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INVESTIGATIONS OF RETARDED PDEs OF SECOND ORDER IN TIME USING THE METHOD OF INERTIAL MANIFOLDS WITH DELAY

by **Alexander V. REZOUNENKO**

To Louis Boutet de Monvel

1. Introduction.

Since the time when H. Poincaré and A. Lyapunov published their famous results on the qualitative analysis of differential equations, this field constantly attracts much attention. Investigations of asymptotic behaviour of solutions belong to this field. In the study of infinite-dimensional dynamical systems (constructed by partial differential and/or delay equations [17, 20, 21]) the possibility of reducing the dimension and obtain a finite-dimensional system which captures all the asymptotic properties of the initial one brings a valuable simplification. In this context the notions of finite-dimensional global attractor and invariant manifolds were introduced. One of the direct ways to get the finite-dimensional system is to construct an inertial manifold (IM) [1, 2, 9, 8, 10] and consider the flow restricted on the manifold. IMs for retarded nonlinear partial differential equations (PDEs) were considered for the first time in [11] for the case of semilinear parabolic equations (see also [26] for systems of second order in time and references on articles where invariant manifolds are studied for retarded ordinary equations). Unfortunately, most of the cases in which the

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existence of an IM known require a big gap in the spectrum of the leading linear part of the system. To consider cases when a system does not possess such a gap, the notion of an approximate inertial manifold (AIM) was introduced [3, 5] (see also [6, 29, 7] and references therein).

Recently, another interesting infinite-dimensional invariant surface called an inertial manifold with delay (IMD) was introduced by A. Debussche and R. Temam [14] (see also [15] and for the case of retarded parabolic equations [25]). It was found [27] that this surface can be effectively used in the construction of new families of AIMs (see also [28] for the extension to retarded parabolic equations).

The aim of this article is to construct IMD for dissipative systems of second order in time and apply this result to the study of different asymptotic properties of solutions. We give the proof (Section 3) for the retarded case and notice that even for the nonretarded case it has not been done before in the context of systems of second order in time. Using IMD, we construct AIMs containing all the stationary solutions (Section 4), improve the result of [23] on the finite number of essential modes (Section 5) and give a new characterization of the $\mathcal{K}$-invariant manifold constructed in [16] (Section 6). In contrast to the case of parabolic equations with delay [11, 25, 28], our investigations are crucially based upon a special choice of the norm introduced in [12].

2. Assumptions and preliminaries.

In this work we study the dynamical system generated by the evolution equation

$$\partial_t^2 u + 2\varepsilon \partial_t u + Au = B(u_t) \quad \text{for} \quad t > 0,\ \varepsilon > 0 \tag{2.1}$$

with initial data

$$u(\theta) = u^0(\theta) \quad \text{for} \quad \theta \in [-r, 0],\ \partial_t u\big|_{t=0} = u^1. \tag{2.2}$$

In (2.1) and below, if z is a continuous function from R into a space Y, then as in [17] $z_t \equiv z_t(\theta) \equiv z(t+\theta), \theta \in [-r, 0]$ denotes the element of $C(t-r, t; Y)$, while $r > 0$ presents the retardation time.

We assume that:

(A1) *In* (2.1)*, A is a positive operator with a discrete spectrum in a separable Hilbert space H.* Hence there exists an orthonormal basis $\{e_k\}$ of H such that

$$Ae_k = \mu_k e_k, \quad \text{with} \quad 0 < \mu_1 \leqslant \mu_2 \leqslant \cdots, \quad \lim_{k\to\infty} \mu_k = \infty.$$

For $r > 0$, we denote for short $C_\alpha = C(-r, 0; D(A^\alpha))$ which is a Banach space with the following norm: $|v|_{C_\alpha} \equiv \sup\{\| A^\alpha v(\theta) \|: \theta \in [-r, 0]\}$. Here and below, $\| \cdot \|$ is the norm of H, and $(\cdot, \cdot)$ the corresponding hermitian product.

(A2) *The nonlinear operator B is a mapping from C_α to H $(0 \leqslant \alpha \leqslant \frac{1}{2})$*

$$v \to B(v) = B_0(v(0)) + B_1(v) \tag{2.3}$$

where B_0 and B_1 are maps from $D(A^\alpha)$ (resp. C_α) to H such that

$$\| B_0(w_1) - B_0(w_2) \| \leqslant M_0 \ \| A^\alpha(w_1 - w_2) \|, \quad \text{for } \ w_1, w_2 \in D(A^\alpha) \tag{2.4}$$

and

$$\| B_1(v_1) - B_1(v_2) \| \leqslant M_1 \ |v_1 - v_2|_{C_\alpha}, \quad \text{for } \ v_1, v_2 \in C_\alpha, \tag{2.5}$$

where M_0 and M_1 are positive constants.

We can rewrite (2.1), (2.2) as follows:

$$\partial_t U + \mathcal{A}U = \mathcal{B}(U_t), \quad U\big|_{\theta\in[-r,0]} = U_0, \tag{2.6}$$

where $U(t) = (u(t); \dot{u}(t)), U_0 = (u^0; u^1)$. Here the operator $\mathcal{A}$ and the map $\mathcal{B}$ are defined by $\mathcal{A}U = (-u^1; Au^0 + 2\varepsilon u^1)$, $\mathcal{B}(U_t) = (0; B(u_t^0))$ for $U = (u^0; u^1)$.

It is easy to verify that the eigenvalues and eigenvectors of $\mathcal{A}$ are $\lambda_n^\pm = \varepsilon \pm \sqrt{\varepsilon^2 - \mu_n}$, $f_n^\pm = (e_n; -\lambda_n^\pm e_n)$, $n = 1, 2, \ldots$

We need the condition $\varepsilon^2 > \mu_{N+1}$ which is the most restrictive one for applications but usual in our framework (see e.g. [12, 6, 26]). We set $E = D(A^{1/2}) \times H$ and consider the splitting $E = E_1 \oplus E_2$, where $E_1 = \text{Lin}\,\{(e_k; 0), (0; e_k) : k = 1, \ldots, N\}, E_2 = \text{Cl Lin}\,\{(e_k; 0), (0; e_k) : k \geqslant N + 1\}$.

