ANNALES

DE

L'INSTITUT FOURIER

Lars HÖRMANDER

The null space of the $\bar{\partial}$-Neumann operator

Tome 54, nº 5 (2004), p. 1305-1369.

<http://aif.cedram.org/item?id=AIF_2004__54_5_1305_0>

Ann. Inst. Fourier, Grenoble
54, 5 (2004), 1305–1369

THE NULL SPACE OF THE $\bar\partial$-NEUMANN OPERATOR

by Lars HÖRMANDER

1. Introduction.

Let Ω be a relatively compact open subset with C^∞ boundary of a complex analytic manifold of dimension n with a hermitian metric. As usual we denote by $\bar\partial$ the part of the exterior differential operator which maps forms of type (p,q) to forms of type $(p,q+1)$, and we denote by $\mathfrak{d}$ the formal adjoint from forms of type $(p,q+1)$ to forms of type (p,q), defined with respect to the L^2 scalar products given by the metric. We shall also use the notation $\bar\partial$ for the closure in L^2 of the $\bar\partial$ operator initially defined for forms in $C^\infty(\overline{\Omega})$ and $\mathfrak{d}_c$ for the closure of $\mathfrak{d}$ defined initially on forms in $C_0^\infty(\Omega)$. The $\bar\partial$-Neumann operator is then the self-adjoint operator in the space $L^2_{p,q}(\Omega)$ of L^2 forms of type (p,q)

$$\Box = \bar\partial\mathfrak{d}_c + \mathfrak{d}_c\bar\partial. \tag{1.1}$$

If the Levi form of $\partial\Omega$ has everywhere at least $n-q$ positive or at least $q+1$ negative eigenvalues, then the null space $\operatorname{Ker}\Box$ is a finite dimensional subspace of $C^\infty_{p,q}(\overline{\Omega})$, and for every form $f\in L^2_{p,q}(\Omega)$ which is orthogonal to $\operatorname{Ker}\Box$ the equation $\Box u = f$ has a solution in $L^2_{p,q}(\Omega)$ with first order derivatives in L^2. (Here the Levi form of $\partial\Omega$ is defined in local coordinates by $\sum_{j,k=1}^n \partial^2\varrho/\partial z_j\partial\bar z_k t_j\bar t_k$, $\sum_1^n \partial\varrho/\partial z_j t_j = 0$, where ϱ is a *defining function* of Ω, that is, $\varrho<0$ in Ω, $\varrho=0$ and $d\varrho\neq 0$ on $\partial\Omega$.) If the derivatives of f of order $\leqslant s$ are in L^2 then those of u of order $\leqslant s+1$ are in L^2. (See e.g. [H1], [K], [KN], [CS].)

Keywords: $\bar\partial$-Neumann operator – Reproducing kernel.
Math. classification: 32W05 – 32A25.

The starting point for this paper was the observation that by expansion in spherical harmonics as done for the ball already in [KS] one can obtain an explicit formula for $\operatorname{Ker}\square$ in $L^2_{0,n-1}(\Omega)$ when $\Omega \subset \mathbb{C}^n$ is a spherical shell (Theorem 2.2); the signature of the Levi form is then $(0, n-1)$ on the inner boundary. (The referee has given a very short and elegant proof which only relies on an elementary integration by parts and avoids spherical harmonics expansions completely.) Using this formula we determine the kernel of the orthogonal projection on $\operatorname{Ker}\square$ modulo kernels which are real analytic in a neighborhood of $\overline{\Omega}$. In order to find a similar model for manifolds such that the Levi form of the boundary has signature $n-q-1, q$ for a general q we also study the open set

$$\Omega = \{z \in \mathbb{C}^n; R_0^2 < -|z'|^2 + |z''|^2 < R_1^2\},$$

$$z' = (z_1, \ldots, z_{n-q-1}), \ z'' = (z_{n-q}, \ldots, z_n).$$

Using spherical harmonics expansions in z' and in z'' separately we determine $\operatorname{Ker}\square$ in $L^2_{0,q}(\Omega)$ when $n \neq 2q+1$. (We are indebted to the referee for pointing out that the discussion does not only work when $n \geqslant 2q+2$.) The result motivates the proof in Section 3 that the null space of $\square$ in $L^2_{0,q}(\Omega)$, where $0 < q < n$, has infinite dimension for any complex hermitian manifold Ω such that the range is closed and the signature of the Levi form has the excluded signature $(n-q-1, q)$ at some smooth boundary point.

For a spherical shell $\Omega \subset \mathbb{C}^n$ Theorem 2.2 shows the remarkable fact that $\operatorname{Ker}\square \subset L^2_{0,n-1}(\Omega)$ has n independent multipliers. In Section 4 we prove that the only open bounded subsets Ω of $\mathbb{C}^n$ with this property are shells bounded by two confocal ellipsoids (Theorems 4.1 and 4.3). For these sets it follows that there is a description of $\operatorname{Ker}\square \subset L^2_{0,n-1}(\Omega)$ which is close to that for a spherical shell (Theorem 4.1). This allows us to determine the boundary behavior of the kernel of the orthogonal projection on $\operatorname{Ker}\square \subset L^2_{0,n-1}(\Omega)$ when the range of $\square$ is closed, at a boundary point z^0 of a complex hermitian manifold Ω of dimension n where $\partial\Omega$ is smooth and the Levi form has signature $(0, n-1)$. (If only Theorem 2.2 had been available it would have been necessary to assume some compatibility between the hermitian metric and the curvature of $\partial\Omega$.) However, the results in Section 4 are incomplete in the sense that we only give asymptotics of the kernel of the projection in a distribution sense. This is probably all that can be extracted from the explicit formulas for confocal elliptic shells. A careful microlocal analysis along the lines of [BMS] should remove this flaw and also give analogues for $\operatorname{Ker}\square \subset L^2_{0,q}(\Omega)$ when the Levi form has signature $(n-q-1, q)$ at some boundary point. We end Section 4 with conjectures on the boundary behavior of the projection kernel in that case. By a crude microlocal analysis we prove in Section 5 bounds of the right order of magnitude for the kernel but a proof of these conjectures must require a much more refined analysis where the Levi form of $\partial\Omega$ enters decisively.

There are three appendices. Appendix A contains some elementary observations on how orthogonal projections in a Hilbert space change if the norm is changed. They are needed at the end of Section 4. Appendix B contains some remarks on pseudodifferential calculus used in Section 5. In Appendix C finally we discuss for the sake of comparison the kernels of projections on harmonic functions or forms. They are probably well known although not easy to locate in the literature.

2. Spherical and hyperbolic shells.

In the first part of this section Ω will be a spherical shell $\Omega = \{z \in \mathbb{C}^n; R_0 < |z| < R_1\}$ where $0 < R_0 < R_1$ and $|z|$ is the euclidean norm which we also take as hermitian metric in Ω. The Levi form of $\partial\Omega$ has $n-1$ positive eigenvalues at the outer boundary but $n-1$ negative eigenvalues at the inner one, so it has there the signature excluded in the results on the $\bar\partial$-Neumann operator on $(0, n-1)$ forms mentioned in the introduction. Before discussing this operator we shall first prove that the range of the $\bar\partial$ operator acting on functions in $L^2(\Omega)$ is closed.

Let Y be a harmonic polynomial in $\mathbb{R}^{2n} = \mathbb{C}^n$ which is homogeneous of degree m. We have a unique decomposition $Y = \sum_{p+q=m} Y_{p,q}$ where $Y_{p,q}$ has bidegree p, q in $z, \bar z$. Since $\Delta Y_{p,q} = 4\sum \partial^2 Y_{p,q}/\partial z_j \partial \bar z_j$ has bidegree $p-1, q-1$ we have $\Delta Y_{p,q} = 0$ because $\Delta Y = 0$. If $Y_{p,q}$ and $Y_{p',q'}$ are bihomogeneous, then the substitution $z \mapsto e^{i\theta} z$ gives

$$\int_{|z|=1} Y_{p,q}\overline{Y_{p',q'}}\, dS = e^{i\theta(p-q-p'+q')} \int_{|z|=1} Y_{p,q}\overline{Y_{p',q'}}\, dS,$$

so the integral vanishes unless $p - q = p' - q'$. If $p + q = p' + q'$ this implies $p = p'$ and $q = q'$. Since spherical harmonics of different degrees are orthogonal it follows that we can find a basis Y_ν, $\nu = 1, 2, \ldots$, for harmonic polynomials such that every Y_ν is bihomogeneous, of bidegree p_ν, q_ν, and they yield an orthonormal basis for $L^2(S^{2n-1})$.

A general function $u \in C^\infty(\overline{\Omega})$ can now be written uniquely as a sum $u = \sum_\nu \varphi_\nu(r) Y_\nu$ where $r = |z|$, thus $\partial r/\partial x_j = x_j/r$ if x_j are the real coordinates, and $\partial r/\partial \bar z_j = z_j/2r$. Since $\sum_1^{2n} x_j \partial Y_\nu/\partial x_j = (p_\nu + q_\nu)Y_\nu$ and $\Delta Y_\nu = 0$, it follows that

$$\Delta u = \sum_\nu (\varphi_\nu''(r) + (2n-1+2p_\nu+2q_\nu)\varphi_\nu'(r)/r)Y_\nu.$$

A partial integration noting that $-\partial H(R_1^2-|z|^2)/\partial z_j = \bar z_j/(2R_1)\,\delta(R_1-|z|)$ and the analogue for $H(|z|^2-R_0^2)$ (where H is the Heaviside function) gives

$$\int_\Omega |\bar\partial u|^2\,d\lambda = -\frac14\int_\Omega (\Delta u)\bar u\,d\lambda + \frac{1}{2R_1}\int_{|z|=R_1}\sum_1^n \bar z_j\partial u/\partial\bar z_j\,\bar u\,dS$$
$$-\frac{1}{2R_0}\int_{|z|=R_0}\sum_1^n \bar z_j\partial u/\partial\bar z_j\,\bar u\,dS,$$

where $d\lambda$ denotes the Lebesgue measure and dS the Euclidean surface measure. Introducing the spherical harmonics expansion we obtain

$$\int_\Omega |\bar\partial u|^2\,d\lambda = -\frac14\sum_\nu\int_{R_0}^{R_1}(\varphi_\nu''(r)$$
$$+ (2n+2p_\nu+2q_\nu-1)\varphi_\nu'(r)/r)\overline{\varphi_\nu(r)}r^{2p_\nu+2q_\nu+2n-1}\,dr$$
$$+\Big[\frac14\sum_\nu(2q_\nu\varphi_\nu(r)+r\varphi_\nu'(r))\overline{\varphi_\nu(r)}\,r^{2p_\nu+2q_\nu+2n-2}\Big]_{R_0}^{R_1}.$$

This complete separation of the different terms in the expansion makes it easy to proceed. With $r=e^t$, $dr = e^t\,dt$, and writing $\Phi_\nu(t) = \varphi_\nu(r)$, we have $\Phi_\nu'(t) = r\varphi_\nu'(r)$, $\Phi_\nu''(t) = r^2\varphi_\nu''(r) + r\varphi_\nu'(r)$, and the formula becomes

$$\int_\Omega |\bar\partial u|^2\,d\lambda = -\frac14\sum_\nu\int_{t_0}^{t_1}(\Phi_\nu''(t)$$
$$+ 2(n+p_\nu+q_\nu-1)\Phi_\nu'(t))\overline{\Phi_\nu(t)}e^{2(p_\nu+q_\nu+n-1)t}\,dt$$
$$+\Big[\frac14\sum_\nu(2q_\nu\Phi_\nu(t)+\Phi_\nu'(t))\overline{\Phi_\nu(t)}e^{2(p_\nu+q_\nu+n-1)t}\Big]_{t_0}^{t_1},$$

where $R_j = e^{t_j}$. If we set $\psi_\nu(t) = \Phi_\nu(t)e^{(p_\nu+q_\nu+n-1)t}$, it simplifies to

$$\begin{aligned}\int_\Omega |\bar\partial u|^2\,d\lambda &= -\frac{1}{4}\sum_\nu \int_{t_0}^{t_1} (\psi_\nu''(t) - (p_\nu+q_\nu+n-1)^2\psi_\nu(t))\overline{\psi_\nu(t)}\,dt \\ &\quad + \frac{1}{4}\Big[\sum_\nu((q_\nu-p_\nu-n+1)\psi_\nu(t)+\psi_\nu'(t))\overline{\psi_\nu(t)}\Big]_{t_0}^{t_1} \\ &= \frac{1}{4}\sum_\nu\Big(\int_{t_0}^{t_1}(|\psi_\nu'(t)|^2+(p_\nu+q_\nu+n-1)^2|\psi_\nu(t)|^2)\,dt \\ &\quad + \Big[(q_\nu-p_\nu-n+1)|\psi_\nu(t)|^2\Big]_{t_0}^{t_1}\Big) \\ &= \frac{1}{4}\sum_\nu\Big(\int_{t_0}^{t_1}\big(|\psi_\nu'(t)+(q_\nu-p_\nu-n+1)\psi_\nu(t)|^2 \\ &\quad + 4q_\nu(p_\nu+n-1)|\psi_\nu(t)|^2\Big)\,dt.\end{aligned}$$

When $q_\nu \neq 0$ we get at once an estimate of ψ_ν. However, the case $q_\nu = 0$ is exceptional, for then Y_ν is holomorphic so $\bar\partial(\varphi_\nu(r)Y_\nu) = Y_\nu\varphi_\nu'(r)\langle z, d\bar z\rangle/2r$, and the square of the L^2 norm is

$$\frac{1}{4}\int_{R_0}^{R_1}|\varphi_\nu'(r)|^2 r^{2p_\nu+2n-1}\,dr.$$

If u is orthogonal to holomorphic functions, in particular to Y_ν, then

$$\int_{R_0}^{R_1}\varphi_\nu(r)r^{\mu-1}\,dr = 0, \text{ hence } \int_{R_0}^{R_1}|\varphi_\nu(r)|^2 r^{\mu-1}\,dr \leqslant \int_{R_0}^{R_1}|\varphi_\nu^0(r)|^2 r^{\mu-1}\,dr,$$

where $\mu = 2p_\nu+2n$ and $\varphi_\nu^0(r) = \varphi_\nu(r)-\varphi_\nu(R_1)$. Integration by parts gives

$$\mu\int_{R_0}^{R_1}|\varphi_\nu^0(r)|^2 r^{\mu-1}\,dr = -|\varphi_\nu^0(R_0)|^2R_0^\mu - 2\,\mathrm{Re}\int_{R_0}^{R_1}\varphi_\nu^0(r)\overline{\varphi_\nu'(r)}r^\mu\,dr,$$

and using the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}\int_{R_0}^{R_1}|\varphi_\nu(r)|^2r^{2p_\nu+2n-1}\,dr &\leqslant (p_\nu+n)^{-2}\int_{R_0}^{R_1}|\varphi_\nu'(r)|^2r^{2p_\nu+2n+1}\,dr \\ &\leqslant (p_\nu+n)^{-2}R_1^2\int_{R_0}^{R_1}|\varphi_\nu'(r)|^2r^{2p_\nu+2n-1}\,dr.\end{aligned}$$

If the decomposition of u only involves holomorphic harmonic polynomials Y_ν, that is, $q_\nu = 0$, and u is orthogonal to holomorphic functions, we conclude that

$$\int_\Omega |u|^2\,d\lambda \leqslant 4n^{-2}R_1^2\int_\Omega|\bar\partial u|^2\,d\lambda.$$

On the other hand, if $q_\nu \neq 0$ for all terms then

$$(n-1)\int_\Omega |u|^2|z|^{-2}\,d\lambda \leqslant \int_\Omega |\bar\partial u|^2\,d\lambda.$$

Combining the two estimates and using the orthogonality we obtain since $n-1 \leqslant n^2/4$

$$(n-1)\int_\Omega |u|^2\,d\lambda \leqslant R_1^2 \int_\Omega |\bar\partial u|^2\,d\lambda, \tag{2.1}$$

when u is orthogonal to the holomorphic functions in Ω. Smooth functions are dense in this space so we have proved:

THEOREM 2.1. — *For the spherical shell Ω the range of the $\bar\partial$ operator acting in $L^2(\Omega)$ is closed, and (2.1) is valid when u is in the domain of $\bar\partial$ and orthogonal to the null space.*

When $q \geqslant 1$ and $n \geqslant 2$ then $q(p+n-1) \geqslant q+p$, which easily leads even to a bound for the norm $\|u\|_{(\frac12)}$ in the Sobolev space $H_{(\frac12)}(\Omega)$. We omit the details since this is not a new result (cf. [S1], [S2], [BS], [CS]). We gave a proof of Theorem 2.1 just to prepare for the use of spherical harmonics to study the $\bar\partial$-Neumann operator, and nothing is new in Theorem 2.1 except possibly the rather precise constant.

We shall now study the $\bar\partial$-Neumann operator on $(0,Q)$ forms f where $0<Q<n$. We shall write

$$f = \sum_{|J|=Q}{}' f_J(z)d\bar z^J, \quad d\bar z^J = d\bar z_{j_1}\wedge\ldots\wedge d\bar z_{j_Q},\ J=(j_1,\ldots,j_Q).$$

The components f_J are defined for all Q tuples J and are antisymmetric in the indices; $\sum'$ means summation for increasing indices. By a basic identity due to Morrey and Kohn we have for smooth $(0,Q)$ forms in the domain of $\mathfrak{d}_c$ (cf. [H1, Prop. 2.1.2])

$$\int_\Omega (|\bar\partial f|^2 + |\mathfrak{d}_c f|^2)\,d\lambda(z) = \sum_{j=1}^n \sum_{|J|=Q}{}' \int_\Omega |\partial f_J/\partial\bar z_j|^2\,d\lambda(z) + \sum_{|K|=Q-1}{}' \int_{\partial\Omega} \sum_{j,l=1}^n \partial^2\varrho/\partial z_j\partial\bar z_l f_{jK}\overline{f_{lK}}\,dS,$$

where $\varrho(z) = \max(|z|-R_1, R_0-|z|)$; since $\sum_j f_{jK}\partial\varrho/\partial z_j = 0$ on $\partial\Omega$ this gives

$$\|\mathfrak{d}_c f\|^2 + \|\bar\partial f\|^2 = \sum_{|J|=Q}{}'\Big(\int_\Omega |\bar\partial f_J|^2\,d\lambda + \frac{Q}{2R_1}\int_{|z|=R_1} |f_J|^2\,dS - \frac{Q}{2R_0}\int_{|z|=R_0}|f_J|^2\,dS\Big).$$

As in the proof of Theorem 2.1 we expand f_J in spherical harmonics as $f_J = \sum_\nu \varphi_{J\nu}(r)Y_\nu$, and obtain with $\varphi_{J\nu}(e^t)e^{(p_\nu+q_\nu+n-1)t} = \psi_{J\nu}(t)$ that

$$4(\|\mathfrak{d}_c f\|^2 + \|\bar\partial f\|^2) = \sum_\nu\sum_J{}'\Big(\int_{t_0}^{t_1} (|\psi_{J\nu}'(t)|^2 + (p_\nu+q_\nu+n-1)^2|\psi_{J\nu}(t)|^2\Big)\,dt$$

$$+\Big[(q_\nu - p_\nu - n + 1)|\psi_{J\nu}(t)|^2\Big]_{t_0}^{t_1} + 2Q\Big[|\psi_{J\nu}(t)|^2\Big]_{t_0}^{t_1}\Big).$$

Here the last term comes from the integrals over $\partial\Omega$ involving the Levi form. Since

$$(p_\nu + q_\nu + n - 1)^2 - (q_\nu - p_\nu - n + 1 + 2Q)^2 = 4(q_\nu + Q)(p_\nu + n - 1 - Q)$$

we obtain

$$\|\mathfrak{d}_c f\|^2 + \|\bar\partial f\|^2 = \sum_{\nu,J}{}' \int_{t_0}^{t_1} \Big(\frac{1}{4}|\psi'_{J\nu}(t) + (q_\nu - p_\nu - n + 1 + 2Q)\psi_{J\nu}(t)|^2$$
$$+ (q_\nu + Q)(p_\nu + n - 1 - Q)|\psi_{J\nu}(t)|^2\Big)\, dt.$$

This remains true for all f in the intersection of the domains of $\bar\partial$ and of $\mathfrak{d}_c$ in $L^2_{0,Q}(\Omega)$, for forms in $C^\infty_{0,Q}(\overline{\Omega})$ are dense in the graph norm. Hence

$$Q(n-1-Q)\|f\|^2 \leqslant R_1^2(\|\mathfrak{d}_c f\|^2 + \|\bar\partial f\|^2) \tag{2.2}$$

when f is in the domain of $\bar\partial$ and in the domain of $\mathfrak{d}_c$. When $Q = n-1$ this is vacuous for a good reason. Then the right-hand side of (2.2) vanishes if and only if $p_\nu = 0$ and $\psi_{J\nu}(t) = c_{J\nu}e^{-(n-1+q_\nu)t}$, that is, $\varphi_{J\nu}(e^t) = c_{J\nu}e^{-2(n-1+q_\nu)t}$ or $\varphi_{J\nu}(r) = c_{J\nu}r^{-2(n-1+q_\nu)}$ when $\varphi_{J\nu} \neq 0$. This means that $\varphi_{J\nu}(r)Y_\nu = c_{J\nu}|z|^{-2(n-1)}Y_\nu(\bar z/|z|^2)$ where Y_ν is regarded as a polynomial of degree q_ν in $\bar z$.

Before proceeding we change notation and write

$$f = \sum_1^n (-1)^j f_j\, d\bar z_1 \wedge \ldots \wedge \widehat{d\bar z_j} \wedge \ldots \wedge d\bar z_n.$$

where ^ denotes a factor which should be omitted. This means that $f_j = (-1)^j f_J$ when $J = 1, \ldots, \hat\jmath, \ldots, n$ with our earlier notation. We have proved that

$$f_j(z) = |z|^{-2(n-1)} \sum_{\mu=0}^\infty g_{j\mu}(\zeta), \quad \zeta = \bar z/|z|^2,$$

where $g_{j\mu}(\zeta)$ is a homogeneous polynomial of degree μ in ζ. The sum is L^2 convergent in Ω, so $g_j(\zeta) = \sum_{\mu=0}^\infty g_{j\mu}(\zeta)$ is L^2 convergent in $\{\zeta; 1/R_1 < |\zeta| < 1/R_0\}$. By Hartogs' theorem it follows that g_j is holomorphic in the ball $\check\Omega = \{\zeta; |\zeta| < 1/R_0\}$ and that the series converges locally uniformly there. That f is in the domain of $\mathfrak{d}_c$ implies that $\bar z_j f_k - \bar z_k f_j$ vanishes on $\partial\Omega$ in a weak sense, so it follows that $\zeta_k g_j(\zeta) - \zeta_j g_k(\zeta) = 0$ when $|\zeta| = 1/R_1$ and therefore by analytic continuation when $\zeta \in \check\Omega$. Since $g_j(\zeta)$ vanishes when $\zeta_j = 0$, it follows that $g_j(\zeta)/\zeta_j$ is a holomorphic function independent of j, so we have proved that a form $f \in L^2_{0,q}(\Omega)$ is in the null space of $\bar\partial$ and $\mathfrak{d}_c$, that is, in the null space of the $\bar\partial$-Neumann operator if and only if

$$f = \sum_1^n (-1)^j \bar z_j \, d\bar z_1 \wedge \ldots \wedge \widehat{d\bar z_j} \wedge \ldots \wedge d\bar z_n F(\bar z/|z|^2)|z|^{-2n}, \tag{2.3}$$

where F is a holomorphic function in $L^2(\check\Omega)$, $\check\Omega = \{\zeta \in \mathbb{C}^n; |\zeta| < 1/R_0\}$. When f is orthogonal to this null space we have

$$\max(n-2,1)\|f\|^2 \leqslant R_1^2(\|\bar\partial f\|^2 + \|\mathfrak{d}_c f\|^2), \tag{2.4}$$

provided that f is in the domain of the operators in the right-hand side. This follows from elementary functional analysis (cf. [H1], Section 1.1) if we prove that for forms $g \in L^2_{0,n}(\Omega)$ we have

$$(n-1)\|g\|^2 \leqslant R_1^2\|\mathfrak{d}_c g\|^2 \tag{2.5}$$

when g is in the domain of $\mathfrak{d}_c$, and that

$$\max(n-2,1)\|h\|^2 \leqslant R_1^2\|\bar\partial h\|^2 \tag{2.6}$$

when $h \in L^2_{0,n-2}(\Omega)$ is in the domain of $\bar\partial$ and orthogonal to its null space. The estimate (2.6) follows from (2.1) when $n = 2$ and from (2.2) when $n > 2$. It suffices to prove (2.5) when $g = u\, d\bar z_1 \wedge \ldots \wedge d\bar z_n$ and $u \in C^\infty(\overline{\Omega})$ vanishes on the boundary. Then the inequality means that

$$(n-1)\int_\Omega |u|^2\, d\lambda \leqslant R_1^2 \int_\Omega |\partial u|^2\, d\lambda = \frac14 R_1^2 \int_\Omega (-\Delta u)\bar u\, d\lambda,$$

that is, that the lowest eigenvalue of the Dirichlet problem in Ω is $\geqslant 4(n-1)/R_1^2$. The lowest eigenvalue is simple so the eigenfunction has rotational symmetry, which means that it suffices to prove the estimate when u is a function of $r = |z|$. Then

$$\int_{R_0}^{R_1} |u(r)|^2 r^{2n-1}\, dr = -n^{-1}\int_{R_0}^{R_1} \operatorname{Re} u'(r)\overline{u(r)} r^{2n}\, dr,$$

and by the Cauchy-Schwarz inequality it follows that $n^2\|u\|^2 \leqslant R_1^2\|u'\|^2$, and since $n^2 \geqslant 4(n-1)$ we obtain (2.5), hence (2.4). (If the boundary conditions for $\mathfrak{d}_c$ were satisfied by the sum of the terms with $p_\nu = 0$ in the expansion, we would obtain (2.4) with the constant on the left-hand side replaced by $n-1$. However, this splitting cannot be expected in general, for if φ is a polynomial in $\bar z$ of degree $q+1$ and $j \neq k$, then $Y = z_j\partial\varphi/\partial\bar z_k - z_k\partial\varphi/\partial\bar z_j$ is harmonic, and $\bar z_j Y - |z|^2\partial\varphi/\partial\bar z_k/(n+q)$ is harmonic of bidegree $1, q+1$. In the boundary conditions $\bar z_k f_j - \bar z_j f_k = 0$ the terms of bidegree $1, q$ will therefore be mixed with those of bidegree $0, q$.)

We have now proved:

THEOREM 2.2. — *When $\Omega = \{z \in \mathbb{C}^n; R_0 < |z| < R_1\}$ is a spherical shell then the $\bar\partial$-Neumann operator in $L^2_{0,Q}(\Omega)$ is invertible if $0 < Q < n-1$, and the norm of the inverse is $\leqslant R_1/\sqrt{Q(n-1-Q)}$. When $Q = n-1$ the null space consists of the forms (2.3) where F is a square integrable holomorphic function in the ball $\check\Omega = \{\zeta \in \mathbb{C}^n; |\zeta| < 1/R_0\}$. The $\bar\partial$-Neumann operator is invertible on the orthogonal complement and the norm of the inverse is $\leqslant R_1/\max(\sqrt{n-2}, 1)$.*

The only novelty in this theorem is the explicit determination of the null space and possibly the explicit constants given: If $\Omega_0 \Subset \Omega_1$ are bounded open strictly pseudoconvex sets in $\mathbb{C}^n$ with smooth boundaries, the same qualitative results are known for $\Omega = \Omega_1 \setminus \overline{\Omega_0}$, even with estimates in the Sobolev space $H_{(\frac12)}$ (see [S1], [S2], [BS], [CS]).

