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Aleksandra NOWEL

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TOPOLOGICAL INVARIANTS OF ANALYTIC SETS ASSOCIATED WITH NOETHERIAN FAMILIES

by **Aleksandra NOWEL** (1)

Introduction.

In [12] Parusiński and Szafraniec proved, that for any regular morphism $\phi : X \longrightarrow W$ of real algebraic sets there exist real polynomials $g_1, g_2, \ldots, g_s$ on W such that for every $w \in W$

$$\chi(\phi^{-1}(w)) = \operatorname{sgn} g_1(w) + \operatorname{sgn} g_2(w) + \ldots + \operatorname{sgn} g_s(w),$$

where sgn $g(w)$ denotes the sign of $g(w)$, $\chi(A)$ denotes the Euler characteristic of the set A (compare also the result of Coste and Kurdyka [4]).

Let $\Omega \subset \mathbb{R}^n$ be a compact semianalytic set and let $\mathcal{F}$ be a collection of real analytic functions defined in some neighbourhood of Ω. With each $\omega \in \Omega$ we can associate an analytic germ $Y_\omega = \bigcap_{f \in \mathcal{F}} f^{-1}(0)$ at ω and an analytic germ $X_\omega = \{x \mid x + \omega \in Y_\omega\}$ at 0. Using arguments similar to Parusiński and Szafraniec, and the properties of Noetherian families, we will show (Theorem 4.11) that there exist analytic functions $v_1, v_2, \ldots, v_s$ defined in a neighbourhood of Ω such that for each $\omega \in \Omega$ there exists $0 < \epsilon_\omega \ll 1$ such that for each $0 < \epsilon < \epsilon_\omega$

$$\frac{1}{2}\chi(S_\epsilon^{n-1} \cap X_\omega) = \sum_{i=1}^{s} \operatorname{sgn} v_i(\omega),$$

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where S_ϵ^{n-1} denotes a sphere in $\mathbb{R}^n$ centered at the origin with the radius ϵ.

This result is proven in section 4. In fact it holds in the more general case, where $\mathcal{F}$ is a family of analytic functions from an Ω–Noetherian algebra satisfying some additional assumptions (see Remark after Theorem 4.11). The Ω–Noetherian algebras were defined by El Khadiri and Tougeron in [5].

Let Ω be a locally closed subset of $\mathbb{R}^n$, and let $\mathcal{O}(\Omega)$ be a subalgebra of the algebra of analytic functions on Ω (or on a neighbourhood of Ω) to $\mathbb{R}$. Let us identify Ω with a subspace of the maximal spectrum $SM(\mathcal{O}(\Omega))$. With each point from Ω we associate the maximal ideal of $\mathcal{O}(\Omega)$ consisting of the functions which vanish at this point. The subalgebra $\mathcal{O}(\Omega)$ is called *Ω–Noetherian* if it is closed under derivation, $\mathbb{R}[x] \subset \mathcal{O}(\Omega)$ and Ω, identified as above with a subspace of the maximal spectrum $SM(\mathcal{O}(\Omega))$, is a Noetherian space. El Khadiri and Tougeron have given other examples of Ω–Noetherian algebras, for instance

- the algebra of Nash functions (i.e. analytic semialgebraic functions) on Ω, where Ω is open semialgebraic in $\mathbb{R}^n$,
- the algebra $\mathbb{R}[x][f_1, \ldots, f_q]$, where $\mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n]$ is the ring of polynomials on $\mathbb{R}^n$, $f_i = e^{Q_i}$, $Q_i \in \mathbb{R}[x]$.

In sections 1–3 we recall the definition and properties of Noetherian families, proved by El Khadiri and Tougeron in [5] and prove some useful properties of germs of some special complex analytic sets and of Noetherian families. Finally, in section 5, we show some consequences of the main result.

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1. Preliminaries.

Let A be a commutative algebra with an identity element over a commutative field $\mathbf{k}$ of characteristic zero, and let Γ be a subset of the maximal spectrum $SM(A)$ of A. In Γ we have the topology induced by the topology of $SM(A)$, i.e. F is closed in Γ if $F = \{\gamma \in \Gamma \mid B \subset \gamma\}$ for some $B \subset A$.

Following El Khadiri and Tougeron [5] we assume that A and Γ satisfy the following conditions:

(a) for all $\gamma \in \Gamma$ the canonical mapping $\mathbf{k} \longrightarrow A/\gamma$ is an isomorphism.

(b) Γ equipped with the topology of $SM(A)$ is a Noetherian space.

This means that every decreasing sequence of closed sets in Γ is stationary. Consequently any closed set in Γ is a union of finitely many irreducible closed sets.

If $a \in A$ and $\gamma \in \Gamma$, let $a(\gamma) \in \mathbf{k}$ denote the image of a under the mapping $A \longrightarrow A/\gamma \cong \mathbf{k}$. If F is a subset of Γ, let $I(F) = \{a \in A \mid a(\gamma) = 0$ for all $\gamma \in F\}$. If S is a subset of A, let $V(S) = \{\gamma \in \Gamma \mid a(\gamma) = 0$ for all $a \in S\}$. Then closed sets in Γ are the sets $V(S)$, where $S \subset A$. A closed set F in Γ is irreducible if and only if $I(F)$ is a prime ideal.

Let $x = (x_1, \ldots, x_n)$, $\mathbf{k} = \mathbb{R}$ or $\mathbb{C}$, and denote by $A[[x]]$ (resp. $\mathbf{k}[[x]]$) the ring of formal power series in x with coefficients in A (resp. in $\mathbf{k}$), and by $\mathbf{k}\{x\}$ the ring of formal power series which are convergent in some neighbourhood of the origin. If $\gamma \in \Gamma$ and $f = \sum_\beta a_\beta x^\beta \in A[[x]]$, let $f_\gamma = \sum_\beta a_\beta(\gamma) x^\beta \in \mathbf{k}[[x]]$. If $f = (f_1, \ldots, f_p) \in A[[x]]^p$, we write $f_\gamma = (f_{1,\gamma}, \ldots, f_{p,\gamma})$. Finally if N is a submodule of $A[[x]]^p$ generated by f_α, let N_γ be the submodule of $\mathbf{k}[[x]]^p$ generated by $f_{\alpha,\gamma}$.

El Khadiri and Tougeron have proved a lot of properties of submodules of $A[[x]]^p$ (see [5]). We recall some of them.

Theorem 1.1 ([5], Proposition 6.2.1). — *Let N be a submodule of $A[[x]]^p$. There exists a submodule $N' \subset N$, generated by finitely many elements, such that $N_\gamma = N'_\gamma$ for all $\gamma \in \Gamma$.*

Theorem 1.2 ([5], Proposition 6.8). — *Let I be an ideal in $A[[x]]$. There exists a positive integer μ such that*

$$\forall_{\gamma \in \Gamma} \, (\mathrm{rad}(I_\gamma))^\mu \subset I_\gamma.$$

Denote by $A_c[[x]]$ the subring of the ring $A[[x]]$ such that

$$f \in A_c[[x]] \Leftrightarrow \forall_{\gamma \in \Gamma} \; f_\gamma \in \mathbf{k}\{x\}.$$

Theorems 1.1 and 1.2 are valid if we replace $A[[x]]$ by $A_c[[x]]$.

Definition. — *A collection $\mathcal{N}$ of submodules of $\mathbf{k}[[x]]^p$ (resp. of $\mathbf{k}\{x\}^p$) is called a Noetherian family (parameterized by (A, Γ)) if there* *exists a couple* (A,Γ) *satisfying the conditions (a) and (b) given above, and a submodule* N *of* $A[[x]]^p$ *(resp.* $A_c[[x]]^p$*) such that* $\mathcal{N} = (N_\gamma)_{\gamma\in\Gamma}$.

Each subcollection of a Noetherian family is a Noetherian family, a union of two Noetherian families is a Noetherian family (if $\mathcal{N}_1$ and $\mathcal{N}_2$ are Noetherian families parametrized resp. by (A_1,Γ_1) and (A_2,Γ_2) then $\mathcal{N}_1 \cup \mathcal{N}_2$ is parametrized by $(A_1 \oplus A_2, \Gamma_1 \cup \Gamma_2)$).

Definition. — *Let* I *be an ideal in* $\mathbb{R}\{x\}$ *generated by* $f_1,\dots,f_p$ *and let* $V(I)$ *be the germ of the set of zeros of* I *at the origin. The Łojasiewicz exponent of* I *is the infimum of all the positive real numbers* α *for which there exists a constant* $c>0$ *such that*

$$\sum_{i=1}^{p} |f_i(x)| \geqslant c\, d(x,V(I))^\alpha$$

in some neighbourhood of the origin (d *denotes the Euclidean distance and we put* $d(x,\emptyset)=1$*).*

Theorem 1.3 ([5], Proposition 8.3). — *Let* $(I_\gamma)_{\gamma\in\Gamma}$ *be a Noetherian family of ideals of* $\mathbb{R}\{x\}$*. Then the family of the Łojasiewicz exponents* $\mathcal{L}(I_\gamma)$ *of* I_γ *is bounded.*

Let $(\overline{A},\overline{\Gamma})$ be a second couple satisfying conditions (a) and (b). *A change of parametrization* is a morphism of $\mathbf{k}$-algebras $\phi : A \longrightarrow \overline{A}$ such that $\phi_* : \operatorname{Spec}\overline{A} \longrightarrow \operatorname{Spec} A$ induces a morphism from $\overline{\Gamma}$ onto Γ. If $\mathcal{N} = (N_\gamma)_{\gamma\in\Gamma}$ is a Noetherian family and $\overline{N}$ is the submodule of $\overline{A}[[x]]^p$ (resp. $\overline{A}_c[[x]]^p$) generated by $\tilde{\phi}(N)$ then $\mathcal{N} = (\overline{N}_{\bar\gamma})_{\bar\gamma\in\overline{\Gamma}}$ and $(\overline{A},\overline{\Gamma})$ is a new parametrization of this family (here $\tilde{\phi} : A[[x]]^p \longrightarrow \overline{A}[[x]]^p$ is a natural extension of ϕ). A composition of changes of parametrization is a change of parametrization.