We will use the following hermitian product [12] in E_1 and E_2 :

$$\langle U, V\rangle_1 = \varepsilon^2(u^0, v^0) - (Au^0, v^0) + (\varepsilon u^0 + u^1, \varepsilon v^0 + v^1),$$
$$\langle U, V\rangle_2 = (Au^0, v^0) - (\varepsilon^2 - 2\mu_{N+1})(u^0, v^0) + (\varepsilon u^0 + u^1, \varepsilon v^0 + v^1).$$

Here, $U = (u^0; u^1), V = (v^0; v^1)$ belong to the corresponding subspace E_i.

Now, the hermitian product in E reads

$$\langle U, V\rangle = \langle U^1, V^1\rangle_1 + \langle U^2, V^2\rangle_2,$$

where $U = U^1 + U^2$, $V = V^1 + V^2$; $V^i, U^i \in E_i$, $i = 1, 2$. We will use $|U|_E \equiv |U| \equiv \langle U, U\rangle^{1/2}$.

An easy calculation gives (see e.g. [6]) for any $U = (u; v) \in E$:

(2.7) $$\| A^\alpha u \| \leqslant \mu_{N+1}^\alpha \delta_{N,\varepsilon}^{-1} |U|, \quad \text{for } 0 \leqslant \alpha \leqslant \frac{1}{2},$$

where

(2.8) $$\delta_{N,\varepsilon} \equiv \sqrt{\mu_{N+1}} \min \left\{ 1, \sqrt{\frac{\varepsilon^2 - \mu_{N+1}}{\mu_{N+1}}} \right\}.$$

Using this and (2.4), (2.5), one easily sees that

(2.9) $$|\mathcal{B}(U_t^1) - \mathcal{B}(U_t^2)| \leqslant \widehat{M}_0 |U^1(t) - U^2(t)| + \widehat{M}_1 \max_{[-r,0]} |U^1(t+\theta) - U^2(t+\theta)|,$$

where

(2.10)
$$\widehat{M}_i \equiv M_i \cdot \mu_{N+1}^\alpha \delta_{N,\varepsilon}^{-1} = M_i \cdot \mu_{N+1}^{\alpha - 1/2} \max \left\{ 1, \sqrt{\frac{\mu_{N+1}}{\varepsilon^2 - \mu_{N+1}}} \right\}, \ i = 0, 1.$$

A *weak solution* of (2.6) is defined as a function $U(t) \equiv (u(t); \dot{u}(t)) \in C(-r, T; E)$ that satisfies $u(\theta) = u^0(\theta), \theta \in [-r, 0]$; $\dot{u}(0) = u^1$ and

$$U(t) = e^{-t\mathcal{A}} U(0) + \int_0^t e^{-(t-s)\mathcal{A}} \mathcal{B}(U_s) ds.$$

The proof of the existence and uniqueness of a weak solution is standard (see e.g. [19]) as well as the following

LEMMA 2.1. — *Consider* $U(t) = V(t) - W(t)$, *where* $V(t)$ *and* $W(t)$ *are solutions of* (2.6). *Then*

$$|U_t|_{C_E} \leqslant a_1 \cdot e^{a_2(t-s)} |U_s|_{C_E}, \quad t \geqslant s.$$

Here a_1, a_2 *are positive constants.*

Now, we can define an evolution semigroup S_t in $C_E \equiv C(-r, 0; E)$ by

(2.11) $$S_t U_0 \equiv [S_t U_0](\theta) = \begin{cases} U(t+\theta), & t+\theta > 0\ , \\ U_0(t+\theta), & t+\theta \leqslant 0 \end{cases}$$

where $U(t)$ is the weak solution of (2.6). We fix an integer N and consider the following subspaces $E_1^\pm = \text{ Lin } \{f_k^\pm : k \leqslant N\}$, which are orthogonal for the hermitian product $\langle \cdot, \cdot \rangle$, so $E_1 = E_1^+ \oplus E_1^-$. If we denote by $P_{E_\imath}$ the orthoprojectors on the corresponding subspaces, one has (see e.g. [6])

(2.12)
$$|e^{-\mathcal{A}t} P_{E_2}| = e^{-\lambda_{N+1}^- t}, \quad |e^{\mathcal{A}t} P_{E_1^-}| = e^{\lambda_N^- t}, \quad |e^{-\mathcal{A}t} P_{E_1^+}| = e^{-\lambda_N^+ t}, \quad t \geqslant 0.$$

We set $P \equiv P_{E_1^-}$ and $Q = I - P$. Now and later, we reserve the notation P for the projector on E_1^-, and P^0, P^1, P^2 for elements of E_1^-. We also define the N-dimensional projector $\widehat{P}$ in $C_E \equiv C(-r, 0; E)$ by

$$\widehat{P}U = (\widehat{P}U)(\theta) = \sum_{k=1}^{N} e^{-\lambda_k^- \theta} \langle U(0), f_k^- \rangle f_k^- \equiv e^{-\mathcal{A}\theta} PU(0),$$

where $-r \leqslant \theta \leqslant 0$ and $U = U(\theta)$ is an element of C_E. We also set $\widehat{Q} \equiv I - \widehat{P}$. A simple computation shows that $\widehat{P}$ is the spectral projector of the infinitesimal generator of the linear semigroup T_t in C_E defined by formulae similar to (2.11) for problem (2.1) and (2.2) with $B(u) \equiv 0$. Indeed,

$$(T_t u_0)(\theta) = e^{-(t+\theta)\mathcal{A}} U_0(0), \text{ if } t + \theta > 0;$$

and

$$(T_t u_0)(\theta) = U_0(t + \theta), \text{ if } t + \theta \leqslant 0.$$

3. Existence of inertial manifolds with delay.

The main result of this section is the following assertion on the existence of inertial manifolds with delay for the problem (2.6).