For every open set $\Omega \subset \mathbb{C}^n$ (or any complex hermitian manifold) the null space of the $\bar\partial$-Neumann operator $\Box$ in $L^2_{p,q}(\Omega)$ is continuously embedded in $C^\infty_{p,q}(\Omega)$, so it has a reproducing kernel, the kernel of the orthogonal projection on $\operatorname{Ker}\Box$. Using Theorem 2.2 we shall now calculate the kernel when $p = 0$ and $q = n-1$ for the spherical shell

$$\Omega = \{z \in \mathbb{C}^n; R_0 < |z| < R_1\}$$

where $0 < R_0 < R_1$. Every $f \in \operatorname{Ker}\Box$ can then be written in the form (2.3) with F holomorphic in $\check\Omega = \{\zeta \in \mathbb{C}^n; 1/R_1 < |\zeta| < 1/R_0\}$, and

$$(2.7)\quad \|f\|^2 = \int_\Omega |F(\zeta)|^2 |z|^{2-4n} d\lambda(z) = \int_{\check\Omega} |F(\zeta)|^2 |\zeta|^{-2}\, d\lambda(\zeta),\ \ \zeta = \bar z/|z|^2.$$

Here we have used that $d\lambda(\zeta) = |z|^{-4n}\, d\lambda(z)$ since the inversion $z \mapsto \zeta$ is conformal with dilation factor $|z|^{-2}$ in the radial direction. We can easily calculate the Bergman kernel K of $\check\Omega$ with the norm in the right-hand side of (2.7), for if $F(\zeta)$ is holomorphic when $|\zeta| < 1/R_0$ then $F(\zeta) = \sum_\alpha c_\alpha \zeta^\alpha$ and

$$\int_{\check\Omega} |F(\zeta)|^2 |\zeta|^{-2}\, d\lambda(\zeta) = \sum_\alpha |c_\alpha|^2 I_\alpha,$$

$$I_\alpha = \int_{\check\Omega} |\zeta^\alpha|^2 |\zeta|^{-2} d\lambda(\zeta)$$
$$= \pi^n \int t^\alpha \Big(\sum_1^n t_j\Big)^{-1} dt = \pi^n \int_{1/R_1^2}^{1/R_0^2} r^{|\alpha|+n-2}\, dr \int t^\alpha \delta\Big(\sum_1^n t_j - 1\Big)\, dt$$

where $t = (|\zeta_1|^2, \dots, |\zeta_n|^2) \in \mathbb{R}^n$ and $1/R_1^2 < \sum_1^n t_j < 1/R_0^2$ in the middle integral. The inner integral on the right is equal to $\alpha!/(|\alpha| + n - 1)!$, a higher dimensional beta function. Since the monomials $\zeta^\alpha/\sqrt{I_\alpha}$ are an orthonormal basis in the Hilbert space of holomorphic functions F with the norm defined by (2.7), it follows that

$$\begin{aligned} K(\zeta,\theta) &= \sum_\alpha \zeta^\alpha\bar\theta^\alpha/I_\alpha \\ &= \pi^{-n}\sum_{k=0}^\infty \frac{(k+n)!-(k+n-1)!}{k!}\langle\zeta,\bar\theta\rangle^k R_0^{2(k+n-1)}/(1-(R_0^2/R_1^2)^{k+n-1}), \\ &\qquad\qquad \zeta,\theta\in\check\Omega. \end{aligned}$$

The series can be summed explicitly if one omits the last denominator, and since

$$1/(1-(R_0^2/R_1^2)^{k+n-1})-1=(R_0^2/R_1^2)^{k+n-1}/(1-(R_0^2/R_1^2)^{k+n-1}),$$

the error then committed is real analytic when $R_0^2|\langle\zeta,\bar\theta\rangle|<R_1^2/R_0^2$. Hence

$$K(\zeta,\theta)-\pi^{-n}R_0^{2n-2}\big(n!(1-R_0^2\langle\zeta,\bar\theta\rangle)^{-n-1}-(n-1)!(1-R_0^2\langle\zeta,\bar\theta\rangle)^{-n}\big)$$

is real analytic in a neighborhood of the closure of $\check\Omega\times\check\Omega$.

For every holomorphic function $F\in L^2(\check\Omega)$ we have

$$F(\zeta)=\int_{\check\Omega}K(\zeta,\theta)F(\theta)|\theta|^{-2}\,d\lambda(\theta).$$

If we write $\zeta=\bar z/|z|^2$ and $\theta=\bar w/|w|^2$, recall that $d\lambda(\theta)=|w|^{-4n}\,d\lambda(w)$, and multiply by $|z|^{-2n}$, we obtain

$$\begin{aligned} &|z|^{-2n}F(\bar z/|z|^2) \\ &= \int_\Omega |z|^{-2n}K(\bar z/|z|^2,\bar w/|w|^2)|w|^{-2n}F(\bar w/|w|^2)|w|^{-2n}\sum_1^n w_k\bar w_k\,d\lambda(w). \end{aligned}$$

Set

$$K_{jk}(z,w)=\bar z_j|z|^{-2n}K(\bar z/|z|^2,\bar w/|w|^2)w_k|w|^{-2n}.$$

Writing $(0,n-1)$ forms f as

$$f=\sum_{j=1}^n(-1)^jf_jd\bar z_1\wedge\cdots\wedge\widehat{d\bar z_j}\wedge\cdots\wedge d\bar z_n$$

and identifying f with $(f_1,\dots,f_n)$ we conclude that

$$f\mapsto\Big(\sum_{k=1}^n\int_\Omega K_{jk}(z,w)f_k(w)\,d\lambda(w)\Big)_{j=1}^n \tag{2.8}$$

reproduces $\operatorname{Ker}\Box$, since all its elements are of the form (2.3). The range is in $\operatorname{Ker}\Box$ since this is true for $(K_{jk}(z,w))_{j=1}^n$ for fixed w and k, and the operator is hermitian symmetric so it is the orthogonal projection on $\operatorname{Ker}\square$. The difference

$$(2.9)\quad K_{jk}(z,w) - \pi^{-n}R_0^{2n-2}\bar z_j\big(n!|z|^2(|z|^2|w|^2 - R_0^2\langle\bar z, w\rangle)^{-n-1}|w|^2 - (n-1)!(|z|^2|w|^2 - R_0^2\langle\bar z, w\rangle)^{-n}\big)w_k$$

is real analytic in a neighborhood of the closure of $\Omega\times\Omega$ which means that we have a complete description of the singularities of the reproducing kernel in the case of the spherical shell. In Section 4 it will serve as a model for more general manifolds Ω.

As a preparation for discussing manifolds where the Levi form of the boundary has signature $(n-q-1, q)$ for some $q < n-1$ we shall now try to adapt the study of the spherical shell to the set

$$(2.10)\qquad \Omega = \{z\in\mathbb{C}^n; R_0^2 < -|z'|^2 + |z''|^2 < R_1^2\},$$

where $z' = (z_1,\dots,z_{n-q-1})$, $z'' = (z_{n-q},\dots,z_n)$. The signature of the Levi form is $(n-q-1, q)$ on the inner boundary and $(q, n-q-1)$ on the outer boundary. We want to study the $\bar\partial$-Neumann operator in $L^2_{0,q}(\Omega)$ which is in the exceptional case on the inner boundary and also on the outer boundary if $2q = n-1$.

In the computations we introduce polar coordinates $r = |z'|$ and $s = |z''|$ so that Ω is defined by $R_0^2 < -r^2 + s^2 < R_1^2$ with $0\leqslant r < s$. Note that on the boundaries $s = \sqrt{r^2 + R_j^2} = r + R_j^2/(s+r) = r + R_j^2/2r + O(1/r^2)$ so for large r the two bounding hyperbolas are very close.

Choose spherical harmonics $Y_\nu(z')$ and $Z_\mu(z'')$ as before, and write a function $u\in C^\infty(\bar\Omega)$ with compact support as

$$u = \sum_{\nu,\mu} u_{\nu\mu},\quad u_{\nu\mu} = c_{\nu\mu}(r,s)Y_\nu(z')Z_\mu(z'');$$

the bidegree of Y_ν is p'_ν, q'_ν and that of Z_μ is p''_μ, q''_μ. Since spherical harmonics are orthogonal to all polynomials of lower degree it follows that $c_{\nu\mu}(r,s)$ is smooth even when $r = 0$. To examine the range of $\bar\partial$ on scalars we write with the notation $n' = n-q-1$, $n'' = q+1 = n-n'$

$$\|\bar\partial u\|^2 = \int_\Omega \sum_1^{n'} |\partial u/\partial\bar z_j|^2\, d\lambda(z) + \int_\Omega \sum_{n'+1}^{n} |\partial u/\partial\bar z_j|^2\, d\lambda(z).$$

In the first sum we integrate first with respect to z' for fixed z'' and then with respect to z'', and in the second sum we integrate in the opposite order. Introducing the notation

$$\psi_{\nu\mu}(t', t'') = c_{\nu\mu}(r,s)r^{p'_\nu+q'_\nu+n'-1}s^{p''_\mu+q''_\mu+n''-1},\quad t' = \log r,\ t'' = \log s,$$

we obtain by the old identities in the case where $q = n-1$ that $\|\bar\partial u\|^2 = \sum_{\nu,\mu} \|\bar\partial u_{\nu\mu}\|^2$,

$$\begin{aligned}(2.11)\qquad 4\|\bar\partial u_{\nu\mu}\|^2 = \int \big(&|\partial\psi_{\nu\mu}/\partial t' + (q'_\nu - p'_\nu - n' + 1)\psi_{\nu\mu}|^2 \\ &+ 4q'_\nu(p'_\nu + n' - 1)|\psi_{\nu\mu}|^2\big)e^{2t''}\,dt'\,dt'' \\ + \int \big(&|\partial\psi_{\nu\mu}/\partial t'' + (q''_\mu - p''_\mu - n'' + 1)\psi_{\nu\mu}|^2 \\ &+ 4q''_\mu(p''_\mu + n'' - 1)|\psi_{\nu\mu}|^2\big)e^{2t'}\,dt'\,dt''.\end{aligned}$$

Assume to exclude an exceptional case that $n' \geqslant 2$ and that $n'' \geqslant 2$. Since $e^{2t''} \geqslant e^{2t'} = e^{2(t'+t'')}/e^{2t''}$, it follows that

$$\|\bar\partial u_{\nu\mu}\|^2 \geqslant \max(q'_\nu, q''_\mu) \int |z''|^{-2}|u_{\nu\mu}|^2\,d\lambda(z).$$

When $q'_\nu = q''_\mu = 0$ then $Y_\nu(z')Z_\mu(z'')$ is holomorphic, and since $|\partial r/\partial z'| = |\partial s/\partial z''| = \frac{1}{2}$, we have then

$$4\|\bar\partial u_{\nu\mu}\|^2 = \iint_\omega (|\partial c_{\nu\mu}/\partial r|^2 + |\partial c_{\nu\mu}/\partial s|^2) r^{2p'_\nu+2n'-1} s^{2p''_\mu+2n''-1}\,dr\,ds,$$

where $\omega = \{(r,s); R_0^2 < -r^2 + s^2 < R_1^2,\ 0 < r < s\}$. We introduce new coordinates (r,S) with $s = \sqrt{S^2+r^2}$ which makes ω defined by $R_0 < S < R_1$ and $r > 0$; we have $D(r,s)/D(r,S) = \partial s/\partial S = S/s$. If $f(r,s) = F(r,S)$ then $\partial F/\partial r = \partial f/\partial r + (r/s)\partial f/\partial s$, for $ds = r dr/s$ when $dS = 0$. Hence $|\partial F/\partial r|^2 \leqslant |f'|^2$, so

$$\int_\omega |f'(r,s)|^2 r^{\lambda'} s^{\lambda''} dr ds \geqslant \int_{R_0}^{R_1} S dS \int_0^\infty |\partial F/\partial r|^2 r^{\lambda'} s^{\lambda''-1} dr,\ s = \sqrt{S^2+r^2}.$$

With a positive increasing function M to be defined in a moment we observe that when $F(r,S) = 0$ for large r we have

$$\begin{aligned}\int_0^\infty |F|^2 M'\,dr &= \big[M|F|^2\big]_0^\infty - 2\int_0^\infty M\,\mathrm{Re}\,\bar F F'\,dr \\ &\leqslant 2\Big(\int_0^\infty |F|^2 M'\,dr \int_0^\infty (M^2/M')|F'|^2\,dr\Big)^{\frac{1}{2}},\end{aligned}$$

where S is regarded as a parameter, hence

$$\int_0^\infty |F|^2 M'\,dr \leqslant 4\int_0^\infty (M^2/M')\,|F'|^2\,dr.$$

Assuming that $\lambda' + \lambda'' > 2$ we define M by

$$M(r)^{-1} = \int_r^\infty \tau^{-\lambda'}(S^2+\tau^2)^{(1-\lambda'')/2}\,d\tau, \quad \text{thus } M^2/M' = r^{\lambda'} s^{\lambda''-1}.$$

Since $M'(r) = M(r)^2 r^{-\lambda'} s^{1-\lambda''}$ we need to estimate M from below, that is, $1/M$ from above. We claim that

$$M(r)^{-1} \leqslant r^{-\lambda'}(S^2+r^2)^{(2-\lambda'')/2}/\lambda'$$

if $\lambda' > 0$. It suffices to prove the opposite inequality for the derivatives, that is,

$$r^{-\lambda'}(S^2+r^2)^{(1-\lambda'')/2} \leqslant r^{-\lambda'-1}(S^2+r^2)^{(2-\lambda'')/2} + r^{1-\lambda'}(S^2+r^2)^{-\lambda''/2}(\lambda''-2)/\lambda',$$

or equivalently $1 \leqslant A + A^{-1}(\lambda''-2)/\lambda'$ where $A = \sqrt{S^2+r^2}/r \geqslant 1$. This is obvious if $\lambda'' \geqslant 2$, and then it follows that

$$M'(r) \geqslant r^{-\lambda'} s^{1-\lambda''} r^{2\lambda'} s^{2(\lambda''-2)} \lambda'^2$$

which proves that

$$\lambda'^2 \int_\omega |f(r,s)|^2 r^{\lambda'} s^{\lambda''-2}\, dr\, ds \leqslant 4 \iint_\omega |f'(r,s)|^2 r^{\lambda'} s^{\lambda''}\, dr\, ds.$$

Applying this to $c_{\nu\mu}$ with $\lambda' = 2p'_\nu + 2n' - 1$ and $\lambda'' = 2p''_\mu + 2n'' - 1$ we have $\lambda' \geqslant 3$ so $4/\lambda'^2 < \frac{1}{2}$, and we obtain

$$\int_\Omega |u|^2 |z''|^{-2}\, d\lambda(z) \leqslant 2\|\bar\partial u\|^2$$

without using the minimizing condition but with u smooth and of compact support. It extends right away to all $u \in L^2_{\text{loc}}$ for which the left-hand side is finite and $\bar\partial u \in L^2(\Omega)$, first by cutting off with a function $\chi(\varepsilon|z|)$ with $\varepsilon \to 0$ and then smoothing as usual. (Note that $|z''|^2 \leqslant |z|^2 < 2|z''|^2$.)

We shall now study the $\bar\partial$-Neumann operator in $L^2_{0,q}(\Omega)$, first for forms $f \in C^\infty_{0,q}(\overline{\Omega})$ with compact support. Let $\partial_0\Omega$ denote the inner boundary where $\varrho(z) = |z'|^2 - |z''|^2 + R_0^2$ is a defining function and let $\partial_1\Omega$ be the outer boundary where $\varrho(z) = -|z'|^2 + |z''|^2 - R_1^2$ is one. We write as usual

$$f = \sum_{|I|=q}{}' f_I(z) d\bar z^I, \quad d\bar z^I = d\bar z_{i_1} \wedge \ldots \wedge d\bar z_{i_q}, \quad I = (i_1, \ldots, i_q),$$

where $\sum'$ indicates summation for increasing sequences I of indices but f_I is defined for all I and antisymmetric in I. The basic estimate of Morrey and Kohn for the $\bar\partial$-Neumann operator states that for an arbitrary defining function ϱ

(2.12)

$$\|\bar\partial f\|^2 + \|\mathfrak{d}_c f\|^2 = \sum_{|I|=q}{}' \|\bar\partial f_I\|^2 + \sum_{|K|=q-1}{}' \int_{\partial\Omega} \sum_{j,l=1}^n \partial^2\varrho/\partial z_j \partial\bar z_l f_{jK} \overline{f_{lK}} \delta(\varrho) d\lambda,$$

where δ is the Dirac measure at the origin and

$$\sum_{j=1}^{n} \partial\varrho(z)/\partial z_j f_{jK}(z) = 0, \quad z \in \partial\Omega,$$

since f is assumed to be in the minimal domain of $\mathfrak{d}$. Let $I = I', I''$ where the indices in I' are $\leqslant n'$ and those in I'' are $> n'$, so that $d\bar{z}^I = d\bar{z}'^{I'} \wedge d\bar{z}''^{I''}$. Then

$$\sum_{|K|=q-1}' \sum_{j=1}^{n} \partial^2\varrho/\partial z_j \partial\bar{z}_l f_{jK}\overline{f_{lK}} = (-1)^\sigma \sum_{|I|=q}'(|I'| - |I''|)|f_I|^2 \quad \text{on } \partial_\sigma\Omega,$$

for defining K by omitting an index $j \in I$ gives a contribution $\pm|f_{jK}|^2$ with the sign depending on whether $j \leqslant n'$ or $j > n'$. If f_I is decomposed in the same way as u above we obtain

$$\begin{aligned}\int_{\partial_\sigma\Omega} &|f_I|^2\delta(\varrho)\, d\lambda \\ &= \sum_{\nu,\mu} \int |c_{\nu\mu}(r,s)|^2 r^{2(p'_\nu+q'_\nu+n')-1} s^{2(p''_\mu+q''_\mu+n'')-1}\delta(-r^2+s^2-R_\sigma^2)\, dr\, ds \\ &= \sum_{\nu,\mu} \int |\psi_{\nu\mu}(t',t'')|^2 rs\delta(-r^2+s^2-R_\sigma^2)\, dr\, ds,\end{aligned}$$

where $c_{\nu\mu}$ and $\psi_{\nu\mu}$ depend on I also. If we use r as parameter then

$$\begin{aligned} rs\delta(-r^2+s^2-R_\sigma^2) &= rs\delta(s-\sqrt{r^2+R_\sigma^2})/(s+\sqrt{r^2+R_\sigma^2}) \\ &= \frac{1}{2} r\delta(s-\sqrt{r^2+R_\sigma^2}),\end{aligned}$$

and if we use s as parameter we obtain similarly $rs\delta(-r^2+s^2-R_\sigma^2) = \frac{1}{2}s\delta(r-\sqrt{s^2-R_\sigma^2})$. Hence

$$\begin{aligned}\int_{\partial_\sigma\Omega} |f_I|^2\delta(\varrho)\, d\lambda &= \sum_{\nu,\mu} \frac{1}{2}\int_{\partial_\sigma\omega} |\psi_{\nu\mu}(t',t'')|^2 e^{2t'}\, dt' \\ &= \sum_{\nu,\mu} \frac{1}{2}\int_{\partial_\sigma\omega} |\psi_{\nu\mu}(t',t'')|^2 e^{2t''}\, dt''\end{aligned}$$

where $\omega = \{(t',t''); R_0^2 < -e^{2t'} + e^{2t''} < R_1^2\}$ and $\partial_0\omega$, $\partial_1\omega$ are the two parts of the boundary. Since t' is a decreasing function of R_σ for fixed t'' while t'' is increasing for fixed t', we have

$$\begin{aligned}\int_{\partial_1\Omega} &|f_I|^2\delta(\varrho)\, d\lambda - \int_{\partial_0\Omega} |f_I|^2\delta(\varrho)\, d\lambda \\ &= \sum_{\nu,\mu} \int_\omega \operatorname{Re} \partial\psi_{\nu\mu}(t',t'')/\partial t''\overline{\psi_{\nu\mu}(t',t'')} e^{2t'}\, dt'\, dt'' \\ &= -\sum_{\nu,\mu} \int_\omega \operatorname{Re} \partial\psi_{\nu\mu}(t',t'')/\partial t'\overline{\psi_{\nu\mu}(t',t'')} e^{2t''}\, dt'\, dt''.\end{aligned}$$

Note that $\psi_{\nu\mu}(t',t'') = O(e^{t'})$ as $t' \to -\infty$ unless $n' = 1$, $q'_\nu = p'_\nu = 0$; the last equality may not be valid then. If $|I'| - |I''| = a'_{\nu\mu} - a''_{\nu\mu}$ and $a'_{\nu\mu} = 0$ if $n' - 1 = q'_\nu = p'_\nu = 0$, the contribution of $\psi_{\nu\mu}$ to the last term in (2.12) is equal to

$$\int_\omega \big(a'_{\nu\mu} \operatorname{Re} \partial\psi_{\nu\mu}(t',t'')/\partial t' \overline{\psi_{\nu\mu}(t',t'')} e^{2t''} + a''_{\nu\mu} \partial\psi_{\nu\mu}(t',t'')/\partial t'' \overline{\psi_{\nu\mu}(t',t'')} e^{2t'}\big)\, dt'dt'',$$

and we add the contribution to the preceding term in (2.12) expressed by (2.11). Altogether the contribution becomes, if we now also temporarily drop the subscripts ν and μ of $\psi_{\nu\mu}$ and $a'_{\nu\mu}$, $a''_{\nu\mu}$ in addition to the subscript I which we already suppressed,

$$\begin{aligned}&\int_\omega \Big(\frac14 |\partial\psi(t',t'')/\partial t' + (q'_\nu - p'_\nu - n' + 1 + 2a')\psi(t',t'')|^2 \\ &+ (a' + q'_\nu)(p'_\nu + n' - 1 - a')|\psi(t',t'')|^2\Big) e^{2t''}\, dt'\, dt'' \\ &+ \int_\omega \Big(\frac14 |\partial\psi(t',t'')/\partial t'' + (q''_\mu - p''_\mu - n'' + 1 + 2a'')\psi(t',t'')|^2 \\ &+ (a'' + q''_\mu)(p''_\mu + n'' - 1 - a'')|\psi(t',t'')|^2\Big) e^{2t'}\, dt'\, dt''.\end{aligned}$$

Next we shall examine if, for arbitrary I with $|I| = q$ and arbitrary $q'_\nu, p'_\nu, q''_\mu, p''_\mu$, it is possible to choose a', a'' so that the coefficients of $|\psi|^2$ are at least non-negative, for if that is the case we should be able to find the null space of the $\bar\partial$-Neumann operator by examining each such term (depending on I, ν, μ) separately. Note that the preceding decomposition extends by continuity to all f in the domain of the $\bar\partial$-Neumann operator in $L^2_{0,q}(\Omega)$.

The first problem is to decide if it is possible for arbitrary $p'_\nu, q'_\nu, p''_\mu, q''_\mu$ to choose a' and a'' with $a' - a'' = |I'| - |I''|$ so that

$$(2.13)\qquad -q'_\nu \leqslant a' \leqslant p'_\nu + n' - 1, \quad -q''_\mu \leqslant a'' \leqslant p''_\mu + n'' - 1.$$

These inequalities imply that

$$|I'| - |I''| = a' - a'' \leqslant p'_\nu + n' - 1 + q''_\mu,$$

and when $p'_\nu = q''_\mu = 0$ this requires that

$$2|I'| - q = |I'| - |I''| \leqslant n' - 1 = n - q - 2,$$

that is, $2|I'| \leqslant n-2$. Since $|I'| \leqslant \min(q, n-q-1)$ this is true if $n \geqslant 2q+2$, that is, $n' \geqslant n''$, or $n \leqslant 2q$, that is, $n' \leqslant n'' - 2$. If $n \neq 2q + 1$, as we shall assume from now on, it follows that $|I'| - |I''| \leqslant n' - 1$, and then (2.13) is fulfilled with $a' = |I'| - |I''|$, $a'' = 0$ if $|I'| - |I''| \geqslant 0$ and $a' = 0$, $a'' = |I''| - |I'|$ if $|I''| \geqslant |I'|$. We shall now examine when one can add the same quantity ε to these values of a', a'' so that the inequalities (2.13) remain valid and one of them is fulfilled with strict inequalities. This will yield an estimate of $\int |\psi(t',t'')|^2 e^{2t'}\, dt'\, dt''$ so that the component in question is absent for elements in the null space of the $\bar\partial$-Neumann operator $\square$. At first we shall assume that $n' > 1$.

(i) If $0 \leqslant |I'| - |I''| < n' - 1$ and $a' = |I'| - |I''| + \frac{1}{2}$, $a'' = \frac{1}{2}$ then all inequalities (2.13) become strict so that $\psi = 0$ if f is in the null space of $\square$. If $|I'| - |I''| = n' - 1$ then $|I'| = q$, $|I''| = 0$ and $n = 2 + 2q$, or $|I'| = n - q - 1 = n'$, $|I''| = 1$, thus $n = 2q$. We postpone discussion of these cases.

(ii) If $|I'| < |I''| < q$ then we can take $a' = \frac{1}{2}$, $a'' = |I''| - |I'| + \frac{1}{2}$ and conclude that $\psi = 0$ if f is in the null space of $\square$.

(iii) If $|I'| = 0$, $|I''| = q$ we get strict inequalities (2.13) if $a' = \pm\frac{1}{2}$ and $a'' = q \pm \frac{1}{2}$ provided that $p''_\mu > 0$ or $q'_\nu \neq 0$ respectively, so $\psi = 0$ unless $q'_\nu = p''_\mu = 0$. Then we must take $a' = 0$ and $a'' = q$ and observe that if f is in the null space of $\square$ then

$$\partial\psi/\partial t' = (p'_\nu + n' - 1)\psi, \quad \partial\psi/\partial t'' = -(q''_\mu + q)\psi,$$

where we have used that $n'' - 1 = q$. This means that there is a constant C such that

$$\psi(t',t'') = Ce^{(p'_\nu + n' - 1)t' - (q''_\mu + q)t''}.$$

If we reintroduce the subscripts ν, μ this means that

$$c_{\nu\mu}(r,s) r^{p'_\nu + n' - 1} s^{q''_\mu + q} = C_{\nu\mu} r^{p'_\nu + n' - 1} s^{-(q''_\mu + q)},$$

that is, $c_{\nu\mu}(r,s) = C_{\nu\mu} s^{-2q''_\mu - 2q}$, so that

$$c_{\nu\mu}(r,s) Y_\nu(z') Z_\mu(z'') = C_{\nu\mu} Y_\nu(z') Z_\mu(z''/|z''|^2)|z''|^{-2q}.$$

Here Y_ν is a holomorphic homogeneous polynomial of degree p'_ν and Z_μ is an antiholomorphic homogeneous polynomial of degree q''_μ. The L^2 norm is finite if and only if

$$\iint_{R_0^2 + r^2 < s^2 < R_1^2 + r^2} r^{2p'_\nu + 2n' - 1} s^{-2q''_\mu - 4q + 2n'' - 1}\, dr\, ds < \infty. \tag{2.14}$$

Since $n'' = q+1$ the exponent of s is $-2q''_\mu - 2q + 1$ so the integral with respect to s is $(s_0^{-2q''_\mu - 2q + 2} - s_1^{-2q''_\mu - 2q + 2})/(2q''_\mu + 2q - 2)$ where $s_j = (R_j^2 + r^2)^{\frac{1}{2}}$, hence $s_1 - s_0 = (R_1^2 - R_0^2)/(s_1 + s_0)$ is asymptotic to $(R_1^2 - R_0^2)/2r$. In the remaining integral with respect to r the integrand is asymptotic to a constant times r^τ when $r \to +\infty$ where $\tau = -2q''_\mu - 2q + 2p'_\nu + 2(n-q-1) - 1$. Convergence requires that $\tau < -1$, that is, $\tau \leqslant -3$ since τ is odd, which means that $p'_\nu + n \leqslant q''_\mu + 2q$. Hence, introducing also the dependence on I in the notation now, we have when $|I'| = 0$ and $|I''| = q$, after some changes of the labels ν, μ,

$$f_I(z) = |z''|^{-2q} \sum_{\nu,\mu} g_{I\nu\mu}(\zeta), \quad \zeta = (\zeta', \zeta'') = (z', \bar{z}''/|z''|^2),$$

where $g_{I\nu\mu}$ is a bihomogeneous holomorphic polynomial in $\zeta = (\zeta', \zeta'')$ of bidegree ν, μ where $\nu + n \leqslant \mu + 2q$ and the sum is L^2 convergent in Ω. By Hartogs' theorem it follows that

$$g_I(\zeta) = \sum_{\nu,\mu} g_{I\nu\mu}(\zeta)$$

is locally convergent to a function g_I which is holomorphic in $\check{\Omega}_0$,

$$\check{\Omega}_\sigma = \{\zeta \in \mathbb{C}^n; |\zeta''|^2(R_\sigma^2 + |\zeta'|^2) - 1 < 0\}, \quad \sigma = 0, 1. \tag{2.15}$$

This is a strictly pseudoconvex domain, for the Levi form of the defining function

$$|\zeta''|^2|t'|^2 + |\zeta'|^2|t''|^2 + 2\,\mathrm{Re}(t', \zeta')(\zeta'', t'') + R_\sigma^2|t''|^2 \geqslant (|\zeta''||t'| - |\zeta'||t''|)^2 + R_\sigma^2|t''|^2$$

can only vanish when $t'' = 0$ and $|\zeta''|t' = 0$, and $\zeta'' \neq 0$ on the boundary. As in the proof of Theorem 2.2 we can now use the minimal boundary condition for ∂ at the outer boundary to conclude that $g_I(\zeta) = (-1)^j \zeta_j g(\zeta)$ if $n' < j \notin I$, where g is also holomorphic in $\check{\Omega}_0$ and is a locally uniformly convergent sum of monomials $\zeta'^\alpha \zeta''^\beta$ with $|\alpha| + n \leqslant |\beta| + 1 + 2q$. We have

$$f = \sum_{n'+1}^{n} (-1)^j \bar{z}_j d\bar{z}_1 \wedge \ldots \wedge \widehat{d\bar{z}_j} \wedge \ldots \wedge d\bar{z}_n g(z', \bar{z}''/|z''|^2)|z''|^{-2n''}, \tag{2.16}$$

where we recall that $n'' = q + 1$. As in (2.7) we have

$$\|f\|^2 = \int_{\check{\Omega}_0 \setminus \check{\Omega}_1} |g(\zeta)|^2 |\zeta''|^{-2}\, d\lambda(\zeta). \tag{2.17}$$

Every monomial in the Taylor expansion has a norm $\leqslant \|f\|$ which again gives the condition $|\alpha| + n - 2q - 1 \leqslant |\beta|$ for the monomials $\zeta'^\alpha \zeta''^\beta$ which can occur in the expansion of g.

