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DETERMINATION OF THE PLURIPOLAR HULL OF GRAPHS OF CERTAIN HOLOMORPHIC FUNCTIONS

by Armen EDIGARIAN & Jan WIEGERINCK

1. Introduction.

Let f be a holomorphic function on its *domain of existence* $D \subset \mathbb{C}$ and let Γ_f be its graph in $D \times \mathbb{C}$. Answering a question of Levenberg, Martin and Poletsky [6], we showed in [2] that it is possible that Γ_f is *not* a complete pluripolar subset of $\mathbb{C}^2$, but that the pluripolar hull of Γ_f is strictly larger than Γ_f. In a subsequent paper [3] we studied the pluripolar hull $(\Gamma_f)^*_{D_0}$ relative to a domain D_0 in the following setup: $D \subset D_0$ are domains in $\mathbb{C}$, $K = D_0 \setminus D$ is a closed polar subset of D, and $z \in K$. We showed that a necessary and sufficient conditions for $\{z\} \times \mathbb{C} \cap (\Gamma_f)^*_{D_0 \times \mathbb{C}} = \emptyset$ is that z be a regular boundary point for the Dirichlet problem on $D_M = \{\zeta \in D_0 : |f(\zeta)| < M\}$.

In the present paper we continue our study of $(\Gamma_f)^*_{D_0 \times \mathbb{C}}$. Our main results in that direction are Theorem 5.10 and Theorem 5.11 in Section 5, stating that if $(\{z\} \times \mathbb{C}) \cap (\Gamma_f)^*_{D_0 \times \mathbb{C}}$ is not empty, then it consists of exactly one point. Thus a complete description is obtained of the pluripolar hulls of graphs of holomorphic functions that have a polar singularity set.

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As an important tool we introduce in this section the notion of *interior values* of holomorphic mappings. These give rise to non-trivial points in the pluripolar hull of graphs of holomorphic mappings. In the one-dimensional case we show that the interior values of f – if they exist – are unique and coincide with the value of a *distinguished homomorphism* as introduced by Gamelin and Garnett, cf. [4]. In [2, 3] we gave a sufficient condition for graphs of holomorphic functions to have a non-trivial pluripolar hull; Theorem 3.4 provides the natural generalization to pluripolar sets.

As a preparation we study in Section 2 pluriharmonic measure and extend work of Levenberg and Poletsky, [7], as well as some results in [3] on this topic. Noteworthy is Theorem 2.3, which leads rapidly to the just mentioned Theorem 3.4. As one may expect, knowledge of pluriharmonic measure can be translated to pluripolar hulls. This is done in Section 3.

In Section 4 we prove a localization principle for pluriharmonic measure. This turns out to be strong enough to explain qualitatively Siciak's [13] extension of our example in [2] of a holomorphic function $f \in A^\infty(\mathbb{D})$ with domain of existence the unit disc $\mathbb{D}$, which has $(\Gamma_f)^*$ extending over most of $\mathbb{C}$. We also show that the pluripolar hull of a connected F_σ-pluripolar set is connected; this may be of independent interest.

Throughout the paper $\mathbb{B}(a,r)$ denotes the ball in $\mathbb{C}^n$, centered at a with radius r.

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2. Pluriharmonic measure.

Let Ω be an open set in $\mathbb{C}^n$ and let $E \subset \overline{\Omega}$ be any subset. The *pluriharmonic measure* of E relative to Ω (or, *relative extremal function*) is defined as follows (see e.g. [5])

$$(2.1)\qquad \omega(z,E,\Omega) = -\sup\{u(z) : u \in \mathrm{PSH}(\Omega), u \leqslant 0 \text{ on } \Omega,$$
$$\text{and } \limsup_{\Omega\ni w\to\zeta} u(w) \leqslant -1 \text{ for } \zeta \in E\}, \quad z \in \Omega.$$

Note that the function $-\omega$ need not be in PSH, but if E is open then $-\omega \in \mathrm{PSH}(\Omega)$.

Let $h : \Omega_1 \to \Omega_2$ be holomorphic. A straightforward consequence of the definition is, see [7],

$$\omega(z, h^{-1}(E), \Omega_1) \leqslant \omega(h(z), E, \Omega_2), \quad z \in \Omega_1, E \subset \Omega_2. \tag{2.2}$$

For a subset E of $\mathbb{C}^n$ and for a $\delta > 0$ we put

$$E_\delta = \{z \in \mathbb{C}^n : \mathrm{dist}\,(\mathrm{z}, \mathrm{E}) < \delta\}.$$

PROPOSITION 2.1. — *Let $\Omega' \Subset \Omega$ be open sets in $\mathbb{C}^n$ and let $V \Subset U \subset \Omega$ be open subsets. Fix a $\zeta_0 \in \overline{\Omega'}$. Then there exists a neighborhood W of ζ_0 such that*

$$\sup_{z \in W \cap \Omega'} \omega(z, V \cap \Omega', \Omega') \leqslant \omega(\zeta_0, U, \Omega). \tag{2.3}$$

Proof. — There exists an $\varepsilon > 0$ such that $V_\varepsilon \subset U$ and that $\Omega'_\varepsilon \subset \Omega$. Put $W = \mathbb{B}(\zeta_0, \varepsilon)$. Then $V - w \subset U$ for any $w \in \mathbb{B}(0, \varepsilon)$. So $\omega(\zeta_0 + w, V \cap \Omega', \Omega') = \omega(\zeta_0, V \cap \Omega' - w, \Omega' - w) \leqslant \omega(\zeta_0, U, \Omega)$ for any $w \in \mathbb{B}(0, \varepsilon)$ such that $\zeta_0 + w \in W \cap \Omega'$. □

Recall the following very useful result (see [7])

PROPOSITION 2.2. — *Let Ω be an open set in $\mathbb{C}^n$ and let $E \subset \Omega$ be any subset. Then*

$$\omega(z, E, \Omega) = \inf\{\omega(z, U, \Omega) : U \subset \Omega \textit{ is open and } E \subset U\}. \tag{2.4}$$

Combination of the above Propositions yields immediately the following.

