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HOMOMORPHIC EXTENSIONS OF JOHNSON HOMOMORPHISMS VIA FOX CALCULUS

by Bernard PERRON

0. Introduction: review of Johnson's and Morita's results.

0.1. — In a remarkable series of papers, Johnson [J1], [J2], [J3], then Morita [Mo1], [Mo2], [Mo3], [Mo4], proved a lot of results enlightening the structure of the mapping class group $\mathcal{M}_{g,1}$ of a surface $S_{g,1}$ and some of its subgroups. The purpose of this paper is to reprove some of their results in a simpler and unified way. The emphasis will be made on the construction by Morita [Mo3], [Mo4] of extensions to the whole mapping class group of the first two Johnson's homomorphisms τ_2, τ_3. Morita's methods are sophisticated and use deep tools, such as Malcev completion of nilpotent groups, Sullivan's minimal models, Levi-Chevalley decomposition theorem, etc.

0.2. — The main tool in our approach is a new way to define the Johnson-Morita homomorphisms. In fact, for each integer $k \in N^*$, using Fox differential calculus, we define a map $A_k : \mathcal{M}_{g,1} \to \otimes^{k+2} H$ (where $H = H_1(S_{g,1};\mathbb{Z})$). The map A_k, when restricted to an appropriate subgroup of $\mathcal{M}_{g,1}$, becomes a homomorphism closely related to the Johnson-Morita homomorphism τ_{k+1}.

This natural extension (as map) of the second (resp third) Johnson-Morita homomorphism τ_2 (resp τ_3) allows us to construct in an elementary way, a homomorphic extension of τ_2 (resp τ_3) to the whole mapping class

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group $\mathcal{M}_{g,1}$. We thus recover the main result of [Mo3] (resp. Theorem 2.2 of [Mo4], which has been stated without proof).

The relevance of Fox differential calculus is not a surprise, and was known of Johnson and Morita (see for example [Mo5]) but never exploited. Of course, most of the beautiful results of Morita (concerning cohomology of the mapping class group) are out of the scope of our methods.

0.3. — First we make a brief review of Johnson's and Morita's results. Let $S_{g,1}$ denote a compact, connected, oriented surface of genus g with one boundary component. Let Γ (resp. H) denote the fundamental group (resp. the first homology group with integer coefficients) of $S_{g,1}$, based at a point $*$ of $\partial S_{g,1}$.

Let $\mathcal{M}_{g,1}$ denote the mapping class group of $S_{g,1}$, that is, the group of isotopy classes of homeomorphisms of $S_{g,1}$ inducing identity on the boundary, the isotopy being fixed on the boundary.

0.4. — The homology group $H = H_1(S_{g,1};\mathbb{Z})$ is equipped with the bilinear form, denoted (.), given by algebraic intersection number in $S_{g,1}$. This bilinear form is antisymmetric and nondegenerate.

The mapping $B_0 : \mathcal{M}_{g,1} \to \mathrm{GL}(H)$ which assigns to a (class of) homeomorphism f the isomorphism induced on H has, in fact, its image in the symplectic group $\mathrm{Sp}(H, (.))$.

By fixing a symplectic basis this group is identified to $\mathrm{Sp}(2g,\mathbb{Z})$. A classical result (see [KMS], Theorem N13, Section 3.7) says that the map

$$B_0 : \mathcal{M}_{g,1} \longrightarrow \mathrm{Sp}(2g,\mathbb{Z})$$

is surjective.

0.5. — Let $\{\Gamma_k\}$ denote the lower central series of Γ, defined by $\Gamma_1 = \Gamma$, $\Gamma_k = [\Gamma_{k-1}, \Gamma]$, where $[,]$ denotes the normal subgroup of Γ generated by the commutators $[a,b] = aba^{-1}b^{-1}$, where $a \in \Gamma_{k-1}$ and $b \in \Gamma$.

Let $N_k = \Gamma_1/\Gamma_k$ be the k-th nilpotent quotient of Γ. Set $\mathcal{L}_k = \Gamma_k/\Gamma_{k+1}$, so that we have a central extension

$$1 \to \mathcal{L}_k \longrightarrow N_{k+1} \longrightarrow N_k \to 1.$$

The natural action of $\mathcal{M}_{g,1}$ on $\Gamma = \Gamma_1$ induces an action on each N_k, so there is a representation $\rho_k : \mathcal{M}_{g,1} \to \mathrm{Aut}(N_k)$, where $\mathrm{Aut}(N_k)$ denotes the group of automorphisms of N_k.

This defines a filtration $\{\mathcal{M}(k)\}$ of $\mathcal{M}_{g,1}$ by setting $\mathcal{M}(k) = \mathrm{Ker}(\rho_k)$. In particular, $\mathcal{M}(2)$ is the normal subgroup of $\mathcal{M}_{g,1}$ consisting of homeomorphisms of $S_{g,1}$ inducing identity on the homology group H. This subgroup $\mathcal{M}(2)$, usually called the Torelli group of $S_{g,1}$, will be also denoted by $\mathcal{I}_{g,1}$.

0.6. — Using the centrality of the above extension, one can define as follows, for each $k \geq 2$ a homomorphism

$$\tau_k : \mathcal{M}(k) \longrightarrow \mathrm{Hom}(H, \mathcal{L}_k) \simeq \mathcal{L}_k \otimes H^* \simeq \mathcal{L}_k \otimes H.$$

Let f_k denote the automorphism of N_k induced by f. Since $f \in \mathcal{M}(k)$, $f_k = \mathrm{id}$. For each $x \in N_{k+1}$, $f_{k+1}(x)x^{-1} \in \mathcal{L}_k$. Let $\varphi : N_{k+1} \to \mathcal{L}_k$ denote the map defined by $\varphi(x) = f_{k+1}(x)\, x^{-1}$. Using the centrality of $\mathcal{L}_k$, it is easy to see that φ is a homomorphism. Since $\mathcal{L}_k$ is abelian, φ induces a homomorphism $\overline{\varphi} : H \to \mathcal{L}_k$.

Then define $\tau_k : \mathcal{M}(k) \longrightarrow \mathrm{Hom}(H, \mathcal{L}_k) \simeq \mathcal{L}_k \otimes H^*$ by $\tau_k(f) = \overline{\varphi}$. It is easy to see that τ_k is a homomorphism, called the k-th Johnson homomorphism.

The case $k = 2$ is easy to handle. The group $\mathcal{L}_2$ is identified with $\wedge^2 H$, the second exterior power of H, by sending $[x, y] \in \mathcal{L}_2$ onto $[x] \wedge [y] \in \wedge^2 H$, where $x, y \in \Gamma$ and $[x], [y]$ are the corresponding classes in H. So τ_2 sends $\mathcal{M}(2) = \mathcal{I}_{g,1}$ into $(\wedge^2 H) \otimes H$.

0.7. — Johnson [J1] identified the image of τ_2. Before stating his results, consider the oriented circles in $S_{g,1}$, x_i, y_i, y'_i, f_i, $i = 1, 2 \cdots g$, equipped with paths joining them to the base point $* \in \partial S_{g,1}$, defined by Figure 0.1 below:

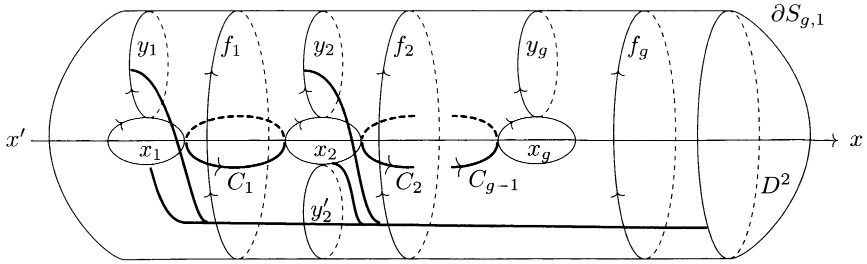

Figure 0.1

The oriented circles and the elements of the fundamental group $\Gamma = \pi_1(S_{g,1},*)$ they represent will be denoted by the same letter x_i, y_i, etc. Clearly the set $\{x_i, y_i\ ;\ i = 1, \ldots, g\}$ is a basis of the free group Γ.

Let a_i (resp. b_i) denote the homology class of x_i (resp. y_i). Then $\{a_i, b_i\ ;\ i = 1, \ldots, g\}$ is a symplectic basis of H since $a_i \cdot a_j = b_i \cdot b_j = 0$ and $a_i \cdot b_j = -b_j \cdot a_i = \delta_{i,j}$ (the Kronecker symbol).

For a simple closed curve c in $S_{g,1}$, let $D(c)$ denote the Dehn twist along c (see [B], §4.3 for the definition).

Then we can state some of Johnson's and Morita's results.

THEOREM 0.1 (see [J1], Theorem 1). — *For $g \geq 2$:*

(i) *The image of τ_2 is $\wedge^3 H \subset (\wedge^2 H) \otimes H$, where $\wedge^3 H$ is identified as a $\mathbb{Z}$-submodule of $(\wedge^2 H) \otimes H$ by sending $a \wedge b \wedge c$ on $(a \wedge b) \otimes c + (b \wedge c) \otimes a + (c \wedge a) \otimes b$).*

(ii) *$\tau_2 : \mathcal{M}(2) = \mathcal{I}_{g,1} \to \wedge^3 H$ respects the actions of $\mathcal{M}_{g,1}$ on $\mathcal{M}(2)$ (by conjugation) and on $\wedge^3 H$ (by $\varphi \cdot a \wedge b \wedge c = B_0(\varphi)(a) \wedge B_0(\varphi)(b) \wedge B_0(\varphi)(c)$).*

(iii) *$\tau_2(D(f_1)) = \tau_2(D(f_2)) = 0$ and $\tau_2(D(y_2)D(y_2')^{-1}) = a_1 \wedge b_1 \wedge b_2$ (where f_1, f_2, y_2, y_2' are the closed curves indicated in Figure 0.1).*

THEOREM 0.2 (see [J3], Theorem 5). — *$\mathcal{M}(3) = \operatorname{Ker} \tau_2$ is the normal subgroup of $\mathcal{M}_{g,1}$, normally generated by the Dehn twists $D(f_1)$, $D(f_2)$.*

THEOREM 0.3 (see [Mo3], Theorem 4.8). — *The Johnson's homomorphism $\tau_2 : \mathcal{I}_{g,1} \to \wedge^3 H$ extends to a homomorphism*

$$\bar{\tau}_2 : \mathcal{M}_{g,1} \longrightarrow \left(\tfrac{1}{2}\wedge^3 H\right) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$$

making the following diagram commutative:

$$\begin{array}{ccccccc}
1 \longrightarrow & \mathcal{I}_{g,1} & \longrightarrow & \mathcal{M}_{g,1} & \xrightarrow{B_0} & \mathrm{Sp}(2g,\mathbb{Z}) & \to 1 \\
& \downarrow \tau_2 & & & & & \\
& \wedge^3 H & & \Big\downarrow \bar{\tau}_2 & & \Big\downarrow \mathrm{id} & \\
& \downarrow & & & & & \\
1 \to & \frac{1}{2}\wedge^3 H & \longrightarrow & \left(\frac{1}{2}\wedge^3 H\right) \rtimes \mathrm{Sp}(2g,\mathbb{Z}) & \longrightarrow & \mathrm{Sp}(2g,\mathbb{Z}) & \to 1.
\end{array}$$

Here $\left(\frac{1}{2}\wedge^3 H\right) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$ denotes the semi-direct product of $\left(\frac{1}{2}\wedge^3 H\right)$ by $\mathrm{Sp}(2g,\mathbb{Z})$, the action of $\mathrm{Sp}(2g,\mathbb{Z})$ on $\left(\frac{1}{2}\wedge^3 H\right)$ being given by

$$\varphi_0 \cdot a \wedge b \wedge c = \varphi_0(a) \wedge \varphi_0(b) \wedge \varphi_0(c).$$

Moreover the image of $\bar{\tau}_2$ is of finite index (in fact a power of 2) in $\left(\frac{1}{2}\wedge^3 H\right) \rtimes \mathrm{Sp}(Zg,\mathbb{Z})$.

The proof of Morita is long and sophisticated: he uses tools such as Malcev completion of nilpotent groups, Sullivan's minimal models, Levi-Chevalley decomposition theorem, etc.

0.8. *Remark.* — The homomorphism $\overline{\tau}_2 : \mathcal{M}_{g,1} \to \left(\frac{1}{2}\wedge^3 H\right) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$ is equivalently given by a map $\varphi : \mathcal{M}_{g,1} \to \frac{1}{2}\wedge^3 H$ satisfying

$$\varphi(fg) = \varphi(f) + B_0(f)\cdot\varphi(g)$$

where $B_0(f) \in \mathrm{Sp}(2g,\mathbb{Z})$ and $(.)$ is the action of $\mathrm{Sp}(2g,\mathbb{Z})$ on $\left(\frac{1}{2}\wedge^3 H\right)$. Such a map will be called a crossed homomorphism. The map $\overline{\tau}_2$ is then defined by $\overline{\tau}_2(f) = (\varphi(f), B_0(f))$. Such a map φ defines a 1-cocycle belonging to $C^1\left(\frac{1}{2}\wedge^3 H, \mathrm{Sp}(2g,\mathbb{Z})\right)$.

0.9. *Remark.* — Our extension of τ_2 is slightly different from Morita's extension, but the two corresponding 1-cocycles define the same element of $H^1\left(\frac{1}{2}\wedge^3 H, \mathrm{Sp}(2g,\mathbb{Z})\right)$.

