ANNALES

DE

L'INSTITUT FOURIER

Pascal LAMBRECHTS & Don STANLEY

The rational homotopy type of configuration spaces of two points

THE RATIONAL HOMOTOPY TYPE
OF CONFIGURATION SPACES OF TWO POINTS

by Pascal LAMBRECHTS[1] & Don STANLEY

———

1. Introduction.

The configuration space of k points in an m-dimensional manifold M is the space

$$F(M,k) := \{(x_1,\ldots,x_k) \in M \times \cdots \times M : x_i \neq x_j \ \text{ if } \ i \neq j\}.$$

As a special case we have

$$F(M,2) = M \times M \setminus \Delta(M)$$

where $\Delta \colon M \to M \times M$ is the diagonal map.

In the present paper we study the rational homotopy type of $F(M,2)$ when the manifold M is closed (that is compact and without boundary); we will always suppose that our manifold is piecewise linear. We will prove in particular that when M is 2-connected then the rational homotopy type of $F(M,2)$ is completely determined by the rational homotopy type of M. In fact we will exhibit a description of the rational homotopy type of $F(M,2)$ in terms of a CDGA-model in the sense of Sullivan's theory ([15] and see also below for a summary of this theory).

Before describing such a model it is enlightening to consider the following description of the cohomology algebra $H^*(F(M,2))$ implicit in

the work of Cohen and Taylor [1]. When M is closed and oriented, $H^*(M)$ is a Poincaré duality algebra and there is a preferred generator $[M] \in H_m(M)$. Choose a homogeneous basis $\{a_i\}_{1 \leqslant i \leqslant N}$ of $H^*(M)$. Then there exists a Poincaré dual basis $\{a_i^*\}_{1 \leqslant i \leqslant N}$ characterized by the equations

$$\langle a_i \cup a_j^*, [M] \rangle = \delta_{ij}.$$

Define the diagonal class

$$\Delta := \sum_{i=1}^{N} (-1)^{|a_i|} a_i \otimes a_i^* \in (H^*(M) \otimes H^*(M))^m,$$

and consider the ideal (Δ) generated by Δ in $H^*(M) \otimes H^*(M)$. We have then the following

THEOREM 1.1 (Cohen-Taylor). — *If M is a closed oriented manifold then there is an isomorphism of algebras*

$$H^*(F(M, 2)) \cong \frac{H^*(M) \otimes H^*(M)}{(\Delta)}.$$

We come now to a description of the results of this paper. Recall that Sullivan [15] has associated to each connected space X a commutative differential graded algebra (a CDGA for short), $A_{PL}(X)$. By definition a *CDGA-model* of X is any CDGA (A, d) that is weakly equivalent to $A_{PL}(X)$ in the sense that there exists a chain of quasi-isomorphisms of CDGA's connecting (A, d) and $A_{PL}(X)$. The main result of Sullivan's theory is that if X is simply-connected (or nilpotent) then any CDGA-model of X determines the rational homotopy type of X (see [3] for an extensive presentation of this theory.) We will prove the following

THEOREM 1.2 (Theorem 5.6). — *Let M be an oriented connected closed manifold of dimension m such that $H^1(M; \mathbb{Q}) = H^2(M; \mathbb{Q}) = 0$. Suppose that (A, d) is a CDGA-model of M such that A is a connected Poincaré duality algebra of formal dimension m. Then there is a well defined (up to a multiplicative unit) diagonal class*

$$\Delta := \sum_{i=1} (-1)^{|a_i|} a_i \otimes a_i^* \in (A \otimes A)^m,$$

the ideal $(\Delta) = \Delta.(A \otimes A)$ is a differential ideal, and the quotient CDGA

$$\frac{A \otimes A}{(\Delta)}$$

is a CDGA model of $F(M, 2)$.

A first class of examples of manifolds admitting such a Poincaré duality model (A,d) is given by formal manifolds: a space X is called *formal* if $(H^*(X;\mathbb{Q}),0)$ is a CDGA model of X. In that case our theorem implies that the configuration space $F(M,2)$ is formal and a CDGA model for it is given by the CDGA $((H^*(M) \otimes H^*(M))/(\Delta),0)$. Important examples of formal spaces are given by smooth projective compact complex varieties [2]. For such a projective manifold M, Fulton-Mac Pherson [4] and Kriz [7] have given an explicit CDGA model for the configuration space $F(M,k)$, for any k (see also [13]). Their model is determined by the cohomology algebra $H^*(M;\mathbb{Q})$ and the diagonal class Δ. In the special case $k=2$ their model is indeed equivalent to ours as we will see at the end of Section 5.