Theorem 1.4 ([6], Proposition 6.6). — *Let* N *be a submodule of* $A[[x]]^p$*. There exist a change of parametrization* $\phi : (A,\Gamma) \longrightarrow (\overline{A},\overline{\Gamma})$*, a finite partition* $(\overline{\Gamma}_i)_{i\in I}$ *of* $\overline{\Gamma}$*, ideals* $p_1,\dots p_s$ *of* $\overline{A}[[x]]$*, submodules* $N_1,\dots,N_s$ *of* $\overline{A}[[x]]^p$ *and constants* $s_i \leqslant s$*,* $i\in I$*, such that for all* $\bar\gamma \in \overline{\Gamma}_i$ *if* $\gamma = \phi_*(\bar\gamma)$*:*

(1) $p_{1,\bar\gamma},\dots,p_{s_i,\bar\gamma}$ *are prime ideals of* $\mathbf{k}[[x]]$ *and if* $j > s_i$ *then* $p_{j,\bar\gamma} = \mathbf{k}[[x]]$.

(2) $N_{j,\bar\gamma}$ *is* $p_{j,\bar\gamma}$*– primary if* $1 \leqslant j \leqslant s_i$ *and* $N_{j,\bar\gamma} = \mathbf{k}[[x]]^p$ *if* $j > s_i$.

(3) $N_\gamma = N_{1,\bar\gamma} \cap \dots \cap N_{s_i,\bar\gamma}$ *and it is a reduced primary decomposition of* N_γ.

THEOREM 1.5 ([5], Proposition 6.4). — *Let N, N' be submodules of $A[[x]]^p$. There exist a change of parametrization $\phi : (A,\Gamma) \longrightarrow (\overline{A},\overline{\Gamma})$ and a submodule $\overline{N}$ of $\overline{A}[[x]]^p$ such that for all $\bar\gamma \in \overline{\Gamma}$ if $\gamma = \phi_*(\bar\gamma)$:*

$$\overline{N}_{\bar\gamma} = N_\gamma \cap N'_\gamma.$$

2. Germs of analytic sets.

Let $f : (\mathbb{C}^n, 0) \longrightarrow (\mathbb{C}, 0)$ be a germ of a holomorphic function at the origin and let $r(z) = z_1^2 + \ldots + z_n^2$ for $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. Denote by $\mathcal{G}$ the germ at the origin of the analytic set

$$\bigcap_{i<j} \left\{ z \in \mathbb{C}^n \mid \det \begin{bmatrix} \frac{\partial r}{\partial z_i} & \frac{\partial r}{\partial z_j} \\ \frac{\partial f}{\partial z_i} & \frac{\partial f}{\partial z_j} \end{bmatrix} = 0 \right\} = \bigcap_{i<j} \left\{ z \in \mathbb{C}^n \mid \det \begin{bmatrix} z_i & z_j \\ \frac{\partial f}{\partial z_i} & \frac{\partial f}{\partial z_j} \end{bmatrix} = 0 \right\},$$

i.e. $z \in \mathcal{G}$ if and only if $\nabla r(z) = \left(\frac{\partial r}{\partial z_1}(z), \ldots, \frac{\partial r}{\partial z_n}(z)\right)$ and $\nabla f(z) = \left(\frac{\partial f}{\partial z_1}(z), \ldots, \frac{\partial f}{\partial z_n}(z)\right)$ are linearly dependent.

Denote by $\mathcal{G}'$ the germ of the set $\overline{\mathcal{G} \setminus f^{-1}(0)}$ at the origin. We will show, that $\mathcal{G}' \cap f^{-1}(0) = \mathcal{G}' \cap r^{-1}(0)$.

LEMMA 2.1. — $\mathcal{G} \cap r^{-1}(0) \subset f^{-1}(0)$.

Proof. — Assume that $(\mathcal{G} \cap r^{-1}(0)) \setminus (\mathcal{G} \cap f^{-1}(0)) \neq \emptyset$. According to the curve selection lemma there exists an analytic curve $\gamma = (\gamma_1, \ldots, \gamma_n)$ such that $\gamma(0) = 0$ and $\gamma \setminus \{0\} \subset (\mathcal{G} \cap r^{-1}(0)) \setminus (\mathcal{G} \cap f^{-1}(0))$. Then we have $r(\gamma(t)) \equiv 0$. Hence

$$(1) \qquad \frac{d}{dt}\, r(\gamma(t)) = \frac{\partial r}{\partial z_1}\,(\gamma(t))\,\frac{d\gamma_1}{dt}\,(t) + \ldots + \frac{\partial r}{\partial z_n}(\gamma(t))\frac{d\gamma_n}{dt}(t) \equiv 0.$$

Since $\nabla r(z) \neq 0$ for $z \neq 0$ and $\gamma(t) \in \mathcal{G}$,

$$\forall_t\ \exists_{c(t)}\ \nabla f(\gamma(t)) = c(t)\nabla r(\gamma(t)).$$

Thus by (1) we have $\frac{d}{dt} f(\gamma(t)) \equiv 0$, so $f \circ \gamma = \mathit{const}$. Since $(f \circ \gamma)(0) = 0$, $\gamma \subset f^{-1}(0)$ — a contradiction. □

LEMMA 2.2. — *$\mathcal{G}'$ is a germ of an analytic set.*

Proof. — Germs of sets $\mathcal{G}$, $\mathcal{G} \cap f^{-1}(0)$ are analytic, so the representative of $\mathcal{G} \setminus f^{-1}(0)$ is an analytically constructible set. The complex closure of an analytically constructible set is analytic, so the representative of the germ $\mathcal{G}'$ is an analytic set ([8], Proposition IV 8.3.5). □

Lemma 2.3. — *$\mathcal{G}' = \mathcal{G}_1 \cup \ldots \cup \mathcal{G}_p$, where $\mathcal{G}_1, \ldots, \mathcal{G}_p$ are the irreducible components of $\mathcal{G}$ such that $\mathcal{G}_i \setminus f^{-1}(0) \neq \emptyset$ for $i = 1, \ldots, p$. Moreover, $\mathcal{G}_i \setminus f^{-1}(0)$ is dense in $\mathcal{G}_i$.*

Proof. — According to [8], Theorem IV 2.10.5, $\mathcal{G}' = \mathcal{G}_1 \cup \ldots \cup \mathcal{G}_p$. Since each germ $\mathcal{G}_i$ is irreducible, [8], Proposition IV 2.8.3, implies that $\mathcal{G}_i \cap f^{-1}(0)$ is nowhere dense in $\mathcal{G}_i$, so $\mathcal{G}_i \setminus f^{-1}(0) = \mathcal{G}_i \setminus (\mathcal{G}_i \cap f^{-1}(0))$ is dense in $\mathcal{G}_i$. □

Lemma 2.4. — *Let $\mathcal{G}_1, \ldots, \mathcal{G}_p$ be defined as in Lemma 2.3. Let $\mathcal{G}_i \setminus r^{-1}(0) = \bigcup A_{i,k}$ be a decomposition into finitely many disjoint analytic submanifolds. Then for each i, k the restriction of r to the set $A_{i,k}$ has no critical points in some neighbourhood of the origin.*

Proof. — Fix i, k and assume that the set of critical points of $r|_{A_{i,k}}$ is nonempty. Then it is analytically constructible. According to the curve selection lemma there is a curve γ such that $\gamma(0) = 0$ and $\gamma \setminus \{0\}$ is contained in the set of critical points of $r|_{A_{i,k}}$. Then the function $r|_{A_{i,k}} \circ \gamma$ is constant. We have $r(\gamma(0)) = r(0) = 0$, so $r|_{A_{i,k}} \circ \gamma \equiv 0$. But it contradicts $\gamma \cap r^{-1}(0) = \emptyset$. So the set of critical points of $r|_{A_{i,k}}$ is empty. □

We will say that an analytic set has a *Whitney stratification*, if it has such a stratification whose every two strata satisfy Whitney conditions **a** and **b**.

Theorem 2.5 (see e.g. [18] Theorem 19.2, [1] Theorem 9.7.11). *Any analytic set has a Whitney stratification. Any stratification $(E_i)_{i\in I}$ of this set has a Whitney refinement, i.e. there exists a Whitney stratification $(F_j)_{j\in J}$ such that each stratum E_i is a union of some strata of $(F_j)_{j\in J}$.*

Lemma 2.6. — $\mathcal{G}' \cap f^{-1}(0) \setminus r^{-1}(0) = \emptyset.$

Proof. — Fix $i \in \{1, \ldots, p\}$. We will show that

$$\mathcal{G}_i \cap f^{-1}(0) \setminus r^{-1}(0) = \emptyset.$$

The set $\mathcal{G}_i$ admits a Whitney stratification such that $\mathcal{G}_i \cap f^{-1}(0)$, as well as $\mathcal{G}_i \setminus r^{-1}(0)$ is a union of strata. According to Lemma 2.4 the restriction $r|_{A_{i,k}}$ is a submersion for each k.

Assume that $z_0 \in \mathcal{G}_i \cap f^{-1}(0) \setminus r^{-1}(0)$. Let A be the stratum such that $z_0 \in A$ and let $\bigcup_j B_j$ be the union of all strata $B_j \subset \mathcal{G}_i \setminus f^{-1}(0)$ such that $A \subset \overline{B_j}$. According to Lemma 2.3 there is at least one nonempty stratum satisfying this condition. Denote $Z = A \cup \bigcup_j B_j$.

