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REGULARITY PROPERTIES OF THE EQUILIBRIUM DISTRIBUTION

by Hans WALLIN

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1. Let R^m, $m \geqslant 1$, be the m-dimensional Euclidean space with points $x = (x^1, \ldots, x^m)$. It is well known that the equilibrium distribution belonging to a compact set F and the kernel r^{2-m} if $m > 2$ and the kernel $-\log r$ if $m = 2$ is concentrated on the boundary of F. This is no longer true if the interior of F is non-empty and if we, instead of r^{2-m} or $-\log r$, consider the kernel $r^{-(m-\alpha)}$ where $0 < \alpha < 2$ if $m \geqslant 2$ and $0 < \alpha < 1$ if $m = 1$. In fact, since $|x|^{-(m-\alpha)}$ in this case is a strictly subharmonic function of x when $x \neq 0$ it is easy to prove that the support of the equilibrium distribution μ_α^F contains every interior point of F, a fact which is also a consequence of the theorems below.

We shall here give some properties of μ_α^F in the interior of F and examine its behaviour near the boundary ∂F of F when $0 < \alpha < 2$ if $m \geqslant 2$ and $0 < \alpha < 1$ if $m = 1$. We intend to prove that the restriction of μ_α^F to the interior of F is absolutely continuous and has a density which is analytic and may be expressed by an explicit formula [the Theorems 1 and 2] and which, when we approach ∂F, tends to infinity in the same way as the distance to ∂F raised to the power $-\frac{\alpha}{2}$, if certain conditions of regularity are satisfied [the Theorems 1, 3 and 4].

The methods of the proofs will be based on the sweeping-out process and a kind of inversion formula, [see the formula (11) below]. The formula (6) for the energy integral is related to (11). A formula similar to (6) has been used by Beurling and Beurling-Deny on several occasions. Compare [2] and [5]. Beurling has also indicated the usefulness of an inversion formula of the type of (11) for the treatment of the problem considered in this paper. Finally it may be noted that some of the statements of Theorem 1 which we shall deduce by means of the sweeping-out process can be obtained from the formula (11) too.

2. We introduce some notations and definitions. For an arbitrary set E we denote the complement by $\complement E$ and the interior by $\mathring{E}$. F is a compact set with boundary ∂F, μ a positive measure with compact support S_μ and α a number satisfying $0 < \alpha < 2$ if $m \geqslant 2$ and $0 < \alpha < 1$ if $m = 1$. $S(x_0, r)$ is the sphere determined by the set of points x which satisfy $|x - x_0| \leqslant r$.

The α-potential of μ is denoted by u_α^μ and defined by

$$u_\alpha^\mu(x) = \int \frac{1}{|x-y|^{m-\alpha}}\, d\mu(y),$$

and the energy integral of order α of μ is denoted by $I_\alpha(\mu)$,

$$I_\alpha(\mu) = \iint \frac{1}{|x-y|^{m-\alpha}}\, d\mu(y)\, d\mu(x).$$

Here and elsewhere, the integration is to be extended over the whole space, if no limits of integration are indicated. The α-capacity of a bounded Borel set E, $C_\alpha(E)$, is defined as

$$C_\alpha(E) = \{\inf I_\alpha(\nu)\}^{-1},$$

where the infimum is taken over the class of all positive measures ν with total mass 1 and $S_\nu \subset E$.

If $C_\alpha(F) > 0$ we denote the equilibrium distribution of F belonging to the kernel $r^{-(m-\alpha)}$ by μ_α^F and the equilibrium potential by u_α^F. We assume throughout the paper that μ_α^F is normalized so that $\mu_\alpha^F(R^m) = 1$. ν_α^F is the restriction of μ_α^F to $\mathring{F}$. We shall prove below that ν_α^F is absolutely continuous; the density of ν_α^F is denoted by f_α^F. We put $v_\alpha(F) = \{C_\alpha(F)\}^{-1}$. M denotes different constants.

3. The following lemma is a consequence of the sweeping-out process.

LEMMA 1. — *Let* F_1 *and* F_2 *be two compact sets with* $F_1 \subset F_2$, $C_\alpha(F_1) > 0$. *Then, for every Borel set* E *with* $E \subset F_1$,

$$\mu_\alpha^{F_1}(E) \geqslant \mu_\alpha^{F_2}(E). \tag{1}$$

Proof. — Let τ_1 and τ_2 be the restrictions of $\mu_\alpha^{F_2}$ to F_1 and $\complement F_1$ respectively. Then there exists a positive measure τ_2^* with $\tau_2^*(R^m) \leqslant \tau_2(R^m)$, $S_{\tau_2^*} \subset F_1$, such that $u_\alpha^{\tau_2^*}(x) = u_\alpha^{\tau_2}(x)$ for every $x \in F_1$ except on a subset of F_1 of α-capacity zero and $u_\alpha^{\tau_2^*}(x) \leqslant u_\alpha^{\tau_2}(x)$ everywhere. Since the α-potential of $\mu_\alpha^{F_2}$ is constant on F_1 except on a subset of F_1 of α-capacity zero, we have

$$\mu_\alpha^{F_1} = \{\tau_1(R^m) + \tau_2^*(R^m)\}^{-1}(\tau_1 + \tau_2^*). \tag{2}$$

$\tau_2^*(R^m) \leqslant \tau_2(R^m)$ and (2) give, if E is a Borel set, $E \subset F_1$,

$$\mu_\alpha^{F_1}(E) \geqslant \tau_1(E) = \mu_\alpha^{F_2}(E),$$

which proves the lemma.

4. By means of Lemma 1 we prove the following theorem:

THEOREM 1. — *Let* F *be a compact set such that* $\mathring{F}$ *is non-empty. Then* ν_α^F *is absolutely continuous and* f_α^F — *properly defined on a set of Lebesgue measure zero — is bounded from below by a positive constant on* $\mathring{F}$.

