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MAPPING CLASS GROUP AND THE CASSON INVARIANT

by Bernard PERRON

We use freely notations and results of [Pe].

0. Introduction.

0.1. — Let S_g (resp $S_{g,1}$) be a closed oriented surface (resp. with one boundary component) of genus g. Let $\mathcal{M}_g$ (resp. $\mathcal{M}_{g,1}$) denote the mapping class group of S_g (resp. $S_{g,1}$), that is the group of isotopy classes of homeomorphisms of S_g (resp. $S_{g,1}$; in this case we consider homeomorphisms equal to the identity on the boundary, the isotopies being also identity on $\partial S_{g,1}$). Since $S_{g,1}$ can be seen as a submanifold of S_g such that $S_g - \overset{\circ}{S}_{g,1} = D^2$, we have a natural (surjective) map $\mathcal{M}_{g,1} \to \mathcal{M}_g$ by extending a homeomorphism of $S_{g,1}$ by identity on the 2-disc D^2.

0.2. — Consider the standard embedding of S_g in $\mathbb{R}^3$ given by Figure 0.1 and let H_g denote the oriented handlebody of genus g bounded by S_g. Let $i_g : S_g \to S_g$ be a homeomorphism which exchanges x_i and y_i $(i = 1, 2, \ldots, g)$, where x_i and y_i are the oriented circles defined by Figure 0.1. One can take for i_g the composition $\rho_1 \circ \cdots \circ \rho_g$ where

$$\rho_i = D(x_i)D(y_i)D(x_i)$$

($D(x)$ is the Dehn twist along the circle x). Then $H_g \bigcup_{i_g} (-H_g)$ is homeomorphic to $\mathbb{S}^3$, $(-H_g)$ being the handlebody H_g with opposite orientation.

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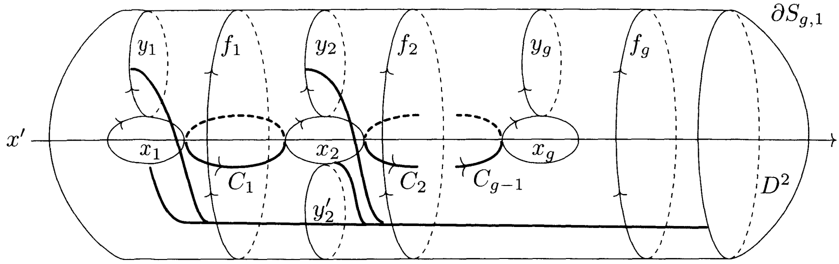

Figure 0.1

0.3. — For $f \in \mathcal{M}_{g,1}$ let $\bar{f}$ denote its extension (by identity) to S_g. Let $M_{\bar{f}}$ denote the 3-manifold obtained by gluing two copies of H_g by the homeomorphism $i_g \circ \bar{f}$. It is obvious that, if f induces the identity at the homological level (*e.g.* f belongs to the Torelli group $\mathcal{I}_{g,1}$ of $S_{g,1}$), then $M_{\bar{f}}$ is a $\mathbb{Z}$-homology sphere. Morita [Mo1] shows that any $\mathbb{Z}$-homology sphere is homeomorphic to $M_{\bar{f}}$ for some f belonging to $\mathcal{T}_{g,1} = \mathcal{M}(3) \subset \mathcal{I}_{g,1}$, where $\mathcal{M}(3)$ is defined in [J1] (see also [Pe], Lemma 3.4).

Let $\mathcal{N}_{g,1}$ (resp. $\mathcal{N}^t_{g,1}$) denote the subgroup of $\mathcal{M}_{g,1}$ consisting of homeomorphisms f of $S_{g,1}$ such that $\bar{f}$ extends to a homeomorphism of H_g (resp. $\mathbb{S}^3 - H_g$).

It is well-known that if $f,\ g \in \mathcal{M}_{g,1}$ are such that $f = \xi\, g\, \eta$, with $\xi \in \mathcal{N}^t_{g,1}$ and $\eta \in \mathcal{N}_{g,1}$, then the manifolds $M_{\bar{f}}$ and $M_{\bar{g}}$ are homeomorphic.

0.4. — Now, for any $\mathbb{Z}$-homology sphere Σ, Casson [C] (see also [GM]) defines an invariant belonging to $\mathbb{Z}$, denoted by $\lambda(M)$. This allows us to define a map $\lambda^* : \mathcal{I}_{g,1} \to \mathbb{Z}$ by setting

$$\lambda^*(f) = \lambda(M_{\bar{f}}).$$

0.5. — We want to express $\lambda^*(f)$ using Johnson's homomorphisms (see [Pe], Chap. 4). Recall from [Mo1], §1, or [Pe], 6.2, that T denotes the subgroup of $(\wedge^2 H)\otimes H\otimes H$ (where $H = H_1(S_{g,1};\mathbb{Z})$) generated by elements of the following form

$$(a\wedge b)^2 = a\wedge b\otimes a\wedge b \quad \text{and} \quad (a\wedge b) \leftrightarrow (c\wedge d)$$

where

$$(a\wedge b) \leftrightarrow (c\wedge d) = (a\wedge b)\otimes(c\wedge d) + (c\wedge d)\otimes(a\wedge b),$$
$$a\wedge b = a\otimes b - b\otimes a \quad \text{when} \quad a\wedge b \in H\otimes H.$$

Then Morita [Mo1], §4, defines a homomorphism $\theta_0 : T \to \mathbb{Z}$ by setting

$$\theta_0\big((a\wedge b)^2\big) = \ell(a,a)\,\ell(b,b) - \ell(a,b)\,\ell(b,a),$$

$$\theta_0(a\wedge b \leftrightarrow c\wedge d) = \ell(a,c)\,\ell(b,d)+\ell(c,a)\,\ell(d,b)-\ell(a,d)\,\ell(b,c)-\ell(d,a)\,\ell(c,b)$$

where

$$\ell(a,b) = \mathrm{link}(a,b^+)$$

is defined as follows. Let S_g be standardly embedded in $\mathbb{R}^3$ (Figure 0.1), ν a non singular normal vector field on S_g, pointing outside H_g. For $b \in H = H_1(S_g;\,\mathbb{Z})$, let b^+ be the 1-chain pushed out of S_g along ν. Then $\ell(a,b)$ is the linking number in $\mathbb{R}^3$ of a and b^+. It is easy to see that

$$\theta_0(a_i\wedge a_j \leftrightarrow b_i\wedge b_j) = 1 \quad (i\neq j)$$

and $\theta_0 = 0$ for the other basis elements of T.

0.6. — Recall the main result of [J3], Theorem 5: the subgroup $\mathcal{T}_{g,1} = \mathcal{M}(3) \subset \mathcal{I}_{g,1}$ is normally generated by the Dehn twists $D(f_1)$ and $D(f_2)$, where the circles f_1, f_2 are defined by Figure 0.1. So any element f of $\mathcal{T}_{g,1}$ can be written, up to order

$$\Big(\prod_{i=1}^{n} \varphi_i D(f_1)^{\varepsilon_i}\varphi_i^{-1}\Big)\prod_{j=1}^{m}\psi_j D(f_2)^{e_j}\psi_j^{-1} \quad (\varphi_i,\psi_j\in\mathcal{M}_{g,1}).$$

Our first main result is:

THEOREM 0.1. — *For $f\in\mathcal{T}_{g,1}$ we have*

$$\lambda^*(f) = -\frac{1}{12}\theta_0\big(\sigma\circ A_2'(f)\big) + \frac{1}{3}\sum_{j=1}^{m} e_j$$

where A_2' (resp. σ) is the map

$$A_2' : \mathcal{M}_{g,1} \xrightarrow{A_2} (\otimes^2 H)\otimes H\otimes H\otimes \xrightarrow{\pi\otimes \mathrm{id}} (\wedge^2 H)\otimes H\otimes H$$

defined in [Pe, (6.1)], *resp. $\sigma : (\wedge^2 H)\otimes H\otimes H \to T$ is the map defined by*

$$\sigma(a\wedge b\otimes c\otimes d) = a\wedge b \leftrightarrow c\wedge d$$

(*see* [Pe], (7.1)).

COROLLARY 0.2. — *The map $\delta : \mathcal{T}_{g,1}\to\mathbb{Z}$ defined by $\delta(f) = \sum_{i=1}^{m} e_j$ is a well-defined homomorphism, where up to order,*

$$f = \Big(\prod_{i=1}^{n}\varphi_i D(f_1)^{\varepsilon_i}\varphi_i^{-1}\Big)\Big(\prod_{j=1}^{m}\psi_j D(f_2)^{e_j}\psi_j^{-1}\Big).$$

0.7. *Remark.* — The above homomorphism $\delta:\mathcal{T}_{g,1}\to\mathbb{Z}$ has a more intrinsic definition. In fact, Morita [Mo1], §5, using Meyer's 2-cocycle (see [Me1]), defines a map d: $\mathcal{M}_{g,1}\to\mathbb{Z}$, such that, when restricted to $\mathcal{T}_{g,1}$, it becomes a homomorphism and such that if $\psi\in\mathcal{T}_{g,1}$ is a Dehn twist along a simple closed curve in $S_{g,1}$ bounding a surface of genus h, then $d(\psi)=4h(h-1)$.

It follows from the definition of δ, that $d_{|\mathcal{T}_{g,1}}=8\delta$. So Theorem 0.1 becomes:

$$\lambda^*(f)=-\frac{1}{12}\theta_0\big(\sigma\circ A_2'(f)\big)+\frac{1}{24}d(f)\quad\text{for } f\in\mathcal{T}_{g,1}.$$

0.8. *Remark.* — Formula above is a rephrasing (in a simpler way) of Morita's formula [Mo1], Theorem 6.1:

$$\lambda^*(f)=\Big(\theta_0+\frac{1}{3}\bar{d}\Big)\big(\tau_3(f)\big)+\frac{1}{24}d(f)\quad\text{for } f\in\mathcal{T}_{g,1}$$

(here $\tau_3(f)\in\overline{T}$ is the third Johnson's homomorphism, and $\overline{T}$ a certain quotient of T).

0.9. — Next we want to compute $\lambda^*(f)$ for any $f\in\mathcal{I}_{g,1}$, so extending the formula of Theorem 0.1 (or equivalently Morita's formula (0.8)). For any $f\in\mathcal{M}_{g,1}$, set

$$\Delta(f)=-\frac{1}{12}\theta_0\big(\sigma\circ A_2'(f)\big)+\frac{1}{24}d(f).$$

For $f\in\mathcal{I}_{g,1}$, defined in [Pe], Corollary 4.5, an element $A_1(f)\in\widetilde{\wedge^3 H}\subset\otimes^3H$, where $\widetilde{\wedge^3 H}$ is the injective image of the homomorphism $\wedge^3H\to\otimes^3H$ given by $x_1\wedge x_2\wedge x_3\mapsto\sum_{\sigma\in\mathcal{G}_3}\varepsilon(\sigma)\,x_{\sigma(1)}\otimes x_{\sigma(2)}\otimes x_{\sigma(3)}$, where $\mathcal{G}_3$ is the group of permutations of the set $\{1,2,3\}$. Moreover $A_1(f)=-\tau_2(f)$ where τ_2 is the second Johnson's homomorphism. Write for $f\in\mathcal{I}_{g,1}$:

$$A_1(f)=\sum_{1\le i<j<k\le g}\alpha^f_{ijk}a_i\wedge a_j\wedge a_k+\sum_{1\le i<j<k\le g}\beta^f_{ijk}b_i\wedge b_j\wedge b_k+R^f,$$

where $(a_i,b_i,\ i=1,\dots,g)$ is the symplectic basis of $H=H_1(S_{g,1};\mathbb{Z})$, respectively equal to the homology class of the oriented circles x_i, y_i of Figure 0.1, and R^f is a sum of terms of the form $a\wedge b\wedge b$ and $a\wedge a\wedge b$. This basis verifies $a_i\cdot a_j=b_i\cdot b_j=0$, $a_i\cdot b_j=-b_j\cdot a_i=\delta_{ij}$ (the Kronecker symbol). Then we have the following:

THEOREM 0.3. — *Let* $f \in \mathcal{I}_{g,1}$, *with* $A_1(f) \in \widetilde{\wedge^3 H} \subset \otimes^3 H$ *written as above. Then*

$$\lambda^*(f) = \Delta(f) + \sum_{1 \le i<j<k \le g} \alpha_{ijk}^f \beta_{ijk}^f .$$

The proof of Theorem 0.3 will use the main result of [Mo2], Theorem 4.3:

THEOREM 0.4. — *For* $f, g \in \mathcal{I}_{g,1}$ *we have*

$$\lambda^*(fg) = \lambda^*(f) + \lambda^*(g) + 2 \sum_{1 \le i<j<k \le g} \beta_{ijk}^f \alpha_{ijk}^g .$$

(Be aware that in [Mo2], the role of β_{ijk} and α_{ijk} have been interchanged.)