There are also many non formal spaces admitting a Poincaré duality CDGA model (see the last section of this paper and in particular Example 6.2). In fact it is conjectured (in folklore of Rational Homotopy Theory) that every closed manifold admits such a model.

Even if M does not admit a CDGA Poincaré duality model, we can still build a CDGA model of $F(M,2)$:

THEOREM 1.3. — *Let M be a connected orientable closed manifold and suppose that $H^1(M;\mathbb{Q}) = H^2(M;\mathbb{Q}) = 0$. Then a CDGA-model of $F(M,2)$ can be explicitly determined out of any CDGA-model of M.*

This model will be described in the proof of Theorem 1.3. According to the theory of Sullivan the last theorem implies the following

COROLLARY 1.4. — *Let M be a simply-connected closed manifold such that $H^2(M;\mathbb{Q}) = 0$. Then the rational homotopy type of $F(M,2)$ depends only on the rational homotopy type of M.*

Note that Levitt [11] has proved that if M is a 2-connected closed manifold then the homotopy type of $F(M,2)$ depends only on the homotopy type of M, and John Klein has an analogous result for 3-connected Poincaré duality spaces ([5] and [6]). But these results do not imply directly Corollary 1.4. Added in proof: very recently R. Longoni and P. Salvatore have given an example of two homotopy equivalent manifolds M_1 and M_2 such that $F(M_1,2)$ and $F(M_2,2)$ are not rationnaly homotopy equivalent. This shows that some connectivity hypothesis is necessary in our Theorem 1.3 and Corollary 1.4.

As a last application we will study the formality of configuration spaces. We will prove that when M is 2-connected and closed then $F(M,2)$

is formal if and only if M is formal. More generally for any $k \geqslant 1$ and for M a simply connected closed manifold, if M is not formal then the same is true for $F(M, k)$.

2. Notation.

In all the paper we work over the field $\mathbb{Q}$ of rational numbers. We will consider commutative non negatively graded differential algebras, or CDGA for short. The degrees are written as superscripts and the differential increases the degree. If R is a CDGA we will consider also left differential graded modules over R (R-dgmodules for short). A CDGA R is called *connected* if $R^0 = \mathbb{Q}$ and of *finite type* if $\dim R^i < \infty$ for all i. If R is a CDGA we make $R \otimes R$ a CDGA in the standard way. See [3] for the precise definitions of all these objects.

The *k-th suspension* of an R-dgmodule M is the R-dgmodule $s^k M$ defined by
$(s^k M)^i = M^{i+k}$ as vector spaces,
$r.(s^k x) = (-1)^{k \cdot |r|} s^k(r.x)$ for $x \in M, r \in R$,
$d(s^k x) = (-1)^k s^k(dx)$ for $x \in M$.

The *dual* of a graded vector space V is the graded vector space $\#V$ defined by
$$(\#V)^k = \hom(V^{-k}, \mathbb{Q}).$$
If (M, d) is a left R-dgmodule then $\#M$ inherits an obvious *right* R-module structure. Using the graded commutativity of R we can turn it into a left R-module structure by the rule
$$r.\phi := (-1)^{\deg(\phi) \cdot \deg(r)} \phi.r, \text{ for } r \in R, \phi \in \#M.$$
Moreover there is a differential $d^\#$ on $\#M$ defined by, for $\phi \in \#M$ and $x \in M$,
$$(d^\# \phi)(x) = -(-1)^{|x|} \phi(dx).$$
It is straightforward to check that this makes $(\#M, d^\#)$ an R-dgmodule.

If $f \colon (M, d_M) \to (N, d_N)$ is a morphism of R-dgmodule, the *mapping cone* of f is the R-dgmodule
$$C(f) := (N \oplus_f sM, d)$$
defined by
$C(f) = N \oplus sM$ as R-module, and
$d(y, sx) = (d_N(y) + f(x), -s(d_M(x)))$ for $x \in M, y \in N$.

3. CDGA-model of the complement of the diagonal.

In this section we prove Theorem 1.3 and Corollary 1.4.

Let W be a closed oriented manifold of dimension n and let $f : P \hookrightarrow W$ be an embedding of a polyhedron. We review Theorem 6.2 of [9] which describes a CDGA model of the complement $W \setminus P$ under some connectivity-dimension restrictions. Suppose that the following data are given:

(i) a CDGA-model $\phi : R \to Q$ of $A_{PL}(f) : A_{PL}(W) \to A_{PL}(P)$;

(ii) an R-dgmodule D weakly equivalent to $s^{-n} \# Q$;

(iii) an R-dgmodule morphism $\phi^! : D \to R$ such that $H^n(\phi^!)$ is an isomorphism (such a map is called a *shriek map* in [9] Definition 3.1).