0.10. — In [Mo1], §1, Morita identified, up to finite index, the image of the third Johnson's homomorphism $\tau_3 : \mathcal{M}(3) \longrightarrow \mathcal{L}_3 \otimes H$. First, it is not too difficult to identify $\mathcal{L}_3$ with $(\wedge^2 H)\otimes H/\wedge^3 H$. So τ_3 maps $\mathcal{M}(3)$ into $(\wedge^2 H)\otimes H\otimes H/(\wedge^3 H)\otimes H$.

Let T denote the subgroup of $(\wedge^2 H)\otimes H\otimes H$ generated by elements

$$a\wedge b\otimes a\wedge b \quad \text{and} \quad a\wedge b \leftrightarrow c\wedge d = a\wedge b\otimes c\wedge d + c\wedge d\otimes a\wedge b,$$

for any a, b, c, d in H (here when we write $a\wedge b\otimes c\wedge d \in (\wedge^2 H)\otimes H\otimes H$, $c\wedge d$ is understood to be equal to $c\otimes d - d\otimes c$).

Let $\overline{T}$ be the image of T under the projection

$$p : (\wedge^2 H)\otimes H\otimes H \longrightarrow (\wedge^2 H)\otimes H\otimes H/(\wedge^3 H)\otimes H.$$

THEOREM 0.4 (Morita [Mo1], Proposition 1.2). — *The image of τ_3 is contained in $\overline{T}$ and is of finite index in $\overline{T}$.*

The next theorem of Morita has been announced in [Mo4] but the proof has not yet appeared.

THEOREM 0.5 (Extension of τ_3, [Mo4]). — *Johnson's homomorphism $\tau_3 : \mathcal{M}(3) \to \overline{T}$ extends to a homomorphism*

$$\overline{\tau}_3 : \mathcal{M}_{g,1} \longrightarrow \left(\overline{T} \mathbin{\tilde{\times}} \left(\tfrac{1}{2}\wedge^3 H\right)\right) \rtimes \mathrm{Sp}(2g,\mathbb{Z}),$$

where $\overline{T} \mathbin{\tilde{\times}} (\frac{1}{2}\wedge^3 H)$ *is a central extension of* $(\frac{1}{2}\wedge^3 H)$ *by* $\overline{T}$. *The semi-direct product* $\rtimes$ *is defined by the natural action of* $\mathrm{Sp}(2g,\mathbb{Z})$ *on* $\overline{T} \mathbin{\tilde{\times}} (\frac{1}{2}\wedge^3 H)$. *Moreover the image of* $\overline{\tau}_3$ *is of finite index in* $(\overline{T} \mathbin{\tilde{\times}} (\frac{1}{2}\wedge^3 H)) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$.

0.11. — In Chapter 7, we will construct a homomorphism $\sigma \circ A_2' : \mathcal{M}(3) \to T$, such that $\widetilde{A_2} = p \circ \sigma \circ A_2' = -12\tau_3$, where $p: T \to \overline{T}$ is the canonical projection. The extension of τ_3 (or equivalently of $\widetilde{A_2}$) we construct here is a little bit different from Morita's one. Precisely we construct two different extensions, each having its own advantage.

The first one, given in Proposition 7.4, is obtained as an easy exercice of group theory. The image of this extension is in a group $(\overline{T} \circ_{\widetilde{\mathcal{F}}} \wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$, but the image is not clearly understood.

The second extension, given in Propositions 7.5 and 7.5 bis, is slightly more difficult to define, but its image is of finite index in an explicitly defined group $\frac{1}{4}\overline{T} \circ_{\widetilde{G}} (3\otimes^3 H \rtimes \mathrm{Sp}(2g,\mathbb{Z}))$.

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1. An algebraic lemma.

1.1. — Let (H,ω) be a symplectic space of dimension $2g$ and $\{a_i, b_i\,;\ i = 1,\dots,g\}$ a symplectic basis, e.g., $\omega(a_i,a_j) = \omega(b_i,b_j) = 0$, $\omega(a_i,b_j) = -\omega(b_j,a_i) = \delta_{i,j}$ (the Kronecker symbol). We will usually write $x\cdot y$ for $\omega(x,y)$. Let $\mathrm{Sp}(H,\omega)$ be the space of symplectic isomorphisms of H, identified with $\mathrm{Sp}(2g,\,\mathbb{Z})$, using the symplectic basis $\{a_i, b_i\,;\ i = 1,\dots,g\}$. Let c_i $(i = 1,\dots,2g)$ denote a_i (resp. b_i) for $1 \le i \le g$ (resp. $g+1 \le i \le 2g$).

1.2. — Let $\mathcal{M}_{2g}(\mathbb{Z})$ denote the additive group of $2g \times 2g$ matrices with integer coefficients. Using the basis $c = \{c_i\,;\ i = 1,\dots,2g\}$ above and the intersection form ω, we have well-known isomorphisms

$$(*)\qquad \mathcal{M}_{2g}(\mathbb{Z}) \xrightarrow{\varphi_c} \mathrm{Hom}(H,H) \simeq H\otimes H^* \xrightarrow{\mathrm{id}\otimes d_\omega^{-1}} H\otimes H,$$

where $d_\omega : H \longrightarrow H^*$ is defined by $d_\omega(y)(x) = x\cdot y$.

Remark. — Note the following explicit computation of the above isomorphisms:

a) $\varphi_c(M) = \varphi_c(\alpha_{ij}) = \sum_{i,j=1}^{2g} \alpha_{ij}\ c_i \otimes c_j^*$, where (c_j^*) is the dual basis of c.

b) $\mathrm{id} \otimes d_\omega^{-1} : \mathrm{Hom}(H, H) \to H \otimes H$ is given by the following formula:

$$\mathrm{id} \otimes d_\omega^{-1}(\delta) = \sum_{i=1}^{g} \delta(a_i) \otimes b_i - \delta(b_i) \otimes a_i.$$

1.3. — The symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$ acts on the left by conjugation on $\mathcal{M}_{2g}(\mathbb{Z})$ and on $H \otimes H$ by $B \cdot (x \otimes y) = B(x) \otimes B(y)$. The isomorphism $(\mathrm{id}_H \otimes d_\omega^{-1}) \circ \varphi_c$ respects these actions: to see it, it is enough to show that $d_\omega(B(y)) = d_\omega y \circ B^{-1}$, which is obvious since $\mathrm{Sp}(2g,\mathbb{Z})$ preserves ω.

Now let A be a free abelian group on which $\mathrm{Sp}(2g,\mathbb{Z})$ acts linearly. Then $\mathrm{Sp}(2g,\mathbb{Z})$ acts on $\mathcal{M}_{2g}(A) = A \otimes \mathcal{M}_{2g}(\mathbb{Z})$ by

$$B \cdot M = B \times {}^{B}M \times B^{-1},$$

where $B \in \mathrm{Sp}(2g,\mathbb{Z})$, $M \in \mathcal{M}_{2g}(A)$ and ${}^{B}M = {}^{B}(a_{ij}) = (B \cdot a_{ij})$.

LEMMA 1.1. — *The tensor product by A of the sequence (*) above produces an isomorphism of abelian groups*

$$\varphi : \mathcal{M}_{2g}(A) \longrightarrow A \otimes H \otimes H$$

which respects the action of $\mathrm{Sp}(2g,\mathbb{Z})$ *on $\mathcal{M}_{2g}(A)$ described above and the obvious action on $A \otimes H \otimes H$ given by $B \cdot (a \otimes x \otimes y) = B \cdot a \otimes B(x) \otimes B(y)$.*

Remark. — The isomorphism φ can be explicited as follows: let e_{ij} be the $2g \times 2g$ matrix, the entries of which are zero, except the one at the place (i,j), which is equal to 1. Then $\varphi(ae_{ij}) = a \otimes \widetilde{e}_{ij}$ $(a \in A)$, where $\widetilde{e}_{ij}$ is the element of $H \otimes H$ defined in the following way. Consider the matrix M_0 with entries in $H \otimes H$, the j-th column of which is, for $1 \leq j \leq g$ (resp. $g < j < 2g$):

$$\begin{pmatrix} a_1 \\ \vdots \\ a_g \\ b_1 \\ \vdots \\ b_g \end{pmatrix} \otimes b_j \quad \Bigg(\text{resp.} \begin{pmatrix} a_1 \\ \vdots \\ a_g \\ b_1 \\ \vdots \\ b_g \end{pmatrix} \otimes (-a_{j-g})\Bigg).$$

Then $\widetilde{e}_{ij}$ is the (i,j) entry of M_0.

2. Review of Fox differential calculus (see [F], or [B]).

2.1. — Let Γ denote the free group generated by $z_1, \dots, z_n$. Let $\mathbb{Z}[\Gamma]$ denote the group ring of Γ, that is the set of finite linear combinations $\sum_i n_i g_i$ ($n_i \in \mathbb{Z}$, $g_i \in \Gamma$). The set $\mathbb{Z}[\Gamma]$ has an obvious structure of non commutative ring. Let $\varepsilon : \mathbb{Z}[\Gamma] \to \mathbb{Z}$ denote the evaluation map defined by

$$\varepsilon\Big(\sum_i n_i g_i\Big) = \sum_i n_i.$$

2.2. — We define the partial derivatives $\partial/\partial z_i : \mathbb{Z}[\Gamma] \to \mathbb{Z}[\Gamma]$ by

$$\frac{\partial z_j}{\partial z_i} = \delta_{ij}, \quad \frac{\partial(u+v)}{\partial z_i} = \frac{\partial u}{\partial z_i} + \frac{\partial v}{\partial z_i}, \quad \frac{\partial(uv)}{\partial z_i} = \varepsilon(v)\frac{\partial u}{\partial z_i} + u\frac{\partial v}{\partial z_i}\cdot$$

As an immediate consequence we have

$$\frac{\partial u^{-1}}{\partial z_i} = -u^{-1}\frac{\partial u}{\partial z_i}\cdot$$

2.3. — The fundamental formula of Fox differential calculus is (see [F], or [B], Prop. 3.4):

$$\alpha = \varepsilon(\alpha) + \sum_{i=1}^{n} \frac{\partial \alpha}{\partial z_i}(z_i - 1) \quad \text{for } \alpha \in \mathbb{Z}[\Gamma].$$

2.4. — Let $\omega(x)$ be a word in the letters $x_1^{\pm 1}, \dots, x_n^{\pm 1}$, each x_i being a word in the variables $z_1^{\pm 1}, \dots, z_n^{\pm 1}$. Then we have the following formula (derivation of a composition, see [B], Prop. 3.3):

$$\frac{\partial \omega(x_i(z))}{\partial z_j} = \sum_{k=1}^{n} \frac{\partial \omega}{\partial x_k}\big(x(z)\big) \cdot \frac{\partial x_k}{\partial z_j}\cdot$$

2.5. — We have the obvious definition of derivations of higher order, by setting

$$\frac{\partial^2 \omega}{\partial z_i \partial z_j} = \frac{\partial}{\partial z_i}\Big(\frac{\partial \omega}{\partial z_j}\Big).$$

The fundamental formula of 2.3 can be generalized as follows (see [F], §3). For any $\alpha \in \mathbb{Z}[\Gamma]$ and $k \in \mathbb{N}$:

$$\alpha = \varepsilon(\alpha) + \sum_{\delta_1} \frac{\partial \alpha}{\partial z_{\delta_1}}(1)(z_{\delta_1} - 1)$$
$$+ \cdots + \sum_{\delta_k,\ldots,\delta_1 \in \{1,\ldots,n\}} \frac{\partial^k \alpha}{\partial z_{\delta_k} \cdots \partial z_{\delta_1}}(1)(z_{\delta_k} - 1) \cdots (z_{\delta_1} - 1)$$
$$+ \sum_{\delta_{k+1},\delta_k,\ldots,\delta_1 \in \{1,\ldots,n\}} \frac{\partial^{k+1} \alpha}{\partial z_{\delta_{k+1}} \cdots \partial z_{\delta_1}}(z_{\delta_{k+1}} - 1) \cdots (z_{\delta_1} - 1)$$

where $\partial^k \alpha / \partial z_{\delta_k} \cdots \partial z_{\delta_1}(1) = \varepsilon(\partial^k \alpha / \partial z_{\delta_k} \cdots \partial z_{\delta_1})$.

2.6. *Magnus representation.* — Let $F(\omega_1, \ldots, \omega_n)$ denote the ring of formal series in the non-commutative variables $\omega_1, \ldots, \omega_n$. Then, using formula of 2.5, we have a representation $\mathbb{Z}[\Gamma] \to F(\omega_1, \omega_2, \cdots \omega_n)$ given by

$$\alpha \in \mathbb{Z}[\Gamma] \longmapsto \varepsilon(\alpha) + \sum_{\delta_1} \frac{\partial \alpha}{\partial z_{\delta_1}}(1)\, \omega_{j_1}$$
$$+ \cdots + \sum_{\delta_1,\ldots,\delta_k \in \{1,\ldots,n\}} \frac{\partial^k \alpha}{\partial z_{\delta_k} \cdots \partial z_{\delta_1}}(1)\, \omega_{\delta_k} \omega_{\delta_{k-1}} \cdots \omega_{\delta_1} + \cdots.$$

It is well known that this representation is the same as Magnus representation defined by sending $z_i \in \Gamma$ on $1 + \omega_i \in F(\omega)$ and z_i^{-1} onto $(1+\omega_i)^{-1} = 1 - \omega_i + \omega_i^2 \cdots + (-1)^k \omega_i^k + \cdots$. This representation is known to be injective [M].