0.10. — Finally we want to restrict our attention to a special subgroup of $\mathcal{M}_{g,1}$, the hyperelliptic mapping class group, denoted by $\mathcal{H}_{g,1}$. This is the subgroup of $\mathcal{M}_{g,1}$ generated by the Dehn twists along the circles $x_1, \ldots, x_g, y_1, C_1, , \ldots, C_{g-1}$ defined by Figure 0.1. Remark that these circles are invariant by the symmetry s_g along the axis $x'x$ of Figure 0.1.

In [PV], it is proved that $\mathcal{H}_{g,1}$ is isomorphic to the usual braid group B_{2g+1}. The isomorphism can be described as follows.

Let $\{\sigma_i\,;\ i = 1, \ldots, 2g\}$ be the canonical generators of B_{2g+1}: send σ_{2i} on $D(x_i)$ $(i \le i \le g)$, σ_1 on $D(y_1)$ and σ_{2i+1} on $D(C_i)$ $(1 \le i \le g-1)$.

Moreover a homeomorphism $f \in \mathcal{M}_{g,1}$ belongs to $\mathcal{H}_{g,1}$ if and only if f commutes (up to isotopy) with the symmetry s_g.

LEMMA 0.5. — *The second Johnson's homomorphism*

$$\tau_2 = -\frac{1}{6} \widetilde{A}_{1 | \mathcal{H}_{g,1} \cap \mathcal{I}_{g,1}} : \mathcal{H}_{g,1} \cap \mathcal{I}_{g,1} \longrightarrow \wedge^3 H$$

is zero. So $\mathcal{H}_{g,1} \cap \mathcal{I}_{g,1} = \mathcal{H}_{g,1} \cap \mathcal{T}_{g,1}$ *(see* [J1] *for the definition of* τ_2 *and* [Pe], (4.6), *for the definition of* $\widetilde{A}_1$*).*

Remark. — $\mathcal{H}_{g,1} \cap \mathcal{I}_{g,1}$, when identified to $B_{2g+1} \cap \mathcal{I}_{g,1}$, is the kernel of the reduced Burau representation (see [B], § 3.3) when evaluated at $t = -1$.

0.11. — Finally, we have a simple geometric interpretation of the mapping $d: \mathcal{M}_{g,1} \to \mathbb{Z}$, when restricted to $\mathcal{H}_{g,1}$. As we have seen above, d is the core of Casson's invariant, and quoting Morita, d is a rather mysterious invariant. Morita [Mo6] gives a geometric interpretation of $d(\varphi)$ in terms of Hirzebruch's signature defect of the mapping torus of φ, with respect to a certain canonical framing of it tangent bundle. But it seems that this interpretation does not help for computations. Pitsch [Pi] gives a purely cohomological construction of the mapping d.

Our last result gives a very simple geometric interpretation of d, when restricted to $\mathcal{H}_{g,1}$, using a nice formula of Gambaudo-Ghys [GG].

PROPOSITION 0.6. — *For $f \in \mathcal{H}_{g,1} \simeq B_{2g+1}$, we have the following formula*

$$d(f) = 3s(\widehat{f}\,) + u(f) + 2\pi(f)$$

where:

1) $\widehat{f}$ is the link in $\mathbb{R}^3$ obtained by closing the braid f, and s is the classical signature of a link.

2) $u(f) = B_0(f)(\delta_g) \cdot \delta_g$, where $(\cdot)$ is the symplectic intersection form on $H = H_1(S_{g,1};\mathbb{Z})$, $B_0(f)$ is the isomorphism induced by f at the homological level and $\delta_g = (g-1)a_g + (g-2)a_{g-1} + \cdots + a_2 \in H$, where a_j is the homology class of the circle x_j of Figure 0.1.

3) $\pi: B_{2g+1} \to \mathbb{Z}$ is the abelianization homomorphism sending each generator $\sigma_i (i = 1, \ldots, 2g)$ on $1 \in \mathbb{Z}$.

THEOREM 0.7. — *One has $d(D(x_i)) = d(D(y_i)) = 2$ for $i = 1, \ldots, g$ and $d(D(C_i)) = 3$ for $i = 1, \ldots, g-1$, where x_i, y_i, C_i are the circles defined by Figure 0.1.*

Notation. — In the remainder of this paper, for $f \in \mathcal{M}_{g,1}$, we will denote by f_* the isomorphism of $H = H_1(S_{g,1}; \mathbb{Z})$ induced by f. We used the notation $B_0(f)$ instead of f_* in [Pe].

1. The mapping $d: \mathcal{M}_{g,1} \to \mathbb{Z}$.

1.1. — We use the notations of [Pe], Chapter 3. For f in $\mathcal{M}_{g,1}$, let $B(f)$ denote the Fox matrix of f (Definition 3.1 of [Pe]). This belongs to $\mathrm{GL}_{2g}(\mathbb{Z}[\Gamma])$ where $\Gamma = \pi_1(S_{g,1}, *)$. Applying the abelianization homomorphism $\Gamma \to H$, we get a matrix $B(f)^{\mathrm{ab}} \in \mathrm{GL}_{2g}(\mathbb{Z}[H])$. We set

$$\widetilde{k}(f) = \det\big[B(f)^{\mathrm{ab}}\big].$$

LEMMA 1.1. — *$\widetilde{k}$ is a crossed homomorphism, that is satisfies*

$$\widetilde{k}(fg) = \widetilde{k}(f) \times f_* \cdot \widetilde{k}(g) \in \mathbb{Z}[H],$$

where $f, g \in \mathcal{M}_{g,1}$ and $\times$ is the operation in $\mathbb{Z}[H]$ induced by the law in H.

Proof. — From Lemma 3.2 of [Pe], we see that

$$B(fg)^{\mathrm{ab}} = B(f)^{\mathrm{ab}} \times {}^{f_*}\big[B(g)^{\mathrm{ab}}\big],$$

where ${}^{f_*}(a_{ij}) = (f_*(a_{ij}))$. Lemma 1.1 follows. □

Recall that we have defined in (0.9) an embedding $\wedge^3 H \xrightarrow{i} \otimes^3 H$, whose image is denoted by $\widetilde{\wedge^3 H}$. We also have the canonical projection $\pi\colon \otimes^3 H \to \wedge^3 H$. It is obvious that $\pi \circ i = 6\,\mathrm{id}(\wedge^3 H)$. On $\wedge^3 H$ we define the *contraction map* $C\colon \wedge^3 H \to H$ by the formula

$$C(a \wedge b \wedge c) = 2\big[(b \cdot c)a + (c \cdot a)b + (a \cdot b)c\big].$$

In [Pe], Chapter 4, we have defined a map $A_1\colon \mathcal{M}_{g,1} \to \otimes^3 H$ such that $A_{1|\mathcal{I}_{g,1}}$ is a homomorphism whose image is $\widetilde{\wedge^3 H}$. Also, in [Pe], (4.6), we have set $\widetilde{A_1} = \pi \circ A_1$. The maps A_1, $\widetilde{A_1}$ satisfy the following property (crossed product, see Lemma 4.1 of [Pe]):

$$A_1(fg) = A_1(f) + f_* A_1(g), \quad \widetilde{A_1}(fg) = \widetilde{A_1}(f) + f_* \widetilde{A_1}(g).$$

We then have:

LEMMA 1.2. — *One has:*

(a) *For $f \in \mathcal{M}_{g,1}$, $\widetilde{k}(f)$ belongs to H (a priori, it belongs to $\mathbb{Z}[H]$).*

(b) *For $f \in \mathcal{I}_{g,1}$, $\widetilde{k}(f) = C(A_1(f))$ $(A_1(f) \in \widetilde{\wedge^3 H})$.*

(c) *For $f \in \mathcal{M}_{g,1}$, $\widetilde{k}(f) = \frac{1}{6} C(\widetilde{A_1}(f))$.*

Proof. — By Lemma 1.1, $\widetilde{k}(f)$ is a unit of $\mathbb{Z}[H]$, sent by the augmentation homomorphism $\varepsilon\colon \mathbb{Z}[H] \to \mathbb{Z}$ onto det $(B_0(f)) = 1$. So $\widetilde{k}(f)$ belongs to H, proving (a). Point (b) is proved in Proposition 6.15 of [Mo5] (remark that $A_1 = -\tau_2$, where τ_2 is the second Johnson's homomorphism, Proposition 4.4 of [Pe]).

1.2. — To prove point (c) of Lemma 1.1, using computations in the proof of Proposition 5.1 of [Pe] (or the proof of Proposition 6.15 of [Mo5]), we can verify that $\widetilde{k}(D(x_i)) = \widetilde{k}(D(y_i)) = 0$ and $\widetilde{k}(D(c_i)) = b_{i+1} - b_i$, using additive notations for H. On the other hand, again using computations in the proof of Proposition 5.1 of [Pe], we have

$$\widetilde{A_1}\big(D(x_i)\big) = \widetilde{A_1}\big(D(y_i)\big) = 0, \quad \widetilde{A_1}\big(D(c_i)\big) = -3(a_i + a_{i+1}) \wedge b_{i+1} \wedge b_i.$$

By the definition of C, point (c) is true for $D(x_i), D(y_i), D(C_i)$. Since these Dehn twists generate $\mathcal{M}_{g,1}$, and since $\widetilde{k}$ and $\frac{1}{6}C \circ \widetilde{A_1}$ are both crossed products (that is satisfy $\varphi(fg) = \varphi(f) + f_*\varphi(g)$), point (c) follows. □

Remark. — In the remainder of this paper, considering Lemma 1.2 (a), formula of Lemma 1.1 will be written $\widetilde{k}(fg) = \widetilde{k}(f) + f_*\widetilde{k}(g)$ where $+$ is the law in H.

1.3. — Now we can define a 2-cocycle on $\mathcal{M}_{g,1}$ with values in $\mathbb{Z}$ (the action of $\mathcal{M}_{g,1}$ on $\mathbb{Z}$ being trivial):

$$c(f,g) = \widetilde{k}(f^{-1}) \cdot \widetilde{k}(g) = -f_*^{-1} \cdot \big(\widetilde{k}(f)\big) \cdot \widetilde{k}(g) = -\widetilde{k}(f) \cdot f_*\big(\widetilde{k}(g)\big)$$

where $(\, . \,)$ is the symplectic form on H.

Remark that this 2-cocycle coincides with the 2-cocycle of Morita [Mo1], § 5, since $\widetilde{k}(f) = k(f^{-1})$ by definition.

1.4. — We now come to Meyer 2-cocycle on the symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$ (see [Me1] or [Me2]). For a pair of symplectic matrices $A, B \in \mathrm{Sp}(2g, \mathbb{Z})$, define a $\mathbb{R}$-vector space $V_{A,B}$ by

$$V_{A,B} = \big\{(x,y) \in \mathbb{R}^{2g} \times \mathbb{R}^{2g} \, ; \; (A^{-1} - I)x + (B - I)y = 0\big\}.$$

Consider the (possibly degenerated) symmetric bilinear form:

$$\langle \, , \rangle_{A,B} : V_{A,B} \times V_{A,B} \longrightarrow \mathbb{R}$$

given by $\langle (x_1,y_1),(x_2,y_2)\rangle = (x_1 + y_1) \cdot (I - B) y_2$ where $(\, . \,)$ is the symplectic form, whose matrix is $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Then we set:

$$\tau(A,B) = \mathrm{signature}\big(V_{A,B}; \langle \, , \rangle_{A,B}\big).$$

LEMMA 1.3 (see [Me1], [Me2]). — *The signature 2-cocycle satisfies the following properties:*

1) $\tau(A,B)+\tau(AB,C)=\tau(A,BC)+\tau(B,C)$ (*2-cocycle property*),

2) $\tau(A,I)=\tau(A,A^{-1})=0$,

3) $\tau(A,B)=\tau(B,A)$,

4) $\tau(A^{-1},B^{-1})=-\tau(A,B)$,

5) $\tau(CAC^{-1},CBC^{-1})=\tau(A,B)$.