Suppose that there exist integers $m > r \geqslant 1$ such the following dimension-connectivity hypotheses are satisfied:

(iv) $R^0 = \mathbb{Q}$, i.e. R is connected;

(v) $\tilde{H}^{\leqslant r}(P; \mathbb{Q}) = 0$, $H^{>m}(P; \mathbb{Q}) = 0$, and $\tilde{H}^{\leqslant r+1}(W; \mathbb{Q}) = 0$;

(vi) $D^{<n-m} = 0$;

(vii) $r > 2m - n$.

Condition (vii) is essential and is called the *unknotting condition*. It imposes a lower bound on the codimension of the embedding in terms of the connectivities and the dimension of the ambient manifold. It is because of this hypothesis that we require some 2-connectivity restrictions in the present paper.

Let I_0 be a complement of the cocycles in R^{n-r-2}, let K_0 be a complement of the cocycles in D^{n-r-1}, set $I = I_0 \oplus R^{>n-r-2}$ and $K = K_0 \oplus D^{>n-r-1}$. Consider the following quotient of mapping cone $\bar{C}(\phi^!) := (R \oplus_{\phi^!} sD)/(I \oplus sK)$. For degree reasons there exists a unique graded commutative multiplication on $\bar{C}(\phi^!)$ that extends the R-module structure. Moreover the differential on $\bar{C}(\phi^!)$ satisfies the Leibnitz rule. Thus this defines a CDGA structure on $(R \oplus_{\phi^!} sD)/(I \oplus sK)$ which is called the *semi-trivial CDGA structure* ([9] Definition 2.20).

We have then the following

THEOREM 3.1 ([9], Theorem 6.2). — *With all the hypotheses above,*
$$\bar{C}(\phi^!) = \frac{R \oplus_{\phi^!} sD}{I \oplus sK}$$

is a CDGA-model of the complement $W \setminus P$.

To determine the CDGA model of $F(M, 2)$ we apply Theorem 3.1 with $W = M \times M$ and $P = \Delta(M)$.

Proof of Theorem 1.3 and Corollary 1.4. — Let M be a connected closed orientable manifold of dimension m such that $H^1(M; \mathbb{Q}) = H^2(M; \mathbb{Q}) = 0$. Suppose that Q is a CDGA-model M. If Q is not connected we replace it by a minimal Sullivan model which is connected.

The multiplication map

$$\phi \colon Q \otimes Q \to Q, x \otimes y \mapsto x.y$$

is a CDGA-model of the diagonal $A_{PL}(\Delta) \colon A_{PL}(M \times M) \to A_{PL}(M)$. Set $m = \dim M$, $n = 2m$, and $R = Q \otimes Q$ which is connected. The CDGA-morphism ϕ induces a structure of R-dgmodule on Q and therefore also on $s^{-n} \# Q$.

Take a minimal semi-free R-dgmodule model D of $s^{-n} \# Q$ in the sense of [3] § 6. Since $H^{<n-m}(s^{-n} \# Q) = 0$ we have by minimality that $D^{<n-m} = 0$. Note that $H^n(D) \cong \# H^0(Q) \cong \mathbb{Q}$. Hence by [9], Proposition 3.2, there exists a shriek map of R-dgmodule $\phi^! \colon D \to R$ characterized by the fact that $H^n(\phi^!) \colon H^n(D) \to H^n(R)$ is an isomorphism.

Set $r = 1$ and let $I \oplus sK$ be a suitable differential ideal in $R \oplus_{\phi^!} sD$ as constructed earlier in this section. Then the hypotheses of [9], Theorem 6.2, hold and $(R \oplus_{\phi^!} sD)/(I \oplus sK)$ is a CDGA-model of $F(M, 2) = M \times M \setminus \Delta(M)$. This proves Theorem 1.3.

We can suppose that $m = \dim M \neq 0$. Since M is simply connected and $H^2(M; \mathbb{Q}) = 0$ we have $m \geqslant 3$. Thus $\Delta(M)$ is of codimension greater than 3 in $M \times M$ and by a general position argument or by the Van Kampen theorem $M \times M \setminus \Delta(M)$ is also simply-connected. Therefore the rational homotopy types of M and $F(M, 2)$ are determined by any of their CDGA-models. Thus Corollary 1.4 follows.   □

4. Differential Poincaré duality algebras and the diagonal class.

In this section we study CDGA's that satisfies Poincaré duality and we define the diagonal class of a Poincaré duality algebra in order to prepare the proof of Theorem 1.2 in the next section.