THEOREM 2.3. — *Let $\Omega' \Subset \Omega$ be open sets in $\mathbb{C}^n$ and let $E \subset \Omega'$ be a compact subset. Assume that a sequence $\{z_n\}_{n=1}^\infty \subset \Omega'$ tends to $\zeta_0 \in \overline{\Omega'}$. Then*

$$\limsup_{n \to \infty} \omega(z_n, E_{\frac{1}{n}}, \Omega') \leqslant \omega(\zeta_0, E, \Omega). \tag{2.5}$$

Proof. — Indeed, from Proposition 2.1 we have

$$\limsup_{n\to\infty} \omega(z_n, E_{\frac{1}{n}}, \Omega') \leqslant \omega(\zeta_0, E_{\frac{1}{m}}, \Omega) \tag{2.6}$$

for any fixed m. Now apply Proposition 2.2 to obtain (2.5). □

Recall the following result (see e.g. [5], Corollary 4.5.11).

THEOREM 2.4. — *Let D be a hyperconvex domain in $\mathbb{C}^n$ and let $K \subset D$ be a compact set. Then $\omega(\cdot, K, D)$ is upper semi-continuous.*

As a corollary of Theorem 2.3 we next present a variant of Theorem 2.4 that gives a little less than upper semi-continuity, but is valid for arbitrary open sets in $\mathbb{C}^n$.

COROLLARY 2.5. — *Let $\Omega' \Subset \Omega$ be open sets in $\mathbb{C}^n$ and let $K \subset \Omega'$ be a compact subset. Then for any $\zeta_0 \in \overline{\Omega'}$ we have*

$$\limsup_{z\to\zeta_0} \omega(z, K, \Omega') \leqslant \omega(\zeta_0, K, \Omega). \tag{2.7}$$

Proof. — Note that $\omega(z, K, \Omega') \leqslant \omega(z, K_{\frac{1}{n}}, \Omega')$ for any $n \in \mathbb{N}$. □

Using similar methods we give an alternative proof of a result of N. Levenberg and E. Poletsky.

COROLLARY 2.6 [Levenberg-Poletsky [7]]. — *Let $\Omega' \Subset \Omega$ be open sets in $\mathbb{C}^n$ and let $E \subset \Omega'$ be a compact subset. Assume that $V \subset \Omega \setminus E$ is an open set and that $\zeta_0 \in V \cap \Omega'$. Put $K = \overline{(\partial V) \cap \Omega'}$. Then there exists a $\zeta \in K$ such that*

$$\omega(\zeta_0, E, \Omega') \leqslant \omega(\zeta, E, \Omega). \tag{2.8}$$

Proof. — Fix n so large that $E_{\frac{1}{n}} \subset \Omega'$. We claim that there exists a sequence $\{z_n\} \subset \partial V \cap \Omega'$ with $\omega(z_n, E_{\frac{1}{n}}, \Omega') > \omega(\zeta_0, E, \Omega') - \frac{1}{n}$.

Indeed, assume that for every $z \in \partial V \cap \Omega'$ we have

$$\omega(z, E_{\frac{1}{n}}, \Omega') \leqslant \omega(\zeta_0, E, \Omega') - \frac{1}{n}. \tag{2.9}$$

Because E is open, the function $-\omega(\cdot, E_{\frac{1}{n}}, \Omega')$ is plurisubharmonic. Therefore

$$v(z) = \begin{cases} -\omega(z, E_{\frac{1}{n}}, \Omega'), & \text{for } z \in \Omega' \setminus V, \\ \max\{-\omega(z, E_{\frac{1}{n}}, \Omega'), -\omega(\zeta_0, E, \Omega') + \frac{1}{n}\}, & \text{for } z \in V \cap \Omega', \end{cases}$$

is in PSH (Ω'). We have $v \leqslant 0$ on Ω', $v \leqslant -1$ on E. Hence,

$$-\omega(\zeta_0, E, \Omega') \geqslant v(\zeta_0) \geqslant -\omega(\zeta_0, E, \Omega') + \frac{1}{n},$$

a contradiction.

The conclusion is that there exists a subsequence $\{z_{n_k}\}$ converging to $\zeta \in \overline{\partial V \cap \Omega'}$ such that

$$\omega(\zeta, E, \Omega) \geqslant \limsup_{k\to\infty} \omega(z_{n_k}, E_{\frac{1}{n_k}}, \Omega') \geqslant \omega(\zeta_0, E, \Omega'). \tag{2.10}$$

□

The next theorem is very important in our theory. It extends Theorem 3.7 in [3].

THEOREM 2.7. — *Let D be a bounded open set in $\mathbb{C}$ and let $\Delta \Subset D$ be a closed disc. Assume that $K \subset \partial D$ is a compact polar set. Then for any $z_0 \in K$ we have*

$$\limsup_{\zeta\to z_0} \omega(\zeta, \Delta, D) = \inf\{\omega(z_0, \Delta, D \cup U) : K \subset U \text{ open}\}.$$

In particular, if $z_0 \in K$ is a regular boundary point of D then

$$\inf\{\omega(z_0, \Delta, D \cup U) : K \subset U \text{ open}\} = 0.$$

Proof. — Observe that $\leqslant$ is evident. For the inequality $\geqslant$ we take an open neighborhood U of K and note that for every $0 < \varepsilon \leqslant 1$ the set

$$F_U^\varepsilon = \{z \in \overline{D \cup U} : \limsup_{\zeta\to z} \omega(\zeta, \Delta, D \cup U) \geqslant \varepsilon\}$$

is a compact connected subset of $\overline{D \cup U}$ that contains Δ. Moreover, if $U_1 \subset U_2$ then $F_{U_1}^\varepsilon \subset F_{U_2}^\varepsilon$. We set $F^\varepsilon = \cap_U F_U^\varepsilon$. Then F^ε is a compact connected subset of $\overline{D}$.

Now let

$$\limsup_{D\ni\zeta\to z_0}\omega(\zeta,\Delta,D)=\alpha\geqslant 0. \tag{2.11}$$

As $F^\varepsilon\cap\partial D$ is a subset of the union of K and the irregular boundary points of D, the set $F^\varepsilon\cap\partial D$ is thin at z_0 and therefore totally disconnected.

To reach a contradiction, suppose that

$$\inf\{\omega(z_0,\Delta,D\cup U):K\subset U\text{ open}\}=\varepsilon>\alpha,$$

then $z_0\in F^\varepsilon$.