Let x_0 *be a boundary point of* F *belonging to the boundary of a sphere* $S(x_1, r_1)$ *such that* $S(x_1, r_1) \subset F$ *and let* $V(x_0)$ *be a bounded right circular cone with vertex at* x_0, *the line through* x_0 *and* x_1 *as axis and* $V(x_0) \subset S(x_1, r_1)$. *Then* [1]

$$\limsup f_\alpha^F(x).|x - x_0|^{\frac{\alpha}{2}} \leqslant M < \infty,\ x \to x_0,\ x \in V(x_0), \tag{3}$$

where M *is a constant which may be chosen only depending on* α, m, r_1 *and the generating angle of* $V(x_0)$ *and where* f_α^F *is properly defined on a set of Lebesgue measure zero.*

Proof. — According to a result by M. Riesz ([7], p. 16) the equilibrium distribution of the sphere $S_2 = S(x_2, r_2)$ belonging

[1] $V(x_0)$ is a line segment when $m = 1$.

to the kernel $r^{-(m-\alpha)}$ is absolutely continuous with density

$$(4) \quad f_\alpha^{S_2}(x) = M\cdot(r_2^2 - |x-x_2|^2)^{-\frac{\alpha}{2}}, \qquad |x-x_2| < r_2,$$

for a certain constant M depending on α, m and r_2.

We choose x_2 and r_2 such that $S_2 \subset F$ and use Lemma 1 with $F_1 = S_2$ and $F_2 = F$. If E is a Borel set, $E \subset S_2$, we obtain

$$0 \leqslant \mu_\alpha^F(E) \leqslant \int_E f_\alpha^{S_2}(x)\, dx,$$

which shows that the restriction of μ_α^F to S_2 is absolutely continuous. Hence ν_α^F is absolutely continuous. The inequality also proves that if f_α^F is properly defined on a set of Lebesgue measure zero, then

$$(5) \qquad f_\alpha^F(x) \leqslant f_\alpha^{S_2}(x) \qquad \text{for every} \qquad x \in S_2.$$

From (4) and (5) we conclude, by an elementary calculation, that (3) is true.

To show that f_α^F is bounded from below by a positive constant in the interior of F we choose a sphere S_3, $S_3 \supset F$, and use Lemma 1 with $F_1 = F$ and $F_2 = S_3$. This proves the theorem.

5. We now prove the following formula : *if*

$$A(\alpha, m) = 2^{-2}\pi^{-(m+1)}\cdot\alpha\cdot\sin\frac{\pi\alpha}{2}\cdot\Gamma\left(\frac{m-\alpha}{2}\right)\cdot\Gamma\left(\frac{m+\alpha}{2}\right),$$

then

$$(6) \quad I_\alpha(\mu) = A(\alpha, m)\iint \frac{|u_\alpha^\mu(x+y) - u_\alpha^\mu(x)|^2}{|y|^{m+\alpha}}\, dy\, dx,$$

in the sense that if one member is finite then the other member is finite too and the equality holds true (²).

We shall prove (6) by means of the Fourier transformation. Let $\hat{T} = \mathcal{F}T$ denote the Fourier transform of a temperated distribution T normed so that

$$\hat{f}(\xi) = \int e^{-2\pi i(x,\xi)} f(x)\, dx, \qquad (x, \xi) = \sum_1^m x^j \xi^j,$$

(²) The formula (6) is proved for the sake of completeness and due to its independent interest in spite of the fact that it is not indispensable for our purpose.

if f is in the Lebesgue class $L^1(R^m)$. Since u_α^μ is a convolution,

$$u_\alpha^\mu = |x|^{-(m-\alpha)} * \mu,$$

and (Schwartz [8], p. 113)

$$\mathcal{F}|x|^{-(m-\alpha)} = A_1(\alpha, m)|x|^{-\alpha}, \qquad A_1(\alpha, m) = \frac{\pi^{\frac{m}{2}-\alpha} \cdot \Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{m-\alpha}{2}\right)},$$

we have, in the distribution sense,

$$\hat{u}_\alpha^\mu = A_1(\alpha, m) \cdot |x|^{-\alpha} \cdot \hat{\mu}, \tag{7}$$

which shows that $\hat{u}_\alpha^\mu$ is a function which is absolutely integrable over every compact set. We also have (cf. Cartan-Deny [4] and Deny [6]):

$$I_\alpha(\mu) = A_1(\alpha, m) \int |\xi|^{-\alpha} |\hat{\mu}(\xi)|^2 \, d\xi. \tag{8}$$

According to the Parseval relation we get

$$\begin{aligned} \int \frac{dy}{|y|^{m+\alpha}} \int |u_\alpha^\mu(x+y) - u_\alpha^\mu(x)|^2 \, dx &= \int \frac{dy}{|y|^{m+\alpha}} \int |\hat{u}_\alpha^\mu(\xi)|^2 \cdot |e^{2\pi i(y,\xi)} - 1|^2 \, d\xi \\ &= \int |\hat{u}_\alpha^\mu(\xi)|^2 \left\{ \int \frac{|e^{2\pi i(y,\xi)} - 1|^2}{|y|^{m+\alpha}} \, dy \right\} d\xi. \end{aligned}$$

A substitution in the integral shows that

$$\int \frac{|e^{2\pi i(y,\xi)} - 1|^2}{|y|^{m+\alpha}} \, dy = A_2(\alpha, m) \cdot |\xi|^\alpha, \tag{9}$$

where $A_2(\alpha, m)$ is a constant which can be calculated explicitly (cf. [1], p. 402):

$$A_2(\alpha, m) = \frac{2\pi^{(m+2+2\alpha)/2}}{\Gamma\left(\frac{\alpha+2}{2}\right)\Gamma\left(\frac{m+\alpha}{2}\right)\sin\frac{\pi\alpha}{2}} \text{ (3)}.$$

(3) It may be noted that an easy calculation proves that the left member of (9) is infinite if $\alpha \geqslant 2$ or $\alpha \leqslant 0$.

An application of (7) now yields

$$\iint \frac{|u_\alpha^\mu(x+y) - u_\alpha^\mu(x)|^2}{|y|^{m+\alpha}}\, dy\, dx$$
$$= A_2(\alpha, m) \cdot \{A_1(\alpha, m)\}^2 \cdot \int |\xi|^{-\alpha} |\hat{\mu}(\xi)|^2\, d\xi.$$

From this formula and (8) we conclude that (6) is valid.

6. Let φ be a function, defined everywhere in R^m, which is infinitely differentiable and has a compact support and suppose that $I_\alpha(\mu) < \infty$. By means of Schwarz's inequality and (6) we get

$$(10) \qquad \iint \frac{|u_\alpha^\mu(x+y) - u_\alpha^\mu(x)| \cdot |\varphi(x+y) - \varphi(x)|}{|y|^{m+\alpha}}\, dy\, dx < \infty.$$

We are going to prove that

$$(11) \qquad \int \overline{\varphi(x)}\, d\mu(x)$$
$$= A(\alpha, m) \iint \frac{(u_\alpha^\mu(x+y) - u_\alpha^\mu(x))\, (\overline{\varphi(x+y)} - \overline{\varphi(x)})}{|y|^{m+\alpha}}\, dy\, dx,$$

where $A(\alpha, m)$ *is the constant occurring in* (6). We observe that the right member of (11) is absolutely convergent according to (10).

