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Antonio BOVE & François TREVES

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ON THE GEVREY HYPO-ELLIPTICITY OF SUMS OF SQUARES OF VECTOR FIELDS

by **Antonio BOVE** and **François TREVES**

1. Introduction.

This article consists of two parts. Part I is devoted to the Poisson stratification of an analytic variety defined by the vanishing of r real-valued, real-analytic, functions in an open subset $\mathcal{U}$ of a symplectic manifold $\mathcal{M}$. This concept is then specialized to the case where $\mathcal{M} = T^*\Omega\backslash 0$, the cotangent bundle of an analytic manifold with the zero section elided (actually, since our viewpoint is strictly local we take Ω to be an open subset of $\mathbb{R}^n$) and to functions that are (after division by $\sqrt{-1}$) the symbols of real vector fields X_i in Ω. It is used to conjecture a necessary and sufficient condition for the sum-of-squares operator $-L = X_1^2 + \cdots + X_r^2$ to be analytic hypo-elliptic.

A $\mathcal{C}^\omega$ map $\mathbf{F} = (F_1, ..., F_r) : \mathcal{U} \longrightarrow \mathbb{R}^n$ defines the Poisson stratification of the variety $\mathbf{V} = \overset{-1}{\mathbf{F}}(0)$ in three steps. *Step 1*: Through the inductive use of $\mathbf{F}$ and of its differentials of every order, $D^m\mathbf{F}$, an analytic stratification of $\mathbf{V}$ is defined. *Step 2*: By using the rank of the pullback of the fundamental symplectic form σ of $\mathcal{M}$ to every analytic leaf (and repeating Step 1 as many times as needed), one gets a stratification by (disjoint) analytic submanifolds (the "symplectic leaves") on each of which σ has constant rank. *Step 3*: In each symplectic leaf one looks at the submanifolds defined

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by the vanishing of the Poisson multibrackets $\{F_0, \{F_1, \dots \{F_{\ell-1}, F_\ell\}, \dots\}\}$ of a given length $\ell = 0, 1, \dots$ (and the nonvanishing of at least one multibracket of length $\ell + 1$). By repeating Steps 1 and 2 as many times as needed, one gets the final *Poisson stratification*, in which each stratum is a real-analytic Poisson submanifold of $\mathbf{V}$.

CONJECTURE. — *The sum of squares $-L$ is analytic hypo-elliptic if and only if every Poisson stratum of the set* $\operatorname{Char} L$ *of common zeros of the symbols $F_j = \sqrt{-1}\,\mathbf{symb}\,(X_j)$ in phase space $\mathcal{M} = \Omega \times (\mathbb{R}^n \backslash \{0\})$ is symplectic (i.e., the pullback to the stratum of the fundamental symplectic form is nondegenerate).*

If a stratum Σ is not symplectic it has a foliation by bicharacteristic curves. Generically, such a bicharacteristic curve $\mathfrak{c}$ is either a ray, *i.e.*, the base projection of $\mathfrak{c}$ is a single point $x^\circ \in \Omega$, or else projects onto a "true" curve $\gamma \subset \Omega$. In Part II we describe an approach which, we hope, will lead to the determination of the precise *Gevrey hypoellipticity* of L, *i.e.*, of the smallest number $s \geqslant 1$ such that if Lu is analytic, then u is of Gevrey class G^s (in the vicinity of some point of $T^*\Omega\backslash 0$). In this context the microlocal neighborhoods are to be understood in terms of higher-order microlocalization (the order is linked to the depth of the stratum). We associate a precise Gevrey threshold to bicharacteristics that are rays (*i.e.*, "vertical") providing a lower bound for the Gevrey index of the solutions of $Lu \in \mathcal{C}^\omega$. When the base projection of the bicharacteristic curve $\mathfrak{c}$ is a "true" curve (*i.e.*, $\mathfrak{c}$ is "horizontal") we limit ourselves to describing our approach and a couple of examples. We hope to complete this work in a future publication, by providing a precise algorithm that will yield the Gevrey threshold associated to horizontal bicharacteristics.

We wish to thank N. Hanges for pointing out an error in our original definition of the analytic stratification (Subsection 2.1).

2. The Poisson stratification and the analytic hypo-ellipticity conjecture.

2.1. Analytic stratification of an analytic set.

Throughout this subsection $\mathbf{V}$ shall denote the set of common zeros of a finite family $F_1, \dots, F_r$ of real-valued $\mathcal{C}^\omega$ functions in an open subset $\mathcal{U}$

of Euclidean space $\mathbb{R}^N$. Needless to say, $N \geqslant 1$, $r \geqslant 1$ and $\emptyset \neq \mathbf{V} \neq \mathcal{U}$. The purpose of this subsection is to define an analytic stratification of $\mathbf{V}$ of a very specific nature: we want to decompose $\mathbf{V}$ as a (locally finite) union of *disjoint*, connected $\mathcal{C}^\omega$ submanifolds of $\mathcal{U}$, the *analytic strata*, each of which is defined in an open (and dense) subset of $\mathcal{U}$ that contains it, by the vanishing of a finite set of polynomials with respect to $F_1, ..., F_r$ and their partial derivatives (of any order).

To achieve this we proceed as follows. Let m be the maximum value of $\operatorname{rank} \mathbf{F}$ in $\mathbf{V}$; $0 \leqslant m \leqslant \min(N, r)$. We define a "functor" $\mathfrak{T}$ which assigns to $\mathbf{V}$ three analytic "objects":

(1) a $\mathcal{C}^\omega$ submanifold $\mathfrak{R}_0(\mathbf{V})$, whose connected components constitute our first batch of strata;

(2) a subvariety $\mathbf{V}_1 \subset \mathbf{V}$ defined by the vanishing of the minors of rank m of the Jacobian matrix $\frac{\partial(F_1,...,F_r)}{\partial(x_1,...,x_N)}$;

(3) a subvariety $\mathbf{V}_2 \subset \mathbf{V} \backslash (\mathbf{V}_1 \cup \mathfrak{R}_0(\mathbf{V}))$ defined by the vanishing of the minors of rank $m+1$ of $\frac{\partial(F_1,...,F_r)}{\partial(x_1,...,x_N)}$.

Having done this we apply the operation $\mathfrak{T}$ to $\mathbf{V}_1$ and $\mathbf{V}_2$ separately by using their defining functions, the F_j's and the specified minors, thus obtaining two new batches of analytic strata $\mathfrak{R}_0(\mathbf{V}_i)$ $(i = 1, 2)$ and four analytic subvarieties $\mathbf{V}_{i,j}$ $(i, j = 1, 2)$. Next we apply the functor $\mathfrak{T}$ to the latter, and so on and so forth.

We define $\mathfrak{R}_0(\mathbf{V})$ as the subset of $\mathbf{V}$ consisting of the points x° having an open neighborhood $N(x^\circ) \subset \mathcal{U}$ with the following property:

- *There is an an open neighborhood $\mathcal{N}(x^\circ) \subset \mathcal{U}$ and a set of indices j_α $(\alpha = 1, ..., m,\ 1 \leqslant j_1 < \cdots < j_m \leqslant N)$ such that*

$$\mathcal{N}(x^\circ) \cap \mathbf{V} = \{x \in \mathcal{N}(x^\circ);\ F_{j_\alpha}(x) = 0,\ \alpha = 1, ..., m\}$$

and $dF_{j_1}(x^\circ) \wedge \cdots \wedge dF_{j_m}(x^\circ) \neq 0$, i.e., some minor $D^{i_1,...,i_m}_{j_1,...,j_m} = \det \frac{\partial(F_{j_1},...,F_{j_m})}{\partial(x_{i_1},...,x_{i_m})}$ does not vanish at x°. .

Note that $\mathfrak{R}_0(\mathbf{V})$ is a $\mathcal{C}^\omega$ submanifold of $\mathcal{U}$ of codimension m; it is a relatively open, *possibly empty*, subset of the regular part of $\mathbf{V}$, *i.e.*, the subset of $\mathbf{V}$ consisting of the points in a neighborhood of which $\mathbf{V}$ is a $\mathcal{C}^\omega$ submanifold; but the latter can be strictly larger than $\mathfrak{R}_0(\mathbf{V})$ even if $\mathbf{V}$ is irreducible, as seen in Example 1 below (for an example of a *reducible* algebraic set see Remark 1). The complement $\mathbf{V} \backslash \mathfrak{R}_0(\mathbf{V})$ of $\mathfrak{R}_0(\mathbf{V})$ can be regarded as the union of two subsets:

(1) the subset $\mathbf{V}_1$ of $\mathbf{V}$ in which all the minors $\left\{D^{i_1,\dots,i_m}_{j_1,\dots,j_m}\right\}_{\substack{1\leqslant i_1<\dots<i_m\leqslant N,\\ 1\leqslant j_1<\dots<j_m\leqslant r}}$ of rank m of the Jacobian matrix vanish identically;

(2) the set $\mathbf{V}_2$ of the points $x^\circ \in \mathbf{V}$ having an open neighborhood, $\mathcal{N}(x^\circ) \subset \mathcal{U}$, in which, for some set of indices j_α ($\alpha = 1,\dots,m$, $1 \leqslant j_1 < \cdots < j_m \leqslant N$), $dF_{j_1} \wedge \cdots \wedge dF_{j_m}$ does not vanish at any point and
$$\mathcal{N}(x^\circ) \cap \mathbf{V} \subsetneqq \{x \in \mathcal{N}(x^\circ);\ F_{j_\alpha}(x) = 0,\ \alpha = 1,\dots,m\}.$$

We point out that $\mathbf{V}_2 = \varnothing$ if $m = r$. A crucial aspect of this decomposition lies in the following claim:

The union $\mathbf{V}_1 \cup \mathfrak{R}_0(\mathbf{V})$ *is a closed subset of* $\mathbf{V}$.

Proof. — Indeed, if the neighborhood $\mathcal{N}(x^\circ) \subset \mathcal{U}$ of $x^\circ \in \mathbf{V}_2$ is as in Property #2 then $\mathcal{N}(x^\circ) \cap \mathbf{V}_1 = \varnothing$ by definition. At no one of its points can $\mathcal{N}(x^\circ) \cap \mathbf{V}$ be a $\mathcal{C}^\omega$ submanifold of codimension m; for if it were, this submanifold would have to be identical to $\{x \in \mathcal{N}(x^\circ);\ F_{j_\alpha}(x) = 0,\ \alpha = 1,\dots,m\}$ (assuming the latter submanifold to be connected). Thus $\mathcal{N}(x^\circ) \cap \mathfrak{R}_0(\mathbf{V}) = \varnothing$. □

The zero set of $\mathbf{F}$ in $\mathcal{U}\setminus(\mathbf{V}_1 \cup \mathfrak{R}_0(\mathbf{V}))$ is the analytic subvariety $\mathbf{V}_2$; it is also the zero set in $\mathcal{U}\setminus(\mathbf{V}_1 \cup \mathfrak{R}_0(\mathbf{V}))$ of all the minors $D^{i_1,\dots,i_{m+1}}_{j_1,\dots,j_{m+1}}$ ($1 \leqslant i_1 < \cdots < i_{m+1} \leqslant N$, $1 \leqslant j_1 < \cdots < j_{m+1} \leqslant r$) of the Jacobian matrix $\frac{\partial(F_1,\dots,F_r)}{\partial(x_1,\dots,x_N)}$. We may now repeat, for $\mathbf{V}_2$, the previous decomposition starting with the map
$$\mathbf{F}^{(2)} = \left(F_1,\dots,F_r,\{D^{i_1,\dots,i_{m+1}}_{j_1,\dots,j_{m+1}}\}_{\substack{1\leqslant i_1<\dots<i_{m+1}\leqslant N,\\ 1\leqslant j_1<\dots<j_{m+1}\leqslant r}}\right)\mathcal{U}\setminus(\mathbf{V}_1\cup\mathfrak{R}_0(\mathbf{V})) \longrightarrow \mathbb{R}^{r_{1,2}}$$
with $r_{1,2} = r + \#\left(\text{minors} D^{i_1,\dots,i_{m+1}}_{j_1,\dots,j_{m+1}}\right)$. As for $\mathbf{V}_1$ it is an analytic subset of $\mathcal{U}$ of the same type as $\mathbf{V}$. We may repeat the same decomposition starting with the map
$$\mathbf{F}^{(1)} = \left(F_1,\dots,F_r,\left\{D^{i_1,\dots,i_m}_{j_1,\dots,j_m}\right\}_{\substack{1\leqslant i_1<\dots<i_m\leqslant N,\\ 1\leqslant j_1<\dots<j_m\leqslant r}}\right):\mathcal{U} \longrightarrow \mathbb{R}^{r_{1,1}}$$
with $r_{1,1} = r + \#\left(\text{minors} D^{i_1,\dots,i_m}_{j_1,\dots,j_m}\right)$.