For any decreasing sequence $\{U_i\}$ of neighborhoods of K with $K=\cap U_i$, the functions $\omega(z,\Delta,D\cup U_i)$ form a decreasing sequence of harmonic functions on $D\setminus\Delta$, and hence converge uniformly on compact sets in $D\setminus\Delta$. The limit function is $\omega(z,\Delta,D)$ and hence $\omega(z,\Delta,D)\geqslant\varepsilon$ on $F^\varepsilon\cap D$. In view of (2.11) we infer that there is a neighborhood V of z_0 such that $F^\varepsilon\cap V\subset\partial D$. Thus z_0 is not in the component of Δ, which is a contradiction. □

3. Properties of pluripolar hulls.

We commence by recalling two important definitions. Let Ω be an open set in $\mathbb{C}^n$ and let $E\subset\overline{\Omega}$ be a pluripolar subset. The *pluripolar hull* of E in Ω is defined as

$$E^*_\Omega=\{z\in\Omega:\text{ for all }u\in\mathrm{PSH}\,(\Omega):u|_E=-\infty\Longrightarrow u(z)=-\infty\}. \tag{3.1}$$

For a pluripolar set E in an open set Ω in $\mathbb{C}^n$, Levenberg and Poletsky define the *negative pluripolar hull of* E in Ω as

$$E^-_\Omega:=\{z\in\Omega:\text{ for all }u\in\mathrm{PSH}\,(\Omega),u\leqslant 0:u|_E=-\infty\Longrightarrow u(z)=-\infty\}.$$

We extend the above definition to arbitrary pluripolar sets $E\subset\mathbb{C}^n$ as follows

$$E^-_\Omega=\cap_{U\supset E\text{ open}}E^-_{\Omega\cup U}.$$

We will use the following two important results from [7].

THEOREM 3.1. — *Let Ω be an open set in $\mathbb{C}^n$ and let E be a pluripolar set in Ω. Then*

$$E_\Omega^- = \{z \in \Omega : \omega(z, E, \Omega) > 0\}.$$

THEOREM 3.2. — *Let Ω be a pseudoconvex domain and let $E \subset \Omega$ be pluripolar. Suppose that $\Omega = \cup_j \Omega_j$, where $\Omega_j \subset \Omega_{j+1}$ form an increasing sequence of relatively compact pseudoconvex subdomains of Ω. Then*

$$E_\Omega^* = \cup_{j=1}^\infty (E \cap \Omega_j)_{\Omega_j}^-.$$

From Theorem 3.1 it follows that for a compact pluripolar set K its negative pluripolar hull K_Ω^- is of G_δ-type. And, therefore, if Ω is pseudoconvex then K_Ω^* is of type $G_{\delta\sigma}$. Hence it is a Borel set.

The following theorem is a high-dimensional version of Theorem 2.7

THEOREM 3.3. — *Let Ω be an open set in $\mathbb{C}^n$ and let $E \subset \Omega$ be any subset. Assume that $F \subset \mathbb{C}^n$ is a pluripolar set. Then*

$$\omega(z, E, \Omega) = \inf\{\omega(z, E, \Omega \cup U) : F \subset U \textit{ open}\}, \quad z \in \Omega \setminus F_\Omega^-.$$

Proof. — Note that the inequality "$\leqslant$" is trivial.

Fix a point $z_0 \in \Omega \setminus F_\Omega^-$. There exists a neighborhood U of F and a negative plurisubharmonic function h on $\Omega \cup U$ such that $h = -\infty$ on F and $h(z_0) \neq -\infty$.

Fix an $\varepsilon > 0$ and put $U_\varepsilon = \{z \in U : h(z) < -\frac{1}{\varepsilon}\}$. Note that $U_\varepsilon \subset U$ is an open neighborhood of F. Let v be a negative plurisubharmonic function on Ω such that $v \leqslant -1$ on E. Consider the plurisubharmonic function

$$v_\varepsilon(z) = \begin{cases} \max\{v(z) + \varepsilon h(z), -1\}, & z \in \Omega \setminus U_\varepsilon, \\ -1, & z \in U_\varepsilon. \end{cases}$$

Note that

$$-v_\varepsilon(z) \geqslant \omega(z, E, \Omega \cup U_\varepsilon) \geqslant \inf\{\omega(z, E, \Omega \cup U) : F \subset U \text{ open}\}, \quad z \in \Omega \setminus F_\Omega^-.$$

We let $\varepsilon \to 0$ and get the result. $\square$

THEOREM 3.4. — *Let Ω be an open set in $\mathbb{C}^n$ and let $\Omega' \Subset \Omega$. Assume that $E \subset \Omega'$ is a compact pluripolar subset. Then for any sequence $\{z_n\}_{n=1}^{\infty} \subset \Omega'$ such that $\limsup_{n\to\infty} \omega(z_n, E, \Omega') > 0$ and that $z_n \to w_0$ it follows that $w_0 \in E_\Omega^-$. Moreover, if Ω is pseudoconvex, then $w_0 \in E_\Omega^*$.*

Proof. — Corollary 2.5 gives $\omega(w_0, E, \Omega) > 0$. Thus Theorem 3.1 shows $w_0 \in E_\Omega^-$. Next if Ω is pseudoconvex we apply Theorem 3.2 on a suitable exhaustion $\cup_j \Omega_j$ of Ω and find $w_0 \in E_\Omega^*$. □

4. A localization principle.

The following localization principle is a main tool in our theory. Special cases of it appear in [15] and [3].

THEOREM 4.1 [A localization principle]. — *Let $\Omega \subset \mathbb{C}^n$ be an open set and let E be an F_σ-pluripolar subset of Ω. Then for any open set $\Omega' \Subset \Omega$ and any open set U such that $\partial U \cap E_\Omega^* = \varnothing$ we have*

$$\omega(z, E \cap U \cap \Omega', \Omega') = \omega(z, E \cap U \cap \Omega', U \cap \Omega'), \quad z \in U \cap \Omega'. \tag{4.1}$$

The proof will be based on two lemmas. Their statement and proof are similar to work of Zeriahi (cf. [18], Lemme 2.1).

LEMMA 4.2. — *Let $\Omega \subset \mathbb{C}^n$ be an open set and let $E \subset \Omega$ be a pluripolar subset. Assume that $F \subset E$, $K \subset \Omega \setminus E_\Omega^*$ are compact subsets and that $\Omega' \Subset \Omega$ is an open set. Then for any number $N > 0$ there exists a continuous negative plurisubharmonic function v on Ω' such that $v \leqslant -N$ on $F \cap \Omega'$, $v \geqslant -1$ on $K \cap \Omega'$.*

Proof. — Let $a \in K \subset \Omega \setminus E_\Omega^*$. By the definition of E_Ω^* there exists a plurisubharmonic function u on Ω such that $u|_E = -\infty$ and $u(a) > -\infty$. Put $M = \max_{z \in \Omega' \cup K}\{u(z) - u(a), 0\}$. Then the function

$$v(z) = \frac{1}{2M+1}(u(z) - u(a)) - \frac{1}{2}, \quad z \in \Omega,$$

is a plurisubharmonic function on Ω with $v|_E = -\infty$, $v(a) = -\frac{1}{2}$ and $v \leqslant 0$ on Ω'.