Definition 2.1. — *In formula of* 2.6, *the term*

$$\sum_{\delta_1,\ldots,\delta_k \in \{1,\ldots,n\}} \frac{\partial^k \alpha}{\partial z_{\delta_k} \cdots \partial z_{\delta_1}}(1)\, \omega_{\delta_k} \cdots \omega_{\delta_1}$$

will be called the homogeneous part of degree k *of* α *and*

$$j^k \alpha = \varepsilon(\alpha) + \sum_{\delta_1} \cdots + \sum_{\delta_1,\ldots,\delta_k \in \{1,\ldots,n\}} \frac{\partial^k \alpha}{\partial z_{\delta_k} \cdots \partial z_{\delta_1}}(1)\, \omega_{\delta_k} \cdots \omega_{\delta_1}$$

the k-th jet of α.

2.7. — Let $I \subset \mathbb{Z}[\Gamma]$ denote the augmentation ideal of $\mathbb{Z}[\Gamma]$, e.g., $I = \mathrm{Ker}(\varepsilon : \mathbb{Z}[\Gamma] \to \mathbb{Z})$ and I^k its k-th power. Then for any $k \in \mathbb{N}$, I^k is generated as an ideal by $\{(z_{j_k} - 1) \cdots (z_{j_1} - 1)\ ;\ j_1, \ldots j_k \in \{1, \ldots, n\}\}$.

2.8. *Remark.* — Under the Magnus representation $\mathbb{Z}[\Gamma] \to F(\omega_1, \ldots, \omega_n)$, the ideal I^k is sent into the ideal generated by $\omega_{j_1} \cdots \omega_{j_k}$ for $j_1, \ldots j_k \in \{1, \ldots, n\}$.

2.9. — Set $I_k = I^k / I^{k+1}$. It is an abelian group isomorphic to the free abelian group generated by $\{(z_{j_k} - 1) \cdots (z_{j_1} - 1)\,;\ j_1, \ldots, j_k \in \{1, \ldots, n\}\}$. We can also identify I_k with the tensor product $\otimes^k H$, where H is the abelianization of Γ, by sending $(z_{j_k} - 1) \cdots (z_{j_1} - 1)$ on $c_{j_k} \otimes \cdots \otimes c_{j_1}$, where c_{j_k} is the image of $z_{j_k} \in \Gamma$ in H.

2.10. — Under the Magnus representation, I_k can also be identified with the additive subgroup of $F(\omega_1, \cdots, \omega_n)$ generated by the homogeneous monomials $\omega_{j_1} \cdots \omega_{j_k}$ of degree k, for $j_1, \ldots, j_k \in \{1, \ldots, n\}$.

2.11. — The following proposition establishes a link between the lower central series $\{\Gamma_k\}$ of Γ (see 0.5) and the filtration

$$\cdots \subset I^{k+1} \subset I^k \subset \cdots \subset I \subset \mathbb{Z}[\Gamma].$$

Proposition 2.2 (see [F], 4.5). — *For $\alpha \in \Gamma$ the following propositions are equivalent*:

1) $\alpha \in \Gamma_k$,

2) $\alpha - 1 \in I^k$,

3) $\partial^i \alpha / \partial z_{j_i} \cdots \partial z_{j_1}(1) = 0$ *for any i such that $1 \leq i < k$ and any i-uplets $(j_i, \ldots, j_1)$, $j_1, \ldots, j_i \in \{1, \ldots, n\}$.*

3. The Fox matrix of a homeomorphism $f \in \mathcal{M}_{g,1}$.

3.1. — We now return to the mapping class group $\mathcal{M}_{g,1}$ of a surface $S_{g,1}$ of genus g. Let $\Gamma = \pi_1(S_{g,1}, *)$. It is a free group, equipped with a "symplectic" basis $\{x_i, y_i\,;\ i = 1, \ldots, g\}$ defined in 0.7. Let $\{a_i, b_i\,;\ i = 1, \ldots, g\}$ be the corresponding symplectic basis of $H = H_1(S_{g,1}; \mathbb{Z})$.

3.2. — The same letter f will denote either an element of $\mathcal{M}_{g,1}$ or the induced isomorphism of Γ. For $i = 1, \ldots, 2g$, set

$$z_i = \begin{cases} x_i & \text{if } 1 \leq i \leq g, \\ y_{i-g} & \text{if } g < i \leq 2g, \end{cases} \qquad c_i = \begin{cases} a_i & \text{if } 1 \leq i \leq g, \\ b_{i-g} & \text{if } g < i < 2g. \end{cases}$$

Let $\overline{(\,)} : \mathbb{Z}[\Gamma] \to \mathbb{Z}[\Gamma]$ denote the anti-isomorphism $\overline{\sum_i n_i g_i} \mapsto \sum_i n_i g_i^{-1}$ where $n_i \in \mathbb{Z}$ and $g \in \Gamma$.

DEFINITION 3.1. — *The* Fox matrix *of $f \in \mathcal{M}_{g,1}$ is the $2g \times 2g$ matrix with coefficients in $\mathbb{Z}[\Gamma]$ defined by*

$$B(f) = \begin{pmatrix} \overline{\partial f(z_1)/\partial z_1} & \dots & \overline{\partial f(z_{2g})/\partial z_1} \\ \overline{\partial f(z_1)/\partial z_2} & \dots & \overline{\partial f(z_{2g})/\partial z_2} \\ \vdots & & \vdots \\ \overline{\partial f(z_1)/\partial z_{2g}} & \dots & \overline{\partial f(z_{2g})/\partial z_{2g}} \end{pmatrix}.$$

The reason we apply the anti isomorphism $\overline{(\,)}$ is given by the following lemma:

LEMMA 3.2. — *For $f, g \in \mathcal{M}_{g,1}$ we have $B(f \circ g) = B(f) \times {}^f B(g)$ where $\times$ is the usual multiplication of matrices and ${}^f B(g)$ is defined by*

$${}^f B(g) = {}^f(a_{ij}) = \big(f(a_{ij})\big) \quad (a_{ij} \in \mathbb{Z}[\Gamma]).$$

As a consequence, $B(f)$ belongs to $\mathrm{GL}_{2g}\ (\mathbb{Z}[\Gamma])$, the group of invertible matrices with coefficients in $\mathbb{Z}[\Gamma]$.

Proof. — This follows easily from 2.4. □

3.3. — As in 2.6, we associate to each $\alpha \in \mathbb{Z}[\Gamma]$ a formal series in $F(u_1, \dots u_g, v_1, \dots v_g)$, where u_i (resp. v_i) corresponds to $x_i - 1$ (resp. $y_i - 1$). Doing this for all entries of the Fox matrix $B(f)$, we associate to f the formal series of matrices

$$B_0(f) + \cdots + B_k(f) + \cdots,$$

where $B_k(f)$ is a $2g \times 2g$ matrix with entries in I_k, the abelian group generated by $\{w_{j_k} w_{j_{k-1}} \cdots w_{j_1}\ ;\ 1 \leq j_\imath \leq 2g\}$ where $\{w_{j_\imath}\}$ are non-commutative variables defined by $w_i = u_i$ if $1 \leq i \leq g$ and $w_i = v_{i-g}$ if $g < i \leq 2g$: they correspond to either $x_\imath - 1$ or $y_i - 1$ (see 3.2 and 2.6).

3.4. — By 2.6 the element $a_{ij}^{(k)}$ of $B_k(f)$ is given by

$$a_{ij}^{(k)} = \sum_{1 \leq j_k, \dots, j_1 \leq 2g} \frac{\partial^k}{\partial z_{j_k} \cdots \partial z_{j_1}} \Big(\overline{\frac{\partial f(z_j)}{\partial z_i}}\Big)(1)\, w_{j_k} \cdots w_{j_1}.$$

Hence $B_k(f)$ belongs to $\mathcal{M}_{2g}(I_k)$, the additive group of $2g \times 2g$ matrices with coefficients in I_k.

Recall that we have identified I_k with the tensor product $\otimes^k H$ in 2.7 by sending $(z_{j_k}-1)\cdots(z_{j_1}-1)$ (or equivalently $\omega_{j_k}\cdots\omega_{j_1}$) on $c_{j_k}\otimes\cdots\otimes c_{j_1}$, where c_{j_i} is defined in 3.2. Finally $B_k(f)$ appears as an element of $\mathcal{M}_{2g}(\otimes^k H)$. The following is well-known:

LEMMA 3.3. — *$B_0(f)$ is the matrix of the isomorphism induced on H by f, in the symplectic basis $\{a_i,b_i;\ i=1,\dots,g\}$.*

3.5. — Define the filtration $\cdots\subset\mathcal{M}'(k)\subset\mathcal{M}'(k-1)\subset\cdots\subset\mathcal{M}_{g,1}$ by

$$\mathcal{M}'(1)=\mathcal{M}_{g,1},$$

$$\mathcal{M}'(2)=\{f\in\mathcal{M}_{g,1}\ ;\ B_0(f)=I\}$$

$$\mathcal{M}'(k+2)=\big\{f\in\mathcal{M}_{g,1}\ ;\ B_0(f)=I, B_1(f)=\cdots=B_k(f)=0\big\}\quad (k\geq 0).$$

LEMMA 3.4. — *The filtration $\{\mathcal{M}'(k);\ k\geq 1\}$ coincides with Johnson's filtration defined in* 0.5.

Proof. — This follows immediately from Proposition 2.2. □

3.6. — For $\alpha\in\mathbb{Z}[\Gamma]$, let $\alpha^{(k)}\in I_k\simeq\otimes^k H$ denote its homogeneous part of degree k (Definition 2.1).

LEMMA 3.5. — *For $f\in\mathcal{M}_{g,1}$ and $\alpha\in I^k$, we have*

$$\big(f(\alpha)\big)^{(k)}=B_0(f)\cdot\alpha^{(k)},$$

where '$\cdot$' is the obvious action of $\mathrm{Sp}(2g,\mathbb{Z})$ *on I_k.*

Proof. — It is sufficient to prove the lemma for

$$\alpha=(z_{j_1}-1)\cdots(z_{j_k}-1)\simeq c_{j_1}\otimes\cdots\otimes c_{j_k}.$$

Then by 2.5:

$$\begin{aligned}f(\alpha)&=\big(f(z_{j_1})-1\big)\cdots\big(f(z_{j_k})-1\big)\\&=\Big(\sum_{i_1}\frac{\partial f(z_{j_1})}{\partial z_{i_1}}(1)(z_{i_1}-1)+I^2\Big)\cdots\Big(\sum_{i_k}\frac{\partial f(z_{j_k})}{\partial z_{i_k}}(1)(z_{i_k}-1)+I^2\Big).\end{aligned}$$

Then

$$\begin{aligned}\big(f(\alpha)\big)^{(k)}&=\Big(\sum_{i_1}\frac{\partial f(z_{j_1})}{\partial z_{i_1}}(1)(z_{i_1}-1)\Big)\cdots\Big(\sum_{i_k}\frac{\partial f(z_{j_k})}{\partial z_{i_k}}(1)(z_{i_k}-1)\Big)\\&\simeq\Big(\sum_{i_1}\frac{\partial f(z_{j_1})}{\partial z_{i_1}}(1)c_{i_1}\Big)\otimes\cdots\otimes\Big(\sum_{i_k}\frac{\partial f(z_{j_k})}{\partial z_{i_k}}(1)c_{i_k}\Big)\end{aligned}$$

(under the identification $I_k \simeq \otimes^k H$). Remark that $\partial f(z_{j_1})/\partial z_{i_1}(1)$ is the (i_1, j_1) entry of $B_0(f)$. Then $\sum_{i_e} \partial f(z_{j_e})/\partial z_{i_e}(1) c_{i_e}$ is equal to $B_0(f)(c_{i_e})$. Lemma 3.5 follows. □

DEFINITION 3.6. — *Let ψ denote the bilinear map*

$$\psi : \mathcal{M}_{2g}(H) \times H \longrightarrow I_2 \simeq \otimes^2 H$$

defined by $\psi(M, c_j) = {}^M c_j = \sum_i h_{ij} \otimes c_i$, where c_j is defined in 3.2 and h_{ij} is the (i,j) entry of M.

LEMMA 3.7. — *We have a commutative diagram*

$$\begin{tikzcd}
\mathcal{M}_{2g}(H) \otimes H \arrow[r, "\psi"] \arrow[d, "\varphi \otimes \mathrm{id}"'] & \otimes^2 H \arrow[d, "\mathrm{id}"] \\
\otimes^4 H = \otimes^3 H \otimes H \arrow[r, "-C_{34}"] & \otimes^2 H
\end{tikzcd}$$

where φ is the map given in the fundamental algebraic Lemma 1.1 and C_{34} is the contraction defined by $C_{34}(x \otimes y \otimes z \otimes u) = (z \cdot u) x \otimes y$ (where $(\cdot)$ is the intersection pairing).