Remark 1. — Denote by m_1 (resp., m_2) the maximum value of $\operatorname{rank}\mathbf{F}^{(1)}$ (resp., $\operatorname{rank}\mathbf{F}^{(2)}$) on $\mathbf{V}_1$ (resp., $\mathbf{V}_2$). Whereas we always have $m_2 \geqslant m_1$ it may happen that $m_1 < m$ as shown in the classical example of Milnor: in $\mathbb{R}^2$ take $F_1 = x(1-x)$, $F_2 = y(1-x)$. Here $m = 2$ and $\mathfrak{R}_0(\mathbf{V}) = \{(0,0)\}$ whereas $m_1 = 1$ and $\mathbf{V}_1$ is the vertical line $x = 1$.

As we have said we iterate indefinitely the operation $\mathfrak{T}$; the end product of this procedure is a decomposition

$$\mathbf{V} = \bigcup_{\alpha=0}^{\infty} \Lambda_\alpha \tag{2.1}$$

in which the connected analytic submanifolds Λ_α are pairwise disjoint. Since the ideal of germs of analytic functions at a point is Noetherian we see that every compact subset of $\mathcal{U}$ intersects only finitely many Λ_α.

By our construction, the Λ_α form a "tree" rather than a linear sequence. The following property of the Λ_α will play a simplifying role in the forthcoming: for each $\alpha \in \mathbb{Z}_+$ there are finitely many differential polynomials

$$P_{\alpha,k}[\mathbf{F}] = P_{\alpha,k}(F_1, ..., F_r, ..., \partial_x^{p_1} F_1, ..., \partial_x^{p_r} F_r, ...), \; k = 1, ..., \nu_\alpha,$$

and an open (and dense) subset $\mathcal{U}_\alpha$ of $\mathcal{U}$ such that

$$\Lambda_\alpha = \left\{x \in \mathcal{U}_\alpha; \; P_{\alpha,k}[\mathbf{F}](x) = 0, \; k = 1, ..., \nu_\alpha\right\}. \tag{2.2}$$

Moreover, each point $x^\circ \in \Lambda_\alpha$ has an open neighborhood $\mathcal{N}(x^\circ) \subset \mathcal{U}_\alpha$ in which the rank of the map

$$x \longrightarrow \left(P_{\alpha,1}[\mathbf{F}](x), ..., P_{\alpha,\nu_\alpha}[\mathbf{F}](x)\right)$$

is exactly equal to $\operatorname{codim} \Lambda_\alpha$.

DEFINITION 1. — *The decomposition* (1) *will be called the* analytic stratification *of* $\mathbf{V}$ *and each submanifold* Λ_α *will be referred to as an* analytic stratum *of* $\mathbf{V}$.

Implicit in this definition is the role of the map $\mathbf{F}: \mathcal{U} \longrightarrow \mathbb{R}^r$. But if $\mathbf{G}$ is a $\mathcal{C}^\omega$ diffeomorphism of an open neighborhood of the origin in $\mathbb{R}^r$ onto another such neighborhood, the analytic stratification of $\mathbf{V}$ viewed as the null set of $\mathbf{G} \circ \mathbf{F}$ is the same as its stratification when $\mathbf{V}$ is viewed as the null set of $\mathbf{F}$.

Example 1. — Consider the *Whitney umbrella*, *i.e.*, the variety $\mathbf{V} = \left\{(x,y,z) \in \mathbb{R}^3; \; x^2 - zy^2 = 0\right\}$. We have

$$\mathfrak{R}_0(\mathbf{V}) = \left\{(x,y,z) \in \mathbb{R}^3; \; x^2 - zy^2 = 0, \; x^2 + y^2 \neq 0\right\}$$

and

$$\mathbf{V}_1 = \left\{(x,y,z) \in \mathbb{R}^3; \; x = y = 0\right\}.$$

But the map whose rank we use to define $\mathbf{V}_1$ is the map $\mathbb{R}^3 \longrightarrow \mathbb{R}^4$ defined by

$$(x,y,z) \Longrightarrow \left(x^2 - zy^2, x, yz, y^2\right).$$

Its differential is given by the matrix

$$\begin{pmatrix} 2x & -2yz & -y^2 \\ 1 & 0 & 0 \\ 0 & z & y \\ 0 & 0 & 2y \end{pmatrix}$$

whose restriction to $\mathbf{V}_1$ has rank 2 for $z \neq 0$ and rank 1 at the origin. The analytic stratification of $\mathbf{V}$ consists of 5 strata.

Let $\mathbf{W}$ be an analytic subset of $\mathcal{U}$ defined by the vanishing of finitely many analytic functions $G_1, ..., G_s$. Suppose $\mathbf{W} \subset \mathbf{V}$ and let $\mathbf{W} = \bigcup_{\beta=1}^{\infty} \Lambda'_\beta$ be the analytic stratification of $\mathbf{W}$. In general it is *not* true that to each β there is α such that $\Lambda'_\beta \subset \Lambda_\alpha$, as shown in the following

Example 2. — The analytic stratification of $\mathbf{V} = \{(x, y, z) \in \mathbb{R}^3; z^2 = x^4 + y^4\}$ consists of three strata:

$$\{0\}, \; \mathbf{V}_+ = \left\{(x, y, z) \in \mathbb{R}^3;\, z^2 = x^4 + y^4,\; z > 0\right\},$$

$$\mathbf{V}_- = \left\{(x, y, z) \in \mathbb{R}^3;\; z^2 = x^4 + y^4,\; z < 0\right\}.$$

Now take $\mathbf{W} = \left\{(x, y, z) \in \mathbb{R}^3;\, z = x^2,\; y = 0\right\}$; the analytic stratification of W consists of a single stratum, W itself. But $\mathbf{W} \subset \mathbf{V}_+ \cup \{0\}$ and $0 \in \mathbf{W}$, $\mathbf{W} \cap \mathbf{V}_+ \neq \emptyset$.

2.2. Symplectic stratification of an analytic submanifold.

In this subsection we take the dimension N to be even, $N = 2n$, and we take $\sigma = \sum_{j=1}^{n} dx_{n+j} \wedge dx_j$ as the fundamental symplectic two-form in $\mathbb{R}^{2n}$. If $x \in \mathbb{R}^{2n}$ we denote by σ_x the nondegenerate skew-symmetric bilinear form induced by σ on the tangent space $T_x\mathbb{R}^{2n}$. To each germ of real-valued $\mathcal{C}^\omega$ function f at x there is a unique germ of $\mathcal{C}^\omega$ vector field at x, which we denote by H_f, defined by the property that, for any tangent vector $\mathbf{v}$ to $\mathcal{U}$ at x, $\sigma_x(H_f, \mathbf{v}) = \langle df(x), \mathbf{v}\rangle$. Here $df(x)$ is the differential of f at x and $\langle , \rangle$ is the duality bracket between tangent vectors and cotangent vectors; H_f is usually referred to as the *Hamiltonian vector field* of f. If g is another germ of $\mathcal{C}^\omega$ function at x we denote the *Poisson bracket* of f and g by $\{f, g\} = \sigma(H_f, H_g)$.

Let Σ be a connected submanifold of $\mathcal{U}$ of class $\mathcal{C}^\omega$ endowed with a property similar to that of the submanifolds Λ_α in (2.2):

(2.3) there are functions $G_j \in \mathcal{C}^{\omega}(\mathcal{U})$ $(j = 1, ..., s < +\infty)$ and an open set $\mathcal{U}' \subset \mathcal{U}$ such that $\Sigma = \{x \in \mathcal{U}';\ G_j(x) = 0,\ j = 1, ..., s\}$ and the rank of the map $\mathbf{G} = (G_1, ..., G_s)$ is equal to $\operatorname{codim} \Sigma$ at every point of Σ .

Henceforth we shall assume that (2.3) holds. Then, if $d = \operatorname{codim} \Sigma$ is the rank of $\mathbf{G}(x)$, $x \in \Sigma$, each point x° of Σ has an open neighborhood $\mathcal{N}(x^{\circ}) \subset \mathcal{U}$ in which there are indices $1 \leqslant i_1 < \cdots < i_d \leqslant s$ such that the following is true:

(1) $dG_{i_1} \wedge \cdots \wedge dG_{i_d} \neq 0$ at x°;

(2) $\Sigma \cap \mathcal{N}(x^{\circ}) = \{x \in \mathcal{N}(x^{\circ});\ G_{i_1}(x) = \cdots = G_{i_d}(x) = 0\}$.

We denote by σ_{Σ} the pullback of σ to Σ: for each $x \in \mathcal{U}$ the restriction to the tangent space $T_x\Sigma$ of the nondegenerate skew-symmetric bilinear form σ_x is a skew-symmetric bilinear form $\sigma_x|_{\Sigma}$, possibly degenerate. The rank of the bilinear form $\sigma_x|_{\Sigma}$, *i.e.*, the rank of the linear map $T_x\Sigma \longrightarrow T_x^*\Sigma$ defined by $\sigma_x|_{\Sigma}$, is related to that of the matrix $(\{G_j, G_k\}(x))_{1 \leqslant j,k \leqslant s}$ by the formula

$$\operatorname{rank} \sigma_x|_{\Sigma} + \operatorname{codim} \Sigma = \operatorname{rank}(\{G_j, G_k\}(x))_{1 \leqslant j,k \leqslant s} + \dim \Sigma.$$

(Both ranks are even numbers.) We refer to $\operatorname{rank} \sigma_x|_{\Sigma}$ as the *symplectic rank* of the submanifold Σ at the point x. Denote by Σ_0 the open and dense subset of Σ consisting of the points x at which the symplectic rank of Σ is maximum, say equal to $\mu \geqslant 0$. Each connected component of Σ_0 is a submanifold of $\mathcal{U}$ of class $\mathcal{C}^{\omega}$ whose symplectic rank is everywhere equal to μ.

The subset $\Sigma \backslash \Sigma_0$ is an analytic subset of Σ : it can be defined in $\mathcal{U}'$ as the set of common zeros of $G_1, ..., G_s$ and of all the $\nu \times \nu$ minors of the matrix $(\{G_j, G_k\})_{1 \leqslant j,k \leqslant s}$ where $\nu = \mu + \operatorname{codim} \Sigma - \dim \Sigma$. It is an analytic subset of $\mathcal{U}'$ precisely of the type considered in Subsection 1 and as such it admits an analytic stratification of type (2.1) in $\mathcal{U}'$. The dimension of each analytic stratum of $\Sigma \backslash \Sigma_0$ is strictly less than $\dim \Sigma$. Furthermore these strata also have Property (2.3). This means that we can repeat with each one of them the construction started with Σ; and that it will suffice to repeat this same construction a number of time not exceeding $\dim \Sigma$ to obtain a decomposition

$$\Sigma = \bigcup_{\beta=1}^{\infty} \Sigma_{\beta} \tag{2.4}$$

where each Σ_{β} is a submanifold of an open subset $\mathcal{U}'_{\beta}$ of $\mathcal{U}'$ and satisfies the analogue of Condition (2.3) in $\mathcal{U}'_\beta$. The submanifold Σ_β are pairwise disjoint.

We can carry out the decomposition (2.4) taking Σ to be any of the analytic strata Λ_α of the analytic set $\mathbf{V}$ in Subsection 2.1, thus obtaining a new decomposition into pairwise disjoint $\mathcal{C}^\omega$ submanifolds of $\mathcal{U}$,

$$\mathbf{V} = \bigcup_{\alpha=1}^{\infty} \bigcup_{\beta=1}^{\infty} \Lambda_{\alpha,\beta}. \tag{2.5}$$

DEFINITION 2. — *The decomposition* (2.5) *will be referred to as the* symplectic stratification *of the analytic set* $\mathbf{V}$ *and each stratum* $\Lambda_{\alpha,\beta}$ *will be referred to as a* stratum in the sense of the symplectic stratification *of* $\mathbf{V}$.

In (2.5) the family of $\mathcal{C}^\omega$ submanifolds $\{\Lambda_{\alpha,\beta}\}_{1\leqslant k\leqslant m}$ is locally finite in $\mathcal{U}$. Each submanifold $\Lambda_{\alpha,\beta}$ satisfies Property (2.3) for an appropriate choice of the functions G_j among the differential polynomials $P_{\alpha,k}(F_1, ..., F_r, ..., \partial_x^{p_1} F_1, ..., \partial_x^{p_r} F_r, ...)$ and of the open set $\mathcal{U}' \subset \mathcal{U}$. The symplectic rank of each $\Lambda_{\alpha,\beta}$ is constant.

2.3. Poisson stratification.

We continue to deal with the analytic set $\mathbf{V}$ and with the functions $F_i \in \mathcal{C}^\omega(\mathcal{U})$, $i = 1, ..., r$. For each multi-index $I = (i_1, ..., i_\nu)$ with $1 \leqslant i_1, ..., i_\nu \leqslant r$, $\nu \geqslant 2$, we write

$$F_I = \{F_{i_1}, ..., F_{i_\nu}\} = \{F_{i_1}, ... \{F_{i_{\nu-1}}, F_{i_\nu}\} ...\}.$$

We refer to ν as the *length* of the multi-index I; we also write $|I| = \nu$. When $|I| = 1$, when $I = \{i\}$ for some i, $1 \leqslant i \leqslant r$, we equate F_I to F_i.