(iv) If $n = 2(q + 1)$ so that $n' = n'' = q + 1$ the arguments above allow another possibility with $|I'| = q$, $|I''| = 0$, $p'_\nu = 0$, $q''_\mu = 0$ which leads to $c_{\nu\mu}(r, s) = C_{\nu\mu} r^{-2q'_\nu - 2q}$. Since this does not give a function in L^2

if $C_{\nu\mu} \neq 0$ it follows that (2.16) determines $\operatorname{Ker}\square$ completely even in this case.

(v) If $n = 2q$, $|I'| \geqslant |I''|$ we encountered the possibility that $|I'| = n-q-1 = q-1$, $|I''| = 1$. Then $q \geqslant 3$ since we have assumed that $n' > 1$. With $a' = q-2+\varepsilon$ and $a'' = \varepsilon$ the inequalities (2.13) become

$$-q'_\nu \leqslant q-2+\varepsilon \leqslant p'_\nu + q - 2, \quad -q''_\mu \leqslant \varepsilon \leqslant p''_\mu + q.$$

If $p'_\nu > 0$ we make them strict by taking $\varepsilon = \frac{1}{2}$, and if $q''_\mu > 0$ we can do so by taking $\varepsilon = -\frac{1}{2}$. If $p'_\nu = q''_\mu = 0$, $a' = q-2$, $a'' = 0$ then

$$\psi = Ce^{t'(-q'_\nu + n' - 1 - 2(n'-1)) + t''(p''_\mu + q)} = Cr^{-(q'_\nu + n' - 1)} s^{p''_\mu + q},$$

which gives $c_{\nu\mu}(r,s) = C_{\nu\mu} r^{-2(q'_\nu + n' - 1)}$. The L^2 norm of $c_{\nu\mu}(r,s)Y_\nu(x')$ $Z_\mu(z'')$ is not finite when $C_{\nu\mu} \neq 0$ unless

$$\iint_{R_0^2 + r^2 < s^2 < R_1^2 + r^2} r^{-4(q'_\nu + n' - 1) + 2q'_\nu + 2n' - 1} s^{2p''_\mu + 2n'' - 1}\, dr\, ds < \infty.$$

Since the exponent of r is $\leqslant -1$ when $n' \geqslant 2$, this does not happen, so (2.16) is still a complete description of $\operatorname{Ker}\square$.

(vi) The remaining case $n' = 1$ can be handled along the same lines but requires a detailed and lengthy study of several cases. We shall omit it since the complete description (2.16) of the null space of the $\bar\partial$-Neumann operator for $(0,q)$ forms in (2.10) when $n \neq 2q+1$ and $n' \neq 1$ is an ample motivation for the constructions in Section 3.

3. Infinite dimensionality of the null space of the $\bar\partial$-Neumann operator.

Although the representation (2.3) of the null space of the $\bar\partial$-Neumann operator in a spherical shell is very exceptional, we shall now show that it has quite general consequences:

THEOREM 3.1. — *Let Ω be an open subset of a complex analytic hermitian manifold of dimension n, and let z^0 be a point in the boundary $\partial\Omega$ where it is in C^3 and the Levi form has $n-q-1$ positive and q negative eigenvalues where $1 \leqslant q \leqslant n-1$. Then the null space of the $\bar\partial$-Neumann operator $\mathfrak{d}_c\bar\partial + \bar\partial\mathfrak{d}_c$ in $L^2_{0,q}$ has infinite dimension if the range is closed.*

The crucial assumption that the range is closed is a global condition which in general is very difficult to check, for example if the Levi form does not have a constant signature. Thus the theorem just affirms that when the Levi form has the critical signature at some point, then the $\bar\partial$-Neumann operator is flawed in the sense that either the dimension of the null space is infinite or else the range is not closed (or both). Since the range of a closed linear operator between Banach spaces is closed if it has finite codimension, the theorem means precisely that the critical signature of the Levi form at z^0 implies that the range of the $\bar\partial$ operator from $L^2_{0,q-1}(\Omega)$ to $L^2_{0,q}(\Omega)$ has infinite codimension in the null space of the $\bar\partial$ operator from $L^2_{0,q}(\Omega)$ to $L^2_{0,q+1}(\Omega)$.

We shall prepare the proof with two lemmas. The first is a very special case of the classification of hypersurfaces in [CM] but the proof is so short and elementary that we give it in detail for the convenience of the reader, in a form which will be useful later on but is more precise than necessary now.

LEMMA 3.2. — *At a point $z^0 \in \partial\Omega$ where $\partial\Omega \in C^3$ and the Levi form is non-degenerate one can choose local complex coordinates vanishing at z^0 and a defining function ϱ such that the metric is $\sum_1^n |dz_j|^2$ at z^0 and*

$$\varrho(z) = 2\operatorname{Im} z_n + \sum_1^{n-1} \lambda_j |z_j|^2 + o(|z|^3), \tag{3.1}$$

where $\lambda_j \neq 0$ for $j = 1, \dots, n-1$.

Proof. — Starting from a defining function $\varrho \in C^3$ such that $|\partial\varrho(z^0)| = 1$ we first choose z_n so that $\partial\varrho = dz_n/i$ at z^0 and extend to a local coordinate system orthonormal at z^0, which gives

$$\varrho(z) = 2\operatorname{Im} z_n + H(z') + \operatorname{Re} A(z) + \operatorname{Re}(\bar z_n B(z)) + O(|z|^3),$$

where H is a hermitian symmetric quadratic form in $z' = (z_1, \dots, z_{n-1})$, $A(z)$ is a holomorphic quadratic form, and $B(z)$ is a complex linear form. Since H is nonsingular by assumption we can by a unitary change of the z' variables attain that $H(z') = \sum_1^{n-1} \lambda_j |z_j|^2$ with $\lambda_j \neq 0$. Replacing $\bar z_n$ by $z_n - 2i \operatorname{Im} z_n$ we obtain

$$\varrho(z) = (1 + \operatorname{Im} B(z))\Big(\operatorname{Im}(2z_n + iA(z) + iz_n B(z)) + \sum_1^{n-1} \lambda_j |z_j|^2\Big) + O(|z|^3).$$

If we divide ϱ by $(1 + \operatorname{Im} B(z))$ and take $z_n + \frac12 iA(z) + \frac12 iz_n B(z)$ as a new coordinate instead of z_n, we obtain a C^3 defining function in the new variables of the form

$$\varrho(z) = 2\operatorname{Im} z_n + \sum_1^{n-1} \lambda_j |z_j|^2 + \operatorname{Re} A_0(z) + 2\sum_1^{n} \operatorname{Re}(\bar z_j A_j(z)) + o\left(|z|^3\right)$$

where A_0 is a holomorphic cubic form and $A_1, \dots, A_n$ are holomorphic quadratic forms. Hence

$$\varrho(z) = (1 + 2\operatorname{Im} A_n(z))\big(\operatorname{Im}(2z_n + iA_0(z) + 2iz_nA_n(z)) + \sum_1^{n-1} \lambda_j |z_j + A_j(z)/\lambda_j|^2\big) + o(|z|^3).$$

If we divide $\varrho(z)$ by $1 + 2\operatorname{Im} A_n(z)$ and introduce

$$(z_1 + A_1(z)/\lambda_1, \dots, z_{n-1} + A_{n-1}(z)/\lambda_{n-1}, z_n + \tfrac{1}{2}iA_0(z) + iz_nA_n(z))$$

as new coordinates, we obtain a new C^3 defining function such that (3.1) is valid with the new coordinates. The coordinates have remained orthonormal at z^0 in every step of the proof.

If one allows a change of the hermitian metric at z^0 then one can choose $\lambda_j = \pm 1$, so the quadratic form in the right-hand side is determined by the signature of the Levi form apart from the order of the terms.

It is more natural to state the next lemma in terms of real variables:

LEMMA 3.3. — *Let ϱ be a real valued and ψ a complex valued function, both in C^2 in a neighborhood of $0 \in \mathbb{R}^N$, and assume that at 0*

$$\varrho(x) = 2x_N + A(x) + o(|x|^2), \quad \psi(x) = x_{N-1} + ix_N + B(x) + o(|x|^2),$$

where A and B are quadratic forms and $Q(x) = \frac{1}{2}A(x) - \operatorname{Im} B(x)$ is positive definite when $x_N = 0$. If χ is a continuous function with support close to the origin, $k \geqslant 0$, and Φ is a continuous function with $|\Phi(z)| \leqslant C(1+|z|)^{-k-1-\frac{1}{4}N}$ when $z \in \mathbb{C}$ and $\operatorname{Re} z \geqslant 0$, then

$$\lim_{\tau\to+\infty} \tau^{k+1+\frac{1}{2}N} \int_{\varrho(x)<0} \chi(x)\big||x|^k \Phi(i\tau\psi(x))\big|^2\, dx = \chi(0) \int_{y_N>0} y_N^{k-1+\frac{1}{2}N} |\Phi(iy_{N-1} + y_N)|^2\, dy'' \int_{Q(y',0)<1} |y'|^{2k}\, dy'.$$

where $y' = (y_1, \dots, y_{N-2})$ and $y'' = (y_{N-1}, y_N)$.

Proof. — We can take $x_{N-1} + \operatorname{Re} B(x)$ and $x_N + \operatorname{Im} B(x)$ as new coordinates instead of x_{N-1} and x_N in a neighborhood of the origin. The Jacobian of the change of variables is then equal to one at the origin, so we may assume in what follows that $B = 0$. Then $Q(x) = \frac{1}{2}A(x)$ is positive definite when $x_N = 0$, and the condition $\varrho(x) < 0$ can be written

$$x_N + Q(x', x_{N-1}, 0) + o(|x'|^2 + x_{N-1}^2) < 0,$$

so it implies that $|x|^2 < C(-x_N)$ for some constant C if $|x|$ is sufficiently small. Hence $\operatorname{Re} i\psi(x) = -x_N + o(x_N) > 0$ and $|\psi(x)| \geqslant |x''| + o(x_N) > |x''|/2$ then. Hence the expression in the left-hand side of the statement is well defined if $\operatorname{supp}\chi$ is sufficiently close to the origin, and with the new variables $y' = \sqrt{\tau}x'$ and $y'' = \tau x''$ it becomes

$$\int_{\varrho(x)<0} \chi(x)(|y'|^2 + |y''|^2/\tau)^k |\Phi(i\tau\psi(x)|^2\, dy.$$

Since $|i\tau\psi(x)| \geqslant |y''|/2$ and $|y'|^2 < C(-y_N)$ when $\varrho(x) < 0$ and $\chi(x) \neq 0$, we have an integrable majorant by the assumed bound for Φ, and when $\tau \to \infty$ the limit becomes

$$\int_{y_N+Q(y',0)<0} \chi(0)|y'|^{2k}|\Phi(iy_{N-1} - y_N)|^2\, dy,$$

which proves the lemma if we first integrate with respect to y', changing the sign of y_N.

Before beginning the proof of Theorem 3.1 we shall also rewrite Theorem 2.2 with $R_0 = 1$ by moving the origin to $z^0 = (0, \dots, 0, -i)$. Then the defining function $1 - |z|^2$ at the inner boundary becomes $1 - |z + z^0|^2 = 2\operatorname{Im} z_n - |z|^2$. The nth component of the argument of F in (2.3) becomes

$$(\bar z_n + i)/(1 + |z|^2 - 2\operatorname{Im} z_n) = i + z_n + 2z_n \operatorname{Im} z_n - i|z|^2 + O(|z|^3).$$

If we set

$$\psi(z) = (\bar z_n+i)/(1+|z|^2-2\operatorname{Im} z_n)-i = z_n+2z_n \operatorname{Im} z_n-i|z|^2+O(|z|^3), \tag{3.2}$$

and set $\varrho(z) = 2\operatorname{Im} z_n - |z|^2$ it follows that for every $\tau > 0$ and κ

$$f=\sum_1^n(-1)^j\partial\varrho/\partial z_j\, d\bar z_1\wedge\dots\wedge\widehat{d\bar z_j}\wedge\dots\wedge d\bar z_n(1+i\tau\psi(z))^{-\kappa}(1+|z|^2-2\operatorname{Im} z_n)^{-n} \tag{3.3}$$

is annihilated by $\bar\partial$ and $\mathfrak{d}_c$ when $\varrho < 0$, and in fact annihilated by $\bar\partial$ and $\mathfrak{d}$ in a neighborhood of 0.

Proof of Theorem 3.1. — If T is the maximal $\bar\partial$ operator from $(0, q-1)$ forms to $(0, q)$ forms and S is the maximal $\bar\partial$ operator from $(0, q)$ forms to $(0, q+1)$ forms, in L^2, then the range of the $\bar\partial$-Neumann operator $TT^* + S^*S$ is closed if and only if the range of T and the range of S are closed, and then the null space has the same dimension as the quotient of the null space of S by the range of T. (See e.g. [H1], Section 1.1.) Thus the hypothesis and conclusion in the theorem do not change if we locally replace the hermitian metric by an equivalent one, so we may later on assume that the metric at z^0 is whatever will prove convenient. By [H1], Theorem 1.1.3, what must be proved is that there is a sequence f_k of $(0,q)$ forms which is bounded in L^2 but has no convergent subsequence although $\mathfrak{d}_c f_k \to 0$ and $\bar\partial f_k \to 0$ in L^2.

Assume at first that $q = n-1$ and choose local complex coordinates vanishing at z^0 such that a C^3 defining function ϱ for Ω in a neighborhood of z^0 has the form

$$\varrho(z) = 2\,\mathrm{Im}\, z_n - |z|^2 + r(z), \quad r(z) = o(|z|^3),$$

which is possible by Lemma 3.2. We may assume that the hermitian metric is the standard Euclidean metric in these coordinates. With this function ϱ and a function $\chi \in C_0^\infty$ with support in a small neighborhood of the origin which is equal to 1 in another neighborhood, we now set, following (3.3), with some $\kappa \geqslant \frac{1}{2}n + 3$

$$f_\tau = \tau^{(n+1)/2}\chi(z)\sum_1^n(-1)^j\partial\varrho/\partial z_j\, d\bar z_1 \wedge \ldots \wedge \widehat{d\bar z_j} \wedge \ldots \wedge d\bar z_n \cdot$$
$$\cdot(1 + i\tau\psi(z))^{-\kappa}(1 + |z|^2 - 2\,\mathrm{Im}\, z_n)^{-n},$$

and we define $f_\tau = 0$ outside $\mathrm{supp}\,\chi$. Then the boundary condition for $\mathfrak{d}_c$ is fulfilled since f_τ is the inner product of a $(0,n)$ form with $\partial\varrho$. We can apply Lemma 3.3 with $N = 2n$ and $y_{N-1} = \mathrm{Re}\, z_n$, $y_N = \mathrm{Im}\, z_n$; then $A(z) = -|z|^2$ and $B(z) = 2z_n\,\mathrm{Im}\, z_n - i|z|^2$, thus

$$Q(z) = \tfrac{1}{2}A(z) - \mathrm{Im}\, B(z) = -\tfrac{1}{2}|z|^2 - 2(\mathrm{Im}\, z_n)^2 + |z|^2 = \tfrac{1}{2}|z|^2 - 2(\mathrm{Im}\, z_n)^2$$

which is positive definite when $\mathrm{Im}\, z_n = 0$. Hence $\|f_\tau\|$ has a positive limit when $\tau \to +\infty$, and since $f_\tau \to 0$ uniformly outside any neighborhood of 0 there is no L^2 convergent subsequence. In $\bar\partial f_\tau$ and $\mathfrak{d}_c f_\tau$ the terms where χ is differentiated also converge to 0, and in the other terms we can replace ϱ by the remainder term r. Then the terms where r is differentiated twice have coefficients $o(|z|)$ and those where r is only differentiated once have coefficients $o(|z|^2)\tau$, so it follows from Lemma 3.3, with $k = 1$ or $k = 2$, that their L^2 norms converge to 0. Thus $\|\bar\partial f_\tau\| + \|\mathfrak{d}_c f_\tau\| \to 0$ when $\tau \to +\infty$, which completes the proof when $q = n-1$.

In the general case we can choose the local coordinates so that

$$\varrho(z) = 2\,\mathrm{Im}\, z_n + |z'|^2 - |z''|^2 + r(z), \quad r(z) = o(|z|^3),$$

where $z' = (z_1, \ldots, z_{n-q-1})$, $z'' = (z_{n-q}, \ldots, z_n)$. As suggested by (2.16) we define

$$f_\tau = \tau^{(n+1)/2}\chi(z)\sum_{n-q}^n(-1)^j\partial\varrho/\partial z_j d\bar z_{n-q} \wedge \ldots \wedge \widehat{d\bar z_j} \wedge \ldots \wedge d\bar z_n \cdot$$
$$\cdot(1 + i\tau\psi(z''))^{-\kappa}(1 + |z''|^2 - 2\,\mathrm{Im}\, z_n)^{-q-1}$$

where $\psi(z'') = (\bar z_n + i)/(1 + |z''|^2 - 2\operatorname{Im} z_n) - i$. The only change in the argument for the case $q = n-1$ is that now we have $A(z) = |z'|^2 - |z''|^2$ while $B(z) = 2z_n \operatorname{Im} z_n - i|z''|^2$. Since $Q(z) = \frac{1}{2}|z|^2 - 2(\operatorname{Im} z_n)^2$ is positive definite when $\operatorname{Im} z_n = 0$, it follows as before that f_τ is in the domain of $\mathfrak{d}_c$ and that $\|\mathfrak{d}_c f_\tau\| + \|\bar\partial f_\tau\| \to 0$ while $\|f_\tau\|$ has a positive limit when $\tau \to +\infty$, and there is no L^2 convergent subsequence. The proof is complete.

If the range of the $\bar\partial$-Neumann operator in $L^2_{0,q}(\Omega)$ is closed then one can find g_τ with $\bar\partial g_\tau = \bar\partial f_\tau$ and $\mathfrak{d}_c g_\tau = \mathfrak{d}_c f_\tau$ such that $\|g_\tau\| \to 0$ as $\tau \to \infty$. Then $F_\tau = f_\tau - g_\tau$ is in the null space of the $\bar\partial$-Neumann operator and $\lim_{\tau\to+\infty} \|F_\tau\| = \lim_{\tau\to+\infty} \|f_\tau\| > 0$. If we write

$$F_\tau = \sum_{|I|=q} F_{\tau,I}(z) d\bar z^I$$

in terms of the local coordinates at z^0, then

$$\tau^{-(n+1)/2} \sum_{|I|=q} F_{\tau,I}(z'/\sqrt{\tau}, z_n/\tau)\, d\bar z^I \to \frac{i}{2}(-1)^{n-1} d\bar z_{n-q} \wedge \ldots \wedge d\bar z_{n-1} \big(1 + i(z_n + B_1(z'))\big)^{-\kappa}$$

locally in L^2 when $2\operatorname{Im} z_n + A(z', 0) < 0$ and $\tau \to +\infty$. Here $z' = (z_1, \ldots, z_{n-1})$, and $B_1(z') = -i\sum_{n-q}^{n-1} |z_j|^2$. In fact, the contribution from g_τ tends to 0 since $\|g_\tau\| \to 0$. (Note that we have not pulled F_τ back as a form.) We shall prove in Section 5 that the left-hand side is bounded on compact subsets for every L^2 bounded sequence F_τ in the null space of the $\bar\partial$-Neumann operator.

4. Confocal ellipsoidal shells.

If $\Omega_0 \Subset \Omega_1$ are bounded connected open sets in $\mathbb{C}^n$ with smooth boundaries, then the special case of (2.3) with a constant F has an analogue for $\Omega = \Omega_1 \setminus \overline{\Omega_0}$, for let H be the harmonic function in Ω which is equal to 1 on $\partial\Omega_1$ and 0 on $\partial\Omega_0$, and set

$$f = \sum_1^n (-1)^j \partial H/\partial z_j \, d\bar z_1 \wedge \ldots \wedge \widehat{d\bar z_j} \wedge \ldots \wedge d\bar z_n. \tag{4.1}$$

Then $\bar\partial f = -\sum_1^n \partial^2 H/\partial z_j \partial \bar z_j d\bar z_1 \wedge \ldots \wedge d\bar z_n = 0$ since H is harmonic, and $\mathfrak{d} f = 0$ since $\partial(\partial H/\partial z_j)/\partial z_k = \partial(\partial H/\partial z_k)/\partial z_j$. Since $H(H-1)$ is a defining function for Ω, it follows that f is in the minimal domain of $\mathfrak{d}$, so f is in the null space of the $\bar\partial$-Neumann operator. The formula

(2.3) for the spherical shell, with a constant F, is a special case for $H(z) = (R_0^{2-2n} - |z|^{2-2n})/(R_0^{2-2n} - R_1^{2-2n})$.

However, (2.3) also implies that for the spherical shell there are n functions $\bar{z}_j/|z|^2$, $j = 1, \dots, n$, which are multipliers on the null space of the $\bar{\partial}$-Neumann operator, and this is a very special property. In fact, for $\Omega = \Omega_1 \setminus \overline{\Omega_0}$ and H as above, a multiplier $\varphi \in C^\infty(\overline{\Omega})$ must satisfy

$$(4.2) \qquad \sum_1^n \partial H/\partial z_j \, \partial\varphi/\partial\bar{z}_j = 0,$$
$$\partial H/\partial z_j \, \partial\varphi/\partial z_k - \partial H/\partial z_k \, \partial\varphi/\partial z_j = 0, \quad j, k = 1, \dots, n,$$

and we shall prove that these equations cannot have n solutions with linearly independent differentials unless Ω is a confocal ellipsoidal shell as described in Theorem 4.1 below. There are n linearly independent equations (4.2) for $d\varphi$ at a point where $dH \neq 0$. The second set of equations in (4.2) means that an operator $\sum_1^n a_j \partial/\partial z_j$ annihilates φ if it annihilates H, which amounts to $n-1$ linearly independent conditions. The commutator of two such operators is of the same type since it also annihilates H. However, we cannot have n solutions of (4.2) with linearly independent differentials unless every commutator of $\sum_1^n \partial H/\partial z_j \partial/\partial\bar{z}_j$ and an operator $\partial H/\partial z_j \partial/\partial z_k - \partial H/\partial z_k \partial/\partial z_j$ is a linear combination of the operators in (4.2) (the Frobenius condition). With $\partial/\partial z_j$ and $\partial/\partial\bar{z}_k$ denoted by subscripts j and $\bar{k}$, we have

$$\Big[\sum_{l=1}^n H_l \partial/\partial\bar{z}_l, H_j \partial/\partial z_k - H_k \partial/\partial z_j\Big] = \sum_{l=1}^n H_l (H_{j\bar{l}} \partial/\partial z_k - H_{k\bar{l}} \partial/\partial z_j)$$
$$- \sum_{l=1}^n (H_j H_{lk} - H_k H_{lj}) \partial/\partial\bar{z}_l,$$

which is in the span of the operators in (4.2) if and only if

$$(4.3) \quad \sum_{l=1}^n H_l (H_{j\bar{l}} H_k - H_{k\bar{l}} H_j) = 0, \ (H_j H_{lk} - H_k H_{lj}) H_m = (H_j H_{mk} - H_k H_{mj}) H_l,$$

for all j, k, m, l. If $H_n \neq 0$ for example, it suffices to take $k = n$, $j < n$, and $m = n$, $l < n$ (in the second condition). At a point where $H_j \neq 0$ for every j, the second condition in (4.3) means that $H_j H_{lk}/H_l - H_k H_{lj}/H_l$ is independent of l, which gives if we take $l = j$ and $l = k$ that

$$(4.4) \qquad 2H_{jk}/H_j H_k = H_{jj}/H_j^2 + H_{kk}/H_k^2,$$

Conversely, (4.4) implies that $H_j H_{lk}/H_l - H_k H_{lj}/H_l = \frac{1}{2}(H_j H_{kk}/H_k - H_k H_{jj}/H_j)$ which does not depend on l, so (4.4) is equivalent to the second part of (4.3). We shall use this to integrate the equations (4.3) when Ω is a Reinhardt domain, that is, invariant under rotation in each of the coordinates z_j. Then the harmonic function H which is equal to 1 on $\partial\Omega_1$ and 0 on $\partial\Omega_0$ must also be invariant under these rotations, that is, $H(z) = h(t_1, \dots, t_n)$ where $t_j = |z_j|^2$. With the notation $h_j(t) = \partial h/\partial t_j$, $h_{jk}(t) = \partial^2 h(t)/\partial t_j \partial t_k$ we have $\partial H/\partial z_j = h_j \bar z_j$, and the equations (4.4) become

$$h_{jk}/h_j h_k = \frac{1}{2}(h_{jj}/h_j^2 + h_{kk}/h_k^2), \quad j,k = 1, \dots, n. \tag{4.5}$$

From (4.5) it follows that

$$\sum_{j,k=1}^{n} h_{jk}\, dt_j dt_k = \sum_1^n h_{jj}/h_j\, dt_j \sum_1^n h_k\, dt_k = 0 \quad \text{when } \sum_1^n h_k\, dt_k = 0,$$

which means that the level surfaces of h are hyperplanes. Conversely, this implies (4.5).

Next we shall examine the first condition (4.3). Since

$$\sum_{l=1}^n H_l H_{j\bar l} H_k = \sum_{l=1}^n h_l \bar z_l \partial(\bar z_j h_j)/\partial \bar z_l h_k \bar z_k = h_j \bar z_j h_j h_k \bar z_k + \sum_{l=1}^n h_l \bar z_l \bar z_j h_{jl} z_l h_k \bar z_k$$

the condition reduces to

$$h_k h_j^2 + \sum_{l=1}^n h_k h_l h_{jl} t_l = h_j h_k^2 + \sum_{l=1}^n h_j h_l h_{kl} t_l, \quad j,k = 1, \dots, n.$$

Division by $h_j h_k$ gives the equivalent condition

$$h_j + \sum_{l=1}^n h_l^2 (h_{jl}/h_j h_l) t_l = h_k + \sum_{l=1}^n h_l^2 (h_{kl}/h_k h_l) t_l.$$

If we use (4.5) and drop $\sum_1^n h_{ll} t_l/2$ on both sides, we obtain

$$h_j + \frac{1}{2} h_{jj}/h_j^2 \sum_{l=1}^n h_l^2 t_l = h_k + \frac{1}{2} h_{kk}/h_k^2 \sum_{l=1}^n h_l^2 t_l, \quad j,k = 1, \dots, n. \tag{4.6}$$

Since the level surfaces of h_j are hyperplanes, we have (at least locally)

$$h(t) = s \quad \text{when } \sum_1^n a_j(s) t_j = 1,$$

for some smooth functions a_j of one variable. Differentiation gives

$$h_j = \partial s/\partial t_j = -a_j/N_1,\ N_1 = \sum_1^n a_k' t_k,\ \partial N_1/\partial t_j - a_j' = h_j N_2,\ N_2 = \sum_1^n a_k'' t_k,$$

$$h_{jj} = -a_j'h_j/N_1 + a_j(a_j' + h_jN_2)/N_1^2 = -2a_j'h_j/N_1 - h_j^2N_2/N_1.$$

When we divide by h_j^2, the second term becomes independent of j, and (4.6) means that

$$-N_1(a_j - a_k) + (a_j'/a_j - a_k'/a_k)\sum_{l=1}^n a_l^2 t_l = 0, \quad j,k = 1,\dots,n$$

when $\sum_l a_l(s)t_l = 1$, hence by the homogeneity for all t. The coefficient of t_l divided by a_l is

$$(a_k - a_j)a_l'/a_l + (a_j'/a_j - a_k'/a_k)a_l,$$

so this must vanish for all j,k,l. Thus we must have a relation

$$a_l'(s)/a_l(s) = \gamma(s)a_l(s), \quad l = 1,\dots,n,$$

that is, $d(1/a_l(s))/ds = -\gamma(s)$, and this is also sufficient. If $\Gamma'(s) = -\gamma(s)$ it follows that there are constants C_j such that $1/a_l(s) = C_l + \Gamma(s)$, that is,

$$(4.7)\quad 1 = \sum_1^n t_j/(C_j+\Gamma(s)) = \sum_1^n |z_j|^2/(C_j+\Gamma(s)) \implies h(t) = H(z) = s.$$

Thus the level surfaces of H are ellipsoids in a family of confocal quadrics. We pause a moment to recall some classical facts on this notion.