By the main approximation theorem for plurisubharmonic function (see [5]), there exists a decreasing sequence $\{v_j\}$ of continuous plurisubharmonic functions on Ω' which tends pointwise to v.

Let $N > 0$ be fixed. Dini's lemma on monotone decreasing sequences of continuous functions provides us with a number $j_a > 1$ such that $v_{j_a} \leqslant -N$ on F and $v_{j_a} \leqslant 0$ on Ω'. Since v_{j_a} is continuous on Ω' and since $v_{j_a}(a) \geqslant v(a) = -\frac{1}{2} > -1$, we may find a neighborhood U_a of a such that $v_{j_a} \geqslant -1$ on U_a.

Using a standard compactness argument, we construct a continuous plurisubharmonic function $\widetilde{v} = \max\{v_{j_{a1}}, \dots, v_{j_{am}}\}$ on Ω' such that $v \leqslant 0$ on Ω', $v \leqslant -N$ on $F \cap \Omega'$, and $v \geqslant -1$ on $K \cap \Omega'$. □

An immediate corollary of Lemma 4.2 is

LEMMA 4.3. — *Let $\Omega \subset \mathbb{C}^n$ be an open set and let $E \subset \Omega$ be an F_σ-pluripolar subset. Assume that $K \subset \Omega \setminus E^*_\Omega$ is a compact subset and that $\Omega' \Subset \Omega$ is an open set. Then there exists a negative plurisubharmonic function v on Ω' such that $v = -\infty$ on $E \cap \Omega'$, $v \geqslant -1$ on $K \cap \Omega'$.*

Proof of Theorem 4.1. — Fix an open set $\Omega' \Subset \Omega$. Since $U \cap \Omega' \subset \Omega'$, we have the inequality "$\geqslant$" in (4.1).

Let us show the inequality "$\leqslant$". Note that $K := \partial U \cap \overline{\Omega'}$ is a compact subset of Ω. According to Lemma 4.3 there exists a plurisubharmonic function v on Ω' such that:

- $v \leqslant 0$ on Ω'
- $v = -\infty$ on $E \cap \Omega'$;
- $v \geqslant -1$ on $K \cap \Omega'$.

Let $h \in \mathrm{PSH}(\Omega' \cap U)$ be such that $h \leqslant -1$ on $E \cap U \cap \Omega'$ and $h \leqslant 0$ on $U \cap \Omega'$. Fix an $\varepsilon > 0$. We consider the function

$$v_\varepsilon(z) := \begin{cases} \max\{h(z) - \varepsilon, \varepsilon v(z)\}, & \text{if } z \in \Omega' \cap U, \\ \varepsilon v(z), & \text{if } z \in \Omega' \setminus U. \end{cases}$$

Note that v_ε is a negative plurisubharmonic function on Ω' which satisfies $v_\varepsilon \leqslant -1$ on $E \cap \Omega'$. Hence,

$$-\omega(z, E \cap U \cap \Omega', \Omega') \geqslant v_\varepsilon(z) \geqslant h(z) - \varepsilon, \quad z \in \Omega' \cap U.$$

Let $\varepsilon \to 0$. Then $-\omega(z, E\cap U\cap\Omega', \Omega') \geqslant h(z)$, $z \in \Omega'\cap U$. Therefore,

$$\omega(z, E\cap U\cap\Omega', \Omega') \leqslant \omega(z, E\cap U\cap\Omega', U\cap\Omega').$$

□

Our next Proposition is an easy consequence of Theorem 4.1. We do not know if the condition that E is an F_σ may be omitted.

Proposition 4.4. — *Let Ω be a pseudoconvex open set in $\mathbb{C}^n$ and let $E \subset \Omega$ be an F_σ-pluripolar subset. Assume that E is connected. Then E^*_Ω is also connected.*

Proof. — Assume that $E^*_\Omega \subset U_1 \cup U_2 = U$, where U_1, U_2 are open sets such that $U_1 \cap U_2 = \varnothing$. Since E is connected, $E \subset U_1$ or $E \subset U_2$. Assume that $E \subset U_1$.

Let $\Omega' \Subset \Omega$ be an open set. Then

$$\omega(z, E\cap\Omega', \Omega') = \omega(z, E\cap U\cap\Omega', \Omega') = \omega(z, E\cap\Omega', U\cap\Omega'), \quad z \in U\cap\Omega'.$$

Hence, $\omega(z, E\cap\Omega', \Omega') = 0$ for $z \in U_2\cap\Omega'$. Therefore, $(E\cap\Omega')^-_{\Omega'} \cap U_2 = \varnothing$ and $E^*_\Omega \cap U_2 = \varnothing$. Here we used Theorem 3.2. □

Remark 4.5. — Using Poletsky's theory [9], [10] of holomorphic discs one can give another proof of Proposition 4.4 [11].

Note that if f is a holomorphic function on the unit disc $\mathbb{D}$, then its graph $(\Gamma_f)^*_{\mathbb{C}^2}$ is a connected set and, therefore, $\pi((\Gamma_f)^*_{\mathbb{C}^2})$ is also connected, where $\pi : \mathbb{C}^2 \ni (z,w) \to z \in \mathbb{C}$ is the projection to the first coordinate. In particular, the set $\pi((\Gamma_f)^*_{\mathbb{C}^2})$ is not thin at any point of itself. Here, we show that in some cases it cannot contain boundary points. We obtain this as a corollary of the following more general result.

Theorem 4.6. — *Let $m, n \in \mathbb{N}$ and let $E \subset \mathbb{C}^n$ be an F_σ-pluripolar subset. Assume that $F : \mathbb{C}^n \to \mathbb{C}^m$ is a holomorphic mapping such that*

- $F(E) \subset \mathbb{B}_m$;
- $F(E^*_{\mathbb{C}^n}) \subset \overline{\mathbb{B}_m}$.

Then $F(E^*_{\mathbb{C}^n}) \subset \mathbb{B}_m$.