DEFINITION 3. — *We say that the functions* $F_1, ..., F_r \in \mathcal{C}^\omega(\mathcal{U})$ *satisfy the* Hörmander condition *if for every* $x \in \mathcal{U}$ *there is a multi-index* I, $|I| \geqslant 1$, *such that* $F_I(x) \neq 0$.

We can define the monotone decreasing sequence of analytic subsets of $\mathcal{U}$: for each $\nu \geqslant 2$,

$$\widetilde{\mathbf{V}}^{(\nu)} = \mathbf{V} \cap \{x \in \mathcal{U};\ \forall I,\ |I| \leqslant \nu,\ F_I(x) = 0\}.$$

In particular $\mathbf{V} = \widetilde{\mathbf{V}}^{(1)}$. The Hörmander condition states that $\bigcap_{\nu=1}^{\infty} \widetilde{\mathbf{V}}^{(\nu)} = \emptyset$.

Note that there is a subsequence of integers $1 = \nu_1 < \nu_2 < \cdots$ such that

(1) $\widetilde{\mathbf{V}}^{(\nu_{p+1})} \neq \widetilde{\mathbf{V}}^{(\nu_p)}$;

(2) if $\nu_p < \nu_{p+1}$ then $\widetilde{\mathbf{V}}^{(\nu')} = \widetilde{\mathbf{V}}^{(\nu)}$ for every ν', $\nu_p \leqslant \nu' < \nu_{p+1}$.

Now consider, for any given integer $p \geqslant 1$, the symplectic stratification of the analytic set $\widetilde{\mathbf{V}}^{(\nu_p)}$ (Definition 2):

$$\widetilde{\mathbf{V}}^{(\nu_p)} = \bigcup_{\alpha=1}^{\infty} \bigcup_{\beta=1}^{\infty} \Lambda_{\alpha,\beta}^{(\nu_p)}.$$

In each stratum $\Lambda_{\alpha,\beta}^{(\nu_p)}$ the set $\widetilde{\Lambda}_{\alpha,\beta}^{(\nu_p)}$ of points $x \in \widetilde{\mathbf{V}}^{(\nu_p)} \setminus \widetilde{\mathbf{V}}^{(\nu_{p+1})}$ is either empty or else, it is an open and dense subset of $\Lambda_{\alpha,\beta}^{(\nu_p)}$ (as the latter is a connected $\mathcal{C}^\omega$ submanifold). If $\widetilde{\Lambda}_{\alpha,\beta}^{(\nu_p)} \neq \emptyset$ we denote by $\Lambda_{\alpha,\beta,\gamma}^{(\nu_p)}$ its connected components. We obtain thus the decomposition

$$\widetilde{\mathbf{V}}^{(\nu_p)} = \widetilde{\mathbf{V}}^{(\nu_{p+1})} \cup \bigcup_{\alpha,\beta,\gamma=1}^{\infty} \Lambda_{\alpha,\beta,\gamma}^{(\nu_p)}.$$

Letting p range over the set of positive integers yields a decomposition

$$\mathbf{V} = \bigcup_{p=1}^{\infty} \bigcup_{j=0}^{\infty} \mathbf{\Sigma}_j^{(\nu_p)} \tag{2.6}$$

in which, whatever p and j,

(1) the $\mathcal{C}^\omega$ submanifolds $\mathbf{\Sigma}_j^{(\nu_p)}$ are connected and pairwise disjoint;

(2) at every point of $\mathbf{\Sigma}_j^{(\nu_p)}$ the rank of $\left(T\mathbf{\Sigma}_j^{(\nu_p)}\right) \cap \left(T\mathbf{\Sigma}_j^{(\nu_p)}\right)^{\sigma\perp}$ is equal to one and the same (even) nonnegative integer;

(3) at every point of $\mathbf{\Sigma}_j^{(\nu_p)}$ all Poisson brackets F_I of length $\nu < \nu_{p+1}$ vanish but at least one of length ν_{p+1} does not.

Definition 4. — *The decomposition* (2.6) *will be called the* Poisson stratification *of* $\mathbf{V}$ *defined by the functions* $F_1, ..., F_r$ *and each submanifold* $\mathbf{\Sigma}_j^{(\nu_p)}$ *will be called a* Poisson stratum *of* $\mathbf{V}$ *defined by these functions. We shall refer to* ν_p *as the* depth *of the Poisson stratum* $\mathbf{\Sigma}_j^{(\nu_p)}$.

It follows immediately from the elementary properties of the Poisson bracket that the Poisson stratification of $\mathbf{V}$ defined by the functions $F_1, ..., F_r$ is invariant under nonsingular $\mathcal{C}^\omega$ substitutions, *i.e.*, substitutions

$$F_j^\flat = \sum_{k=1}^{r} a_j^k F_k, \; j = 1, ..., r, \tag{2.7}$$

with $a_j^k \in \mathcal{C}^\omega(\mathcal{U})$ and $\det\left(a_j^k\right)_{1\leqslant j,k\leqslant r} \neq 0$ at every point of $\mathcal{U}$.

If a Poisson stratum Σ of $\mathbf{V}$ is not symplectic (*i.e.*, if the pullback to Σ of the fundamental symplectic form in $T^*\mathbb{R}^n$ is degenerate) then the intersection $T\Sigma \cap (T\Sigma)^{\sigma\perp}$ is a nonvanishing vector bundle over Σ which satisfies the Frobenius condition: the commutation bracket of two of its smooth sections is a section of $T\Sigma \cap (T\Sigma)^{\sigma\perp}$. As a consequence $T\Sigma \cap (T\Sigma)^{\sigma\perp}$ defines a foliation on Σ in which all the leaves have the same dimension. We shall refer to the leaves of this foliation as the *bicharacteristic leaves* and to any analytic curve contained in a bicharacteristic leaf as a *bicharacteristic curve*. The following terminology simplifies a little some forthcoming statements.

DEFINITION 5. — *By a* normal bicharacteristic curve *in the Poisson stratum* Σ *we shall mean the range* $\mathfrak{c}$ *of an analytic embedding* F *of* $\mathbb{R}$ *into* Σ *which is a bicharacteristic of* Σ *and has the property that* $\pi \circ F$ *has constant rank (necessarily equal to* 0 *or to* 1*).*

2.4. Poisson stratification associated to vector fields.

We consider r real vector fields $X_1, ..., X_r$ of class $\mathcal{C}^\omega$ in a connected and open subset Ω of $\mathbb{R}^n$ and the "sum of squares" operator $-L = X_1^2 + \cdots + X_r^2$. Let $T^*\mathbb{R}^n\backslash 0$ denote the cotangent bundle of $\mathbb{R}^n$ from which the zero section has been deleted and π the base projection $T^*\mathbb{R}^n\backslash 0 \to \mathbb{R}^n$. The symplectic manifold $\mathcal{U}$ of the preceding subsections will be the open subset $\overset{-1}{\pi}(\Omega)$ of $T^*\mathbb{R}^n\backslash 0$. The variety $\mathbf{V}$ will be the set of common zeros of the *symbols* $\sigma(X_j)$ of the vector fields X_j; in other words, $\mathbf{V}$ will be the *characteristic variety* of the operator L, $\operatorname{Char} L$. In accordance with established custom the symbol $\sigma(X)$ of a real vector field X is obtained by substituting $\sqrt{-1}\xi_j$ for the partial derivative $\frac{\partial}{\partial x_j}$ and therefore $\sigma(X)$ is purely imaginary. We have equated $X_1^2+\cdots+X_r^2$ to $-L$ to ensure that the principal symbol of L is nonnegative: $\sigma(L) = |\sigma(X_1)|^2 + \cdots + |\sigma(X_n)|^2$.

We apply the concepts of the previous subsections with the choice of $F_j = \sqrt{-1}\sigma(X_j)$, $j = 1, ..., r$. This choice will define once for all the meaning of the Poisson strata of $\operatorname{Char} L$. We recall that a subset of phase-space $T^*\mathbb{R}^n$ is said to be *conic* if it is invariant under the dilations $(x, \xi) \to (x, \lambda\xi)$, $\lambda > 0$. Of course $\operatorname{Char} L$ is conic. We can repeat the constructions in Subsections 1.1, 1.2, 1.3, making use only of functions $F(x,\xi)$ that are homogeneous with respect to ξ, *i.e.*, $F(x,\lambda\xi) = \lambda^m F(x,\xi)$ for some integer m and all $\lambda \in \mathbb{R}$. We obtain

PROPOSITION 1. — *Every Poisson stratum of* Char L *is conic.*

We say that the vector fields $X_1, ..., X_r$ and the differential operator L satisfy the Hörmander condition if the set of functions $\sqrt{-1}\sigma(X_j)$, $j = 1, ..., r$, does (Definition 3). That they do will be our hypothesis throughout (unless otherwise specified).

We list here a few examples that are going to guide us in the sequel. In the first example the characteristic variety is smooth but nonsymplectic:

Example 3 ([B-G, 1972]). — The characteristic set of the *Baouendi-Goulaouic operator*

$$L = D_1^2 + D_2^2 + x_1^2 D_3^2$$

consists of the two open half-spaces $x_1 = \xi_1 = \xi_2 = 0$, $\xi_3 > 0$ (resp., $\xi_3 < 0$). The (maximal) bicharacteristic curves are the x_2-lines.

The next example generalizes the Baouendi-Goulaouic operator in that the characteristic variety is smooth, but now it is symplectic.

Example 4 ([O, 1973]). — The characteristic set of the *Oleinik operator*

$$L = D_1^2 + x_1^{2(p-1)} D_2^2 + x_1^{2(q-1)} D_3^2 \ (1 < p < q)$$

can be identified to the phase-space (x_2, x_3, ξ_2, ξ_3) with the null section excised. It admits the following stratification: two symplectic strata $x_1 = \xi_1 = 0$, $\xi_2 > 0$ (resp., $\xi_2 < 0$) at depth 1; two nonsymplectic strata $x_1 = \xi_1 = \xi_2 = 0$, $\xi_3 > 0$ (resp., $\xi_3 < 0$) at depth p. The latter remains true down to depth $q-1$. At depth q we encounter the zero section (which, by our convention, is not part of Char L). The (maximal) bicharateristic curves in the nonsymplectic strata are the x_2-lines.

In the next example at depth 1 the Poisson strata are symplectic; nonsymplectic strata occur at depth $p > 1$; symplectic strata re-appear at depth $r > p$.

Example 5. — At depth one the characteristic set of the operator

$$L = D_1^2 + x_1^{2(p-1)} D_2^2 + x_1^{2(r-1)} \left(x_1^{2(q-r)} + x_2^{2\ell} \right) D_3^2$$

($p < r < q \leqslant \ell p + r,\ \ell > 1$) admits the symplectic strata $x_1 = \xi_1 = 0,\ \xi_2 > 0$ (resp., $\xi_2 < 0$). It admits the strata $x_1 = \xi_1 = \xi_2 = 0,\ \xi_3 > 0$ (resp., $\xi_3 < 0$) at depth p and the strata $x_1 = x_2 = \xi_1 = \xi_2 = 0,\ \xi_3 > 0$ (resp., $\xi_3 < 0$) at depth r. We encounter the zero section at depth q. The only nonsymplectic strata occur at depth p; the bicharateristic leaves in the nonsymplectic strata are the x_2 -lines.

In a variant of the preceding example the strata at depth r are isotropic, not symplectic (yet their base projections are lines).

Example 6. — The characteristic set of the operator

$$L = D_1^2 + x_1^{2(p-1)} D_2^2 + x_1^{2(r-1)} \left(x_1^{2(q-r)} + x_3^{2\ell} \right) D_3^2$$

($p < r < q,\ \ell > 1$) admits the symplectic strata $x_1 = \xi_1 = 0,\ \xi_2 > 0$ (resp., $\xi_2 < 0$) at depth one. It admits the strata $x_1 = \xi_1 = \xi_2 = 0,\ \xi_3 > 0$ (resp., $\xi_3 < 0$) at depth p and the strata $x_1 = x_3 = \xi_1 = \xi_2 = 0,\ \xi_3 > 0$ (resp., $\xi_3 < 0$) at depth r. We encounter the zero section at depth q . The only symplectic strata occur at depth one. In the strata at depth p the bicharacteristic leaves are the x_2 -lines. The strata at depth r are the $(x_2, \xi_3 \gtrless 0)$ half-planes; they are isotropic.

The next example shows that the characteristic set may lie above a single point.

Example 7 ([M, 1981]). — The characteristic set of the *Métivier operator*

$$L = D_1^2 + \left(x_1^2 + x_2^2 \right) D_2^2$$

consists of the two rays $x_1 = x_2 = \xi_1 = 0,\ \xi_2 > 0$ or $\xi_2 < 0$.

The following variants of the Métivier example shows how the rays can occur as singularities of the characteristic variety. In the first example they lie in intersecting symplectic planes entirely contained in the characteristic set; in the second example they lie on the boundary of symplectic surfaces also entirely contained in the characteristic set; but the surfaces cannot be smoothly continued beyond the rays.

