{k=0}^{i-1} \frac{(-1)^{k-i}}{k!} \frac{x^{\mu+i-k-1}}{(\mu+i-k-1)!} + b \sum_{k=i}^{\mu+i-1} \frac{(-1)^{k-i}}{k!} \frac{x^{\mu+i-k-1}}{(\mu+i-k-1)!} \\
&= (-)^i \frac{(x-1)^{\mu+i-1}}{(\mu+i-1)!} + (-)^{\mu-1}(b-1) \sum_{j=0}^{\mu-1} \frac{(-x)^j}{j!(\mu+i-j-1)!}.
\end{aligned}$$

Combining all the previous results we find:

THEOREM 8.1. — *The expressions*

$$y(x) = x\frac{\pm(x-1)\mu\omega_1\omega_2\omega_3 + x\omega_1^2\omega_2^2 + \omega_1^2\omega_3^2}{(x-1)^2\omega_2^2\omega_3^2 + x^2\omega_1^2\omega_2^2 + \omega_1^2\omega_3^2}$$

and

$$y(x) = -x\frac{x(\omega_1\omega_2 \mp \mu\omega_3)^2 + (\omega_1\omega_3 \pm \mu\omega_2)^2}{\left(\omega_3^2 + \mu^2 + x(\omega_2^2 + \mu^2)\right)^2 + 4x\mu^2\omega_1^2}$$

for $\mu = 1, 2, \ldots$, *with*

$$\omega_1(x) = \sqrt{-b}(1-x)\frac{W\left(R_2^\mu(x), R_3^\mu(x), \ldots, R_\mu^\mu(x)\right)}{W\left(R_1^\mu(x), R_2^\mu(x), \ldots, R_\mu^\mu(x)\right)}, \tag{8.43}$$

$$\omega_2(x) = -\sqrt{-b(b-1)}\frac{W\left(R_2^\mu(x), R_3^\mu(x), \ldots, R_\mu^\mu(x), \frac{x^{\mu-1}}{(\mu-1)!}\right)}{W\left(R_1^\mu(x), R_2^\mu(x), \ldots, R_\mu^\mu(x)\right)},$$

$$\omega_3(x) = \sqrt{b-1}x\frac{W\left(R_1^\mu(x), R_2^\mu(x), \ldots, R_\mu^\mu(x), \frac{x^{\mu-1}}{(\mu-1)!}\right)}{W\left(R_1^\mu(x), R_2^\mu(x), \ldots, R_\mu^\mu(x)\right)}$$

are rational solutions of the Painlevé VI equation (1.5) for the parameters

$$(\alpha,\beta,\gamma,\delta) = \left(\frac{(1\mp\mu)^2}{2}, -\frac{\mu^2}{2}, \frac{\mu^2}{2}, \frac{1-\mu^2}{2}\right), \quad \textit{respectively}$$

$$(\alpha,\beta,\gamma,\delta) = \left(\frac{(1\pm 2\mu)^2}{2}, 0, 0, \frac{1}{2}\right).$$

The ω_i *separately satisfy the time dependent Euler top equations (1.11).*

The above results are clearly valid for $\mu > 1$. We will now treat the case $\mu = 1$ separately. In that case W_1 corresponds to the 1-vector 1-constrained KP hierarchy and

$$\tau_1(t) = R_1^1(t) = -S_1(t) + bS_0(t) = b - x.$$

We use the same expressions for the $\beta_{ij}(x)$ in terms of the Wronskian determinants as in the case $\mu > 1$, viz.,

$$\beta_{23}(x) = \frac{1}{\sqrt{-b}}\frac{-b}{b-x}, \quad \beta_{12}(x) = \sqrt{b-1}\frac{1}{b-x},$$

$$\beta_{13}(x) = \sqrt{-b(b-1)}\frac{1}{b-x}.$$

This leads to the ω_i's (1.35) for $\mu^2 = 1$.

Remark 8.1. — From the rational solutions (8.43) for the time dependent Euler top equations for the values $\mu = 1, 2, 3, \ldots$ one can recover the expression of the ω_i in the u_i, $i = 1, 2, 3$, by just substituting:

$$x = \frac{u_2 - u_1}{u_3 - u_1}$$

in $V(x)$, *i.e.*, in all $\omega_i(x)$. Using (2.8) one finds expressions for the rotation coefficients $\beta_{ij}(u)$ that satisfy (2.1)–(2.3).

Finally we give as an example the explicit ω_i's for $\mu = 3$:

$$\omega_i = \frac{N_i(x)}{D(x)}$$

where

$$\begin{aligned} N_1(x) &= 3\sqrt{-b}(1-x)(b^2-8b^2x+18bx^2+10b^2x^2-56bx^3+70bx^4-56bx^5 \\ &\quad +10x^6+18bx^6-8x^7+x^8) \\ N_2(x) &= -3\sqrt{-b(b-1)}\left(b^2-18bx^2+52bx^3-60bx^4+24bx^5+10x^6-12x^7+3x^8\right) \\ N_3(x) &= 3\sqrt{b-1}x\left(3b^2-12b^2x+10b^2x^2+24bx^3-60bx^4+52bx^5-18bx^6+x^8\right) \\ D(x) &= b^3-9b^2x+36b^2x^2-84b^2x^3+36bx^4+90b^2x^4-90bx^5-36b^2x^5 \\ &\quad +84bx^6-36bx^7+9bx^8-x^9. \end{aligned}$$

Acknowledgement. — JvdL wishes to thank Pierre van Moerbeke for valuable conversations.

BIBLIOGRAPHY

[1] H. Aratyn, J.F. Gomes, J.W. van de Leur, A.H. Zimerman, WDVV equations, Darboux-Egoroff metric and the dressing method, contribution to the UNESP2002 workshop on Integrable Theories, Solitons and Duality, http://jhep.sissa.it or [arXiv:math-ph/0210038].