This defines a 2-cocycle on $\mathcal{M}_{g,1}$ via the representation

$$B_0:\mathcal{M}_{g,1}\longrightarrow \mathrm{Sp}(2g,\mathbb{Z}).$$

The remarkable fact, noted in [Mo1], § 5, is that the 2-cocycle $c+3\tau$ on $\mathcal{M}_{g,1}$ is in fact a coboundary. So there exists a 1-cochain $d:\mathcal{M}_{g,1}\to\mathbb{Z}$ (necessarily unique, since $\mathcal{M}_{g,1}$ is perfect for $g\geq 3$ by [Po]) such that $\delta d=c+3\tau$.

1.5. — The mapping d satisfies the following properties.

PROPOSITION 1.4 (see [Mo1], Proposition 5.1). — *For any $f,g\in\mathcal{M}_{g,1}$ we have:*

(i) $d(fg)=d(f)+d(g)+\widetilde{k}(f)\cdot f_*(\widetilde{k}(g))-3\tau(f_*,g_*)$,

(ii) $d(f^{-1})=-d(f)$,

(iii) $d(fgf^{-1})=d(g)+\widetilde{k}(f)\cdot\big[f_*g_*^{-1}(\widetilde{k}(g))+f_*(\widetilde{k}(g))-f_*g_*f_*^{-1}(\widetilde{k}(f))\big]$.

Having in mind that $\widetilde{k}(\alpha)=k(\alpha^{-1})$ this is exactly Proposition 5.1 of [Mo1].

PROPOSITION 1.5 (see [Mo1], Proposition 5.3). — *Let $\mathcal{T}_{g,1}=\mathcal{M}(3)$ be the subgroup of $\mathcal{M}_{g,1}$ generated by all Dehn twists on bounding simple closed curves. Then the mapping $d_{|\mathcal{T}_{g,1}}:\mathcal{T}_{g,1}\to\mathbb{Z}$ is a homomorphism. Moreover if $f\in\mathcal{T}_{g,1}$ is a Dehn twist on a bounding simple closed curve of genus h, then $d(f)=4h(h-1)$.*

1.6. — In Chapter 3, we will need the following result.

LEMMA 1.6. — 1) *Let u be the simple closed curve given by Figure* 1.1. *Then $d(D(u))=d(D(x_2))+4$, where x_2 is the curve defined by Figure* 0.1.

2) *One has $d(D(y_2'))=d(D(y_2))+4$, where y_2,y_2' are curves defined by Figure* 0.1.

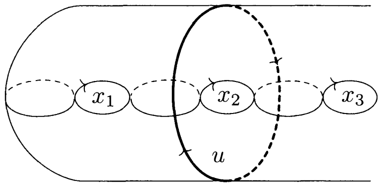

Figure 1.1

Proof of 1). — Recall that $\{a_i, b_i\,;\ i=1,\ldots,g\}$ is the symplectic basis of $H = H_1(S_{g,1};\mathbb{Z})$ defined in 0.9, verifying $a_i\cdot a_j = b_i\cdot b_j = 0$, $a_i\cdot b_j = -b_j\cdot a_i = \delta_{ij}$ (the Kronecker symbol).

It is easy to see that

$$u = D(y_2')\circ D(y_2)^{-1}[x_2].$$

Setting $\delta = D(y_2)\circ D(y_2')^{-1}$, we then have $D(u) = \delta^{-1}\circ D(x_2)\circ\delta$. By (1.2), $\widetilde{k}(D(x_2)) = 0$. By Proposition 1.4 (iii), we get

$$d\big(D(u)\big) = d\big(D(x_2)\big) - \widetilde{k}(\delta^{-1})\cdot D(u)_*\widetilde{k}(\delta^{-1}).$$

Since $\delta\in\mathcal{I}_{g,1}$, by Lemma 1.1, Lemma 1.2 (b) and Lemma 4B of [J1], we have

$$\widetilde{k}(\delta^{-1}) = -\widetilde{k}(\delta) = C(a_1\wedge b_1\wedge b_2) = 2b_2.$$

So $d(D(u)) = d(D(x_2)) - 4b_2\cdot u_*(b_2)$. But $u_*(b_2) = [u] + b_2 = a_2 + b_2$ (where $[u]$ denotes the homology class of u) and the result follows.

1.7. *Proof of* 2). — Let s_g be the symmetry of $S_{g,1}$ along the axis $x'x$ (Figure 0.1). Let $S'_{g,1}$ be the surface obtained from $S_{g,1}$ by adding the collar $\partial S_{g,1}\times[0,1]$ along $\partial S_{g,1}\times\{0\}$. Extend the map $s_g : S_{g,1}\to S_{g,1}$ by the map $S:\partial S_{g,1}\times[0,1]\to\partial S_{g,1}\times[0,1]$ defined by $S(\mathrm{e}^{i\theta},t) = (\mathrm{e}^{i\theta+\pi(1-t)},t)$, for $\mathrm{e}^{i\theta}\in S^1\simeq\partial S_{g,1}$ and $t\in[0,1]$. Then the map $s_g\coprod S$ represents an element of $\mathcal{M}_{g,1}$, denoted Δ_g^2.

It is well known that Δ_g^2 can be expressed as the composition

$$\big[D(y_1)D(x_1)D(C_1)D(x_2)\cdots D(C_{2g-1})D(x_g)\big]^{2g+1}$$

(see Figure 0.1 for the definition of y_i, x_i, C_j).

The reason for the notation Δ_g^2 is that Δ_g^2 is the square of a homeomorphism $\Delta_g \in \mathcal{M}_{g,1}$ which will be used later.

Since $y'_2 = \Delta_g^2(y_2)$ we have $D(y'_2) = \Delta_g^2\ D(y_2)\ (\Delta_g^2)^{-1}$. Using again (1.2) and Proposition 1.4 (iii) we obtain

$$d\big(D(y'_2)\big) = d\big(D(y_2)\big) - \widetilde{k}(\Delta_g^2) \cdot D(y'_2)_*\big(\widetilde{k}(\Delta_g^2)\big).$$

We will see that $\widetilde{k}(\Delta_g^2) = 2[(g-1)a_g + (g-2)a_{g-1} + \cdots + a_2]$ in Chapter 4, §4.2. The Dehn twist $D(y'_2)$ act non trivially only on a_2 by

$$D(y'_2)_*(a_2) = a_2 + [y'_2] = a_2 - b_2.$$

So $D(y'_2)_*(\widetilde{k}(\Delta_g^2)) = \widetilde{k}(\Delta_g^2) - 2b_2$. Lemma 1.6, 2) follows. □

2. Proof of Theorem 0.1.

Theorem 0.1 depends on two results of Morita.

PROPOSITION 2.1 (see [Mo1], Proposition 3.5). — *The mapping*

$$\lambda^*/\mathcal{T}_{g,1} : \mathcal{T}_{g,1} \longrightarrow \mathbb{Z}$$

defined in 0.4 *is a homomorphism.*

PROPOSITION 2.2 (see [Mo1], Proposition 4.5). — *Let $\psi \in \mathcal{T}_{g,1}$ be a Dehn twist along a bounding simple closed curve γ of genus h of $S_{g,1}$. Let $(u_1, \ldots, u_h; v_1, \ldots v_h)$ be a symplectic basis of the homology of the compact surface bounded by γ. Then*

$$\lambda^*(\psi) = -\theta_0\Big(\Big(\sum_{i=1}^{h} u_i \wedge v_i\Big)^2\Big),$$

where $\left(\sum_{i=1}^{h} u_i \wedge v_i\right)^2$ is seen in T (see 0.3) *and θ_0 has been defined in* 0.5.

2.1. — Let f_h be the simple closed curve of genus h, given by Figure 0.1. By a fundamental result of Johnson [J3], Theorem 5, any element f of $\mathcal{T}_{g,1}$ can be written, up to order, as

$$\Big(\prod_{i=1}^{n} \varphi_i D(f_1)^{\varepsilon_i} \varphi_i^{-1}\Big)\Big(\prod_{j=1}^{m} \psi_j D(f_2)^{e_j} \psi_j^{-1}\Big), \quad \varphi_i, \psi_j \in \mathcal{M}_{g,1}.$$

By Corollary 4.3 and Lemma 6.2 of [Pe] we have

$$A_2'\big(\varphi_i D(f_1)^{\varepsilon_i}\varphi_i^{-1}\big) = \varepsilon_i\varphi_{i*}\big(A_2'\big(D(f_1)\big)\big) = 3\varepsilon_i\varphi_{i*}\big[(a_1\wedge b_1)^2\big] \in T,$$
$$A_2'\big(\psi_j D(f_2)^{e_j}\psi_j^{-1}\big) = e_j\big[3\psi_{j*}\big((a_1\wedge b_1 + a_2\wedge b_2)^2\big) - \psi_{j*}(s_1)\big] \in T,$$

where s_1 is the following element of T (see [Pe], (6.2))

$$s_1 = (a_1\wedge b_1) \leftrightarrow (a_2\wedge b_2) - (a_1\wedge a_2) \leftrightarrow (b_1\wedge b_2) + (a_1\wedge b_2) \leftrightarrow (b_1\wedge a_2).$$

Using Propositions 2.1 and 2.2 we obtain

$$\lambda^*(f) = -\frac{1}{3}\theta_0\big(A_2'(f)\big) - \frac{1}{3}\sum_{j=1}^{m} e_j\theta_0\big(\psi_{j*}(s_1)\big).$$

We claim that $\theta_0(\psi_{j*}(s_1)) = -1$. To see this, set $a_i' = \psi_{j*}(a_i)$ and $b_i' = \psi_{j*}(b_i)$. By the definition of θ_0 given in 0.5 and the well-known formula $\ell(u,v) - \ell(v,u) = -u\cdot v$ we find

$$\theta_0\big(\psi_{j*}(s_1)\big) = \ell(a_1',b_1')(a_2'\cdot b_2') - \ell(b_1',a_1')(a_2'\cdot b_2').$$

Since the symplectic form (.) is invariant by elements of $\mathcal{M}_{g,1}$, it follows that

$$\theta_0\big(\psi_{j*}(s_1)\big) = (b_1'\cdot a_1') \times (a_2', b_2') = (b_1\cdot a_1)(a_2\cdot b_2) = -1.$$

Now recall that we have defined in [Pe], (7.1), a homomorphism $\sigma : (\wedge^2 H)\otimes H\otimes H \to T$ by setting $\sigma((a\wedge b)\otimes c\otimes d) = (a\wedge b) \leftrightarrow (c\wedge d) \in T$. When restricted to T, $\sigma_{|T}$ is $4\,\mathrm{id}_T$. So we have proved Theorem 0.1.

Corollary 0.2 is obvious, since $\lambda^*(f)$ and $A_2'(f)$ do not depend on a particular writing of f as a product (up to order) $\big(\prod_i \varphi_i D(f_1)^{\varepsilon_i}\varphi_i^{-1}\big)\cdot \prod_j\big(\psi_j D(f_2)^{e_j}\psi_j^{-1}\big)$.

As observed in 0.7, the homomorphism $\delta : \mathcal{T}_{g,1} \to \mathbb{Z}$ defined by $\delta(f) = \sum_{j=1}^m e_j$ is the restriction of $\frac{1}{8}d$, where d is the map defined in Chapter 1 (use Proposition 1.5).