Example 8. — The characteristic set of the operator

$$L = D_1^2 + \left(x_1^2 - x_2^2 \right)^2 D_2^2$$

consists of the symplectic surfaces $\xi_1 = 0$, $x_1 = \pm x_2$, $x_2\xi_2 \neq 0$, and of the two rays $x_1 = x_2 = \xi_1 = 0$, $\xi_2 \gtrless 0$.

Example 9. — The characteristic set of the operator

$$L = D_1^2 + \left(x_1^2 - x_2^3\right)^2 D_2^2$$

consists of the symplectic surfaces $\xi_1 = 0$, $x_1^2 = x_2^3$, $x_1\xi_2 \neq 0$, and of the two rays $x_1 = x_2 = \xi_1 = 0$, $\xi_2 \gtrless 0$.

2.5. The analytic hypo-ellipticity conjecture.

We recall that a linear differential operator L in Ω with $\mathcal{C}^\infty$ coefficients is said to be *hypo-elliptic* in Ω if, given any open subset U of Ω and any distribution u in U, $Lu \in \mathcal{C}^\infty(U) \Longrightarrow u \in \mathcal{C}^\infty(U)$. We recall the classical theorem of Hörmander (see [H, 1967]):

THEOREM. — *For the "sum of squares" operator $X_1^2 + \cdots + X_r^2$ to be hypo-elliptic in Ω it is necessary and sufficient that the real vector fields $X_1, ..., X_r$ satisfy the Hörmander condition.*

In our present set-up the vector fields X_j are defined and analytic in Ω. A theorem of Nagano (see [N, 1966]) states that the base Ω is foliated by immersed analytic submanifolds defined by the following property: the tangent space to any leaf at any of its points is equal to the "freezing" at that point of the Lie algebra $\mathfrak{g}(X_1, ..., X_r)$ generated (through commutation) by the vector fields $X_1, ..., X_r$. To say that $L = -\left(X_1^2 + \cdots + X_r^2\right)$ satisfies the Hörmander condition is to say that there is only one Nagano leaf, Ω itself (since Ω is connected); to say that L does *not* satisfy the Hörmander condition at some point $x^\circ \in \Omega$ is to say that the Nagano leaf Λ through x° is a proper submanifold of Ω and thus $\dim \Lambda < n$. In passing note that $\Lambda = \{x^\circ\}$ if and only if all the vector fields $X_1, ..., X_r$ vanish at x°. The hypo-ellipticity of L precludes this.

A differential operator L in Ω (with $\mathcal{C}^\omega$ coefficients) is said to be *analytic hypo-elliptic* if, given any open subset U of Ω and any distribution u in U, $Lu \in \mathcal{C}^\omega(U) \Longrightarrow u \in \mathcal{C}^\omega(U)$. By virtue of the Cauchy-Kovalewski theorem, the analytic hypo-ellipticity of L in Ω is equivalent to the following:

- *given any open subset U of Ω and any function $u \in \mathcal{C}^\infty(U)$, $Lu = 0 \Longrightarrow u \in \mathcal{C}^\omega(U)$.*

We restate the conjecture first formulated in [T, 1999]:

CONJECTURE 1. — *For $-L = X_1^2 + \cdots + X_r^2$ to be analytic hypo-elliptic in Ω it is necessary and sufficient that every Poisson stratum of* $\operatorname{Char} L$ *be symplectic.*

When the Nagano foliation defined by $X_1, ..., X_r$ admits a leaf Λ with $\dim \Lambda < n$ the conormal bundle of Λ is a Lagrangian submanifold of $T^*\mathbb{R}^n \backslash 0$ entirely contained in $\operatorname{Char} L$. A nonempty, relatively open subset of this Lagrangian submanifold must be contained in a Poisson stratum of $\operatorname{Char} L$. Since this stratum has dimension $< 2n$ it cannot be symplectic.

Remark 2. — The linear differential operator L in Ω is said to be *germ-analytic hypo-elliptic* at a point $x_\circ \in \Omega$ if to every open neighborhood $U \subset \Omega$ of $x_\circ$ and to every function $u \in \mathcal{C}^\infty(U)$ such that $Lu \in \mathcal{C}^\omega(U)$ there is an open neighborhood $V \subset U$ of $x_\circ$ in which u is analytic. Obviously L is analytic hypo-elliptic in Ω if and only if L is germ-analytic hypo-elliptic at every point of Ω. But a sum of squares operator can be germ-analytic hypo-elliptic at a point $x_\circ$ without being analytic hypo-elliptic in any open neighborhood of $x_\circ$ (see [Hanges, 2003]).

We wish to mention two consequences of the analytic hypo-ellipticity conjecture. We have already pointed out that the Poisson stratification is invariant under nonsingular substitutions of the type (2.7). In particular, the Poisson stratification of $\operatorname{Char} L$ is invariant under substitutions

$$\widetilde{X}_j = \sum_{k=1}^{r} a_j^k(x) X_k,\ j = 1, ..., r,$$

with $a_j^k \in \mathcal{C}^\omega(\Omega)$, $\det\left(a^k_{j1 \leqslant j,k \leqslant r}\right)$ nowhere zero in Ω. But then,

- **if the conjecture 1 is correct**, *the analytic hypo-ellipticity of $-L = X_1^2 + \cdots + X_r^2$ and that of $-\widetilde{L} = \widetilde{X}_1^2 + \cdots + \widetilde{X}_r^2$ are equivalent properties.*

Now let us use the following notation. For each multi-index $I = (i_1, ..., i_\ell)$ with $1 \leqslant i_1, ..., i_\ell \leqslant r$, $\ell = |I| \geqslant 2$, we write

$$X_I = [X_{i_1}, ..., X_{i_\nu}] = [X_{i_1}, ...[X_{i_{\nu-1}}, X_{i_\nu}]....],$$

where $[\ ,\ ]$ is the Lie bracket. For each length $\ell \geqslant 1$ we form the sum of squares operator

$$-L^{(\ell)} = \sum_{|I| \leqslant \ell} X_I^2.$$

It is clear that each stratum of $\operatorname{Char} L$ is a stratum of $\operatorname{Char} L^{(\ell)}$ for some $\ell \geqslant 1$; and conversely, each stratum of $\operatorname{Char} L^{(\ell)}$, for any $\ell = 1, 2, \ldots$, is a stratum of $\operatorname{Char} L$. The following ensues:

- **if the conjecture 1 is correct**, *L is analytic hypo-elliptic if and only if the same is true of $L^{(\ell)}$ for every positive integer ℓ.*

It is often convenient to reason microlocally. The sum of squares $-L$ is said to be *analytic hypo-elliptic at the point* $(x^\circ, \xi^\circ) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$ if given any function $u \in \mathcal{C}^\infty(\Omega)$,

$$(x^\circ, \xi^\circ) \notin WF_a(Lu) \Longrightarrow (x^\circ, \xi^\circ) \notin WF_a(u).$$

Below, we say that L is analytic hypo-elliptic in a conic open subset $\mathcal{U}$ of $\Omega \times (\mathbb{R}^n \setminus \{0\})$ if L is analytic hypo-elliptic at every point of $\mathcal{U}$. In this language, for L to be analytic hypo-elliptic in Ω it is necessary and sufficient that L be analytic hypo-elliptic in $\Omega \times (\mathbb{R}^n \setminus \{0\})$. This allows us to state the microlocal version of Conjecture 1:

CONJECTURE 2. — *For $-L = X_1^2 + \cdots + X_r^2$ to be analytic hypo-elliptic in a conic and open subset $\mathcal{U}$ of $\Omega \times (\mathbb{R}^n \setminus \{0\})$ it is necessary and sufficient that every Poisson stratum of $\mathcal{U} \cap \operatorname{Char} L$ be symplectic.*

Returning to the sums of squares operators $L^{(\ell)}$ we define the following sequence of open and conic subsets $\mathcal{U}^{(\ell)}$ of $\Omega \times (\mathbb{R}^n \setminus \{0\})$:

$$\mathcal{U}^{(\ell)} = \Omega \times (\mathbb{R}^n \setminus \{0\}) \setminus \operatorname{Char} L^{(\ell+1)}.$$

We have $\mathcal{U}^{(\ell)} \subset \mathcal{U}^{(\ell+1)}$ for each ℓ, and $\mathcal{U}^{(\ell)} = \Omega \times (\mathbb{R}^n \setminus \{0\})$ for ℓ sufficiently large (we are tacitly assuming that the X_i are defined and analytic in an open neighborhood of the closure of Ω). Note that the set $\mathcal{U}^{(\ell)} \cap \operatorname{Char} L^{(\ell)}$ might be empty, when $\operatorname{Char} L^{(\ell)} = \operatorname{Char} L^{(\ell+1)}$. If it is not empty, then at each one of its points all multibrackets $\sigma(X_I)$, $|I| \leqslant \ell$, vanish, but at least one multibracket of length $\ell + 1$ does not. This leads to the microlocal version of one of the statements above:

- **If the conjecture 2 is correct**, *then for L to be analytic hypo-elliptic in Ω it is necessary and sufficient that $L^{(\ell)}$ be analytic hypo-elliptic in $\mathcal{U}^{(\ell)}$ for every positive integer ℓ.*

In a sense, at the microlocal level this observation reduces the study of the analytic hypo-ellipticity of sums of squares to that of sums of squares $-L = X_1^2 + \cdots + X_r^2$ having the property that, at each point of $\operatorname{Char} L$, at least one of the brackets $\{\sigma(X_i), \sigma(X_j)\}$ does not vanish.

3. A partial Gevrey hypo-ellipticity conjecture.

3.1. Gevrey hypo-ellipticity.

We continue to deal with the real vector fields $X_1, ..., X_r$ of class $\mathcal{C}^\omega$ in the domain $\Omega \subset \mathbb{R}^n$ and with the "sum of squares" operator $-L = X_1^2 + \cdots + X_r^2$. In what follows s will always denote a real number $s \geqslant 1$. We recall that a complex-valued function $F \in \mathcal{C}^\infty(\Omega)$ is said to be of *Gevrey class* s if to each compact set $K \subset \Omega$ there are constants C, $M > 0$ such that, for all multi-indices $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}_+^n$ and all $x \in K$,

$$|\partial^\alpha F(x)| \leqslant CM^{|\alpha|} (\alpha!)^s . \tag{3.1}$$

We have used multi-index notation: $|\alpha| = \alpha_1 + \cdots + \alpha_n$; $\alpha! = \alpha_1! \cdots \alpha_n!$; $\partial^\alpha F = \partial_1^\alpha F \cdots \partial_n^\alpha F$ where $\partial_j = \frac{\partial}{\partial x_j}$. We denote by $G^s(\Omega)$ the space of functions $F \in \mathcal{C}^\infty(\Omega)$ of Gevrey class s. Of course $G^1(\Omega) = \mathcal{C}^\omega(\Omega)$. If $s > 1$ the differential operator L is said to be *Gevrey-s hypo-elliptic* when the following is true:

- *Given any open subset Ω' of Ω and any distribution u in Ω', $Lu \in G^s(\Omega') \Longrightarrow u \in G^s(\Omega')$.*

Gevrey hypo-ellipticity requires that the Hörmander property be valid (Definition 3). Indeed the existence of a Nagano leaf Λ with $\dim \Lambda < n$ through a point $x^\circ \in \Omega$ allows one to construct a solution h of the *homogeneous* equation $Lh = 0$ in a neighborhood of x° which is not of class $\mathcal{C}^\infty$ at x°. In other words Gevrey hypo-ellipticity implies hypo-ellipticity and, as a consequence, the Gevrey-s hypo-ellipticity of L is equivalent to the following property:

- *Given any open subset Ω' of Ω and any $\mathcal{C}^\infty$ function u in Ω', $Lu \in G^s(\Omega') \Longrightarrow u \in G^s(\Omega')$.*

PROPOSITION 2. — *Let the differential operator $-L = X_1^2 + \cdots + X_r^2$ satisfy Hörmander's condition and let the real number $s \geqslant 1$ be arbitrary. For L to be Gevrey-s hypo-elliptic it is necessary and sufficient that the following properties be valid:*

(1) Let $x^\circ \in \Omega$ be arbitrary. To each open neighborhood $U \subset \Omega$ of x° there is another open neighborhood $V \subset U$ of x° such that, given any $F \in G^s(U)$, there is a solution $u \in G^s(V)$ of the equation $Lu = F$ in V.