[2] H. Aratyn and J. van de Leur, Integrable structures behind WDVV equations, Teor. Math. Phys., 134, Issue 1 (2003), 14–26 [arXiv:hep-th/0111243].

[3] H. Aratyn and J. van de Leur, Solutions of the WDVV Equations and Integrable Hierarchies of KP Type Commun. Math. Phys., 239 (2003), 155–182.

[4] H. Aratyn, E. Nissimov and S. Pacheva, Multi-component matrix KP hierarchies as symmetry-enhanced scalar KP hierarchies and their Darboux-Bäcklund solutions, in Bäcklund and Darboux transformations. The geometry of solitons (Halifax, NS, 1999), CRM Proc. Lecture Notes, 29, Amer. Math. Soc., Providence, RI (2001), 109–120.

[5] E. Date, M. Jimbo, M. Kashiwara and T. Miwa,, Transformation groups for soliton equations. 6. KP hierarchies of orthogonal and symplectic type, J. Phys. Soc. Japan ,50 (1981), 3813–3818.

[6] B. Dubrovin, Integrable systems and classification of 2-dimensional topological field theories, in: Integrable Systems, proceedings of Luminy 1991 conference dedicated to the memory of J.-L. Verdier, eds. O. Babelon, P. Cartier, Y. Kosmann-Schwarzbach, Birkhäuser (1993), 313–359.

[7] B. Dubrovin, Geometry on 2D topological field theories, in: Integrable Systems and Quantum Groups (Montecatini Terme, 1993), Lecture Notes in Math. 1620, Springer Berlin (1996), 120–348.

[8] B. Dubrovin and M. Mazzocco, Monodromy of certain Painlevé VI trascendents and reflection groups, Invent. Math., 141 (2000), 55–147.

[9] B. Dubrovin, Y.J. Zhang, Frobenius manifolds and Virasoro constraints. Selecta Math. (N.S.), 5-4 (1999), 423–466.

[10] G. F. Helminck and J. W. van de Leur, Geometric Bäcklund-Darboux transformations for the KP hierarchy, Publ. Res. Inst. Math. Sci., 37-4 (2001), 479–519.

[11] G. F. Helminck and J. W. van de Leur, An analytic description of the vector constrained KP hierarchy, Commun. Math Phys., 193 (1998), 627–641.

[12] G. F. Helminck and J. W. van de Leur, Constrained and Rational Reductions of the KP hierarchy, in: "Supersymmetry and Integrable Models", (H. Aratyn. T.D. Imbo, W.-Y. Keung, U. Sukhatme eds.), Springer Lecture Notes in Physics, 502 (1998), 167–182.

[13] N. J. Hitchin, Twistor spaces, Einstein metrics and isomonodromic deformations, J. Diff. Geom., 42 (1995), 30–112.

[14] N. J. Hitchin, Poncelet polygons and the Painlevé transcendents, Geometry and Analysis, Oxford University Press, Bombay (1996) 151–185.

[15] N. J. Hitchin, A new family of Einstein metrics, manifolds and geometry, Manifolds and geometry (Pisa, 1993), Sympos. Math., XXXVI, Cambridge Univ. Press, Cambridge (1996), 190–222.

[16] M. Jimbo, T. Miwa, K. Ueno, Monodromy preserving deformations of linear ordinary differential equations with rational coefficients I, Physica 2D, 2 (1981), 306–352.

[17] M. Jimbo, T. Miwa, Monodromy preserving deformations of linear ordinary differential equations with rational coefficients II, Physica 2D, 3 (1981), 407–448.

[18] M. Jimbo, T. Miwa, Monodromy preserving deformations of linear ordinary differential equations with rational coefficients III, Physica 2D, 4 (1981), 26–46.

[19] M. Jimbo, M. and T. Miwa, Solitons and Infinite Dimensional Lie Algebras, Publ. RIMS, Kyoto Univ., 19 (1983), 943–1001.

[20] V.G. Kac and J.W. van de Leur, The n-component KP hierarchy and representation theory, Integrability, topological solitons and beyond. J. Math. Phys., 44-8 (2003), 3245–3293.

[21] J.W. van de Leur, Twisted GL_n Loop Group Orbit and Solutions of WDVV Equations. Internat. Math. Res. Notices, 11 (2001), 551–573.

[22] J.W. van de Leur and R. Martini, The construction of Frobenius Manifolds from KP tau-Functions, Commun. Math. Phys., 205 (1999), 587–616.

[23] I.G. Macdonald, Symmetric functions and Hall polynomials. Second edition. Oxford Mathematical Monographs, xford University Press, New York, 1995.

[24] G. Mahoux, Introduction to the theory of isomonodromic deformations of linear ordinarydifferential equations with rational coefficients, The Painlevé property,

one century later, ed. R. Conte, CRM series in mathematical physics, Springer, New York (1999) 35–76.

[25] M. Mazzocco, Picard and Chazy solutions to the Painlevé VI equation equation, Math. Annalen, 321 (2001), 131–169. math.AG/9901054.

[26] T. Shiota, Prym varieties and soliton equations, in: Infinite-dimensional Lie algebras and groups (Luminy-Marseille, 1988), Adv. Ser. Math. Phys., 7, World Sci. Publishing, Teaneck, NJ (1989), 407–448.

Henrik ARATYN,
University of Illinois at Chicago
Department of Physics
845 W. Taylor St.
Chicago, IL 60607-7059 (USA)
aratyn@uic.edu

Johan van de LEUR,
University of Utrecht
Mathematical Institute
P.O. Box 80010,
3508 TA Utrecht (The Netherlands)
vdleur@math.uu.nl