DEFINITION 4.1. — *A Poincaré duality algebra (over $\mathbb{Q}$) of formal dimension m is a non-negatively graded commutative algebra A of finite type for which there exists a linear form $\omega\colon A \to \mathbb{Q}$ of degree $-m$ such that the bilinear form*

$$\langle .,. \rangle\colon A \otimes A \to \mathbb{Q}, a \otimes b \mapsto \langle a,b \rangle := \omega(a.b)$$

is non degenerate, that is it induces an isomorphism

(4.1) $$\theta\colon A \xrightarrow{\cong} s^{-m}\#A\,, \ a \mapsto \theta(a) := s^{-m}\langle a, - \rangle.$$

The linear form $\omega \in \#A^m$ is called a *fundamental class* or an *orientation class* and the couple (A, ω) is called an *oriented Poincaré duality algebra*.

If $\{a_i\}_{1 \leqslant i \leqslant N}$ is a homogeneous basis of A then the unique basis $\{a_i^*\}_{1 \leqslant i \leqslant N}$ of A characterized by the equations

$$\langle a_i, a_j^* \rangle = \delta_{ij},$$

where δ_{ij} is the Kronecker symbol, is called the *Poincaré dual basis* of $\{a_i\}_{1 \leqslant i \leqslant N}$.

For example if M is an oriented closed manifold of dimension m then $H^*(M; \mathbb{Q})$ is a Poincaré duality algebra of formal dimension m.

If A is a Poincaré duality algebra of formal dimension m then $A^k \cong A^{m-k}$ and $A^i = 0$ for $i > m$. Since A is of finite type, this implies that A is finite-dimensional as a vector space and therefore it always admits a homogeneous finite basis $\{a_i\}_{1 \leqslant i \leqslant N}$. Notice also that the Poincaré dual basis $\{a_i^*\}_{1 \leqslant i \leqslant N}$ is also homogeneous with $\deg(a_i^*) = m - \deg(a_i)$.

Note that the bilinear form $\langle .,. \rangle$ depends on the choice of a fundamental class ω. If A is connected then this choice is unique up to a multiplicative unit in $\mathbb{Q} \setminus \{0\}$. This bilinear form is graded symmetric in the sense that $\langle b, a \rangle = (-1)^{|a|.|b|}\langle a, b \rangle$.

LEMMA 4.2. — *Suppose that (A, ω) is an oriented Poincaré duality algebra. Let $\{a_i\}_{1 \leqslant i \leqslant N}$ be a homogeneous basis of A and let $\{a_i^*\}_{1 \leqslant i \leqslant N}$ be its Poincaré dual basis. If $\mu_{ij}^k \in \mathbb{Q}$ are defined by the equations $a_i.a_j = \sum_{1 \leqslant k \leqslant N} \mu_{ij}^k a_k$ then $a_j.a_k^* = \sum_{1 \leqslant i \leqslant N} \mu_{ij}^k a_i^*$.*

Proof. — For each $j, k, l \in \{1, \cdots, N\}$ we have, using the equations

$$\langle a_r\,,\,a_s^*\rangle = \delta_{rs},$$

$$\langle a_l\,,\,a_j.a_k^*\rangle = \langle a_l.a_j\,,\,a_k^*\rangle = \left\langle \sum_{r=1}^{N}\mu_{lj}^r a_r\,,\,a_k^*\right\rangle = \sum_{r=1}^{N}\mu_{lj}^r\langle a_r\,,\,a_k^*\rangle$$

$$= \mu_{lj}^k = \sum_{i=1}^{N}\mu_{ij}^k\langle a_l\,,\,a_i^*\rangle = \left\langle a_l\,,\,\sum_{i=1}^{N}\mu_{ij}^k a_i^*\right\rangle.$$

Since these equations are true for any $l = 1,\dots,N$, the non degeneracy of the bilinear form $\langle.,.\rangle$ implies that $a_j.a_k^* = \sum_{1\leqslant i\leqslant N}\mu_{ij}^k a_i^*$. $\qquad\square$

PROPOSITION 4.3. — *Let (A,ω) be an oriented Poincaré duality algebra of formal dimension m. Let $\{a_i\}_{1\leqslant i\leqslant N}$ be a homogeneous basis of A and let $\{a_i^*\}_{1\leqslant i\leqslant N}$ be its Poincaré dual basis. Then the element*

$$(4.2)\qquad \Delta = \sum_{i=1}^{N}(-1)^{\deg(a_i)}a_i\otimes a_i^* \in (A\otimes A)^m$$

does not depend on the choice of the basis $\{a_i\}_{1\leqslant i\leqslant N}$.