(2) *Given any open subset Ω' of Ω and any $\mathcal{C}^\infty$ function u in Ω', if $Lu \equiv 0$ in Ω' then $u \in G^s(\Omega')$.*

Proof. — That Properties 1 and 2 entail the Gevrey-s hypo-ellipticity of L is self-evident. The Gevrey-s hypo-ellipticity of L implies trivially Property 2. Since L satisfies Hörmander's condition the classical subelliptic inequality

$$c' \|\varphi\|_\delta^2 \leqslant c \sum_{j=1}^{r} \|X_j\varphi\|_0^2 \leqslant (L^*\varphi, \varphi)_0 + \|\varphi\|_0^2 \tag{3.2}$$

is valid for suitable positive constants δ, c, c' and all $\varphi \in \mathcal{C}^\infty(\Omega)$ $[L^*$: adjoint of L; $\|\cdot\|_\kappa$: norm in the Sobolev space $H^\kappa(\mathbb{R}^n)]$. The estimate (3.2) implies the local solvability of the equation $Lu = F$. If moreover L is Gevrey-s hypo-elliptic, Property 1 ensues. □

When $s = 1$ Property 1 is a direct consequence of the Cauchy-Kovalewski theorem. In the remainder of this paper we shall focus on Property 2 of Proposition 2. We shall restrict our attention, in the sequel, to the operators $-L = X_1^2 + \cdots + X_r^2$ that satisfy the Hörmander condition but are not analytic hypo-elliptic. This allows us to assume that *not* all the Poisson strata of $\operatorname{Char} L$ are symplectic.

Before proceeding let us clarify the two-variables case, meaning that, after division by a nonvanishing factor, we are looking at a differential operator

$$L = D_1^2 + \sum_{j=1}^{\nu} \left(\lambda_j(x_1, x_2) D_1 + A_j(x_1, x_2) D_2\right)^2 \tag{3.3}$$

where the λ_j and the A_j are analytic functions in an open neighborhood Ω of the origin in $\mathbb{R}^2$. Moreover we assume that the vector fields $\frac{\partial}{\partial x_1}$ and $A_j(x_1, x_2)\frac{\partial}{\partial x_2}$ $(j = 1, ..., \nu)$ satisfy the Hörmander condition.

PROPOSITION 3 (cf. Theorem 3.11, [T, 1999]). — *If the differential operator* (3.3) *is not analytic hypo-elliptic in some open neighborhood of the origin in $\mathbb{R}^2$ then the ray $\mathfrak{c}_\circ^+$ (resp., $\mathfrak{c}_\circ^-$) consisting of the points $((0,0),(0,\tau))$, $\tau > 0$ (resp., $\tau < 0$) is a Poisson stratum of L.*

Proof. — The characteristic variety of L is defined by the equations $\xi_1 = 0$, $A_j(x_1, x_2) = 0$ $(j = 1, ..., \nu)$. Let us write $\mathbf{V} = \pi(\operatorname{Char} L)$. If the origin is either a singular point or an isolated point of $\mathbf{V}$ then the ray $\mathfrak{c}_\circ$ is a Poisson stratum of L.

Now suppose the origin is a nonisolated regular point of $\mathbf{V}$. Let p be the smallest positive integer such that $\frac{\partial^p A_j}{\partial x_1^p}(0,0) \neq 0$ for some j; Hörmander's condition demands $p < +\infty$. Define, for each $j = 1, ..., \nu$,

$$\mathbf{V}'_j = \left\{ (x_1, x_2) \in \mathbf{V};\ \frac{\partial^{p-1} A_j}{\partial x_1^{p-1}}(x_1, x_2) = 0 \right\}.$$

Since $\mathbf{V}$ is an analytic curve and $\mathbf{V}'_j$ is an analytic subvariety of $\mathbf{V}$ there are only two possibilities: either the origin is an isolated point of $\mathbf{V}'_j$ for some j, in which case the ray $\mathfrak{c}_\circ$ is a Poisson stratum of L; or else there is an open neighborhood $\Omega' \subset \Omega$ of the origin such that $\Omega' \cap \mathbf{V}'_j = \Omega' \cap \mathbf{V}$ *for every* $j = 1, ..., \nu$. In the latter case the implicit function theorem applied to $\frac{\partial^{p-1} A_j}{\partial x_1^{p-1}}(x_1, x_2) = 0$ enables us to define $\mathbf{V}$ in Ω' (possibly contracted about the origin) by an analytic equation $x_1 = \varphi(x_2)$. For every $j = 1, ..., \nu$, $\frac{\partial^{p-1} A_j}{\partial x_1^{p-1}}(x_1, x_2)$ is divisible by $x_1 - \varphi(x_2)$; and for at least one j we have $\frac{\partial^{p-1} A_j}{\partial x_1^{p-1}}(x_1, x_2) = E(x_1, x_2)(x_1 - \varphi(x_2))$ with $E \in \mathcal{C}^\omega(\Omega')$, $E > 0$. Since

$$\frac{\partial^{k-1} A_j}{\partial x_1^{k-1}}(x_1, x_2) = \int_{\varphi(x_2)}^{x_1} \frac{\partial^k A_j}{\partial x_1^k}(s, x_2)\, ds$$

for $k = 1, ..., p-1$, we conclude that for every $j = 1, ..., \nu$, $\frac{\partial^{p-1} A_j}{\partial x_1^{p-1}}(x_1, x_2)$ is divisible by $(x_1 - \varphi(x_2))$; and for at least one j we have $A_j(x_1, x_2) = F_j(x_1, x_2)(x_1 - \varphi(x_2))^p$ with $F_j \in \mathcal{C}^\omega(\Omega')$, $F_j > 0$. We reach the conclusion that in a sufficiently small open neighborhood of the origin

$$L = D_1^2 + \sum_{j=1}^{\nu} \left(\lambda_j(x_1, x_2) D_1 + F_j(x_1, x_2)(x_1 - \varphi(x_2))^p D_2\right)^2 \tag{3.4}$$

with $\sum_{j=1}^{\nu} F_j^2(x_1, x_2) > 0$. The analytic hypo-ellipticity of the differential operator in (3.4) can be proved by the same methods used to prove the analytic hypo-ellipticity of $D_1^2 + x_1^{2p} D_2^2$ in $\mathbb{R}^2$ (see e.g. the forthcoming paper [A-B, 2004]). □

3.2. Basic symplectic submanifolds.

As before π denotes the base projection $T^*\mathbb{R}^n \backslash 0 \to \mathbb{R}^n$. Below m will be a positive integer; we denote by $L(\mathbb{R}^m; \mathbb{R}^n)$ the space of linear maps $\mathbb{R}^m \longrightarrow \mathbb{R}^n$.

DEFINITION 6. — *By a* basic symplectic submanifold *of* $\overset{-1}{\pi}(\Omega)$ *we shall mean a* $\mathcal{C}^\omega$ *submanifold* $\mathcal{E}$ *of* $\overset{-1}{\pi}(\Omega)$ *such that the following is true, for every* $(x^\circ, \xi^\circ) \in \mathcal{E}$*:*

- *There are a $\mathcal{C}^\omega$ embedding F of an open subset U of $\mathbb{R}^m$ into Ω, a $\mathcal{C}^\omega$ map $\Phi: U \longrightarrow L(\mathbb{R}^m; \mathbb{R}^n)$ and an open cone $\Gamma \neq \emptyset$ in $\mathbb{R}^m$ such that the map $(y, \eta) \longrightarrow (F(y), \Phi(y)\eta)$ is a symplectomorphism of $U \times \Gamma$ onto a conic neighborhood of (x°, ξ°) in $\mathcal{E}$.*

It follows immediately from ($\bullet$) that $\mathcal{E}$ is symplectic and conic, and furthermore, that $m = \frac{1}{2} \dim \mathcal{E} = \dim \pi(\mathcal{E})$. We recall that a symplectomorphism is a diffeomorphism which preserves the fundamental symplectic two-form. Here because of "conicity" it preserves the fundamental symplectic *one*-form. The latter is equivalent to saying that ${}^\top\!\left(\frac{\partial F}{\partial y}(y)\right) \Phi(y) = I$, the identity map of $\mathbb{R}^m$, for every $y \in U$ $[{}^\top\!\left(\frac{\partial F}{\partial y}(y)\right)$: transpose of the differential $\frac{\partial F}{\partial y}(y)]$.

PROPOSITION 4. — *For a $\mathcal{C}^\omega$ submanifold $\mathcal{E}$ of $\overset{-1}{\pi}(\Omega)$ to be basic symplectic it is necessary and sufficient that to each point $(x^\circ, \xi^\circ) \in \mathcal{E}$ there be an open neighborhood $\widetilde{U} \subset \Omega$ of x°, analytic coordinates $x_1, ..., x_n$ in $\widetilde{U}$ and an open cone $\widetilde{\Gamma}$ in $\mathbb{R}^n \backslash \{0\}$ containing ξ° such that $\mathcal{E} \cap \left(\widetilde{U} \times \widetilde{\Gamma}\right)$ is defined by the equations*

$$x_i = \xi_i = 0, \ i = 1, ..., n - m, \tag{3.5}$$

where $m = \frac{1}{2} \dim \mathcal{E}$.

In the preceding statement $\widetilde{U} \times \widetilde{\Gamma}$ is viewed as a conic subset of $\overset{-1}{\pi}(\Omega)$.

Proof. — The sufficiency of the condition is self-evident. Suppose ($\bullet$), Definition 6, holds. There is no loss of generality in assuming that U is an open ball centered at the origin in $\mathbb{R}^m$ and that $x^\circ = F(0)$, $\xi^\circ = \Phi(0)\eta^\circ$ with $\eta^\circ \in \Gamma$. There is an open neighborhood W of x° in Ω in which we can select analytic coordinates $x_1, ..., x_n$ vanishing at x° such that $W \cap \pi(\mathcal{E})$ is defined by the equations $x_i = 0, \ i = 1, ..., n - m$. This means that $F_i \equiv 0$ for $i = 1, ..., n - m$, and that the Jacobian matrix $\left(\frac{\partial F_i}{\partial x_j}\right)_{i,j=n-m+1,...,n}$ is nonsingular. In other words we can take $x'' = (x_{n-m+1}, ..., x_n)$ as parameters. Let ξ_j $(j = 1, ..., n)$ denote the dual coordinates. The pullback of the one-form $\sum_{j=1}^n \xi_j dx_j$ to $\mathcal{E}$ or, equivalently, to (x'', ξ'') -space, must be the standard form $\sum_{j=n-m+1}^n \xi_j dx_j$. This demands that we have, on $\mathcal{E} \cap \overset{-1}{\pi}(W)$,

$$\xi_i = \sum_{j=n-m+1}^{n} \varphi_{ij}(x'') \xi_j, \ i = 1, ..., n - m,$$

for suitable $\mathcal{C}^\omega$ functions φ_{ij} in U (possibly contracted about 0). Let us then carry out the following change of variables:

$$\widetilde{x}_i = x_i,\ i = 1, ..., n-m;$$

$$\widetilde{x}_j = x_j + \sum_{i=1}^{n-m} \varphi_{ij}(x'')\, x_i,\ j = n-m+1, ..., n.$$

The contragredient change of variables is given by

$$\xi_i = \widetilde{\xi}_i + \sum_{j=n-m+1}^{n} \varphi_{ij}(x'')\, \widetilde{\xi}_j,\ i = 1, ..., n-m;$$

$$\xi_j = \widetilde{\xi}_j + \sum_{i=1}^{n-m} \sum_{k=n-m+1}^{n} \frac{\partial \varphi_{ik}}{\partial x_j}(x'')\, \widetilde{x}_i \widetilde{\xi}_k,\ j = n-m+1, ..., n,$$

where x'' is viewed as a function of $\widetilde{x}$ equal to $\widetilde{x}''$ when $\widetilde{x}_i = x_i = 0$ for all $i = 1, ..., n-m$. After deleting the tildas we see that there is a conic and open neighborhood of (x°, ξ°) in $\overset{-1}{\pi}(\Omega)$ in which the submanifold $\mathcal{E}$ is defined by the equations (3.5). $\square$

The linear model is self-evident: a basic simplectic *linear subspace* $\mathcal{E}$ of $\mathbb{C}^n \cong \mathbb{R}^n + i\mathbb{R}^n$ (equipped with the natural symplectic form $\omega = x \cdot y' - x' \cdot y$) is the complexification of a real linear subspace $\mathcal{E}^{\mathbb{R}}$ of $\mathbb{R}^n$.