THEOREM 3.1. — *Let $\varepsilon^2 > \mu_{N+1}$. There exists T_0 such that for any $T \in (0, T_0]$, any $p \in PE \equiv E_1^-$ and $\psi \in \widehat{Q}C_E$ there exists a unique solution $U(t)$ of (2.6) defined on $[-T, \infty)$ such that*

$$PU(0) = p, \quad \widehat{Q}U_{-T} = \psi.$$

Moreover if we set $\Phi(p, \psi) \equiv \widehat{Q}u_0$, this defines a Lipschitz mapping from $E_1^- \times \widehat{Q}C_E$ to $\widehat{Q}C_E$ i.e., for any $(p^i, \psi^i) \in E_1^- \times \widehat{Q}C_E, i = 1, 2$ we have:

$$(3.1) \quad |\Phi(p^1, \psi^1) - \Phi(p^2, \psi^2)|_{C_E} \leqslant L_1(T)|p^1 - p^2| + L_2(T)|\psi^1 - \psi^2|_{C_E}.$$

We say that Φ defines a manifold $\mathcal{M}$ in $E_1^- \times \widehat{Q}C_E \times \widehat{Q}C_E$. This manifold is invariant i.e., if $U(t)$ is a solution of (2.6), then

$$\widehat{Q}U_t = \Phi\left(PU(t), \widehat{Q}U_{t-T}\right), \quad t \geqslant 0.$$

The following additional conditions give bounds for the Lipschitz constants L_i.

Assume that

$$(3.2) \qquad \liminf_{n\to\infty} \frac{\mu_n}{\mu_{n+1}} \equiv \eta_0 > 0.$$

Take any $\delta_1, \delta_2 \in (0,1]$ and any $c > \ln(2\delta_2^{-1})$.

Then (3.2) implies (see Lemma 3.3 for details) the existence of big enough N_0 satisfying

$$\lambda^-_{N_0} > (\widehat{M}_0 + \widehat{M}_1 e^c) \cdot e^c(e^c - e^{-c}) \left(1 + \delta_1^{-1}\right),$$

such that for any $N \geqslant N_0$, any $\varepsilon^2 \in [2\mu_{N+1}, 2\mu_{N+1}+\mu_N]$ and T, r satisfying

$$r + \frac{\ln(2\delta_2^{-1})}{\lambda^-_{N+1}} < T \leqslant \frac{c}{\lambda^-_{N+1}}, \tag{3.3}$$

we get $L_1(T) < \delta_1$ and $L_2(T) < \delta_2$.

Remark 3.1. — This manifold with delay exists for any N, without any restriction on the spectrum of A and for any time retardation r (r is large as well as small).

Remark 3.2. — In fact (see Corollary below), for $L_i < \delta_i$ we need the following condition

$$\varepsilon - \sqrt{\varepsilon^2 - \mu_N} > \mu_{N+1}^{\alpha-1/2}(M_0 + M_1 e^c) \max\left\{1, \sqrt{\frac{\mu_{N+1}}{\varepsilon^2 - \mu_{N+1}}}\right\} D_c \cdot \frac{1}{4}\left(1 + \frac{1}{\delta_1}\right),$$

together with (3.3). Here $D_c \equiv 4e^c(e^c - e^{-c})$. We choose (3.2) in Theorem 3.1 for the simplicity.

Proof of Theorem 3.1. — We follow line of arguments given in [25] and introduce for fixed $T > 0$ and $\psi \in \widehat{Q}C_E$ the following spaces $Y_1 \equiv \{y \in C(-T,0;E) : Q\,y(-T) = 0\}$, $Y_2 = Y_2(\psi) \equiv \{y \in C(-T-r,0;E) : \widehat{Q}_N\, y_{-T} = \psi\}$ with the sup-norm.

Now for any $\psi \in \widehat{Q}C_E$ we introduce the following shift-continuation function $\mathcal{E} : Y_1 \to Y_2$ which is a key tool in our considerations:

$$\mathcal{E}(y,\psi)(s) \equiv \begin{cases} y(s) + e^{-(s+T)\mathcal{A}}\psi(0), & \text{if } s \in [-T,0]\ , \\ \psi(s+T) + e^{-(s+T)\mathcal{A}} P y(-T), & \text{if } s \in [-T-r,-T]. \end{cases} \tag{3.4}$$

As in [25], we prove the following

LEMMA 3.1. — *For any $y^i \in Y_1$, $\psi^i \in \widehat{Q}C_E, i = 1,2$ and any $s \in [-T,0]$ we have*

$$\begin{aligned} &|\mathcal{B}(\mathcal{E}(y^1,\psi^1)_s) - \mathcal{B}(\mathcal{E}(y^2,\psi^2)_s)| \\ &\qquad \leqslant (\widehat{M}_0 + \widehat{M}_1)|\psi^1 - \psi^2|_{C_E} + (\widehat{M}_0 + \widehat{M}_1 e^{\lambda^-_N r})|y^1 - y^2|_{Y_1}. \end{aligned} \tag{3.5}$$

Let us fix $p \in E_1^-$ and $\psi \in \widehat{Q}C_E$. We define the map $\mathcal{F} : Y_1 \to Y_1$ as follows

(3.6)

$$\mathcal{F}(y)(t) \equiv e^{-t\mathcal{A}}p + \int_0^t e^{-(t-\tau)\mathcal{A}} P\mathcal{B}(\mathcal{E}(y)_\tau)\, d\tau + \int_{-T}^t e^{-(t-\tau)\mathcal{A}} Q\mathcal{B}(\mathcal{E}(y)_\tau)\, d\tau$$

where $t \in [-T, 0]$ and $\mathcal{E}(y) = \mathcal{E}(y, \psi)$.

If we find a fixed point of $\mathcal{F}$ *i.e.*, $\mathcal{F}(\bar{y}) = \bar{y}$, then $U(s) \equiv \mathcal{E}(\bar{y})(s)$ is a solution of (2.6) for $s \in [-T, 0]$ with the properties $PU(0) = p$, $\widehat{Q}\,U_{-T} = \psi$.