If $C_1 < C_2 < \dots < C_n$ are real numbers then the quadrics in $\mathbb{R}^n$ defined by

$$\sum_1^n x_j^2/(C_j + \Lambda) = 1$$

with a fixed Λ are said to be confocal, for reasons which are obvious when $n = 2$. If $x \in \mathbb{R}^n$ and all coordinates are different from 0, this equation of order n for Λ has n different real roots for there is one with precisely k negative denominators for $k = 0,\dots,n-1$. Taking these n roots as local coordinates one obtains an orthogonal coordinate system which implies that the Laplacian has a simple expression in terms of them.

Here we shall only use the coordinate which corresponds to ellipsoids. Let $b_1,\dots,b_n$ be positive numbers and let

$$\Omega = \{z \in \mathbb{C}^n; \sum_1^n |z_j|^2/(b_j^2 + \gamma) < 1 < \sum_1^n |z_j|^2/b_j^2\}$$

where $\gamma > 0$ be the shell between two confocal ellipsoids. In a neighborhood of $\overline{\Omega}$ the equation

$$\sum_1^n |z_j|^2/(b_j^2 + \Lambda) = 1$$

defines Λ as an analytic function of $t_j = |z_j|^2$, $j = 1, \dots, n$, which is equal to 0 and to γ on the two boundaries. With the modified notation

$$N_\nu = \sum_1^n t_j/(b_j^2 + \Lambda)^\nu,$$

differentiation of the equation $N_1 = 1$ gives if $\Lambda_j = \partial\Lambda/\partial t_j$ and $\Lambda_{jj} = \partial^2\Lambda/\partial t_j^2$

$$1/(b_j^2 + \Lambda) = N_2\Lambda_j, \quad -(b_j^2 + \Lambda)^{-2}\Lambda_j = (b_j^2 + \Lambda)^{-2}\Lambda_j - 2N_3\Lambda_j^2 + N_2\Lambda_{jj},$$

hence

$$N_2\sum_1^n t_j\Lambda_j^2 = 1, \quad N_2\sum_1^n t_j\Lambda_{jj} = 2N_3/N_2 - 2N_3/N_2 = 0.$$

The harmonic function H must have the same level surfaces as Λ, so we have $H(z) = \psi(\Lambda(|z_1|^2, \dots, |z_n|^2))$ for some ψ, which gives

$$\partial H/\partial z_j = \psi'(\Lambda)\Lambda_j\bar z_j, \quad \partial^2 H/\partial z_j\bar z_j = \psi'(\Lambda)(\Lambda_j + t_j\Lambda_{jj}) + \psi''(\Lambda)\Lambda_j^2 t_j.$$

Hence

$$\begin{aligned}\frac{1}{4}\Delta H = \psi''(\Lambda)\sum_1^n t_j\Lambda_j^2 + \psi'(\Lambda)\sum_1^n(t_j\Lambda_{jj} + \Lambda_j)& \\ = \big(\psi''(\Lambda) + \psi'(\Lambda)\sum_1^n(b_j^2 + \Lambda)^{-1}\big)/N_2,&\end{aligned}$$

so H is harmonic if and only if for some constant C

$$\psi'(\Lambda) = C\prod_1^n(b_j^2 + \Lambda)^{-1}.$$

Since we require that $H = 0$ and $H = 1$ on the two boundaries, we have the boundary condition $\psi(0) = 0$, and $C^{-1} = \int_0^\gamma \prod_1^n (b_j^2 + \Lambda)^{-1}\, d\Lambda$.

Next we shall determine the multipliers on the null space of $\Box$ which we know must exist. For $j = 1, \dots, n$ it is clear that $\varphi(z) = \bar z_j\chi(\Lambda)$ satisfies the second set of equations (4.2), and φ satisfies the first equation (4.2) also if

$$\sum_{l=1}^n \Lambda_l\bar z_l\partial(\bar z_j\chi(\Lambda))/\partial\bar z_l = \Lambda_j\bar z_j\chi(\Lambda) + \sum_{l=1}^n \Lambda_l\bar z_j\chi'(\Lambda)\Lambda_l t_l = 0,$$

which simplifies to

$$\chi'(\Lambda)/\chi(\Lambda) = -\Lambda_j/\sum_{l=1}^n t_l\Lambda_l^2 = -1/(b_j^2 + \Lambda)$$

with the solution $\chi(\Lambda) = 1/(b_j^2+\Lambda)$. Thus $\zeta_j = \bar z_j/(b_j^2+\Lambda)$ is a multiplier. Note that $\sum_1^n z_j\zeta_j = 1$, that $\partial H/\partial z_j = \psi'(\Lambda)\Lambda_j\bar z_j$, and that $\Lambda_j(b_j^2+\Lambda) = 1/N_2 = 1/|\zeta|^2$. Since solutions of (4.2) form an algebra, this proves finally that

$$(4.8)\; f = F(\zeta)\prod_1^n (b_l^2+\Lambda)^{-1}\sum_1^n(-1)^j\zeta_j/|\zeta|^2\, d\bar z_1\wedge\ldots\wedge\widehat{d\bar z_j}\wedge\ldots\wedge d\bar z_n,$$
$$\zeta_j = \bar z_j/(b_j^2+\Lambda),$$

is in the null space of $\Box$ acting on forms in $L^2_{0,n-1}(\Omega)$ if F is a holomorphic polynomial or just a holomorphic square integrable function in $\{\zeta\in\mathbb{C}^n; \sum_1^n b_j^2|\zeta_j|^2<1\}$. The formula becomes even simpler if one introduces the ζ variables throughout. In fact, we have

$$d\bar z_j = (b_j^2+\Lambda)d\zeta_j + \zeta_j\, d\Lambda,$$

and in working out the exterior product in (4.8) we can only choose the $d\Lambda$ term in one of the factors. The product of $d\Lambda$ with all $(b_l^2+\Lambda)d\zeta_l$ in increasing order for $l\neq j$ and $l\neq k$ can be obtained in two ways. If $j<k$ it will appear with a factor $(-1)^j\zeta_j(-1)^{k-2}\zeta_k$ from the jth term and with the factor $(-1)^k\zeta_k(-1)^{j-1}\zeta_j$ from the kth term, and they cancel. Hence

$$(4.8)' \qquad f = F(\zeta)\sum_1^n(-1)^j\zeta_j/((b_j^2+\Lambda)|\zeta|^2)d\zeta_1\wedge\ldots\wedge\widehat{d\zeta_j}\wedge\ldots\wedge d\zeta_n$$

which is even simpler than (4.8). The function Λ is hidden in the transformation $\bar z_j = \zeta_j(b_j^2+\Lambda)$ where $\sum_1^n|\zeta_j|^2(b_j^2+\Lambda) = 1$, that is, $\Lambda|\zeta|^2 = 1-\sum_1^n b_j^2|\zeta_j|^2$, which implies that $(b_j^2+\Lambda)|\zeta|^2 = 1+\sum_{l=1}^n(b_j^2-b_l^2)|\zeta_l|^2$. This is equal to 1 in the spherical case.

It remains to prove that every element in the null space is of the form (4.8). To do so we must prove that every $f\in L^2_{0,n-1}(\Omega)\cap\mathcal{D}_{\bar\partial}\cap\mathcal{D}_{\mathfrak{d}_c}$ is a sum of an element in the range of $\bar\partial$ acting on $L^2_{0,n-2}(\Omega)$, an element in the range of $\mathfrak{d}_c$ acting on $L^2_{0,n}(\Omega)$, and an element of the form (4.8). In the proof we may assume that $f\in C^\infty_{0,n-1}(\bar\Omega)$ for this is a dense subset in the graph norm. The component of f in the range of $\mathfrak{d}_c$ is equal to $\mathfrak{d}_c(\bar\partial\mathfrak{d}_c)^{-1}\bar\partial f$ where $(\bar\partial\mathfrak{d}_c)^{-1}$ is the Green operator for the Dirichlet problem in Ω. Hence it is in $C^\infty_{0,n-1}(\bar\Omega)$, and replacing f by $f-\mathfrak{d}_c(\bar\partial\mathfrak{d}_c)^{-1}\bar\partial f$ we may assume in what follows that $\bar\partial f = 0$. The representation of the orthogonal projection on the space of forms (4.8) by a Bergman kernel given below shows that the component of f there is also in $C^\infty_{0,n-1}(\bar\Omega)$. What remains to prove is thus that if $f\in C^\infty_{0,n-1}(\bar\Omega)$, $\bar\partial f = 0$, and f is orthogonal to all elements in the null space of the form (4.8), then f is in the range of $\bar\partial$. This will follow if we prove that f can be extended to a $\bar\partial$ closed form in the ellipsoid which is the convex hull of Ω. Such an extension is possible if and only if for every multiindex α we have

$$\int_{\Lambda=0} z^\alpha\, dz_1 \wedge \ldots \wedge dz_n \wedge f = 0. \tag{4.9}$$

(Cf. [CS], Theorem 9.2.2 and the references to earlier literature given there.) Now orthogonality of $f = \sum_1^n (-1)^j f_j d\bar z_1 \wedge \ldots \wedge \widehat{d\bar z_j} \wedge \ldots \wedge d\bar z_n$ to the form (4.8) with $F(\zeta) = \zeta^\alpha$ means that

$$\int_\Omega \sum_1^n f_j \psi'(\Lambda)\bar\zeta_j/|\zeta|^2 z^\alpha/(b^2+\Lambda)^\alpha\, d\lambda(z) = 0.$$

Since $\bar\zeta_j/|\zeta|^2 = z_j/((b_j^2+\Lambda)N_2) = \Lambda_j z_j = \partial\Lambda/\partial\bar z_j$, this is equivalent to

$$\int_\Omega \sum_1^n f_j \partial\Lambda/\partial\bar z_j \psi'(\Lambda)/(b^2+\Lambda)^\alpha\, z^\alpha\, d\lambda(z) = 0.$$

If we define $m(\Lambda)$ so that $m'(\Lambda) = \psi'(\Lambda)/(b^2+\Lambda)^\alpha = C\prod_1^n (b_j^2+\Lambda)^{-\alpha_j-1}$ and $m(\gamma) = 0$, then $m(0) \neq 0$ and the integral differs only by a constant factor from

$$\int_\Omega z^\alpha dz_1 \wedge \ldots \wedge dz_n \wedge f \wedge \bar\partial m(\Lambda),$$

which by Stokes' formula is a non-zero constant times (4.9). This completes the proof of the following extension of the main point in Theorem 2.2:

THEOREM 4.1. — *Let Ω be a shell bounded by two confocal ellipsoids,*

$$\Omega = \Big\{z \in \mathbb{C}^n; \sum_1^n |z_j|^2/(b_j^2+\gamma) < 1 < \sum_1^n |z_j|^2/b_j^2\Big\}. \tag{4.10}$$

Then the null space of the $\bar\partial$-Neumann operator in $L^2_{0,n-1}(\Omega)$ consists of the forms (4.8) where F is a square integrable holomorphic function in the ellipsoid

$$\check\Omega = \Big\{\zeta \in \mathbb{C}^n; \sum_1^n b_j^2|\zeta_j|^2 < 1\Big\}.$$

There is a natural extension with ellipsoids replaced by paraboloids such as

$$\sum_1^{n-1} |z_j|^2/b_j^2 = 2\,\mathrm{Im}\, z_n.$$

Adding a term $\varepsilon^2|z_n|^2$ with a small $\varepsilon > 0$ we get the ellipsoid

$$\sum_1^{n-1} |z_j|^2/b_j^2 + |z_n - i/\varepsilon^2|^2/\varepsilon^{-2} = 1/\varepsilon^2.$$

The confocal ellipsoids are defined by

$$\sum_1^{n-1} |z_j|^2/(b_j^2/\varepsilon^2 + \Lambda_\varepsilon) + |z_n - i/\varepsilon^2|^2/(1/\varepsilon^4 + \Lambda_\varepsilon) = 1.$$

When $\varepsilon \to 0$ and $\varepsilon^2 \Lambda_\varepsilon \to \Lambda$ this converges to the paraboloid

$$\sum_1^{n-1} |z_j|^2/(b_j^2 + \Lambda) = 2 \operatorname{Im} z_n + \Lambda.$$

Note that this equation determines Λ as a decreasing function of $\operatorname{Im} z_n$. Our ellipsoidal multipliers $\bar z_j/(b_j^2/\varepsilon^2 + \Lambda_\varepsilon)$ converge to $\bar z_j/(b_j^2 + \Lambda)$ after division by ε^2, when $1 \leqslant j < n$. The last multiplier

$$\begin{aligned}(\bar z_n + i/\varepsilon^2)/(1/\varepsilon^4 + \Lambda_\varepsilon) &= \varepsilon^2(\varepsilon^2 \bar z_n + i)/(1 + \varepsilon^4 \Lambda_\varepsilon) \\ &= \varepsilon^2\big(i + \varepsilon^2(\bar z_n - i\varepsilon^2\Lambda_\varepsilon)/(1 + \varepsilon^4 \Lambda_\varepsilon)\big)\end{aligned}$$

converges to $\bar z_n - i\Lambda$ after subtraction of the constant $i\varepsilon^2$ and division by ε^4. The harmonic function converges after appropriate normalization to $\psi(\Lambda)$ where

$$\psi'(\Lambda) = C \prod_1^{n-1} (b_j^2 + \Lambda)^{-1}.$$

We have $\partial\Lambda/\partial z_j = \zeta_j/|\zeta|^2$ where $\zeta_j = \bar z_j/(b_j^2 + \Lambda)$ when $1 \leqslant j < n$ and $\zeta_n = i$. We denote by $\zeta_n^* = \bar z_n - i\Lambda$ the substitute for the trivial multiplier ζ_n found above and write $\zeta_j^* = \zeta_j$ when $1 \leqslant j < n$. In the paraboloidal shell

$$\Omega = \Big\{z \in \mathbb{C}^n; \sum_1^{n-1} |z_j|^2/b_j^2 > 2 \operatorname{Im} z_n > \sum_1^{n-1} |z_j|^2/(b_j^2 + \gamma) - \gamma\Big\},$$

where $\gamma > 0$, the preceding arguments suggest and it is easy to verify that the form

$$f = F(\zeta^*) \prod_1^{n-1} (b_l^2 + \Lambda)^{-1} \sum_1^n (-1)^j \zeta_j/|\zeta|^2 d\bar z_1 \wedge \ldots \wedge \widehat{d\bar z_j} \wedge \ldots \wedge d\bar z_n$$

is in the null space of the $\bar\partial$-Neumann operator in Ω if it is in L^2 and F is holomorphic in

$$\check\Omega = \Big\{\zeta \in \mathbb{C}^n; \sum_1^{n-1} |\zeta_j|^2 b_j^2 < -2 \operatorname{Im} \zeta_n < \sum_1^{n-1} |\zeta_j|^2 (b_j^2 + \gamma) + \gamma\Big\}.$$

It seems plausible that this describes the null space completely, but we leave this question aside in order to return to a general discussion of the integrability condition (4.3).

Recall that (4.3) describes the Frobenius integrability condition for the system (4.2). To find the most general solutions we shall examine a neighborhood of a point where H is real analytic, the differential is not equal to 0, and the level surface of H has a definite Levi form. We can place the point in question at the origin and choose the orthonormal coordinate system in $\mathbb{C}^n$ so that $H_j(0) = 0$ when $j \neq n$, thus $H_n(0) \neq 0$; as above, subscript k (resp. $\bar k$) will denote differentiation with respect to z_k(resp. $\bar z_k$). In a neighborhood of 0 the conditions (4.3) with $k = m = n$ can be written

(4.11)
$$H_{j\bar n} = -\sum_{m=1}^{n-1} H_m H_{j\bar m}/H_n + H_j H_{n\bar n}/H_n + \sum_{m=1}^{n-1} H_{n\bar m} H_j H_m/H_n^2, \quad j < n,$$

(4.12) $$H_{lj} = H_j H_{ln}/H_n + H_l H_{jn}/H_n - H_j H_l H_{nn}/H_n^2, \quad j < n,\ l < n,$$

and since H is harmonic we also have

(4.13) $$H_{n\bar n} = -\sum_{m=1}^{n-1} H_{m\bar m}.$$

In particular this gives $H_{j\bar n}(0) = 0$ when $j < n$ and $H_{lj}(0) = 0$ when $l < n$, $j < n$, so the last sum in (4.11) vanishes of third order at the origin. By Taylor's formula

$$H(z) - H(0) = H_n(0)z_n + H_{\bar n}(0)\bar z_n + \sum_{j,k=1}^{n} H_{j\bar k}(0) z_j \bar z_k + \mathrm{Re}\sum_{j,k=1}^{n} H_{jk}(0) z_j z_k + O(|z|^3).$$

Here $\sum_{j,k=1}^n H_{jk}(0)z_j z_k = z_n(H_{nn}(0)z_n + 2\sum_{j=1}^{n-1} H_{jn}(0)z_j)$ by (4.12). Multiplication by $\sigma(z) = 1 - \mathrm{Re}\Big((2\sum_1^{n-1} H_{jn}(0)z_j + H_{nn}(0)z_n)/H_n(0)\Big)$ gives

$$\sigma(z)(H(z) - H(0)) = H_n(0)z_n + H_{\bar n}(0)\bar z_n + \sum_{j,k=1}^{n} a_{jk} z_j \bar z_k + O(|z|^3),$$

where a_{jk} is hermitian symmetric and

$$a_{jk} = \begin{cases} H_{j\bar k}(0), & \text{when } j < n,\ k < n \\ -H_{jn}(0)H_{\bar n}(0)/H_n(0), & \text{when } j < n,\ k = n \\ -H_{\bar n\bar k}(0)H_n(0)/H_{\bar n}(0), & \text{when } j = n,\ k < n, \\ H_{n\bar n}(0) - \mathrm{Re}(H_{nn}(0)H_{\bar n}(0)/H_n(0)), & \text{when } j = k = n. \end{cases}$$

Thus the level surface $\{z; H(z) = H(0)\}$ is tangent of second order to the ellipsoid (or paraboloid) Σ defined by

$$2\,\mathrm{Re}(H_n(0)z_n) + \sum_{j,k=1}^{n} a_{jk} z_j \bar{z}_k = 0,$$

for this is $O(|z|^3)$ on the surface. It is not true for any other hermitian symmetric matrix $(a_{jk} + b_{jk})$, for if $\sum_{j,k=1}^{n} b_{jk} z_j \bar{z}_k = 0$ in the real hyperplane where $\mathrm{Re}(H_n(0)z_n) = 0$ then this is true for all z. Note that, in view of (4.11) – (4.13), (a_{jk}) and $H_n(0)$ determine the second order Taylor expansion of $H - H(0)$ at 0 apart from the fact that only the real part and not the imaginary part of $H_{nn}(0)/H_n(0)^2$ is determined.

By a unitary change of variables and a translation we can transform Σ to an ellipsoid or paraboloid Σ_0 for which we have already found a harmonic function satisfying (4.3) and vanishing on Σ_0. Going back to the original variables this gives a solution $\tilde{H}$ of the equations (4.11)–(4.13) vanishing on Σ. After multiplying $\tilde{H}$ by a suitable real constant we conclude that the derivatives of $H - H(0)$ and of $\tilde{H}$ of order $\leqslant 2$ are the same at the origin with the possible exception that $\mathrm{Im}(H_{nn}(0)/H_n(0)^2)$ and $\mathrm{Im}(\tilde{H}_{nn}(0)/\tilde{H}_n(0)^2)$ might differ. The following proposition will prove that $H - H(0)$ is equal to $\tilde{H}$ though.

PROPOSITION 4.2. — *Suppose that H is a real valued solution of* (4.11), (4.12), (4.13) *in a neighborhood of* $0 \in \mathbb{C}^n$ *such that* $H_j(0) = 0$ *for $j < n$ but $H_n(0) \neq 0$, and the Levi form of the level surface where $H = H(0)$ is non-degenerate at the origin. Then $H - H(0)$ is uniquely determined by $H_n(0)$, $\mathrm{Re}(H_{nn}(0)/H_n(0)^2)$, $H_{jn}(0)$ for $j < n$, and $H_{j\bar{k}}(0)$ for $j, k < n$.*

Proof. — As a harmonic function H is real analytic so it suffices to prove that all derivatives can be calculated at the origin. When doing so it is no restriction to assume that the Levi form at the origin is diagonalized, that is, that $H_{j\bar{k}}(0) = 0$ when $j < k < n$, which implies that the diagonal elements $H_{j\bar{j}}$ with $j < n$ are not equal to 0. We shall write

$$H^{\alpha,\beta} = \partial^{|\alpha|+|\beta|} H(0)/\partial z^\alpha \partial \bar{z}^\beta, \quad \alpha = (\alpha_1, \ldots, \alpha_n),\ \beta = (\beta_1, \ldots, \beta_n),$$

and use the notation $\alpha' = (\alpha_1, \ldots, \alpha_{n-1})$, $\beta' = (\beta_1, \ldots, \beta_{n-1})$. Taking into account the consequences of (4.11), (4.12) at 0 we know $H^{\alpha,\beta}$ when $1 \leqslant |\alpha| + |\beta| \leqslant 2$ except when $\alpha_n = 2$ or $\beta_n = 2$; and we also know $\mathrm{Re}(H_{nn}(0)/H_n(0)^2)$. We shall prove recursively that $H^{\alpha,\beta}$ can be calculated for arbitrary α, β. By the hermitian symmetry we may of course always assume that $|\alpha| \geqslant |\beta|$.

First let $|\alpha| + |\beta| = 3$. Then $H^{\alpha,\beta}$ is obtained by differentiation of (4.12) if $|\alpha'| \geqslant 2$. If $|\alpha'| = 1$ then $|\beta| \leqslant 1$, and if $|\beta| = 1$ then $\alpha_n = 1$ and we can calculate $H^{\alpha,\beta}$ by differentiating (4.11) or its complex conjugate instead when $\beta_n = 1$ or $|\beta'| = 1$. If $\alpha' = 0$ and $\alpha_n = 2$ we can find $H^{\alpha,\beta}$ by differentiating the complex conjugate of (4.11) when $|\beta'| = 1$. If $\beta_n = 1$ we can use (4.13) to reduce to cases with $|\alpha'| = 1$ and $|\beta'| = 1$ which we have already studied. This gives all $H^{\alpha,\beta}$ with $|\alpha| + |\beta| = 3$ and $|\alpha| \geqslant |\beta|$ except those with $\beta = 0$ and $\alpha_n \geqslant 2$.

Calculating $H^{\alpha,\beta}$ when $|\alpha| + |\beta| = 4$ by two differentiations of (4.11), (4.12), (4.13) requires some care since we do not even know $H_{nn}(0)$ completely and have not yet found $H_{jnn}(0)$. Taking $l = j$ in (4.12) and noting that $H_{n\bar{j}}(0) = 0$, we obtain at the origin

$$(4.14) \qquad H_{jj\bar{j}\bar{j}} = 2H_{j\bar{j}\bar{j}}H_{jn}/H_n + 4H_{j\bar{j}}H_{jn\bar{j}}/H_n - 2H_{j\bar{j}}{}^2 H_{nn}/H_n^2.$$

Differentiation of (4.11) gives at the origin

$$H_{j\bar{n}\bar{j}} = -\sum_{m=1}^{n-1} |H_{m\bar{j}}|^2/H_n + H_{n\bar{n}}H_{j\bar{j}}/H_n,$$

so $H_nH_{j\bar{n}\bar{j}}$ is real, hence $H_{\bar{j}nj}/H_n$ is real at the origin. By (4.12) we have at the origin

$$H_{jj\bar{j}} = 2H_{j\bar{j}}H_{jn}/H_n,$$

so $2H_{j\bar{j}\bar{j}}H_{jn}/H_n = 4H_{j\bar{j}}|H_{jn}|^2/|H_n|^2$ is real. Using (4.14) we can now conclude that $H_{nn}(0)/H_n(0)^2$ is real which means that $H_{nn}(0)$ is also known. Next we shall calculate $H_{jnn}(0)$ when $j < n$ by observing that $H_{jj\bar{j}n} = (H_{\bar{j}n})_{jj}$ can be calculated at the origin by two differentiations of (4.11). On the other hand, if we differentiate

$$(4.15) \qquad H_{jj} = 2H_jH_{jn}/H_n - H_j^2H_{nn}/H_n^2$$

with respect to z_n and $\bar{z}_j$ then third derivatives will only appear at the origin in the terms

$$2H_{j\bar{j}n}H_{jn}/H_n + 2H_{j\bar{j}}H_{jnn}/H_n + 2H_{jn}H_{jn\bar{j}}/H_n,$$

and only the middle term is not already calculated. Since $H_{j\bar{j}}(0) \neq 0$ this means that we have now calculated $H^{\alpha,\beta}$ when $|\alpha| + |\beta| = 3$ except when $\alpha_n = 3$ or $\beta_n = 3$. This implies that only already known terms will appear in the right-hand side of (4.11) or (4.12) when we differentiate twice, for one derivative must fall on each factor H_k with $k < n$.

If $|\alpha|+|\beta| = 4$, $|\beta| \leqslant |\alpha|$ we can thus calculate $H^{\alpha,\beta}$ by differentiation of (4.12) or its complex conjugate unless $|\alpha'| \leqslant 1$ and $|\beta'| \leqslant 1$. Then $\alpha_n \geqslant 1$ and we can use the complex conjugate of (4.11) unless $\beta' = 0$, and if $\beta_n \neq 0$ we can use (4.11) unless $\alpha' = 0$. This leaves only the case $|\alpha'| = 1$, $\beta = 0$, $\alpha_n = 3$, and the case $\alpha' = 0$, $\beta' = 0$, $\alpha_n \geqslant 2$. If $\alpha_n < 4$ in the latter case then $\beta_n > 0$ and we can use (4.13) to reduce to the cases already handled. Hence we have calculated $H^{\alpha,\beta}$ for all α, β with $|\alpha| + |\beta| \leqslant 4$ and $|\beta| \leqslant |\alpha|$ except those with $\beta = 0$ and $\alpha_n \geqslant 3$.

We claim now that *using the equations* (4.11)–(4.13), *their complex conjugates, and the derivatives at the origin of order* $\leqslant N$, *we can for every* $N \geqslant 2$ *calculate all* $H^{\alpha,\beta}$ *with* $|\alpha| + |\beta| \leqslant N + 2$, $|\beta| \leqslant |\alpha|$ *and* $\alpha_n \leqslant N$ *if* $\beta = 0$. This is what we have just done for $N = 2$, so we may assume that $N > 2$ and that the statement has been verified with N replaced by $N - 1$. Then we know $H^{\alpha,\beta}$ when $|\alpha| + |\beta| = N$ except when $\alpha_n = N$ (or $\beta_n = N$), and when $|\alpha| + |\beta| = N + 1$ except when $\beta = 0$ and $\alpha_n \geqslant N$ (or the complex conjugate). Our first task is to calculate $H^{\alpha,\beta}$ when $\beta = 0$ and $\alpha' = 0$, $\alpha_n = N$ by differentiating (4.15) twice with respect to $\bar{z}_j$ and $N-2$ times with respect to z_n. To produce a non-zero term at least one derivative must fall on the factor H_j in the first term, and if the remaining $N - 1$ all fall on the factor H_{jn} then one of them is a derivative with respect to $\bar{z}_j$ and we obtain a derivative of order $N + 1$ which is already known; if only the $N-2$ derivatives with respect to z_n fall on H_{jn} we get a derivative of order N which is already known. In the second term in (4.15) two derivatives must fall on H_j^2 to produce a non-zero term, so all terms obtained are already calculated except $-2H_{j\bar{j}}^2 \partial^N H/\partial z_n^N/H_n^2$. Expressing $H_{\bar{j}n}$ as the complex conjugate of (4.11) we find that $\partial^{N+2}H(0)/\partial z_j^2 \partial \bar{z}_j^2 \partial z_n^{N-2}$ can be calculated, hence we have found that $\partial^N H(0)/\partial z_n^N$ can be calculated.

Next we shall calculate $\partial^{N+1}H(0)/\partial z_j \partial z_n^N$ when $j < n$ by differentiating (4.15) once with respect to $\bar{z}_j$ and $N - 1$ times with respect to z_n. On one hand, this can be calculated using (4.11) as a derivative of $H_{\bar{j}n}$, on the other hand we see from (4.15) that at the origin the only term obtained which involves a derivative of order $N + 1$ with no $\bar{z}$ derivative is $2H_{j\bar{j}}\partial^{N+1}H/\partial z_j \partial z_n^N/H_n$, so we have calculated $H^{\alpha,\beta}$ when $\beta = 0$ and $|\alpha'| = 1$, $\alpha_n = N$. Thus $H^{\alpha,\beta}$ is now known when $|\alpha| + |\beta| \leqslant N + 1$ except when $\alpha_n = N + 1$ (or $\beta_n = N + 1$).