Proof. — Consider the following diagram

$$\begin{tikzcd}
\mathcal{M}_{2g}(H) \otimes H \arrow[r, "\psi"] \arrow[d, "\mathrm{id} \otimes \varphi_c \otimes \mathrm{id}"'] \arrow[dr, phantom, "(1)"] & \otimes^2 H \arrow[d, equal] \\
H \otimes (H \otimes H^*) \otimes H \arrow[r, "C'_{34}"] \arrow[d, "\mathrm{id} \otimes d_\omega^{-1} \otimes \mathrm{id}"'] \arrow[dr, phantom, "(2)"] & \otimes^2 H \arrow[d, equal] \\
H \otimes (H \otimes H) \otimes H \arrow[r, "-C_{34}"] & \otimes^2 H
\end{tikzcd}$$

where the vertical sequence is the sequence (*) of 1.2 tensored by H on the right and the left, and C'_{34} is defined by

$$C'_{34}(x \otimes y \otimes z^* \otimes u) = z^*(u) x \otimes y.$$

By definition of d_ω (in 1.2), diagram (2) commutes. So, to prove Lemma 3.7, it is sufficient to prove that diagram (1) commutes. By bilinearity, it is sufficient to prove commutativity for $h \otimes e_{ij} \otimes c_k$. By the remark in 1.2:

$$\mathrm{id} \otimes \varphi_c \otimes \mathrm{id}(h \otimes e_{ij} \otimes c_k) = h \otimes c_i \otimes c_j^* \otimes c_k.$$

Applying C'_{34} we get $\delta_{jk} h \otimes c_i$, which is precisely $\psi(h \otimes e_{ij} \otimes c_k)$. □

COROLLARY 3.8. — *Let Ψ denote the bilinear map*

$$\Psi\colon \mathcal{M}_{2g}(H)\otimes\mathcal{M}_{2g}(H)\longrightarrow\mathcal{M}_{2g}(\otimes^2 H)$$

defined by $\Psi(M,A)={}^{M}A={}^{M}(a_{ij})=({}^{M}a_{ij})$, *where* ${}^{M}a_{ij}$ *has been defined in Definition* 3.6. *Then we have a commutative diagram*

$$\begin{tikzcd}
\mathcal{M}_{2g}(H)\otimes\mathcal{M}_{2g}(H) \arrow[rr, "\Psi"] \arrow[d, "\varphi\otimes\varphi"'] & & \mathcal{M}_{2g}(\otimes^2 H) \arrow[d, "\varphi"] \\
\big(H\otimes(\otimes^2 H)\big)\otimes\big(H\otimes(\otimes^2 H)\big)=\otimes^6 H \arrow[rr, "-C_{34}"] & & (\otimes^2 H)\otimes(\otimes^2 H)
\end{tikzcd}$$

where φ is the isomorphism given in Lemma 1.1 *and C_{34} is the contraction*

$$C_{34}(x_1\otimes x_2\cdots\otimes x_6)=(x_3\cdot x_4)x_1\otimes x_2\otimes x_5\otimes x_6.$$

LEMMA 3.9. — *Let* $\widetilde{\psi}\colon\mathcal{M}_{2g}(H)\otimes\mathcal{M}_{2g}(H)\to\mathcal{M}_{2g}(\otimes^2 H)$ *denote the bilinear map defined by the usual multiplication of matrices. Then we have a commutative diagram:*

$$\begin{tikzcd}
\mathcal{M}_{2g}(H)\otimes\mathcal{M}_{2g}(H) \arrow[r, "\widetilde{\psi}"] \arrow[d, "\varphi\otimes\varphi"'] & \mathcal{M}_{2g}(\otimes^2 H) \arrow[d, "\varphi"] \\
(\otimes^3 H)\otimes(\otimes^3 H) \arrow[r, "-\tau_{23}\circ C_{35}"] & (\otimes^2 H)\otimes(\otimes^2 H)
\end{tikzcd}$$

where C_{35} is the contraction $C_{35}(x_1\otimes\cdots\otimes x_6)=(x_3\cdot x_5)x_1\otimes x_2\otimes x_4\otimes x_6$ *and τ_{23} is the permutation* $\tau_{23}(x_1\otimes x_2\otimes x_3\otimes x_4)=x_1\otimes x_3\otimes x_2\otimes x_4$.

Proof. — Consider the following diagram

$$\begin{tikzcd}
\mathcal{M}_{2g}(H)\otimes\mathcal{M}_{2g}(H) \arrow[rr, "\widetilde{\psi}"] \arrow[d, "\varphi_c\otimes\varphi_c"'] & (1) & \mathcal{M}_{2g}(\otimes^2 H) \arrow[d, "\varphi_c"] \\
H\otimes(H\otimes H^*)\otimes H\otimes(H\otimes H^*) \arrow[rr, "+\tau_{23}\circ C'_{35}"] \arrow[d, "\mathrm{id}\otimes\mathrm{id}\otimes d_\omega^{-1}\ \ \mathrm{id}\otimes\mathrm{id}\otimes d_\omega^{-1}"] & (2) & \otimes^2 H\otimes(H\otimes H^*) \arrow[d, "\mathrm{id}\otimes\mathrm{id}\ \ \mathrm{id}\otimes d_\omega^{-1}"] \\
H\otimes(H\otimes H)\otimes H\otimes(H\otimes H) \arrow[rr, "-\tau_{23}\circ C_{35}"] & & \otimes^2 H\otimes(H\otimes H)
\end{tikzcd}$$

where the vertical lines come from 1.2 and C'_{35} is defined by

$$C'_{35}(x_1\otimes x_2\otimes x_3^*\otimes x_4\otimes x_5\otimes x_6^*)=x_3^*(x_5)x_1\otimes x_2\otimes x_4\otimes x_6^*.$$

Diagram (1) commutes by definition of the composition. Diagram (2) commutes by definition of d_ω. $\square$

PROPOSITION 3.10. — *Let $\alpha \in I \subset \mathbb{Z}[\Gamma]$, and $f \in \mathcal{M}_{g,1}$. Then, with the notations of* 3.6, *we have*

$$\big(f(\alpha)\big)^{(2)} = -\psi\big(B_1(f), \alpha^{(1)}\big) + B_0(f) \cdot \alpha^{(2)} \in I_2.$$

(See 3.6 for the definition of $\alpha^{(1)}$, $\alpha^{(2)}$).

Proof. — By Lemma 3.5, it is sufficient to prove Proposition 3.10 for $\alpha = z_j - 1$. Consider $f(\overline{z_j} - 1) = \overline{f(z_j)} - 1 = \sum_{i=1}^{2g} \overline{\partial f(z_j)/\partial z_i (z_i - 1)}$ (by Fox formula 2.3). Then

$$\begin{aligned} f(\overline{z_j} - 1) &= \sum_{i=1}^{2g} (\overline{z_i} - 1) \frac{\overline{\partial f(z_j)}}{\partial z_i} \\ &= \sum_{i=1}^{2g} (\overline{z_i} - 1) \Big\{ \varepsilon \Big(\frac{\overline{\partial f(z_j)}}{\partial z_i} \Big) + \sum_{k=1}^{2g} \frac{\partial}{\partial z_k} \Big(\frac{\overline{\partial f(z_j)}}{\partial z_i} \Big)(1)(z_k - 1) \Big\} + I^3 \\ &= \sum_{i=1}^{2g} (\overline{z_i} - 1) \varepsilon \Big(\frac{\overline{\partial f(z_j)}}{\partial z_i} \Big) + \sum_{i=1}^{2g} (\overline{z_i} - 1) b_{ij}^{(1)} + I^3, \end{aligned}$$

where $b_{ij}^{(1)}$ is the (i,j) entry of $B_1(f)$ (see 3.4).

So the term of order 2 in $f(z_j) - 1$ is $\sum_{i=1}^{2g} (\overline{b_{ij}^{(1)}})^{(1)} (z_i - 1)$. But it is easy to see that $(\bar{\beta})^{(1)} = -\beta^{(1)}$, for any $\beta \in \mathbb{Z}[\Gamma]$. By definition, the term $-\sum_{i=1}^{2g} b_{ij}^{(1)} (z_i - 1)$ is exactly $-\psi(B_1(f), z_j - 1)$. □

4. The maps A_k and their relations to Johnson homomorphisms.

4.1. — For $f \in \mathcal{M}_{g,1}$, set

$$A_k(f) = B_k(f) \times B_0(f)^{-1} \in \mathcal{M}_{2g}(I_k),$$

where $B_k(f)$ has been defined in 3.3 (the product makes sense, since I_k is a $\mathbb{Z}$-module). By the fundamental algebraic Lemma 1.1, this defines maps, still denoted A_k:

$$A_k : \mathcal{M}_{g,1} \longrightarrow \mathcal{M}_{2g}(I_k) \simeq I_k \otimes H \otimes H \simeq (\otimes^k H) \otimes H \otimes H.$$

The next three lemmas and corollary present properties of the first two maps A_1 and A_2.

LEMMA 4.1. — *For any $f,g \in \mathcal{M}_{g,1}$ we have the following fundamental formulas:*

(i) $A_1(fg) = A_1(f) + B_0(f) \times {}^{B_0(f)}A_1(g) \times B_0(f)^{-1} \in \mathcal{M}_{2g}(H)$, *where the symbol* ${}^{B_0}A$ *has been defined in* 1.2, *for* $B_0 \in \mathrm{Sp}(2g,\mathbb{Z})$.

(ii) $A_1(fg) = A_1(f) + B_0(f).A_1(g) \in \otimes^3 H$, *where* (.) *is the obvious action of* $\mathrm{Sp}(2g,\ \mathbb{Z})$ *on* $\otimes^3 H$.

(iii) *The restriction of* A_1 *to* $\mathcal{I}_{g,1} = \mathcal{M}(2)$ *is a homomorphism into* $\otimes^3 H$ *satisfying* $A_1(f\varphi f^{-1}) = B_0(f) \cdot A_1(\varphi)$ *for any* $f \in \mathcal{M}_{g,1}$ *and* $\varphi \in \mathcal{I}_{g,1}$.

Proof. — By Lemma 3.2, we have

$$B(fg) = B(f) \times {}^f B(g) = \big(B_0(f) + B_1(f) + I^2\big)\, {}^f(B_0(g) + B_1(g) + I^2).$$

By definition ${}^f B_0(g) = B_0(g)$. By Lemma 3.5, the term of degree 1 of ${}^f B_1(g)$ is ${}^{B_0(f)}B_1(g)$. On the other hand, ${}^f B_k(g) \in \mathcal{M}_{2g}(I^k)$. It follows that $B_1(fg) = B_0(f) \times {}^{B_0(f)}B_1(g) + B_1(f) \times B_0(g)$. Point (i) follows.

Point (ii) is equivalent to (i), using the fundamental algebraic Lemma 1.1. Point (iii) is a direct consequence of (ii). □

Remark. — In the terminology of Morita, $A_1 : \mathcal{M}_{g,1} \to \otimes^3 H$ is a crossed homomorphism, by formula (ii) (the action of $\mathcal{M}_{g,1}$ on $\otimes^3 H$ is the obvious one, going through $\mathrm{Sp}(2g,\mathbb{Z})$).

LEMMA 4.2. — *For $f,g \in \mathcal{M}_{g,1}$ we have*

$$A_2(fg) = A_2(f) + B_0(f) \cdot A_2(g) + F\big(A_1(f), B_0(f) \cdot A_1(g)\big) \in \otimes^4 H,$$

where $B_0(f) \cdot (\ )$ *is the usual action of* $\mathrm{Sp}(2g,\mathbb{Z})$ *on* $\otimes^3 H$ *and* F *is the bilinear map* $(\otimes^3 H) \otimes (\otimes^3 H) \to \otimes^4 H$ *defined by* $F = C_{34} - \tau_{23} \circ C_{35}$, *where* C_{34} *and* $\tau_{23} \circ C_{35}$ *have been defined in Lemmas* 3.7 *and* 3.9, *respectively.*

Proof. — By Lemma 3.2 we have

$$B(fg) = \big(B_0(f) + B_1(f) + B_2(f) + I^3\big) \times {}^f\big(B_0(g) + B_1(g) + B_2(g) + I^3\big).$$

The term of degree 2 in $B(fg)$ is given by

$$\begin{aligned} B_2(fg) = B_0(f) \times \big({}^f B_2(g)\big)^{(2)} \\ + B_2(f) \times B_0(g) + B_1(f) \times \big({}^f B_1(g)\big)^{(1)} + B_0(f) \times \big({}^f B_1(g)\big)^{(2)}, \end{aligned}$$

where $(\ )^{(i)}$ denote the term of degree i.

By Lemma 3.5, $({}^{f}B_i(g))^{(i)} = {}^{B_0(f)}B_i(g)$ and by Proposition 3.10, $({}^{f}B_1(g))^{(2)} = -\Psi(B_1(f), B_1(g)) = -{}^{B_1(f)}B_1(g)$. Then we get

$$\begin{aligned}A_2(fg) &= B_2(fg) \times B_0(g)^{-1} \times B_0(f)^{-1}\\ &= A_2(f) + B_0(f)\cdot A_2(g) + A_1(f) \times B_0(f)\cdot A_1(g)\\ &\qquad - B_0(f) \times \Psi\big(B_1(f), B_1(g)\big) \times B_0(g)^{-1} \times B_0(f)^{-1}.\end{aligned}$$

We claim that the last term above is equal to $\Psi(A_1(f), B_0(f)\cdot A_1(g))$. This follows from the three formulas below where B_0 is an integer matrix and $A_1, B_1 \in \mathcal{M}_{2g}(H)$:

(i) $B_0 \times \Psi(B_1, A_1) = \Psi(B_1, B_0 \times A_1)$,

(ii) $\Psi(B_1, A_1) \times B_0 = \Psi(B_1, A_1 \times B_0)$,

(iii) $\Psi(B_1, {}^{B_0}A_1) = \Psi(B_1 \times B_0, A_1)$,

where ${}^{B_0}A_1$ has been defined by ${}^{B_0}(a_{ij}) = (B_0(a_{ij}))$.