So for the moment, our goal is to find a fixed point of $\mathcal{F}$. Using the estimates (2.12) and Lemma 3.1 with $\psi^1 = \psi^2 = \psi$ we get

$$|\mathcal{F}(y^1) - \mathcal{F}(y^2)|_{Y_1} \leqslant \widehat{\gamma}_N(T)\left(\widehat{M_0} + \widehat{M_1} e^{\lambda_N^- r}\right)|y^1 - y^2|_{Y_1},$$

where

$$\widehat{\gamma}_N(T) \equiv \frac{e^{\lambda_N^- T} - 1}{\lambda_N^-} + \frac{1 - e^{-\lambda_{N+1}^- T}}{\lambda_{N+1}^-}. \tag{3.7}$$

It is evidently that for any N, we have $\widehat{\gamma}_N(T) \to 0$, when $T \to 0$. So if we choose T_0 such that

$$\delta \equiv \widehat{\gamma}_N(T_0)\left(\widehat{M_0} + \widehat{M_1} e^{\lambda_N^- r}\right) < 1, \tag{3.8}$$

then for any $T \in (0, T_0]$ we get $|\mathcal{F}(y^1) - \mathcal{F}(y^2)|_{Y_1} \leqslant \delta |y^1 - y^2|_{Y_1}$. Hence there exists a unique fixed point $\bar{y}$ of a strict contraction $\mathcal{F}$.

Now let us define the map Φ as follows. For fixed $p \in E_1^-$ and $\psi \in \widehat{Q}C_E$ denote by $\bar{y}$ the fixed point of $\mathcal{F}$ constructed by p and ψ. Hence

$$\Phi(p, \psi) \equiv \widehat{Q}\mathcal{E}(\bar{y})_0 \equiv \mathcal{E}(\bar{y})(\theta) - e^{-\mathcal{A}\theta}p, \quad \theta \in [-r, 0]. \tag{3.9}$$

We also prove that Φ is a Lipschitz mapping *i.e.*, we get (3.1) with

$$L_1(T) \equiv e^{\lambda_N^- r}\left[1 + \frac{e^{\lambda_N^- T}}{1 - \widehat{\gamma}_N(T)(\widehat{M_0} + \widehat{M_1} e^{\lambda_N^- r})}\right],$$

$$L_2(T) \equiv 1 + \frac{\gamma_N(T) e^{\lambda_N^- r}(\widehat{M_0} + \widehat{M_1})}{1 - \widehat{\gamma}_N(T)(\widehat{M_0} + \widehat{M_1} e^{\lambda_N^- r})}. \tag{3.10}$$

In the most interesting case $r < T$ we can get more refined estimate (see [25] for the technical details in the parabolic case)

$$L_1(T) = \widehat{\gamma}_N(r)\frac{e^{\lambda_N^- T}(\widehat{M_0} + \widehat{M_1} e^{\lambda_N^- r})}{1 - \widehat{\gamma}_N(T)(\widehat{M_0} + \widehat{M_1} e^{\lambda_N^- r})},$$

$$L_2(T) = e^{-\lambda_{N+1}^-(T-r)} + \widehat{\gamma}_N(r)(\widehat{M_0} + \widehat{M_1}) \cdot \left[1 + \frac{\widehat{\gamma}_N(T)(\widehat{M_0} + \widehat{M_1} e^{\lambda_N^- r})}{1 - \widehat{\gamma}_N(T)(\widehat{M_0} + \widehat{M_1} e^{\lambda_N^- r})}\right]. \tag{3.11}$$

We are interested in the case when the both Lipschitz constants $L_i(T), i = 1, 2$, of Φ (see (3.1)) are less than 1. We need the following refinement of the algorithm [25, Lemma 3.2] of choosing T and r with respect to N to guarantee that L_i are small enough.

LEMMA 3.2. — *Take any $\delta_1, \delta_2 \in (0,1]$. For any $c > 0$ denote $D_c \equiv 4e^c(e^c - e^{-c})$.*

(i) Let $\lambda_N^- > \left(1 + \frac{1}{\delta_1}\right)(\widehat{M}_0 + \widehat{M}_1 e^c)\frac{1}{4}D_c$. Then if r and T are such that $r < T$ and $\lambda_{N+1}^- T \leqslant c$, then $L_1(T) < \delta_1$.

(ii) Let $\lambda_N^- > \frac{1}{\delta_2}(\widehat{M}_0 + \widehat{M}_1 e^c)D_c e^{-c}$. Then the condition $r + \frac{\ln(2\delta_2^{-1})}{\lambda_{N+1}^-} < T \leqslant \frac{c}{\lambda_{N+1}^-}$, implies $L_2(T) < \delta_2$.

COROLLARY. — *Take any $\delta_1, \delta_2 \in (0,1]$. Then for any $c > \ln(2\delta_2^{-1})$ the conditions* (3.3) *and*

$$\lambda_N^- > (\widehat{M}_0 + \widehat{M}_1 e^c)D_c \cdot \frac{1}{4}\Big(1 + \frac{1}{\delta_1}\Big), \tag{3.12}$$

imply that $L_1(T) < \delta_1$ and $L_2(T) < \delta_2$. Here D_c is defined as in Lemma 3.2.

Proof of Lemma 3.2. — We need the explicit formulas for L_i (see (3.11) and also [25, (3.12)]). It is easy to see that $\widehat{\gamma}_N(T)e^{\lambda_N^- T}(\widehat{M}_0 + \widehat{M}_1 e^{\lambda_N^- r}) < \delta_1(\delta_1+1)^{-1}$ implies $L_1(T) < \delta_1$. Using (3.7), $r < T, \lambda_{N+1}^- T \leqslant c$ and $\widehat{\gamma}_N(T) \leqslant \frac{D_c}{4e^c\lambda_N^-}$ we get the condition $D_c(\widehat{M}_0 + \widehat{M}_1 e^{\lambda_N^- r})(4\lambda_N^-)^{-1} < \delta_1(\delta_1 + 1)^{-1}$. This proves (i).

Let us prove (ii). One has $e^{-\lambda_{N+1}^-(T-r)} < \frac{\delta_2}{2}$ iff $r + \frac{\ln(2\delta_2^{-1})}{\lambda_{N+1}^-} < T$. As in (i), we get $\widehat{\gamma}_N(r)(\widehat{M}_0 + \widehat{M}_1) < \frac{\delta_2}{4}$ if $D_c(\widehat{M}_0 + \widehat{M}_1)(4e^c\lambda_N^-)^{-1} < \frac{\delta_2}{4}$ and $\lambda_{N+1}r \leqslant c$. Hence, we need $\lambda_N^- > \frac{1}{\delta_2}(\widehat{M}_0 + \widehat{M}_1)D_c e^{-c}$. On the other hand, for $\frac{\widehat{\gamma}_N(T)(\widehat{M}_0+\widehat{M}_1 e^{\lambda_N^- r})}{1-\widehat{\gamma}_N(T)(\widehat{M}_0+\widehat{M}_1 e^{\lambda_N^- r})} < 1$ it is sufficient if $\widehat{\gamma}_N(T)(\widehat{M}_0 + \widehat{M}_1 e^{\lambda_N^- r}) < \frac{1}{2}$. It is possible if $\lambda_N^- T \leqslant c$ and $\lambda_N^- > \frac{1}{2}(\widehat{M}_0 + \widehat{M}_1 e^c)D_c e^{-c}$. Since $\delta_2 \leqslant 1$, we get (ii). The proof of Lemma 3.2 is complete.