We are now free to differentiate (4.11) and (4.12) as we did before when $N = 2$, so the argument made then can be repeated. If $|\alpha|+|\beta| = N+2$ and $|\beta| \leqslant |\alpha|$, differentiation of (4.12) gives $H^{\alpha,\beta}$ unless $|\alpha'| \leqslant 1$ and

$|\beta'| \leqslant 1$. Then $\alpha_n \geqslant N/2$ and we can use the complex conjugate of (4.11) instead unless $\beta' = 0$. If $\beta_n = 0$ too, then $\alpha_n \geqslant N+1$, and if $\beta_n > 0$ we can use (4.13) to pass to terms with $\alpha' \neq 0$ and $\beta' \neq 0$ which we have already handled. This completes the inductive proof.

Summing up, we have proved:

THEOREM 4.3. — *If $\Omega_0 \Subset \Omega_1 \Subset \mathbb{C}^n$ are open sets with smooth boundary, $\Omega = \Omega_1 \setminus \overline{\Omega_0}$, and the null space of the $\bar\partial$-Neumann operator acting in $L^2_{0,n-1}(\Omega)$ admits n independent multipliers, then Ω is with suitable orthonormal coordinates a confocal ellipsoidal shell as described in Theorem 4.1.*

Our next goal is to prove an analogue of (2.9) for the confocal ellipsoidal shell Ω defined by (4.10). By Theorem 4.1 the null space of $\Box$ in $L^2_{0,n-1}(\Omega)$ consists of the forms (4.8) where F is a holomorphic square integrable function in

$$\check\Omega = \Big\{\zeta \in \mathbb{C}^n; \sum_1^n b_j^2|\zeta_j|^2 < 1 < \sum_1^n (b_j^2+\gamma)|\zeta_j|^2\Big\},$$

hence also in its convex hull. We have

$$\|f\|^2 = \int_\Omega |f(z)|^2\,d\lambda(z) = \int_\Omega |F(\zeta)|^2|\zeta|^{-2}\prod_1^n (b_j^2+\Lambda)^{-2}\,d\lambda(z),$$

$$\zeta_j = \bar z_j/(b_j^2+\Lambda(z)).$$

To pass entirely to the ζ variables we observe that since $\bar z_j = \zeta_j(b_j^2+\Lambda)$ we have

$$d\bar z_j = (b_j^2+\Lambda)d\zeta_j + \zeta_j d\Lambda, \quad dz_j = (b_j^2+\Lambda)d\bar\zeta_j + \bar\zeta_j d\Lambda,$$
$$dz_j \wedge d\bar z_j = (b_j^2+\Lambda)^2 d\bar\zeta_j \wedge d\zeta_j + (b_j^2+\Lambda)d\Lambda \wedge (\bar\zeta_j d\zeta_j - \zeta_j d\bar\zeta_j),$$

and since $\sum_1^n |\zeta_l|^2(b_l^2+\Lambda) = 1$ we have

$$|\zeta|^2\,d\Lambda = -\sum_1^n (b_l^2+\Lambda)d|\zeta_l|^2, \quad d|\zeta_j|^2 \wedge (\bar\zeta_j d\zeta_j - \zeta_j d\bar\zeta_j) = 2|\zeta_j|^2\,d\bar\zeta_j \wedge d\zeta_j.$$

This gives

$$dz_1 \wedge d\bar z_1 \wedge \cdots \wedge dz_n \wedge d\bar z_n = -\prod_1^n (b_j^2+\Lambda)^2 d\bar\zeta_1 \wedge d\zeta_1 \wedge \cdots \wedge d\bar\zeta_n \wedge d\zeta_n,$$

for the factor $d\bar\zeta_j \wedge d\zeta_j$ annihilates $d|\zeta_j|^2$, and $1 - 2\sum_1^n |\zeta_j|^2/|\zeta|^2 = -1$. Hence $d\lambda(z) = \prod_1^n (b_j^2+\Lambda)^2 d\lambda(\zeta)$, and (cf. (2.7))

$$\|f\|^2 = \int_{\check\Omega} |F(\zeta)|^2|\zeta|^{-2}\,d\lambda(\zeta). \tag{4.16}$$

For the norm (4.16) in $L^2(\check\Omega)$ the Bergman kernel $K(\zeta,\theta)$ can be computed along the same lines as for the spherical shell in Section 2 if one introduces $b_j\zeta_j = w_j$ as new variables. With these variables $\check\Omega$ is defined by

$$\sum_1^n |w_j|^2 < 1 < \sum_1^n |w_j|^2(1+\gamma/b_j^2).$$

The computation of the reproducing kernel K depends as in the spherical case on evaluation of integrals I_α and summation of a power series with coefficients $1/I_\alpha$. The details of this are given in [H2], and we content ourselves here with listing the main singularities of the kernel obtained after returning to the ζ variable:

$$K(\zeta,\theta) - \pi^{-n}\prod_1^n b_j^2\Big(\frac{n!\langle\zeta,\bar\theta\rangle}{(1-\sum_1^n b_j^2\zeta_j\bar\theta_j)^{n+1}} \\ + \frac{(n-1)!}{(1-\sum_1^n b_j^2\zeta_j\bar\theta_j)^n}\sum_1^n(1-\zeta_j\bar\theta_j/\langle\zeta,\bar\theta\rangle)/b_j^2\Big)$$

is $O\big((1-\sum_1^n b_j^2\zeta_j\bar\theta_j)^{1-n}\big)$ in a neighborhood of the diagonal of the outer boundary of $\check\Omega$ and smooth elsewhere. (The leading term on the diagonal follows from [H1], Theorem 3.5.1, and the results of [BMS] imply that the remainder is $O\big((1-\sum_1^n b_j^2\zeta_j\bar\theta_j)^{-n}\big)$.) When $F\in L^2(\check\Omega)$ is holomorphic we have

$$F(\zeta) = \int_{\check\Omega} K(\zeta,\theta)F(\theta)|\theta|^{-2}\,d\lambda(\theta).$$

With $\zeta_j = \bar z_j/(b_j^2+\Lambda(z))$ and $\theta_j = \bar w_j/(b_j^2+\Lambda(w))$, and the shorthand notation $\zeta = \bar z/(b^2+\Lambda(z))$, $\theta = \bar w/(b^2+\Lambda(w))$, it follows that

$$F(\zeta)\prod_{l=1}^n(b_l^2+\Lambda(z))^{-1}\zeta_j|\zeta|^{-2} \\ = \sum_{k=1}^n\int_\Omega K_{jk}(z,w)F(\theta)\prod_{l=1}^n(b_l^2+\Lambda(w))^{-1}\theta_k|\theta|^{-2}\,d\lambda(w),$$

$$K_{jk}(z,w) = \zeta_j|\zeta|^{-2}\prod_{l=1}^n(b_l^2+\Lambda(z))^{-1}K(\zeta,\theta)\prod_{l=1}^n(b_l^2+\Lambda(w))^{-1}\bar\theta_k|\theta|^{-2}.$$

As in the spherical case it follows that (K_{jk}) is the kernel of the orthogonal projection on $\operatorname{Ker}\square$ in $L^2_{0,n-1}(\Omega)$. Since

$$1-\sum_{l=1}^n b_l^2|\zeta_l|^2 = \sum_{l=1}^n\big(|z_l|^2/(b_l^2+\Lambda(z)) - b_l^2|z_l|^2/(b_l^2+\Lambda(z))^2\big) \\ = \Lambda(z)\sum_{l=1}^n|z_l|^2/(b_l^2+\Lambda(z))^2 = \Lambda(z)|\zeta|^2$$

we obtain the main singularities of $K_{jk}(z,z)$ at the inner boundary of the annulus Ω

$$(4.17)\qquad K_{jk}(z,z) = \pi^{-n} n! \Big(\sum_{l=1}^{n} |z_l|^2/b_l^4\Big)^{-n-2} \cdot \prod_{l=1}^{n} b_l^{-2}\, \bar z_j b_j^{-2} z_k b_k^{-2} \Lambda(z)^{-n-1} + O(\Lambda(z)^{-n}).$$

For the sake of brevity we have only used the leading singularity of $K(z,z)$ at the diagonal, which as already pointed out is easy to justify. We shall now see that (4.17) has an interesting interpretation in terms of the curvature of the inner ellipsoidal boundary, where $\Lambda = 0$. Since $\sum_1^n |z_l|^2/(b_l^2+\Lambda(z)) = 1$ we have

$$\sum_1^n |z_l|^2/b_l^2 - 1 = \Lambda(z)\sum_1^n |z_l|^2/(b_l^2(b_l^2+\Lambda(z)))$$
$$= \Lambda(z)\sum_1^n |z_l|^2/b_l^4 - \Lambda(z)^2 \sum_1^n |z_l|^2/(b_l^4(b_l^2+\Lambda(z))).$$

When $\Lambda = 0$ it follows that

$$N_2 \partial\Lambda/\partial z_j = \bar z_j/b_j^2, \quad N_2 = \sum_1^n |z_l|^2/b_l^4,$$

so (4.17) can be written

$$K_{jk}(z,z) = \pi^{-n} n! N_2^{-n} \prod_1^n b_l^{-2} \partial\Lambda(z)/\partial z_j \partial\Lambda(z)/\partial \bar z_k \Lambda(z)^{-n-1} + O(\Lambda(z)^{-n}).$$

We shall compare this with the product of the eigenvalues of the Levi form for Λ in the complex tangent plane of $\{z; \Lambda(z)=0\}$, that is, the product of the eigenvalues of

$$(4.18)\qquad \frac{1}{N_2}\sum_1^n |f_j|^2/b_j^2 \quad\text{when}\quad \sum_1^n f_j \bar z_j/b_j^2 = 0,$$

for $N_2\Lambda(z+\tilde z) = \sum_1^n |z_l+\tilde z_l|^2/b_l^2 - 1 + O(|\tilde z|^3)$ when $\Lambda(z+\tilde z) = O(|\tilde z|^2)$.

Quite generally, if H is a hermitian symmetric $n\times n$ matrix and $0 \neq w \in \mathbb{C}^n$, then the product of the eigenvalues of the form $(Hz,z) = \langle Hz, \bar z\rangle$ restricted to the orthogonal space of w is equal to

$$(\det H)(H^{-1}w,w)/(w,w), \quad \text{if } \det H \neq 0.$$

In the proof we may assume that w is the unit vector e_n along the z_n axis so that the orthogonal space is defined by $z_n = 0$. When $z_n = 0$ and $t \in \mathbb{C}$ then

$$(H(z+tH^{-1}e_n), z+tH^{-1}e_n) = (Hz,z) + |t|^2(H^{-1}e_n,e_n),$$

and as a hermitian form in $\mathbb{C}^{n-1}\oplus\mathbb{C}=\mathbb{C}^n$ the determinant of the corresponding matrix is equal to the product k of the eigenvalues of (Hz,z) when $z_n=0$ and $(H^{-1}e_n,e_n)$. The determinant of the map

$$\mathbb{C}^n\ni z\mapsto(z_1,\dots,z_{n-1},0)+z_nH^{-1}e_n\in\mathbb{C}^n$$

is $(H^{-1}e_n,e_n)$, so it follows that $(H^{-1}e_n,e_n)^2\det H=k(H^{-1}e_n,e_n)$, which proves the claim. Since $H^{-1}\det H$ is the algebraic complement of H which is always well defined, it makes sense as a limiting case for arbitrary H. Hence it follows that the product κ of the eigenvalues of the Levi form (4.18) is

$$\kappa=N_2^{1-n}\prod_1^n b_l^{-2}\sum_1^n b_j^2|z_j|^2/b_j^4\Big/\sum_1^n|z_k|^2/b_k^4=N_2^{-n}\prod_1^n b_l^{-2}.$$

Since $|\partial\Lambda/\partial z|^2=1/N_2$ it follows that

$$K_{jk}(z,z)\Lambda(z)^{n+1}-\pi^{-n}n!\kappa\partial\Lambda(z)/\partial z_j\partial\Lambda(z)/\partial\bar z_k=O(\Lambda(z)).\tag{4.19}$$

Note that multiplication of Λ by a positive constant γ would change both terms by the same factor γ^{n+1}, so (4.19) remains valid with Λ replaced by any function ϱ such that $-\varrho$ is a defining function for Ω at the boundary point considered. On the diagonal the matrix $(K_{jk}(z,z))_{j,k=1}^n$ defines a hermitian symmetric form on $(0,n-1)$ forms at z. Since (Kg,g) is the supremum of $|(f,g)|^2$ when $f\in L^2_{0,n-1}(\Omega)$, $\square f=0$, and $\|f\|\leqslant 1$, we have as a limiting case when $\operatorname{supp} g$ tends to z

$$\sum_{j,k=1}^n K_{jk}(z,z)t_k\bar t_j=\sup\{|(f(z),t)_z|^2;f\in L^2_{0,n-1}(\Omega),\ \square f=0,\ \|f\|\leqslant 1\},\qquad t\in\mathbb{C}^n,$$

where f is identified with $(f_1,\dots,f_n)$. Thus $\sqrt{(K(z,z)t,t)}$ is the supporting function of the range of $f(z)$ when $f\in\operatorname{Ker}\square$ and $\|f\|\leqslant 1$, which is obviously convex. By (4.19) the product of the form by $\varrho(z)^{n+1}$ is equal to

$$\pi^{-n}n!\kappa(z)|\langle t,\partial\varrho(z)/\partial\bar z\rangle|^2+O(\varrho(z)|t|^2),$$

where $\kappa(z)$ is the product of the eigenvalues of $\sum_{j,k=1}^n\partial^2\varrho(z)/\partial z_j\partial\bar z_k s_j\bar s_k$ restricted to the plane where $\sum_1^n\partial\varrho(z)/\partial z_j s_j=0$. Changing notation we conclude that

$$\varrho(z)^{n+1}|(f(z),t)_z|^2\leqslant\pi^{-n}n!\kappa(z)|t\wedge\bar\partial\varrho(z)|^2+O(\varrho(z)|t|^2)\tag{4.20}$$

when $f\in L^2_{0,n-1}(\Omega)$ is in $\operatorname{Ker}\square$, $\|f\|\leqslant 1$, and t is now a form of type $(0,n-1)$ at z. As a first step toward a localization we shall prove:

PROPOSITION 4.4. — *Let z^0 be on the inner boundary of the ellipsoidal shell $\Omega\subset\mathbb{C}^n$ defined by (4.10) ($n>2$), and let $\omega_0\Subset\omega_1$ be* *open neighborhoods of* z^0. *If* $f \in L^2_{0,n-1}(\Omega\cap\omega_1)$, $\bar\partial f = 0$, $\mathfrak{d} f = 0$ *in* $\Omega\cap\omega_1$, *and the minimal boundary conditions for* $\mathfrak{d}$ *are satisfied in* $\omega_1\cap\partial\Omega$, *then* (4.20) *is valid when* $z\in\Omega\cap\omega_0$ *if* $\|f\|_{\Omega\cap\omega_1}\leqslant 1$.

Proof. — Choose $\chi\in C_0^\infty(\omega_1)$ equal to 1 in a neighborhood $\omega_{\frac12}$ of $\bar\omega_0$ so that $0\leqslant\chi\leqslant 1$. Then χf, defined as 0 outside $\operatorname{supp}\chi$, is in the domain of $\mathfrak{d}_c$ and that of $\bar\partial$ in Ω, $\|\chi f\|\leqslant 1$, and $\bar\partial(\chi f)=0$, $\mathfrak{d}_c(\chi f)=0$ in $\omega_{\frac12}$. If $\tilde f$ is the orthogonal projection in $L^2_{0,n-1}(\Omega)$ of χf on $\operatorname{Ker}\Box$, then $\|\tilde f\|\leqslant 1$, and (4.20) is valid with $f(z)$ replaced by $\tilde f(z)$. Since $n>2$ by assumption we have

$$\tilde f = \chi f - \bar\partial\mathcal{N}_{n-2}\mathfrak{d}_c(\chi f) - \mathfrak{d}_c\mathcal{N}_n\bar\partial(\chi f) \tag{4.21}$$

where $\mathcal{N}_{n-2}$ and $\mathcal{N}_n$ denote the inverses of the $\bar\partial$-Neumann operator in $L^2_{0,n-2}(\Omega)$ and in $L^2_{0,n}(\Omega)$; the latter is essentially just the solution operator for the Dirichlet problem. The two terms subtracted from χf are respectively the orthogonal projection on the range of the $\bar\partial$ operator from $(0,n-2)$ forms and the $\mathfrak{d}_c$ operator from $(0,n)$ forms. The operators $\bar\partial\mathcal{N}_{n-2}$ and $\mathfrak{d}_c\mathcal{N}_n$ are both L^2 continuous, and we have bounds for $\mathfrak{d}_c(\chi f)$ and $\bar\partial(\chi f)$ in L^2; both vanish in $\Omega\cap\omega_{\frac12}$. By the local regularity theory of the $\bar\partial$-Neumann problem (Theorems 4 and 5 in [KN]) we conclude that the terms subtracted from χf in (4.21) are bounded in $C^\infty(\Omega\cap\omega_{\frac12})$, hence uniformly bounded in $\Omega\cap\omega_0$. Thus we have a uniform bound for $f(z)-\tilde f(z)$ when $z\in\Omega\cap\omega_0$, so $|(f(z),t)_z|\leqslant|(\tilde f(z),t)_z|+C|t|$ for some constant C, and since

$$\varrho(z)^{n+1}(|(\tilde f(z),t)_z|+C|t|)^2\leqslant\varrho(z)^{n+1}(1+\varrho(z))|(\tilde f(z),t)_z|^2 \\ +C^2|t|^2\varrho(z)^n(1+\varrho(z))$$

the statement follows.

Remark. — When $n=2$ we still have a decomposition like (4.21); the range of $\bar\partial$ acting in $L^2(\Omega)$ is closed and $\mathcal{N}_0$ should be interpreted as the inverse of $\mathfrak{d}_c\bar\partial$ on the orthogonal space of the holomorphic functions. However, we lack a reference to a local regularity theorem which could replace those of [KN] used in the preceding argument.

Proposition 4.4 gives an upper bound for the kernel of the orthogonal projection in $L^2_{0,n-1}(\tilde\Omega)$ on $\operatorname{Ker}\Box$ when $\tilde\Omega\subset\mathbb{C}^n$ is an arbitrary open set which is equal to the ellipsoidal shell Ω defined by (4.3) near a boundary point. When the range of $\Box$ in $L^2_{0,n-1}(\tilde\Omega)$ is closed, there is a lower bound of the same kind:

PROPOSITION 4.5. — *Let z^0 be a point on the inner boundary of the ellipsoidal shell $\Omega \subset \mathbb{C}^n$ defined by (4.10) ($n > 2$), and let $\tilde{\Omega} \subset \mathbb{C}^n$ be another open subset of $\mathbb{C}^n$ such that the range of $\Box$ in $L^2_{0,n-1}(\tilde{\Omega})$ is closed. If $\Omega \cap \omega = \tilde{\Omega} \cap \omega$ for some open neighborhood ω of z^0, then*

$$(4.22)\quad \varrho(z)^{n+1} \sup\{|(f(z),t)_z|^2;\ f \in L^2_{0,n-1}(\tilde{\Omega}),\ \Box f = 0,\ \|f\| \leqslant 1\} = \pi^{-n} n! \kappa(z) |t \wedge \bar{\partial}\varrho(z)|^2 + O(\varrho(z)|t|^2)$$

in a neighborhood of z^0.Here $-\varrho$ is a defining function for $\tilde{\Omega}$ and $\kappa(z)$ is the product of the eigenvalues of $\sum_{j,k=1}^n \partial^2\varrho(z)/\partial z_j \partial \bar{z}_k s_j \bar{s}_k$ restricted to the plane where $\sum_1^n \partial\varrho(z)/\partial z_j s_j = 0$.

Proof. — By Proposition 4.4 we have an upper bound of the form (4.22), for if $f \in L^2_{0,n-1}(\tilde{\Omega})$ and $\|f\| \leqslant 1$, then the L^2 norm of the restriction to $\tilde{\Omega} \cap \omega = \Omega \cap \omega$ is $\leqslant 1$, so (4.20) is applicable. The lower bound in (4.22) is trivial unless $|t \wedge \bar{\partial}\varrho(z)|^2 \gg \varrho(z)|t|^2$.

Choose $\chi \in C_0^\infty(\omega)$ equal to 1 in an open neighborhood ω_1 of z^0 so that $0 \leqslant \chi \leqslant 1$, and let $\omega_0 \Subset \omega_1$ be another open neighborhood of z^0. With $w \in \omega_0$ and a $(0, n-1)$ form t at w with $|t| = 1$ let

$$f_{w,t}(z) = \mathcal{K}(z,w)t$$

where $\mathcal{K}$ is the kernel of the orthogonal projection on $\operatorname{Ker}\Box$ in $L^2_{0,n-1}(\Omega)$, which also acts on distribution forms of compact support. Since $\mathcal{K}$ is the kernel of a self-adjoint projection we have

$$\|f_{w,t}\|^2 = (\mathcal{K}(w,w)t,t), \quad \Box f_{w,t} = 0.$$

The support of the form $\Box(\chi(z) f_{w,t}(z))$ is contained in $\omega \setminus \overline{\omega}_1$, and it has a uniform bound independent of $w \in \omega_0$ since $\mathcal{K}$ is smooth outside the diagonal, which implies a uniform bound for the norm in $L^2_{0,n-1}(\tilde{\Omega})$. Since by hypothesis the range of the $\bar{\partial}$-Neumann operator $\Box$ is closed there, it follows that we can find $g_{w,t} \in L^2_{0,n-1}(\tilde{\Omega})$ with $\Box g_{w,t}(z) = \Box(\chi(z) f_{w,t}(z))$ and $\|g_{w,t}\| \leqslant C$. Hence $\tilde{f}_{w,t}(z) = \chi(z) f_{w,t}(z) - g_{w,t}(z)$ is in the null space of $\Box$ in $L^2_{0,n-1}(\tilde{\Omega})$, and since $\Box g_{w,t}(z) = 0$ when $z \in \omega_1 \cap \tilde{\Omega} = \omega_1 \cap \Omega$, it follows from Proposition 4.4 that $|g_{w,t}(w)| \leqslant C' \varrho(w)^{-\frac{1}{2}(n+1)}$ for some constant C'. Hence

$$\|\tilde{f}_{w,t}\| \leqslant (\mathcal{K}(w,w)t,t)^{\frac{1}{2}} + C,\ |(\tilde{f}_{w,t}(w),t)| \geqslant (\mathcal{K}(w,w)t,t) - C'\varrho(w)^{-\frac{1}{2}(n+1)};$$

here

$$(\mathcal{K}(w,w)t,t) = \varrho(w)^{-n-1}\pi^{-n} n! \kappa(w) |t \wedge \bar{\partial}\varrho(w)|^2 + O(\varrho(w)^{-n}).$$

As observed at the beginning of the proof there is nothing to prove unless $(\mathcal{K}(w,w)t,t) \gg \varrho(w)^{-n}$, and then we have

$$|(\tilde f_{w,t}(w),t)|^2 \geqslant (\mathcal{K}(w,w)t,t)^2 - 2C'\varrho(w)^{-\frac12(n+1)}(\mathcal{K}(w,w)t,t),$$
$$\|\tilde f_{w,t}\|^{-2} \geqslant (1-2C/\sqrt{(\mathcal{K}(w,w)t,t)})/(\mathcal{K}(w,w)t,t),$$

hence

$$|(\tilde f_{w,t}(w),t)|^2/\|\tilde f_{w,t}\|^2 \geqslant (\mathcal{K}(w,w)t,t) - 2C\sqrt{(\mathcal{K}(w,w)t,t)} - 2C'\varrho(w)^{-\frac12(n+1)}, \quad w\in\omega_0,$$

which implies a lower bound as in (4.22) and completes the proof.

The supremum in the left-hand side of (4.22) is equal to $(\tilde{\mathcal{K}}(z,z)t,t)$ if $\tilde{\mathcal{K}}$ is the kernel of the orthogonal projection on $\operatorname{Ker}\square$ in $L^2_{0,n-1}(\tilde\Omega)$. We would like to extend the conclusion to an arbitrary open relatively compact subset Ω of a complex hermitian manifold of dimension $n>2$ such that the $\bar\partial$-Neumann operator $\square$ in $L^2_{0,n-1}(\Omega)$ has a closed range and z^0 is a boundary point where $\partial\Omega\in C^\infty$ and the Levi form is negative definite. However, the methods used in this paper only allow us to give corresponding asymptotics for $\mathcal{K}(z,w)$ in the distribution sense:

THEOREM 4.6. — *Under the preceding hypotheses on Ω and z^0 let $z_1,\dots,z_n$ be complex analytic coordinates at z^0 vanishing at z^0 such that $\partial z_1,\ \dots,\ \partial z_n$ are orthonormal at z^0 and the complex tangent plane is defined by $\partial z_n = 0$. Set $z'=(z_1,\dots,z_{n-1})$, write $(0,n-1)$ forms f as*

$$f=\sum_1^n(-1)^j f_j d\bar z_1\wedge\ldots\wedge\widehat{d\bar z_j}\wedge\ldots\wedge d\bar z_n$$

in a neighborhood of z^0, and let $K_{jk}(z,w)$ be the corresponding matrix of the projection $\mathcal{K}$ in $L^2_{0,n-1}(\Omega)$ on $\operatorname{Ker}\square$. If $-\varrho$ is a defining function of Ω at z^0 then

$$(4.23)\quad \varepsilon^{2(n+1)}K_{jk}((\varepsilon z',\varepsilon^2 z_n),(\varepsilon w',\varepsilon^2 w_n)) \to \pi^{-n}n!\kappa\partial\varrho(0)/\partial z_j\partial\varrho(0)/\partial\bar z_k F(z,w)^{-n-1} \quad \text{in } \mathcal{D}'(\Omega_0\times\Omega_0)$$

when $\varepsilon\to+0$. Here

(4.24)

$$F(z,w)=\partial\varrho(0)/\partial z_n\, z_n+\frac12\sum_{j,k=1}^{n-1}\partial^2\varrho(0)/\partial z_j\partial z_k\, z_jz_k+\partial\varrho(0)/\partial\bar z_n\,\bar w_n$$
$$+\frac12\sum_{j,k=1}^{n-1}\partial^2\varrho(0)/\partial\bar z_j\partial\bar z_k\,\bar w_j\bar w_k+\sum_{j,k=1}^{n-1}\partial^2\varrho(0)/\partial z_j\partial\bar z_k(z_j\bar z_k+w_j\bar w_k-w_j\bar z_k),$$

κ is the product of the eigenvalues of $\sum_{j,k=1}^{n-1} \partial^2\varrho(0)/\partial z_j\partial\bar z_k t_j\bar t_k$, *and*
(4.25)

$$\Omega_0 = \Big\{ z\in\mathbb{C}^n; 2\,\mathrm{Re}(\partial\varrho(0)/\partial z_n\, z_n) + \mathrm{Re}\sum_{j,k=1}^{n-1}\partial^2\varrho(0)/\partial z_j\partial z_k\, z_j z_k + \sum_{j,k=1}^{n-1}\partial^2\varrho(0)/\partial z_j\partial\bar z_k\, z_j\bar z_k > 0\Big\}.$$

Proof. — The first step is to show that the statement is independent of the choice of the local coordinates $z_1,\dots,z_n$ and the defining function $-\varrho$. If we multiply ϱ by a positive constant c then κ is replaced by $c^{n-1}\kappa$ and F by cF, so the statement is unchanged. If ϱ is multiplied by $1+\sigma$ where $\sigma(0)=0$ then F is unchanged since $\sigma\varrho$ vanishes of second order at the origin, of third order when $z_n=0$, which proves the independence of ϱ. The Taylor expansion of order two $\partial\varrho(0)/\partial z_n\, z_n + \frac12\sum_{j,k=1}^{n-1}\partial^2\varrho(0)/\partial z_j\partial z_k\, z_j z_k$ as well as its complex conjugate are invariant under analytic changes of coordinates, and so is the Levi form in the complex tangent plane, which completes the proof of the invariance of the statement in Theorem 4.6. The right-hand side of (4.23) is well defined in $\Omega_0\times\Omega_0$, for

$$2\,\mathrm{Re}\, F(z,w) = 2\,\mathrm{Re}(\partial\varrho(0)/\partial z_n\, z_n) + \mathrm{Re}\sum_{j,k=1}^{n-1}\partial^2\varrho(0)/\partial z_j\partial z_k\, z_j z_k$$
$$+\sum_{j,k=1}^{n-1}\partial^2\varrho(0)/\partial z_j\partial\bar z_k z_j\bar z_k + 2\,\mathrm{Re}(\partial\varrho(0)/\partial z_n\, w_n) + \mathrm{Re}\sum_{j,k=1}^{n-1}\partial^2\varrho(0)/\partial z_j\partial z_k w_j w_k$$
$$+\sum_{j,k=1}^{n-1}\partial^2\varrho(0)/\partial z_j\partial\bar z_k\, w_j\bar w_k + \sum_{j,k=1}^{n-1}\partial^2\varrho(0)/\partial z_j\partial\bar z_k(z_j-w_j)(\bar z_k-\bar w_k)$$

is positive if $z,w\in\Omega_0$.