Proof. — Assume that $z_0 \in E^*_{\mathbb{C}^n}$ is such that $F(z_0) \in \partial\mathbb{B}_m$. Fix an $\varepsilon > 0$ and $r \in (0,1)$. Put $U_\varepsilon = F^{-1}(\mathbb{B}_{1+\varepsilon})$. Note that U_ε is an open neighborhood of $E^*_{\mathbb{C}^n}$. By Theorem 4.1 we have for any $R > 1$

$$\begin{aligned}(4.2)\quad \omega\big(z_0, E\cap F^{-1}(\mathbb{B}_r)\cap\mathbb{B}_R,\mathbb{B}_R\big) &= \omega\big(z_0, E\cap F^{-1}(\mathbb{B}_r)\cap\mathbb{B}_R,\mathbb{B}_R\cap U_\varepsilon\big)\\ &\leqslant \omega\big(F(z_0),\mathbb{B}_r,\mathbb{B}_{1+\varepsilon}\big).\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\omega\big(z_0, E\cap F^{-1}(\mathbb{B}_r)\cap\mathbb{B}_R,\mathbb{B}_R\big) = 0.$$

So, $z_0 \notin (E\cap F^{-1}(\mathbb{B}_r)\cap\mathbb{B}_R)^-_{\mathbb{B}_R}$. Since $R > 1$ is arbitrary, it follows that $z_0 \notin F\Big(\big((E\cap F^{-1}(\mathbb{B}_r)\big)^*_{\mathbb{C}^n}\Big)$. From [1] we obtain that $z_0 \notin F(E^*_{\mathbb{C}^n})$. □

Remark 4.7. — The first condition in Theorem 4.6 (i.e. $F(E) \subset \mathbb{B}_m$) is essential. Indeed, in [6] a function $f \in \mathcal{O}(\mathbb{D})\cap C^\infty(\overline{\mathbb{D}})$ is constructed such that the graph

$$\Gamma_f = \{(z, f(z)) : z \in \overline{\mathbb{D}}\}$$

is complete pluripolar in $\mathbb{C}^2$. Hence, $\pi((\Gamma_f)^*_{\mathbb{C}^2}) = \overline{\mathbb{D}}$, where π is the projection.

COROLLARY 4.8. — *Let $f \in \mathcal{O}(\mathbb{D})$ be a holomorphic function such that $(\Gamma_f)^*_{\mathbb{C}^2} \subset \overline{\mathbb{D}}_\rho \times \mathbb{C}$, where $\rho \geqslant 1$. Then $(\Gamma_f)^*_{\mathbb{C}^2} \subset \mathbb{D}_\rho \times \mathbb{C}$.*

In [2], the authors constructed an example of a smooth holomorphic function f on the unit disc such that $(\Gamma_f)^*_{\mathbb{C}^2}\setminus\Gamma_f \neq \varnothing$. From Proposition 4.4 (see the discussion after the Proposition) and Corollary 4.8 we see that the set $(\Gamma_f)^*_{\mathbb{C}^2}\setminus\Gamma_f$ is actually quite big. See also Siciak [13].

COROLLARY 4.9. — *Let $f \in \mathcal{O}(\mathbb{D})$ be a holomorphic function. Assume that $r_n \searrow 1$ is a sequence of radii such that $(\Gamma_f)^*_{\mathbb{C}^2}\cap(\partial\mathbb{D}_{r_n}\times\mathbb{C}) = \varnothing$. Then Γ_f is complete pluripolar.*

Proof. — From Corollary 4.8 we see that $(\Gamma_f)^*_{\mathbb{C}^2} \subset \mathbb{D}\times\mathbb{C}$.

Fix a closed disc $S \subset \mathbb{D}$, denote the graph of f over S by Γ_S, and put $R_0 = \sup_S |f| + 1$. Then for any $R > R_0$ from Theorem 4.1 (take $U = \mathbb{D}\times\mathbb{C}$ and $\Omega' = \mathbb{D}^2_R$) we get

$$\omega\big((z,w),\Gamma_S,\mathbb{D}^2_R\big) = \omega\big((z,w),\Gamma_S,\mathbb{D}\times\mathbb{D}_R\big),\quad (z,w)\in\mathbb{D}\times\mathbb{D}_R.$$

So, $(\Gamma_S)^-_{\mathbb{D}^2_R} \cap (\mathbb{D} \times \mathbb{D}_R) = (\Gamma_S)^-_{\mathbb{D}\times\mathbb{D}_R}$. But $\mathbb{D} \times \mathbb{D}_R$ is a hyperconvex domain, hence $(\Gamma_S)^-_{\mathbb{D}\times\mathbb{D}_R} = (\Gamma_S)^*_{\mathbb{D}\times\mathbb{D}_R} = \Gamma_f \cap (\mathbb{D} \times \mathbb{D}_R)$. From this we get $(\Gamma_f)^*_{\mathbb{C}^2} = (\Gamma_S)^*_{\mathbb{C}^2} = \cup_{R>R_0}(\Gamma_S)^-_{\mathbb{D}^2_R} = \cup_{R>R_0}\Gamma_f \cap (\mathbb{D} \times \mathbb{D}_R) = \Gamma_f$. □

As a simple corollary of the localization principle we have the following

COROLLARY 4.10.— *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ and let $E \subset \Omega$ be an F_σ-pluripolar set such that $E^*_\Omega \Subset \Omega$. Then for any open set $\Omega' \Subset \Omega$ such that $E^*_\Omega \subset \Omega'$ we have $E^-_{\Omega'} = E^*_\Omega$.*

Proof.— Let Ω'' be a pseudoconvex domain such that $\Omega' \Subset \Omega'' \Subset \Omega$. From the localization principle we have $\omega(z, E, \Omega'') = \omega(z, E, \Omega')$ for $z \in \Omega'$. Hence, $E^-_{\Omega''} = E^-_{\Omega'}$. Since Ω'' is arbitrary, $E^*_\Omega = E^-_{\Omega'}$. □

5. The set of interior values.

In the study of boundary behavior of a holomorphic function the properties of its cluster set are very important (see e.g. [8]). In connection with the pluripolar hull a certain subset of the cluster set is very useful.

DEFINITION 5.1.— *Let $\Omega \subset \mathbb{C}^n$ be an open set and let $f : \Omega \to \mathbb{C}^m$ be a* bounded *holomorphic mapping. Assume that z_0 is a boundary point of Ω. An interior value of f at z_0 is a limit point of a sequence $f(z_k)$, where $z_k \in \Omega$ tend to z_0 in such a way that for some closed non-empty ball $B \subset \Omega$ and some positive number α we have*

$$\omega(z_k, B, \Omega) \geqslant \alpha \quad \text{for any } k \geqslant 1. \tag{5.1}$$

We denote the set of interior values of f at $z_0 \in \partial\Omega$ by $L_{z_0}(f;\Omega)$.