LEMMA 3.3. — *Assume that* (3.2) *is satisfied. Take any $\delta_1, \delta_2 \in (0,1]$ and any $c > \ln(2\delta_2^{-1})$. Then there exists N_0 big enough such that for any $N \geqslant N_0$, any $\varepsilon^2 \in [2\mu_{N+1}, 2\mu_{N+1} + \mu_N]$ and T, r satisfying* (3.3), *we get $L_1(T) < \delta_1$ and $L_2(T) < \delta_2$.*

Proof of Lemma 3.3. — Corollary gives $L_1(T) < \delta_1$ and $L_2(T) < \delta_2$ if

$$\lambda_N^- \equiv \varepsilon - \sqrt{\varepsilon^2 - \mu_N} > \mu_{N+1}^{\alpha-1/2}(M_0 + M_1 e^c) \max\left\{1, \sqrt{\frac{\mu_{N+1}}{\varepsilon^2 - \mu_{N+1}}}\right\} D_c \cdot \frac{1}{4}\Big(1 + \frac{1}{\delta_1}\Big),$$

and (3.3) are satisfied (see also the definition of $\widehat{M_i}$). Let us multiply the last estimate by $\varepsilon + \sqrt{\varepsilon^2 - \mu_N}$

$$\mu_N > (\varepsilon + \sqrt{\varepsilon^2 - \mu_N}) \cdot \mu_{N+1}^{\alpha-1/2}(M_0 + M_1 e^c) \max\left\{1, \sqrt{\frac{\mu_{N+1}}{\varepsilon^2 - \mu_{N+1}}}\right\} D_c \cdot \frac{1}{4}\left(1 + \frac{1}{\delta_1}\right),$$

For any $\varepsilon^2 \in [2\mu_{N+1}, 2\mu_{N+1} + \mu_N]$ one has $\max\left\{1, \sqrt{\frac{\mu_{N+1}}{\varepsilon^2 - \mu_{N+1}}}\right\} = 1$ and $\varepsilon + \sqrt{\varepsilon^2 - \mu_N} \leqslant \sqrt{2\mu_{N+1} + \mu_N} + \sqrt{2\mu_{N+1}} \leqslant (\sqrt{3} + \sqrt{2}) \cdot \mu_{N+1}^{1/2}$. Hence it is sufficient if

$$\mu_N > (\sqrt{3} + \sqrt{2}) \cdot \mu_{N+1}^{\alpha}(M_0 + M_1 e^c) D_c \cdot \frac{1}{4}\Big(1 + \frac{1}{\delta_1}\Big).$$

Now the property $\inf \lim_{n\to\infty} \frac{\mu_n}{\mu_{n+1}} \equiv \eta_0 > 0$ gives that for any $q \in (0,1)$ there exists n_0 big enough such that $\mu_n \geqslant \eta_0 q \mu_{n+1}$ for $n \geqslant n_0$. Finally, the condition

$$\mu_{N+1}^{1-\alpha} > (\eta_0 q)^{-1}(\sqrt{3} + \sqrt{2}) \cdot (M_0 + M_1 e^c) D_c \cdot \frac{1}{4}\Big(1 + \frac{1}{\delta_1}\Big).$$

provides the existence of N_0 we need. The proof of Lemma 3.3 is complete.

4. AIMs containing all the stationary solutions.

Consider N, r and T satisfying (3.12), (3.3). For the moment, it is sufficient to take $\varepsilon = \delta = 1$ and for any $c > \ln 2$, assume $\lambda_N^- > (M_0 + M_1 e^c) D_c$ and $\lambda_{N+1}^- r + \ln 2 < \lambda_{N+1}^- T \leqslant c$ (see [25, Lemma 3.2]). Since $L_2 < 1$ we can use the mapping Φ (for any fixed $p \in E_1^-$) as a strict contraction in $\widehat{Q}C_E$:

$$|\Phi(p, \psi^1) - \Phi(p, \psi^2)|_{C_E} \leqslant L_2 |\psi^1 - \psi^2|_{C_E}, \quad L_2 < 1.$$

Consider the unique fixed point $\psi \in \widehat{Q}C_E$ of Φ constructed for $p \in P_N H$. We define the mapping $\Psi^T \equiv \Psi^{T,N} : E_1^- \to \widehat{Q}C_E$ as follows

$$\Psi^T(p) \equiv \Phi(p, \psi), \text{ where } \psi = \Phi(p, \psi). \tag{4.1}$$

Our goal is to prove that the graph of Ψ^T

$$\mathcal{M} \equiv \{e^{-A\theta} p + \Psi^T(p)(\theta) : p \in PE \equiv E_1^-\} \subset C_E. \tag{4.2}$$

is an approximate inertial manifold. Note that the manifold $\mathcal{M}$ contains all the steady states of (2.6). For this reason, following the terminology in [29] we call $\mathcal{M}$ the *steady approximate inertial manifold.*

Remark 4.1. — By definition, $\mathcal{M}$ contains also all the T-periodic solutions of (2.6).

We will suppose that (2.6) is dissipative *i.e.*, it possesses an absorbing ball in C_E. Using the existence of an absorbing ball for the equation we can classically truncate the nonlinear term $\mathcal{B}$ outside the ball so that it will be replaced by a function which is equal to $\mathcal{B}$ inside the absorbing ball but which has bounded support. We denote by R_d the radius of a ball containing this support. For concrete examples of dissipative retarded PDEs of second order in time see e.g., [22, 23, 24].

The following properties of Ψ^T and $\mathcal{M}$ will be used in the sequel.