We will show, that z_0 is not isolated in $\bigcup_j B_j \cap r^{-1}(r(z_0))$, using the following Thom-Mather theorem:

Theorem 2.7 ([16] Theorem 4.3.1). — *Let $X = \bigcup X_\alpha$ be an analytic space admitting a Whitney stratification. For each $x \in X_\alpha$, each local embedding $X \subset \mathbb{C}^n$ in a neighbourhood of x, and each local retraction $\rho : \mathbb{C}^n \longrightarrow X_\alpha$ there exist an open neighbourhood U of x in $\mathbb{C}^n$ and a homeomorphism compatible with ρ such that, denoting $V = U \cap X_\alpha$ and $\Pi_2 : (\rho^{-1}(x) \cap X \cap U) \times V \longrightarrow V$ — the projection on the second variable, we have*

$$\begin{array}{ccc} X \cap U & \simeq & (\rho^{-1}(x) \cap X \cap U) \times V \\ \rho|_{X \cap U} \searrow & & \swarrow \Pi_2 \\ & V & \end{array}$$

inducing for each $\overline{X}_\beta$ containing X_α the analogous homeomorphism

$$\begin{array}{ccc} \overline{X}_\beta \cap U & \simeq & (\rho^{-1}(x) \cap \overline{X}_\beta \cap U) \times V \\ \rho|_{\overline{X}_\beta \cap U} \searrow & & \swarrow \Pi_2 \\ & V & \end{array} .$$

The set Z satisfies the assumptions of the theorem. Fix $B_j \neq \emptyset$ and denote $k = \dim_{\mathbb{C}} A$. Since $\tilde{r} := r|_A$ has no critical points, there exist $r_2, \ldots, r_k : \mathbb{C}^n \longrightarrow \mathbb{C}$ defined in some neighbourhood of z_0 such that, denoting $\tilde{r_i} = r_i|_A$, $\mathrm{d}\,\tilde{r}(z_0), \mathrm{d}\,\tilde{r_2}(z_0), \ldots, \mathrm{d}\,\tilde{r_k}(z_0)$ are linearly independent. Take $R = (r, r_2, \ldots, r_k) : \mathbb{C}^n \longrightarrow \mathbb{C}^k$. A is transversal to $R^{-1}(R(z_0))$ and crosses it at z_0. Denote $\tilde{R} = R|_A$, then $\operatorname{rank} \mathrm{D}\,\tilde{R}(z_0) = k$. So $\tilde{R} : (A, z_0) \longrightarrow (\mathbb{C}^k, R(z_0))$ is an analytic diffeomorphism. Denote by $S : (\mathbb{C}^k, R(z_0)) \longrightarrow (A, z_0)$ the inverse of $\tilde{R}$.

Let define a local retraction $\rho : \mathbb{C}^n \longrightarrow A$, $\rho(z) = (S \circ R)(z)$. According to Theorem 2.7 there exist a neighbourhood U of z_0 and a homeomorphism h such that, for $V = U \cap A$

$$\begin{array}{ccc} \overline{B_j} \cap U & \overset{h}{\simeq} & (\rho^{-1}(z_0) \cap \overline{B_j} \cap U) \times V \\ \rho|_{\overline{B_j} \cap U} \searrow & & \swarrow \Pi_2 \\ & V & \end{array} .$$

We have $(\rho^{-1}(z_0) \cap \overline{B_j} \cap U) \times V = (R^{-1}(R(z_0)) \cap \overline{B_j} \cap U) \times V \subset (r^{-1}(r(z_0)) \cap \overline{B_j} \cap U) \times V$. Since $A \subset \overline{B_j}$, there exist a sequence $(z_n) \subset B_j$

such that $z_n \to z_0$. Let $(y_n) \subset (R^{-1}(R(z_0)) \cap \overline{B_j} \cap U)$ be such that $z_n = h^{-1}(y_n, \rho(z_n))$. Then $y_n \to z_0$ and $(y_n) \subset r^{-1}(r(z_0))$.

Hence z_0 is not isolated in $\bigcup_j B_j \cap r^{-1}(r(z_0))$, so by the curve selection lemma there is a curve γ such that $\gamma(0) = z_0$ and $\gamma \setminus \{z_0\} \subset \bigcup_j B_j \cap r^{-1}(r(z_0))$.

Because $\gamma \subset \mathcal{G}_i \subset \mathcal{G}$ and $r|_{A_{i,k}}$ are submersions, we can deduce as above, using arguments from the proof of Lemma 2.1, that f is constant along γ and $f(\gamma(0)) = f(z_0) = 0$, so $f \equiv 0$ along γ. But $\gamma \setminus \{z_0\} \subset \mathcal{G}_i \setminus f^{-1}(0)$, a contradiction. Then $\mathcal{G}' \cap f^{-1}(0) \setminus r^{-1}(0) = \emptyset$. □

Hence we obtain

Corollary 2.8. — $\mathcal{G}' \cap f^{-1}(0) = \mathcal{G}' \cap r^{-1}(0)$.

3. Properties of Noetherian families.

Assume that $\Omega \subset \mathbb{R}^n$ is a semianalytic compact subset and denote by $\mathcal{A}(\Omega)$ the algebra of real analytic functions defined in a neighbourhood of Ω. We can treat $\mathbb{R}^n$ as a subspace of $\mathbb{C}^n$, so $\Omega \subset \mathbb{C}^n$ and we denote by $\mathcal{H}(\Omega)$ the algebra of complex analytic functions defined in a neighbourhood of Ω.

El Khadiri and Tougeron have proven (see [5]), that if $\mathcal{O}(\Omega) = \mathcal{A}(\Omega)$ or $\mathcal{H}(\Omega)$, then $\mathcal{O}(\Omega)$ is an Ω-Noetherian algebra, so Ω is a Noetherian space with the topology induced from $SM(\mathcal{O}(\Omega))$ (by identifying $\omega \in \Omega$ with the ideal $p_\omega = \{f \in \mathcal{O}(\Omega) \,|\, f(\omega) = 0\}$, $\{\bigcap_{f\in B} f^{-1}(0) \cap \Omega\}_{B\subset\mathcal{O}(\Omega)}$ is the family of closed sets in Ω), and the pair $(\mathcal{O}(\Omega), \Omega)$ satisfies conditions (a) and (b) from the section 1. Notice that since Ω is a Noetherian space, for every closed (with respect to the topology induced by the topology on the maximal spectrum) subset D of Ω there exist $f_1, \ldots, f_p \in \mathcal{O}(\Omega)$ such that $D = \bigcap_{i=1}^{p} f_i^{-1}(0) \cap \Omega$, so D is an intersection of Ω and an analytic set.

The result of Frisch [7] says, that $\mathcal{A}(\Omega)$ is Noetherian and if Ω admits a fundamental system of Stein neighborhoods, then $\mathcal{H}(\Omega)$ is also Noetherian.

If $f \in \mathcal{A}(\Omega)$ and $\omega \in \Omega$, we denote $\tilde{f} = \sum_\alpha \frac{1}{\alpha!} \mathrm{D}^\alpha f x^\alpha$, $\tilde{f}_\omega = \sum_\alpha \frac{1}{\alpha!} \mathrm{D}^\alpha f(\omega) x^\alpha$. Of course $\tilde{f} \in \mathcal{A}(\Omega)_c[[x]]$.

Define $\tilde{f}^C_\omega : (\mathbb{C}^n, 0) \longrightarrow \mathbb{C}$ as $\tilde{f}^C_\omega = \sum_\alpha \frac{1}{\alpha!} \mathrm{D}^\alpha f(\omega) z^\alpha$, then $\tilde{f}^C = \sum_\alpha \frac{1}{\alpha!} \mathrm{D}^\alpha f z^\alpha \in \mathcal{H}(\Omega)_c[[x]]$.

THEOREM 3.1. — *Let $f \in \mathcal{A}(\Omega)$. There is $N_0 > 0$ such that for each $N \geqslant N_0$, $\omega \in \Omega$ there exist $\epsilon_\omega > 0$ and $c_\omega > 0$ such that if $\epsilon \in (0;\epsilon_\omega)$ and $x \in S^{n-1}_\epsilon \setminus \tilde{f_\omega}^{-1}(0)$ is a critical point of $\tilde{f}_\omega|_{S^{n-1}_\epsilon}$ then*

$$|\tilde{f}_\omega(x)| \geqslant \frac{1}{c_\omega}||x||^{2N}.$$

Proof. — Let $r(z) = z_1^2 + \ldots + z_n^2$ for $z \in \mathbb{C}^n$. Let define $M^{ij} = \det \begin{bmatrix} \frac{\partial r}{\partial z_i} & \frac{\partial r}{\partial z_j} \\ \frac{\partial \tilde{f}^C}{\partial z_i} & \frac{\partial \tilde{f}^C}{\partial z_j} \end{bmatrix}$. Then $M^{ij} \in \mathcal{H}(\Omega)_c[[x]]$, M^{ij}_ω are germs of complex analytic functions at the origin. Let $\mathcal{G}_\omega = V((M^{ij}_\omega)_{i<j})$ for $\omega \in \Omega$. According to Lemma 2.3 for each $\omega \in \Omega$ there exist $p(\omega)$, $l(\omega)$ and a decomposition into irreducible components $\mathcal{G}_\omega = \mathcal{G}_{1,\omega} \cup \ldots \cup \mathcal{G}_{p(\omega),\omega} \cup \ldots \cup \mathcal{G}_{l(\omega),\omega}$ such that $\mathcal{G}'_\omega := \overline{\mathcal{G}_\omega \setminus (\tilde{f}^C_\omega)^{-1}(0)} = \mathcal{G}_{1,\omega} \cup \ldots \cup \mathcal{G}_{p(\omega),\omega}$.

We have $I(\mathcal{G}_\omega) = I(\mathcal{G}_{1,\omega}) \cap \ldots \cap I(\mathcal{G}_{p(\omega),\omega}) \cap \ldots \cap I(\mathcal{G}_{l(\omega),\omega})$ and $I(\mathcal{G}'_\omega) = I(\mathcal{G}_{1,\omega}) \cap \ldots \cap I(\mathcal{G}_{p(\omega),\omega})$. Denote $J_{j,\omega} = I(\mathcal{G}_{j,\omega})$. $\mathcal{G}_{j,\omega}$ are irreducible components of a complex analytic germ $\mathcal{G}_\omega$, so $J_{j,\omega}$ are prime and $I(\mathcal{G}_\omega) = J_{1,\omega} \cap \ldots \cap J_{l(\omega),\omega}$ is a reduced prime decomposition.