PROPOSITION 5. — *If a two-dimensional symplectic submanifold $\mathcal{E}$ of $\overset{-1}{\pi}(\Omega)$ is such that $\dim \pi(\mathcal{E}) = \frac{1}{2}\dim \mathcal{E} = 1$ then $\mathcal{E}$ is basic.*

Proof. — A two-dimensional symplectic submanifold $\mathcal{E}$ of $\overset{-1}{\pi}(\Omega)$, of class $\mathcal{C}^\omega$, conic and symplectic, containing the ray $x = 0$, $\xi = (0, ..., 0, \tau)$, $\tau > 0$, is defined by equations

$$x_i = f_i(x_n),\ \xi_i = \varphi_i(x_n)\, \xi_n,\ i = 1, ..., n-1,$$

with x_n varying in some interval $(-\delta, \delta)$, $\delta > 0$, $\xi_n > 0$ and real-valued analytic functions f_i, φ_i such that

$$f_i(0) = \varphi_i(0) = 0,\ i = 1, ..., n-1.$$

The pullback to $\mathcal{E}$ of the fundamental one-form $\sum_{j=1}^{n} \xi_j dx_j$ is equal to

$$\left(1 + \sum_{i=1}^{n-1} \varphi_i(x_n)\, f_i'(x_n)\right) dx_n.$$

The change of variables $x_n \longrightarrow x_n + \sum_{i=1}^{n-1} \int_0^{x_n} \varphi_i(t) f_i'(t)\, dt$ reduces us to the case in which $\sum_{i=1}^{n-1} \varphi_i(x_n) f_i'(x_n) \equiv 0$. Then the change of variables $\widetilde{x}_i = x_i - f_i(x_n)$ $(i = 1, ..., n-1)$, $\widetilde{x}_n = x_n + \sum_{i=1}^{n-1} \varphi_i(x_n)(x_i - f_i(x_n))$, and the associated change of dual variables

$$\xi_i = \widetilde{\xi}_i + \varphi_i(x_n)\widetilde{\xi}_n,\ i = 1, ..., n-1,$$

$$\xi_n = \left(1 + \sum_{i=1}^{n-1} \varphi_i'(x_n)(x_i - f_i(x_n))\right)\widetilde{\xi}_n - \sum_{i=1}^{n-1} f_i'(x_n)\widetilde{\xi}_i,$$

show that the following equations are satisfied on $\mathcal{E}$: $\widetilde{x}_i = 0$ $(i = 1, ..., n-1)$ and

$$\begin{aligned} \xi_i &= \widetilde{\xi}_i + \varphi_i(x_n)\widetilde{\xi}_n,\ i = 1, ..., n-1, \\ \xi_n &= \widetilde{\xi}_n - \sum_{i=1}^{n-1} f_i'(x_n)\left(\xi_i - \varphi_i(x_n)\widetilde{\xi}_n\right) \\ &= \widetilde{\xi}_n - \sum_{i=1}^{n-1} f_i'(x_n)\xi_i = \widetilde{\xi}_n - \sum_{i=1}^{n-1} f_i'(x_n)(\xi_i - \varphi_i(x_n)\xi_n) = \widetilde{\xi}_n. \end{aligned}$$

on $\mathcal{E}$. It follows at once that $\mathcal{E}$ can be defined by the equations (3.5) with $m = 1$. □

Symplectic submanifolds $\mathcal{E}$ of $\overset{-1}{\pi}(\Omega)$ such that $\dim \pi(\mathcal{E}) = \frac{1}{2} \dim \mathcal{E} > 1$ are generally not basic.

3.3. Symplectic slicing along a bicharacteristic.

We return to the "sum of squares" operator $-L = X_1^2 + \cdots + X_r^2$ in the open subset Ω of $\mathbb{R}^n$. Throughout the sequel, by the term "bicharacteristic" we shall refer to a normal bicharacteristic curve $\mathfrak{c}$ in some Poisson stratum Σ of $\operatorname{Char} L$ (Definition 5); (x°, ξ°) will be a point of $\mathfrak{c}$.

Definition 7. — *By a symplectic slicing along* $\mathfrak{c}$ *about the point* $(x^\circ, \xi^\circ) \in \mathfrak{c}$ *we shall mean a basic symplectic submanifold* $\mathcal{E}$ *of* $\overset{-1}{\pi}(\Omega)$ *such that* $\mathcal{E} \cap \mathfrak{c}$ *is an arc of curve containing* (x°, ξ°) *and such, moreover, that* $\frac{1}{2} \dim \mathcal{E} = 1 + \dim \pi(\mathfrak{c})$.

If $\mathcal{E} \cap \mathfrak{c}$ is an arc of curve, $1 + \dim \pi(\mathfrak{c})$ is the smallest possible value of $m = \frac{1}{2} \dim \mathcal{E}$; $m = 1$ or 2 depending on whether $\pi(\mathfrak{c})$ is a single point or a true curve. The conic span $\widehat{\mathfrak{c}}$ of $\mathfrak{c}$, *i.e.*, the set of points $(x, \lambda\xi)$ with

$(x, \xi) \in \mathfrak{c}$ and $\lambda > 0$, is isotropic since it is contained in the Poisson stratum Σ and perpendicular to $T\Sigma$ for the symplectic form; $\widehat{\mathfrak{c}}$ is a conic *Lagrangian* submanifold of $\mathcal{E}$. If $m = 1$ we must have $\mathcal{E} \cap \Sigma = \widehat{\mathfrak{c}}$. If $m = 2$ then $\widehat{\mathfrak{c}} \subset \mathcal{E} \cap \Sigma$ but we may also have $\dim \mathcal{E} \cap \Sigma = 3$. This does not preclude $\mathcal{E} \subset \operatorname{Char} L$ (cf. Example 10).

As before let $(U, x_1, ..., x_n)$ be a local chart in Ω centered at x°, with $U = U' \times U''$, $U' \subset \mathbb{R}^{n-m}$, $U'' \subset \mathbb{R}^m$, and an open and convex cone $\Gamma \subset \mathbb{R}^n \backslash \{0\}$ with $\xi^\circ \in \Gamma$, such that $\mathcal{E} \cap (U \times \Gamma)$ is defined by the equations (3.5). Let $\xi^\circ = (0, ..., 0, 1)$. When $m = 1$ the bicharacteristic $\mathfrak{c} = \widehat{\mathfrak{c}}$ is the ray $x = 0$, $\xi = (0, ..., 0, \tau)$.

When $m = 2$ $\mathfrak{c}$ is a curve

$$x_1 = \cdots = x_{n-2} = 0,\ \xi_1 = \cdots = \xi_{n-2} = 0,$$

$$x_{n-1} = f(t),\ x_n = g(t),\ \xi_{n-1} = \varphi(t),\ \xi_n = \psi(t),$$

with $t \in \mathbb{R}$ and

$$f(0) = g(0) = \varphi(0) = 0,\ \psi(0) = 1,$$

$$\varphi' + \psi g' \equiv 0,\ f'^2 + g'^2 > 0,\ \varphi^2 + \psi^2 > 0.$$

The latter conditions entail $f'(0) \neq 0$, $g'(0) = 0$. Therefore, after contracting U'' we can take x_{n-1} as parameter and write

$$x_n = g(x_{n-1}),\ \xi_{n-1} = \varphi(x_{n-1}),\ \xi_n = \psi(x_{n-1}).$$

Now we have $\varphi + \psi g' \equiv 0$ which requires $\psi(t) > 0$ for all t (lest $\mathfrak{c}$ meet the null section of $T^*\Omega$). If we carry out the change of variables $\widetilde{x}_{n-1} = x_{n-1}$, $\widetilde{x}_n = x_n - g(x_{n-1})$ [with no effect on the equations defining $\mathcal{E} \cap (U \times \Gamma)$] the dual change of variables reads

$$\widetilde{\xi}_{n-1} = \xi_{n-1} + g'(x_{n-1})\,\xi_n,\ \widetilde{\xi}_n = \xi_n.$$

On $\mathfrak{c}$ we get

$$\widetilde{x}_{n-1} = t,\ \widetilde{x}_n = 0,\ \widetilde{\xi}_{n-1} = 0,\ \widetilde{\xi}_n = \psi(t).$$

Clearly $\widehat{\mathfrak{c}}$ is equal to the conic span of the curve $t \longrightarrow ((0, ..., 0, t, 0), (0, ..., 0, 1))$ which is also a bicharacteristic and can be taken to be $\mathfrak{c}$ without loss of generality.

Example 10. — Let L be the Oleinik operator (Example 4):

$$L = D_1^2 + x_1^{2(p-1)} D_2^2 + x_1^{2(q-1)} D_3^2\ (1 < p < q).$$

The open half-subspace defined by $x_1 = \xi_1 = \xi_2 = 0$, $\xi_3 > 0$, is a Poisson stratum Σ of $\operatorname{Char} L$ of depth p; Σ is foliated by the lines parallel to the

x_2−axis, which are normal bicharacteristic curves in Σ. We select $\mathfrak{c}$ to be the curve in Σ defined by $x_3 = 0,\ \xi_3 = 1$. The submanifold

$$\mathcal{E} = \left\{(x,\xi) \in \mathbb{R}^3 \times \mathbb{R}^3;\ x_1 = \xi_1 = 0,\ \xi_3 > 0\right\}$$

is a symplectic slicing along $\mathfrak{c}$ about any one of its points. Note that $\mathcal{E} \subset \operatorname{Char} L$.

Example 11. — We consider the following generalization of the Métivier operator (Example 6):

$$L = D_1^2 + x_1^{2(q-1)} D_2^2 + x_1^{2(p-1)} x_2^{2k} D_2^2$$

with p, q, k integers and $1 < p < q$ and $k \geqslant 1$. The rays $x = 0,\ \xi_1 = 0,\ \xi_2 \gtrless 0$, are the bicharacteristic curves that make up the Poisson stratum at depth p . We take

$$\mathfrak{c} = \left\{(x,\xi) \in \mathbb{R}^2 \times \mathbb{R}^2;\ x = 0,\ \xi_1 = 0,\ \xi_2 > 0\right\}$$

and $\mathcal{E}$ to be the submanifold defined by $x_1 = 0,\ \xi_1 = 0,\ \xi_2 > 0$. We have $\mathfrak{c} \subset \mathcal{E}$ and $\frac{1}{2} \dim \mathcal{E} = 1 + \pi(\mathfrak{c}) = 1$; $\mathcal{E}$ is a symplectic slicing along the bicharacteristic $\mathfrak{c}$ about the point $x = 0, \xi_1 = 0, \xi_2 = 1$.

Example 12. — If $L = D_1^2 + \left(x_1^2 - x_2\right)^2 D_2^2$ the characteristic variety,

$$\operatorname{Char} L = \left\{(x,\xi) \in \mathbb{R}^2 \times \mathbb{R}^2;\ x_2 - x_1^2 = \xi_1 = 0,\ \xi_2 \neq 0\right\},$$

is symplectic for $x \neq 0$. The rays $x = 0,\ \xi_1 = 0,\ \xi_2 \gtrless 0$, are the bicharacteristic curves that make up the Poisson stratum at depth 3. We may take $\mathfrak{c}$ and $\mathcal{E}$ as in the preceding example; $\mathcal{E}$ is a symplectic slicing along the bicharacteristic $\mathfrak{c}$ about the point $x = 0, \xi_1 = 0, \xi_2 = 1$. But note that here we cannot find a basic symplectic submanifold $\mathcal{E} \supset \mathfrak{c}$ such that $\mathcal{E} \subset \operatorname{Char} L$.

In what follows $\mathcal{E}$ will be a symplectic slicing along the bicharacteristic $\mathfrak{c}$ about the point $(x^\circ, \xi^\circ) \in \mathfrak{c}$ (Definition 7) and let $m = \frac{1}{2} \dim \mathcal{E}$. We have always $m < n$; this is obvious if $n > 2$ and follows from Proposition 3 when $n = 2$.

By $(U, x_1, ..., x_n)$ and Γ we shall mean respectively an analytic local chart in Ω centered at x° (*i.e.*, all the analytic coordinates x_i vanish at x°) and an open and convex cone in $\mathbb{R}^n \setminus \{0\}$ with $\xi^\circ = (0, ..., 0, 1) \in \Gamma$, such that

(1) $\mathfrak{c} \cap (U \times \Gamma)$ is either the ray of points $(0, (0, ..., 0, \tau))$, $\tau > 0$ (case $m = 1$), or else (case $m = 2$) the curve $t \to ((0, ..., 0, t, 0), (0, ..., 0, 1))$, $|t| < \delta$ $(\delta > 0)$.

(2) $\mathcal{E} \cap (U \times \Gamma)$ is defined by the equations (3.5).

We shall indicate that all these conditions, including 1 and 2, are satisfied by saying that the local chart $(U, x_1, ..., x_n)$ and the cone Γ are *adapted to the pair* $(\mathcal{E}, \mathfrak{c})$ at (x°, ξ°). Possibly after contracting Γ about the ray through $(0, ..., 0, 1)$ we may also assume that $C\sqrt{\xi_1^2 + \cdots \xi_{n-1}^2} \leqslant \xi_n$ for all $\xi \in \Gamma$ and some $C > 0$. Here and throughout the sequel we use the Euclidean norms for vectors as well as covectors. We shall also use the notation $\overline{\xi}^\circ$ to mean the projection of ξ° into the subspace of the coordinates ξ_{n-m+1}, ξ_n : when $m = 1$, $\overline{\xi}^\circ = 1$ viewed as a point in the "cone" $\mathbb{R}_+$; when $m = 2$ we have $\overline{\xi}^\circ = (0, 1) \in \mathbb{R}^2$. Note also that when $m = 1$ it is pointless to mention (x°, ξ°) as this point determines and is completely determined (up to a positive dilation of ξ°) by $\mathfrak{c}$. In the case $m = 1$ we shall to refer to $\mathfrak{c}$ either as a *ray* or as a *vertical* bicharacteristic; when $m = 2$ we refer to $\mathfrak{c}$ as a *horizontal* bicharacteristic.