For an unbounded *holomorphic mapping f defined on an open set Ω we put $L_{z_0}(f;\Omega) = \cup_{R>0}L_{z_0}(f;\Omega_R)$, where*

$$\Omega_R = \{z \in \Omega : \|f(z)\| < R\}$$

and $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{C}^m$.

In case for some (and, therefore, for any) closed non-empty ball $B \subset \Omega$ we have $\lim_{z\to z_0}\omega(z, B, \Omega) = 0$, we put $L_{z_0}(f;\Omega) = \varnothing$. This happens if and only if z_0 is a regular boundary point of Ω for the Dirichlet problem.

Example 5.2. — Let $f(z) = \exp(1/z)$. Note that f is a holomorphic function on $\mathbb{C} \setminus \{0\}$. It is well-known that the cluster set of f at 0 is the set $\mathbb{C} \cup \{\infty\}$. On the other hand $L_0(f; \mathbb{C} \setminus \{0\}) = \varnothing$. Indeed, since f is unbounded, we have $L_0(f; \mathbb{C} \setminus \{0\}) = \cup_{R>0} L_0(f; \Omega_R)$, where

$$\Omega_R = \{z \in \mathbb{C} \setminus \{0\} : |\exp(1/z)| < R\}.$$

It is easy to check that

$$\Omega_R = \begin{cases} \{z \in \mathbb{C} \setminus \{0\} : |z - \frac{1}{2\log R}| < |\frac{1}{2\log R}|\}, & \text{for } R \in (0;1), \\ \{z \in \mathbb{C} \setminus \{0\} : \operatorname{Re} z < 0\}, & \text{for } R = 1, \\ \{z \in \mathbb{C} \setminus \{0\} : |z - \frac{1}{2\log R}| > |\frac{1}{2\log R}|\}, & \text{for } R > 1. \end{cases}$$

We see that Ω_R is regular at 0 for any $R > 0$, hence $L_0(f; \Omega_R) = \varnothing$ and $L_0(f; \mathbb{C} \setminus \{0\}) = \varnothing$. So, the set of interior values being similar in definition to the cluster set, in some cases is very different from it.

The following little lemma shows that in $\mathbb{C}$ interior value is a "local property".

LEMMA 5.3. — *Let Ω be an open set in $\mathbb{C}$ and let $z_0 \in \partial\Omega$. Assume that $f : \Omega \to \mathbb{C}^m$ is a holomorphic mapping. Then there exists an $r > 0$ such that*

$$L_{z_0}(f; \Omega) = L_{z_0}(f; \Omega \cap \mathbb{D}(z_0, r)), \tag{5.2}$$

where $\mathbb{D}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}$.

Proof. — This follows from Bouligand's lemma (see [12]). □

From Theorem 3.4 we have the following.

COROLLARY 5.4. — *Let $\Omega' \Subset \Omega$ be open sets in $\mathbb{C}^n$ and let $f : \Omega' \to \mathbb{C}^m$ be a holomorphic mapping. Assume that $\zeta_0 \in \partial\Omega'$. Then $\{\zeta_0\} \times L_{\zeta_0}(f; \Omega') \subset (\Gamma_f)^*_{\Omega \times \mathbb{C}^m}$.*

In particular, if $L_{\zeta_0}(f; \Omega')$ is non-pluripolar, then

$$(\Gamma_f)^*_{\Omega \times \mathbb{C}^m} \cap \{\zeta_0\} \times \mathbb{C}^m = \{\zeta_0\} \times \mathbb{C}^m.$$

For $n = 1$ we have a little bit stronger result.

COROLLARY 5.5. — *Let $\Omega' \subset \Omega$ be open sets in $\mathbb{C}$ and let $f : \Omega' \to \mathbb{C}$ be a holomorphic function. Assume that $\zeta_0 \in \partial\Omega' \cap \Omega$. Then $\{\zeta_0\} \times L_{\zeta_0}(f;\Omega') \subset (\Gamma_f)^*_{\Omega\times\mathbb{C}}$.*

Proof of both corollaries. — Fix a ball $B \subset \Omega'$. The inequality (2.2) with the map $h : z \mapsto (z, f(z))$ and the estimate (5.1) provide us with points $w_k = (z_k, h(z_k)) \in \Gamma_f \cap \Omega' \times \mathbb{C}^m$ converging to $(\zeta_0, \eta_0) \in \{\zeta_0\} \times L_{\zeta_0}(f, \Omega')$ such that $\limsup_{k\to\infty} \omega(w_k, h(B), \Omega' \times \mathbb{B}(\eta_0, R)) > 0$. Now Theorem 3.4 applies. Use Lemma 5.3 for Corollary 5.4. $\square$

Let D be a domain in $\mathbb{C}$ and let $f \in \mathcal{O}(D)$. Assume that $z_0 \in \partial D$. We want to show that $\#L_{z_0}(f;D) \leqslant 1$ and, therefore, the set $L_{z_0}(f;D)$ is always polar. The crucial ingredient is work of Gamelin and Garnett [4], which extends earlier work of Zalcman [17]. We recall it here for a small part. Consider $H^\infty(D)$, the algebra of bounded holomorphic functions on D. A *distinguished homomorphism* at z_0 is a homomorphism above z_0 that admits a representing measure supported on D. Distinguished homomorphisms need not exist, but it is shown in [4] that there can at most be one distinguished homomorphism above z_0.

LEMMA 5.6. — *Let D be a domain in $\mathbb{C}$ and let $f \in \mathcal{O}(D)$. Assume that $z_0 \in \partial D$. Then $\#L_{z_0}(f;D) \leqslant 1$.*

Remark 5.7. — It is well possible that a regular boundary point admits a distinguished homomorphism. Existence of distinguished homomorphisms can be characterized in terms of analytic capacity (Melnikov type condition), cf. [4], while regularity is characterized in terms of logarithmic capacity (Wiener's criterion), cf. [12].

The proof of the lemma will be based on the connection between distinguished homomorphisms and interpolating sequences. A sequence $\{z_n\}_{n=1}^\infty \subset D$ is called an *interpolating sequence* for $H^\infty(D)$ if for every bounded sequence $\{s_n\}_{n=1}^\infty \in \ell^\infty$, there is $f \in H^\infty(D)$ such that $f(z_n) = s_n$ for any $n \geqslant 1$.