Denote by $\mathcal{J}$ the ideal in $\mathcal{H}(\Omega)_c[[x]]$ generated by M^{ij}, $i < j$, so $\mathcal{J}_\omega = (M^{ij}_\omega)_{i<j}$. Then, by the local Hilbert Nullstellensatz, $\text{rad}(\mathcal{J}_\omega) = I(\mathcal{G}_\omega)$ and then $J_{1,\omega}, \ldots, J_{l(\omega),\omega}$ are minimal prime ideals associated with the ideal $\mathcal{J}_\omega$. According to Theorem 1.4, there exist a change of parametrization $\phi : (\mathcal{H}(\Omega), \Omega) \longrightarrow (A, \Gamma)$, a finite partition $(\Gamma_i)_{i\in I}$ of Γ, ideals $p_1, \ldots p_s$ of $A_c[[x]]$ and constants $s_i \leqslant s$, $i \in I$, such that for all $\gamma \in \Gamma_i$, if $\omega = \phi_*(\gamma)$ then $p_{1,\gamma}, \ldots, p_{s_i,\gamma}$ are minimal prime ideals associated with $\mathcal{J}_\omega$. Because of uniqueness of such ideals, for each $j \in \{1, \ldots, l(\omega)\}$ there exists $q \in \{1, \ldots, s_i\}$ such that $J_{j,\omega} = p_{q,\gamma}$.

According to Theorem 1.5 there exist a change of parametrization $\phi : (A, \Gamma) \longrightarrow (\overline{A}, \overline{\Gamma})$ and ideals $\overline{N}^Q$ of $\overline{A}_c[[x]]$, $Q \subset \{1, \ldots, s\}$, such that for all $\bar{\gamma} \in \overline{\Gamma}_i$, if $\gamma = \phi_*(\bar{\gamma})$ then $\overline{N}^Q_{\bar{\gamma}} = \bigcap_{j\in Q} p_{j,\gamma}$.

A finite union of Noetherian families is a Noetherian family, so let $\mathcal{K} = (K_{\bar{\gamma}})_{\bar{\gamma}\in\overline{\Gamma}}$ be a Noetherian family containing all families $(\overline{N}^Q_{\bar{\gamma}})_{\bar{\gamma}\in\overline{\Gamma}}$, $Q \subset \{1, \ldots, s\}$. Then $\mathcal{K}$ contains all $I(\mathcal{G}'_\omega)$ for $\omega \in \Omega$. Let $(M_{\bar{\gamma}})_{\bar{\gamma}\in\overline{\Gamma}}$ denote the Noetherian family $(\tilde{f}^C_\omega)_{\omega\in\Omega}$ after the change of parametrization $\phi' : (\mathcal{H}(\Omega), \Omega) \longrightarrow (\overline{A}, \overline{\Gamma})$ which is a composition of changes of parametrization. According to Theorem 1.2

$$\exists_{N_0>0}\ \forall_{N\geqslant N_0}\ \forall_{\bar{\gamma}\in\overline{\Gamma}}\ (\text{rad}(K_{\bar{\gamma}} + M_{\bar{\gamma}}))^N \subset (K_{\bar{\gamma}} + M_{\bar{\gamma}}).$$

According to Corollary 2.8, for each $\omega \in \Omega$ we have $V(I(\mathcal{G}'_\omega)+(r)) = \mathcal{G}'_\omega \cap r^{-1}(0) = \mathcal{G}'_\omega \cap (\tilde{f}^C_\omega)^{-1}(0) = V(I(\mathcal{G}'_\omega)+(\tilde{f}^C_\omega))$. By the local Hilbert Nullstellensatz, $\mathrm{rad}(I(\mathcal{G}'_\omega)+(r)) = \mathrm{rad}(I(\mathcal{G}'_\omega)+(\tilde{f}^C_\omega))$. For each $\omega \in \Omega$ there exists $\bar{\gamma} \in \overline{\Gamma}$ such that $I(\mathcal{G}'_\omega) = K_{\bar{\gamma}}$, and then

$$\begin{aligned}(I(\mathcal{G}'_\omega)+(r))^{N_0} &\subset (\mathrm{rad}(I(\mathcal{G}'_\omega)+(r)))^{N_0} = (\mathrm{rad}(I(\mathcal{G}'_\omega)+(\tilde{f}^C_\omega)))^{N_0}\\ &= (\mathrm{rad}(K_{\bar{\gamma}}+M_{\bar{\gamma}}))^{N_0} \subset (K_{\bar{\gamma}}+M_{\bar{\gamma}}) = (I(\mathcal{G}'_\omega)+(\tilde{f}^C_\omega)).\end{aligned}$$

Let $g_{i,\omega}$ be the generators of $I(\mathcal{G}'_\omega)$. Then $r^{N_0} = a_\omega \tilde{f}^C_\omega + \sum_i c_{i,\omega} g_{i,\omega}$ for some germs of complex analytic functions a_ω, $c_{i,\omega}$.

Let $0 < \epsilon_\omega \ll 1$ be such that representatives of the germs $\tilde{f}_\omega^{\,C}$, a_ω and all $c_{i,\omega}$, $g_{i,\omega}$ are defined on $\{z \in \mathbb{C}^n \mid ||z|| < \epsilon_\omega\}$. If $0 < \epsilon < \epsilon_\omega$ and x is a critical point of $\tilde{f}_\omega|_{S^{n-1}_\epsilon}$ such that $x \notin \tilde{f}_\omega^{\,-1}(0)$ then $x \in \mathcal{G}'_\omega$ and for each i we have $g_{i,\omega}(x) = 0$. Then $r^{N_0}(x) = a_\omega(x)\tilde{f}_\omega(x)$, so

$$\exists_{c_\omega>0}\ \forall_{N \geqslant N_0}\ r^N(x) \leqslant r^{N_0}(x) = |a_\omega(x)||\tilde{f}_\omega(x)| \leqslant c_\omega|\tilde{f}_\omega(x)|.$$

Thus

$$|\tilde{f}_\omega(x)| \geqslant \frac{1}{c_\omega} r^N(x) = \frac{1}{c_\omega}||x||^{2N}.$$

□

COROLLARY 3.2. — *Let $f \in \mathcal{A}(\Omega)$. Then there is $\alpha = 2N_0+1$ such that for each $\omega \in \Omega$ there exists $0 < \epsilon_\omega \ll 1$ such that if $0 < \epsilon < \epsilon_\omega$ and $x \in S^{n-1}_\epsilon \setminus \tilde{f}_\omega^{\,-1}(0)$ is a critical point of $\tilde{f}_\omega|_{S^{n-1}_\epsilon}$ then*

$$|\tilde{f}_\omega(x)| \geqslant ||x||^\alpha.$$

4. Families of germs of real analytic functions.

Let $\mathbf{k} = \mathbb{R}$ or $\mathbf{k} = \mathbb{C}$ and let $\mathbf{m}$ be the maximal ideal of $\mathbf{k}[[x]] = \mathbf{k}[[x_1,\dots,x_n]]$. Let $\mathcal{F}_p = \oplus_p \mathbf{m} \subset \mathbf{k}[[x]]^p$. If $g \in \mathcal{F}_p$, then $g = (g_1,\dots,g_p)$, where

$$g_j = \sum_{|\alpha|\geqslant 1} \frac{a_j^\alpha}{\alpha!} x^\alpha \quad (\text{i.e. } a_j^\alpha = \mathrm{D}^\alpha g_j(0)).$$

Let $\Psi_1,\dots\Psi_s$ be formal power series in x with coefficients which depend polynomially on a_j^α, where $|\alpha| \geqslant 1$ and $1 \leqslant j \leqslant p$. If $g = (g_1,\dots,g_p) \in \mathcal{F}_p$, we denote by $\Psi_{i,g}$ the formal power series obtained by putting $a_j^\alpha = \mathrm{D}^\alpha g_j(0)$ in Ψ_i. Let I_g be the ideal of $\mathbf{k}[[x]]$ generated by $\Psi_{1,g}, \ldots, \Psi_{s,g}$.

Denote by W_h the set $\{g \in \mathcal{F}_p \mid \dim_{\mathbf{k}}(\mathbf{k}[[x]]/I_g) > h\}$. Then, by [17], Corollary II.5.2, we have $W_h = \Big\{g \in \mathcal{F}_p \mid \dim_{\mathbf{k}}(I_g + \mathbf{m}^{h+1}/\mathbf{m}^{h+1}) < \binom{n+h}{n} - h\Big\}$. We consider $\mathbf{k}[[x]] + \mathbf{m}^{h+1}/\mathbf{m}^{h+1}$ which is an affine space of finite dimension. The space $I_g + \mathbf{m}^{h+1}/\mathbf{m}^{h+1}$ is its linear subspace generated by $x^\alpha \Psi_{i,g}$, where $\alpha \in \mathbb{N}^n$, $0 \leqslant |\alpha| \leqslant h$. The above description of W_h involves only finitely many coefficients of the series.

Let $\Psi_{i,g}^{\alpha,\beta}$, $|\beta| \leqslant h, |\alpha| \leqslant h$ be the coefficients at x^β in the series $x^\alpha \Psi_{i,g}$. Then the set W_h is the set of such $g \in \mathcal{F}_p$, for which all the minors of the matrix $(\Psi_{i,g}^{\alpha,\beta})$ of degree $\binom{n+h}{n} - h$ vanish ((i, α) is a row index, β is a column index).

THEOREM 4.1 ([17], Lemma VII.5.3]). — *The sets W_h are algebraic and*

$$\{g \in \mathcal{F}_p \mid \dim_{\mathbf{k}}(\mathbf{k}[[x]]/I_g) < \infty\} = \mathcal{F}_p \setminus \bigcap_{h=0}^{\infty} W_h.$$

We will say that a germ of an analytic mapping $F = (F^1, \ldots, F^n) : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$ has an algebraically isolated zero at the origin if $\dim_{\mathbb{R}} \mathbb{R}[[x]]/(P_1, \ldots, P_n) < \infty$, where $P_i = \sum_\alpha \frac{1}{\alpha!} \mathrm{D}^\alpha F^i(0) x^\alpha$. If $0 \in \mathbb{C}^n$ is isolated in the inverse image of 0 for the complexification of F then the origin is an algebraically isolated zero of F.