The next step is to verify the statement when Ω is the ellipsoidal shell (4.10), z^0 is a point on the inner boundary and the metric is the standard Euclidean metric in $\mathbb{C}^n$. With $z_1,\dots,z_n$ now denoting the coordinates in $\mathbb{C}^n$ we have when $|z-z^0|=O(\varepsilon)$ and $|w-z^0|=O(\varepsilon)$ and K_{jk} is defined accordingly, since $b_j^2\zeta_j/|\zeta|^2 = \bar z_j/N_2 = b_j^2\partial\Lambda/\partial z_j$ when $\Lambda=0$

$$K_{jk}(z,w) = \Big(\partial\Lambda(z^0)/\partial z_j\partial\Lambda(z^0)/\partial\bar z_k\prod_1^n b_l^{-4} + O(\varepsilon)\Big)K(\zeta,\theta)$$

$$K(\zeta,\theta) = \pi^{-n}n!\prod_1^n b_l^2(1+O(\varepsilon))N_2\Big(1-\sum_1^n b_l^2\zeta_l\bar\theta_l\Big)^{-n-1}\cdot$$
$$\cdot\Big(1+O(1-\sum_1^n b_l^2\zeta_l\bar\theta_l)\Big).$$

Here $N_2 = \sum_1^n |z_l^0|^2/b_l^4 = |\partial\Lambda(z^0)/\partial z|^{-2}$. If $z = z^0 + Z$ and $w = z^0 + W$ and $\Lambda(z) + \Lambda(w) = O(\varepsilon^2)$ then

$$\begin{aligned}1 - \sum_1^n b_l^2 \zeta_l \bar\theta_l &= \sum_1^n |z_l^0|^2/b_l^2 - \sum_1^n b_l^2(\bar z_l^0 + \bar Z_l)(z_l^0 + W_l)/\big((b_l^2 + \Lambda(z))(b_l^2 + \Lambda(w))\big)\\ &= \sum_1^n |z_l^0|^2 b_l^{-4}(\Lambda(z) + \Lambda(w)) - \sum_1^n z_l^0 b_l^{-2} \bar Z_l - \sum_1^n \bar z_l^0 b_l^{-2} W_l\\ &\qquad - \sum_1^n b_l^{-2} \bar Z_l W_l + O(\varepsilon^3).\end{aligned}$$

The first order terms are

$$\begin{aligned}N_2\big(2\,\mathrm{Re}\langle\partial\Lambda(z^0)/\partial z, Z\rangle + 2\,\mathrm{Re}\langle\partial\Lambda(z^0)/\partial z, W\rangle - \langle\partial\Lambda(z^0)/\partial\bar z, \bar Z\rangle\\ -\langle\partial\Lambda(z^0)/\partial z, W\rangle\big) = N_2\big(\langle\partial\Lambda(z^0)/\partial z, Z\rangle + \langle\partial\Lambda(z^0)/\partial\bar z, \overline{W}\rangle\big),\end{aligned}$$

and the Taylor expansion of $\sum_1^n |z_l|^2/b_l^2 - 1$ in the complex tangent plane at z^0 is N_2 times that of Λ. If we recall that $\kappa = N_2^{-n}\prod_1^n b_l^{-2}$ it follows that

$$\begin{aligned}K_{jk}(z,w) = (\pi^{-n} n! \kappa \partial\Lambda(z^0)/\partial z_j \partial\Lambda(z^0)/\partial\bar z_k + O(\varepsilon))\big(\langle\partial\Lambda(z^0)/\partial z, Z\rangle\\ +\langle\partial\Lambda(z^0)/\partial\bar z, \overline{W}\rangle + \sum_{\ell,m=1}^n \partial^2\Lambda(z^0)/\partial z_\ell\partial\bar z_m(Z_\ell\overline{Z}_m + W_\ell\overline{W}_m - W_\ell\overline{Z}_m)\big)^{-n-1},\end{aligned}$$

which gives (4.23) when $Z = \varepsilon Z' + \varepsilon^2 Z''$ where Z' is in the complex tangent plane at z^0 and Z'' is in the orthogonal direction $\partial\Lambda(z^0)/\partial\bar z$. (Note that cross product terms involving both Z' and Z'' are of higher order and play no role, and that the second order tangential derivatives of $\Lambda(z)$ with respect to z vanish for $\sum \Lambda_{jk} s_j s_k = 0$ if $\sum \Lambda_j s_j = 0$ since the level surfaces are hyperplanes, and $\partial\Lambda/\partial z_j = \bar z_j \Lambda_j$, $\partial^2\Lambda/\partial z_j\partial z_k = \bar z_j \bar z_k \Lambda_{jk}$.)

Now we turn to the general proof. By Lemma 3.2 we can choose local complex coordinates at z^0 so that the coordinates of z^0 give a point on the unit sphere and there is a defining function $-\varrho$ of Ω at z^0 such that $\varrho(z) = |z|^2 - 1 + O(|z - z^0|^4)$. The hermitian metric at z^0 is a positive definite quadratic form. By a unitary transformation we can bring it to diagonal form $\sum_1^n b_l^2 |dz_l|^2$, so we may assume that the metric at z^0 has this form. If we take $b_l z_l$ as new coordinates the hermitian metric becomes the standard Euclidean metric, z^0 becomes a point on the ellipsoid $\sum_1^n |z_l|^2/b_l^2 = 1$, and Ω has a defining function $-\varrho$ at z^0 such that

$\varrho(z) = \sum_1^n |z_l|^2/b_l^2 - 1 + O(|z - z^0|^4)$. What remains is to prove that the limit (4.23) will be the same for the projection in $L^2_{0,n-1}(\Omega)$ as for that in $L^2_{0,n-1}(\tilde{\Omega})$ where $\tilde{\Omega}$ is the ellipsoidal shell defined by (4.3) with some $\gamma > 0$.

Let f be a $(0, n-1)$ form with constant coefficients in $\mathbb{C}^n$ and norm 1 (at a fixed point), let $\varphi \in C_0^\infty(\Omega_0)$ and set

$$f_\varepsilon(z) = f\varphi_\varepsilon(z)\varepsilon^{-n-1}; \quad \varphi_\varepsilon(z^0 + \varepsilon Z' + \varepsilon^2 Z'') = \varphi(Z' + Z''),$$

where Z' is in the complex tangent space at z^0 and Z'' is orthogonal to it. Then $\operatorname{supp} f_\varepsilon \subset \Omega \cap \tilde{\Omega}$ when $\varepsilon > 0$ is sufficiently small, and the norm $\|f_\varepsilon\|_{\tilde{\Omega}}$ in $L^2_{0,n-1}(\tilde{\Omega})$ is equal to the L^2 norm of φ, which is also equal to the limit as $\varepsilon \to 0$ of $\|f_\varepsilon\|_\Omega$. Here we use the given hermitian metric in Ω and the standard Euclidean metric in $\tilde{\Omega}$. Recall that they coincide at z^0. Let P (resp. $\tilde{P}$) be the orthogonal projection in $L^2_{0,n-1}(\Omega)$ (resp. $L^2_{0,n-1}(\tilde{\Omega})$) on $\operatorname{Ker}\square$. By the case of (4.23) already proved we have

$$\lim_{\varepsilon\to+0}(\tilde{P}f_\varepsilon, f_\varepsilon)_{\tilde{\Omega}}$$
$$= \pi^{-n} n!\kappa |\langle \partial\varrho(0)/\partial\bar{z}, f\rangle|^2 \iint_{\Omega_0\times\Omega_0} F(z,w)^{-n-1}\overline{\varphi(z)}\varphi(w)\, d\lambda(z)\, d\lambda(w),$$

if f is considered as a vector in $\mathbb{C}^n$. Conversely, this implies (4.23) for $\tilde{\Omega}$ at z^0, for by polarization it follows that (4.23) is valid for the scalar product with $\overline{\psi(z)}\varphi(z)$ when $\varphi, \psi \in C_0^\infty(\Omega_0)$, and this implies convergence in $\mathcal{D}'(\Omega_0 \times \Omega_0)$ by the Schwartz kernel theorem. In the same way (4.23) will follow for Ω if we prove that

$$(Pf_\varepsilon, f_\varepsilon)_\Omega - (\tilde{P}f_\varepsilon, f_\varepsilon)_{\tilde{\Omega}} \to 0 \quad \text{when } \varepsilon \to 0.$$

In the proof we may assume that the hermitian metric of Ω expressed in the local coordinates at z^0 is equal to the euclidean metric in a neighborhood of z^0 and not only at z^0. In fact, for every $\delta > 0$ we can choose a metric in Ω with this property which lies between $1+\delta$ and $1-\delta$ times the given metric. If P^δ is the corresponding projection and $(\cdot,\cdot)^\delta_\Omega$ the corresponding scalar product in $L^2_{0,n-1}(\Omega)$ we shall then have

$$(P^\delta f_\varepsilon, f_\varepsilon)^\delta_\Omega - (\tilde{P}f_\varepsilon, f_\varepsilon)_{\tilde{\Omega}} \to 0, \quad \varepsilon \to 0,$$

and $|(P^\delta f_\varepsilon, f_\varepsilon)^\delta_\Omega - (Pf_\varepsilon, f_\varepsilon)_\Omega| \leqslant C\delta$ by Lemma A.1. Hence it will follow that $(Pf_\varepsilon, f_\varepsilon)_\Omega - (\tilde{P}f_\varepsilon, f_\varepsilon)_{\tilde{\Omega}} \to 0$ when $\varepsilon \to 0$. (From Lemma A.2 it follows that the inverse of the $\bar{\partial}$-Neumann operator in $L^2_{0,n-1}(\Omega)$ on the orthogonal space of the null space, defined with the modified metric, has a bound independent of δ.) To simplify notation we assume in what follows that the given metric in Ω is itself the Euclidean metric in a neighborhood of z^0 with the fixed local coordinates.

In the local coordinates $\varrho(z) = \sum_1^n |z_l|^2/b_l^2 - 1 + r(z)$ where $D^\alpha r(z) = O(|z-z^0|^{4-|\alpha|})$ when $|\alpha| < 4$. We have $\partial\varrho/\partial z_j = \bar z_j/b_j^2 + \partial r/\partial z_j$, and when $\varrho(z) = 0$ then $\Lambda(z) = r(z)\Lambda(z)/(1-\sum_1^n |z_l|^2/b_l^2) = r_0(z)$ where $D^\alpha r_0(z) = O(|z-z^0|^{4-|\alpha|})$ when $|\alpha| < 4$. Since $\zeta_j = \bar z_j/(b_j^2+\Lambda)$ it follows that

$$\partial\varrho/\partial z_j = \zeta_j(1+r_0/b_j^2) + \partial r/\partial z_j = \zeta_j + r_j \quad \text{when } \varrho = 0.$$

Here $D^\alpha r_j = O(|z-z^0|^{3-|\alpha|})$ when $|\alpha| < 3$.

The representation of $\tilde P f_\varepsilon$ by the kernel shows that it is $O(\varepsilon^{n+1})$ outside any neighborhood of z^0 and well defined in $\Omega \cap \omega$ for small ε if ω is a sufficiently small neighborhood of z^0. If $\psi \in C_0^\infty(\omega)$ is equal to 1 in another neighborhood of z^0 then $(1-\psi)\tilde P f_\varepsilon$ and its derivatives are $O(\varepsilon^{n+1})$. If we replace ζ_j by $\zeta_j + r_j$ in the kernel of $\tilde P$ then $\psi\tilde P f_\varepsilon$ becomes an element g_ε in the domain of the $\bar\partial$-Neumann operator in Ω. To estimate $\bar\partial g_\varepsilon$ and $\mathfrak{d}_c g_\varepsilon$ we note that the terms where derivatives fall on the cutoff function ψ are $O(\varepsilon^{n+1})$. In the others we may replace $\zeta_j + r_j$ by r_j because $\bar\partial\tilde P = 0$ and $\mathfrak{d}\tilde P = 0$. (Here we assume that the metric for Ω is the Euclidean metric in $\operatorname{supp}\psi$.) When r_j is differentiated we have a bound $O(|z-z^0|^2)$ for the derivatives, and derivatives falling on the other factors in the kernel of $\tilde P$ give at most a loss of a factor $|z-z^0|^2$ in the estimates while $r_j = O(|z-z^0|^3)$. Hence it follows that $\|\bar\partial g_\varepsilon\|_\Omega = O(\varepsilon)$ and that $\|\mathfrak{d}_c g_\varepsilon\|_\Omega = O(\varepsilon)$; the subscript Ω refers to L^2 norms in Ω with the given hermitian metric. (Cf. the proof of Theorem 3.1.) The hypothesis that the range of $\Box$ in $L^2_{0,n-1}(\Omega)$ is closed implies that the $\bar\partial$ operators from $L^2_{0,n-2}(\Omega)$ to $L^2_{0,n-1}(\Omega)$ and from $L^2_{0,n-1}(\Omega)$ to $L^2_{0,n}(\Omega)$ have a closed range, so it follows that $\|Pg_\varepsilon - g_\varepsilon\|_\Omega = O(\varepsilon)$. Hence

$$|(g_\varepsilon, f_\varepsilon)_\Omega| = |(g_\varepsilon, Pf_\varepsilon)_\Omega + (g_\varepsilon - Pg_\varepsilon, f_\varepsilon)_\Omega| \leqslant \|g_\varepsilon\|_\Omega\|Pf_\varepsilon\|_\Omega + O(\varepsilon).$$

Since $\|g_\varepsilon\|_\Omega \leqslant \|\psi\tilde P f_\varepsilon\|_{\tilde\Omega} + O(\varepsilon) = \|\tilde P f_\varepsilon\|_{\tilde\Omega} + O(\varepsilon)$ and

$$\|\tilde P f_\varepsilon\|^2_{\tilde\Omega} = (\tilde P f_\varepsilon, f_\varepsilon)_{\tilde\Omega} = (\psi\tilde P f_\varepsilon, f_\varepsilon)_{\tilde\Omega} = (g_\varepsilon, f_\varepsilon)_\Omega + O(\varepsilon) \leqslant \|g_\varepsilon\|_\Omega\|Pf_\varepsilon\|_\Omega + O(\varepsilon),$$

it follows that

$$\|\tilde P f_\varepsilon\|^2_{\tilde\Omega} \leqslant \|\tilde P f_\varepsilon\|_{\tilde\Omega}\|Pf_\varepsilon\|_\Omega + O(\varepsilon) \leqslant \frac12\|\tilde P f_\varepsilon\|^2_{\tilde\Omega} + \frac12\|Pf_\varepsilon\|^2_\Omega + O(\varepsilon),$$

hence that

$$(\tilde P f_\varepsilon, f_\varepsilon)_{\tilde\Omega} = \|\tilde P f_\varepsilon\|^2_{\tilde\Omega} \leqslant \|Pf_\varepsilon\|^2_\Omega + O(\varepsilon) = (Pf_\varepsilon, f_\varepsilon)_\Omega + O(\varepsilon),$$

which is the desired lower bound for $(Pf_\varepsilon, f_\varepsilon)_\Omega$.

To give an upper bound for $(Pf_\varepsilon, f_\varepsilon)_\Omega$ we assume at first that $\Omega \cap \omega \subset \tilde\Omega \cap \omega$ for some neighborhood ω of z^0. As before we choose $\psi \in C_0^\infty(\omega)$ equal to 1 in another neighborhood of z^0. In $\tilde\Omega$ we have

$$f_\varepsilon = \tilde P f_\varepsilon + \bar\partial g_\varepsilon + \mathfrak{d}_c h_\varepsilon$$

where $g_\varepsilon \in L^2_{0,n-2}(\tilde\Omega)$, $\vartheta_c g_\varepsilon = 0$, and $h_\varepsilon \in L^2_{0,n}(\tilde\Omega)$ vanishes on $\partial\tilde\Omega$. When $\varepsilon \to 0$ we have $g_\varepsilon \to 0$ weakly in $L^2_{0,n-2}(\tilde\Omega)$ and $h_\varepsilon \to 0$ weakly in $L^2_{0,n}(\tilde\Omega)$, for $f_\varepsilon \to 0$ weakly in $L^2_{0,n-1}(\tilde\Omega)$. Since $\bar\partial\vartheta_c h_\varepsilon = \bar\partial f_\varepsilon$ it follows from the regularity of solutions of the Dirichlet problem for the Laplacian that $h_\varepsilon \to 0$ in $C^\infty(\tilde\Omega \setminus \omega_1)$ if ω_1 is any neighborhood of z^0, and that h_ε is bounded in the Sobolev space $H_{(1)}(\tilde\Omega)$ which implies that $\|h_\varepsilon\|_{\tilde\Omega} \to 0$ when $\varepsilon \to 0$. Since $\vartheta_c g_\varepsilon = 0$ and $\bar\partial g_\varepsilon = f_\varepsilon - \tilde P f_\varepsilon - \vartheta_c h_\varepsilon$ we also have $g_\varepsilon \to 0$ in $C^\infty(\tilde\Omega \setminus \omega_1)$ by Theorems 4 and 5 in [KN]. Hence

$$f_\varepsilon = \psi\tilde P f_\varepsilon + \bar\partial(\psi g_\varepsilon) + \vartheta_c(\psi h_\varepsilon) + r_\varepsilon, \quad r_\varepsilon = [\psi, \bar\partial]g_\varepsilon + [\psi, \vartheta_c]h_\varepsilon,$$

where $r_\varepsilon \to 0$ in $C^\infty(\tilde\Omega)$. Here $\bar\partial(\psi g_\varepsilon)$ is orthogonal to Pf_ε, and since $\bar\partial P f_\varepsilon = 0$ we can estimate $(Pf_\varepsilon, \vartheta_c(\psi h_\varepsilon))$ by the product of the norm in $H_{(-\frac12)}$ of the restriction to $\partial\Omega$ of Pf_ε and that of the restriction of ψh_ε in $H_{(\frac12)}$. The former is bounded since $\|Pf_\varepsilon\|_{\tilde\Omega} \leqslant 1$ and Pf_ε satisfies an elliptic differential equation. An elementary but tedious estimate of h_ε using local estimates of Green's function in $\tilde\Omega$ shows that

$$|h_\varepsilon(z)| \leqslant C\tilde\varrho(z)\varepsilon^{-n-1}\varepsilon^{2n}(|z-z^0|^2+\varepsilon^2)^{-n} \quad \text{when } \tilde\varrho(z) \ll |z-z^0|^2,$$

where $\tilde\varrho(z)$ denotes the distance from $z \in \tilde\Omega$ to $\partial\tilde\Omega$. This is applicable on $\partial\Omega$ when $|z-z^0|$ is small since $\tilde\varrho(z) = O(|z-z^0|^4)$ then. For first order tangential derivatives of $\varepsilon h_\varepsilon$ along $\partial\Omega$ we have similar estimates which implies that the norm in $H_{(\frac12)}$ of the restriction of h_ε to $\partial\Omega$ is $O(\varepsilon^2)$. Thus

$$\|Pf_\varepsilon\|^2_\Omega = (Pf_\varepsilon, f_\varepsilon)_\Omega = (Pf_\varepsilon, \psi\tilde P f_\varepsilon)_\Omega + o(1) \leqslant \|Pf_\varepsilon\|_\Omega\|\tilde P f_\varepsilon\|_{\tilde\Omega} + o(1)$$

which proves that $\|Pf_\varepsilon\|^2_\Omega \leqslant \|\tilde P f_\varepsilon\|^2_{\tilde\Omega} + o(1)$ as claimed.

If there is no neighborhood ω of z^0 with $\Omega\cap\omega \subset \tilde\Omega\cap\omega$ we can replace $\tilde\Omega$ by a slightly larger $\tilde\Omega$ and repeat the argument above to complete the proof. The only difference then is that we can only achieve a small positive bound δ for the $H_{(\frac12)}$ norm of h_ε on $\partial\Omega$, but since δ is an arbitrary positive number the conclusion is not affected.

The preceding proof is admittedly very sketchy but there seems to be no point in elaborating it further, for the right approach must be a detailed microlocal analysis which should yield Theorem 4.6 in full generality with C^∞ convergence. The restriction $n > 2$ above should then disappear, and an analogue for $(0,q)$ forms when the Levi form has signature $(n-q-1, q)$ at some boundary point should emerge. We shall now give a formal argument suggesting what the analogue of the results on the boundary behavior when $q = n-1$ might be in general.

Let Ω be a relatively compact open subset of a complex hermitian manifold of dimension n and let z^0 be a boundary point where $\partial\Omega$ is smooth and the Levi form has signature $(n-q-1,q)$. By Lemma 3.2 we can choose local coordinates vanishing at z^0 and a defining function ϱ there such that

$$\varrho(z)=2\operatorname{Im} z_n+\sum_1^{n-1}\lambda_j|z_j|^2+O(|z|^4),$$

the hermitian metric is the Euclidean metric at z^0 and $\lambda_j>0$ when $j\leqslant n-q-1$ while $\lambda_j<0$ when $n-q\leqslant j\leqslant n-1$. For reasons which will be explained later we assume that $\lambda_{n-q}=\ldots=\lambda_{n-1}$, and we set $\lambda_n=\lambda_{n-1}=-1/R$. Adding $\lambda_n|z_n|^2$ does not change this local normal form, which means that we can as well assume that

$$\varrho(z)=\sum_1^{n-1}\lambda_j|z_j|^2+\lambda_n|z_n+i/\lambda_n|^2-1/\lambda_n+O(|z|^4).$$

If we take z_n+i/λ_n as a new variable instead of z_n then $z^0=(0,\ldots,0,i/\lambda_n)=(0,\ldots,0,-iR)$ and

$$\varrho(z)=\sum_1^{n-q-1}\lambda_j|z_j|^2+R-|z''|^2/R+O(|z-z^0|^4),\quad z''=(z_{n-q},\ldots,z_n),$$

at z^0 with the new coordinates. Thus Ω is locally well approximated by

$$\tilde\Omega=\Big\{z\in\mathbb{C}^n;R<-\sum_1^{n-q-1}\lambda_j|z_j|^2+|z''|^2/R\Big\}$$

which is essentially of the form discussed in Section 2. Set $\zeta_j=z_j$ for $1\leqslant j\leqslant n-q-1$ and $\zeta_j=\bar z_j/|z''|^2$ when $n-q\leqslant j\leqslant n$. As suggested by (2.16) and (4.8) it follows that

$$f=g(\zeta)|z''|^{-2(q+1)}\sum_{n-q}^{n}(-1)^j\bar z_j d\bar z_{n-q}\wedge\ldots\wedge\widehat{d\bar z_j}\wedge\ldots\wedge d\bar z_n \tag{4.26}$$

is in the null space of the $\bar\partial$-Neumann operator $\square$ in $L^2_{0,q}(\tilde\Omega)$ if g is holomorphic in

$$\check\Omega=\Big\{\zeta\in\mathbb{C}^n;\big(R+\sum_1^{n-q-1}\lambda_j|\zeta_j|^2\big)R|\zeta''|^2<1\Big\},$$

and there is some reason to believe that every element in the null space is of this form.

If $\mathcal{K}$ is the reproducing kernel of $\operatorname{Ker}\square$ in $L^2_{0,q}(\tilde\Omega)$ and t is a $0,q$ form then

$$(\mathcal{K}(z,z)t,t)=\sup_{0\neq f\in\operatorname{Ker}\square}|(f,t)_z|^2/\|f\|^2. \tag{4.27}$$

When $z = (1+\delta)z^0 = (0,\dots,0,-iR-\delta iR)$, $0<\delta<1$, and f is of the form (4.26) then only the part of t which is a form in $d\bar z''$ will contribute, and (f,t) does not change if ζ_k is replaced by $e^{i\theta_k}\zeta_k$ when $1\leqslant k\leqslant n-q-1$ and $0\leqslant\theta_k\leqslant 2\pi$. The average over all such rotations is independent of the first $n-q-1$ variables, and the norm is at most equal to that of f. In examining the supremum in (4.27) it is therefore enough to consider forms (4.26) where g is a holomorphic function of ζ'' when $|\zeta''|<1/R$. In computing the norm of f we have to take into account that

$$m\Big\{(z_1,\dots,z_{n-q-1})\in\mathbb{C}^{n-q-1};\ \sum_1^{n-q-1}\lambda_j|z_j|^2<|z''|^2/R-R\Big\}$$
$$=\frac{\pi^{n-q-1}}{(n-q-1)!}\prod_1^{n-q-1}\lambda_j^{-1}(|z''|^2/R-R)^{n-q-1}.$$

This gives that if f is defined by (4.26) then

$$\|f\|^2=\frac{\pi^{n-q-1}}{(n-q-1)!}\prod_1^{n-q-1}\lambda_j^{-1}\int_{|\zeta''|<1/R}|g(\zeta'')|^2|\zeta''|^{-2}\cdot$$
$$\cdot\big(1/(R|\zeta''|^2)-R\big)^{n-q-1}\,d\lambda(\zeta'').$$

We can compare this with

$$\int_{|\zeta|<1/R}|g(\zeta'')|^2\,d\lambda(\zeta)$$
$$=\frac{\pi^{n-q-1}}{(n-q-1)!}\int_{|\zeta''|<1/R}|g(\zeta'')|^2(R^{-2}-|\zeta''|^2)^{n-q-1}\,d\lambda(\zeta'').$$

Since

$$|\zeta''|^{-2}\big(1/(R|\zeta''|^2)-R\big)^{n-q-1}=|\zeta''|^{-2(n-q)}R^{n-q-1}(R^{-2}-|\zeta''|^2)^{n-q-1},$$

and $|\zeta''|^{-2(n-q)}R^{n-q-1}=R^{3(n-q)-1}$ when $|\zeta''|=1/R$, and since

$$|g(\zeta'')|^2\leqslant\pi^{-n}n!R^{2n}(1-R^2|\zeta''|^2)^{-n-1}\int_{|\zeta|<1/R}|g(\zeta'')|^2\,d\lambda(\zeta),$$

by the well known formula for the Bergman kernel in a ball, we are led to the bound

$$|g(\zeta'')|^2\leqslant\|f\|^2R^{1-3(n-q)}\prod_1^{n-q-1}\lambda_j\ \pi^{-n}n!R^{2n}(1-R^2|\zeta''|^2)^{-n-1}(1+o(1)),$$

when $z=(1+\delta)z^0\to z^0$. Here $1-R^2|\zeta''|^2=1-R^2/|z''|^2=(|z''|^2/R-R)R/|z''|^2$, so this can be rewritten as

$$|g(\zeta'')|^2\leqslant\|f\|^2R^{3q+2}\prod_1^{n-q-1}\lambda_j\pi^{-n}n!|\varrho(z)|^{-n-1}(1+o(1)).$$

We have

$$|(f(z),t)|^2 = |g(\zeta'')|^2|z''|^{-4(q+1)}|z''|^2|(d\bar z_{n-q}\wedge\ldots\wedge d\bar z_{n-1},t)|^2$$

which finally suggests that when $z=(1+\delta)z^0\to z^0$,

$$|\varrho(z)|^{n+1}(\mathcal{K}(z,z)t,t)\leqslant\pi^{-n}n!\kappa(z^0)|t\wedge\bar\partial\varrho(z^0)|_-^2+o(1)|t|^2, \tag{4.28}$$

where $\kappa(z^0)=R^{-q}\prod_1^{n-q-1}\lambda_j=\prod_1^{n-1}|\lambda_j|$ is the product of the absolute values of the eigenvalues of the Levi form and $|\gamma|_-$ stands for the norm of the restriction of a form γ at z^0 to the orthogonal complement of the space spanned by the $n-q-1$ eigenvectors corresponding to positive eigenvalues of the Levi form of ϱ at z^0. (The formal arguments above give no real support for this unless all the negative eigenvalues are equal or $q=n-1$, for otherwise $\operatorname{Ker}\Box$ contains no element which is a form in $d\bar z''$ since confocal quadrics are never homothetic except in the spherical case.) If (4.28) is true one might also ask if the inequality can be replaced by an equality when the range of $\Box$ is closed.