For a fixed y we introduce the functions ν_y and ψ_y by

$$\nu_y(x) = u_\alpha^\mu(x+y) - u_\alpha^\mu(x), \qquad \psi_y(x) = \varphi(x+y) - \varphi(x).$$

Since ν_y defines a temperated distribution we obtain from the definition of the Fourier transform:

$$\int \nu_y(x)\overline{\psi_y(x)}\, dx = \int \hat{\nu}_y(\xi)\overline{\hat{\psi}_y(\xi)}\, d\xi = \int \hat{u}_\alpha^\mu(\xi)\overline{\hat{\varphi}(\xi)}\, |e^{2\pi i(y,\xi)} - 1|^2\, d\xi.$$

If we divide the first and the last members of this formula by $|y|^{m+\alpha}$ and integrate over y the first member is — except for the constant $A(\alpha, m)$ — transformed into the right member of (11); the last member becomes by means of (7) and (9), after simplification,

$$A_1(\alpha, m) A_2(\alpha, m) \int \overline{\hat{\varphi}(\xi)}\hat{\mu}(\xi)\, d\xi,$$

where $A_1(\alpha, m)$ and $A_2(\alpha, m)$ are the constants occurring in (7) and (9). By using the fact that

$$\int \overline{\varphi(x)}\, d\mu(x) = \int \overline{\hat{\varphi}(\xi)}\, \hat{\mu}(\xi)\, d\xi,$$

we finally obtain that (11) holds true.

7. We now use (11) to study the equilibrium distribution μ_α^F of a compact set F with non-empty interior.

Suppose that $x_0 \in \mathring{F}$. Choose r such that $S(x_0, r)$ is a subset of $\mathring{F}$ and let ψ be the characteristic function of $S(x_0, r)$. Let $\{\varphi_n\}$ be a uniformly bounded sequence of real- valued infinitely differentiable functions with supports in a prescribed neighborhood G of $S(x_0, r)$ such that

$$\lim_{n \to \infty} \varphi_n(x) = \psi(x), \quad \text{for every } x.$$

If we choose the closure of G as a subset of $\mathring{F}$ and use (11) with φ replaced by φ_n and μ by μ_α^F we obtain from Lebesgue's theorem of dominated convergence, when n tends to infinity, that (11) is true also with φ replaced by ψ and μ by μ_α^F. By a substitution in the integral this can be written

$$\int_{S(x_0, r)} f_\alpha^F(x)\, dx = A(\alpha, m) \iint \frac{(u_\alpha^F(y) - u_\alpha^F(x))\,(\psi(y) - \psi(x))}{|y - x|^{m+\alpha}}\, dx\, dy,$$

where f_α^F as usual denotes the density of the restriction of μ_α^F to $\mathring{F}$. Since ψ is the characteristic function of $S(x_0, r)$ we can simplify this formula and get

$$(12) \qquad \int_{S(x_0, r)} f_\alpha^F(x)\, dx = 2A(\alpha, m) \int_{S(x_0, r)} dy \int \frac{u_\alpha^F(y) - u_\alpha^F(x)}{|x - y|^{m+\alpha}}\, dx.$$

The fact that u_α^F is constant in $\mathring{F}$ means that the function g defined by

$$g(y) = \int \frac{u_\alpha^F(y) - u_\alpha^F(x)}{|y - x|^{m+\alpha}}\, dx$$

is continuous and even analytic in $\mathring{F}$. By dividing both members of (12) by the Lebesgue measure of $S(x_0, r)$ and letting r tend to zero we consequently obtain, if f_α^F is suitably defined on a set of Lebesgue measure zero,

$$f_\alpha^F(x_0) = 2A(\alpha, m)\int \frac{u_\alpha^F(x_0) - u_\alpha^F(x)}{|x_0 - x|^{m+\alpha}}\, dx.$$

This formula shows that f_α^F is analytic in $\mathring{F}$. As $u_\alpha^F(x_0) = \nu_\alpha(F)$, where $\nu_\alpha(F) = \{C_\alpha(F)\}^{-1}$, we may sum up the results in the following way:

THEOREM 2. — *Let* F *be a compact set with interior points. If the density* f_α^F *of the restriction to* $\mathring{F}$ *of the equilibrium distribution of order* α *of* F *is properly defined on a set of Lebesgue measure zero, then* f_α^F *is an analytic function in* $\mathring{F}$ *which is given by*

$$f_\alpha^F(x) = 2A(\alpha, m)\int \frac{\nu_\alpha(F) - u_\alpha^F(y)}{|x - y|^{m+\alpha}}\, dy \tag{13}$$

for every $x \in \mathring{F}$.

From now on we assume that f_α^F is defined and satisfies (13) for all $x \in \mathring{F}$.

8. In order to be able to use (13) to get an estimate of f_α^F near ∂F we have to study $\nu_\alpha(F) - u_\alpha^F(y)$ when y is a point of $\complement F$ which is situated near ∂F.

Suppose that the interior of the sphere $S(x_1, r_1)$ belongs to $\complement F$ and that y is an interior point of $S(x_1, r_1)$. The result of sweeping the measure consisting of the mass 1 at the point y onto the closure S^* of $\complement S(x_1, r_1)$ is, according to a result by M. Riesz, ([7], p. 17) the measure $\lambda_y(x)\, dx$, where

(14)

$$\lambda_y(x) = A_3(\alpha, m)\, (r_1^2 - |y - x_1|^2)^{\frac{\alpha}{2}} . (|x - x_1|^2 - r_1^2)^{-\frac{\alpha}{2}} |y - x|^{-m},$$
$$|y - x_1| < r_1, \qquad x \in S^*,$$

and

$$A_3(\alpha, m) = \pi^{-\left(\frac{m}{2}+1\right)} . \Gamma\left(\frac{m}{2}\right) \sin\frac{\pi\alpha}{2}.$$

We have

$$\int_{S^*} \lambda_y(x)\, dx = 1, \tag{15}$$

which may, for instance, be proved in the following way: if $y = x_1$ (15) easily follows by the introduction of polar coordinates and a direct integration. If $y \neq x_1$ we introduce the sphere $S(y, r_0)$ and choose r_0 so that $S(y, r_0) \supset S(x_1, r_1)$. The measure $\tau_y(x)\, dx$ which we obtain by sweeping the measure consisting of the mass 1 at the point y onto the closure of $\complement S(y, r_0)$ has total mass 1, according to the above, and it may be obtained from $\lambda_y(x)\, dx$ by sweeping to the closure of $\complement S(y, r_0)$ the restriction to $S(y, r_0)$ of $\lambda_y(x)\, dx$.