By $\deg_0 F$ we denote the local topological degree at the origin of the mapping F which has an isolated zero at the origin.

Recall that a closed subset of Ω has to be understood with respect to the topology induced from $SM(\mathcal{A}(\Omega))$. We will say that a closed subset of Ω is irreducible if it is not a union of two its proper closed subsets. Every closed subset D of a Noetherian space Ω has a decomposition into finitely many irreducible components, i.e. $D = \bigcup_{i=1}^k D_i$, where every D_i is a closed irreducible subset of D and $D_i \not\subset \bigcup_{j \neq i} D_j$.

Let $D \subset \Omega$ be a closed subset. Denote $J = \{f \in \mathcal{A}(\Omega) \mid f|_D \equiv 0\}$, and define

$$\mathcal{A}(D) := \mathcal{A}(\Omega)/J.$$

If D is irreducible then J is a prime ideal and $\mathcal{A}(D)$ is an integral domain.

Denote by $\mathcal{S}_n(D)$ the set of families $\{F_\omega = (F_\omega^1, \ldots, F_\omega^n) : (\mathbb{R}^n, 0) \longrightarrow$

$(\mathbb{R}^n, 0)\}_{\omega \in D}$ of analytic germs at the origin such that

$$\forall_{1 \leqslant i \leqslant n} \exists_{f_i \in \mathcal{A}(\Omega)_c[[x]]} \forall_{\omega \in D} F^i_\omega(x) = f_i(\omega, x).$$

In particular if

$$\forall_{1 \leqslant i \leqslant n} \exists_{h_i \in \mathcal{A}(\Omega)} \forall_{\omega \in D} F^i_\omega(x) = h_i(x + \omega),$$

then $\{F_\omega\}_{\omega \in D} \in \mathcal{S}_n(D)$.

LEMMA 4.2. — *Assume that a closed subset $D \subset \Omega$ is irreducible, $\{F_\omega\}_{\omega \in D} \in \mathcal{S}_n(D)$ and $0 \in \mathbb{R}^n$ is isolated in $F_\omega^{-1}(0)$ for all $\omega \in D$. Then there exist a proper closed subset $\Sigma \subset D$, and a family $\{G_\omega\}_{\omega \in D} \in \mathcal{S}_n(D)$ such that*

(i) $\forall_{\omega \in D \setminus \Sigma}$ G_ω *has an algebraically isolated zero at the origin,*

(ii) $\forall_{\omega \in D}$ $\deg_0 F_\omega = \deg_0 G_\omega$.

Proof. — For $\omega \in D$ we define the germ G_ω:

$$G_\omega(x) = F_\omega(x) + a(x_1^k, \dots, x_n^k),$$

where k is a positive integer, $a \neq 0$. We have $G^i_\omega(x) = f_i(\omega, x) + ax_i^k$, so G^i_ω is a real analytic germ. Let $c_{i\alpha} \in \mathcal{A}(D)$ be residue classes of $\frac{1}{\alpha!} \mathrm{D}^\alpha G^i_\omega(0) \in \mathcal{A}(\Omega)$, and let associate with G^i_ω the formal power series

$$P_i(\omega, x) = \sum_\alpha c_{i\alpha}(\omega) x^\alpha \in \mathcal{A}(D)_c[[x]].$$

According to Theorem 4.1 the set $\{\omega \in D \mid \dim_{\mathbb{R}}(\mathbb{R}[[x]]/(P_1(\omega, \cdot), \dots, P_n(\omega, \cdot))) < \infty\} = D \setminus \bigcap_{h=0}^\infty \Sigma_h$, where $\Sigma_h = \{\omega \in D \mid \dim_{\mathbb{R}}(\mathbb{R}[[x]]/(P_1(\omega, \cdot), \dots, P_n(\omega, \cdot))) > h\}$ are closed in D. Indeed, Σ_h is the intersection of the zero sets of some compositions of $c_{i\alpha}$ and polynomials. So $\Sigma = \bigcap_{h=0}^\infty \Sigma_h$ is a closed subset of D such that the origin is algebraically isolated in $G_\omega^{-1}(0) \subset \mathbb{R}^n$ for $\omega \in D \setminus \Sigma$.

Using arguments similar as in the proof of [15], Lemma 1.3, we can show, that Σ is a proper subset of D. We have

$$P_i(\omega, x) = G^i_\omega(x) = F^i_\omega(x) + ax_i^k = f_i(\omega, x) + ax_i^k$$

for x sufficiently close to the origin. Fix $\omega_0 \in D$. The set

$$A = \{a \in \mathbb{R} \setminus \{0\} \mid \dim_{\mathbb{R}} \left(\mathbb{R}[[x]]/(f_1(\omega_0, x) + ax_1^k, \dots, f_n(\omega_0, x) + ax_n^k)\right) > h\}$$

is finite for h sufficiently large.

Indeed, denote $H_a^i(x) = af_i(\omega_0, x) + x_i^k$ for $a \in \mathbb{R}$. Then $H_0^i = x_i^k$ and we have $\dim_{\mathbb{R}}(\mathbb{R}[[x]]/(x_1^k, \dots, x_n^k)) = k^n$. Then according to Theorem 4.1 the set

$$A' = \left\{a \in \mathbb{R} \mid \dim_{\mathbb{R}} \left(\mathbb{R}[[x]]/(H_a^1, \dots, H_a^n)\right) > h\right\}$$

is algebraic and $0 \notin A'$ for $h > k^n$, so A' is finite for $h > k^n$. If $a \neq 0$ then we have $H_{\frac{1}{a}}(x) = \frac{1}{a}P_i(\omega_0, x)$, so A is also finite for $h > k^n$.

Take $a \notin A$ in the definition of G_ω, then

$$\omega_0 \notin \Sigma_h = \{\omega \in D \mid \dim_{\mathbb{R}} (\mathbb{R}[[x]]/(P_1(\omega, \cdot), \dots, P_n(\omega, \cdot))) > h\},$$

so $\Sigma_h \neq D$ for h sufficiently large and Σ is a proper subset of D.

Let $I_\omega \subset \mathbb{R}\{x\}$ be the ideal generated by germs $F_\omega^1, \dots, F_\omega^n$. Theorem 1.3 implies that the Łojasiewicz exponent of I_ω is bounded:

$$\exists_M \ \forall_{\omega \in D} \ \alpha_\omega = \inf\{\alpha \mid \exists_{c>0} \sum_{i=1}^{n} |F_\omega^i(x)| \geqslant c\, d(x, V_0(I_\omega))^\alpha\} \leqslant M.$$

The origin is isolated in the zero set of F_ω, so

$$\exists_M \ \forall_{\omega \in D} \ \exists_{c_\omega > 0} \sum_{i=1}^{n} |F_\omega^i(x)| \geqslant c_\omega d(x, V_0(I_\omega))^{\alpha_\omega} = c_\omega d(x, \{0\})^{\alpha_\omega} \geqslant c_\omega ||x||^M$$

for x near 0.

Hence if we take $k > M$ in the definition of G_ω then there exists $c_\omega > 0$ such that

$$\begin{aligned} ||tG_\omega(x) + (1-t)F_\omega(x)|| &= ||F_\omega(x) + at(x_1^k, \dots, x_n^k)|| \\ &\geqslant c_\omega ||x||^M - at||(x_1^k, \dots, x_n^k)|| \geqslant \frac{c_\omega}{2} ||x||^M, \end{aligned}$$

where $0 \leqslant t \leqslant 1$, x near 0 (see [12]).

Then $\deg_0 F_\omega = \deg_0 G_\omega$. □

LEMMA 4.3. — *Under the assumptions of Lemma 4.2 there exist* $q_1, \dots, q_t \in \mathcal{A}(\Omega)$ *and a proper closed subset* $\Sigma \subset D$ *such that for* $\omega \in D \setminus \Sigma$

$$\deg_0 F_\omega = \operatorname{sgn} q_1(\omega) + \ldots + \operatorname{sgn} q_t(\omega).$$

Proof. — According to Lemma 4.2 we can assume that $\{F_\omega\}_{\omega \in D}$ is a family in $\mathcal{S}_n(D)$ for which there exists a proper closed subset $\Sigma' \subset D$ such that F_ω has an algebraically isolated zero at the origin for $\omega \in D \setminus \Sigma'$.

Taking $\mathcal{A} = \mathcal{A}(D)$ (an integral domain) we can follow the arguments of [12], Lemma 3.3 (in particular studying $\deg_0 F_\omega$ in the context of

Eisenbud and Levine Theorem). They imply that there exist a proper closed subset $\Sigma \subset D$ such that $\Sigma' \subset \Sigma$, and a symmetric matrix T whose entries belong to $\mathcal{A}(D)$ such that for every $\omega \in D \setminus \Sigma$ the matrix $T(\omega)$ is non-degenerate and $\deg_0 F_\omega = \text{signature } T(\omega)$. Let $\tilde{q}_1, \dots, \tilde{q}_t \in \mathcal{A}(D)$ be the elements of the diagonal of T after making T diagonal by a change of variables over the rational fractions on $\mathcal{A}(D)$ and multiplying by the squares of the denominators of the entries. Then, if we enlarge Σ in such a way, that the zeros of the denominators belong to Σ, and take $q_i \in \mathcal{A}(\Omega)$ such that $\tilde{q}_i$ is the residue class of q_i, $i = 1, \dots t$, we have

$$\deg_0 F_\omega = \operatorname{sgn} q_1(\omega) + \ldots + \operatorname{sgn} q_t(\omega)$$

for $\omega \in D \setminus \Sigma$. □

Lemma 4.4. — *Assume that $\tilde{\Omega} \subset \Omega$ is a closed subset and $0 \in \mathbb{R}^n$ is isolated in $F_\omega^{-1}(0)$ for $\omega \in \tilde{\Omega}$. Then there exist $v_1, \dots, v_s \in \mathcal{A}(\Omega)$ and a proper closed subset $\Sigma \subset \tilde{\Omega}$ such that for $\omega \in \tilde{\Omega} \setminus \Sigma$ we have*

$$\deg_0 F_\omega = \operatorname{sgn} v_1(\omega) + \ldots + \operatorname{sgn} v_s(\omega).$$

Proof. — Induction on the number of irreducible components of $\tilde{\Omega}$.