Proof. — Let $\{b_i\}_{1\leqslant i\leqslant N}$ be another homogeneous basis of A and let $\{b_i^*\}_{1\leqslant i\leqslant N}$ be its Poincaré dual basis. There exists an invertible matrix $(\gamma_j^i)_{1\leqslant i,j\leqslant N} \in \mathbb{Q}^{N\times N}$ such that $b_j = \sum_i \gamma_j^i.a_i$. Let $(\hat\gamma_k^j)_{1\leqslant j,k\leqslant N}$ be the inverse of the matrix $(\gamma_j^i)_{1\leqslant i,j\leqslant N}$, that is $\sum_j \gamma_j^i.\hat\gamma_k^j = \delta_{ik}$. Then $a_k = \sum_j \hat\gamma_k^j.b_j$. An elementary computation shows that $a_i^* = \sum_j \gamma_j^i.b_j^*$ and $b_j^* = \sum_k \hat\gamma_k^j.a_k^*$.

Since the bases are homogeneous we have $\gamma_i^j = 0$ if $\deg(a_i)\neq\deg(b_j)$. Therefore

$$\sum_{j=1}^{N}(-1)^{\deg(b_j)}b_j\otimes b_j^* = \sum_{j=1}^{N}(-1)^{\deg(b_j)}\Big(\sum_{i=1}^{N}\gamma_j^i.a_i\Big)\otimes\Big(\sum_{k=1}^{N}\hat\gamma_k^j.a_k^*\Big)$$

$$= \sum_{1\leqslant i,j,k\leqslant N}(-1)^{\deg(a_i)}\gamma_j^i a_i\otimes\hat\gamma_k^j a_k^*$$

$$= \sum_{1\leqslant i,k\leqslant N}(-1)^{\deg(a_i)}\sum_{j=1}^{N}\gamma_j^i\hat\gamma_k^j a_i\otimes a_k^*$$

$$= \sum_{1\leqslant i,k\leqslant N}(-1)^{\deg(a_i)}\delta_{ik}a_i\otimes a_k^* = \sum_{i=1}^{N}(-1)^{\deg(a_i)}a_i\otimes a_i^*.$$

$\square$

DEFINITION 4.4. — Let (A, ω) be an oriented Poincaré duality algebra of formal dimension m. The element

$$\Delta \in (A \otimes A)^m$$

defined by Equation (4.2) in Proposition 4.3 is called the diagonal class of (A, ω).

Notice that the diagonal class *does depend* on the orientation ω. In fact it is easy to compute that if $r \in \mathbb{Q} \setminus \{0\}$ then the diagonal class of $(A, r.\omega)$ is $(1/r)$ times the diagonal class of (A, ω).

There is an obvious $A \otimes A$-module structure on $s^{-m}A$ defined by $(x \otimes y).(s^{-m}a) = (-1)^{(m.\deg(x)+m.\deg(y)+\deg(a).\deg(y))} s^{-m}(x.a.y)$ for homogeneous elements $a, x, y \in A$.

LEMMA 4.5. — Let (A, ω) be an oriented Poincaré duality algebra of finite dimension and of formal dimension m, and let $\Delta \in (A \otimes A)^m$ be its diagonal class. Then the map

$$\hat{\Delta} \colon s^{-m}A \to A \otimes A \ , \ s^{-m}a \mapsto \Delta.(1 \otimes a)$$

is a morphism of $A \otimes A$-modules.

Proof. — Let $a, x, y \in A$ be homogeneous elements. Set

$$\epsilon = (-1)^{(m.\deg(x)+m.\deg(y)+\deg(a).\deg(y))}.$$

Then

$$\hat{\Delta}\left((x \otimes y).(s^{-n}a)\right) = \epsilon \hat{\Delta}(s^{-n}(x.a.y))$$

(4.3)
$$= \epsilon \Delta.(1 \otimes x.a.y)$$

$$= \epsilon \Delta.(1 \otimes x).(1 \otimes a.y).$$

On the other hand

$$(x \otimes y).\left(\hat{\Delta}(s^{-n}a)\right) = (x \otimes y).\Delta.(1 \otimes a)$$

(4.4)
$$= \epsilon \Delta.(x \otimes a.y)$$

$$= \epsilon \Delta.(x \otimes 1).(1 \otimes a.y).$$

From Equations (4.3) and (4.4), we see that it is enough to prove that $\Delta.(1 \otimes x) = \Delta.(x \otimes 1)$, and by linearity it is sufficient to prove this equation for x an element of a homogeneous basis $\{a_i\}_{1 \leqslant i \leqslant N}$ of A.