As the total mass does not increase by the sweeping-out process, we get

$$1 = \int_{\complement S(y, r_0)} \tau_y(x)\, dx \leqslant \int_{S^*} \lambda_y(x)\, dx \leqslant 1,$$

which proves (15) when $y \neq x_1$.

The measure $\lambda_y(x)\, dx$ can be used to express the value of the potential u_α^F at a point of the interior of $S(x_1, r_1)$ by means of the values of u_α^F in S^* ([7], p. 17):

$$u_\alpha^F(y) = \int_{S^*} u_\alpha^F(x) \lambda_y(x)\, dx \qquad \text{if} \qquad |y - x_1| < r_1.$$

This formula and (15) give together

$$(16) \quad \nu_\alpha(F) - u_\alpha^F(y) = A_3(\alpha, m)\,(r_1^2 - |y - x_1|^2)^{\frac{\alpha}{2}} \cdot \int_{S^*} (\nu_\alpha(F) - u_\alpha^F(x))\,(|x - x_1|^2 - r_1^2)^{-\frac{\alpha}{2}} \,.\, |y - x|^{-m}\, dx$$

if $|y - x_1| < r_1$.

9. By means of (13) and (16) it is easy to prove the following theorem.

THEOREM 3. — *Let x_0 belong to ∂F and the closure of $\mathring{F}$. Suppose that there exists a closed sphere S with $\mathring{S} \subset \complement F$ such that x_0 is a boundary point of S. Then*

$$\liminf f_\alpha^F(x) \,.\, |x - x_0|^{\frac{\alpha}{2}} > 0, \qquad x \to x_0,\ x \in \mathring{F}.$$

Proof. — Let V be a bounded right circular cone having vertex at x_0, altitude r_0, axis along the normal at x_0 of the boundary of S and being contained, except the point x_0, in $\mathring{S}$. Suppose that S has center at the origin and radius r_1, $S = S(0, r_1)$. There exists a constant $M > 0$, [cf. the formula (24) below], such that

$$r_1 - |y| \geqslant M \cdot |y - x_0| \qquad \text{for} \qquad y \in V.$$

This and (16) give, with a new constant $M > 0$ which depends on r_1,

$$v_\alpha(F) - u_\alpha^F(y) \geqslant M \cdot |y - x_0|^{\frac{\alpha}{2}}, \qquad y \in V.$$

Remembering (13) this gives, when $x \in \mathring{F}$, with constants $M > 0$,

$$f_\alpha^F(x) > M \int_V \frac{|y - x_0|^{\frac{\alpha}{2}}}{(|x - x_0| + |x_0 - y|)^{m+\alpha}} dy$$
$$\geqslant |x - x_0|^{-\frac{\alpha}{2}} \cdot M \int_0^a \frac{t^{\frac{\alpha}{2}}}{(1 + t)^{m+\alpha}} \cdot t^{m-1} \, dt,$$

where $a = r_0 \cdot |x - x_0|^{-1}$. This proves the theorem.

10. If we suppose that F satisfies certain conditions of regularity in a neighborhood of a fix boundary point x_0, then the Theorems 1 and 3 show that the expression

$$f_\alpha^F(x) |x - x_0|^{\frac{\alpha}{2}}, \tag{17}$$

takes values between two strictly positive constants when x tends to x_0 inside some cone contained in F and having vertex at x_0. We shall treat the problem to examine under what conditions the limit of (17) exists when x tends to x_0. To keep the calculations comparatively simple we shall be content with the following theorem:

THEOREM 4. — *Let x_0 be a boundary point of* F. *Suppose that there exist positive numbers r_0 and δ_0 such that for every $t_0 \in S(x_0, r_0) \cap \partial F$ we can find two closed spheres with radii δ_0 — spheres which we denote by $S^i(t_0, \delta_0)$ and $S^e(t_0, \delta_0)$ — which have t_0 as a boundary-point, the same tangent plane at t_0 and are such that $S^i(t_0, \delta_0)$ is contained in* F *and the interior of $S^e(t_0, \delta_0)$ in $\complement$* F. *Let $N(x_0)$ be the common normal of $S^i(x_0, \delta_0)$ and $S^e(x_0, \delta_0)$.*

Then

$$(18) \quad \lim f_\alpha^F(x) . |x - x_0|^{\frac{\alpha}{2}}, \qquad x \to x_0, \qquad x \in N(x_0) \cap F,$$

exists and is strictly positive and finite.

The fact that the limit is strictly positive and finite if it exists, is an immediate consequence of the Theorems 1 and 3. The limit depends on m, α, F and the position of x_0 in a way that will appear from the proof [cf. the end of § 14]. When $m = 1$ the assumption shall be interpreted to mean that r_0 and δ_0 may be chosen such that $S(x_0, r_0) \cap \partial F = \{x_0\}$, $S^i(x_0, \delta_0) \cap S^e(x_0, \delta_0) = \{x_0\}$, $S^i(x_0, \delta_0)$ is a subset of F and the interior of $S^e(x_0, \delta_0)$ of $\complement F$.

We start the proof of Theorem 4 with some preliminary considerations after which the proof is completed in §§ 11-14 using the same notations in all the paragraphs.

Let φ_0 be a number satisfying $0 < \varphi_0 < \frac{\pi}{2}$ and let K be the infinite, two-sided cone of revolution — including the interior of the cone — with vertex at x_0 axis $N(x_0)$ and generating angle φ_0 [see Figure 1]. The plan of the proof is as follows. Using (13) we shall estimate $f_\alpha^F(x)$, $x \in N(x_0) \cap F$. By means of the results of § 11 we show in § 12 that u_α^F satisfies a Lipschitz condition at x_0 which is then used to prove that the contribution to the integral occurring in (13), coming from the integration over $\complement K$ may be neglected if φ_0 is chosen near $\frac{\pi}{2}$. In §§ 13-14 we estimate the contribution to (13) coming from the integration over K, a contribution which consequently determines the limit (18).