5. A crude microlocal approach.

If u is a square integrable harmonic function in an open set $\Omega\subset\mathbb{R}^N$ then by the mean value property

$$|u(x)|^2\leqslant\|u\|^2d(x)^{-N}/C_N,\quad x\in\Omega, \tag{5.1}$$

where $\|u\|$ is the norm of u in $L^2(\Omega)$, C_N is the volume of the unit ball in $\mathbb{R}^N$, and $d(x)$ is the distance from x to $\partial\Omega$. The best bound for

$$K(x,x)=\sup_{u\neq0}|u(x)|^2/\|u\|^2$$

is smaller by a factor $2^{2-N}(N-1)/N$ at a C^1 boundary by Proposition C.1. A crude bound of the same order of magnitude can be obtained as follows when $\partial\Omega$ is smooth. The restriction u° to $\partial\Omega$ of a harmonic function u in Ω with $\|u\|\leqslant 1$ is bounded in the Sobolev space $H_{(-\frac12)}(\partial\Omega)$, and $u=Ku^\circ$ where K is a Poisson operator. In terms of local coordinates at a point in $\partial\Omega$ such that Ω is defined by $x_N>0$ the Poisson operator is of the form $K_{x_N}(x',D')$ where the symbol $K_{x_N}(x',\xi')$ satisfies the hypotheses of Proposition B.2 with N replaced by $N-1$. Taking $s=-\frac12$ and $a=0$ we have $\mu=N-1$ there and obtain a bound of the form (5.1).

The justification for this more complicated proof of a weaker result is that it can be adapted to other situations. If $\Omega\subset\mathbb{C}^n=\mathbb{R}^{2n}$ and $\partial\Omega\in C^2$

then the simplest estimate of the Bergman kernel gives an estimate of the form

$$|u(z)|^2 \leqslant C\|u\|^2 d(z)^{-n-1}, \quad z \in \Omega,$$

when u is holomorphic in Ω, and the exponent $n+1$ of $1/d$ here is smaller than the exponent $2n$ provided by (5.1) if $n > 1$. This can be also be derived from Proposition B.2 as follows. If u is a harmonic function in Ω with boundary value u° then the boundary value of $\partial u/\partial \bar{z}_j$ is equal to $Z_j u^\circ$ where Z_j is a first order pseudodifferential operator in $\partial\Omega$. If u is holomorphic then $Z_j u^\circ = 0$ for $j = 1, \dots, n$, hence

$$Au^\circ = 0, \quad A = \sum_1^n Z_j^* Z_j. \tag{5.2}$$

We shall prove that the hypotheses of Proposition B.2 (with $N = 2n-1$) are fulfilled for this operator with $\mu = 1$, and then $2s - (N+\mu)/2 = -1-n$ which gives an estimate of the desired form. It suffices to calculate the principal symbol of the operator in (5.2) at one point in $\partial\Omega$, which we place at the origin so that $\operatorname{Im} z_n \geqslant O(|z|^2)$ in Ω. In a neighborhood of the origin we can take $x_j = \operatorname{Re} z_j$ and $y_k = \operatorname{Im} z_k$ as local coordinates on $\partial\Omega$, with $j \leqslant n$ and $k < n$; we use the notation x and y' for them and let ξ and η' be the dual variables. In terms of these coordinates $u^\circ \mapsto \partial(Ku^\circ)/\partial y_n|_{\partial\Omega}$ is a pseudodifferential operator in $\partial\Omega$ with principal symbol $-\Xi$ where $\Xi = \sqrt{|\xi|^2 + |\eta'|^2}$ at the origin. The principal symbol of $2Z_j$ is $i\xi_j - \eta_j$ when $j < n$ and $i\xi_n - i\Xi$ when $j = n$, so the principal symbol of the operator in (5.2) is at the origin

$$\sum_1^{n-1} |i\xi_j - \eta_j|^2 + |\xi_n - \Xi|^2 = \sum_1^{n-1} (\xi_j^2 + \eta_j^2) + |\xi_n - \Xi|^2 \geqslant |\xi'|^2 + |\eta'|^2.$$

The form on the right vanishes only on the ξ_n axis which proves the claim. (The full principal symbol vanishes only on the positive ξ_n axis.)

Again this is a much too complicated proof of an elementary fact. However, it has the virtue that it can adapted to the study of the null space of the $\bar\partial$-Neumann problem. Let $f \in L^2_{0,q}(\Omega)$ and $\bar\partial f = 0$, $\vartheta_c f = 0$, and write as usual

$$f = \sum_{|I|=q}{}' f_I \, d\bar{z}^I.$$

Since $\vartheta\bar\partial + \bar\partial\vartheta = -\Delta/4$, the coefficients f_I are harmonic functions so they have boundary values $f_I^\circ \in H_{(-\frac12)}(\partial\Omega)$. Let

$$f^\circ = \sum_{|I|=q}{}' f_I^\circ d\bar{z}^I \in H_{(-\frac12)}(\partial\Omega, \wedge^{0,q}(\mathbb{C}^n)).$$

That $\bar\partial f = 0$ in Ω means that

$$\sum_{j=1}^{n} \sum_{|I|=q}{}' Z_j f_I^\circ \, d\bar z_j \wedge d\bar z^I = 0. \tag{5.3}$$

To express the condition $\partial_c f = 0$ in terms of f° we need a lemma.

LEMMA 5.1. — *If $u \in L^2(\Omega)$ is a harmonic function with boundary value u° then the boundary value of $\partial u/\partial z_j$ is $\tilde Z_j u^\circ$ where $\tilde Z_j$ is a first order pseudodifferential operator and*

$$\tilde Z_j u^\circ + Z_j^* u^\circ - 2\Lambda(\partial\varrho/\partial z_j u^\circ) = R_j u^\circ.$$

Here R_j is a pseudodifferential operator of order 0, ϱ is a defining function of Ω with $|\varrho'| = 1$ on $\partial\Omega$, and Λ is the square root of $-\Delta_{\partial\Omega}$.

Proof. — It suffices to examine the principal symbol at $0 \in \partial\Omega$ assuming that $d\varrho = -dy_n$ there. With the notation used before in this section the principal symbol of $2\tilde Z_j$ is there $i\xi_j + \eta_j$ for $j < n$ and $i\xi_n + i\Xi$ for $j = n$ while that of $2Z_j^*$ is $-i\xi_j - \eta_j$ for $j < n$ and $-i\xi_n + i\Xi$ for $j = n$. Hence the principal symbol of $2(\tilde Z_j + Z_j^*)$ is equal to 0 at the origin when $j < n$ but equal to $2i\Xi$ when $j = n$. Since $\partial\varrho(0)/\partial z_j = 0$ when $j < n$ and $\partial\varrho(0)/\partial z_n = i/2$ the lemma is proved.

The equation $\partial f = 0$ can be written

$$-\sum_{j=1}^{n} \partial f_{jK}/\partial z_j = 0, \quad |K| = q-1,$$

which means that $-\sum_{j=1}^n \tilde Z_j f_{jK}^\circ = 0$ on $\partial\Omega$. If $\partial_c f = 0$ then $\sum_{j=1}^n \partial\varrho/\partial z_j f_{jK}^\circ = 0$, so it follows from Lemma 5.1 that

$$\sum_{j=1}^{n} Z_j^* f_{jK}^\circ = \sum_{j=1}^{n} R_j f_{jK}^\circ, \quad |K| = q-1. \tag{5.4}$$

From (5.3) and (5.4) it follows by the usual proof that $\partial\bar\partial + \bar\partial\partial = -\Delta/4$ that

$$Af^\circ = Rf^\circ, \tag{5.5}$$

where A is the operator in (5.2) acting on each component f_I° and R is a first order $\binom{n}{q} \times \binom{n}{q}$ matrix of first order pseudodifferential operators. The proof of Proposition B.2 remains valid for such a weakly coupled system, so we obtain:

THEOREM 5.2. — *If $\Omega \subset \mathbb{C}^n$ is bounded and $\partial\Omega$ is smooth then*

$$|f(z)|^2 \leqslant Cd(z)^{-n-1}\|f\|^2_{L^2}, \quad z \in \Omega, \tag{5.6}$$

when f is in the null space of the $\bar\partial$-Neumann operator in $L^2_{0,q}(\Omega)$. Here $d(z)$ denotes the distance from z to $\partial\Omega$.

Only minor modifications would be required to prove that Theorem 5.2 remains valid when Ω is an open relatively compact subset with smooth boundary of a complex hermitian manifold. The arguments in Section 4 suggest that $d(z)^{-n-1}$ can be replaced by $o(d(z)^{-n-1})$ when z approaches a boundary point where the signature of the Levi form is not $n-q-1, q$. We have also suggested in (4.28) an upper bound depending on the eigenvalues of the Levi form at such points which might be optimal when the range of the $\bar\partial$-Neumann operator is closed. A study of these questions must obviously require a much more detailed discussion of the boundary reduction than the calculation of principal symbols above, for the eigenvalues of the Levi form only occur at the next level in the symbols.

Appendix A. Generalities on orthogonal projections.

In this appendix we shall present two elementary functional analytic lemmas needed in the main text.

LEMMA A.1. — *Let H be a Hilbert space, let H_0, H_1 be two closed subspaces, $H_0 \subset H_1 \subset H$, and denote by P the orthogonal projection in H on $H_1 \ominus H_0$. Let $\tilde H$ be the space H with an equivalent norm,*

$$(1-\delta)\|u\|_H \leqslant \|u\|_{\tilde H} \leqslant (1+\delta)\|u\|_H, \quad u \in H,$$

where $0 < \delta < 1$, and let $\tilde P$ be the $\tilde H$ orthogonal projection in $\tilde H$ on the $\tilde H$ orthogonal complement of H_0 in H_1. Then

$$\left|\|\tilde P u\|^2_{\tilde H} - \|Pu\|^2_H\right| \leqslant 2\delta\|u\|^2_H, \quad u \in H. \tag{A.1}$$

Proof. — If P_j and $\tilde P_j$ denote the H and $\tilde H$ orthogonal projections of H on H_j, $j = 1, 2$, then

$$\|u\|^2_{\tilde H} - \|\tilde P_j u\|^2_{\tilde H} = \inf_{v \in H_j} \|u-v\|^2_{\tilde H} \begin{cases} \leqslant (1+\delta)\inf_{v\in H_j}\|u-v\|^2_H \\ \quad = (1+\delta)(\|u\|^2_H - \|P_j u\|^2_H), \\ \geqslant (1-\delta)\inf_{v\in H_j}\|u-v\|^2_H \\ \quad = (1-\delta)(\|u\|^2_H - \|P_j u\|^2_H). \end{cases}$$

Hence

$$\|\tilde{P}u\|_{\tilde{H}}^2 = \|\tilde{P}_1 u\|_{\tilde{H}}^2 - \|\tilde{P}_0 u\|_{\tilde{H}}^2 \begin{cases} \leqslant 2\delta\|u\|_H^2 + \|Pu\|_H^2, \\ \geqslant -2\delta\|u\|_H^2 + \|Pu\|_H^2, \end{cases}$$

which proves (A.1).

The next lemma concerns a short complex of operators in Hilbert spaces. Let H_1, H_2, H_3 be Hilbert spaces and let $T : H_1 \to H_2$ and $S : H_2 \to H_3$ be closed linear operators with $ST = 0$. Then (see e.g. [H1, Section 1.1]) the range of T and the range of S are both closed if and only if there is a constant C such that

$$\|f\|_{H_2}^2 \leqslant C^2(\|T^*f\|_{H_1}^2 + \|Sf\|_{H_3}^2),\ f \in \mathcal{D}_{T^*} \cap \mathcal{D}_S,$$
$$f \perp N = \{g \in H_2; T^*g = 0, Sg = 0\},$$

which is equivalent to the estimate

$$\|f\|_{H_2} \leqslant C^2\|\Box f\|_{H_2}, \quad f \in \mathcal{D}_\Box,\ \Box = TT^* + S^*S,\ f \perp N = \operatorname{Ker}\Box,$$

and that $\Box$ has a closed range. Another equivalent condition is that

$$\|u\|_{H_1} \leqslant C\|Tu\|_{H_2},\ u \in \mathcal{D}_T \cap \mathcal{R}_{T^*}, \quad \|f\|_{H_2} \leqslant C\|Sf\|_{H_3},\ f \in \mathcal{D}_S \cap \mathcal{R}_{S^*},$$

which again is equivalent to

$$\inf_{Tu=f} \|u\|_{H_1} \leqslant C\|f\|_{H_2},\ f \in \mathcal{R}_T, \quad \inf_{Sf=g} \|f\|_{H_2} \leqslant C\|g\|_{H_3},\ g \in \mathcal{R}_S.$$

In this formulation there are no orthogonal spaces involved. If $\tilde{H}_j$ are the spaces H_j with equivalent norms,

$$(A.2) \qquad (1-\delta)\|u\|_{H_j} \leqslant \|u\|_{\tilde{H}_j} \leqslant (1+\delta)\|u\|_{H_j}, \quad u \in H_j,$$

then the last estimates remains valid with C replaced by $C(1+\delta)/(1-\delta)$. If $\tilde{\Box}$ is the analogue of $\Box$ defined with respect to the norms in $\tilde{H}_j$, it follows that

$$\|f\|_{\tilde{H}_2} \leqslant \big(C(1+\delta)/(1-\delta)\big)^2\|\tilde{\Box}f\|_{\tilde{H}_2}, \quad f \in \mathcal{D}_{\tilde{\Box}}, f \perp N = \operatorname{Ker}\tilde{\Box} \text{ in } \tilde{H}_2.$$

Thus we have proved:

LEMMA A.2. — *Suppose that the selfadjoint operator $\Box = TT^* + S^*S$ on H_2 has a closed range and let M be a bound for the inverse on the H_2 orthogonal space of* $\operatorname{Ker}\Box$. *Denote by $\tilde{\Box}$ the analogous operator defined with Hilbert spaces $\tilde{H}_j$ equal to H_j but with equivalent norms satisfying* (A.2). *Then it follows that $\tilde{\Box}$ also has a closed range, and the inverse in the $\tilde{H}_2$ orthogonal space of the null space has the bound $M((1+\delta)/(1-\delta))^2$.*

Appendix B. Some metrics and weights.

In this appendix we shall discuss some metrics and weights which are useful in the microlocal study of reproducing kernels in Section 5. Let $p \in S^2_{1,0}(\mathbb{R}^N \times \mathbb{R}^N)$ and assume that

$$(B.1) \qquad \langle \xi \rangle \leqslant p(x,\xi), \quad x, \xi \in \mathbb{R}^N,$$

where $\langle \xi \rangle = (1+|\xi|^2)^{\frac{1}{2}}$. (We use the notation of [H3], Chapter 18.) From the inequality

$$|g'(0)|^2 \leqslant g(0)(g(0) + 2 \sup_{|t|<1} |g''(t)|), \quad g \in C^2(-1,1),\ g \geqslant 0,$$

applied with $g(t) = p(x, \xi + t\eta)$, $|\eta| = \langle \xi \rangle$, or $g(t) = p(x+ty, \xi)$, $|y| = 1$, it follows then that

$$(B.2) \quad |\partial p(x,\xi)/\partial \xi| \leqslant C\sqrt{p(x,\xi)},\ |\partial p(x,\xi)/\partial x| \leqslant C\sqrt{p(x,\xi)}\langle \xi \rangle,\ x,\xi \in \mathbb{R}^N.$$

Since $\sqrt{p}\, d\sqrt{p} = dp/2$ we obtain

$$|\sqrt{p(x,\xi)} - \sqrt{p(y,\eta)}| \leqslant \frac{1}{2} C(|x-y|\langle \xi \rangle + |\xi - \eta|), \quad \text{hence}$$

$$\frac{1}{2}\sqrt{p(x,\xi)} \leqslant \sqrt{p(y,\eta)} \leqslant \frac{3}{2}\sqrt{p(x,\xi)},\ \text{if } |x-y|\langle\xi\rangle + |\xi-\eta| < \varepsilon\sqrt{p(x,\xi)},\ \varepsilon C \leqslant 1.$$

Since $\sqrt{p(x,\xi)} \leqslant C'\langle \xi \rangle$ we also have

$$\frac{1}{2}\langle \xi \rangle \leqslant \langle \eta \rangle \leqslant \frac{3}{2}\langle \xi \rangle, \quad \text{if } |\xi - \eta| < \varepsilon\sqrt{p(x,\xi)}, \varepsilon C' < \frac{1}{2}.$$

Hence the metric

$$(B.3) \qquad G = \frac{\langle \xi \rangle^2}{p(x,\xi)}\Big(|dx|^2 + \frac{|d\xi|^2}{\langle \xi \rangle^2}\Big) = \frac{\langle \xi \rangle^2 |dx|^2}{p(x,\xi)} + \frac{|d\xi|^2}{p(x,\xi)}$$

is slowly varying (see [H3], Def. 18.4.1), and $p(x,\xi)$ as well as $\langle \xi \rangle$ is G continuous (see [H3], Def. 18.4.2). Since

$$G^\sigma_{x,\xi} = p(x,\xi)|dx|^2 + p(x,\xi)\langle \xi \rangle^{-2}|d\xi|^2$$

we have $H(x,\xi)^2 = G_{x,\xi}/G^\sigma_{x,\xi} = (\langle \xi \rangle / p(x,\xi))^2 \leqslant 1$, and since $p(x,\xi)/\langle \xi \rangle^2$ is bounded it follows from [H3], Prop. 18.5.6 that G is σ temperate (see [H3], Def. 18.5.1). However, we shall give a direct verification in the simple case at hand that more precisely

$$(B.4) \qquad G_{y,\eta} \leqslant C G_{x,\xi}(1 + G^\sigma_{y,\eta}(x-y, \xi-\eta)),$$

$$G^\sigma_{y,\eta}(x-y,\xi-\eta) = p(y,\eta)|x-y|^2 + \frac{p(y,\eta)}{\langle \eta \rangle^2}|\xi - \eta|^2.$$

Since G is slowly varying this is true if $G_{x,\xi}(x-y,\xi-\eta)$ is small enough, so we may assume that for some constant $c>0$

$$c \leqslant G_{x,\xi}(x-y,\xi-\eta) = \frac{\langle\xi\rangle^2}{p(x,\xi)}\Big(|x-y|^2 + \frac{|\xi-\eta|^2}{\langle\xi\rangle^2}\Big).$$

a) If $\frac{1}{2}\langle\eta\rangle \leqslant \langle\xi\rangle \leqslant 2\langle\eta\rangle$ then

$$c \leqslant 4\frac{\langle\xi\rangle^2}{p(x,\xi)}\Big(|x-y|^2 + \frac{|\xi-\eta|^2}{\langle\eta\rangle^2}\Big) = 4\frac{\langle\xi\rangle^2}{p(x,\xi)}G^\sigma_{y,\eta}(x-y,\xi-\eta)/p(y,\eta).$$

Since $\langle\xi\rangle^2 \leqslant p(x,\xi)^2$ and $\langle\eta\rangle^2 \leqslant p(y,\eta)^2$, this implies

$$cp(y,\eta) \leqslant 4p(x,\xi)G^\sigma_{y,\eta}(x-y,\xi-\eta), \quad c\frac{\langle\eta\rangle^2}{p(y,\eta)} \leqslant 4\frac{\langle\xi\rangle^2}{p(x,\xi)}G^\sigma_{y,\eta}(x-y,\xi-\eta),$$

hence

$$c\langle\eta\rangle \leqslant 4\langle\xi\rangle G^\sigma_{y,\eta}(x-y,\xi-\eta), \quad \frac{c}{p(y,\eta)} \leqslant \frac{16}{p(x,\xi)}G^\sigma_{y,\eta}(x-y,\xi-\eta),$$

which verifies (B.4) as well as its analogues for $p(x,\xi)$, $1/p(x,\xi)$ and $\langle\xi\rangle$ in this case.

b) If $\langle\xi\rangle > 2\langle\eta\rangle$ then $\langle\xi\rangle \leqslant \langle\eta\rangle + |\xi-\eta| \leqslant \langle\xi\rangle/2 + |\xi-\eta|$, hence $|\xi-\eta| \geqslant \langle\xi\rangle/2$ and $G^\sigma_{y,\eta}(x-y,\xi-\eta) \geqslant \frac{1}{4}p(y,\eta)\langle\xi\rangle^2/\langle\eta\rangle^2$, which implies that

$$(B.5) \qquad p(y,\eta)^{-1} \leqslant Cp(x,\xi)^{-1}G^\sigma_{y,\eta}(x-y,\xi-\eta),$$

for $p(x,\xi) \leqslant \frac{1}{4}Cp(y,\eta)2\langle\xi\rangle^2/\langle\eta\rangle^2$ when $p(x,\xi) \leqslant \frac{1}{4}C\langle\xi\rangle^2$. The estimate (B.5) remains valid with another factor $\langle\eta\rangle^2$ in the left-hand side and $\langle\xi\rangle^2$ in the right-hand side, which gives (B.4) in this case.

c) If $\langle\eta\rangle > 2\langle\xi\rangle$ then $|\xi-\eta| \geqslant \langle\eta\rangle/2$, hence $G^\sigma_{y,\eta}(x-y,\xi-\eta) \geqslant p(y,\eta)/4$. Since $\langle\xi\rangle^2/p(x,\xi)$ has a positive lower bound and $\langle\eta\rangle^2/p(y,\eta) \leqslant p(y,\eta)$ it follows that

$$\langle\eta\rangle^2/p(y,\eta) \leqslant C\langle\xi\rangle^2/p(x,\xi)\, G^\sigma_{y,\eta}(x-y,\xi-\eta),$$

which completes the proof of (B.4).

In the course of the proof we have also verified that $1/p$ is σ, G temperate (see [H3], Def. 18.5.1). This is also obvious for $\xi \mapsto \langle\xi\rangle$, in case c) because $\langle\eta\rangle \leqslant p(y,\eta)$. Summing up, we have proved:

PROPOSITION B.1. — *When $p \in S^2_{1,0}(\mathbb{R}^N \times \mathbb{R}^N)$ satisfies* (B.1) *it follows that the metric G defined by* (B.3) *is σ-temperate. Moreover, the weights $p(x,\xi)$ and $\xi \mapsto \langle\xi\rangle$ are σ, G temperate.*

Let $a \in S^2_{1,0}(\mathbb{R}^N \times \mathbb{R}^N)$ be a symbol with real non-negative positively homogeneous principal symbol a_2. To study the solutions of the equation

$$(B.6) \qquad a(x, D)u = 0$$

we choose $p \in S^2_{1,0}(\mathbb{R}^N \times \mathbb{R}^N)$ satisfying (B.1) with principal symbol a_2. Then $a \in S(p, G)$ since $a - p \in S^1_{1,0}(\mathbb{R}^N \times \mathbb{R}^N) \subset S(p, G)$, $p \in S(p, G)$, and $q = p^{-1} \in S(p^{-1}, G)$. The equation (B.6) implies $q(x, D)a(x, D)u = 0$, and $q(x, D)a(x, D) = \mathrm{Id} - R(x, D)$ where $R \in S(H, G)$ and $H(x, \xi) = \langle\xi\rangle/p(x, \xi)$, for $q(x, \xi)a(x, \xi) - 1 = (a(x, \xi) - p(x, \xi))/p(x, \xi)$ can be estimated by H, and it follows from the calculus that $1 - R(x, \xi) - q(x, \xi)a(x, \xi) \in S(H, G)$. Hence it follows from (B.6) that for every positive integer ν

$$(B.7) \qquad u = R(x, D)^\nu u, \quad R(x, D)^\nu \in \mathrm{Op}\, S(H^\nu, G).$$

Assuming that $u \in H_{(s)}(\mathbb{R}^N)$ we want to estimate $K_t(x, D)u$ where

$$(B.8) \qquad K_t(x, \xi) \in S((1 + t|\xi|)^{-\nu}, |dx|^2 + |d\xi|^2/\langle\xi\rangle^2)$$

for every ν, uniformly in the parameter $t \in (0, 1)$. (This is true for the Poisson operator for the Laplacian, with t equal to the boundary distance.) Since

$$K_t(x, D)u(x) = (2\pi)^{-N} \int e^{i\langle x,\xi\rangle} K_t(x, \xi)\hat{u}(\xi)\, d\xi,$$

$$\|u\|^2_{(s)} = (2\pi)^{-N} \int |\hat{u}(\xi)|^2 \langle\xi\rangle^{2s}\, d\xi,$$

it follows that

$$|K_t(x, D)u(x)|^2 \leqslant (2\pi)^{-N} \int |K_t(x, \xi)|^2 \langle\xi\rangle^{-2s}\, d\xi\, \|u\|^2_{(s)}.$$

If u satisfies (B.6), hence (B.7), then

$$K_t(x, D)u = K_t(x, D)R(x, D)^\nu u = K_t^{(\nu)}(x, D)u$$

where $K_t^{(\nu)}$ is bounded in $S((1+t|\xi|)^{-\nu} H(x, \xi)^\nu, G)$ for fixed ν, so we have

$$|K_t(x, D)u(x)|^2/\|u\|^2_{(s)} \leqslant C_\nu \int (1 + t|\xi|)^{-2\nu} H(x, \xi)^{2\nu} \langle\xi\rangle^{-2s}\, d\xi.$$

The integral on the right for $|\xi| < 1$ is finite even for $t = 0$, and when $\xi = r\omega$ with $r \geqslant 1$ and $|\omega| = 1$ we have

$$H(x, \xi) \leqslant C|\xi|/(|\xi| + a_2(x, \xi)) = 1/(1 + ra_2(x, \omega)),$$

so the integral can be estimated by a constant times

$$\iint \bigl(1 + r(t + a_2(x, \omega))\bigr)^{-2\nu} r^{N-1-2s}\, dr d\omega$$
$$= \int_0^\infty (1 + r)^{-2\nu} r^{N-1-2s}\, dr \int_{|\omega|=1} (t + a_2(x, \omega))^{2s-N}\, d\omega.$$

The integral converges if $2\nu + 2s > N$. We have proved:

PROPOSITION B.2. — *Let $a \in S^2_{1,0}(\mathbb{R}^N \times \mathbb{R}^N)$ be a symbol with real non-negative positively homogeneous principal symbol a_2, and let K_t be symbols depending on a parameter $t \in (0,1)$ satisfying* (B.8) *uniformly in t for every ν. Then*

$$(B.9) \qquad |K_t(x,D)u(x)|^2/\|u\|^2_{(s)} \leqslant C_s \int_{|\omega|=1} (t + a_2(x,\omega))^{2s-N}\, d\omega$$

for all $u \in H_{(s)}(\mathbb{R}^N)$ such that $a(x,D)u = 0$. If $a_2(x,\xi)$ is bounded below by a positive semidefinite quadratic form in ξ with null space of dimension μ then the right-hand side is $O(t^{2s-(N+\mu)/2})$ if $2s - (N+\mu)/2 < 0$.

Appendix C. Reproducing kernels for harmonic functions and forms.

If u is a square integrable harmonic function in an open set $\Omega \subset \mathbb{R}^N$ then

$$|u(x)|^2 \leqslant \|u\|^2 d(x)^{-N}/C_N, \quad x \in \Omega,$$

where $\|u\|$ is the norm of u in $L^2(\Omega)$, C_N is the volume of the unit ball in $\mathbb{R}^N$, and $d(x)$ is the distance from x to $\partial\Omega$. This follows at once from the mean value property, and implies that the orthogonal projection P in $L^2(\Omega)$ on harmonic functions has a kernel K_Ω,

$$Pu(x) = \int_\Omega K_\Omega(x,y)u(y)\, dy, \quad u \in L^2(\Omega),$$

where K_Ω is harmonic in x and in y, $K_\Omega(x,y) = K_\Omega(y,x)$, so $K_\Omega \in C^\infty(\Omega \times \Omega)$. We have

$$\sup_{u\neq 0} |Pu(x)|^2/\|u\|^2 = \int_\Omega K_\Omega(x,y)^2 dy = \int_\Omega K_\Omega(x,y)K_\Omega(y,x)\, dy = K_\Omega(x,x),$$

and the supremum is attained when $u(y) = K_\Omega(x,y)$. It is easy to compute the reproducing kernel $K_\Omega(x,y)$ when Ω is the half space $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N; x_N > 0\}$; we shall denote it by K_+ instead of $K_{\mathbb{R}^N_+}$. To do so we observe using Fourier transforms in $x' = (x_1, \dots, x_{N-1})$ that if $u \in L^2(\mathbb{R}^N_+)$ and $\Delta u = 0$ then

$$u(x) = (2\pi)^{1-N} \int e^{i\langle x',\xi'\rangle - x_N|\xi'|} F(\xi')\, d\xi',$$

$$\int_\Omega |u(x)|^2\,dx = (2\pi)^{1-N}\int_{x_N>0} dx_N \int |F(\xi')|^2 e^{-2x_N|\xi'|}\,d\xi'$$
$$= (2\pi)^{1-N}\int |F(\xi')|^2\,d\xi'/2|\xi'|.$$

Hence by Cauchy-Schwarz

$$|u(x)|^2 \leqslant (2\pi)^{1-N}\int 2|\xi'|e^{-2x_N|\xi'|}\,d\xi'\|u\|^2;$$

if $x'=0$ there is equality only if $F(\xi')=C|\xi'|e^{-x_N|\xi'|}$. To calculate the inverse Fourier transform we note that

$$U(x)=(2\pi)^{1-N}\int e^{i\langle x',\xi'\rangle-|x_N||\xi'|}\,d\xi'/|\xi'|$$

is harmonic when $x_N\neq 0$ and a continuous function of x_N with distribution values while the limit of $\partial U/\partial x_N$ is $\mp\delta(x')$ when $x_N\to\pm 0$. Hence $\Delta U=-2\delta_0$, and since U is homogeneous we conclude that $-U/2$ is the standard fundamental solution,

$$U(x)=2|x|^{2-N}/((N-2)c_N),\quad \partial U(x)/\partial x_N=-2x_N|x|^{-N}/c_N,$$

where $c_N=2\pi^{N/2}/\Gamma(N/2)=NC_N$ is the area of the unit sphere in $\mathbb{R}^N$. This gives $\partial^2U(x)/\partial x_N^2=-2(|x|^{-N}-Nx_N^2|x|^{-N-2})/c_N$ when $x_N>0$, and in particular

$$K_+(x,x)=(2\pi)^{1-N}\int 2|\xi'|e^{-2x_N|\xi'|}\,d\xi'=4(N-1)(2x_N)^{-N}/c_N,\ x\in\mathbb{R}^N_+.$$

More generally, $K_+(x,y)=2\partial^2U(x'-y',x_N+y_N)/\partial x_N^2$, that is,

$$(C.1)\qquad K_+(x,y)=\frac{4}{c_N}\frac{(N-1)(x_N+y_N)^2-|x'-y'|^2}{(|x'-y'|^2+(x_N+y_N)^2)^{(N+2)/2}},\quad x,y\in\mathbb{R}^N_+.$$

It is now easy to prove a well known asymptotic result for more general open sets Ω:

PROPOSITION C.1. — *If $\Omega\subset\mathbb{R}^N$ is an open set with C^1 boundary and $d(x)$ denotes the distance from $x\in\Omega$ to $\partial\Omega$ then*

$$(C.2)\qquad d(x)^N K_\Omega(x,x)\to 2^{2-N}(N-1)/c_N,$$

when $x\in\Omega$ and $|x|$ is bounded while $d(x)\to 0$.