If $\tilde{\Omega}$ is irreducible then Lemma 4.3 implies the result.

Assume that $\tilde{\Omega} = D_1 \cup D_2 \cup \ldots \cup D_m$ is a decomposition of $\tilde{\Omega}$ into irreducible components. Denote $\Omega' = D_2 \cup \ldots \cup D_m$. Let $h_1 \in \mathcal{A}(\Omega)$, $h_2 \in \mathcal{A}(\Omega)$ be non-negative and such that

$$h_1 \equiv 0 \text{ on } D_1,\ h_1 \not\equiv 0 \text{ on } \Omega',$$

$$h_2 \equiv 0 \text{ on } \Omega',\ h_2 \not\equiv 0 \text{ on } D_1.$$

According to Lemma 4.3 and the inductive assumption, there exist $q_1, \dots, q_t, p_1, \dots, p_{t'} \in \mathcal{A}(\Omega)$ and proper closed subsets $\Sigma_1 \subset D_1$, $\Sigma_2 \subset \Omega'$ such that for $\omega \in D_1 \setminus \Sigma_1$ we have

$$\deg_0 F_\omega = \operatorname{sgn} q_1(\omega) + \ldots + \operatorname{sgn} q_t(\omega)$$

and for $\omega \in \Omega' \setminus \Sigma_2$ we have

$$\deg_0 F_\omega = \operatorname{sgn} p_1(\omega) + \ldots + \operatorname{sgn} p_{t'}(\omega).$$

Let $\Sigma = \Sigma_1 \cup \Sigma_2 \cup (h_1^{-1}(0) \cap \Omega') \cup (h_2^{-1}(0) \cap D_1)$, then

$$\deg_0 F_\omega = \sum_{i=1}^{t} \operatorname{sgn} h_2(\omega) q_i(\omega) + \sum_{j=1}^{t'} \operatorname{sgn} h_1(\omega) p_j(\omega)$$

for $\omega \in \tilde{\Omega} \setminus \Sigma$. We take $s = t + t'$, $v_i(\omega) = h_2(\omega)q_i(\omega)$ for $i = 1, \dots t$ and $v_i(\omega) = h_1(\omega)p_{i-t}(\omega)$ for $i = t+1, \dots s$. □

We will use below the following fact (see [12]):

Let $h \in \mathcal{A}(\Omega)$ be non-negative and such that $h^{-1}(0) \cap \Omega = \Sigma$ (such h exists because Ω is a Noetherian space). Then

$$\sum \operatorname{sgn} h(\omega)v_i(\omega) = \sum \operatorname{sgn} v_i(\omega)$$

for $\omega \in \Omega \setminus \Sigma$ and

$$\sum \operatorname{sgn} h(\omega)v_i(\omega) = 0$$

for $\omega \in \Sigma$.

Similarly, let $p_1, \dots, p_r \in \mathcal{A}(\Omega)$, then

$$\sum \operatorname{sgn} p_j(\omega) + \sum \operatorname{sgn}(-h(\omega)p_j(\omega)) = 0$$

for $\omega \in \Omega \setminus \Sigma$ and

$$\sum \operatorname{sgn} p_j(\omega) + \sum \operatorname{sgn}(-h(\omega)p_j(\omega)) = \sum \operatorname{sgn} p_j(\omega)$$

for $\omega \in \Sigma$.

So we have

$$\begin{aligned}&\sum \operatorname{sgn} h(\omega)v_i(\omega) + \sum \operatorname{sgn} p_j(\omega) + \sum \operatorname{sgn}(-h(\omega)p_j(\omega))\\ &= \begin{cases} \sum \operatorname{sgn} v_i(\omega), & \omega \in \Omega \setminus \Sigma \\ \sum \operatorname{sgn} p_j(\omega), & \omega \in \Sigma. \end{cases}\end{aligned}$$

THEOREM 4.5. — *Let $\{F_\omega\}_{\omega \in \Omega} \in \mathcal{S}_n(\Omega)$ and let $0 \in \mathbb{R}^n$ be isolated in $F_\omega^{-1}(0)$ for each $\omega \in \Omega$. Then there exist $v_1, \dots, v_s \in \mathcal{A}(\Omega)$ such that for $\omega \in \Omega$*

$$\deg_0 F_\omega = \operatorname{sgn} v_1(\omega) + \ldots + \operatorname{sgn} v_s(\omega).$$

Proof. — According to Lemma 4.4 there exist a proper closed subset $\Sigma_1 \subset \Omega$ and $u_1, \dots, u_{s(1)} \in \mathcal{A}(\Omega)$ such that for $\omega \in \Omega \setminus \Sigma_1$

$$\deg_0 F_\omega = \operatorname{sgn} u_1(\omega) + \ldots + \operatorname{sgn} u_{s(1)}(\omega).$$

Let $\Omega_1 = \Sigma_1$; using Lemma 4.4 again, we obtain $\Sigma_2 \subset \Sigma_1$ and $w_1, \dots, w_{s(2)} \in \mathcal{A}(\Omega)$ such that for $\omega \in \Omega_1 \setminus \Sigma_2$

$$\deg_0 F_\omega = \operatorname{sgn} w_1(\omega) + \ldots + \operatorname{sgn} w_{s(2)}(\omega).$$

Continuing this construction we obtain a descending family of proper closed subsets

$$\Omega \supset \Sigma_1 \supset \Sigma_2 \supset \ldots$$

Ω is a Noetherian space, so this family has to be finite and for some positive integer k we have $\Sigma_k = \emptyset$.

Now we apply the above fact and the proof is complete. □

Let us recall that if $f \in \mathcal{A}(\Omega)$ then we denote $\tilde{f} = \sum_\alpha \frac{1}{\alpha!} \mathrm{D}^\alpha f x^\alpha \in \mathcal{A}(\Omega)_c[[x]]$, and if $h = \sum_\alpha h_\alpha x^\alpha \in \mathcal{A}(\Omega)[[x]]$ then we denote $h_\omega = \sum_\alpha h_\alpha(\omega) x^\alpha$.

Let $\mathcal{F} \subset \mathcal{A}(\Omega)$. For each $\omega \in \Omega$ let $I_\omega \subset \mathbb{R}\{x\} = \mathbb{R}\{x_1, \ldots, x_n\}$ denote the ideal generated by $\left\{ \tilde{f}_\omega \mid f \in \mathcal{F} \right\}$, and let X_ω denote a representative of $V_0(I_\omega)$. We will show, that there exist $v_1, v_2, \ldots, v_s \in \mathcal{A}(\Omega)$ such that

$$\forall_{\omega \in \Omega}\ \exists_{0 < \epsilon_\omega \ll 1}\ \forall_{0 < \epsilon < \epsilon_\omega}\ \frac{1}{2} \chi(S^{n-1}_\epsilon \cap X_\omega) = \sum_{i=1}^s \operatorname{sgn} v_i(\omega),$$

where $S^{n-1}_\epsilon = \{x \in \mathbb{R}^n \mid ||x|| = \epsilon\}$ and $\chi(A)$ is the Euler characteristic of the set A.

Lemma 4.6. — *There exist $h_1, h_2, \ldots, h_q \in \mathcal{A}(\Omega)_c[[x]]$ such that for $\omega \in \Omega$*

$$X_\omega = V_0(h_{1,\omega}, \ldots h_{q,\omega}).$$

Proof. — Denote by I the ideal in $\mathcal{A}(\Omega)_c[[x]]$ generated by the set $\{\tilde{h} \mid h \in \mathcal{F}\}$. Theorem 1.1 implies that there is an ideal $I' = (h_1, \ldots, h_q) \subset \mathcal{A}(\Omega)_c[[x]]$ generated by finitely many elements such that

$$\forall_{\omega \in \Omega}\ I_\omega = I'_\omega,$$

where $I'_\omega = (h_{1,\omega}, \ldots, h_{q,\omega})$. We have

$$X_\omega = V_0(I_\omega) = V_0(I'_\omega) = V_0(h_{1,\omega}, \ldots, h_{q,\omega}).$$

□

Remark. — Since $\mathcal{A}(\Omega)$ is Noetherian, this lemma is clear for $\mathcal{A}(\Omega)$, but it is valid for any Ω–Noetherian algebra instead of $\mathcal{A}(\Omega)$.

Corollary 4.7. — *There exists $h = h_1^2 + \ldots + h_q^2 \in \mathcal{A}(\Omega)_c[[x]]$ such that $X_\omega = V_0(h_\omega)$ for each $\omega \in \Omega$.*

Now we will show that for any $h \in \mathcal{A}(\Omega)_c[[x]]$ such that $h(0) = 0$ there exists such $k > 0$ that for all $\omega \in \Omega$ there exists $\epsilon_\omega > 0$ such that

$$g_\omega(x) = h_\omega(x) - (x_1^2 + \ldots + x_n^2)^k$$

has an isolated critical point at the origin and for $0 < \epsilon < \epsilon_\omega$

$$\chi(S_\epsilon^{n-1} \cap \{h_\omega \leqslant 0\}) = 1 - \deg_0 \nabla g_\omega,$$

where $\nabla g_\omega = \left(\frac{\partial g_\omega}{\partial x_1}, \ldots, \frac{\partial g_\omega}{\partial x_n}\right) : (\mathbb{R}^n, 0) \longrightarrow (\mathbb{R}^n, 0)$.

We will strictly follow the proof of [14], Theorem 1.