We carry through the proof of the theorem only for the case $m \geqslant 2$. However, when $m = 1$ a proof follows almost immediately from (13) and (16) depending on the fact that in this case the complement of $F \cup S^e(x_0, \delta_0)$ is situated at a positive distance from x_0. Thus, from now on we assume that $m \geqslant 2$. We also suppose that $\delta_0 < r_0/2$ which we clearly may without limitation. This will guarantee that all points from ∂F with which we shall be concerned are situated in $S(x_0, r_0)$.

The part of ∂F which is situated in $S(x_0, r_0)$ has the following properties :

1° There exists a constant c such that if $y_0 \in \partial F \cap S(x_0, r_0)$ and φ is the angle between $N(x_0)$ and the common normal of $S^i(y_0, \delta_0)$ and $S^e(y_0, \delta_0)$, then

$$\varphi < c|x_0 - y_0|.$$

2° If r_1 is strictly positive and less than a certain number — which may be chosen equal to min $\{\delta_0/2, \pi/4c\}$ where c is the constant in 1° — then, for every $t \in S(x_0, r_1/2)$, the intersection between $\partial F \cap S(x_0, r_1)$ and the line through t parallel to $N(x_0)$ consists of exactly one point, t_0.

1° is proved in [9], p. 112, for $m = 3$ and a certain class of surfaces without being stated explicitly. However, the same proof is valid for a general m and with the assumptions we have made. To prove 2° we suppose that $r_1 < \min\{\delta_0/2, \pi/4c\}$ and let l be the line through t, $t \in S(x_0, r_1/2)$, which is parallel to $N(x_0)$. $l \cap S(x_0, r_1)$ contains points both from the interior of $S^i(x_0, \delta_0)$ and the interior of $S^e(x_0, \delta_0)$, i.e. points both from the interior of F and from $\complement F$ and, consequently, also at least one point t_0 from ∂F. But the angle between l and the common normal at t_0 of $S^i(t_0, \delta_0)$ and $S^e(t_0, \delta_0)$ is, according to 1°, less than $\frac{\pi}{4}$ as $cr_1 < \frac{\pi}{4}$. This means that at least every point from $l \cap S(t_0, \delta_0)$, except t_0, is situated in the interior of $S^i(t_0, \delta_0)$ or the interior of $S^e(t_0, \delta_0)$. Hence

$$l \cap \partial F \cap S(t_0, \delta_0) = \{t_0\}.$$

The distance from x_0 to a point from $l \cap \partial F$, different from t_0, is thus larger than

$$\delta_0 - |t_0 - x_0| \geqslant \delta_0 - r_1 > r_1,$$

which shows that

$$l \cap \partial F \cap S(x_0, r_1) = \{t_0\}$$

and so 2° is proved.

It is clear that ∂F has a unique tangent plane at every point of $\partial F \cap S(x_0, r_0)$. $N(x_0)$ is the normal of ∂F at x_0.

11. We start the proper proof of Theorem 4 by deducing an upper bound of f_α^F in a neighborhood of x_0. Let t be a fixed point belonging to $\dot{F} \cap S(x_0, \delta_0)$. Let $t'_0 \in \partial F$ be such that $|t - t'_0| = d(t, \partial F)$, where $d(t, \partial F)$ denotes the distance between t and ∂F. The facts that $|t - x_0| \leqslant \delta_0$ and $\delta_0 < r_0/2$ imply that $t'_0 \in \partial F \cap S(x_0, r_0)$ and consequently that $S^i(t'_0, \delta_0) \subset F$. As the line through t'_0 and t contains the center of $S^i(t'_0, \delta_0)$ we may use § 4 to conclude that there exists a constant M, only depending on m, α and δ_0, such that

$$(19) \quad f^F_\alpha(t) \leqslant M\{d(t, \partial F)\}^{-\frac{\alpha}{2}} \quad \text{for all} \quad t \in S(x_0, \delta_0) \cap \dot{F}.$$

(19) will be used to prove the existence of a constant M such that if $\mu^F_\alpha(x_0, r)$ is the value of μ^F_α for the sphere $S(x_0, r)$, then

$$(20) \qquad \mu^F_\alpha(x_0, r) \leqslant Mr^{\frac{2m-\alpha}{2}}, \qquad \text{for all} \qquad r > 0.$$

It is enough to prove (20) for all r less than an arbitrarily chosen positive number. We suppose that $2r < \min\{\delta_0/2, \pi/4c\}$, where c is the constant in 1°. Let $t \in F \cap S(x_0, r)$. According to 2° the intersection between $\partial F \cap S(x_0, 2r)$ and the line l through t parallel to $N(x_0)$ consists of exactly one point t_0. The angle between l and the normal of ∂F at t_0 is less than $\pi/4$ according to 1°. Combined with

$$|t - t_0| \leqslant |t - x_0| + |x_0 - t_0| \leqslant r + 2r < \frac{3\delta_0}{4},$$

this gives that t belongs to the right circular cone which is contained in $S^i(t_0, \delta_0)$, has axis along the normal to ∂F at t_0, vertex at t_0, altitude $\frac{3\delta_0}{4}$ and generating angle $\pi/4$. As the distance from t to the boundary of $S^i(t_0, \delta_0)$ is less than or equal to $d(t, \partial F)$, we conclude, [cf. the formula (24) below], that there exists a number $M > 0$, only depending on δ_0, such that

$$(21) \qquad d(t, \partial F) \geqslant M|t - t_0|, \qquad t \in F \cap S(x_0, r).$$

In order to estimate $\mu^F_\alpha(x_0, r)$ we suppose for a moment that the coordinate-system is chosen with the origin at x_0 and the x^1-axis along $N(x_0)$. (19) and (21) give then

$$\nu^F_\alpha(x_0, r) = \int_{F \cap S(x_0, r)} f^F_\alpha(t)\, dt \leqslant M \int_{S(x_0, r)} |t - t_0|^{-\frac{\alpha}{2}}\, dt$$

and if we evaluate the last integral by means of repeated one-dimensional integration, we obtain

$$\nu_\alpha^{\mathrm{F}}(x_0, r) \leqslant \mathrm{M}r^{m-1}. \int_{-2r}^{2r} |x^1|^{-\frac{\alpha}{2}} dx^1 = \mathrm{M}r^{\frac{2m-\alpha}{2}}.$$