Corollary 2.3. — *For $f\in\mathcal{T}_{g,1}$ we have*

$$\lambda^*(f) = -\frac{1}{12}\theta_0\big(\sigma\circ A_2'(f)\big) + \frac{1}{24}d(f).$$

2.2. *Remark.* — This is a reformulation of a formula of Morita [Mo1], Theorem 6.1, put in a simpler way. Morita's formula is

$$\lambda^*(f) = \Big(\theta_0 + \frac{1}{3}\bar{d}\Big)\big(\tau_3(f)\big) + \frac{1}{24}d(f),$$

where $\tau_3 : \mathcal{T}_{g,1} \to \overline{T} = T/_{T_0}$ is the third Johnson homomorphism. Here T_0 is the subgroup of T generated by elements of the form $(u \wedge v) \leftrightarrow (w \wedge t) - (u \wedge w) \leftrightarrow (v \wedge t) + (u \wedge t) \leftrightarrow (v \wedge w)$ (remark that $s_1 \in T_0$). Since the homomorphism $\theta_0 : T \to \mathbb{Z}$ does not factor through $\overline{T}$, Morita has to correct θ_0 by a homomorphism $\bar{d} : T \to \mathbb{Z}$ such that $\theta_0 + \frac{1}{3}\bar{d}$ factors through $\overline{T}$. The main advantage of the method of [Pe] is that we have an invariant $\sigma \circ A_2'(f)$ at the T level, and so a unified formula.

Of course we have

$$\Big(\theta_0 + \frac{1}{3}\bar{d}\Big)\big(\tau_3(f)\big) = -\frac{1}{12}\theta_0\big[\sigma \circ A_2'(f)\big]$$

since we have proved in [Pe], Lemma 7.1, that $p \circ \sigma \circ A_2' = -12\tau_3$, where $p : T \to \overline{T} = T/T_0$ is the canonical projection.

3. Proof of Theorem 0.3.

3.1. — For f belonging to $\mathcal{I}_{g,1}$, set

$$G(f) = \lambda^*(f) - \Delta(f) - \sum_{1 \le i < j < k \le g} \alpha^f_{ijk}\,\beta^f_{ijk} \in \mathbb{Q}$$

(the difference between the first and second member of the desired equality of Theorem 0.3). Thus, we have to show that $G = 0$ on $\mathcal{I}_{g,1}$. We will first prove that $G : \mathcal{I}_{g,1} \to \mathbb{Q}$ is a homomorphism. For this we need some computations.

3.2. — Let $\mathcal{W}_a$, $\mathcal{W}_{ab}$, $\mathcal{W}_b$ denote the subgroups of $\widetilde{\wedge^3 H}$ ($\simeq \wedge^3 H$, see 0.9) generated respectively by $\{a_i \wedge a_j \wedge a_k\}$ (a only), $\{c \wedge a_i \wedge b_j\}$ (at least one a and one b), $\{b_i \wedge b_j \wedge b_k\}$ (b only). Of course we have a decomposition

$$\widetilde{\wedge^3 H} = \mathcal{W}_a \oplus \mathcal{W}_{ab} \oplus \mathcal{W}_b$$

Notation. — Set, for $f \in \mathcal{I}_{g,1}$, $f_1 = A_1(f) \in \widetilde{\wedge^3 H}$ (see [Pe], Corollary 4.5) and decompose f_1 as $f_{1a} + f_{1ab} + f_{1b}$, where $f_{1a} \in \mathcal{W}_a$, $f_{1ab} \in \mathcal{W}_{ab}$ and $f_{1,b} \in \mathcal{W}_b$.

3.3. — In [Pe], Lemma 4.2, we have defined a bilinear map $F:(\otimes^3 H)\otimes(\otimes^3 H)\to\otimes^4 H$ by setting $F=C_{34}-\tau_{23}\circ C_{35}$ where

$$C_{34}(x_1\otimes\cdots\otimes x_6)=(x_3\cdot x_4)x_1\otimes x_2\otimes x_5\otimes x_6,$$
$$C_{35}(x_1\otimes\cdots\otimes x_6)=(x_3\cdot x_5)x_1\otimes x_2\otimes x_4\otimes x_6,$$
$$\tau_{23}(x_1\otimes x_2\otimes x_3\otimes x_4)=x_1\otimes x_3\otimes x_2\otimes x_4.$$

We also consider the map $\widetilde{\sigma}=\sigma\circ(\pi\otimes\mathrm{id})\colon\otimes^4 H\xrightarrow{\pi\otimes\mathrm{id}}\wedge^2 H\otimes H\otimes H\xrightarrow{\sigma}T$, where π is the canonical projection and σ is the map defined in Theorem 0.1. We will need to compute $\theta_0\circ\widetilde{\sigma}\circ F$ on the subspace $(\widetilde{\wedge^3 H})\otimes(\widetilde{\wedge^3 H})$ of $\otimes^6 H$.

3.4. — Recall (see 0.5) that $\theta_0\colon T\to\mathbb{Z}$ is defined by $\theta_0(a_i\wedge a_j\leftrightarrow b_i\wedge b_j)=1$ for $i,j\in\{1,\dots,g\}$, $i\neq j$ and $\theta_0=0$ on the other basis elements of T.

Two subspaces A, B of $\widetilde{\wedge^3 H}$ are said to be *orthogonal for* $\theta_0\circ\widetilde{\sigma}\circ F$ if $\theta_0\circ\widetilde{\sigma}\circ F(\alpha,\beta)=0$ for any $(\alpha,\beta)\in A\times B\cup B\times A$.

Lemma 3.1.

1) *The subspace* $\mathcal{W}_a$ *(resp.* $\mathcal{W}_b$*) is orthogonal to* $\mathcal{W}_a\oplus\mathcal{W}_{ab}$ *(resp.* $\mathcal{W}_b\oplus\mathcal{W}_{ab}$*).*

2) *If the sets of indices* $\{i,j,k\}$, $\{i',j',k'\}$ *are different,*

$$\theta_0\circ\widetilde{\sigma}\circ F(a_i\wedge a_j\wedge a_k,b_{i'}\wedge b_{j'}\wedge b_{k'})=\theta_0\circ\widetilde{\sigma}\circ F(b_{i'}\wedge b_{j'}\wedge b_{k'},a_i\wedge a_j\wedge a_k)=0.$$

3) *For* i,j,k *different,*

$$\theta_0\circ\widetilde{\sigma}\circ F(a_i\wedge a_j\wedge a_k,b_i\wedge b_j\wedge b_k)=-\theta_0\circ\widetilde{\sigma}\circ F(b_i\wedge b_j\wedge b_k,a_i\wedge a_j\wedge a_k)=12.$$

4) *For* $\alpha,\beta\in\mathcal{W}_{ab}$, *we have* $\theta_0\circ\widetilde{\sigma}\circ F(\alpha,\beta)=\frac{1}{2}C(\alpha)\cdot C(\beta)$ *where* C *is the contraction on* $\widetilde{\wedge^3 H}$ *defined by* $C(x\wedge y\wedge z)=2[(x\cdot y)z+(y\cdot z)x+(z\cdot x)y]$, *and* $(\,.\,)$ *the symplectic intersection form on* H.

3.5. *Proof.* — By definition of F and θ_0, it is clear that

$$\theta_0\circ\widetilde{\sigma}\circ F(c_1\wedge c_2\wedge c_3,d_1\wedge d_2\wedge d_3)=0$$

unless the set $\{c_i,\,d_j\,;\;i,j=1,2,3\}$ is the union of three pairs $\{a_k,\,b_k\}$, $k\in\{1,2,\cdots,g\}$. This proves points 1) and 2).

3.6. — The construction of the term corresponding to C_{34} in $\theta_0 \circ \widetilde{\sigma} \circ F(a_i \wedge a_j \wedge a_k, b_i \wedge b_j \wedge b_k)$ is non zero only when a term a on the left is coupled with a term b on the right with same index. This contribution is easily seen to be equal to 12.

The contribution of the term $-\tau_{23} \circ C_{35}$ is easily seen to be zero. This proves 3).

3.7. — To prove 4), we have only to consider the case $\alpha = a_i \wedge a_j \wedge b_k$ and $\beta = b_{i'} \wedge b_{j'} \wedge a_{k'}$ and the case when α and β are permuted. This amounts to exchange a and b: this changes the sign of $\theta_0 \circ \widetilde{\sigma} \circ F(\alpha, \beta)$ and $\frac{1}{2} C(\alpha) \cdot C(\beta)$ since $(\,.\,)$ is antisymmetric.

3.8. — Using 3.4 we have only to consider the following three cases:

1) $\theta_0 \circ \widetilde{\sigma} \circ F(a_i \wedge a_j \wedge b_k, b_i \wedge b_j \wedge a_k)$ for i, j, k distinct;

2) $\theta_0 \circ \widetilde{\sigma} \circ F(a_i \wedge a_j \wedge b_j, b_i \wedge b_k \wedge a_k)$ for i, j, k distinct;

3) $\theta_0 \circ \widetilde{\sigma} \circ F(a_i \wedge a_j \wedge b_j,\, b_i \wedge b_j \wedge a_j)$ for i, j distinct.

3.9. — The contribution of the term corresponding to C_{34} in each case is non zero only if the b term on the left is coupled with the a term on the right with same index. This contribution is respectively $-4, 0, -4$.

3.10. — The contribution of the term corresponding to $-\tau_{23} \circ C_{35}$ in each case is non zero only if an a term on the left is coupled with the b term on the right with same index. For the cases 1), 2), 3), the contribution is respectively $4, -2, 2$.

Then point 4) of Lemma 3.1 follows immediately. □

3.11. — We are now ready to prove:

LEMMA 3.2. — *$G \colon \mathcal{I}_{g,1} \to \mathbb{Q}$ is a homomorphism, equal to 0 on $\mathcal{T}_{g,1} = \mathcal{M}(3)$.*

Proof. — By definition, for $f, g \in \mathcal{I}_{g,1}$

$$G(fg) = \lambda^*(fg) - (\Delta(fg) - \sum_{1 \le i < j < k \le g} \alpha_{ijk}^{fg} \beta_{ijk}^{fg}).$$

By Theorem 0.4, $\lambda^*(fg) = \lambda^*(f) + \lambda^*(g) + 2 \sum_{1 \le i < j < k \le g} \beta_{ijk}^{f} \alpha_{ijk}^{g}$. By additivity of A_1 on $\mathcal{I}_{g,1}$, $\alpha_{ijk}^{fg} = \alpha_{ijk}^{f} + \alpha_{ijk}^{g}$ and $\beta_{ijk}^{fg} = \beta_{ijk}^{f} + \beta_{ijk}^{g}$.

3.12. — From the properties of A_2' (see [Pe], Lemma 4.2) and d (see Proposition 1.4) we have:

$$\begin{aligned}\Delta(fg) &= -\frac{1}{12}\big(\theta_0\circ\sigma\circ A_2'(fg)\big)+\frac{1}{24}d(fg)\\ &= -\frac{1}{12}\theta_0\circ\widetilde{\sigma}\left[A_2(fg)\right]+\frac{1}{24}d(fg)\quad\text{(see Theorem 0.1).}\\ &= -\frac{1}{12}\theta\,\widetilde{\sigma}\big[A_2(f)+A_2(g)+F(A_1(f),A_1(g))\big]\\ &\qquad\qquad +\frac{1}{24}\big[d(f)+d(g)+\widetilde{k}(f)\cdot\widetilde{k}(g)\big]\\ &= \Delta(f)+\Delta(g)-\frac{1}{12}\theta_0\circ\widetilde{\sigma}\circ F\big(A_1(f),A_1(g)\big)+\frac{1}{24}\widetilde{k}(f)\cdot\widetilde{k}(g).\end{aligned}$$

So $G(fg)-G(f)-G(g)$ is equal to

$$\sum_{1\le i<j<k\le g}\beta^f_{ijk}\,\alpha^g_{ijk}-\alpha^f_{ijk}\,\beta^g_{ijk}+\frac{1}{12}\theta_0\circ\widetilde{\sigma}\circ F\big(A_1(f),A_1(g)\big)-\frac{1}{24}\widetilde{k}(f)\cdot\widetilde{k}(g).$$

3.13. — With the notations of 3.2 and Lemma 3.1 we get

$$\begin{aligned}\theta_0\circ\widetilde{\sigma}\circ F\big(A_1(f),A_1(g)\big) &= \theta_0\circ\widetilde{\sigma}\circ F(f_{1a}+f_{1b}+f_{1ab},g_{1a}+g_{1b}+g_{1ab})\\ &= \theta_0\circ\widetilde{\sigma}\circ F(f_{1a},g_{1b})+\theta_0\circ\widetilde{\sigma}\circ F(f_{1b},g_{1a})\\ &\quad+\theta_0\circ\widetilde{\sigma}\circ F(f_{1ab},g_{1ab}).\end{aligned}$$

By definition of f_{1a}, α^f_{ijk} and Lemma 3.1:

$$\begin{aligned}\theta_0\circ\widetilde{\sigma}\circ F(f_{1a},g_{1b}) &= 12\sum_{1\le i<j<k\le g}\alpha^f_{ikj}\,\beta^g_{ijk},\\ \theta_0\circ\widetilde{\sigma}\circ F(f_{1b},g_{1a}) &= -12\sum_{1\le i<j<k\le g}\beta^f_{ijk}\,\alpha^g_{ikj},\\ \theta_0\circ\widetilde{\sigma}\circ F(f_{1ab},g_{1ab}) &= \frac{1}{2}C(f_{1ab})\cdot C(g_{1ab}).\end{aligned}$$

By the definition of C (Lemma 3.1) and Lemma 1.2 it is easy to see that $C(f_{1ab})=C(f_1)=\widetilde{k}(f)$ and $C(g_{1ab})=C(g_1)=\widetilde{k}(g)$.