3.4. The Gevrey threshold of L at vertical bicharacteristics.

In this subsection we focus on the case $m = 1$: our bicharacteristic $\mathfrak{c}$ is the ray of points $x = 0$, $\xi = (0, ..., 0, \xi_n)$ with $\xi_n > 0$. We describe an algorithm that associates to the the bicharacteristic ray $\mathfrak{c}$ a *Gevrey threshold* for the "sum of squares" operator $-L = X_1^2 + \cdots + X_r^2$.

We reason within the framework described in the preceding paragraph. We make use of the two-dimensional symplectic slicing

$$\mathcal{E} = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n;\ x_i = \xi_i = 0, i = 1, ..., n-1;\ |x_n| < T,\ \xi_n > 0\};$$

the local chart $(U, x_1, ..., x_n)$ and the cone Γ are adapted to the pair $(\mathcal{E}, \mathfrak{c})$.

We define a parametrized family of deformations of $\mathcal{E}$: the parameter will be a point $a = (a_1, ..., a_{n-1}) \in \mathbb{Q}^{n-1}$, $0 < a_j < 1$ for every $j = 1, ..., n-1$. We define

$$\mathcal{E}(a) = \Big\{(x, \xi) \in U \times \mathbb{R}^n;\ x_i = \xi_n^{-a_i},\ \xi_i = \xi_n^{a_i}, i = 1, ..., n-1; \\ |x_n| < \rho^{-1},\ \xi_n > \rho\Big\}$$

with $\rho > 0$ suitably large. The submanifolds $\mathcal{E}(a)$ are asymptotic to $\mathcal{E}$ in the sense that if $(x, \xi) \in \mathcal{E}(a)$ then $\left(x, \frac{\xi}{|\xi|}\right) \longrightarrow ((0, ..., x_n), (0, ..., 0, 1)) \in \mathcal{E}$ when $|\xi| \approx \xi_n \longrightarrow \infty$. We are going to look at the restriction to $\mathcal{E}(a)$ of the symbol $\sigma(L) = \sum_{j=1}^r |\sigma(X_j)|^2$.

We can write, in the open neighborhood $U \subset \mathbb{R}^n$ of the origin,

$$X_i = \sum_{j=1}^{n} a_{i,j}(x) \frac{\partial}{\partial x_j} \ (i = 1, ..., r). \tag{3.6}$$

Since $\sigma(X_i)((0, ..., 0), (0, ..., 0, 1)) = 0$ we must have $a_{i,n}(0) = 0$ for every $i = 1, ..., r$. By the Hörmander condition not all vector fields $X_1, ..., X_r$ vanish at the origin: the rank of the matrix $(a_{i,j}(0))_{1\leqslant i\leqslant r, 1\leqslant j\leqslant n-1}$ must be strictly positive. After a linear change of the variables $x_1, ..., x_{n-1}$ which replaces the local chart $(U, x_1, ..., x_n)$ and the cone Γ by a local chart and a cone also adapted to the pair $(\mathcal{E}, \mathfrak{c})$ at (x°, ξ°) we may assume that $\sigma(X_j)(0, \xi) = \xi_j$ for $j = 1, ..., \nu$, and $\sigma(X_j)(0, \xi) = \sum_{i=1}^{\nu} c_j^i \xi_i$ for $j = \nu + 1, ..., r$, whence

$$\sigma(L)(0, \xi) = \sum_{i=1}^{\nu} \xi_i^2 + \sum_{j=\nu+1}^{r} \left(\sum_{i=1}^{\nu} c_j^i \xi_i \right)^2. \tag{3.7}$$

The analyticity of the coefficients of the vector fields X_i allows us to write

$$\begin{aligned} \sigma(L)|_{\mathcal{E}(a)} &= \sum_{i=1}^{r} \left| \sigma(X_i)\left(\xi_n^{-a_1}, ..., \xi_n^{-a_{n-1}}, x_n, \xi_n^{a_1}, ..., \xi_n^{a_{n-1}}, \xi_n\right) \right|^2 \\ &= \sum_{i=1}^{\nu} \xi_n^{2a_i} + \sum_{j=\nu+1}^{r} \left(\sum_{i=1}^{\nu} c_j^i \xi_n^{a_i} \right)^2 \\ &\quad + \sum_{j,k=1}^{n} \sum_{\substack{(p_1, ..., p_n) \in \mathbb{Z}_+^n \\ |p| \geqslant 1}} \gamma_{j,k,p} x_n^{p_n} \xi_n^{a_j + a_k - (p_1 a_1 + \cdots + p_{n-1} a_{n-1})} \end{aligned} \tag{3.8}$$

with the understanding that $a_n = 1$ (and $|p| = p_1 + \cdots + p_n$). The infinite series in (3.8) is absolutely convergent for $|x_n| < \varepsilon$, $|\xi_n| > \varepsilon^{-1}$ ($\varepsilon > 0$ sufficiently small).

Denote by $\mathcal{O}$ the subset of $\mathbb{Q}^{n-1}$ consisting of the points $(a_1, ..., a_{n-1})$ such that $0 < a_i < 1$ for $1 \leqslant i \leqslant n - 1$, verifying the following condition:

(3.9) For all $j, k, p_1, ..., p_{n-1} \in \mathbb{Z}_+$, $1 \leqslant j, k \leqslant n$, $p_1 + \cdots + p_{n-1} \neq 0$, $p_1 a_1 + \cdots + p_{n-1} a_{n-1} \neq a_j + a_k - 2 \max_{1\leqslant i\leqslant \nu} a_i$ with $a_n = 1$.

The subset $\mathcal{O}$ is open and dense in $\mathbb{Q}^{n-1} \cap (0, 1)^{n-1}$. Indeed, the set of multiplets $(a_1, ..., a_{n-1}) \in \mathbb{Q}^{n-1} \cap (0, 1)^{n-1}$ such that $\min_{1\leqslant i\leqslant n-1} a_i > \frac{1}{N}$ ($N = 1, 2, ...$) which do *not* belong to $\mathcal{O}$ satisfy one of the following finitely many linear equations:

$$p_1 a_1 + \cdots + p_{n-1} a_{n-1} + 2a_i = a_j + a_k$$

where $1 \leqslant i \leqslant \nu$, $1 \leqslant j,k \leqslant n$, $p_1, ..., p_{n-1} \in \mathbb{Z}_+$, $p_1 + \cdots + p_{n-1} \leqslant 2(N-1)$.

Taking $a \in \mathcal{O}$ ensures that, if $\mu = \max\limits_{1\leqslant i\leqslant\nu} a_i$, then

$$(3.10) \qquad \sum_{i=1}^{\nu} \delta_{a_i,\mu} + \sum_{j,k=1}^{n} \sum_{\substack{a_j+a_k-(p_1a_1+\cdots+p_{n-1}a_{n-1})=2\mu \\ p_n=0}} \gamma_{j,k,p} \neq 0$$

since the left-hand side in (3.10) is actually equal to the number of indices $h, 1 \leqslant h \leqslant \nu$, such that $a_h = \max\limits_{1\leqslant i\leqslant\nu} a_i$. If $\mu > \max\limits_{1\leqslant i\leqslant\nu} a_i$, then (3.10) is equivalent to

$$\sum_{j,k=1}^{n} \sum_{\substack{a_j+a_k-(p_1a_1+\cdots+p_{n-1}a_{n-1})=2\mu \\ p_1+\cdots+p_{n-1}\neq 0,\ p_n=0}} \gamma_{j,k,p} \neq 0.$$

The set of rational numbers $\frac{1}{2}(a_j + a_k - (p_1a_1 + \cdots + p_{n-1}a_{n-1})) > \max\limits_{1\leqslant i\leqslant\nu} a_i$ is finite, implying the finiteness of the set of rational numbers μ such that (3.10) is valid, and consequently, the existence of a maximum element, from now on taken to be μ. We underline the fact that, thus defined, μ depends on L, $\mathcal{E}$ and $a \in \mathcal{O}$; moreover, $\mu < 1$ since $\max\limits_{1\leqslant i\leqslant\nu} a_i < 1$ and

$$\frac{1}{2}(a_i + a_j - (p_1a_1 + \cdots + p_{n-1}a_{n-1})) \leqslant a_n - \frac{1}{2} \min_{1\leqslant j\leqslant n-1} a_j.$$

We can now rewrite (3.8) as

$$(3.11) \quad \sigma(L)|_{\mathcal{E}(a)} = A\xi_n^{2\mu} + \sum_{(0,\lambda)\in S(a),\lambda<2\mu} A_{0,\lambda}\xi_n^{\lambda} + \sum_{(\ell,\lambda)\in S(a),\ \ell\geqslant 1} A_{\ell,\lambda} x_n^{\ell}\xi_n^{\lambda}$$

where $S(a)$ consists of pairs $(\ell, \lambda) \in \mathbb{Z}_+ \times \mathbb{Q}$, with $\lambda = a_i + a_j - (p_1a_1 + \cdots + p_{n-1}a_{n-1})$ for some pairs of integers $i, j \in [1, ..., n]$ and some $(p_1, ..., p_{n-1}) \in \mathbb{Z}_+^{n-1}$, such that $A_{\ell,\lambda} \neq 0$. Note that $\lambda \leqslant 2$. The number A is positive, since $A \neq 0$ by definition of μ, and since

$$A = \lim_{\xi_n \longrightarrow +\infty} \xi_n^{-2\mu} \left(\sigma(L)|_{\mathcal{E}(a), x_n=0} \right) \geqslant 0.$$

We make use of the following order on $\mathbb{Z}_+ \times \mathbb{Q}$: $(\ell, \lambda) \prec (\ell', \lambda')$ means that either $\ell > \ell'$ or else $\lambda < \lambda'$. We denote by $S^\circ(a)$ the subset of *maximal* pairs $(\ell, \lambda) \in S(a)$ for the order $\prec$; $S^\circ(a)$ *is a finite set.* This allows us to isolate a "dominant part",

$$P_a(x_n, \xi_n) = A\xi_n^{2\mu} + \sum_{(\ell,\lambda)\in S^\circ(a),\ \ell\geqslant 1} A_{\ell,\lambda} x_n^{\ell}\xi_n^{\lambda},$$

such that

$$\left|\sigma(L)|_{\mathcal{E}(a)} - P_a(x_n,\xi_n)\right| \leqslant \text{const.} \sum_{(\ell,\lambda)\in S_\imath(a)\setminus S_\imath^\circ(a)} |x_n|^\ell \xi_n^\lambda.$$

We underline the fact that to any $\ell \in \mathbb{Z}_+$ there is at most one rational number λ_ℓ such that $(\ell,\lambda_\ell) \in S^\circ(a)$; and if $\ell \geqslant 1$, then $2\mu < \lambda_\ell \leqslant 2$. Since $P_a(x_n,\xi_n) \geqslant 0$ for large ξ_n we must have

$$A_{1,\lambda_1}^2 \xi_n^{2\lambda_1} \leqslant \text{const. } A A_{2,\lambda_2} \xi_n^{2\mu+\lambda_2}.$$

This implies that if $A_{1,\lambda_1} \neq 0$, *i.e.*, if $(1,\lambda_1) \in S^\circ(a)$, then $\lambda_1 \leqslant 1+\mu$, which is the same as saying that, for $\ell = 1$,

$$(\ell,\lambda) \in S^\circ(a) \Longrightarrow \ell - \lambda + 2\mu \geqslant \ell\mu. \tag{3.12}$$

But (3.12) is also true if $\ell \geqslant 2$ since then $\ell - \lambda + 2\mu \geqslant \ell - 2 + 2\mu$. This enables us to attach a *Gevrey index* $s(L,\mathcal{E},a)$ to the operator L relative to the symplectic submanifold $\mathcal{E}(a)$. In order to do this we attach a Gevrey index to each binomial $\xi_n^{2\mu} + \gamma_{\ell,\lambda} x_n^\ell \xi_n^\lambda$ for $(\ell,\lambda) \in S^\circ(a)$, $\ell \geqslant 1$. Heuristically we think of the symbol

$$\xi_n^{2\mu} + x_n^\ell \xi_n^\lambda = \xi_n^{2\mu}\left(1 + x_n^\ell \xi_n^{\lambda-2\mu}\right),\ x_n \geqslant 0,\ \xi_n > 0,$$

which, after division by the elliptic symbol $\xi_n^{2\mu}$ and after extraction of the $(\lambda-2\mu)^{\text{th}}$ root, is akin to the symbol of the differential operator

$$P_{\ell,\lambda,\mu} = 1 - H(x)\, x^c \frac{d}{dx},$$

where H is the Heaviside function and $c = \frac{\ell}{\lambda-2\mu} > 1$. The solution $H(x)\exp\left(-\frac{1}{c-1}x^{1-c}\right)$ of the homogeneous equation $P_{\ell,\lambda,\mu}u = 0$ is of Gevrey class $s = \frac{c}{c-1}$. In view of this it is natural to attach the Gevrey index $\frac{\ell}{\ell-\lambda+2\mu}$ to $(\ell,\lambda) \in S^\circ(a)$, $\ell \geqslant 1$; we have $\frac{\ell}{\ell-\lambda+2\mu} \leqslant \frac{1}{\mu}$ by (3.12). We attach the Gevrey index:

$$s(L,\mathcal{E},a) = \max\left\{\frac{\ell}{\ell-\lambda+2\mu};\ (\ell,\lambda) \in S^\circ(a)\ ,\ \ell \geqslant 1\right\}$$

to the operator L, relative to the symplectic slicing $\mathcal{E}$ along $\mathfrak{c}$ and to the multiplet $a = (a_1,\ldots,a_{n-1}) \in \mathcal{O}$.