(20) now follows from this estimate and the following lemma:

LEMMA 2. — *Let the assumptions in Theorem 4 be satisfied. Then*

$$\mu_\alpha^{\mathrm{F}}(\partial\mathrm{F} \cap \mathrm{S}(x_0, r)) = 0, \qquad \textit{for some} \qquad r > 0.$$

Proof of Lemma 2. — If r is small enough the following discussion is valid for all sufficiently small ε. Let $t_0 \in \partial\mathrm{F} \cap \mathrm{S}(x_0, r)$. For a fixed δ_1, $0 < \delta_1 < \delta_0$, we introduce the closed sphere $\mathrm{S}^i(t_0, \delta_1)$ which is a subset of $\mathrm{S}^i(t_0, \delta_0)$, has radius δ_1 and t_0 as a boundary point. Let $\mathrm{S}_\varepsilon(t_0, \delta_1)$, $\varepsilon > 0$, be the translation the distance ε of $\mathrm{S}^i(t_0, \delta_1)$ along the outer normal of $\partial\mathrm{F}$ at t_0. The union of the interior $\mathring{\mathrm{S}}_\varepsilon(t_0, \delta_1)$ of all the spheres $\mathrm{S}_\varepsilon(t_0, \delta_1)$, $t_0 \in \partial\mathrm{F} \cap \mathrm{S}(x_0, r)$, covers $\partial\mathrm{F} \cap \mathrm{S}(x_0, r)$. We choose a finite subcover of open spheres $\mathring{\mathrm{S}}_\varepsilon(t_0, \delta_1)$ and let $\mathrm{K}(\varepsilon)$ be the closure of the union of these spheres. Let $\mathrm{F}(\varepsilon) = \mathrm{K}(\varepsilon) \cup \mathrm{F}$. Then $\mu_\alpha^{\mathrm{F}(\varepsilon)}$ does not distribute any mass on $\partial\mathrm{F} \cap \mathrm{S}(x_0, r)$. As we, except for a constant factor, obtain μ_α^{F} from $\mu_\alpha^{\mathrm{F}(\varepsilon)}$ by sweeping to F the restriction to $\complement\mathrm{F}$ of $\mu_\alpha^{\mathrm{F}(\varepsilon)}$, it is clearly enough to prove that

$$\mu_\alpha^{\mathrm{F}(\varepsilon)}\left(\complement\mathrm{F}\right) \to 0 \qquad \text{when} \qquad \varepsilon \to 0.$$

However, by another application of the sweeping-out process we can realize that $\mu_\alpha^{\mathrm{F}(\varepsilon)}(\partial\mathrm{F}(\varepsilon)\setminus\partial\mathrm{F}) = 0$. In fact, if this was not the case we could, due to the construction of $\mathrm{K}(\varepsilon)$, find a closed sphere $\mathrm{S} \subset \mathrm{K}(\varepsilon)$ with $\mu_\alpha^{\mathrm{F}(\varepsilon)}(\partial\mathrm{S}) > 0$ and by sweeping to S the restriction to $\complement\mathrm{S}$ of $\mu_\alpha^{\mathrm{F}(\varepsilon)}$ we would get $\mu_\alpha^{\mathrm{S}}(\partial\mathrm{S}) > 0$ which is wrong [cf. the formula (4)]. By combining the above facts we obtain that if $\mathrm{G}(\varepsilon) = \mathring{\mathrm{F}}(\varepsilon)\setminus\mathrm{F}$ it is enough to prove that

$$\mu_\alpha^{\mathrm{F}(\varepsilon)}(\mathrm{G}(\varepsilon)) \to 0 \qquad \text{when} \qquad \varepsilon \to 0.$$

Let l be a line parallel to $\mathrm{N}(x_0)$ such that $l \cap \mathrm{G}(\varepsilon)$ is non-empty. We obserwe that if $\mathrm{S}_\varepsilon(t_0, \delta_1) \subset \mathrm{K}(\varepsilon)$ then we have, as $\delta_1 < \delta_0$

and $S^i(t_0, \delta_0) \subset F$, that the part of $S_\varepsilon(t_0, \delta_1)$ which belongs to $\complement F$ is situated inside any chosen, fixed neighborhood of t_0. This means that if r_1 is a fixed number chosen as indicated in 2° in § 10 then $l \cap S(x_0, r_1/2)$ is non-empty and, according to 2°, that the intersection between l and $\partial F \cap S(x_0, r_1)$ consists of exactly one point y_0. It also means that there exists at least one point $y_1 \in l \cap (\partial F(\varepsilon) \setminus \partial F)$. y_1 is situated on the boundary of at least one sphere $S_\varepsilon(t_1, \delta_1) \subset K(\varepsilon)$, $t_1 \in \partial F \cap S(x_0, r)$. Finally it means that the angle between the normal $n(y_1)$ of $S_\varepsilon(t_1, \delta_1)$ at y_1 and the normal of ∂F at t_1 is arbitrarily small and hence, according to the property 1° in § 10, also that the angle between $n(y_1)$ and l (or $N(x_0)$) is less than any prescribed, fixed positive number. Accordingly we get that $y_0 \in V(y_1)$ where $V(y_1) \subset S_\varepsilon(t_1, \delta_1)$ is a bounded right circular cone with vertex at y_1 and axis $n(y_1)$ having an altitude only depending on δ_1 and a generating angle which is an absolute constant. Simple arguments now prove that y_1 is the only intersection between l and $\partial F(\varepsilon) \setminus \partial F$, that there exists a function h, defined on the positive numbers, such that $h(\varepsilon)$ tends to zero when ε tends to zero and

$$|y_0 - y_1| \leqslant h(\varepsilon),$$

and, finally, that there exists a number M not depending on y_0, y_1 and ε such that (cf. § 4) for all y lying on the line-segment between y_0 and y_1,

$$f_\alpha^{F(\varepsilon)}(y) \leqslant M|y - y_1|^{-\alpha/2}.$$

If for a moment we suppose that the coordinate-system is chosen with the x^1-axis along $N(x_0)$ and use the estimates obtained we get by repeated one-dimensional integration

$$\mu_\alpha^{F(\varepsilon)}(G(\varepsilon)) \leqslant Mr^{m-1} \cdot \int_0^{h(\varepsilon)} |x^1|^{-\alpha/2}\, dx^1,$$

which tends to zero when ε tends to zero. The proof of the lemma is complete.