This proves that $G(fg)-G(f)-G(g)=0$.

The last part of Lemma 3.2 follows from Theorem 0.1. □

3.14. — By Lemma 3.2, the homomorphism G factors through a homomorphism $\mathcal{G}\colon\widetilde{\wedge^3 H}\to\mathbb{Q}$ such that $G=\mathcal{G}\circ A_1$, because of the exact sequence

$$1\to\mathcal{M}(3)=\mathcal{T}_{g,1}\longrightarrow\mathcal{I}_{g,1}\xrightarrow{A_1}\widetilde{\wedge^3 H}\to 1.$$

3.15. — So, to prove that $G = 0$, it is sufficient to show that $\mathcal{G} = 0$. For this purpose we will study some symmetries of G.

LEMMA 3.3. — *Let $S_{g,1} \subset \mathbb{R}^3$ be the surface of genus g, with one boundary component, standardly embedded in $\mathbb{R}^3$ as shown by Figure 3.2 below. Then:*

(a) *For each pair (s,t) $(1 \leq s < t \leq g)$, there exists a homeomorphism $\rho_{s,t} \in \mathcal{N}_{g,1} \cap \mathcal{N}^t_{g,1} \subset \mathcal{M}_{g,1}$ (see 0.3 for the definition of $\mathcal{N}_{g,1}$ and $\mathcal{N}^t_{g,1}$), which exchanges the handles h_s and h_t of Figure 3.2. More precisely, at the fundamental group level the action of $\rho_{s,t}$ is*

$$\rho_{s,t}(z_i) = \alpha(s,t,z_i)\, z_{(s,t)(i)}\, \alpha(s,t,z_i)^{-1}$$

where $z_i = x_i$ or y_i, (s,t) is the transposition of s and t, and $\alpha(s,t,z_i)$ is a product of commutators.

(b) *$A_1(\rho_{s,t}) = 0$ and $\widetilde{k}(\rho_{s,t}) = 0$, for any pair (s,t).*

3.16. — *Proof*: It is easy to construct an isotopy $\tau_{i,u}$ $(i = 1,2,$ $u \in [0,1])$ of $\mathbb{R}^3$ fixed outside a big compact set such that $\tau_i = \tau_{i,1}$ has the following properties:

(i) τ_i leaves invariant the surface $S_{2,1}$ of genus 2 of Figure 3.1;

(ii) $\tau_{i|S_{2,1}}$ is the identity outside the disk $2D_i$;

(iii) $\tau_{i|D_i \cup h_i}$ is the rotation of angle π around the axis $\mathbb{Z}_i$.

3.17. — It is also easy to construct an isotopy ρ'_u $(u \in [0,1])$ of $\mathbb{R}^3$, fixed outside a big compact of $\mathbb{R}^3$ such that ρ'_1 verifies:

(i) ρ'_1 leaves the surface $S_{2,1}$ of Figure 3.1 invariant;

(ii) $\rho'_{1|S_{2,1}}$ is the identity on $\partial(2D)$;

(iii) $\rho'_{1|D \cup h_1 \cup h_2}$ is the rotation of angle π around the axis $\mathbb{Z}$.

3.18. — Now set $\rho = \tau_1 \circ \tau_2 \circ \rho_1$. Then ρ is time 1 of an isotopy of $\mathbb{R}^3$ which has the following properties:

(i) ρ leaves invariant the surface $S_{2,1}$;

(ii) ρ is the identity on $\partial S_{2,1}$;

(iii) ρ exchanges the handles h_1 and h_2.

At the fundamental group level, ρ satisfies the formula of Lemma 3.3, with $s=1$ and $t=2$. We can even arrange things such that

$$\rho(x_1)=f_1^{-1}x_2f_1,\quad \rho(x_2)=x_1,\quad \rho(y_1)=f_1^{-1}y_2f_1,\quad \rho(y_2)=y_1$$

where f_1 is the homotopy class of $2D_1$ (with suitable orientation and path).

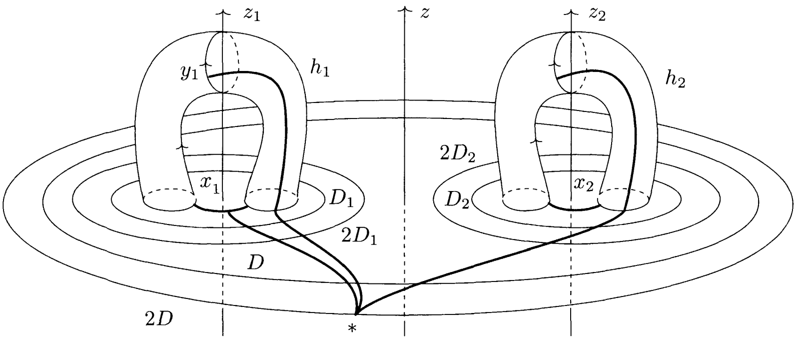

Figure 3.1

3.19. — Now let (s,t) be a pair of integers such that $1\le s<t\le g$, and γ be an embedded circle on $S_{g,1}$ surrounding only the feet of the handles h_s, h_u (see Figure 3.2). Then γ is the boundary of a genus 2 subsurface Σ of $S_{g,1}$. Then there is an isotopy H_u $(u\in[0,1])$ of $\mathbb{R}^3$ such that H_1 preserves $S_{g,1}$ and sends Σ onto $S_{2,1}$ seen as a subsurface of $S_{g,1}$. Then $\rho_{s,t}=H_1^{-1}\circ\rho_1\circ H_1$ satisfies point 1) of Lemma 3.3.

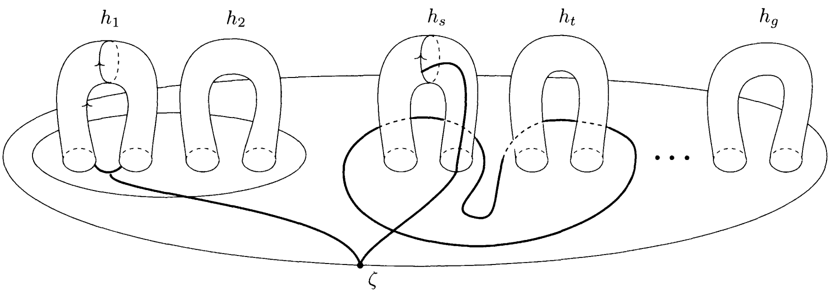

Figure 3.2

3.20. — Then, using the definition of A_1 (see [Pe], Chapter 4) and the fact that $\alpha(s,t,z_i)$ is a product of commutators, it is easy to see that $A_1(\rho_{s,t})=0$. Then $\widetilde{k}(\rho_{s,t})=0$ by Lemma 1.2 (c). This finishes the proof of Lemma 3.3. □

LEMMA 3.4. — *For $\varphi \in \mathcal{I}_{g,1}$, $f \in \mathcal{M}_{g,1}$ such that $A_1(f) = 0$ (and so $\widetilde{k}(f) = 0$) we have:*

1) $A_2(f\varphi f^{-1}) = f_*.\,A_2(\varphi)$;

2) $d(f\varphi f^{-1}) = d(\varphi)$.

Proof. — Part 1) comes from Lemma 7.1 of [Pe] and 2) from Proposition 1.4. □

LEMMA 3.5. — *For $f \in \mathcal{I}_{g,1}$ we have*

1) $G(\rho_{st} f \rho_{st}^{-1}) = G(f)$;

2) $\mathcal{G} = \mathcal{G} \circ (\rho_{\mathrm{st}*}.)$ *where $\rho_{\mathrm{st}*} = B_0(\rho_{\mathrm{st}})$ and $\rho_{\mathrm{st}*}.$ stands for the action of $\rho_{\mathrm{st}*}$ on $\wedge^3 H$.*

Proof. — By Lemma 3.4 we have $A_2(\rho_{\mathrm{st}} f \rho_{\mathrm{st}}^{-1}) = \rho_{\mathrm{st}*} A_2(f)$. Therefore

$$\Delta(\rho_{\mathrm{st}} \circ f \circ \rho_{\mathrm{st}}^{-1}) = -\frac{1}{12}\theta_0\big(\rho_{\mathrm{st}*}\widetilde{\sigma}\big(A_2(f)\big)\big) + \frac{1}{24} d(f).$$

Since the effect of $\rho_{\mathrm{st}*}$ on H is to permute a_s with a_t and b_s with b_t, it follows from the definition of θ_0 (see 0.5) that $\Delta(\rho_{\mathrm{st}} \circ f \circ \rho_{\mathrm{st}}^{-1}) = \Delta(f)$.

Since $\rho_{\mathrm{st}} \in \mathcal{N}_{g,1} \cap \mathcal{N}_{g,1}^t$ (Lemma 3.3), from 0.3, it follows that

$$\lambda^*(\rho_{\mathrm{st}} \circ f \circ \rho_{\mathrm{st}}^{-1}) = \lambda^*(f).$$

On the other hand, it is easy to see that

$$\sum_{1 \le i < j < k \le g} \alpha_{ijk}^f \beta_{ijk}^f = \sum_{1 \le i < j < k \le g} \alpha_{ijk}^g \beta_{ijk}^g,$$

where $g = \rho_{\mathrm{st}} \circ f \circ \rho_{\mathrm{st}}^{-1}$.

This proves point 1) of Lemma 3.5. Point 2) follows from the definition of $\mathcal{G}$ and the formula: $A_1(\rho_{\mathrm{st}} \circ f \circ \rho_{s,t}^{-1}) = \rho_{\mathrm{st}*}\, A_1(f)$ (see [Pe], Lemma 4.1).

From Lemma 3.5, we deduce that $\mathcal{G}(a_i \wedge a_j \wedge b_k) = \mathcal{G}(a_j \wedge a_i \wedge b_k) = 0$, for $k \neq i, j$. We have the same result by replacing a by b, and when we have three a or three b. So, to prove that $\mathcal{G}$ is identically 0 on $\wedge^3 H$, we have just to show that $\mathcal{G}(a_1 \wedge a_2 \wedge b_1) = \mathcal{G}(a_1 \wedge b_1 \wedge b_2) = 0$.

3.21. *Computation of $\mathcal{G}(a_1 \wedge a_2 \wedge b_1)$.*

LEMMA 3.6.

1) $\mathcal{G}(a_1 \wedge a_2 \wedge b_1) = G(D(x_2)^{-1} D(u))$, *where x_2 and u are the simple closed curves given by Figure 3.3;*

2) $\mathcal{G}(a_1 \wedge a_2 \wedge b_1) = 0$.

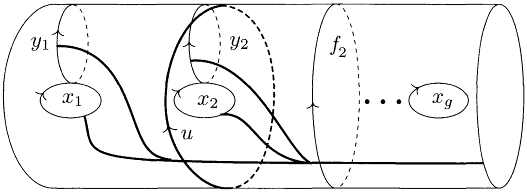

Figure 3.3

Proof. — The same letter u, x, y, f will denote either the closed path or the element of the fundamental group $\Gamma = \pi_1(S_{g,1}, *)$, equipped with paths as indicated in Figure 3.3.

3.22. — Then straightforward computations show that

$$f_2 = [y_2, x_2] \cdot [y_1, x_1] \in \Gamma$$

(where $[a, b]$ denotes the commutator $aba^{-1}b^{-1}$) and $u = x_2 f_2 \in \Gamma$

$$\big[D(x_2)^{-1}D(u)\big](x_1) = ux_1u^{-1}, \quad \big[D(x_2)^{-1}D(u)\big](y_1) = uy_1u^{-1},$$
$$\big[D(x_2)^{-1}D(u)\big](x_2) = x_2, \quad \big[D(x_2)^{-1}D(u)\big](y_2) = f_2x_2y_2x_2^{-1},$$

(composition of paths is written from left to right).