Denote $r(x) = x_1^2 + \ldots + x_n^2$. Assume that h_ω, r are the representatives of germs defined on an open neighbourhood U of the origin. Define

$$V_\omega = \{(x, \epsilon, y) \in U \times \mathbb{R} \times \mathbb{R} \mid r(x) = \epsilon^2, \operatorname{rank}(\mathrm{d}\, r(x), \mathrm{d}\, h_\omega(x)) \leqslant 1, y = h_\omega(x)\}.$$

Let $\pi : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}$ be the projection. V_ω is an analytic set and $\pi : V_\omega \longrightarrow \pi(V_\omega)$ is proper in some neighbourhood of the origin. Hence $\pi(V_\omega)$ is closed and subanalytic in some neighbourhood of the origin.

Denote $Y_1 = \mathbb{R} \times \{0\}$, $Y_2^\omega = \overline{\pi(V_\omega) \setminus Y_1}$. Then Y_2^ω is subanalytic. If $\epsilon \neq 0$ then

$$\pi(V_\omega) \cap \{\epsilon\} \times \mathbb{R} = \{\epsilon\} \times \{\text{the set of critical values of } h_\omega|_{S_\epsilon^{n-1}}\}.$$

Since h_ω is analytic, $\pi(V_\omega) \cap \{\epsilon\} \times \mathbb{R}$ is finite. Hence $\dim \pi(V_\omega) = \dim Y_2^\omega = 1$, and then 0 is an isolated point of $Y_1 \cap Y_2^\omega$.

According to Corollary 3.2. there exists a constant $\alpha > 0$ such that for $\omega \in \Omega$

$$|y| = |h_\omega(x)| \geqslant ||x||^\alpha = \epsilon^\alpha$$

for each $(\epsilon, y) \in Y_2^\omega$ such that $\epsilon < \epsilon_\omega$ and y is sufficiently close to the origin. Let $k > \alpha$ be an integer. Define $g_\omega(x) = h_\omega(x) - r^k(x)$.

Set

$$V'_\omega = \{(x, \epsilon, y) \in U \times \mathbb{R} \times \mathbb{R} \mid r(x) = \epsilon^2, \operatorname{rank}(\mathrm{d}\, r(x), \mathrm{d}\, g_\omega(x)) \leqslant 1, y = g_\omega(x)\}.$$

Because $\operatorname{rank}(\mathrm{d}\, r(x), \mathrm{d}\, g_\omega(x)) = \operatorname{rank}(\mathrm{d}\, r(x), \mathrm{d}\, h_\omega(x))$,

$$V'_\omega = \{(x, \epsilon, y) \in U \times \mathbb{R} \times \mathbb{R} \mid r(x) = \epsilon^2, \operatorname{rank}(\mathrm{d}\, r(x), \mathrm{d}\, h_\omega(x)) \leqslant 1, \\ y = h_\omega(x) - \epsilon^{2k}\}.$$

Define $G(\epsilon, y) = (\epsilon, y - \epsilon^{2k})$. Then $\pi(V'_\omega) = G(\pi(V_\omega))$, so we have

$$\pi(V'_\omega) \cap \mathbb{R} \times \{0\} = \{(0, 0)\}$$

in some neighbourhood of the origin. Hence, if $\epsilon \neq 0$ is sufficiently close to the origin, 0 is a regular value of $g_\omega|_{S_\epsilon^{n-1}}$ and then g_ω has an isolated critical point at the origin.

According to [14], Lemma 1, we have

$$\chi(S_\epsilon^{n-1} \cap \{h_\omega \leqslant 0\}) = 1 - \deg_0 \nabla g_\omega.$$

Hence, applying Theorem 4.5 and the fact, that for $\omega \in \Omega$ and sufficiently small $\epsilon > 0$ if $h(0) \neq 0$ then $\chi(S^{n-1}_\epsilon \cap \{h_\omega \leqslant 0\})$ is equal to 0 or 2, we obtain:

THEOREM 4.8. — *If $f \in \mathcal{A}(\Omega)$ then there exist $v_1, v_2, \dots, v_s \in \mathcal{A}(\Omega)$ such that for $\omega \in \Omega$ there exists $0 < \epsilon_\omega \ll 1$ such that for $0 < \epsilon < \epsilon_\omega$*

$$\chi(S^{n-1}_{\omega,\epsilon} \cap \{f \leqslant 0\}) = \chi(S^{n-1}_\epsilon \cap \{\tilde{f}_\omega \leqslant 0\}) = \sum_{i=1}^s \operatorname{sgn} v_i(\omega),$$

where $S^{n-1}_{\omega,\epsilon}$ denotes a sphere in $\mathbb{R}^n$ centered at ω with the radius ϵ.

LEMMA 4.9. — *If $f \in \mathcal{A}(\Omega)$ then there exist $h_1, h_2, \dots, h_s \in \mathcal{A}(\Omega)$ such that for $\omega \in \Omega$ there exists $0 < \epsilon_\omega \ll 1$ such that for $0 < \epsilon < \epsilon_\omega$*

$$\frac{1}{2}(\chi(S^{n-1}_{\omega,\epsilon} \cap \{f \geqslant 0\}) \pm \chi(S^{n-1}_{\omega,\epsilon} \cap \{f \leqslant 0\}))$$
$$= \frac{1}{2}(\chi(S^{n-1}_\epsilon \cap \{\tilde{f}_\omega \geqslant 0\}) \pm \chi(S^{n-1}_\epsilon \cap \{\tilde{f}_\omega \leqslant 0\})) = \sum_{i=1}^s \operatorname{sgn} h_i(\omega).$$

Proof. — Let define $g(\omega, t) = tf(\omega)$, where ω belongs to some neighbourhood of Ω, $t \in [-1; 1]$. The set $\Omega \times [-1, 1]$ is compact and semianalytic, so $g \in \mathcal{A}(\Omega \times [-1, 1])$.

Then $g \geqslant 0$ if $f \geqslant 0$ and $t \geqslant 0$ or if $f \leqslant 0$ and $t \leqslant 0$. Hence for $t > 0$

$$\chi(S^{n-1}_{\omega,\epsilon} \cap \{f \geqslant 0\}) = 2 - \chi(S^n_{(\omega,t),\epsilon} \cap \{g \geqslant 0\})$$

and

$$\chi(S^{n-1}_{\omega,\epsilon} \cap \{f \leqslant 0\}) = 2 - \chi(S^n_{(\omega,-t),\epsilon} \cap \{g \geqslant 0\})$$

for ϵ sufficiently small.

According to Theorem 4.8 there exist $g_1, g_2, \dots, g_s$ in $\mathcal{A}(\Omega \times [-1; 1])$ such that

$$\forall_{(\omega,t)\in\Omega\times[-1;1]}\ \exists_{0<\epsilon_{(\omega,t)}\ll 1}\ \forall_{0<\epsilon<\epsilon_{(\omega,t)}}\ \chi(S^n_{(\omega,t),\epsilon} \cap \{g \geqslant 0\}) = \sum_{i=1}^s \operatorname{sgn} g_i(\omega, t).$$

For $0 < \epsilon < \epsilon_{(\omega,t)}$ we obtain

$$\begin{aligned}
&\frac{1}{2}(\chi(\{f \geqslant 0\} \cap S^{n-1}_{\omega,\epsilon}) - \chi(\{f \leqslant 0\} \cap S^{n-1}_{\omega,\epsilon}))\\
&= \frac{1}{2} \lim_{t\to 0^+} (2 - \chi(S^n_{(\omega,t),\epsilon} \cap \{g \geqslant 0\}) - 2 + \chi(S^n_{(\omega,-t),\epsilon} \cap \{g \geqslant 0\}))\\
&= \frac{1}{2} \lim_{t\to 0^+} (\chi(S^n_{(\omega,-t),\epsilon} \cap \{g \geqslant 0\}) - \chi(S^n_{(\omega,t),\epsilon} \cap \{g \geqslant 0\}))\\
&= \frac{1}{2} \lim_{t\to 0^+} \sum_{i=1}^s (\operatorname{sgn} g_i(\omega, -t) - \operatorname{sgn} g_i(\omega, t)).
\end{aligned}$$

Let $\Omega = D_1 \cup \ldots \cup D_m$ be the decomposition into irreducible components. Fix j. We can assume, that $g_i \not\equiv 0$ on $D_j \times [-1;1]$. For all $i = 1, 2, \ldots, s$ there exists $h_i \in \mathcal{A}(\Omega \times [-1;1])$ and a non–negative integer k_i such that $g_i(\omega, t) = t^{k_i} h_i(\omega, t)$, and $h_i \not\equiv 0$ on $\mathrm{D}_j \times \{0\}$. Let $\Sigma := \{\omega \in \mathrm{D}_j \mid \exists_{i=1,\ldots,s}\ h_i(\omega, 0) = 0\}$, then Σ is proper and closed subset of D_j. For $\omega \in \mathrm{D}_j \setminus \Sigma$

$$\frac{1}{2} \lim_{t \to 0^+} \sum_{i=1}^{s} (\operatorname{sgn} g_i(\omega, -t) - \operatorname{sgn} g_i(\omega, t)) = \sum_{i=1}^{s} \operatorname{sgn} h_i'(\omega),$$

where $h_i'(\omega) = -h_i(\omega, 0)$ if k_i is odd, and $h_i'(\omega) = 0$ if k_i is even. Obviously $h_i' \in \mathcal{A}(\Omega)$.