Remark. — Lemma 2 gives us a general class of sets F such that $\mu_\alpha^F(\partial F) = 0$.

12. We need the following lemma:

LEMMA 3. — *μ is a positive measure with compact support.*

Suppose that, for some point x_1 in R^m and for some β, $0<\beta<1$,

$$\mu(x_1, r) \leqslant \text{const. } r^{m-\alpha+\beta}, \qquad \textit{for all} \qquad r>0.$$

Then

$$u_\alpha^\mu(x_1) - u_\alpha^\mu(x) \leqslant \text{const. } |x_1 - x|^\beta, \qquad \textit{for all} \qquad x \in R^m.$$

Remark. — We have in this paper throughout assumed that $0<\alpha<2$, but the lemma is true for all α satisfying $0<\alpha\leqslant m$. It is also true for $m=1$.

The lemma has been proved by Carleson ([3], pp. 15-16) for $\alpha=2$ in a somewhat different form. It is, however, possible to use his method of proof also for a general α and in the form we have formulated the lemma.

The formula (20) and Lemma 3, used with $x_1 = x_0$ and $\mu = \mu_\alpha^F$, give, as $u_\alpha^F(x_0) = \nu_\alpha(F)$,

$$(22) \qquad \nu_\alpha(F) - u_\alpha^F(y) \leqslant \text{const. } |x_0 - y|^{\frac{\alpha}{2}} \quad \text{for all } y.$$

Put

$$B(\nu_0) = \limsup |x - x_0|^{\frac{\alpha}{2}} \cdot \int_{\complement K} \frac{\nu_\alpha(F) - u_\alpha^F(y)}{|x-y|^{m+\alpha}}\, dy,$$
$$x \to x_0, \qquad x \in N(x_0) \cap F.$$

Using (22) we shall prove that

$$(23) \qquad \lim_{\nu_0 \to \pi/2} B(\nu_0) = 0.$$

To have a suitable reference in § 14 we give a detailed proof of (23). For an arbitrary $y \neq x_0$, let θ be the angle between a vector directed along the outer normal to ∂F at x_0 and the vector from x_0 to y [see figure 1]. If $x \in S^i(x_0, \delta_0) \cap N(x_0)$ we have

$$|x-y|^2 = |x-x_0|^2 + 2|x-x_0|\,.\,|x_0-y|\cos\theta + |x_0-y|^2.$$

Using (22) and introducing polar coordinates $(r, \theta, \theta^1, \ldots, \theta^{m-2})$, $r = |x_0 - y|$, in the integral occurring in the expression $B(\nu_0)$ we obtain, if $x \in S^i(x_0, \delta_0) \cap N(x_0)$,

$$0 \leqslant B(\nu_0)$$
$$\leqslant M\cdot\limsup_{x\to x_0}|x-x_0|^{\frac{\alpha}{2}}\int_0^\infty\int_{\nu_0}^{\pi-\nu_0}\frac{r^{\frac{\alpha}{2}}}{[r^2+2r|x-x_0|\cos\theta+|x-x_0|^2]^{(m+\alpha)/2}}$$
$$\cdot\, r^{m-1}\sin^{m-2}\theta\, dr\, d\theta,$$

and putting $r = |x - x_0| \cdot t$ we get the majorant $M(\pi - 2\varphi_0)$ when $\varphi_0 \to \pi/2$. This proves (23).

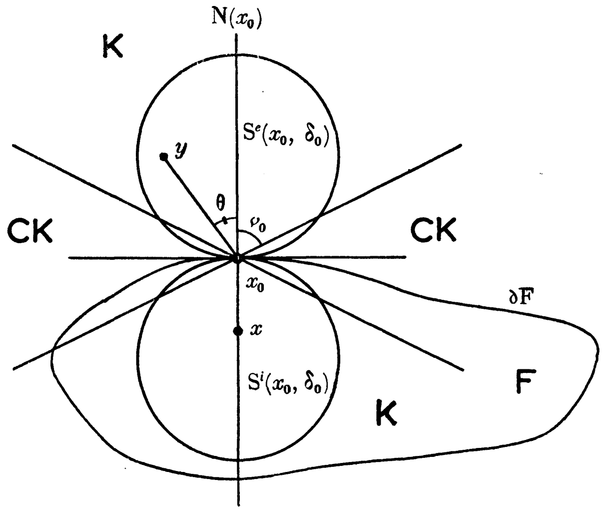

FIG. 1.

13. The object of this paragraph is to put $\varphi_\alpha(F) - u_\alpha^F(y)$, $y \in K \cap \complement F$, in a form [see (26)] which is suitable for the final estimation of f_α^F.

We suppose from now on that the coordinate system is chosen so that the origin is the center of the sphere $S^e(x_0, \delta_0)$, i.e. $S^e(x_0, \delta_0) = S(0, \delta_0)$ and $|x_0| = \delta_0$. Simple geometric considerations show that, for a fixed φ_0,

$$(24) \quad \lim \frac{|x_0| - |y|}{|x_0 - y| \cdot \cos \theta} = 1, \qquad y \to x_0, \qquad y \in K \cap S(0, \delta_0),$$

where θ is defined immediately after the formula (23).

If we introduce the notation

$$J(y) = \int_{|x| > \delta_0} (\varphi_\alpha(F) - u_\alpha^F(x)) \, (|x|^2 - \delta_0^2)^{-\frac{\alpha}{2}} \cdot |y - x|^{-m} \, dx,$$

for all y, then we have, according to (16),

$$(25) \quad \varphi_\alpha(F) - u_\alpha^F(y) = A_3(\alpha, m) \, (|x_0|^2 - |y|^2)^{\frac{\alpha}{2}} \cdot J(y) \quad \text{for} \quad |y| < \delta_0.$$

Using (25), (24) and (22) we find

$$\limsup J(y) < \infty, \qquad y \to x_0, \qquad y \in K \cap S(0, \delta_0).$$

However, by a standard argument,

$$\limsup \mathrm{J}(y) \geqslant \mathrm{J}(x_0), \qquad y \to x_0, \qquad y \in \mathrm{K} \cap \mathrm{S}(0, \delta_0)$$

and accordingly the integral $\mathrm{J}(x_0)$ is convergent. From (24) it follows that the integrand of $\mathrm{J}(y)$ is majorized by a constant times the integrand of the convergent integral $\mathrm{J}(x_0)$ when $y \in \mathrm{K} \cap \mathrm{S}(0, \delta_0)$ and $|y - x_0|$ is small enough. Hence, by Lebesgue's theorem of dominated convergence,

$$\lim \mathrm{J}(y) = \mathrm{J}(x_0), \qquad y \to x_0, \qquad y \in \mathrm{K} \cap \mathrm{S}(0, \delta_0).$$