LEMMA 4.1. — *Let r, T and N satisfy* (3.12), (3.3) *for some* $\delta_1, \delta_2 \in (0,1)$. *Then the mapping Ψ^T defined in* (4.1) *satisfies*

$$|\Psi^T(p^1) - \Psi^T(p^2)|_{C_E} \leqslant L_1(1-L_2)^{-1}|p^1 - p^2|, \tag{4.3}$$

where L_1, L_2 are Lipschitz constants of Φ. Moreover for all $0 \leqslant t \leqslant T$ one has

$$|\widehat{Q}U_t - \Psi^T(PU(t))|_{C_E} \leqslant (1-\bar{q})^{-1}\left[C_R^1 \cdot e^{-\gamma T} + C_R^2 \cdot e^{-\gamma t}\right], \tag{4.4}$$

for any solution of (2.6) $U = U(t)$ *with the initial condition* $U_0 = e^{-\mathcal{A}\theta}p + \Psi^T(p) \in C_E$ *when $t = 0$. Here γ is any number from the interval* $[\lambda_N, \lambda_{N+1}]$, *the positive constants C_R^1 and C_R^2 depend on the dissipativity radius R_d only, the positive constant $\bar{q}$ is defined as*

$$\bar{q} \equiv (\widehat{M_0} + \widehat{M_1}e^{\bar{\lambda}_{N+1}r}) \cdot T < \frac{1}{2}. \tag{4.5}$$

The proof of Lemma 4.1 uses similar arguments to [28].

The main result of this section is the following

THEOREM 4.1. — *Take any $\eta > 0$. There exist N and r_0 such that for any $r \in [0, r_0]$ and any $\varepsilon^2 \in [2\mu_{N+1}, 2\mu_{N+1} + \mu_N]$ there exists an N-dimensional approximate inertial manifold which contains all the steady states of* (2.6) *and the thickness of its attractive neighborhood is η. More precisely, we have*

$$\left|\widehat{Q}U_t - \Psi^T(PU(t))\right|_{C_E} \leqslant C_R \cdot \exp\left\{-\frac{2}{T}(t - t_*)\ln 2\right\} + \eta,$$

for all $t \geqslant t_* + \frac{T}{2}$ *and any solution* $U = U(t)$ *of* (2.6) *such that* $|U_t|_{C_E} \leqslant R$ *for* $t_* \leqslant t < \infty$.

Proof of Theorem 4.1. — We follow line of arguments given in [7] and use Lemmas 4.1 and 4.2. One can notice that we cannot directly apply the arguments of [7] to our retarded case since the considerations in [7] are in an essential manner based on the estimate (see [7, Lemma 2.2])

$$\|QA^\alpha u(t)\| \leqslant \Big[e^{-\lambda_{N+1}(t-s)} + M(1+k)a_1\lambda_{N+1}^{-1+\alpha} \cdot e^{a_2(t-s)}\Big] \, \|A^\alpha u(s)\|, \; t > s.$$

Unfortunately, there is no analogue of the last estimate in the retarded case for the projection $\widehat{Q}$ (see [25, Remark 3.8]). Instead, our proof is based on the Lipschitz properties of mappings Φ and Ψ. We follow arguments of [28] to prove

LEMMA 4.2. — *Assume that* N, T *and* r *satisfy*

$$\lambda_N^- > (\widehat{M}_0 + \widehat{M}_1 e^c) D_c \frac{1}{4}\left(1 + \frac{28}{3} a_1 e^{a_2 T}\right), \tag{4.6}$$

$$r + \frac{\ln 8}{\lambda_{N+1}^-} < \frac{T}{2} < T \leqslant \frac{c}{\lambda_{N+1}^-}, \tag{4.7}$$

for some $c > 2\ln 8$ *and the constants* a_1, a_2 *defined as in Lemma 2.1. Then the mapping* Ψ^T *defined in* (4.1) *possesses the following property*

$$|\widehat{Q}U_t - \Psi^T(PU(t))|_{C_\alpha} \leqslant C_R^1 \cdot \exp\Big\{-\frac{2}{T}(t-t_*)\ln 2\Big\} + C_R^2 \cdot \exp\Big\{-\frac{1}{2}\lambda_{N+1}T\Big\}, \tag{4.8}$$

for all $t \geqslant t_* + \frac{T}{2}$ *and any solution* $U = U(t)$ *of* (2.6) *such that* $|U_t|_{C_E} \leqslant R$ *for* $t_* \leqslant t < \infty$.

Now to complete the proof of Theorem 4.1 we need to choose the parameters c, N, T and r_0 satisfying the conditions of Lemma 4.2 in such a way that the last term in (4.8) is less than η. Let us do it. Take any $\eta > 0$ (the thickness of the attractive neighborhood). Choose $c > 2\ln 8$ such that $C_R^2 e^{-c/2} \leqslant \eta$. Here C_R^2 is defined in (4.8). Then choose N to satisfy (4.6). Now we take $T \equiv c\lambda_{N+1}^{-1}$ to satisfy (see (4.7)) $\frac{\ln 8}{\lambda_{N+1}} < \frac{T}{2} < T \leqslant \frac{c}{\lambda_{N+1}}$. It is now easy to see that r_0 can be chosen less than $\frac{T}{2} - \frac{\ln 8}{\lambda_{N+1}}$ to satisfy (4.7). Lemma 4.2 completes the proof of Theorem 4.1.

Remark 4.2. — To construct the AIM (4.2) we found fixed points of two contractions: $\mathcal{F}$ (see (3.6)) to construct $\Phi(p,\psi)$ and $\Phi(p,\cdot)$, to get Ψ^T.