The first two points are obvious from the definition of Ψ and ψ (Definition 3.6 and Corollary 3.8). For the third one, we have for $B_0 = (b^{(0)}_{k\ell})$ and $B_1 = (b^{(1)}_{ij})$:

$$\begin{aligned}\psi\big(B_1, B_0(c_j)\big) &= \psi\Big(B_1, \sum_k b^{(0)}_{kj} c_k\Big)\\ &= \sum_k b^{(0)}_{kj}\psi(B_1, c_k) = \sum_k b^{(0)}_{kj}\Big(\sum_i b^{(1)}_{ik} \otimes c_i\Big)\\ &= \sum_i\Big(\sum_k b^{(0)}_{kj} b^{(1)}_{ik}\Big) \otimes c_i = \psi(B_1 \times B_0, c_j).\end{aligned}$$

This completes the proof of Lemma 4.2, using Corollary 3.8 and Lemma 3.9. □

Corollary 4.3. — *The map A_2 restricted to $\mathcal{M}(3)$ is a homomorphism into $\otimes^4 H$, satisfying $A_2(\varphi f \varphi^{-1}) = B_0(\varphi)\cdot A_2(f)$, for any f in $\mathcal{M}(3)$ and φ in $\mathcal{M}_{g,1}$.* □

4.2. — We have reviewed in 0.6 the definition of the k-th Johnson homomorphism:

$$\tau_k : \mathcal{M}(k) \longrightarrow \mathrm{Hom}(H, \mathcal{L}_k) \simeq \mathcal{L}_k \otimes H^*$$

4.3. — Remark that $\mathcal{L}_k$ imbeds in $I_k \simeq \otimes^k H$ (the embedding is induced by the map $\Gamma_k \to I^k$ which sends $\alpha \in \Gamma_k$ to $\alpha - 1 \in I^k$, by Proposition 2.2). So τ_k appears as a homomorphism

$$\tau_k : \mathcal{M}(k) \longrightarrow (\otimes^k H) \otimes H^* \xrightarrow{\mathrm{id} \otimes d_\omega^{-1}} (\otimes^k H) \otimes H.$$

4.4. — The relation beetween A_k and τ_ℓ is given by

PROPOSITION 4.4. — *For any $k \geq 2$,*

$$\big(A_{k-1} \mid \mathcal{M}(k)\big) = (-1)^{k-1}(P \otimes \mathrm{id}) \circ \tau_k : \mathcal{M}(k) \longrightarrow \otimes^{k+1} H,$$

where $P \colon \otimes^{k-1} H \to \otimes^{k-1} H$ is the reversing isomorphism, *defined by $P(u_1 \otimes \cdots \otimes u_{k-1}) = u_{k-1} \otimes \cdots \otimes u_1$.*

Proof. — Let $f \in \mathcal{M}(k)$. Using (2.3) we can write

$$f(z_j)z_j^{-1} = 1 - (z_j - 1) + \sum_{i=1}^{2g} \frac{\partial f(z_j)}{\partial z_i}(z_i - 1) - \big(f(z_j)z_j^{-1} - 1\big)(z_j - 1).$$

4.5. — Since $f(z_j)z_j^{-1} - 1 \in I^k$ it follows that

$$f(z_j)z_j^{-1} - \Big(1 - (z_j - 1) + \sum_{i=1}^{2g} \frac{\partial f(z_j)}{\partial z_i}(z_i - 1)\Big) \in I^{k+1}$$

and $\partial f(z_j)/\partial z_i - \varepsilon(\partial f(z_j)/\partial z_i) \in I^{k-1}$. Using notation of 3.6, it is then easy to see that

$$\Big(\frac{\partial f(z_j)}{\partial z_i}\Big)^{(k-1)} = (-1)^{k-1} P\Big(\Big(\overline{\frac{\partial f(z_j)}{\partial z_i}}\Big)^{(k-1)}\Big),$$

where $P \colon \otimes^{k-1} H \to \otimes^{k-1} H$ is defined by $P(u_1 \otimes \cdots \otimes u_{k-1}) = u_{k-1} \otimes \cdots \otimes u_1$. We thus obtain

$$\big(f(z_j)z_j^{-1}\big)^{(k)} = (-1)^{k-1} \sum_{i=1}^{2g} P\Big(\Big(\overline{\frac{\partial f(z_j)}{\partial z_i}}\Big)^{(k-1)}\Big)(z_i - 1).$$

But $(\overline{\partial f(z_j)}/\partial z_i)^{(k-1)}$ is by definition $(B_{k-1}(f))_{i,j} = (A_{k-1}(f))_{i,j}$, the (i,j) entry of the matrix $A_{k-1}(f)$.

By definition, $\tau_k(f) \in I_k \otimes H^*$ $(\supset \mathcal{L}_k \otimes H^*)$ is given by

$$\begin{aligned}\tau_k(f) &= \sum_{j=1}^{2g} \tau_k(f)(c_j) \otimes c_j^* = \sum_{j=1}^{2g} \big(f(z_j)z_j^{-1}\big)^{(k)} \otimes c_j^* \\ &= (-1)^{k-1} \sum_{i,j=1}^{2g} P\big(A_{k-1}(f)\big)_{i,j} c_i \otimes c_j^* \\ &= (-1)^{k-1}(P \otimes \mathrm{id}) \cdot A_{k-1}(f) \in (\otimes^{k-1} H) \otimes H \otimes H^*.\end{aligned}$$

Applying the isomorphism $\mathrm{id} \otimes d_\omega^{-1}$ of 1.2, we get the result. $\square$

4.6. *Remarks.* — 1) Note that A_k, for $k \geq 2$, is defined on the whole mapping class group $\mathcal{M}_{g,1}$, compared with τ_k which is defined only on $\mathcal{M}(k)$. This will be a great avantage, used in chapters 5 and 7. But $A_k : \mathcal{M}_{g,1} \to \otimes^{k+2} H$ is no longer a homomorphism.

2) Johnson [J1], Theorem 1, proved that the image of τ_2 is contained in $\widetilde{\wedge^3 H} \subset \otimes^3 H$, the image of the injective homomorphism $\wedge^3 H \hookrightarrow \otimes^3 H$ defined by

$$x_1 \wedge x_2 \wedge x_3 \longmapsto \sum_{\sigma \in \mathcal{G}_3} \varepsilon(\sigma) x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes x_{\sigma(3)},$$

where $\mathcal{G}_3$ is the group of permutations of the set $\{1,2,3\}$, and $\varepsilon(\sigma)$ the signature of the permutation σ.

Let $\widetilde{A_1} : \mathcal{M}_{g,1} \to \wedge^3 H$ denote the composition of A_1 by the canonical homomorphism $\pi : \otimes^3 H \to \wedge^3 H$. It is obvious that the restriction of π to $\widetilde{\wedge^3 H}$ is $6\,\mathrm{id}_{\wedge^3 H}$. We then have from Proposition 4.4 and [J1], Theorem 1:

Corollary 4.5. — (i) *$A_{1|\mathcal{M}(2)} = \mathcal{I}_{g,1} \to \wedge^3 H$ has its image in $\widetilde{\wedge^3 H}$.*

(ii) *The composition $\widetilde{A_1} = \pi \circ A_{1|\mathcal{I}_{g,1}} \to \widetilde{\wedge^3 H} \xrightarrow{\pi} \wedge^3 H$ is equal to $-6\tau_2$.*

(iii) *$A_1(\mathcal{I}_{g,1}) = \widetilde{\wedge^3 H}$ and $\widetilde{A_1}(\mathcal{I}_{g,1}) = 6 \wedge^3 H \subset \wedge^3 H$.*

In fact, the third Johnson homomorphism τ_3 did not appear in the litterature in the form given by 4.3, but only through its composition with the canonical map $(\otimes^2 H) \otimes H \otimes H \to (\wedge^2 H) \otimes H) \otimes H / (\wedge^3 H \otimes H)$, where the inclusion $\wedge^3 H \subset (\wedge^2 H) \otimes H$ is given by

$$x \wedge y \wedge z \longmapsto (x \wedge y) \otimes z + (y \wedge z) \otimes x + (z \wedge x) \otimes y$$

(see [Mo1], [Mo2] and Chapter 6 for more details).

5. Extension of the second Johnson homomorphism.

Recall that, in Chapter 4, for any integer $k \geq 1$, we have defined maps

$$A_k : \mathcal{M}_{g,1} \longrightarrow \otimes^{k+2} H.$$

PROPOSITION 5.1 (Extension of $\widetilde{A_1} = -6\tau_2$).

(i) *The crossed homomorphism* $\widetilde{A_1} : \mathcal{M}_{g,1} \xrightarrow{A_1} \otimes^3 H \xrightarrow{\pi} \wedge^3 H$ *(by Lemma 4.1) has its image contained in* $3\wedge^3 H \subset \wedge^3 H$ *(we have seen in Corollary 4.5 that* $\widetilde{A_1}$ *sends* $\mathcal{M}(2)$ *onto* $6\wedge^3 H$*). This defines a (true) homomorphism* $\widetilde{A_1} \rtimes B_0$ *(the action of* $\mathrm{Sp}(2g,\mathbb{Z})$ *on* $3\wedge^3 H$ *is defined by* $g \cdot (x \wedge y \wedge z) = g(x) \wedge g(y) \wedge g(z)$*), making the following diagram commutative:*

$$\begin{tikzcd}
1 \arrow[r] & \mathcal{I}_{g,1} \arrow[r] \arrow[d, "\widetilde{A_1}"'] & \mathcal{M}_{g,1} \arrow[r] \arrow[dd, "\widetilde{A_1} \rtimes B_0 = \widetilde{A_{10}}"'] & \mathrm{Sp}(2g,\mathbb{Z}) \arrow[r] \arrow[dd, "\mathrm{id}"'] & 1 \\
& 6\wedge^3 H \arrow[d] & & & \\
1 \arrow[r] & 3\wedge^3 H \arrow[r] & (3\wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z}) \arrow[r] & \mathrm{Sp}(2g,\mathbb{Z}) \arrow[r] & 1.
\end{tikzcd}$$

Here $(3\wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$ *denotes the semi-direct product and* $\widetilde{A_1} \rtimes B_0$ *is defined by* $(\widetilde{A_1} \rtimes B_0)(f) = (\widetilde{A_1}(f),\, B_0(f))$.

(ii) *Moreover the image of* $\widetilde{A_1} \rtimes B_0$*, also denoted* $\widetilde{A_{10}}$*, is of finite index in* $(3\wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$.

Proof. — We know (see [H]) that the set of Dehn twists $D(y_1)$, $D(y_2)$, $D(x_i)$ $1 \leq i \leq g$ (defined by Figure 0.1) and $D(C_1), \dots, D(C_{g-1})$ (defined by Figure 5.1, below) is a system of generators for $\mathcal{M}_{g,1}$. Recall that the model for Dehn twist is the homeomorphism of $S^1 \times [-1,+1]$ defined by $D(\mathrm{e}^{i\theta}, t) = (\mathrm{e}^{\imath(\theta+\pi(1+t))}, t)$. To define the Dehn twist $D(c)$ along a simple closed curve c of $S_{g,1}$, choose an orientation preserving embedding $\varphi : S^1 \times [-1,+1] \to S_{g,1}$ such that $\varphi(S^1 \times \{0\}) = c$. On the image of φ we set $D(c) = \varphi \circ D \circ \varphi^{-1}$ and $D(c) =$ identity outside.

Using Lemma 4.1, (ii), to prove that the image of $\widetilde{A_1}$ is contained in $3 \overset{3}{\wedge} H$, it is enough to show it on a system of generators.

5.1. — Consider first $D(x_i)$; we have the following action on Γ: $D(x_i)(y_i) = y_i x_i$ and $D(x_i)(z_j) = z_j$ for $z_j \neq y_i$. It is easy to see, using notations of 1.2 that

$$B\big(D(x_i)\big) = I_{2g} + y_i a_i \otimes b_i^* \quad \text{and} \quad B\big(D(y_i)\big) = I_{2g} - y_i x_i^{-1} b_i \otimes a_i^*.$$

It follows immediately that $\widetilde{A_1}(D(x_i)) = \widetilde{A_1}(D(y_i)) = 0$ for all $i = 1, \dots, g$.

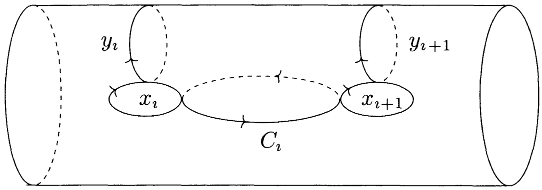

Figure 5.1

5.2. — We have the following formulas:

$$C_i = x_{i+1}y_{i+1}^{-1}x_{i+1}^{-1}y_i \in \Gamma, \quad D(C_i)(x_i) = x_i y_i^{-1}x_{i+1}y_{i+1}x_{i+1}^{-1}$$

$$D(C_i)(y_i) = C_i y_i C_i^{-1}, \qquad D(C_i)(x_{i+1}) = C_i x_{i+1}$$

(the other generators of Γ being fixed by $D(C_i)$). Straightforward computations show that

$$A_1\big(D(C_i)\big) = \begin{array}{c} \begin{array}{cccc} i & i{+}1 & i{+}g & \\ \end{array} \\ \begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \\ b_{i+1} & -b_i & 0 & 0 \\ a_i - b_i & b_{i+1} & b_{i+1} & 0 \\ -a_i + b_i - a_{i+1} & a_{i+1} - b_{i+1} & -b_i & 0 \end{pmatrix} \end{array} \begin{array}{l} \\ \\ i \\ i{+}1 \\ i{+}g \\ i{+}g{+}1 \end{array}$$

So $\widetilde{A_1}\ (D(C_i)) = 3(a_i + a_{i+1}) \wedge b_i \wedge b_{i+1} \in 3\wedge^3 H$. This proves the first part of point (i) of Proposition 5.1.

5.3. — Recall some well-known facts about semi-direct products of groups. Let G be a group acting on a group A. We define the semi-direct product $A \rtimes G$ as the cartesian product $A \times G$ with the following law:

$$(a, g)(a', g') = (a + g \cdot a', gg').$$

We then obtain a split exact sequence:

$$1 \to A \longrightarrow A \rtimes G \longrightarrow G \to 1.$$

Now let B be a group with a homomorphism $B \to G$ and a crossed homomorphism $\varphi : B \to A$ (e.g., $\varphi(xy) = \varphi(x) + \lambda(x) \cdot \varphi(y)$). Then the map $\varphi \rtimes \lambda : B \to A \rtimes G$ defined by $\varphi \rtimes \lambda(x) = (\varphi(x), \lambda(x))$ is a true homomorphism. We apply this construction to $B = \mathcal{M}_{g,1}$, $G = \mathrm{Sp}(2g, \mathbb{Z})$, $A = 3\wedge^3 H$, $\varphi = \widetilde{A_1}$ and $\lambda = B_0$.