3.23. — This proves that $D(x_2)^{-1}\ D(u) \in \mathcal{N}_{g,1} \cap \mathcal{I}_{g,1}$ by a result of [G], Theorem 10.1, which says that a homeomorphism of $S_{g,1}$ belongs to $\mathcal{N}_{g,1}$ if and only if it leaves the normal subgroup of Γ generated by $\{y_1, \ldots, y_g\}$ invariant. This implies that $\lambda^*(D(x_2)^{-1}\ D(u) = 0$.

3.24. — Using Chapter 3 of [Pe] we find:

$$A_1\big(D(x_2)^{-1}D(u)\big) = \left(\begin{array}{cc|cc} -a_2 & 0 & 0 & -b_1 \\ a_1 & 0 & b_1 & 0 \\ \hline 0 & 0 & -\,a_2 & a_1 \\ 0 & 0 & 0 & 0 \end{array}\right) \in \mathcal{M}_{2g}(H)$$

$= a_1 \wedge a_2 \wedge b_1 \in \widetilde{\wedge^3 H} \subset \otimes^3 H$ (using the identification $\mathcal{M}_{2g}(H) \simeq H \otimes (H \otimes H)$; see Lemma 1.1 of [Pe]). This proves point 1) of Lemma 3.6.

3.25. — Remark that in order to compute $\theta_0 \circ \sigma \circ A_2'(f)$, it is only necessary to know the terms of the matrix $A_2'(f)$ on the ascending diagonal:

this comes from the remark of 1.3 of [Pe] and the fact that θ_0 is non-zero only on terms such as $a_i \wedge a_j \leftrightarrow b_i \wedge b_j$. By 3.22 we find

$$A'_2\big(D(x_2)^{-1}D(u)\big) = \begin{pmatrix} \times & \times & \times & 2b_1 \wedge a_1 \\ \times & \times & 2a_2 \wedge b_1 - b_2 \wedge b_1 & \times \\ \times & 0 & \times & \times \\ a_2 \wedge a_1 & \times & \times & \times \end{pmatrix}$$

belonging to $\mathcal{M}_{2g}(\wedge^2 H) \simeq \wedge^2 H \otimes H \otimes H$ (by Lemma 1.1 of [Pe]). Applying the homomorphisms $\sigma : \wedge^2 H \otimes H \otimes H \to T$ (defined in Theorem 0.1) and $\theta_0 : T \to \mathbb{Z}$ (see 0.5), we find that $\theta_0(\sigma \circ A'_2(D(x_2)^{-1}D(u))) = 2$.

By Lemmas 1.3, 1.6, Proposition 1.4 and the fact that $\widetilde{k}(D(x_2)) = 0$ we get $d(D(x_2)^{-1}D(u)) = d(D(u)) - d(D(x_2)) = 4$. This finishes the proof of Lemma 3.6. □

3.26. *Computation of* $\mathcal{G}(a_1 \wedge b_1 \wedge b_2)$.

LEMMA 3.7.

1) $\mathcal{G}(a_1 \wedge b_1 \wedge b_2) = -G(D(y_2)D(y'_2)^{-1})$, *where* y_2, y'_2 *are defined by Figure* 0.1.

2) $\mathcal{G}(a_1 \wedge b_1 \wedge b_2) = 0$.

3.27. — Proof: by Proposition 4.4 of [Pe] and [J1], Lemma 4.B, we have

$$A_1\big(D(y_2)D(y'_2)^{-1}\big) = -\tau_2\big(D(y_2)D(y'_2)^{-1}\big) = -a_1 \wedge b_1 \wedge b_2.$$

Moreover $D(y_2)\ D(y'_2)^{-1} \in \mathcal{N}_{g,1} \cap \mathcal{I}_{g,1}$, since y_2, y'_2 bound a 2-disc in the handlebody H_g. Set $\delta = D(y_2)D(y'_2)^{-1}$. Then we have

$$\begin{aligned} &y'_2 = x_2 y_2^{-1} x_2^{-1} [y_1, x_1] \in \Gamma, && \delta(x_1) = y_2'^{-1} x_1 y'_2, \\ &\delta(y_1) = y_2'^{-1} y_1 y'_2, && \delta(x_2) = [x_1, y_1] x_2, \\ &\delta(y_2) = y_2. \end{aligned}$$

Then $A'_2(D(y_2)D(y'_2)^{-1}) \in \mathcal{M}_{2g}\ (\wedge^2 H)$ is equal to

$$\begin{pmatrix} \times & \times & \times & 0 \\ \times & \times & b_2 \wedge b_1 & \times \\ \times & 2a_1 \wedge b_1 & \times & \times \\ 2b_2 \wedge a_1 + a_1 \wedge a_2 & \times & \times & \times \end{pmatrix}.$$

3.28. — As in the proof of Lemma 3.6, using the isomorphism $\mathcal{M}_{2g}(\wedge^2 H) \simeq (\wedge^2 H)\otimes H\otimes H$, we find that $\theta_0(\sigma \circ A'_2(D(y_2)D(y'_2)^{-1})) = -2$.

Also using properties of d, we can show (using Lemma 1.6) that $d(D(y_2)D(y'_2)^{-1}) = d(D(y_2)) - d(D(y'_2)) = -4$. This shows that $\mathcal{G}(D(y_2)D(y'_2)^{-1}) = 0$.

This finishes the proof of Proposition 0.3. □

Chapter 4. The hyperelliptic mapping class group.

4.1. Proof of Lemma 0.5.

4.1. — In 0.10 we have defined the hyperelliptic mapping class group as the subgroup $\mathcal{H}_{g,1}$ of $\mathcal{M}_{g,1}$ generated by the Dehn twists along the curves $y_1, x_1, \ldots, x_g, C_1, C_2, \ldots, C_{g-1}$ of Figure 0.1. In [PV] we have described an isomorphism between the usual braid group B_{2g+1} and $\mathcal{H}_{g,1}$ as follows: let $\{\sigma_1, \ldots, \sigma_{2g}\}$ be the standard generators of B_{2g+1}. Then send σ_{2i} $(i = 1, \ldots, g)$ onto $D(x_i)$, σ_1 onto $D(y_1)$ and σ_{2i+1} $(i = 1, \ldots, g-1)$ onto $D(C_i)$.

4.2. — Moreover an element $f \in \mathcal{M}_{g,1}$ belongs to $\mathcal{H}_{g,1}$ if and only if f commutes (up to isotopy) with the symmetry s_g along the axis $x'x$ of Figure 0.1. This symmetry can be seen as an element of $\mathcal{M}_{g,1}$ (in fact of $\mathcal{H}_{g,1} \simeq B_{2g+1}$), after composition with a half-twist (see 1.7). As an element of $\mathcal{H}_{g,1}$, s_g is represented by

$$\Delta_g^2 = (\sigma_1\sigma_2\cdots\sigma_{2g})^{2g+1} = \big(D(y_1)D(x_1)D(C_1)D(x_2)\cdots D(C_{g-1})D(x_g)\big)^{2g+1}.$$

The reason of the notation Δ_g^2 is that Δ_g^2 as an element of B_{2g+1} is the square of $\Delta_g = (\sigma_1\ \sigma_2\cdots\sigma_g)(\sigma_1\cdots\sigma_{g-1})\cdots(\sigma_1\sigma_2)\sigma_1$ (see [B], §2.3).

Lemma 4.1. — *For $\beta \in \mathcal{H}_{g,1}$ we have:*

(i) $2A_1(\beta) = A_1(\Delta_g^2) - \beta_* A_1(\Delta_g^2) \in \otimes^3 H$;

(ii) $2\widetilde{k}(\beta) = \widetilde{k}(\Delta_g^2) - \beta_*\widetilde{k}(\Delta_g^2) \in H$, *where $\beta_* = B_0(\beta)$ is the isomorphism induced by β at the homological level (see remark below).*

Proof. — By Lemma 4.1 of [Pe] we have

$$A_1(\beta\Delta_g^2) = A_1(\beta) + \beta_* A_1(\Delta_g^2), \quad A_1(\Delta_g^2\beta) = A_1(\Delta_g^2) + (\Delta_g^2)_* A_1(\beta).$$

But $(\Delta_g^2)_* = -\mathrm{id}_H$ since it represents the symmetry along the axis $x'x$. Point (i) follows from the fact that β and Δ_g^2 commutes. Point (ii) is proved in the same way, using Lemma 1.1. □

Lemma 0.5 is a direct corollary of Lemma 4.1.

4.3. *Remark.* — For $\beta \in \mathcal{H}_{g,1} \simeq B_{2g+1}$, $\beta_* = B_0(\beta)$ is a conjugate of the reduced Burau representation of B_{2g+1} eveluated at $t = -1$ (see [B], §3.2). In fact if we write the matrix of $D(y_1)$, $D(x_i)$, $D(C_j)$ in the basis $([y_1],[x_1],[C_1],\dots,[x_i],[C_i],\dots,[x_{2g}])$ of H (where [ ] represents the homology class), we find exactly the Burau representation when $t=-1$.

As a consequence $\mathcal{H}_{g,1} \cap \mathcal{I}_{g,1}$ is identified with the kernel of the Burau representation of B_{2g+1} when $t=-1$.

4.2. Computation of $\widetilde{k}(\Delta_g^2)$.

Lemma 4.2. — *One has* $\widetilde{k}(\Delta_g^2) = 2[(g-1)a_g + (g-2)a_{g-1} + \cdots + a_2]$, *where a_i is the homology class of the oriented circle x_i (see Figure* 0.1).

Proof. — Recall that Δ_g^2 is the symmetry of $S_{g,1}$ along the axis $x'x$, followed by a half twist. We have first to find the effect of Δ_g^2 on the generators x_i, $y_i (i = 1,2\cdots g)$ of $\Gamma = \pi_1(S_{g,1},*)$.

The image by Δ_g^2 of the oriented curve y_i equipped with path γ_i is the oriented curve y_i' of Figure 4.1 equipped with the path γ_i'.

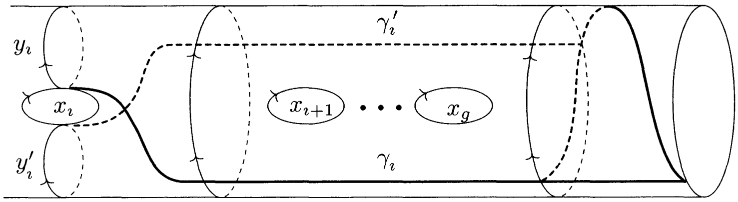

Figure 4.1

4.4. — A careful inspection of Figure 4.1 shows that

$$\Delta_g^2(y_i) = x_g^{-1} \cdots x_{i+1}^{-1}(f_i y_i^{-1}) x_{i+1} \cdots x_g$$

where $f_i = [y_i, x_i][y_{i-1}, x_{i-1}] \cdots [y_1, x_1]$. We set

$$\beta_i = x_g^{-1} x_{g-1}^{-1} \cdots x_{i+1}^{-1},$$

for $1 \le i < g$ and $\beta_g = 1$, so that we get $\Delta_g^2(y_i) = \beta_i f_i y_i^{-1} \beta_i^{-1}$.

4.5. — The image by Δ_g^2 of the oriented curve x_i (with path μ_i) is x_i^{-1} with path μ_i' given by Figure 4.2.

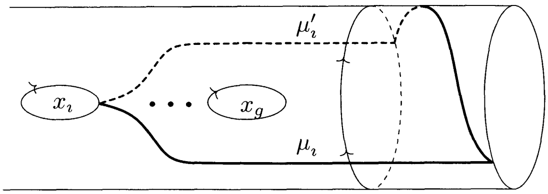

Figure 4.2

We finally get $\Delta_g^2(x_i) = \alpha_i x_i^{-1} \alpha_i^{-1}$ where $\alpha_i = x_g^{-1} \cdots x_{i+1}^{-1} y_i = \beta_i y_i$.