Let us show the following variation of the well-known result related to the Green function g_D of a domain D (see e.g. [12], Corollary 4.5.5).

PROPOSITION 5.8. — *Let D be a domain in $\mathbb{C}$ and let $\{z_n\}_{n=1}^\infty \subset D$ be an interpolating sequence. Then $\lim_{n\to\infty} g_D(z_n, z_1) = 0$.*

Proof. — There exists a bounded holomorphic function f on D such that $f(z_1) = 1$ and $f(z_n) = 0$ for any $n \geqslant 2$. Assume that $\|f\| = M$. Then $g_D(z; \{z_n\}_{n=2}) \geqslant \log|f(z)| - \log M$ and, therefore,

$$\sum_{n=2}^{\infty} g_D(z_1; z_n) = g_D(z_1; \{z_n\}_{n=2}^{\infty}) \geqslant -\log M.$$

Hence, $g_D(z_n, z_1) = g_D(z_1, z_n) \to 0$ when $n \to \infty$. □

PROPOSITION 5.9. — *Let D be a domain in $\mathbb{C}$ and let $f \in \mathcal{O}(D)$. Assume that $z_0 \in \partial D$ is an irregular point. Then there exists $w_0 \in \mathbb{C}$ such that for any sequence $\{z_n\} \subset D$ with $z_n \to z_0$ there exists a subsequence $\{z_{n_k}\}$ such that $f(z_n) \to w_0$ or $g_D(z_{n_k}, z_1) \to 0$.*

Proof. — Theorem 4.5 in [4] states that either $\{z_n\}$ converges in $H^\infty(D)^*$ to the distinguished homomorphism ϕ_{z_0} at z_0 or there exists an interpolating subsequence of $\{z_n\}$. Hence Proposition 5.8 implies that $\{z_n\}$ converges to ϕ_{z_0} and $w_0 = \phi_{z_0}(f) = \lim_n f(z_n)$. □

Proof of Lemma 5.6. — In case f is bounded Proposition 5.9 applies. The general case follows from the definitions. □

THEOREM 5.10. — *Let D be an open set in $\mathbb{C}$ and let $A \subset D$ be a closed polar set. Assume that $f \in \mathcal{O}(D \setminus A)$ and that $z_0 \in A$. Then*

$$(\Gamma_f)^*_{D\times\mathbb{C}} \cap \{z_0\} \times \mathbb{C} = \{z_0\} \times L_{z_0}(f; D \setminus A).$$

And, therefore, $\#\Big((\Gamma_f)^*_{D\times\mathbb{C}} \cap \{z_0\} \times \mathbb{C}\Big) \leqslant 1.$

For the proof, first let us show the following refinement of the main result of [3].

THEOREM 5.11. — *Let D be an open set in $\mathbb{C}$ and let A be a closed polar subset of D. Suppose that $f \in \mathcal{O}(D \setminus A)$ and that $z_0 \in A$. Assume that $U \subset \mathbb{C}$ is an open set. Then the following conditions are equivalent:*

(1) $(\{z_0\} \times \mathbb{C}) \cap (\Gamma_f \cap (D \times U))^*_{D\times U} = \varnothing$;

(2) *there exists a sequence of open sets $V_1 \subset V_2 \subset \ldots \Subset U$ such that $\cup_j V_j = U$ and the set $\{z \in D \setminus A : f(z) \in U \setminus V_j\}$ is not thin at z_0 for any $j \geqslant 1$;*

(3) *for any open set $V \Subset U$ the set $\{z \in D \setminus A : f(z) \in U \setminus V\}$ is not thin at z_0.*

Moreover, if the set $\{z \in D \setminus A : f(z) \notin V\}$ *is thin at* z_0 *for some open set* $V \Subset U$*, then there exists a* $w_0 \in \overline{V}$*, such that* $(z_0, w_0) \in (\Gamma_f \cap D \times U)^*_{D\times U}$.

Proof. — (1) $\Longrightarrow$ (3). Assume that there exists an open set $V \Subset U$ such that $\{z \in D \setminus A : f(z) \notin V\}$ is thin at z_0. Then the set $\{z \in D \setminus A : f(z) \in V\}$ is not regular at z_0. Hence, there exist an open set $G \Subset D$ such that $\partial G \cap A = \varnothing$, $z_0 \in G$, and a sequence $\{z_n\}_n$ in $G \setminus A$ tending to z_0 such that $\limsup_{n\to\infty} \omega(z_n, S, G \setminus A) > 0$ for some closed disc $S \subset G \setminus A$. There is a subsequence $\{z_{n_k}\}$ such that $f(z_{n_k})$ converges to an interior value $w_0 \in \overline{V}$ and, using Theorem 3.4 $(z_0, w_0) \in (\Gamma_f)^*_{D\times U}$. We have also proved the last statement of the theorem.

(3) $\Longrightarrow$ (2). Obvious.

(2) $\Longrightarrow$ (1). Again Γ_S will denote the graph of f over a disc S in D. In view of Theorems 3.1 and 3.2, it suffices to show that for $w \in V_j$ $\omega((z_0, w), \Gamma_S, G \times V_j) = 0$ for any fixed, open set $G \Subset D$ such that $\partial G \cap A = \varnothing$ and some closed disc $S \subset G \setminus A$. To estimate $\omega((z, f(z)), \Gamma_S, G \times V_j)$, let $\varepsilon > 0$ and start with a small neighborhood V of $A \cap G$, to be determined later. Put $\widetilde{V} = V \cup (D \setminus \overline{G})$. Let

$$U = [\{z \in D\setminus A : f(z) \in V_{j+1}\} \cup \widetilde{V}] \times V_{j+1} \cup \{z \in D \setminus A : f(z) \notin \overline{V_j}\} \times (\mathbb{C} \setminus \overline{V_j}).$$

Then U is a neighborhood of Γ_f. It was proved in [3] that

$$(\Gamma_f)^*_{D\times\mathbb{C}} \subset \Gamma_f \cup A \times \mathbb{C}. \tag{5.3}$$

Therefore $\partial U \cap (\Gamma_S)^*_{G\times V_j} = \emptyset$. We may apply the localization principle, Theorem 4.1 and find

$$\begin{aligned} \omega((z,w), \Gamma_S, G \times V_j) &= \omega((z,w), \Gamma_S, G \times V_j \cap U) \\ &= \omega((z,w), \Gamma_S, [\{z \in D \setminus A : f(z) \in V_{j+1}\} \cup \widetilde{V}] \times V_j), \end{aligned} \tag{5.4}$$

for $(z, w) \in U \cap G \times V_j)$. Now we apply (2.2) to the projection $(z, w) \mapsto z$ and find that the right-hand side of (5.4) is

$$\leqslant \omega(z, S, \{z \in D \setminus A : f(z) \in V_{j+1}\} \cup \widetilde{V}). \tag{5.5}$$