Point (ii) follows immediately from the fact that in the diagram of Proposition 5.1, the image of $\widetilde{A_1}$ is of finite index in $3\wedge^3 H$. $\square$

5.4. *Remark.* — Proposition 5.1 defines an extension $\widetilde{A_{10}}$ of Johnson's homomorphism $\widetilde{A_1} = -6\tau_2\colon \mathcal{I}_{g,1} \to 6\wedge^3 H$ to all of $\mathcal{M}_{g,1}$. We thus recover Morita's result [Mo3], Theorem 4.8, in a very simple way.

5.5. *Remark.* — Our extension $\widetilde{A_{10}}\colon \mathcal{M}_{g,1} \to (3\wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$ is not exactly the same as Morita's one, which is denoted $\overline{\tau}_2$. In fact Morita [Mo3], Theorem 6.1, proved that two crossed homomorphisms from $\mathcal{M}_{g,1}$ into $\frac{1}{2}\wedge^3 H$, extending Johnson homomorphism $\tau_2\colon \mathcal{I}_{g,1} \longmapsto \wedge^3 H$ define the same homology class in $H^1(\mathcal{M}_{g,1};\frac{1}{2}\wedge^3 H)$, the action of $\mathcal{M}_{g,1}$ on $\frac{1}{2}\wedge^3 H$ being defined through $\mathrm{Sp}(2g,\mathbb{Z})$. Using computations in 5.1, 5.2, 5.3 and Proposition 4.7 of [Mo3], it is not difficult to see that the two crossed homomorphisms τ_2' and $\widetilde{A_1}$ corresponding to the two extension's $\overline{\tau}_2$ (of Morita) and $\widetilde{A_{10}}$ (ours) are related, for any $f \in \mathcal{M}_{g,1}$, by the formula

$$\tau_2'(f) + \frac{1}{6}\widetilde{A_1}(f) = B_0(f)\cdot\alpha - \alpha,$$

where $\alpha = \frac{1}{2}(\sum_{i=1}^{g} a_i + b_i) \wedge (\sum_{i=1}^{g} a_i \wedge b_i)$, and so are cohomologous.

6. New definition of the third Johnson homomorphism τ_3.

6.1. — First recall a fundamental result of Johnson.

THEOREM 6.1 (see [J3], Theorem 5). — *The kernel of $\tau_2\colon \mathcal{I}_{g,1} \to \wedge^3 H$, which is by definition $\mathcal{M}(3)$(also denoted $\mathcal{T}_{g,1}$) is normally generated by the Dehn twists $D(f_1)$,$D(f_2)$, where f_i are the simple closed curves defined by Figure* 0.1.

This should be compared with the following result due to J. Powell.

PROPOSITION (see [Po]). — *The Torelli group $\mathcal{M}(2) = \mathcal{I}_{g,1}$ is normally generated by $D(f_1)$,$D(f_2)$ and $D(y_2)D(y_2')^{-1}$, where y_2, y_2' are curves defined by Figure* 0.1.

6.1. — Consider the map

$$A_2\colon \mathcal{M}_{g,1} \longrightarrow \mathcal{M}_{2g}(I_2) \simeq I_2 \otimes H \otimes H \simeq (\wedge^2 H) \otimes H \otimes H,$$

defined by $A_2(f) = B_2(f) \times B_0(f)^{-1}$ (see 4.1). Let A_2' denote the composition

$$\mathcal{M}_{g,1} \xrightarrow{A_2} (\otimes^2 H) \otimes H \otimes H \xrightarrow{\pi\otimes\mathrm{id}} (\wedge^2 H) \otimes H \otimes H,$$

where π is the canonical projection.

6.2. — In the sequel, we will consider the following elements of $\wedge^2 H \otimes \widetilde{\wedge^2 H} \subset \wedge^2 \otimes (H \otimes H)$ (where $\widetilde{\wedge^2 H}$ is the subgroup of $\otimes^2 H$ generated by $a \wedge b = a \otimes b - b \otimes a$):

$$(u \wedge v)^2 = (u \wedge v) \otimes (u \wedge v) = (u \wedge v) \otimes (u \otimes v - v \otimes u),$$
$$u \wedge v \leftrightarrow w \wedge t = (u \wedge v) \otimes (w \wedge t) + (w \wedge t) \otimes (u \wedge v),$$
$$s_1 = a_1 \wedge b_1 \leftrightarrow a_2 \wedge b_2 - a_1 \wedge a_2 \leftrightarrow b_1 \wedge b_2 + a_1 \wedge b_2 \leftrightarrow b_1 \wedge a_2.$$

Let T denote the subgroup of $\wedge^2 H \otimes \widetilde{\wedge^2 H}$ generated by $(u \wedge v)^2$ and $u \wedge v \leftrightarrow w \wedge t$, for any u, v, w, t in H.

Lemma 6.2. — *One has*:

(i) $A_2'(D(f_1)) = 3(a_1 \wedge b_1)^2$,

(ii) $A_2'(D(f_2)) = 3(a_1 \wedge b_1 + a_2 \wedge b_2)^2 - s_1$,

Note that $A_2'(D(f_i))$, $i = 1{,}2$, *belongs to* $T \subset \wedge^2 H \otimes \widetilde{\wedge^2 H}$.

Proof. — Recall (see 0.7) that x_i, y_i, f_i, $i = 1, \ldots, g$, are the oriented circles equipped with paths given by Figure 0.1. The same letters will also denote the elements of $\pi_1(S_{g,1}, *)$ they represent. Let $[a, b]$ denote the commutator $ab\, a^{-1}\, b^{-1}$.

6.3. — Then we have $f_1 = [y_1, x_1] \in \Gamma$,

$$D(f_1)(x_1) = [y_1, x_1] x_1 [y_1, x_1]^{-1},$$
$$D(f_1)(y_1) = [y_1, x_1] y_1 [y_1, x_1]^{-1},$$
$$D(f_1)(x_i) = x_i \quad (i \geq 2),$$
$$D(f_1)(y_i) = y_i \quad (i \geq 2).$$

(Here we have made the convention that composition of paths is written from left to right.) Easy computation shows that the Fox matrix of $D(f_1)$ is

$$B\big(D(f_1)\big) = \begin{pmatrix} F_{1,1} & F_{1,2} \\ F_{2,1} & F_{2,2} \end{pmatrix}$$

with

$$F_{1,1} = x_1(1 - y_1) x_1^{-1} y_1^{-1} f_1 (1 - x_1^{-1}) f_1^{-1} + f_1^{-1},$$
$$F_{1,2} = x_1(1 - y_1) x_1^{-1} y_1^{-1} f_1 (1 - y_1^{-1}) f_1^{-1},$$
$$F_{2,1} = y_1(1 - x_1^{-1}) y_1^{-1} f_1 (1 - x_1^{-1}) f_1^{-1},$$
$$F_{2,2} = y_1(1 - x_1^{-1}) y_1^{-1} f_1 (1 - y_1^{-1}) f_1^{-1} + f_1^{-1}.$$

It is easy to see that $A_1(D(f_1)) = \widetilde{A_1}(D(f_1)) = 0$ and

$$A_2(D(f_1)) = \begin{pmatrix} -b_1a_1 + (a_1b_1 - b_1a_1) & -b_1^2 \\ a_1^2 & a_1b_1 + (a_1b_1 - b_1a_1) \end{pmatrix}.$$

From the fundamental algebraic Lemma 1.1, we easily deduce that

$$A_2'\big(D(f_1)\big) = 3a_1 \wedge b_1 \otimes (a_1b_1 - b_1a_1) = 3(a_1 \wedge b_1)^2.$$

6.4. — By the same type of computation, we have for $D(f_2)$: $f_2 = [y_2, x_2][y_1, x_1] \in \Gamma$ and

$$D(f_2)(u) = \begin{cases} f_2 u f_2^{-1} & \text{for } u = x_1, y_1, x_2, y_2, \\ u & \text{for } u = x_i, y_i,\ \ i \geq 3. \end{cases}$$

Then

$$B(D(f_2)) = \begin{pmatrix} f_2^{-1} & 0 & 0 & 0 \\ 0 & f_2^{-1} & 0 & 0 \\ 0 & 0 & f_2^{-1} & 0 \\ 0 & 0 & 0 & f_2^{-1} \end{pmatrix} + (\gamma_1, \gamma_2, \gamma_3, \gamma_4),$$

where γ_i, $i = 1, 2, 3, 4$ are 4-columns defined by $\gamma_i = \gamma\ f_2(1 - z_i^{-1})\ f_2^{-1}$, where $z_i = x_i$ for $i = 1, 2$, $z_i = y_{i-2}$ for $i = 3, 4$ and γ is the column

$$\gamma = \begin{pmatrix} \overline{\partial f_2/\partial x_1} \\ \overline{\partial f_2/\partial x_2} \\ \overline{\partial f_2/\partial y_1} \\ \overline{\partial f_2/\partial y_2} \end{pmatrix} = \begin{pmatrix} x_1(1 - y_1)x_1^{-1}y_1^{-1}[x_2, y_2] \\ x_2(1 - y_2)x_2^{-1}y_2^{-1} \\ y_1(1 - x_1^{-1})y_1^{-1}[x_2, y_2] \\ y_2(1 - x_2^{-1})y_2^{-1} \end{pmatrix}.$$

From this we can conclude that $A_1(D(f_2)) = \widetilde{A_1}(D(f_2)) = 0$ and

$$A_2(D(f_2)) = (a_1b_1 - b_1a_1 + a_2b_2 - b_2a_2)I_4 + \begin{pmatrix} -b_1a_1 & -b_1a_2 & -b_1^2 & -b_1b_2 \\ -b_2a_1 & -b_2a_2 & -b_2b_1 & -b_2^2 \\ a_1^2 & a_1a_2 & a_1b_1 & a_1b_2 \\ a_2a_1 & a_2^2 & a_2b_1 & a_2b_2 \end{pmatrix}.$$

By the remark following Lemma 1.1 and the formula $(a \wedge b + c \wedge d)^2 = (a \wedge b)^2 + (c \wedge d)^2 + a \wedge b \leftrightarrow c \wedge d$, we get point (ii). $\square$

6.5. — Let T_0 denote the subgroup of T generated by elements of the form $u \wedge v \leftrightarrow w \wedge t + u \wedge w \leftrightarrow t \wedge v + u \wedge t \leftrightarrow v \wedge w$. Then Morita [Mo2] shows that $T_0 = T \cap (\wedge^3 H \otimes H) \subset (\wedge^2 H \otimes H) \otimes H$ (where $\wedge^3 H$ is identified with a subgroup of $\wedge^2 H \otimes H$ by the identification $a \wedge b \wedge c \mapsto a \wedge b \otimes c + (b \wedge c) \otimes a + (c \wedge a) \otimes b$). Then the quotient $\overline{T} = T/T_0$ is identified with the image of T in $(\wedge^2 H \otimes H) \otimes H/(\wedge^3 H) \otimes H$. Remark that s_1 defined above belongs to T_0.

LEMMA 6.3. — (i) *A_2' defines a homomorphism*

$$A_2' \colon \mathcal{T}_{g,1} = \mathcal{M}(3) \longrightarrow T \subset (\wedge^2 H) \otimes H \otimes H,$$

respecting the action of $\mathcal{M}_{g,1}$ on $\mathcal{M}(3)$ (by conjugation) and on T (through the action of $\mathrm{Sp}(2g,\mathbb{Z})$*).*

(ii) *Composing with the projection $p \colon T \to \overline{T}$, we get a homomorphism*

$$p \circ A_2' \colon \mathcal{T}_{g,1} \longrightarrow 3\overline{T} \subset \overline{T}.$$

Moreover the image of $p \circ A_2'$ is of finite index in $3\overline{T}$ and $p \circ A_2' = -3\tau_3$, τ_3 being the third Johnson's homomorphism (see [Mo1], §1*).*

Proof. — (i) Follows from Theorem 6.1, Corollary 4.3 and Lemma 6.2. (ii) Follows from Proposition 1.1 and 1.2 of [Mo1] and from Lemma 6.2, since $s_1 \in T_0$. □

6.6. — We can explain the presence of the factor 3 in the formula $p \circ A_2' = -3\tau_3$, which does not appear in Proposition 4.4, in the following way. Morita in [Mo1, §1] or [Mo2, §2] identified $\mathcal{L}_3$ with $\wedge^2 H \otimes H/\wedge^3 H$ through the map $\lambda \colon [[x,y],z] \mapsto (x \wedge y) \otimes z$. Then τ_3 can be seen as a homomorphism from $\mathcal{M}(3)$ into $(\wedge^2 H \otimes H) \otimes H/(\wedge^3 H) \otimes H$.