In the other hand

$$\begin{aligned}
&\frac{1}{2}(\chi(\{f \geqslant 0\} \cap S^{n-1}_{\omega,\epsilon}) + \chi(\{f \leqslant 0\} \cap S^{n-1}_{\omega,\epsilon}))\\
&= \frac{1}{2} \lim_{t \to 0^+} (2 - \chi(S^n_{(\omega,t),t} \cap \{g \geqslant 0\}) + 2 - \chi(S^n_{(\omega,-t),t} \cap \{g \geqslant 0\}))\\
&= \frac{1}{2} \lim_{t \to 0^+} (4 - \chi(S^n_{(\omega,-t),t} \cap \{g \geqslant 0\}) - \chi(S^n_{(\omega,t),t} \cap \{g \geqslant 0\}))\\
&= 2 - \frac{1}{2} \lim_{t \to 0^+} \sum_{i=1}^{s} (\operatorname{sgn} g_i(\omega, -t) + \operatorname{sgn} g_i(\omega, t)).
\end{aligned}$$

As above for $\omega \in \mathrm{D}_j \setminus \Sigma$

$$\frac{1}{2} \lim_{t \to 0^+} \sum_{i=1}^{s} (\operatorname{sgn} g_i(\omega, -t) + \operatorname{sgn} g_i(\omega, t)) = \sum_{i=1}^{s} \operatorname{sgn} h_i''(\omega),$$

where $h_i''(\omega) = h_i(\omega, 0)$ if k_i is even, and $h_i''(\omega) = 0$ if k_i is odd.

We have proven that $\frac{1}{2}(\chi(\{f \geqslant 0\} \cap S^{n-1}_{\omega,\epsilon}) \pm \chi(\{f \leqslant 0\} \cap S^{n-1}_{\omega,\epsilon}))$ is a sum of signs of analytic functions on $\mathrm{D}_j \setminus \Sigma$. As in proofs of Lemma 4.4 and Theorem 4.5, proceeding by induction we can complete the proof. □

Corollary 4.10. — *If $f \in \mathcal{A}(\Omega)$ then there exist $g_1, g_2, \ldots, g_q \in \mathcal{A}(\Omega)$ such that for $\omega \in \Omega$ there exists $0 < \epsilon_\omega \ll 1$ such that for each $0 < \epsilon < \epsilon_\omega$*

$$\frac{1}{2}\chi(S^{n-1}_{\omega,\epsilon} \cap V_0(f)) = \frac{1}{2}\chi(S^{n-1}_{\epsilon} \cap V_0(\tilde{f}_\omega)) = \sum_{i=1}^{q} \operatorname{sgn} g_i(\omega).$$

Proof. — We have

$$\chi(S^{n-1}_{\epsilon} \cap V_0(\tilde{f}_\omega)) = \chi(S^{n-1}_{\epsilon} \cap \{\tilde{f}_\omega \leqslant 0\}) + \chi(S^{n-1}_{\epsilon} \cap \{\tilde{f}_\omega \geqslant 0\}) - \chi(S^{n-1}_{\epsilon}),$$

so according to Lemma 4.9

$$\frac{1}{2}\chi(S_\epsilon^{n-1}\cap V_0(\tilde{f}_\omega)) = \sum_{i=1}^{s} h_i(\omega) - \frac{1+(-1)^{n-1}}{2}.$$

□

Corollary 4.7 and Corollary 4.10 imply:

THEOREM 4.11. — *There exist $v_1, v_2, \dots, v_q \in \mathcal{A}(\Omega)$ such that*

$$\forall_{\omega\in\Omega}\ \exists_{0<\epsilon_\omega\ll 1}\ \forall_{0<\epsilon<\epsilon_\omega}\ \frac{1}{2}\chi(S_\epsilon^{n-1}\cap X_\omega) = \sum_{i=1}^{q} \operatorname{sgn} v_i(\omega).$$

Remark. — Following the proof of the Lemma 4.9 one can check that this result is true also if instead of $\mathcal{A}(\Omega)$ we take any Ω–Noetherian algebra $\mathcal{O}(\Omega)$ (Ω is a locally closed subset of $\mathbb{R}^n$) such that:

1) there exists a subset $I \subset \mathbb{R}$ containing a neighbourhood of 0 such that $\mathcal{O}(\Omega\times I)$ is $\Omega\times I$–Noetherian and there is a natural inclusion $\mathcal{O}(\Omega)\subset\mathcal{O}(\Omega\times I)$.

2) For $g\in\mathcal{O}(\Omega\times I)$ and an irreducible component D of Ω if $g\not\equiv 0$ on $D\times I$ then there exist $h\in\mathcal{O}(\Omega\times I)$ and a non-negative integer k that $g(\omega,t) = t^k h(\omega,t)$ for $\omega\in D$ and t sufficiently close to 0, $h(\cdot,0)\in\mathcal{O}(\Omega)$, and $h\not\equiv 0$ on $D\times\{0\}$.

The algebra of Nash functions on an open semialgebraic set $\Omega\subset\mathbb{R}^n$ satisfies these assumptions.

For the algebra $\mathbb{R}[x][f_1,\dots,f_q]$ defined in the Introduction we can define the algebra $\mathbb{R}[x,t][F_1,\dots,F_q]$, where $F_i : \mathbb{R}^n\times[-1;1]\longrightarrow\mathbb{R}$, $F_i(x,t) = f_i(x)$. It is $\mathbb{R}^n\times[-1;1]$–Noetherian and $F_1,\dots,F_q$ do not depend on the last variable, so it has the property 2).

5. Sums of signs of real analytic functions.

Let $Y\subset\mathbb{R}^n$ be a real compact semianalytic set. Suppose that a function $\phi : Y\longrightarrow\mathbb{Z}$ admits a presentation as a finite sum

$$\phi = \sum_i m_i \mathbf{1}_{Y_i},$$

where the m_i's are integers, the Y_i's are semianalytic subsets of Y and where $\mathbf{1}_{Y_i}$ denotes the characteristic function of the subset Y_i.

We can choose Y_i such that they are compact semianalytic subsets of Y. Following [9] and [2] we define the Euler integral, the link of ϕ, and the duality operator D on ϕ:

$$\int_Y \phi = \sum_i m_i \chi(Y_i),$$

$$\Lambda\phi(y) = \int_Y \phi \mathbf{1}_{S^{n-1}_{y,\epsilon}},$$

where ϵ is sufficiently small,

$$\mathrm{D}\,\phi(y) = \phi(y) - \Lambda\phi(y).$$

Let Ω, as above, be a compact semianalytic subset of $\mathbb{R}^n$. We will say, that a function $g : \Omega \longrightarrow \mathbb{Z}$ is *a sum of signs of analytic functions* if there exist $v_1, v_2, \dots, v_s \in \mathcal{A}(\Omega)$ such that $g(\omega) = \sum_{i=1}^s \operatorname{sgn} v_i(\omega)$. Then in fact g is defined on a compact semianalytic neighbourhood Y of Ω. In that case, for $\omega \in \operatorname{int} Y \supset \Omega$ we have:

$$\Lambda g(\omega) = \int_Y g \mathbf{1}_{S^{n-1}_{\omega,\epsilon}} = \int_{S^{n-1}_{\omega,\epsilon}} g = \sum_{i=1}^s \big(\chi(A_i \cap S^{k-1}_{\omega,\epsilon}) - \chi(B_i \cap S^{k-1}_{\omega,\epsilon})\big)$$

where $A_i = \{v_i \geqslant 0\}$, $B_i = \{v_i \leqslant 0\}$, ϵ is sufficiently small.

Using Theorem 4.11, Lemma 4.9, and arguments like in [12], Corollary 6.3 and Theorem 6.4, we can show similar results as the main result of [4].

Suppose that f is an analytic function defined in a neighbourhood of Ω. Then $X = f^{-1}(0)$ is an analytic set defined in a neighbourhood of Ω.

According to Theorem 4.11, there exist $v_1, v_2, \dots, v_q \in \mathcal{A}(\Omega)$ such that for each $\omega \in \Omega$ there exists $0 < \epsilon_\omega \ll 1$ such that for each $0 < \epsilon < \epsilon_\omega$, $\frac{1}{2}\chi(S^{n-1}_\epsilon \cap X_\omega) = \sum_{i=1}^q \operatorname{sgn} v_i(\omega)$. Let $\Omega = \Omega_1 \cup \dots \Omega_m$ be a decomposition into irreducible components. Assume that v_i does not vanish identically on Ω_1 for $i = 1, \dots, l \leqslant q$. Taking $v = v_1 v_2 \dots v_l$ and $\Sigma = \{\omega \in \Omega_1 \mid v(\omega) = 0\} \cup \bigcup_{i=2}^m \Omega_i$ we obtain:

Corollary 5.1. — *There exist a proper closed subset $\Sigma \subset \Omega$, an integer $\mu = l - 1$, and an analytic function $v \in \mathcal{A}(\Omega)$, such that v does not vanish on $\Omega \setminus \Sigma$ and*

$$\forall_{\omega\in\Omega\setminus\Sigma}\ \exists_{0<\epsilon_\omega\ll 1}\ \forall_{0<\epsilon<\epsilon_\omega}\ \frac{1}{2}\chi(S^{n-1}_\epsilon \cap X_\omega) = \mu + \operatorname{sgn} v(\omega)\ (mod\ 4).$$

In particular, for such ω, $\frac{1}{2}\chi(S^{n-1}_\epsilon \cap X_\omega) = \mu + 1$ (mod 2).

Theorem 5.2. — *If $g : \Omega \longrightarrow \mathbb{Z}$ is a sum of signs of analytic functions $v_1, v_2, \dots, v_s \in \mathcal{A}(\Omega)$ (in particular if $g(\omega) = \frac{1}{2}\chi(S^{n-1}_\epsilon \cap X_\omega)$),* *then the function* $\frac{1}{2}\Lambda g$, *as well as* $\frac{1}{2}(g + \mathrm{D}\, g)$, *is integer–valued and it is a sum of signs of analytic functions.*

Proof. — We have

$$\Lambda g(\omega) = \sum_{i=1}^{s} \big(\chi(\{v_i(\omega) \geqslant 0\} \cap S^{n-1}_{\omega,\epsilon}) - \chi(\{v_i(\omega) \leqslant 0\} \cap S^{n-1}_{\omega,\epsilon})\big)$$

for ϵ sufficiently small, so the theorem is implied by Lemma 4.9. □

So, proceeding the same way as McCrory and Parusiński in [10] one may get a large family of topological invariants associated with $\Omega \subset X$.

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Aleksandra NOWEL,
University of Gdańsk
Institute of Mathematics
Wita Stwosza 57
80–952 Gdańsk (Poland)
Aleksandra.Nowel@math.univ.gda.pl