Let $\mu_{ij}^k \in \mathbb{Q}$ be such that $a_i.a_j = \sum_k \mu_{ij}^k a_k$. A straightforward computation yields

(4.5)
$$\Delta.(a_j \otimes 1) = \sum_{1 \leqslant i, k \leqslant N} (-1)^{\deg(a_i)} (-1)^{\deg(a_i^*)\deg(a_j)} \mu_{ij}^k a_k \otimes a_i^*,$$

and a similar computation using Lemma 4.2 gives

$$(4.6) \qquad \Delta.(1 \otimes a_j) = \sum_{1 \leqslant i,k \leqslant N} (-1)^{\deg(a_k)}(-1)^{\deg(a_j)\deg(a_k^*)} \mu_{ij}^k a_k \otimes a_i^*.$$

Notice that if $\deg(a_k) \neq \deg(a_i) + \deg(a_j)$ then $\mu_{ij}^k = 0$. Therefore we can suppose in each of the terms of the sums (4.5) and (4.6) that $\deg(a_k) = \deg(a_i)+\deg(a_j)$. Using the fact that $\deg(a_l^*) = m-\deg(a_l)$ we deduce that the signs $(-1)^{\deg(a_i)}(-1)^{\deg(a_i^*)\deg(a_j)}$ and $(-1)^{\deg(a_k)}(-1)^{\deg(a_j)\deg(a_k^*)}$ in these two sums agree. This proves that $\Delta.(a_j \otimes 1) = \Delta.(1 \otimes a_j)$. $\qquad \square$

DEFINITION 4.6. — An oriented differential Poincaré duality algebra is a triple (A, d, ω) such that

(i) (A, d) is a CDGA,

(ii) (A, ω) is an oriented Poincaré duality algebra, and

(iii) $\omega(dA) = 0$.

We say also that (A, d) is a differential Poincaré duality algebra.

The motivation for condition (iii) in the last definition is the following

PROPOSITION 4.7. — Let (A, d, ω) be an oriented differential Poincaré duality algebra of formal dimension m. Then

(i) the map $\theta \colon A \to s^{-m}\#A$ defined in (4.1) is an isomorphism of A-dgmodules, and

(ii) $H^*(A, d)$ is a Poincaré duality algebra of formal dimension m with an orientation class

$$[\omega] \colon H^m(A, d) \to \mathbb{Q}, \; [a] \mapsto \omega(a).$$

Proof. — (i) It is straightforward to check that θ is a morphism of A-modules. We will show that it commutes with the differentials. Denote by $d^\#$ the differential on the dual $\#A$ and by d^* the differential on its suspension $s^{-m}\#A$. Let a be an homogeneous element in A. We need to prove that $\theta(da) = d^*(\theta(a))$. By definition of θ we have $\theta(da) = s^{-m}\langle da, - \rangle$ and by the definition of the differential on the suspension we have $d^*(\theta(a)) = (-1)^m s^{-m} d^\#(\theta(a))$. Therefore we have to prove that $\langle da, - \rangle = (-1)^m d^\#(\theta(a))$.

Let x be an homogeneous element in A, we have

$$(4.7) \qquad\qquad \langle da, x \rangle = \omega((da).x).$$

On the other hand by the definition of the differentials on the dual we have
(4.8)
$$(-1)^m \left(d^\# (\theta(a)) \right)(x) = -(-1)^{|x|}(-1)^m \theta(a)(dx) = -(-1)^{|x|}(-1)^m \omega(a.dx).$$

Since $\omega(dA) = 0$ we have

(4.9) $0 = \omega(d(a.x)) = \omega((da).x) + (-1)^{|a|} \omega(a.dx).$

If $|x| + |a| + 1 \neq m$ then the expressions (4.8) and (4.7) are both 0 for degree reasons. If $|x| + |a| + 1 = m$ then a straightforward computation using (4.9) implies that the expressions (4.8) and (4.7) are equal.

This proves that θ commutes with the differentials and is an A-dgmodule morphism.

(ii) Since $\omega(dA) = 0$, the linear form $[\omega]$ is well defined. The differential isomorphism θ induces an isomorphism

$$H^*(\theta) \colon H^*(A, d) \xrightarrow{\cong} s^{-m} \# H^*(A, d).$$

It is immediate to check that $H^*(\theta)$ is the morphism defined by the bilinear form associated to $[\omega]$ following (4.1) in Definition 4.1. Since $H^*(\theta)$ is an isomorphism, this bilinear form is non degenerate and $(H^*(A, d), [\omega])$ is an oriented Poincaré algebra.   □

Note that because of the hypothesis (iii) in Definition 4.6, a CDGA whose underlying algebra is Poincaré duality is not necessarily a differential Poincaré duality algebra. However we have the following easy criterion:

PROPOSITION 4.8. — Suppose that (A, d) is a CDGA and that (A, ω) is a Poincaré duality algebra of formal dimension m. If A is connected then (A, d, ω) is a differential Poincaré duality algebra if and only if $H^m(A, d) \neq 0$.