We can approximate the graph of Ψ^T by a sequence of manifolds given more explicitly. Consider for a fixed $p \in P_N H$ the mapping $\Omega(\psi, y) = (\Omega_1(\psi, y); \Omega_2(\psi, y))$ in the space $\Xi \equiv \widehat{Q}_N C_E \times Y_1$ with the norm $|(\psi, y)|_\Xi \equiv |\psi|_{C_E} + |y|_{Y_1}$, where

$$\Omega_1(\psi, y)(\theta) \equiv e^{-(\theta+T)\mathcal{A}}\psi(0) + \int_0^\theta e^{-(\theta-\tau)\mathcal{A}} P\mathcal{B}\left(\mathcal{E}(y,\psi)_\tau\right) d\tau + \int_{-T}^\theta e^{-(\theta-\tau)\mathcal{A}} Q\mathcal{B}\left(\mathcal{E}(y,\psi)_\tau\right) d\tau,$$

$$\Omega_2(\psi, y)(\theta) \equiv \mathcal{F}(y, \psi)(t).$$

One can easily check that conditions (3.3), (3.12) imply the contraction of Ω : $|\Omega(\psi^1, y^1) - \Omega(\psi^2, y^2)|_\Xi \leqslant \varepsilon \cdot |(\psi^1, y^1) - (\psi^2, y^2)|_\Xi$, with $\varepsilon \equiv e^{-\lambda_{N+1}^-(T-r)} + \delta < 1$, where $\delta < \frac{1}{2}$ (due to (3.3), (3.12)) is defined in (3.8). So we can approximate the fixed point of Ω by the convergent sequence $(\psi^n, y^n) \equiv \Omega(\psi^{n-1}, y^{n-1}), n = 1, 2, \ldots$ If we define $\Psi_n^T(p) \equiv \widehat{Q}\mathcal{E}(y^n, \psi^n)_0 : PE \to \widehat{Q}C_E$, we approximate manifold $\mathcal{M}$ defined in (4.2) by the sequence

$$\mathcal{M}_n \equiv \left\{e^{-\mathcal{A}\theta}p + \Psi_n^T(p)(\theta) : p \in PE \equiv E_1^-\right\} \subset C_E.$$

It is evidently, that $\sup\{\|A^\alpha(\Psi_n^T(p) - \Psi^T(p))\|_{C_E} : \|p\| \leqslant R\} \leqslant \varepsilon^n \cdot C_R$, where $\varepsilon < 1$, and C_R depends on the dissipativity radius R_d only.

5. Finite number of essential modes.

Many interesting results on finite number of determining parameters for infinite-dimensional dynamical systems (see e.g. [6]) have been obtained since the pioneering work of C. Foias and G. Prodi [4].

As in [14] (see also [25] for the case of semilinear retarded parabolic equations), using IMD, we prove the following result on the finite number of essential modes.

THEOREM 5.1. — *Let T, r and N be as in Theorem 3.1 such that the Lipschitz constants $L_i < 1$, $i = 1, 2$. Consider any sequence $\{t_k\}_{k=1}^\infty$ such that $0 = t_1 < t_2 < \cdots < t_k \to +\infty$ and*

$$r + \frac{\ln 2}{\lambda_{N+1}} < t_{i+1} - t_i \leqslant T, \; i = 1, 2, \ldots$$

Then if for any two solutions $U^1(t), U^2(t)$ of (2.6) we have

$$|P_N(u^1(t_k) - u^2(t_k))|_\alpha \to 0, \quad \text{when} \quad k \to \infty \tag{5.1}$$

then

$$|u_t^1 - u_t^2|_{C_\alpha} \to 0, \quad \text{when} \quad t \to \infty.$$

This result extends the one proved in [23], where we used $|P_N(u^1(t) - u^2(t))|_\alpha \to 0$, when $t \to \infty$ instead of (5.1).

6. Characterization of a $\mathcal{K}$-invariant manifold.

In this section using the existence of inertial manifold with delay given by Theorem 3.1, we give a characterization of a $\mathcal{K}$-invariant manifold constructed by M. Taboada and Y.C. You in [16]. This manifold lives in the phase space C_E of retarded system (2.6) and is a locally attracting surface weaved by the solutions of a *nonretarded* system (see (6.1)).

Assume, in addition to **(A1)**, **(A2)**, that the nonlinear term B_0 (which does not involve the delay) satisfies

(A3) *B_0 is Fréchet differentiable on $D(A^{1/2})$ and its Fréchet derivative is locally Lipschitz continuous in u.*

(A4) *For any b such that $0 < b < \infty$ and any function $u \in C([-r, b); D(A^{1/2})) \cap C^1([0, b); H)$, the following holds*

$$\int_0^t (B_0(u(s)), \dot{u}(s))ds \leqslant C(u^0; u^1) < \infty$$

where $(\cdot, \cdot)$ denotes the inner product in H and $C(u^0; u^1)$ is a constant independent of b, but which may depend on the initial data $u^0 = u(0)$ and $u^1 = \dot{u}(0)$.

Consider the following *nonretarded* equation

$$\partial_t U + \mathcal{A}U = \mathcal{K}(U(t)), \tag{6.1}$$

where nonlinear map $\mathcal{K}$ has the form

$$\mathcal{K} = \begin{pmatrix} \mathcal{K}^1(U) \\ B_0(u^0) + \mathcal{K}^2(U) \end{pmatrix}. \tag{6.2}$$

Here, as before, $U = (u^0; u^1) \in E$. We need the following

DEFINITION 6.1 ([16]). — *The retarded evolution equation has the $\mathcal{K}(\Omega)$-Property if there is a strongly continuous mapping $\mathcal{K} : \Omega \subset E \to E$ (of the form (6.2)) such that*

$$\mathcal{K}(\xi) = \mathcal{B}(U(\cdot)), \tag{6.3}$$

where $U(\theta)$ is the solution of (6.1) *for $\theta \in [-r, 0]$ and initial data $U(0) = \xi \in E$.*

Remark 6.1. — We will consider $\mathcal{K}^i, i = 1, 2$, which are locally Lipschitz continuous and locally bounded, so the unique solution, considered in the definition, exists.

The existence of such mapping $\mathcal{K}$ under the above assumptions is the main result of [16].