Recall that $\widetilde{k}(\Delta_g^2) = \det(B(\Delta_g^2)^{\mathrm{ab}})$. By Lemma 1.2, a), we know that $\widetilde{k}(\Delta_g^2)$ belongs to H (rather than $\mathbb{Z}[H]$). Let $\alpha = \sum_{i=1}^{2g} n_i c_i$ be any element of H. Written multiplicatively it becomes $c_1^{n_1} \cdots c_{2g}^{n_{2g}}$. Applying the commutative Magnus representation (see [Pe], 2.6) and taking the term of degree one, we recover the additive writing of α. So $(1 + \widetilde{k}(f))$ is equal to the 1-jet (in the sense of Definition 2.1 of [Pe]), of the determinant of $(B_0(f) + B_1(f))^{\mathrm{ab}} = B_0(f) + B_1(f)$.

Since $B_0(\Delta_g^2) = -I$, we obtain that $\widetilde{k}(\Delta_g^2) = -\mathrm{trace}(B_1(\Delta_g^2))$ using properties of the determinant. Now, by 4.4 and 4.5, trace $(B_1(\Delta_g^2))$ is equal to

$$\text{degree 1 term of } \sum_{j=1}^{g} \overline{\frac{\partial \Delta^2(x_j)}{\partial x_j}} + \overline{\frac{\partial \Delta^2(y_j)}{\partial y_j}}$$

$$= \text{degree 1 term of } \sum_{j=1}^{g} \overline{\frac{\partial \alpha_j}{\partial x_j}}(1 - x_j) - x_j \alpha_j^{-1} + \sum_{j=1}^{g} \overline{\frac{\partial \beta_j}{\partial y_j}}(1 - y_j) + \Big(\overline{\frac{\partial f_j}{\partial y_j}} - y_j\Big)\beta_j^{-1}$$

$$= \text{degree 1 term of } \sum_{j=1}^{g} - x_j \alpha_j^{-1} + \Big(\overline{\frac{\partial f_j}{\partial y_j}} - y_j\Big)\beta_j^{-1} \text{ since } \frac{\partial \alpha_j}{\partial x_j} = \frac{\partial \beta_j}{\partial y_j} = 0$$

$$= \text{degree 1 term of } \sum_{j=1}^{g} -x_j y_j^{-1} x_{j+1} \cdots x_g + (1 - x_j^{-1}) x_{j+1} \cdots x_g - y_j x_{j+1} \cdots x_g$$

$$= \sum_{j=1}^{g} -(a_g + \cdots + a_j - b_j) + a_j - (a_g + \cdots a_{j+1} + b_j) = -2 \sum_{j=1}^{g} \Big(\sum_{i=j+1}^{g} a_i \Big).$$

This proves Lemma 4.2. □

4.3. A formula of Gambaudo and Ghys for the signature of a link.

4.6. — Let $\beta \in B_n$ (the usual braid group with n strings) and $\widehat{\beta}$ be the link in R^3 obtained by closing β according to Figure 4.3:

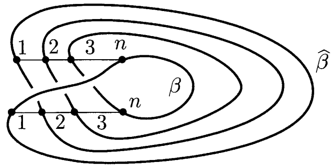

Figure 4.3

For a link k in $\mathbb{S}^3$, we recall the definition of the signature of k, denoted by $s(k)$ (see [GL], §2). Let V be an orientable surface embedded in $\mathbb{S}^3$, bounded by k. Let N denote a closed tubular neighbourhood of V: this is a I-bundle ($I \simeq [0,1]$) over V. Let $\widetilde{V}$ denote the corresponding ∂I-bundle and let $\tau : H_1(V) \to H_1(\widetilde{V})$ be the transfer map. Then define the bilinear form $\mathcal{G}_V$ on $H_1(V)$ by: $\mathcal{G}_V(\alpha,\beta) =$ linking number of $(\alpha, \tau(\beta))$. It is shown in [GL] that $\mathcal{G}_V$ is symmetric. Then $s(k)$ is the signature of $\mathcal{G}_V$.

Proposition 4.3 (see [GG], Theorem 1.1). — *Let* $\alpha,\beta \in B_{2g+1}$. *Then*

$$s(\widehat{\alpha\beta}) = s(\widehat{\alpha}) + s(\widehat{\beta}) - \tau(\alpha_*,\beta_*)$$

where τ *is the Meyer* 2*-cocycle defined in* 1.3, *associated to* α, β *identified to elements of* $\mathcal{H}_{g,1}$ ($\simeq B_{2g+1}$), *by* 4.1.

4.4. Proof of proposition 0.6.

4.7. — We now restrict d to $\mathcal{H}_{g,1}$. The signature defines a map $s : \mathcal{H}_{g,1} \to \mathbb{Z}$ by setting $s(\alpha) = s(\widehat{\alpha})$. From Proposition 1.4 and Proposition 4.3, the mapping $d - 3s : \mathcal{H}_{g,1} \to \mathbb{Z}$ satisfies

$$(d-3s)(\alpha\beta) = (d-3s)(\alpha) + (d-3s)(\beta) + \widetilde{k}(\alpha)\cdot \alpha_*\widetilde{k}(\beta).$$

In cohomological terms, the 1-chain $d - 3s$ on $\mathcal{H}_{g,1}$ has it coboundary equal to the 2-cocycle c on $\mathcal{H}_{g,1}$ defined by $c(\alpha,\beta) = -\widetilde{k}(\alpha)\cdot\alpha_*\widetilde{k}(\beta)$. Using Lemma 4.1, this 2-cocyle is given by

$$c(\alpha,\beta) = -\frac{1}{4}\widetilde{k}(\Delta_g^2)\cdot\left[\alpha_*\widetilde{k}(\Delta_g^2) - \alpha_*\beta_*\widetilde{k}(\Delta_g^2) + \beta_*\widetilde{k}(\Delta_g^2)\right].$$

4.8. — Let $u:\mathcal{H}_{g,1}\to\mathbb{Z}$ be the 1-cochain $u(\alpha)=\frac{1}{4}\alpha_*\widetilde{k}(\Delta_g^2)\cdot\widetilde{k}(\Delta_g^2)$. Obviously u satisfies $u(\alpha\beta)=u(\alpha)+u(\beta)-c(\alpha,\beta)$ by the above formula. Remark that u takes *a priori* its values in $\frac{1}{4}\mathbb{Z}$. But we have seen in Lemma 4.2 that $\widetilde{k}(\Delta_g^2)=2\delta_g$ where $\delta_g=(g-1)a_g+\cdots+a_2$. So $u(\alpha)=\alpha_*(\delta_g)\cdot\delta_g$ belongs to $\mathbb{Z}$.

4.9. — By 4.7 the mapping $d-3s-u:\mathcal{H}_{g,1}\to\mathbb{Z}$ is a homomorphism. From the presentation of $\mathcal{H}_{g,1}\simeq B_{2g+1}$, it is well known that the abelianization of B_{2g+1} is isomorphic to $\mathbb{Z}$, the canonical homomorphism $\pi:B_{2g+1}\to\mathbb{Z}$ sending each generator σ_i $(i=1,\dots,g)$ onto $1\in\mathbb{Z}$. So there exists an integer $n_0\in\mathbb{Z}$ such that $d-3s-u=n_0\pi$.

4.10. — To determine the value of n_0, it is enough to evaluate the two terms of the equality above on the element $D(f_1)$ (f_1 is the circle defined by Figure 0.1). The Dehn twist $D(f_1)$ is known to be equal to $(D(x_1)D(y_1))^6\simeq(\sigma_2\,\sigma_1)^6\in\mathcal{H}_{g,1}\simeq B_{2g+1}$.

By Corollary 0.2, $d(D(f_1))=\delta(D(f_1))=0$. Since $D(f_1)$ belongs to $\mathcal{T}_{g,1}\simeq\mathcal{M}(3)\subset\mathcal{I}_{g,1}$, $u(D(f_1))=0$. Then formula 4.9 gives

$$3s\big((\sigma_2\sigma_1)^6\big)=3s\big((\widehat{\sigma_2\sigma_1})^6\big)=n_0\pi\big(D(f_1)\big)=12n_0.$$

Claim: $s((\widehat{\sigma_2\sigma_1})^6)=-8$ and so $n_0=2$. We can compute the signature of the link $(\widehat{\sigma_2\sigma_1})^6$ by the method of [GL], using the diagram of $(\widehat{\sigma_2\sigma_1})^6$ given by the braid $(\sigma_2\sigma_1)^6$, or use the formula of Proposition 4.3.

COROLLARY 4.4. — *The mapping d takes the following values on the Lickorish generators of $\mathcal{M}_{g,1}$:*

1) $d(D(y_1))=d(D(x_i))=2$, *for* $i=1,\dots,g$.

2) $d(D(C_j))=3$, *for* $i=1,\dots,g-1$.

3) $d(D(y_i))=2$, *for* $i=2,\dots,g$.

4) $d(D(y_i'))=6$, *for* $i=2,\dots,g$.

(*The circles x_i, y_i, y_i', C_j are defined by Figure* 0.1.)

4.11. *Remark.* — *A priori d* depends on the choice of the symplectic basis $\{a_i,b_j;i=1,\dots,g\}$. Morita [Mo1] proved that $d/\mathcal{I}_{g,1}$ is independant of the choices.

Proof. — It is easy to see that $u(D(y_1)) = u(D(x_i)) = 0$ for $i = 1, \ldots, g$ (u is defined in 4.8). So $d(D(y_1)) = d(D(x_i)) = 2$ by Proposition 0.6.

Since, for $i = 1, \ldots, g$, the circle C_i cuts transversally in one point the circle x_i, the corresponding Dehn twists $D(C_i)$ and $D(x_i)$ satisfy the usual braid relation, which is equivalent to

$$D(C_i) = D(x_i)D(C_i)D(x_i)D(C_i)^{-1}D(x_i)^{-1}.$$

By Lemma 1.1, Proposition 1.4 (iii) and the fact that $\widetilde{k}(D(x_i)) = 0$, we have

$$d\big(D(C_i)\big) = d\big(D(x_i)\big) - D(x_i)_*\widetilde{k}\big(D(C_i)\big) \cdot D(C_i)_*D(x_i)_*\widetilde{k}\big(D(C_i)\big).$$

By [Pe], 5.2, and Lemma 1.2 we get $\widetilde{k}(D(C_i)) = b_{i+1} - b_i$. Point 2) follows.

Point 3) follows from the same type of argument, using the fact that the circle y_i cuts transversally in one point the circle x_i. Point 4) follows from Lemma 1.6 (ii). □

4.12. *Remark.* — Corollary 4.4 contradicts an affirmation of [Mo1], §5, line above Proposition 5.1, which says that the values of d on the Lickorish generators is 3. This affirmation cannot be true, since we have proved above that $d(D(C_i)) = d(D(x_i)) + 1$.

Corollary 4.5. — *Let $\beta \in \mathcal{H}_{g,1} \cap \mathcal{I}_{g,1}$ (which is equivalent to say, by Remark* 4.3, *that β belongs to the kernel of the Burau representation when $t = -1$). Then the Casson invariant of the homology-sphere M_β is given by*

$$\lambda^*(\beta) = \lambda(M_\beta) = \frac{1}{12}\Big(\pi(\beta) - \theta_0\big(\sigma \circ A_2'(\beta)\big)\Big) + \frac{1}{8}s(\widehat{\beta}),$$

where π and s are defined in Proposition 0.6.

The object of the next proposition is to describe geometrically the 3-manifold M_β when $\beta \in \mathcal{H}_{g,1} \simeq B_{2g+1}$. For a braid $\gamma \in B_{2n}$ denote by $\widehat{\widehat{\gamma}}$ the closure by "plats" (see Figure 4.4).

Then we have:

Proposition 4.6. — *For $\beta \in \mathcal{H}_{g,1} \simeq B_{2g+1}$, the 3-manifold M_β is homeomorphic to the 2-fold covering of $\mathbb{S}^3$ branched along the link $\widehat{\widehat{\gamma}}$ where $\gamma = \sigma_2 \cdots \sigma_{2g+1}\beta$ (remark that γ belongs to B_{2g+2}).*

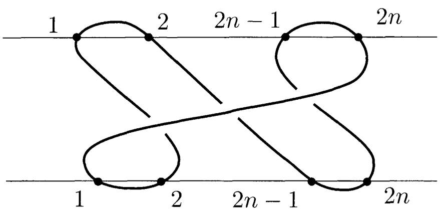

Figure 4.4

Proof. — Recall the construction of the 3-manifold M_β. Let $S_{g,1} \subset S_g$ be standardly embedded in $\mathbb{R}^3 \subset \mathbb{S}^3 = \mathbb{R}^3 \cup \{+\infty\}$, bounding the handlebody H_g. Set $H'_g = \overline{\mathbb{S}^3 - H_g}$. Let $x'x$ be the symmetry axis of H_g, intersecting S_g (resp. H_g) at points P_i, $i = 1, \dots, 2g+2$ (resp. segments $\alpha_i = [P_{2i-1}, P_{2i}]$, $i = 1, \dots, g+1$).