Proof. — Assuming at first that $0\in\partial\Omega$ and that $e_N=(0,\dots,0,1)$ is the inner unit normal at 0 we shall prove (C.2) when $x=\varepsilon e_N$ and $\varepsilon\downarrow 0$. With $u_\varepsilon(x)=\varepsilon^N K_\Omega(\varepsilon x,\varepsilon e_N)$ we have

$$\int_{\Omega/\varepsilon}|u_\varepsilon(x)|^2\,dx=\varepsilon^N\int_\Omega |K_\Omega(x,\varepsilon e_N)|^2\,dx=\varepsilon^N K_\Omega(\varepsilon e_N,\varepsilon e_N),$$

and $u_\varepsilon(e_N) = \varepsilon^N K_\Omega(\varepsilon e_N, \varepsilon e_N)$. Choose a sequence $\varepsilon_j \to 0$ such that

$$u_{\varepsilon_j}(e_N) \to \overline{\lim_{\varepsilon\to+0}}\, \varepsilon^N K_\Omega(\varepsilon e_N, \varepsilon e_N) = S$$

and $u_{\varepsilon_j}(x)$ converges in $\mathbb{R}^N_+$ to a harmonic function u with $u(e_N) = S$. By Fatou's lemma $\int_{\mathbb{R}^N_+} |u(x)|^2\,dx \leqslant S$, so it follows from (C.1) that $S^2 \leqslant 2^{2-N}(N-1)c_N^{-1}S$, that is, $S \leqslant 2^{2-N}(N-1)/c_N$. This proves (C.2) if Ω is convex, for $K_\Omega \geqslant K_+$ then since $\Omega \subset \mathbb{R}^N_+$. To prove (C.2) for a general Ω with a C^1 boundary we also have to give a lower bound. Let $v_\varepsilon(x) = K_+(x, \varepsilon e_N)$ which is well defined except when $x = -\varepsilon e_N$, which is not in Ω for small ε. Then

$$\varepsilon^N \int_\Omega |v_\varepsilon(x)|^2\,dx = \int_{\Omega/\varepsilon} K_+(x, e_N)^2\,dx \to \int_{\mathbb{R}^N_+} K_+(x, e_N)^2\,dx = K_+(e_N, e_N)$$

by dominated convergence, and $v_\varepsilon(\varepsilon e_N) = K_+(\varepsilon e_N, \varepsilon e_N) = \varepsilon^{-N}K_+(e_N, e_N)$. Since

$$\varepsilon^N K_\Omega(\varepsilon e_N, \varepsilon e_N) \geqslant \varepsilon^N v_\varepsilon(\varepsilon e_N)^2 \Big/ \int_\Omega |v_\varepsilon(x)|^2\,dx \to K_+(e_N, e_N),$$

the proof of (C.2)is completed when x approaches $\partial\Omega$ along the normal at a fixed point. A moment's reflection proves that the conclusion is uniform in that point as long as it belongs to a compact set, which completes the proof.

The proof of Proposition C.1 is easily adapted to the case of "harmonic" p forms, $0 < p < N$, that is, p forms u in $\Omega \subset \mathbb{R}^N$ such that $du = 0$ and $d^*u = 0$, where d is the exterior differential operator and d^* its formal adjoint with respect to the L^2 scalar product of forms. These equations imply that $-\Delta u = d^*du + dd^*u = 0$ so the components of u are harmonic and therefore reproduced by the kernel K_Ω above, but it projects on the much larger space of all forms with $\Delta u = 0$. However, we shall now study the kernel of the orthogonal projection $\mathcal{P}$ on the harmonic p forms $H_p(\Omega) \subset L^2_p(\Omega)$. It is clear that $\mathcal{P}$ has a C^∞ kernel $\mathcal{K}_\Omega(x,y)$, $x, y \in \Omega$, where $\mathcal{K}(x,y)$ is a linear map from p forms (at y) to p forms (at x), $\mathcal{K}_\Omega(x,y)^* = \mathcal{K}_\Omega(y,x)$, such that $\mathcal{K}_\Omega(x,y)\varphi$ for fixed $y \in \Omega$ and p form φ at y is in $H_p(\Omega)$. If φ is a p form at x and $0 \neq u \in L^2_p(\Omega)$, then

$$\begin{aligned}
&\sup \big|((\mathcal{P}u)(x), \varphi)\big|^2/\|u\|^2 = \sup \Big|\int (K_\Omega(x,y)u(y), \varphi)\,dy\Big|^2/\|u\|^2 \\
&= \sup \Big|\int (u(y), \mathcal{K}_\Omega(y,x)\varphi)\,dy\Big|^2/\|u\|^2 = \int \big|\mathcal{K}_\Omega(y,x)\varphi\big|^2\,dy \\
&= \int \big(\mathcal{K}_\Omega(y,x)\varphi, \mathcal{K}_\Omega(y,x)\varphi\big)\,dy = \int \big(\varphi, \mathcal{K}_\Omega(x,y)\mathcal{K}_\Omega(y,x)\varphi\big)\,dy \\
&= (\varphi, \mathcal{K}_\Omega(x,x)\varphi),
\end{aligned}$$

and when the supremum is attained then $u(y) = C\mathcal{K}_\Omega(y,x)\varphi$ for some constant C; we have $u(y) = \mathcal{K}_\Omega(y,x)\varphi$ if u is normalized so that $(u(x),\varphi) = (\mathcal{K}_\Omega(x,x)\varphi,\varphi)$.

Again we shall begin by looking at the case of the half space $\Omega = \mathbb{R}^N_+$, and we write $\mathcal{K}_+$ instead of $\mathcal{K}_\Omega$ then. First we shall determine the exponential solutions

$$u = e^{i\langle x,\xi\rangle} f$$

where f is a constant p form, $\xi' = (\xi_1,\dots,\xi_{N-1}) \in \mathbb{R}^{N-1}$ and $\xi_N = i|\xi'|$. Then du is the exterior product $i\langle\xi,dx\rangle \wedge u$, so $du = 0$ means that f is divisible by this one form, $f = \langle\xi,dx\rangle \wedge h$ where h is a $p-1$ form. Since $\xi_n \neq 0$ we can replace dx_n by $(\langle\xi,dx\rangle - \langle\xi',dx'\rangle)/\xi_n$ in h; then h becomes a form in dx' only since the first term drops out. The equation $d^*u = 0$ means that $\langle\xi,dx\rangle\lrcorner(\langle\xi,dx\rangle\wedge h) = 0$, and since $\langle\xi,\xi\rangle = 0$ and $\langle\xi,dx\rangle\lrcorner h = \langle\xi',dx'\rangle\lrcorner h$ this is equivalent to $\langle\xi,dx\rangle \wedge (\langle\xi',dx'\rangle\lrcorner h) = 0$. In particular $\xi_N dx_N \wedge (\langle\xi',dx'\rangle\lrcorner h) = 0$, so $\langle\xi',dx'\rangle\lrcorner h = 0$. Thus the most general exponential solution is

$$u = e^{i\langle x,\xi\rangle}\langle\xi,dx\rangle \wedge h_{\xi'}, \quad \xi_N = i|\xi'|,$$

where $h_{\xi'}$ is a $p-1$ form in the orthogonal space of ξ' in $\mathbb{R}^{N-1}$ lifted to $\mathbb{R}^N$. When $N = 2$ and $p = 1$ the equations for u are equivalent to the Cauchy-Riemann equations so the reproducing kernel can be identified with the reproducing kernel of holomorphic functions, for $u_1 - iu_2$ is a holomorphic function of $x_1 + ix_2$. We shall therefore assume that $N > 2$ in what follows.

For the harmonic p form

$$u(y) = (2\pi)^{1-N}\int e^{i\langle y',\xi'\rangle - y_N|\xi'|}(\langle\xi',dx'\rangle + i|\xi'|\,dx_N)\wedge h_{\xi'}\,d\xi'$$

we have

$$(u(x),\varphi) = (2\pi)^{1-N}\int e^{i\langle x',\xi'\rangle - x_N|\xi'|}\big((\langle\xi',dx'\rangle + i|\xi'|dx_N)\wedge h_{\xi'},\varphi\big)\,d\xi',$$

$$\|u\|^2 = (2\pi)^{1-N}\iint_{y_N>0} e^{-2y_N|\xi'|}2|\xi'|^2|h_{\xi'}|^2\,d\xi'\,dy_N = (2\pi)^{1-N}\int|\xi'||h_{\xi'}|^2\,d\xi'.$$

Here $h_{\xi'}$ is a $p-1$ form in the x' variables orthogonal to ξ'. To determine the supremum of $|(u(x),\varphi)|^2/\|u\|^2$ the first step is to maximize $\big((\langle\xi',dx'\rangle + i|\xi'|dx_N)\wedge h_{\xi'},\varphi\big)$ for fixed $|h_{\xi'}|$. Writing

$$\varphi = \varphi_0 + dx_N\wedge\varphi_N, \quad \varphi_0 = dx_N\lrcorner(dx_N\wedge\varphi),\ \varphi_N = dx_N\lrcorner\varphi,$$

we have

$$\big((\langle\xi',dx'\rangle + i|\xi'|dx_N)\wedge h_{\xi'},\varphi\big) = (h_{\xi'},\langle\xi',dx'\rangle\lrcorner\varphi_0) + i|\xi'|(h_{\xi'},\varphi_N),$$

and in the last term we use that

$$|\xi'|^2\varphi_N = \langle \xi', dx'\rangle \lrcorner (\langle \xi', dx'\rangle \wedge \varphi_N) + \langle \xi', dx'\rangle \wedge (\langle \xi', dx'\rangle \lrcorner \varphi_N)$$

where the second term is orthogonal to $h_{\xi'}$ since $\langle \xi', dx'\rangle \lrcorner h_{\xi'} = 0$. Hence

$$\big((\langle \xi', dx'\rangle + i|\xi'|dx_N) \wedge h_{\xi'}, \varphi\big) = (h_{\xi'}, \Phi_{\xi'}),$$

$$\Phi_{\xi'} = \langle \xi', dx'\rangle \lrcorner (\varphi_0 - i\langle \xi', dx'\rangle \wedge \varphi_N/|\xi'|),$$

and to maximize we must take $h_{\xi'} = C(\xi')\Phi_{\xi'}$, which is indeed annihilated by $\langle \xi', dx'\rangle \lrcorner$. Then we obtain, if $x' = 0$,

$$\|u\|^2 = (2\pi)^{1-N} \int |C(\xi')|^2 |\Phi_{\xi'}|^2 |\xi'|\, d\xi',$$

$$(u(x), \varphi) = (2\pi)^{1-N} \int e^{-x_N|\xi'|} C(\xi') |\Phi_{\xi'}|^2\, d\xi',$$

which gives that

$$(C.3) \qquad (\mathcal{K}_+(x,x)\varphi, \varphi) = (2\pi)^{1-N} \int e^{-2x_N|\xi'|} |\Phi_{\xi'}|^2 d\xi'/|\xi'|,$$

that $C(\xi') = e^{-x_N|\xi'|}/|\xi'|$ for the extremal function, and that

$$\mathcal{K}_+(y,x)\varphi = (2\pi)^{1-n} \int e^{i\langle y',\xi'\rangle - (x_N+y_N)|\xi'|} (\langle \xi', dx'\rangle + i|\xi'|dx_N) \wedge \Phi_{\xi'}\, d\xi'/|\xi'|.$$

What remains is to make this result explicit.

Multiplication by ξ_j on the Fourier transform side corresponds to $-i\partial/\partial y_j$ when $j < n$, and multiplication by $i|\xi'|$ corresponds to $-i\partial/\partial y_N$, so

$$\mathcal{K}_+(y,x)\varphi = -id_y(2\pi)^{1-N} \int e^{i\langle y',\xi'\rangle - (y_N+x_N)|\xi'|} \Phi_{\xi'}\, d\xi'/|\xi'|.$$

If $\varphi_N = 0$ this is equal to $-id_y(id_y^* U(y', y_N + x_N)\varphi_0)$ with the notation used in the proof of Proposition C.1. (Here we regard φ_0 as a form in dy with constant coefficients.) If $\varphi_0 = 0$ we note that

$$\begin{aligned} -i\langle \xi', dx'\rangle \lrcorner (\langle \xi', dx'\rangle \wedge \varphi_N) &= i\langle \xi', dx'\rangle \wedge (\langle \xi', dx'\rangle \lrcorner \varphi_N) - i|\xi'|^2\varphi_N \\ &= i\langle \xi, dx\rangle \wedge (\langle \xi', dx'\rangle \lrcorner \varphi_N) + |\xi'|dx_N \wedge (\langle \xi', dx'\rangle \lrcorner \varphi_N) - i|\xi'|^2\varphi_N. \end{aligned}$$

Since $d^2 = 0$ the first term on the right gives no contribution to $\mathcal{K}_+(y,x)\varphi$, and we obtain

$$\begin{aligned} \mathcal{K}_+(y,x)\varphi &= -id_y\big(dy_N \wedge (id_y^* U(y', y_N + x_N)\varphi_N + i\partial(U(y', y_N + x_N)\varphi_N)/\partial y_N\big) \\ &= -d_y d_y^* (U(y', y_N + x_N) dy_N \wedge \varphi_N), \end{aligned}$$

which differs only by the sign from the result for the other component of φ. Summing up, and exchanging the roles of x and y we obtain for general $x, y \in \mathbb{R}^N_+$ and p form φ at y

$$(C.4) \qquad \mathcal{K}_+(x,y)\varphi = d_x d_x^*\big(U(x'-y', x_N+y_N)(\varphi - 2dx_N \wedge (dx_N \lrcorner \varphi))\big).$$

When $x = y$ we can also calculate $(\mathcal{K}_+(x,x)\varphi, \varphi)$ directly using (C.3) and noting that the cross products between the two terms in $\Phi_{\xi'}$ will give no contributions since they are odd with respect to ξ', which gives

$$(\mathcal{K}_+(x,x)\varphi,\varphi) = (2\pi)^{1-N}\int e^{-2x_N|\xi'|}|\langle\xi', dx'\rangle\lrcorner\varphi_0|^2\, d\xi'/|\xi'|$$
$$+(2\pi)^{1-N}\int e^{-2x_N|\xi'|}|\langle\xi', dx'\rangle\lrcorner(\langle\xi', dx'\rangle\wedge\varphi_N)|^2\, d\xi'/|\xi'|^3.$$

If we expand $|\sum_1^{N-1}\xi_j dx_j\lrcorner\varphi_0|^2$ in the first integral on the right-hand side then the contribution from the cross products vanishes and since $\sum_1^{N-1}|dx_j\lrcorner\varphi_0|^2 = p|\varphi_0|^2$, the integral is equal to

$$(2\pi)^{1-N}\int e^{-2x_N|\xi'|}p|\varphi_0|^2|\xi'|\, d\xi'/(N-1).$$

To evaluate the second integral we observe that

$$|\langle\xi', dx'\rangle\lrcorner(\langle\xi', dx'\rangle\wedge\varphi_N)|^2 = |\xi'|^2|\langle\xi', dx'\rangle\wedge\varphi_N|^2$$
$$= |\xi'|^2(|\xi'|^2|\varphi_N|^2 - |\langle\xi', dx'\rangle\lrcorner\varphi_N|^2),$$

and since φ_N is a $p-1$ form it follows that the integral is equal to

$$(2\pi)^{1-N}\int e^{-2x_N|\xi'|}(N-p)|\varphi_N|^2|\xi'|\, d\xi'/(N-1),$$

hence

$$(C.5) \qquad (\mathcal{K}_+(x,x)\varphi,\varphi) = 2(2x_N)^{-N}(p|\varphi_0|^2 + (N-p)|\varphi_N|^2)/c_N.$$

The proof of Proposition C.1 allows us to carry (C.5) over to more general open sets Ω:

PROPOSITION C.2. — *If $\Omega \subset \mathbb{R}^N$ is an open set with C^1 boundary, $d(x)$ denotes the distance from $x \in \Omega$ to $\partial\Omega$, and $\mathcal{K}_\Omega$ is the kernel of the orthogonal projection in $L^2_p(\Omega)$ on the harmonic forms $H_p(\Omega)$, then*

$$d(x)^N(\mathcal{K}_\Omega(x,x)\varphi,\varphi) \to 2^{1-N}(p|\varphi_0|^2 + (N-p)|\varphi_\nu|^2)/c_N$$

when $x \in \Omega$ converges to a boundary point x_0 with unit normal ν, φ is a constant p form and $\varphi = \varphi_0 + \langle\nu, dx\rangle\wedge\varphi_\nu$ where $\varphi_\nu = \langle\nu, dx\rangle\lrcorner\varphi$. The convergence is uniform when x_0 is in a bounded set.

We shall now discuss a complex analogue where $\Omega \subset \mathbb{C}^n$ is an open set with smooth boundary and the object is to study the kernel of the orthogonal projection in $L^2_{0,q}(\Omega)$ on $H_{0,q} = \{u \in L^2_{0,q}(\Omega); \bar\partial u = 0, \vartheta u = 0\}$, with no boundary conditions. Here $0 < q < n$. This implies $\Delta u = 0$ so it is again clear that the orthogonal projection on $H_{0,q}$ has a C^∞ kernel. As in the real case we begin by examining the half space $\Omega = \mathbb{C}^n_+$ of $\mathbb{C}^n$ where $\operatorname{Im} z_n > 0$, with the Euclidean metric, and denote the kernel of the projection by $\mathcal{K}_+$. We write $z = x + iy$ and let ξ, η be the dual variables. Taking Fourier transforms in ξ and $\eta' = (\eta_1, \dots, \eta_{n-1})$ leads us to look at exponential solutions

$$u = e^{i(\langle x,\xi\rangle + \langle y,\eta\rangle)} f, \quad \xi \in \mathbb{R}^n,\ \eta' \in \mathbb{R}^{n-1}, \eta_n = i\Xi,\ \Xi = \sqrt{|\xi|^2 + |\eta'|^2},$$

where f is a constant form of type $(0,q)$. Since

$$2\partial/\partial z_j(\langle x,\xi\rangle + \langle y,\eta\rangle) = \xi_j - i\eta_j = \tilde\zeta_j,\ 2\partial/\partial \bar z_j(\langle x,\xi\rangle + \langle y,\eta\rangle) = \xi_j + i\eta_j = \zeta_j,$$

where $\tilde\zeta_j = \bar\zeta_j$ when $j < n$ but not when $j = n$, it is clear that $\bar\partial u = \frac{1}{2}\langle \zeta, d\bar z\rangle \wedge u$, so the equation $\bar\partial u = 0$ means that $f = \langle \zeta, d\bar z\rangle \wedge h_\zeta$ where h_ζ is of type $0, q-1$. From now on we assume that $q = 1$ which is a great simplification of the calculations but still allows us to show the essential difference from the real case. Then h is just a complex number and u automatically satisfies the equation $\vartheta u = 0$ since $\langle \zeta, \tilde\zeta\rangle = 0$, and we can write every $u \in H_{0,1}(\mathbb{C}^n_+)$ in the form

$$u(z) = (2\pi)^{1-2n} \iint e^{i(\langle x,\xi\rangle + \langle y',\eta'\rangle) - y_n\Xi} \langle \zeta, d\bar z\rangle h(\xi,\eta')\, d\xi\, d\eta',$$

$$\|u\|^2 = (2\pi)^{1-2n} \iiint_{y_n>0} e^{-2y_n\Xi} |\zeta|^2 |h|^2\, d\xi\, d\eta'\, dy_n$$
$$= (2\pi)^{1-2n} \iint |h|^2 |\zeta|^2/(2\Xi)\, d\xi\, d\eta'.$$

With $\varphi = \sum_1^n \varphi_j\, d\bar z_j$ denoting a $0,1$ form with constant coefficients and $Z = (0,\dots,0,it)$, $t > 0$, we want to determine $(\mathcal{K}_+(Z,Z)\varphi,\varphi) = \sup |(u(Z),\varphi)|^2/\|u\|^2$ where

$$(u(Z),\varphi) = (2\pi)^{1-2n} \iint e^{-t\Xi} (\zeta,\varphi) h\, d\xi\, d\eta'.$$

By Cauchy-Schwarz we obtain

$$(C.6) \qquad (\mathcal{K}_+(Z,Z)\varphi,\varphi) = \int e^{-2t\Xi} |(\zeta,\varphi)|^2 2\Xi |\zeta|^{-2}\, d\xi\, d\eta',$$

and since equality occurs when $h = e^{-t\Xi}(\varphi,\zeta) 2\Xi |\zeta|^{-2}$ it follows that
(C.7)

$$\mathcal{K}_+(z,Z)\varphi = (2\pi)^{1-2n} \int e^{i(\langle x,\xi\rangle + \langle y',\eta'\rangle) - (\operatorname{Im} z_n + t)\Xi} \langle \zeta, d\bar z\rangle (\varphi,\zeta) 2\Xi |\zeta|^{-2}\, d\xi\, d\eta'.$$

If we expand $|\sum \bar\zeta_j\varphi_j|^2$ in (C.6) the cross product terms give no contribution since they will be odd in at least one variable ζ_j with $j<n$. What remains is therefore to evaluate

$$I_j = (2\pi)^{1-2n}\iint e^{-T\Xi}|\zeta_j|^2|\zeta|^{-2}2\Xi\,d\xi\,d\eta', \quad j=1,\dots,n,$$

where $T=2t$. It is clear that $I_1=\dots=I_{n-1}$, and we have

$$I_1+\dots+I_n = (2\pi)^{1-2n}\iint e^{-T\Xi}2\Xi\,d\xi\,d\eta',$$

$$I_n = (2\pi)^{1-2n}\iint e^{-T\Xi}(\Xi-\xi_n)\,d\xi\,d\eta' = (2\pi)^{1-2n}\iint e^{-T\Xi}\Xi\,d\xi\,d\eta',$$

where we have used that $|\zeta|^2=\Xi^2-\xi_n^2+(\Xi-\xi_n)^2=2\Xi(\Xi-\xi_n)$. With $N=2n$ we now obtain using the earlier expressions for the derivatives of U that

$$I_1+\dots+I_n = 2\cdot 2(N-1)/(c_NT^N), \quad I_n=2(N-1)/(c_NT^N),$$

hence $I_1=\dots=I_{n-1}=2(N-1)/((n-1)c_NT^N)$, which gives

$$(\mathcal{K}_+(Z,Z)\varphi,\varphi) = 2c_N^{-1}(2t)^{-N}\Big(\sum_1^{n-1}|\varphi_j|^2+(n-1)|\varphi_n|^2\Big)(N-1)/(n-1),$$

or since $c_{2n}/2n=\pi^n/n!$

$$(C.8)\quad (\mathcal{K}_+(Z,Z)\varphi,\varphi)$$
$$= (n-2)!(2n-1)\pi^{-n}\Big(\sum_1^{n-1}|\varphi_j|^2+(n-1)|\varphi_n|^2\Big)/(2\operatorname{Im} Z_n)^{2n},\ Z\in\mathbb{C}^n_+.$$

The Fourier transform of the distribution limit of the harmonic function $\mathcal{K}_+(z,Z)$ when $\operatorname{Im} z_n\to 0$ has the components $\zeta_j\bar\zeta_k2\Xi|\zeta|^{-2}=\zeta_j\bar\zeta_k(\Xi-\xi_n)^{-1}$, equal to $-\zeta_j$ if $k=n$ and $j<n$, equal to $\Xi+\xi_n$ if $j=k=n$; if $j,k<n$ we can also write them in the form $\zeta_j\bar\zeta_k(\Xi+\xi_n)/(|\xi'|^2+|\eta'|^2)$. They are positively homogeneous of degree 1. When $j,k<n$ the wave front set outside the origin is the conormal bundle of the half line where $\xi'=\eta'=0$ and $\xi_n>0$. By [H3], Theorem 8.1.8 it follows that the wave front set of the boundary values of $\mathcal{K}_+(z,Z)$ outside the origin is equal to one half of the conormal bundle of the complex tangent plane defined by $x_n=0$, so the singular support is equal to that plane. Since $\mathcal{K}_\Omega(z,z)$ increases when Ω decreases the arguments in the first part of the proof of Proposition C.1 can still be applied to give an analogue of (C.8) for the asymptotic behavior of $\mathcal{K}_\Omega$ at the boundary when Ω is convex, but the fact that $\mathcal{K}_\Omega(z,Z)$ cannot be extended across the complex plane $\{z\in\mathbb{C}^n; z_n=0\}$ makes the argument used in the non-convex case break down. It seems plausible that the result remains true if $\partial\Omega \in C^2$ is strictly pseudoconvex, for it should then be possible to localize and change coordinates locally to make the boundary convex. However, we shall not pursue this argument since the main purpose of this appendix has been to emphasize how important the boundary condition for $\mathfrak{d}$ in the $\bar{\partial}$-Neumann problem is for the boundary behavior of the kernel of the orthogonal projection on the null space.

BIBLIOGRAPHY

[BS] H.P. Boas and M.-C. Shaw, Sobolev estimates for the Lewy operator on weakly pseudo-convex boundaries, Math. Ann., 274 (1986), 221–231.

[BMS] L. Boutet de Monvel and J. Sjöstrand, Sur la singularité des noyaux de Bergman et de Szegö, Astérisque, 34–35 (1976), 123–164.

[CM] S.-S. Chern and J.-K. Moser, Real hypersurfaces in complex manifolds, Acta Math., 133 (1974), 219–271.

[CS] S.-C. Chen and M.-C. Shaw, Partial differential equations in several complex variables, Amer. Math. Soc., (2001), AMS/IP Studies in advanced mathematics, Vol. 19.

[H1] L. Hörmander, L^2 estimates and existence theorems for the $\bar{\partial}$ operator, Acta Math., 113 (1965), 89–152.

[H2] L. Hörmander, The multinomial distribution and some Bergman kernels, Geometric analysis of PDE and several complex variables, Contemporary Mathematics Proceedings (to appear).

[H3] L. Hörmander, The analysis of linear partial differential operators I,III, Springer Verlag, Berlin, Heidelberg, ... 1983, 1985.

[K] J.-J. Kohn, Harmonic integrals on strongly pseudo-convex manifolds, I, Ann. of Math., 78 (1963), 112–148; II, Ibid., 79 (1964), 450–472.

[KN] J.-J. Kohn and L. Nirenberg, Non-coercive boundary problems, Comm. Pure Appl. Math., 18 (1965), 443–492.

[KS] J.-J. Kohn and D.C. Spencer, Complex Neumann problems, Ann. of Math., 66 (1957), 89–140.

[S1] M.-C. Shaw, Global solvability and regularity for $\bar{\partial}$ on an annulus between two weakly pseudo-convex domains, Trans. Amer. Math. Soc., 291 (1985), 255–267.

[S2] M.-C. Shaw, L^2 estimates and existence theorems for the tangential Cauchy-Riemann complex, Invent. Math., 82 (1985), 133–150.

Lars HÖRMANDER,
University of Lund
Department of Mathematics
Box 118
SE-221 00 Lund (Sweden).
lvh@maths.lth.se