Proof. — Suppose that $H^m(A, d) \neq 0$. By the connectivity and Poincaré duality we have $A^m \cong \mathbb{Q}$. Since $H^m(A, d) \neq 0$ we get that $A^m \cap \operatorname{im} d = 0$. Therefore $\omega(dA) = 0$ and (A, d, ω) is a differential Poincaré duality algebra. This proves one direction.

The other direction is an immediate consequence of Proposition 4.7 (ii). Indeed since $H^*(A, d)$ is a Poincaré duality algebra of formal dimension m, we have $H^m(A, d) \neq 0$.   □

We want now to prove that the diagonal class Δ of a differential Poincaré duality algebra (A, d) is a cocycle in $A \otimes A$. For this we will

first prove that there exists a suitable decomposition of the vector space A compatible with both the differential and the Poincaré duality pairing (Lemma 4.10).

Let (A, d, ω) be an oriented differential Poincaré duality algebra. We define a *representative subspace of the cohomology* as a vector subspace $H \subset \ker(d)$ such that the composite

$$H \hookrightarrow \ker d \to \ker d / \operatorname{im} d = H(A, d)$$

is an isomorphism. Using the bilinear form associated to the orientation ω we define the *orthogonal* of a subspace $V \subset A$ as

$$V^{\perp} := \{a \in A : \langle a, v \rangle = 0, \forall v \in V\}.$$

LEMMA 4.9. — *Let (A, d, ω) be an oriented differential Poincaré duality algebra and let $H \subset \ker d$ be a representative subspace of the cohomology. Then*

(i) *The restriction of the bilinear form $\langle ., . \rangle$ to $H \otimes H$ is non degenerate.*

(ii) $H \cap H^{\perp} = 0$ *and* $A = H \oplus H^{\perp}$.

(iii) *The restriction of the bilinear form $\langle ., . \rangle$ to $H^{\perp} \otimes H^{\perp}$ is non degenerate.*

Proof. — (i) is a consequence of Proposition 4.7 (ii) and of the following commutative diagram

$$
\begin{array}{ccccc}
H \otimes H & \xrightarrow{\text{mult}} & H & \xrightarrow{\omega} & \mathbb{Q} \\
\downarrow \cong & & \downarrow \cong & & \| \\
H(A, d) \otimes H(A, d) & \xrightarrow{\text{mult}} & H(A, d) & \xrightarrow{[\omega]} & \mathbb{Q}.
\end{array}
$$

(ii) As a consequence of (i), for any non-zero element x in H there exists $y \in H$ such that $\langle x, y \rangle \neq 0$. Therefore $H \cap H^{\perp} = \{0\}$. The fact that $A = H \oplus H^{\perp}$ as well as (iii) can be proved exactly like Lemma I.3.1 in [12].   □

We introduce the following notation. Let A be a differential Poincaré algebra and let $H \subset \ker d$ be a representative subspace of the cohomology. Let $V \subset H^{\perp}$ be a subspace. The orthogonal of V in $H^{\perp}$ will be denoted by

$$V^{\sqcup} := \{y \in H^{\perp} : \langle y, v \rangle = 0, \forall v \in V\} = V^{\perp} \cap H^{\perp}.$$

LEMMA 4.10. — Let (A, d, ω) be an oriented differential Poincaré duality algebra and let $H \subset \ker d$ be a representative subspace of the cohomology. Then

(i) $\ker d \subset (\operatorname{im} d)^{\perp}$;

(ii) $\operatorname{im} d \subset H^{\perp}$;

(iii) $(\operatorname{im} d)^{\sqcup} = \operatorname{im} d$;

(iv) there exists a subspace $T \subset H^{\perp}$ such that

   – $H^{\perp} = T \oplus \operatorname{im} d$

   – $T^{\sqcup} = T$

   – $d \colon T \xrightarrow{\cong} \operatorname{im} d$ is an isomorphism.

In particular we have a decomposition $A = H \oplus \operatorname{im} d \oplus T$.

Proof. — (i) Let $z \in \ker d$ and let $dx \in \operatorname{im} d$. Then

$$\langle z, dx \rangle = \omega(z.(dx)) = \pm \omega(d(z.x)) = 0$$

because $\omega(dA) = 0$.

(ii) $H \subset \ker d \subset (\operatorname{im} d)^{\perp}$ implies that $\operatorname{im} d \subset ((\operatorname{im} d)^{\perp})^{\perp} \subset H^{\perp}$.