Denote by C the circle of $\mathbb{S}^3$ defined by $C = (x'x) \cup \{\infty\}$, and let $\{\gamma_i\,;\ i = 1, \cdots, g+1\}$ the trace of C on H'_g. More precisely $\gamma_i = [P_{2i}, P_{2i+1}]$ for $1 \leq i \leq g$ and γ_{g+1} is the segment of C with ends P_{2g+2}, P_1 containing $\infty \in \mathbb{S}^3$ (see Figure 4.5).

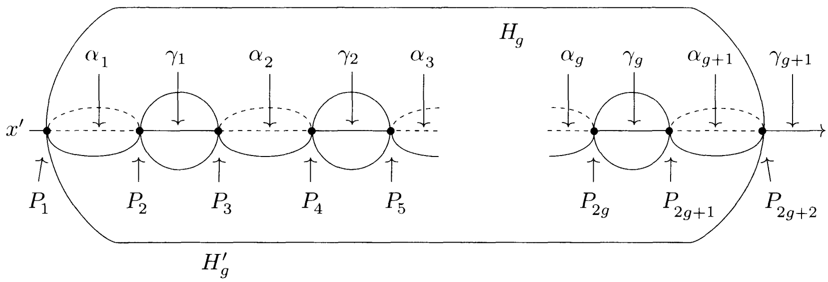

Figure 4.5

The quotient of H_g (resp. H'_g) by the symmetry along xx' (resp C) is homeomorphic to a 3-ball B (resp. B'). The image of the fixed points $\{P_i\,;\ i = 1, \dots, 2g+2\}$ are denoted $\{Q_i\,;\ i = 1, \dots, 2_{g+2}\}$. The image of the set of fixed points $\{\alpha_i\,;\ i = 1, \dots, g+1\}$ (resp. $\{\gamma_i\,;\ i = 1, 2 \cdots, g+1\}$) are denoted $\{\overline{\alpha_i}\} \subset B$ (resp $\{\overline{\gamma_i}\} \subset B'$). They are arcs in the interior of B (resp B') whose ends are the points $\{Q_i\}$ (see Figure 4.6)

The mapping $H_g \xrightarrow{\pi} B$ (resp. $H'_g \xrightarrow{\pi'} B'$) is the 2-fold cyclic covering ramified along the arcs $\{\overline{\alpha_i}\}$ (resp. $\{\overline{\gamma_i}\}$).

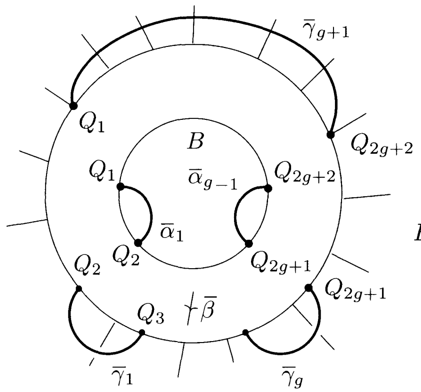

Figure 4.6

By definition, an element β of $\mathcal{M}_{g,1}$ belongs to the hyperelliptic mapping class group $\mathcal{H}_{g,1} \simeq B_{2g+1}$ if β is the lift of the braid $\beta \in B_{2g+1}$ by the 2-fold cyclic covering $\pi_{|S_{g,1}} \subset \partial H_g \to \overline{\partial B - D^2}$ where D^2 is a small disc centered at P_{2g+2}.

Then M_β is the 2-fold cyclic covering over $B \bigcup_\beta B'$, where β is seen as a homeomorphism of $\partial B - D^2$ (we extend it by identity on D^2) leaving $\{Q_i\,;\ i = 1, \dots, g-1\}$ globally invariant and fixing P_{2g+2}, the set of ramification being $\{\bigcup_i \overline{\alpha_i}\} \cup \{\bigcup_i \overline{\gamma_i}\}$. Equivalently, if $\beta \in \beta_{2g+1}$ is represented by $2g+1$ strings in $\overline{(\partial B - D^2)} \times [0,1]$, let β' be the $2g+2$ strings of $\partial B \times [0,1]$ obtained by adding the trivial string $P_{2g+2} \times [0,1]$. Let $\mathcal{L}$ be the link in $\mathbb{S}^3 = B^3 \cup B'^3$ obtained from the braid β' by Figure 4.7 or Figure 4.8.

Clearly the link $\mathcal{L}$ is isotopic to the link $\mathcal{L}'$ of Figure 4.9.

The link $\mathcal{L}'$ is obtained from the braid $\gamma = \sigma_2 \cdots \sigma_{g+1}\, \beta$ by closing by plats. This concludes the proof of Proposition 4.6.

COROLLARY 4.7. — *Let $\beta \in B_{2g+1} \cap \mathcal{I}_{g,1}$ and γ the braid of B_{2g+2} given by $\gamma = \sigma_2 \cdots \sigma_{g+1}\, \beta$. Denote by $M^{(2)}(\widehat{\widehat{\gamma}})$ the two-fold cyclic covering of $\mathbb{S}^3$ ramified along $\widehat{\widehat{\gamma}}$. Then the Casson invariant $\lambda(M^{(2)}(\widehat{\widehat{\gamma}}))$ is given by*

$$\lambda\big(M^{(2)}(\widehat{\widehat{\gamma}})\big) = \frac{1}{12}\big(\pi(\beta) - \theta_0\big(\sigma \circ A_2'(\beta)\big)\big) + \frac{1}{8} s(\widehat{\beta}\,).$$

Proof. — This follows from Corollary 4.5 and Proposition 4.6.

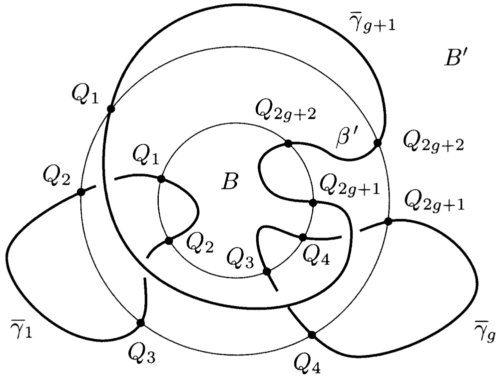

Figure 4.7

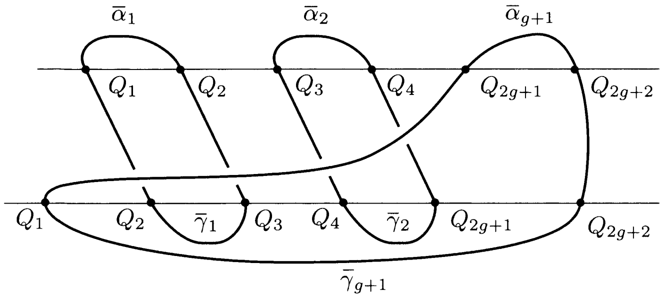

Figure 4.8

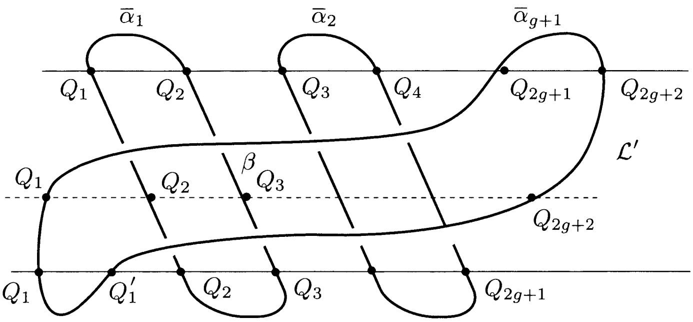

Figure 4.9

Final remark. — One should compare formula of Corollary 4.7 with Mullins formula [Mu] giving the Casson invariant of the 2-fold cyclic covering of $\mathbb{S}^3$ ramified along a link L

$$\lambda_2'(L) = \frac{-(\mathrm{d}V_L/\mathrm{d}t)(-1)}{12V_L(-1)} + \frac{1}{8}s(L),$$

where V_L (t) is the Jones polynomial of L (be aware that Mullins formula in Theorem 5.1 of [Mu] gives two times Casson's invariant).

BIBLIOGRAPHY

[B] J. BIRMAN, Braids, links and mapping class groups, Ann. of Math. Stud., vol 82, Princeton Univ. Press, Princeton, 1974.

[C] A. CASSON, Lectures at MSRI, 1985.

[GG] J.-M. GAMBAUDO, E. GHYS, Braids and Signatures, to appear in Bull. Math. Soc. France.

[GL] C.M.cA. GORDON, R. Litherland, On the signature of a link, Invent. Math., 47 (1978), 53–69.

[G] H.B. GRIFFITHS, Automorphisms of a 3-dimensional handlebody, Abh. Math. Sem. Univ. Hamburg, 26 (1964), 191–210.

[GM] L. GUILLOU, A. MARIN, Notes sur l'invariant de Casson des sphères d'homologie de dimension 3, Enseign. math., 38 (1992), 233–290.

[J1] D. JOHNSON, An abelian quotient of the mapping class group $\mathcal{I}_g$, Math. Ann., 249 (1980), 225–242.

[J2] D. JOHNSON, The structure of the Torelli group I, Annals of Math., 118 (1983), 423–442.

[J3] D. JOHNSON, The structure of the Torelli group II, Topology, 24 (1985), 113–126.

[Me1] W. MEYER, Die signatur von lokalen koeffizientensystemen und Faserbündeln, Bonner Mathematische Schriften, 53 (1972).

[Me2] W. MEYER, Die signatur von Flächenbündeln, Math. Ann., 201 (1973), 239–264.

[Mo1] S. MORITA, Casson's invariant for homology 3–spheres and characteristic classes of surface bundles I, Topology, 28 (1989), 305–323.

[Mo2] S. MORITA, On the structure of the Torelli group and the Casson invariant, Topology, 30 (1991), 603–621.

[Mo3] S. MORITA, The extension of Johnson's homomorphism from the Torelli group to the mapping class group, Invent. Math., 111 (1993), 197–224.

[Mo4] S. MORITA, The structure of the mapping class group and characteristic classes of surface bundles, in 'Mapping class groups and Moduli spaces of Riemann surfaces', Contemp. Math., 150 (1993), 303–315.

[Mo5] S. MORITA, Abelian quotients of subgroups of the mapping class group of surfaces, Duke Math. J., 70 (1993), 699–726.

[Mo6] S. MORITA, Casson invariant, signature defect of framed manifolds and second characteristic classes of surface bundles, J. Diff. Geom., 47 (1997), 560–599.

[Mu] D. MULLINS, The generalized Casson invariant for 2-fold branched covers of $\mathbb{S}^3$ and the Jones polynomial, Topology, 32 (1993), 419–438.

[Pe] B. PERRON, Homomorphic extensions of Johnson homomorphisms via Fox Calculus, this volume, 2004.

[Pi] W. PITSCH, Une construction intrinsèque du cœur de l'invariant de Casson, Ann. Inst. Fourier, 51-6 (2001), 1741–1761.

[PV] B. PERRON, J.-P. VANNIER, Groupe de monodromie géométrique des singularités simples, Math. Ann., 306 (1996), 231–245.

[Po] J. POWELL, Two theorems on the mapping class group of surfaces, Proc. Amer. Math. Soc., 68 (1978), 347–350.

[R] D. ROLFSEN, Knots and links, Publish or Perish Inc., Berkeley CA., 1976.

[Si] J. SINGER, Three dimensional manifolds and their Heegard diagrams, Trans. Amer. Math. Soc., 35 (1933), 88–111.

[S] S. SUZUKY, On homeomorphisms of a 3-dimensional handlebody, Can. J. Math., 29 (1977), 111–124.

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