This, combined with (25) and (24), proves the existence of a function η such that $\eta(y) \to 0$ when $y \to x_0$, $y \in \mathrm{K} \cap \mathrm{S}(0, \delta_0)$ and

$$(26) \quad \varphi_\alpha(\mathrm{F}) - u_\alpha^{\mathrm{F}}(y) = \mathrm{A}_3(\alpha, m) \cdot (2\delta_0)^{\frac{\alpha}{2}} \cdot (|x_0 - y| \cos \theta)^{\frac{\alpha}{2}} \cdot \mathrm{J}(x_0) + |x_0 - y|^{\frac{\alpha}{2}} \cdot \eta(y), \qquad y \in \mathrm{K} \cap \mathrm{S}(0, \delta_0).$$

14. We finish the proof of Theorem 4 by means of (13), (23) and (26). Suppose that $x \in \mathrm{S}^i(x_0, \delta_0) \cap \mathrm{N}(x_0)$ and let ρ_0 be a positive number.

$$f_\alpha^{\mathrm{F}}(x) \cdot |x - x_0|^{\frac{\alpha}{2}} = 2\mathrm{A}(\alpha, m) \cdot |x - x_0|^{\frac{\alpha}{2}} \int \frac{\varphi_\alpha(\mathrm{F}) - u_\alpha^{\mathrm{F}}(y)}{|x - y|^{m+\alpha}}\, dy$$
$$= 2\mathrm{A}(\alpha, m)|x - x_0|^{\frac{\alpha}{2}} \left\{ \int_{\complement \mathrm{K}} + \int_{\mathrm{K} \cap \mathrm{S}(x_0, \rho_0)} + \int_{\mathrm{K} \cap \complement \mathrm{S}(x_0, \rho_0)} \right\} = \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

According to (23) I becomes arbitrarily small when x tends to x_0 if we choose φ_0 sufficiently near $\pi/2$. When φ_0 has been fixed we choose ρ_0 so small that

$$\{\mathrm{K} \cap \mathrm{S}(x_0, \rho_0)\} \setminus \mathrm{F} \subset \mathrm{S}(0, \delta_0).$$

We observe that we only have to integrate over $\complement \mathrm{F}$ in the integrals above. When we use (26) the second term of the right member of (26) gives a contribution to II which has the following majorant: $\mathrm{M} \cdot \sup \eta(y)$ where the supremum is taken over all $y \in \{\mathrm{K} \cap \mathrm{S}(x_0, \rho_0)\} \setminus \mathrm{F}$ and where the constant M does not depend on φ_0. This is proved by calculations which are analogous to those of the proof of (23). Hence, the contribution to II which comes from the second term of the right member of (26) is small, independently of x, if ρ_0 is small. For a fixed ρ_0, III tends to zero when x tends to x_0 and accordingly it only remains to estimate the contribution to II which, using (26), comes from the first term of the right member of (26). This contribution is, if

$$A_4 = 2A(\alpha, m)\cdot A_3(\alpha, m)\,(2\delta_0)^{-\frac{\alpha}{2}}\cdot J(x_0)$$

and

$$R(\rho_0) = \{K \cap S(x_0, \rho_0)\}\setminus F,$$

$$A_4\cdot|x - x_0|^{\frac{\alpha}{2}}\cdot\int_{R(\rho_0)}\frac{|x_0 - y|^{\frac{\alpha}{2}}\cdot\cos^{\frac{\alpha}{2}}\theta}{|x - y|^{m+\alpha}}\,dy.$$

This becomes after simplification if we introduce polar coordinates in the same way as in the proof of (23) and if

$$A_5 = 2A_4\cdot\pi^{\frac{m-1}{2}}\cdot\left\{\Gamma\left(\frac{m-1}{2}\right)\right\}^{-1} \quad \text{and} \quad \rho = \rho_0\cdot|x - x_0|^{-1},$$

$$A_5\int_0^{\rho}\int_0^{\varphi_0}\frac{t^{\frac{\alpha}{2}}\cos^{\frac{\alpha}{2}}\theta}{(t^2 + 2t\cos\theta + 1)^{(m+\alpha)/2}}\cdot t^{m-1}\sin^{m-2}\theta\,dt\,d\theta.$$

If φ_0 was chosen sufficiently near $\pi/2$ this expression is, when $|x - x_0|$ is small enough, arbitrarily near a certain constant. Together with the other estimates in this paragraph this shows that the limit (18) exists.

Added 8/10/65. Prof. L. Hörmander has informed me that the fact that f_α^F is infinitely differentiable in $\dot{F}$ also may be obtained as a consequence of results for certain general classes of operators.

BIBLIOGRAPHY

[1] N. Aronszajn et K. T. Smith, Theory of bessel potentials, Part I. *Ann. Inst. Fourier Grenoble*, 11, (1961), 385-475.

[2] A. Beurling, Ensembles exceptionnels, *Acta Math.*, 72, (1940), 1-13.

[3] L. Carleson, Removable singularities of continuous harmonic functions in R^m, *Math. Scand.*, 12, (1963), 15-18.

[4] H. Cartan et J. Deny, Le principe du maximum en théorie du potentiel et la notion de fonction surharmonique, *Acta Sci. Math. Szeged*, 12, (1950), 81-100.

[5] J. Deny, Sur les espaces de Dirichlet, *Séminaire de théorie du potentiel* (Paris), 1, No 5, (1957), 14 pages.

[6] J. DENY, Sur la définition de l'énergie en théorie du potentiel, *Ann. Inst. Fourier, Grenoble*, 2, (1950), 83-99.
[7] M. RIESZ, Intégrales de Riemann-Liouville et potentiels, *Acta Sci. Math. Szeged*, 9, (1938), 1-42.
[8] L. SCHWARTZ, Théorie des distributions, II, Paris, (1951).
[9] M. TSUJI, On the boundary value of a harmonic function in space, *Japanese J. of Math.*, 19, (1944), 111-137.

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