In 4.3 we have defined an embedding $\mu \colon \mathcal{L}_3 \to I_3 \simeq \otimes^3 H$ which can be explicitly described by

$$\mu[[x,y],z] = (x \otimes y - y \otimes x) \otimes z - z \otimes (x \otimes y - y \otimes x).$$

Let j denote the composition $\mu \circ \lambda^{-1} : (\wedge^2 H \otimes H)/\wedge^3 H \to \otimes^3 H$. We have said in 6.5 that the image of T in $(\wedge^2 H \otimes H) \otimes H/\wedge^3 H \otimes H$ can be identified with $\overline{T} = T/T_0$. Then Morita [Mo1], Proposition 1.1, shows that the image of τ_3 is contained in $\overline{T}$. On the other hand, it is not difficult to see that the map

$$\begin{aligned}\pi \circ (j \otimes \mathrm{id}) : (\wedge^2 H \otimes H) \otimes H/\wedge^3 H \otimes H \\ \longrightarrow (\otimes^3 H) \otimes H \xrightarrow{\pi} (\wedge^2 H \otimes H) \otimes H/\wedge^3 H \otimes H\end{aligned}$$

(where π is the canonical projection), restricted to $\overline{T}$ is $3\,\mathrm{id}_T$. This explains the factor 3 in Lemma 6.3.

6.7. *Remark.* — We will exploit in [Pe] the fact that there exists a homomorphism A_2' above $-3\tau_3$, with values in T ("above" meaning that $-3\tau_3$ factors through T) to simplify Morita's formula for the Casson's invariant.

7. Extensions of the third Johnson homomorphism τ_3.

7.1. — We define a homomorphism

$$\sigma : (\wedge^2 H) \otimes H \otimes H \longrightarrow (\wedge^2 H) \otimes H \otimes H$$

by the formula $\sigma(a \wedge b \otimes (c \otimes d))) = a \wedge b \leftrightarrow c \wedge d$ where $\leftrightarrow$ has been defined in 6.2 (recall that $c \wedge d$ as element of $H \otimes H$ is understood to be equal to $c \otimes d - d \otimes c$).

By definition of T (see 6.2), σ sends $(\wedge^2 H) \otimes H \otimes H$ onto $T \subset (\wedge^2 H) \otimes H \otimes H$. Moreover $\sigma_{|T} : T \to T$ is $4\,\mathrm{id}_T$ since

$$a \wedge b \leftrightarrow c \wedge d = (a \wedge b) \otimes (cd - dc) + (c \wedge d) \otimes (ab - ba).$$

7.2. — Now we define a map $\widetilde{A_2} : \mathcal{M}_{g,1} \to \overline{T}$ as the composition

$$\widetilde{A_2} : \mathcal{M}_{g,1} \xrightarrow{A_2} \otimes^4 H \xrightarrow{\pi \otimes \mathrm{id}} (\wedge^2 H) \otimes H \otimes H \xrightarrow{\sigma} T \xrightarrow{p} \overline{T}.$$

Lemma 7.1. — *The map $\widetilde{A_2} : \mathcal{M}_{g,1} \to \overline{T}$ defined above is such that:*

(i) *its restriction to $\mathcal{M}(3) = \mathcal{T}_{g,1}$ is $-12\tau_3$ (where τ_3 is the third Johnson homomorphism);*

(ii) *for $f, g \in \mathcal{M}_{g,1}$,*

$$\widetilde{A_2}(fg) = \widetilde{A_2}(f) + B_0(f) \cdot \widetilde{A_2}(g) + \widetilde{F}\big(A_1(f), B_0(f) \cdot A_1(g)\big),$$

where $\widetilde{F}$ is defined as the composition

$$(\widetilde{\wedge^3 H}) \otimes (\widetilde{\wedge^3 H}) \xrightarrow{F} \otimes^4 H \xrightarrow{\pi \otimes \mathrm{id}} (\wedge^2 H) \otimes H \otimes H \xrightarrow{\sigma} T \xrightarrow{\rho} \overline{T}.$$

Proof. — (i) Follows from Lemma 6.3 and 7.1. Part (ii) Follows from Lemma 4.2 and the definition of $\widetilde{A_2}$ (see 7.2). □

7.3. — Recall some general facts about abelian extensions of groups. Let A be an abelian group on which G acts as a group of isomorphisms.

Let $\mathcal{F}$ be a normalized 2-cocycle of G with values in A, that is a map $\mathcal{F}: G \times G \to A$ such that for any triple $(g_0, g_1, g_2) \in G^3$ we have

$$g_0\mathcal{F}(g_1, g_2) - \mathcal{F}(g_0g_1, g_2) + \mathcal{F}(g_0, g_1g_2) - \mathcal{F}(g_0, g_1) = 0$$

and $\mathcal{F}(g,1) = \mathcal{F}(1,g) = 0$ for any $g \in G$ (see [Br], Chap. IV, 3). This defines an extension $A \circ_{\mathcal{F}} G$ of G by A as follows: $A \circ_{\mathcal{F}} G$ is the cartesian product with multiplication

$$(a,g)(a',g') = \big(a + g \cdot a' + \mathcal{F}(g,g'), gg'\big).$$

Moreover, if M is any group, $\tau: M \to G$ a homomorphism and $\alpha: M \to A$ a crossed homomorphism (meaning $\alpha(xy) = \alpha(x) + \tau(x)\alpha(y) + \mathcal{F}(\tau(x), \tau(y))$), then the map

$$\alpha \circ_{\mathcal{F}} \tau : M \longrightarrow A \circ_{\mathcal{F}} G, \quad \alpha \circ_{\mathcal{F}} \tau(x) = \big(\alpha(x), \tau(x)\big)$$

is actually a homomorphism.

7.4. — As a particular case, if G acts trivially on A, any bilinear map $\mathcal{F}: G \times G \to A$ will be a normalized 2-cocycle and, as such, defines an extension of G by A, which is in this case central. We apply this last construction in the following case: $A = \overline{T}$, $G = \widetilde{\wedge^3 H} \subset \otimes^3 H$,

$$\mathcal{F} = \widetilde{F} = p \circ \sigma \circ (\pi \otimes \mathrm{id}) \circ F:$$
$$\widetilde{\wedge^3 H} \otimes \widetilde{\wedge^3 H} \to \otimes^4 H \xrightarrow{\pi\otimes\mathrm{id}} (\wedge^2 H) \otimes H \otimes H \xrightarrow{\sigma} T \xrightarrow{\rho} \overline{T},$$

$M = \mathcal{I}_{g,1} = \mathcal{M}(2)$; $\tau = A_1 : \mathcal{I}_{g,1} \to \widetilde{\wedge^3 H}$ and $\alpha = \widetilde{A_2} : \mathcal{I}_{g,1} \longmapsto \overline{T}$ (with $\widetilde{\wedge^3 H}$ acting trivially on $\overline{T}$). Then we get:

Proposition 7.2 (Extension of $\widetilde{A_2} = -12\tau_3$ to $\mathcal{I}_{g,1}$).

(i) *We have a homomorphism*

$$\widetilde{A_2} \circ_{\widetilde{F}} A_1 : \mathcal{I}_{g,1} \longrightarrow \overline{T} \circ_{\widetilde{F}} (\widetilde{\wedge^3 H})$$

making the following diagram commutative:

$$\begin{array}{ccccccc}
1 \to \mathcal{T}_{g,1} & \longrightarrow & \mathcal{I}_{g,1} & \xrightarrow{A_1} & \widetilde{\wedge^3 H} \to 1 \\
\scriptstyle\widetilde{A_2} = -12\tau_3 \downarrow & & & & \\
12\overline{T} & & \Big\downarrow \scriptstyle\widetilde{A_2}\circ_{\widetilde{F}} A_1 & & \Big\downarrow \scriptstyle\mathrm{id} \\
\downarrow & & & & \\
1 \longrightarrow \overline{T} & \longrightarrow & \overline{T} \circ_{\widetilde{F}} (\widetilde{\wedge^3 H}) & \longrightarrow & \widetilde{\wedge^3 H} \to 1,
\end{array}$$

where the extension in the row below is central.

(ii) *The image of* $\widetilde{A_2} \circ_{\widetilde{F}} A_1$ *is of finite index. Let* $\widetilde{A_{21}}$ *denote the homomorphism* $\widetilde{A_2} \circ_{\widetilde{F}} A_1$.

Proof. — (i) is obvious by 7.4, $\widetilde{\wedge^3 H}$ acting trivially on $\overline{T}$.

(ii) is obvious too, since the image of $\widetilde{A_2}: \mathcal{T}_{g,1} \to 12\overline{T}$ is of finite index (by Lemma 6.3, (ii)). $\square$

7.5. *Remark.* — $\overline{T} \circ_{\widetilde{F}} (\widetilde{\wedge^3 H})$ supports an action of $\mathrm{Sp}(2g,\mathbb{Z})$ defined by $f_0 \cdot (x,y) = (f_0 \cdot x,\, f_0 \cdot y)$. To see this, it is sufficient to prove that the 2-cocycle $\widetilde{F} = p \circ \sigma \circ (\pi \otimes \mathrm{id}) \circ F$: $(\widetilde{\wedge^3 H}) \otimes (\widetilde{\wedge^3 H}) \longrightarrow \overline{T}$ preserves the actions of $\mathrm{Sp}(2g,\mathbb{Z})$. But this is obvious since contractions and permutation respect these actions (recall that $F = C_{34} - \tau_{23} \circ C_{35}$).

7.6. *First extension of* $\widetilde{A_2}: \mathcal{M}(3) \to \overline{T}$ *to* $\mathcal{M}_{g,1}$. — Remark that we have defined the extension $\overline{T} \circ_{\widetilde{F}} (\widetilde{\wedge^3 H})$ using the bilinear map $\widetilde{F}: (\widetilde{\wedge^3 H}) \otimes (\widetilde{\wedge^3 H}) \to \overline{T}$. This map $\widetilde{F}$ is in fact defined on $(\otimes^3 H) \otimes (\otimes^3 H)$. So, doing the same as in 7.4, we can define an extension $\overline{T} \circ_{\widetilde{F}} (\otimes^3 H)$ of $\otimes^3 H$ by $\overline{T}$ (the action of $\otimes^3 H$ on $\overline{T}$ being trivial). The maps $\widetilde{A_2}: \mathcal{M}_{g,1} \to \overline{T}$ and $A_1: \mathcal{M}_{g,1} \to \otimes^3 H$ define a map, which we call $\widetilde{A_{21}}$ from $\mathcal{M}_{g,1}$ into $\overline{T} \circ_{\widetilde{F}} (\otimes^3 H)$ by setting

$$\widetilde{A_{21}}(f) = \big(\widetilde{A_2}(f), A_1(f)\big).$$

On $\overline{T} \circ_{\widetilde{F}} (\otimes^3 H)$ we have an action of $\mathrm{Sp}(2g,\mathbb{Z})$ (see remark 7.5) defined by $\varphi_0 \cdot (t,x) = (\varphi_0 \cdot t, \varphi_0 \cdot x)$, where $\varphi_0 \in \mathrm{Sp}(2g,\mathbb{Z})$, $t \in \overline{T}$ and $x \in \otimes^3 H$.

The next lemma is obvious.

LEMMA 7.3. — *The map* $\widetilde{A_{21}}: \mathcal{M}_{g,1} \to \overline{T} \circ_{\widetilde{F}} (\otimes^3 H)$ *is a crossed homomorphism, that is*

$$\widetilde{A_{21}}(fg) = \widetilde{A_{21}}(f) * B_0(f) \cdot \widetilde{A_{21}}(g),$$

where $*$ *is the group operation in* $\overline{T} \circ_{\widetilde{F}} (\otimes^3 H)$ *defined in* 7.3. $\square$

By 5.3, this defines a homomorphism $\widetilde{A_{210}}: \mathcal{M}_{g,1} \to (\overline{T} \circ_{\widetilde{F}} \otimes^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$ by the formula

$$\widetilde{A_{210}}(f) = \big(\widetilde{A_{21}}(f), B_0(f)\big) = \big(\widetilde{A_2}(f), A_1(f), B_0(f)\big).$$

Here $(\rtimes)$ denotes the semi-direct product of $(\overline{T} \circ_{\widetilde{F}} H)$ by $\mathrm{Sp}(2g,\mathbb{Z})$, the action of $\mathrm{Sp}(2g,\mathbb{Z})$ on $\overline{T} \circ_{\widetilde{F}} \otimes^3 H$ being defined as in 7.5. We thus have proved:

PROPOSITION 7.4 (First extension of $\widetilde{A_2} : \mathcal{M}(3) \longrightarrow \overline{T}$ to $\mathcal{M}_{g,1}$). — *We have a commutative diagram of homomorphisms between exact sequences*:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_{g,1} \arrow[r] \arrow[d, "\widetilde{A_2} \circ_{\widetilde{F}} A_1"'] & \mathcal{M}_{g,1} \arrow[r] \arrow[dd, "\widetilde{A_{210}}"] & \mathrm{Sp}(2g,\mathbb{Z}) \arrow[r] \arrow[dd, "\mathrm{id}"'] & 0 \\
& \overline{T} \circ_{\widetilde{F}} \widetilde{\wedge^3 H} \arrow[d, phantom, "\cap"] & & & \\
0 \arrow[r] & \overline{T} \circ_{\widetilde{F}} \otimes^3 H \arrow[r] & (\overline{T} \circ_{\widetilde{F}} \otimes^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z}) \arrow[r] & \mathrm{Sp}(2g,\mathbb{Z}) \arrow[r] & 0.
\end{tikzcd}$$

Together with Proposition 7.2, *this gives the desired extension.*

7.7. *Remark.* — The above extension $\widetilde{A_{210}}$ has the advantage of being simply defined but we do not control its image.