DEFINITION 8. — *By the* Gevrey threshold *of L at the bicharacteristic ray $\mathfrak{c}$ we shall mean the supremum, denoted by $s(L,\mathfrak{c})$, of the numbers $s(L,\mathcal{E},a)$ as $a = (a_1,\ldots,a_{n-1})$ ranges over $\mathcal{O}$ and $\mathcal{E}$ ranges over the set of all symplectic slicings along $\mathfrak{c}$.*

Example 13. — We consider the following three vector fields in $\mathbb{R}^2$: $X_1 = \frac{\partial}{\partial x_1}$, $X_2 = x_1^p \frac{\partial}{\partial x_2}$, $X_3 = x_2^q \frac{\partial}{\partial x_2}$, with $p \geqslant 1$, $q \geqslant 1$. The ray defined by $x = 0$, $\xi_1 = 0$, $\xi_2 > 0$, is taken to be the bicharacteristic $\mathfrak{c}$ and the submanifold defined by $x_1 = \xi_1 = 0$, $\xi_2 > 0$, to be the symplectic slicing $\mathcal{E}$. We carry out the deformation

$$\mathcal{E}(a) = \left\{(x,\xi) \in \mathbb{R}^2 \times \mathbb{R}^2;\ x_1 = \xi_2^{-a},\ \xi_1 = \xi_2^a,\ \xi_2 > 0\right\}$$

where a is an arbitrary rational number such that $0 < a < 1$. If $-L = X_1^2 + X_2^2 + X_3^2$ we have

$$\sigma(L)|_{\mathcal{E}(a)} = \xi_2^{2a} + \xi_2^{2-2pa} + x_2^{2q}\xi_2^2.$$

We take $\mu = \max(a, 1-pa)$; and thus

$$\xi_2^{-2\mu}\ \sigma(L)|_{\mathcal{E}(a)} \approx A + x_2^{2q}\xi_2^{2-2\mu},\ A > 0.$$

We derive directly

$$s(L,\mathcal{E},a) = \frac{q}{q-1+\mu}.$$

The largest possible value of $s(L,\mathcal{E},a)$ is obtained by minimizing μ, *i.e.*, for $a = \frac{1}{p+1}$. We find

$$\max_{0<a<1} s(L,\mathcal{E},a) = \frac{(p+1)\,q}{(p+1)\,q-p}. \tag{3.13}$$

This is consistent with the result of [Ma, 1998], according to which the Gevrey hypo-ellipticity of the generalization of the Métivier operator (see Example 7), $D_1^2 + x_1^{2p}D_2^2 + x_2^{2q}D_2^2$, is precisely equal to the number (3.13).

CONJECTURE 3. — *If L is Gevrey-s hypo-elliptic in Ω then $s \geqslant s(L,\mathfrak{c})$ for every bicharacteristic ray $\mathfrak{c}$.*

3.5. Horizontal bicharacteristics.

We close this article with a few words about the case $m = 2$: the central point in phase-space $\mathbb{R}^n \times \mathbb{R}^n$ is still $(0,(0,...,0,1))$; our bicharacteristic $\mathfrak{c}$ is the line of points (x,ξ) such that $x_1 = \cdots = x_{n-2} = x_n = 0$, $|x_{n-1}| < T$, $\xi = (0,...,0,1)$; necessarily $n \geqslant 3$ (in view of Proposition 3). We can consider the four-dimensional symplectic slicing

$$\mathcal{E} = \left\{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n; x_i = \xi_i = 0,\ i = 1,...,n-2;\right.$$
$$\left. x_{n-1}^2 + x_n^2 < R^{-2},\ R\,|\xi_{n-1}| < \xi_n\right\}$$

with $R > 0$ suitably large. We define a parametrized family of deformations of $\mathcal{E}$: the parameter will be a point $a = (a_1, ..., a_{n-2}) \in \mathbb{Q}^{n-2}$, $0 < a_j < 1$ for every $j = 1, ..., n-2$. We define $\mathcal{E}(a)$ to be the set of points $(x, \xi) \in U \times \mathbb{R}^n$ such that

$$x_i = \xi_n^{-a_i},\ \xi_i = \xi_n^{a_i}, i = 1, ..., n-2;\ x_{n-1}^2 + x_n^2 < R^{-2},\ R|\xi_{n-1}| < \xi_n.$$

The submanifolds $\mathcal{E}(a)$ are asymptotic to $\mathcal{E}$ as $|\xi| \approx \xi_n \longrightarrow \infty$. We look at the restriction to $\mathcal{E}(a)$ of the symbol $\sigma(L) = \sum_{j=1}^r |\sigma(X_j)|^2$.

Here we must allow for the possibility that $\nu = n-2$, as is the case for the Baouendi-Goulaouic operator (Example 3) whose symbol is $\sigma(L) = \xi_1^2 + \xi_2^2 + x_1^2\xi_3^2$. After a linear change of the variables $x_1, ..., x_{n-2}$ which replaces the local chart $(U, x_1, ..., x_n)$ and the cone Γ by a local chart and a cone also adapted to the pair $(\mathcal{E}, \mathfrak{c})$ at $(0, (0, ..., 0, 1))$ we may assume that $\sigma(X_j)(0, \xi) = \xi_j$ for $j = 1, ..., \nu' = \min(\nu, n-2)$ and $\sigma(X_j)(0, \xi) = c_j\xi_{n-1} + \sum_{i=1}^{\nu'} c_j^i \xi_i$ for $j = \nu'+1, ..., r$, whence

$$\sigma(L)(0, \xi) = \sum_{i=1}^{\nu'} \xi_i^2 + \sum_{j=\nu'+1}^{r} \left(c_j\xi_{n-1} + \sum_{i=1}^{\nu'} c_j^i \xi_i\right)^2. \tag{3.14}$$

We can then write

(3.15)

$$\begin{aligned}
\sigma(L)|_{\mathcal{E}(a)} &= \sum_{i=1}^{r} \left|\sigma(X_i)\left(\xi_n^{-a_1}, ..., \xi_n^{-a_{n-2}}, x_{n-1}, x_n, \xi_n^{a_1}, ..., \xi_n^{a_{n-2}}, \xi_{n-1}, \xi_n\right)\right|^2 \\
&= \sum_{i=1}^{\nu'} \xi_n^{2a_i} + \sum_{j=\nu'+1}^{r} \left(c_j\xi_{n-1} + \sum_{i=1}^{\nu'} c_j^i \xi_n^{a_i}\right)^2 \\
&\quad + \sum_{k,\ell=0}^{\infty} \sum_{p=(p_1,...,p_{n-2})\in\mathbb{Z}_+^{n-2}} \alpha_{k,\ell,p} x_{n-1}^k x_n^\ell \xi_{n-1}^2 \xi_n^{-(p_1a_1+\cdots+p_{n-2}a_{n-2})} \\
&\quad + \sum_{j=1}^{n} \sum_{k,\ell=0}^{\infty} \sum_{p=(p_1,...,p_{n-2})\in\mathbb{Z}_+^{n-2}} \beta_{j,k,\ell,p} x_{n-1}^k x_n^\ell \xi_{n-1} \xi_n^{a_j-(p_1a_1+\cdots+p_{n-2}a_{n-2})} \\
&\quad + \sum_{i,j=1}^{n} \sum_{k,\ell=0}^{\infty} \sum_{p=(p_1,...,p_{n-2})\in\mathbb{Z}_+^{n-2}} \gamma_{i,j,k,\ell,p} x_{n-1}^k x_n^\ell \xi_n^{a_i+a_j-(p_1a_1+\cdots+p_{n-2}a_{n-2})},
\end{aligned}$$

here with the understanding that $a_{n-1} = 0$, $a_n = 1$.

A superficial inspection of (3.15) shows the difficulty of isolating a "leading part" in the far right-hand side. But even assuming that such a part has been isolated, one encounters the difficulty of assigning a Gevrey "threshold" to a pseudodifferential operator

$$\sum_{k,\ell\in\mathbb{Z}_+}\sum_{m=0}^{2}\sum_{s\in\mathbb{Q}} A_{k,\ell}x_{n-1}^{k}x_n^{\ell}D_{n-1}^{m}D_n^{s}$$

(the sum is finite). Of course there are cases where the procedure is straightforward; we conclude with mentioning two such cases.

Example 14. — In dealing with the symbol of the Oleinik operator in $\mathbb{R}^3$,

$$\sigma(L)=\xi_1^2+x_1^{2(p-1)}\xi_2^2+x_1^{2(q-1)}\xi_3^2\ (1<p<q\ ),$$

the nonsymplectic stratum Σ is defined by $x_1=\xi_1=\xi_2=0,\ \xi_3>0$. Our bicharacteristic curve $\mathfrak{c}$ is the x_2-line in Σ defined by $x_3=0,\ \xi_3=1$; and the symplectic slicing $\mathcal{E}$ along $\mathfrak{c}$ is defined by $x_1=\xi_1=0,\ \xi_3>0$ (Example 13). The "displacement" $\mathcal{E}(a)$ is defined by $x_1=\xi_3^{-a},\ \xi_1=\xi_3^{a},\ \xi_3>0$. We have

$$\xi_3^{2a(p-1)}\ \sigma(L)|_{\mathcal{E}(a)}=\xi_3^{2pa}+\xi_2^2+\xi_3^{2-2a(q-p)}\approx\xi_2^2+\xi_3^{2\mu}$$

with $\mu=\max(pa,1-a(p-q))$. The Gevrey hypo-ellipticity of the "heat-like" convolution operator $D_2^2+D_3^{2\mu}$ near the point $((0,0,0),(0,0,1))\in T^*\mathbb{R}^3$ is equal to $\max\left(1,\mu^{-1}\right)$. We must therefore choose a to minimize μ (in order to get the "worst" possible Gevrey regularity). This demands $a=q^{-1}$ and yields the Gevrey exponent $s=\frac{q}{p}$, in keep with known results.

Example 15. — Consider the symbol

$$\sigma(L)=\xi_1^2+x_1^{2(p-1)}\left(\xi_2+x_1^{q-p}\xi_3\right)^2+x_1^{2(p+r-1)}\xi_3^2,$$

where $1<p<q$ and $r\geqslant 0$. The stratum Σ , the bicharacteristic curve $\mathfrak{c}$ and the symplectic slicing are the same as in Example 14, defined respectively by $x_1=\xi_1=\xi_2=0,\ \xi_3>0$; $x_1=x_3=\xi_1=\xi_2=0,\ \xi_3=1$; $x_1=\xi_1=0,\ \xi_3>0$. The displacement is also the same:

$$\xi_3^{2a(p-1)}\ \sigma(L)|_{\mathcal{E}(a)}=\xi_3^{2ap}+\xi_2^2+2\xi_3^{1-a(q-p)}\xi_2+\xi_3^{2(1-a(q-p))}+\xi_3^{2(1-ar)}.$$

Here we look at the solutions of the ODE with the large parameter $\xi_3>0$,

$$D_2^2u+2\xi_3^{1-a(q-p)}D_2u+\left(\xi_3^{2ap}+c_1\xi_3^{2(1-a(q-p))}+c_2\xi_3^{2(1-ar)}\right)u=0. \tag{3.16}$$

If $q<p+r$ then, for ξ_3 large, the monomial $\xi_3^{2(1-ar)}$ is negligible compared to $\xi_3^{2(1-a(q-p))}$ and we choose $c_1=1,\ c_2=0$. If $q=p+r$ we take $c_1=2,\ c_2=0$. In the cases $q\leqslant p+r$ the inverse Fourier transform with respect to ξ_3 of the general solution of (3.16) yields immediately a

function of (x_2, x_3) which is of Gevrey class at most $s \leqslant \frac{q}{p}$ with respect to ξ_3 ($s = \frac{q}{p}$ is obtained taking $a = \frac{1}{q}$). If $q > p + r$ then, for ξ_3 large, the monomial $\xi_3^{2(1-a(q-p))}$ is negligible compared to $\xi_3^{2(1-ar)}$ and we choose $c_1 = 0$, $c_2 = 1$. The inverse Fourier transform with respect to ξ_3 of the general solution of (3.16) yields immediately a function of (x_2, x_3) which is of Gevrey class at most $s \leqslant 1 + \frac{r}{p} < \frac{q}{p}$ ($s = 1 + \frac{r}{p}$ is obtained for $a = \frac{1}{p+r}$).

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Antonio BOVE,
Università di Bologna
Dipartimento di Matematica
Piazza di porta S. Donato 5
40127 Bologna (Italia)
Antonio.Bove@bo.infn.it
&
François TREVES,
Rutgers University
Department of Mathematics
110 Frelinghuysen RD
Piscataway, NJ 08854-8019 (USA).
treves@math.rutgers.edu