By Theorem 2.7 we can choose V so small that $\omega(z_0, S, \{z \in D \setminus A : f(z) \in V_{j+1}\} \cup \widetilde{V}) < \varepsilon$. Letting $\varepsilon \to 0$, it follows that $\omega((z_0, w), \Gamma_S, G \times V_j) = 0$. □

As an easy corollary we get

COROLLARY 5.12. — *Let D be an open set in $\mathbb{C}$ and let A be a closed polar subset of D. Suppose that $f \in \mathcal{O}(D \setminus A)$ and that $z_0 \in A$. Then the following conditions are equivalent:*

(1) $(\{z_0\} \times \mathbb{C}) \cap (\Gamma_f \cap (D \times U))^*_{D\times U} = \varnothing$;

(2) *there exists a sequence of bounded open sets $V_1 \subset V_2 \subset \dots$ such that $\cup_j V_j = \mathbb{C}$ and the set $\{z \in D \setminus A : f(z) \in U \setminus V_j\}$ is not thin at z_0 for any $j \geqslant 1$;*

(3) *for any bounded open set V the set $\{z \in D \setminus A : f(z) \notin V\}$ is not thin at z_0.*

Moreover, if the set $\{z \in D \setminus A : f(z) \notin V\}$ is thin at z_0 for some open set $V \Subset U$, then there exists a $w_0 \in \overline{V}$, such that $(z_0, w_0) \in (\Gamma_f \cap D \times U)^*_{D\times U}$.

For the proof of the main result we need the following simple remark related to the pluripolar hull.

LEMMA 5.13. — *Let $D \subset \mathbb{C}^n$ be a pseudoconvex set and let $A \subset D$ be a closed pluripolar subset. Assume that $E \subset D \setminus A$ is a pluripolar compact set. Then $E^*_D \subset E^*_{D\setminus A} \cup A$.*

Proof. — Let $D_1 \Subset D_2 \Subset \dots \Subset D$ be an exhaustion of D by hyperconvex domains. Then by Theorem 3.2 we have $E^*_D = \cup_{j=1}^{\infty}(E \cap D_j)^-_{D_j} = \cup_{j=1}^{\infty} E^-_{D_j}$. Now we apply Lemma 3.1 from [7], saying that $\omega(z, E, D_j) = \omega(z, E, D_j \setminus A)$, and infer that $E^-_{D_j} \cap D_j \setminus A = E^-_{D_j\setminus A}$, and the lemma follows. □

Proof of Theorem 5.10. — Assume that $L_{z_0}(f; D \setminus A) \subset \{w_0\}$. Put $U := \mathbb{C} \setminus \{w_0\}$. Then, by the definition of interior value, for every relative compact subset $W \Subset U$ the set $\{z \in D \setminus A : \ f(z) \in U \setminus W\}$ is not thin at z_0. Hence by Theorem 5.11 $\{z_0\} \times U \cap (\Gamma_S)^*_{D\times U} = \varnothing$. But $(\Gamma_S)^*_{D\times\mathbb{C}} \subset (\Gamma_S)^*_{D\times U} \cup (D \times \{w_0\})$. Therefore, $\{z_0\} \times \mathbb{C} \cap (\Gamma_S)^*_{D\times\mathbb{C}} \subset \{(z_0, w_0)\}$. □

Remark 5.14. — Let $A = \{a_n\}_{n=1}^{\infty} \subset \mathbb{D} \setminus \{0\}$ be a sequence such that $a_n \to 0$ and let $\{c_n\}_{n=1}^{\infty} \subset \mathbb{C} \setminus \{0\}$. Put

$$f(z) = \sum_{n=1}^{\infty} \frac{c_n}{z - a_n}. \tag{5.6}$$

Suppose that $\sum_{n=1}^{\infty}|c_n| < +\infty$ and that $\sum_{n=1}^{\infty}\left|\frac{c_n}{a_n}\right|$ converges. Then $f \in \mathcal{O}(\mathbb{C}\setminus(A\cup\{0\}))$ and $f(0)$ is well-defined.

In [3] the authors gave sufficient conditions on $\{a_n\}$ and $\{c_n\}$ such that $(\Gamma_f)^*_{\mathbb{C}^2} = \Gamma_f \cup \{(0, f(0)\}$.

Theorem 5.10 gives that $\#\Big((\Gamma_f)^*_{\mathbb{C}^2}\setminus\Gamma_f\Big) \leqslant 1$. In case $(\Gamma_f)^*_{\mathbb{C}^2} = \Gamma_f \cup \{(0, w_0)\}$ it seems likely that $w_0 = f(0)$, as defined by the series. Under mild convergence conditions this is easily proved.

Example 5.15. — Suppose that the series (5.6) has the property that $(\Gamma_f)^*_{\mathbb{C}^2}$ contains a point $(0, w_0)$ and suppose that for every M either the series

$$\sum_{n=1}^{\infty}\left|\frac{c_n}{z-a_n}\right| \tag{5.7}$$

is bounded on $\{z : |f(z)| < M\}$, or the function

$$g(z) = \sum_{n=1}^{\infty}\frac{c_n}{a_n(z-a_n)} \tag{5.8}$$

is in $H^\infty(D_M)$. Then $f(0) = w_0$.

From [4] we know that the distinguished homomorphism ϕ_0 at 0 can be represented by a positive measure μ_0 on D_m. Note that $\phi_0(c_n/(z-a_n)) = -c_n/a_n$, because $c_n/(z-a_n)$ is holomorphic in a neighborhood of 0. If (5.7) is satisfied, then by the dominated convergence theorem

$$f(0) = \lim_{N\to\infty}\sum_{n=1}^{N}\frac{-c_n}{a_n} = \lim_{n\to N}\sum_{n=1}^{N}\int_{D_M}\left(\frac{c_n}{z-a_n}\right)d\mu_0 = \int_{D_M} f\,d\mu_0 = w_0.$$

In case of (5.8) we observe that

$$f(z) - f(0) = \sum_{n=1}^{\infty}\frac{zc_n}{a_n(z-a_n)} = zg(z).$$

Hence $\phi_0(f(z)-f(0)) = 0$, or $f(0) = w_0$.

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