THEOREM 6.1 ([16]). — *Under the assumptions* **(A1)–(A4)**, *the retarded evolution equation* (2.6) *has the $\mathcal{K}(\Omega)$-Property, provided the delay $r > 0$ is sufficiently small. More precisely, for any $d > 0$, there exists an $r_0 >$ such that if $0 < r \leqslant r_0$, there is a continuous mapping (of the form* (6.2)*)*

$$\mathcal{K} : \Omega = \{\xi \in E : \|\xi\| \leqslant d\} \to E$$

that satisfies (6.3)*. The same result holds if instead of assuming that r is small we assume that the magnitude and Lipschitz constants of B_1 are sufficiently small (see* (2.3)*).*

DEFINITION 6.2 ([16]). — *A C^0 manifold $\mathcal{M}$ in the Banach space C_E is called a $\mathcal{K}(\Omega)$-*invariant manifold *for the retarded equation* (2.6) *if:*

(i) *For any $U_0 \in \mathcal{M}$, the global solution of* (2.6) *exists and lies always on $\mathcal{M}$.*

(ii) *There is a strongly continuous mapping $\mathcal{K} : \Omega \to E$ such that $\mathcal{M}$ is weaved by the mild solutions of the nonretarded equation* (6.1) *associated with this $\mathcal{K}$ for $t \in [-r, \infty)$.*

Sufficient conditions for the existence of a $\mathcal{K}$-invariant manifold are given in [16, Corollaries 5.4, 5.5]. Using these results we prove

LEMMA 6.1. — *Let $\varepsilon^2 > \mu_{N+1}$. There exists T_0 such that any solution $U(t)$ of* (2.6) *satisfies*

$$\widehat{Q}\, U_t = \Phi_T^r \left(P\, U(t),\ \widehat{Q}\, U_{t-T}\right),$$

for any $T \in (0, T_0]$ and any $t \geqslant 0$. Here the mapping Φ_T^r is constructed in Theorem 3.1 for each $T \in (0, T_0]$.

Assume additionally that **(A1)–(A4)** *are satisfied and there exists r^0, such that for any $r \in (0, r^0]$ there is a $\mathcal{K}$-invariant manifold $\mathcal{M}$ (see [16,* *Corollaries 5.4, 5.5]). Then any solution $U(t)$ of (2.6) such that $U_t \in \mathcal{M}$ satisfies*

$$\widehat{Q}\,U_t = \widehat{\Phi}^0_T\,(P\,U(t),\; Q\,U(t-T))\;,$$

for any $T \in (0, T_0]$ and any $t \geqslant 0$. Here the mapping $\widehat{\Phi}^0_T : PE \times QE \to \widehat{Q}\,C_E$ is Lipschitz.

THEOREM 6.2. — *Let $\varepsilon^2 > \mu_{N+1}$. Assume that* **(A1)–(A4)** *are satisfied and r^0 is small enough such that for any $r \in (0, r^0]$ the invariant manifold $\mathcal{M}$ exists (see Def. 6.2 and [16, Corollaries 5.4, 5.5]).*

Then a solution $U(t)$ of (2.6) satisfies $U_t \in \mathcal{M}$ if and only if *$U(t)$ is uniquely defined by values $PU(t_1)$ and $QU(t_2)$ for any t_1, t_2 provided $t_1 - t_2 \in [0, T_0]$. Here T_0 is defined by IMD for the nonretarded equation (6.1).*

Remarks 6.2.

(a) If $U_{t_1} \in \mathcal{M}$ for some t_1, then $U(t)$ is defined for all $t \in R$ and $U_t \in \mathcal{M}$ for each $t \in R$.

(b) If $U(\cdot)$ is uniquely defined by values $PU(t_1)$ and $QU(t_2)$ for some t_1, t_2 such that $t_1 - t_2 \in [0, T_0]$, then Theorem 6.2 and the previous remark give that $U(\cdot)$ is uniquely defined by values $PU(s_1)$ and $QU(s_2)$ for arbitrary s_1, s_2 such that $s_1 - s_2 \in [0, T_0]$.

(c) We notice that when r changes (in $(0, r^0]$), the manifold $\mathcal{M} = \mathcal{M}^r$ also changes, moreover the space $C_E = C_E^r$ changes, but it does not affect the criterion in Theorem 6.2 since the value T_0 does not depend on r (see the proof of Theorem 6.2).

(d) If we choose $t_1 = t_2$, we arrive to the idea of construction of manifold $\mathcal{M}$ [16] which is weaved by trajectories of the nonretarded evolution equation (6.1).

Proof of Theorem 6.2. — Consider a solution $U(t)$ of (2.6) satisfying $U_t \in \mathcal{M}$. By construction of $\mathcal{M}$ [16], $U(t)$ is a solution of (6.1). We apply Theorem 3.1 to (6.1) and define IMD $\Phi^0_T : PE \times QE \to QE$ for any $T \in (0, T_0^r]$. Here T_0^r depends on r. Hence $Q\,U(t) = \Phi^0_T\,(P\,U(t),\; Q\,U(t-T))$. Using the solution of (6.1) which defines IMD, we can easily introduce the mapping $\widehat{\Phi}^0_T : PE \times QE \to \widehat{Q}C_E$, so $\widehat{Q}\,U_t = \widehat{\Phi}^0_T\,(P\,U(t),\; Q\,U(t-T))$. That means that the solution $U(t)$ is uniquely defined by values $PU(t)$ and $QU(t-T)$. It is easy to see from the proof of Theorem 3.1 that there exists positive $T_0 \equiv \min\{T_0^r : r \in (0, r^0]\} > 0$. The first part of the theorem is proved.

Consider a solution $U(t)$ of (2.6) which is uniquely defined by values $p \equiv PU(t_1)$ and $q \equiv QU(t_2)$ for some t_1, t_2 such that $t_1 - t_2 \in [0, T_0]$. Since $t_1 - t_2 \in [0, T_0]$ we can construct IMD (for (6.1)) $\Phi^0_{t_1-t_2}$ *i.e.*, $T = t_1 - t_2$ by the *unique* solution $\widetilde{U}(t)$ of (6.1) satisfying $P\widetilde{U}(t_1) = p$ and $Q\widetilde{U}(t_2) = q$. By construction, any solution of (6.1) is a solution of (2.6), hence $\widetilde{U}(t)$ also. Since there is only one solution of (2.6) with the property $PU(t_1) = p$, $QU(t_2) = q$, we get $U(t) \equiv \widetilde{U}(t)$. Hence $U(t)$ is a solution of (6.1) and, by definition of $\mathcal{M}$, we conclude that $U_t \in \mathcal{M}$ for all t. The proof of Theorem 6.2 is complete.

As an application of our results we can also consider the dissipative Klein-Gordon equation with a retarded perturbation (see [16, Section 7] for more details).

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