7.8 *Remark.* — In Proposition 7.4 above, one may wonder why we do not define an extension of $\widetilde{A_2} : \mathcal{T}_{g,1} \to \overline{T}$ into $(\overline{T} \circ \wedge^3 H) \rtimes \mathrm{Sp}(H)$ since we have a map $\widetilde{A_1} : \mathcal{M}_{g,1} \to \wedge^3 H$. The reason is that the 2-cocycle $\widetilde{F}$ we use is defined on $\otimes^3 H$ and not on its quotient $\wedge^3 H$. There is no obvious way to define a 2-cocycle on the $\wedge^3 H$ level with the right properties. The first idea coming to mind would be to embed $\wedge^3 H$ into $\otimes^3 H$ as $\widetilde{\wedge^3 H}$, but this would not produce the right formula which has to be

$$\widetilde{A_2}(fg) = \widetilde{A_2}(f) + B_0(f) \cdot \widetilde{A_2}(g) + \widetilde{F}\big(\widetilde{A_1}(f), B_0(f) \cdot \widetilde{A_1}(g)\big).$$

(This formula is true when we remplace $\widetilde{A_1}(f)$ by $A_1(f)$).

7.9. — The object of this section is to prove the following two propositions:

PROPOSITION 7.5 (Second extension of $\widetilde{A_2} : \mathcal{M}(3) \to 12\overline{T}$). — *There exists an extension* $\overline{T} \circ_{\widetilde{F}} (6\wedge^3 H)$ *of* $6\wedge^3 H$ *by* $\overline{T}$ *and a homomorphism* $\widetilde{A_2} \circ_{\widetilde{F}} \widetilde{A_1} : \mathcal{I}_{g,1} \longmapsto \overline{T} \circ_{\widetilde{F}} (6\wedge^3 H)$, *extending* $\widetilde{A_2} = -12\tau_3 : \mathcal{M}(3) \to 12\overline{T}$, *whose image is of finite index. More precisely we have a commutative diagram of exact sequences*:

(I)
$$\begin{tikzcd}
1 \arrow[r] & \mathcal{T}_{g,1} \arrow[r] \arrow[d, "\widetilde{A_2}"] & \mathcal{I}_{g,1} \arrow[r, "\widetilde{A_1}"] \arrow[dd, "\widetilde{A_2} \circ_{\widetilde{F}} \widetilde{A_1}"] & 6\wedge^3 H \arrow[r] \arrow[dd, "\mathrm{id}"] & 1 \\
& 12\overline{T} \arrow[d] & & & \\
1 \arrow[r] & \overline{T} \arrow[r] & \overline{T} \circ_{\widetilde{F}} (6\wedge^3 H) \arrow[r] & 6 \overset{3}{\wedge} H \arrow[r] & 1.
\end{tikzcd}$$

Proposition 7.5 bis. — *To simplify the notations, put*

$$\Upsilon = (3\wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z}).$$

There exists an extension $(\frac{1}{4}\overline{T}) \circ_{\widetilde{F}} (\Upsilon)$ *of* Υ *by* $\frac{1}{4}\overline{T}$ *and a homomorphism*

$$A'_{210}\colon \mathcal{M}_{g,1} \longrightarrow (\tfrac{1}{4}\overline{T}) \circ_{\widetilde{G}} (\Upsilon)$$

extending $\widetilde{A_2}$, *whose image is of finite index and making the following diagrams commutative*:

$$\text{(II)}\qquad \begin{tikzcd}
1 \arrow[r] & \mathcal{I}_{g,1} \arrow[r] \arrow[d, "\widetilde{A}_2 \circ_{\widetilde{F}} \widetilde{A}_1"] & \mathcal{M}_{g,1} \arrow[r, "B_0"] \arrow[dd, "A'_{210}"'] & \mathrm{Sp}(2g,\mathbb{Z}) \arrow[r] \arrow[dd, "\mathrm{id}"'] & 1 \\
 & \overline{T} \circ_{\widetilde{F}} (6\wedge^3 H) \arrow[d] & & & \\
1 \arrow[r] & (\frac{1}{4}\overline{T}) \circ_{\widetilde{G}} (3\wedge^3 H) \arrow[r, "i"] & \frac{1}{4}\overline{T} \circ_{\widetilde{G}} (\Upsilon) \arrow[r] & \mathrm{Sp}(2g,\mathbb{Z}) \arrow[r] & 1,
\end{tikzcd}$$

the map i being induced by the inclusion $3\wedge^3 H \to \Upsilon$.

$$\text{(III)}\qquad \begin{tikzcd}
1 \arrow[r] & \mathcal{T}_{g,1} \arrow[r] \arrow[d, "\widetilde{A}_2"'] & \mathcal{M}_{g,1} \arrow[r, "A_1 \rtimes B_0"] \arrow[dd, "\widetilde{A'_{210}}"'] & \Upsilon \arrow[r, two heads] \arrow[dd, "\mathrm{id}"'] & 1 \\
 & 12\overline{T} \arrow[d] & & & \\
1 \arrow[r] & \frac{1}{4}\overline{T} \arrow[r] & (\frac{1}{4}\overline{T}) \circ_{\widetilde{G}} (\Upsilon) \arrow[r] & \Upsilon \arrow[r] & 1,
\end{tikzcd}$$

the notation $\twoheadrightarrow 0$ *meaning that the image is of finite index.*

7.10. *Remark.* — The extension $\widetilde{A_2} \circ_{\widetilde{F}} \widetilde{A_1} : \mathcal{I}_{g,1} \longmapsto \overline{T} \circ_{\widetilde{F}} 6\wedge^3 H$ is the one of Proposition 7.4 using the identification $\widetilde{\wedge^3 H} \xrightarrow{\pi} 6\wedge^3 H$.

7.11. — In order to define the extension $\frac{1}{4}\overline{T} \circ_{\widetilde{G}} ((3\wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z}))$, we first define the action of $(3\wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$ on $\frac{1}{4}\overline{T}$: this is simply defined by $(x,g)\cdot t = g \cdot t$, where $x \in 3\wedge^3 H$, $g \in \mathrm{Sp}(2g,\mathbb{Z})$ and $t \in \frac{1}{4}\overline{T}$.

To define the 2-cocycle $\widetilde{G}$ on $(3\wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$ with values in $\frac{1}{4}\overline{T}$ we need some lemmas. Recall first that we have a commutative diagram

$$\begin{tikzcd}
\mathcal{M}_{g,1} \arrow[r, "A_1 \rtimes B_0"] \arrow[d, "\widetilde{A}_1 \rtimes B_0"'] & (\otimes^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z}) \arrow[dl, "\pi \rtimes \mathrm{id}"] \\
(\wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z}). &
\end{tikzcd}$$

LEMMA 7.6. — *The map*

$$\pi \rtimes \mathrm{id}_{|\mathrm{Im}(A_1 \rtimes B_0)} : \mathrm{Im}(A_1 \rtimes B_0) \longrightarrow (3\wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$$

is injective, and its image is of finite index.

Proof follows from Corollary 4.5 and Proposition 5.1.

We begin to define the cocycle $\widetilde{G}$ on $\mathrm{Im}(\widetilde{A_1} \rtimes B_0) \subset (3\wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$ and then try to extend it on the whole group $(3\wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$.

7.12. — Let $(\xi, f_0), (\eta, g_0) \in \mathrm{Im}(\widetilde{A_1} \rtimes B_0)$. Then there exist $f, g \in \mathcal{M}_{g,1}$ such that $(\xi, f_0) = (\widetilde{A_1} \rtimes B_0)(f)$ and $(\eta, g_0) = (\widetilde{A_1} \rtimes B_0)(g)$. Then we set

$$G\big((\xi, f_0), (\eta, g_0)\big) = \widetilde{F}\big(A_1(f), B_0(f) \cdot A_1(g)\big) \in \overline{T},$$

where $A_1(f), A_1(g) \in \otimes^3 H$ and $\widetilde{F}$ is defined in 7.4.

LEMMA 7.7. — *G is a well-defined 2-cocyle on $\mathrm{Im}(\widetilde{A_1} \rtimes B_0)$ with values in $\overline{T}$ (recall that the action on $\overline{T}$ is defined by $(\xi, f_0) \cdot t = f_0 \cdot t$, where $(\xi, f_0) \in (3\wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$).*

Proof. — G is well defined by Lemma 7.6. It is a 2-cocycle since $\widetilde{F}$ is. □

LEMME 7.8. — *Let (ξ, f_0) be any element of $(3\wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$. Then there exists $\varphi \in \mathcal{M}(2) = \mathcal{I}_{g,1}$ and $f \in \mathcal{M}_{g,1}$ such that:*

(i) $\widetilde{A_{10}}(f) = \widetilde{A_1} \rtimes B_0(f) = (\alpha, f_0) \in (3\wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$ *for some* $\alpha \in 3\wedge^3 H$,

(ii) $\widetilde{A_{10}}(\varphi) = (2\xi - 2\alpha, I) \in (6\wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$,

(iii) *setting* $\frac{1}{2}(\beta, g_0) = (\frac{1}{2}\beta,\ g_0) \in (3\wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$ *for any* $(\beta, g_0) \in (6\wedge^3 H) \rtimes \mathrm{Sp}(H)$, *we then have*

$$\big(\tfrac{1}{2}\widetilde{A_{10}}(\varphi)\big) \times \widetilde{A_{10}}(f) = (\xi, f_0),$$

where $\times$ is the law in $(3\wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$.

Proof. — (i) Follows from the surjectivity of B_0. Obviously $2\xi - 2\alpha \in 6\wedge^3 H$. Then (ii) follows from the surjectivity of $\widetilde{A_1} : \mathcal{I}_{g,1} \to 6\wedge^3 H$ (see Corollary 4.5). Point (iii) follows from the definition of multiplication in $(3\wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$. □

7.13. — Now we can extend G defined in Lemma 7.7 to the group $(3\wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$. Consider first two elements $(\xi, f_0), (\eta, g_0)$ of $(3\wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$ such that $(\eta, g_0) \in \mathrm{Im}\ \widetilde{A_1} \rtimes B_0$; thus there exists $g \in \mathcal{M}_{g,1}$ such that $\widetilde{A_1} \rtimes B_0(g) = (\widetilde{A_1}(g),\, B_0(g)) = (\eta, g_0)$. On the other hand let $(\varphi, f) \in \mathcal{I}_{g,1} \times \mathcal{M}_{g,1}$ be a pair given by Lemma 7.8, corresponding to (ξ, f_0). Then we set, with the notation of Lemma 7.8, (iii),

$$\begin{aligned}\widetilde{G}\big((\xi, f_0), (\eta, g_0)\big) &= \widetilde{G}\big(\tfrac{1}{2}\widetilde{A_{10}}(\varphi) \times \widetilde{A_{10}}(f), \widetilde{A_{10}}(g)\big)\\ &= \tfrac{1}{2}\widetilde{F}\big(A_1(\varphi), B_0(f)\cdot A_1(g)\big) + \widetilde{F}\big(A_1(f), B_0(f)\cdot A_1(g)\big) \in \tfrac{1}{2}\overline{T}.\end{aligned}$$

Lemma 7.9. — (i) *$\widetilde{G}$ defined above does not depend on the choice of (φ, f) and g.*

(ii) *$\widetilde{G}$ extends G.*

The proof is easy using Lemma 4.1, (ii), Corollary 4.5 and Lemma 7.8.

Point (ii) is obvious since if $(\xi, f_0) = \widetilde{A_{10}}(f)$, then we can take in Lemma 7.8, $\varphi = \mathrm{id} \in \mathcal{I}_{g,1}$ and use the definition of G given in 7.12. □

7.14. — Now we can extend G to a 2-cocycle on $(3\wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$. For any pairs (ξ, f_0), $(\eta, g_0) \in (3\wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$, choose pairs (φ, f), (ψ, g) given by Lemma 7.8 and set, with the notation of Lemma 7.8:

$$\begin{aligned}\widetilde{G}\big((\xi, f_0), (\eta, g_0)\big) &= \widetilde{G}\big((\xi, f_0), (\tfrac{1}{2}\widetilde{A_{10}}(\psi)) \times \widetilde{A_{10}}(g)\big)\\ &= \tfrac{1}{2}\widetilde{G}\big((\xi, f_0), A_{10}(\psi)\big) + \widetilde{G}\big((\xi, f_0), A_1(g)\big)\\ &= \tfrac{1}{2}\left[\tfrac{1}{2}\widetilde{F}(A_1(\varphi), B_0(f)\cdot A_1(\psi)) + \widetilde{F}(A_1(f), B_0(f)\cdot A_1(\psi))\right]\\ &\quad + \tfrac{1}{2}\widetilde{F}\big(A_1(\varphi), B_0(f)\cdot A_1(g)\big) + \widetilde{F}\big(A_1(f), B_0(f)\cdot A_1(g)\big).\end{aligned}$$

Lemma 7.10. — *$\widetilde{G}$ is a well-defined 2-cocycle on $(3\wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$ with values in $\frac{1}{4}\overline{T}$, extending G.*

Proof. — Well-definedness and the fact that it is a 2-cocycle is just a matter of computation, using the bilinearity of $\widetilde{F}$ and the fact that $\widetilde{F}$ preserves the action of $\mathrm{Sp}(2g,\mathbb{Z})$ (see Remark 7.5).

7.15. — The 2-cocycle $\widetilde{G}$ on $(3\wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$ with values in $\frac{1}{4}\,\overline{T}$ in 7.14, produces, by 7.3, an extension of $(3\wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$ by $\frac{1}{4}\overline{T}$, denoted $\frac{1}{4}\overline{T} \circ_{\widetilde{G}} (3\wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$ verifying all the properties of Proposition 7.5 bis.

The fact that the image of $\widetilde{A'_{210}}$ is of finite index comes from the fact that $\widetilde{A_2} : \mathcal{T}_{g,1} \to \frac{1}{4}\overline{T}$ and $\widetilde{A_1} \rtimes B_0 : \mathcal{M}_{g,1} \to (3\wedge^3 H) \rtimes \mathrm{Sp}(2g,\mathbb{Z})$ have images of finite index.

Index of the main symbols

In front of each symbol appears the section in which it first appears.

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