(iii) Since $\operatorname{im} d \subset \ker d$, (i) implies that $\operatorname{im} d \subset (\operatorname{im} d)^{\perp}$. By (ii) we have $\operatorname{im} d \subset H^{\perp}$. Thus $\operatorname{im} d \subset H^{\perp} \cap (\operatorname{im} d)^{\perp} = (\operatorname{im} d)^{\sqcup}$.

The two short exact sequences

$$0 \to \operatorname{im} d \quad \hookrightarrow \ker d \to \quad H \to 0$$
$$0 \to \ker d \quad \hookrightarrow A \xrightarrow{d} \quad \operatorname{im} d \to 0$$

and the fact that $A = H \oplus H^{\perp}$ imply that

$$\dim H^{\perp} = \dim A - \dim H$$

(4.10)
$$= (\dim \operatorname{im} d + \dim \ker d) - (\dim \ker d - \dim \operatorname{im} d)$$
$$= 2 \dim(\operatorname{im} d).$$

Since the pairing $\langle ., . \rangle$ is non degenerate on $H^{\perp}$ and $\operatorname{im} d \subset H^{\perp}$ we have also

$$\dim((\operatorname{im} d)^{\sqcup}) + \dim(\operatorname{im} d) = \dim H^{\perp}.$$

This combined with Equation (4.10) implies that $\dim \operatorname{im} d = \dim(\operatorname{im} d)^{\sqcup}$. From the inclusion $\operatorname{im} d \subset (\operatorname{im} d)^{\sqcup}$ we deduce that $\operatorname{im} d = (\operatorname{im} d)^{\sqcup}$.

(iv) This is essentially the argument of [12] Lemma I.6.3, that we adapt to the case of a graded symmetric bilinear form. Let $\{b_1, \ldots, b_r\}$ be

a homogeneous basis of $\operatorname{im} d$. Since $H^\perp = 2 \dim(\operatorname{im} d)$ we complete this basis of $\operatorname{im} d$ into a homogeneous basis of $H^\perp$:

$$\{b_1, \ldots, b_r, u_1, \ldots, u_r\}.$$

Let $\{b_1^*, \ldots, b_r^*, u_1^*, \ldots, u_r^*\}$ be the dual basis in $H^\perp$ with respect to the non degenerate pairing. Then $\{u_1^*, \ldots, u_r^*\}$ is clearly a basis of $(\operatorname{im} d)^\sqcup$. Since $(\operatorname{im} d)^\sqcup = \operatorname{im} d$, this implies that $\{u_1^*, \ldots, u_r^*, u_1, \ldots, u_r\}$ is a basis of $H^\perp$. Moreover $\langle u_i^*, u_j^* \rangle = 0$ because $\langle \operatorname{im} d, \operatorname{im} d \rangle = 0$.

Let m be the formal dimension of A. Set

$$\epsilon_{ik} = \begin{cases} 1 & \text{if } m \text{ is odd} \\ (-1)^{\deg(u_i)\deg(u_k)} & \text{if } m \text{ is even} \end{cases}$$

and

$$t_i := u_i - \frac{1}{2} \sum_{k=1}^{r} \epsilon_{ik} \langle u_i, u_k \rangle u_k^*.$$

Then $\{u_1^*, \ldots, u_r^*, t_1, \ldots, t_r\}$ is another basis of $H^\perp$ and a computation gives

$$\langle t_i, u_j^* \rangle = \epsilon_{ij} \langle u_j^*, t_i \rangle = \delta_{ij}$$
$$\langle t_i, t_j \rangle = \langle u_i^*, u_j^* \rangle = 0$$

In other words the matrix of the bilinear form $\langle ., . \rangle$ on $H^\perp$ in the latter basis is

$$\begin{pmatrix} 0 & I^\pm \\ I & 0 \end{pmatrix}$$

where I is an identity matrix of rank r and $I^\pm$ is a diagonal matrix of rank r with only ± 1 on the diagonal.

Set $T = \oplus_{k=1}^{r} \mathbb{Q}.t_i \subset H^\perp$. Then $H^\perp = T \oplus \operatorname{im} d$ and the form of the matrix above implies that $T^\sqcup = T$.

Since $H \oplus \operatorname{im} d = \ker d$, the morphism $d: T \to \operatorname{im} d$ is injective. For dimension reasons it is therefore an isomorphism.

The last statement of the lemma is a consequence of Lemma 4.9 (ii).   □

PROPOSITION 4.11. — *Let (A, d, ω) be an oriented differential Poincaré algebra of formal dimension m. Then the diagonal class $\Delta \in (A \otimes A)^m$ is a cocycle.*

Proof. — Let $H \subset \ker d$ be a representative subspace of the cohomology. By Lemma 4.10, there exists a subspace $T \subset H^\perp$ such that

$H^\perp = \operatorname{im} d \oplus T$, $T = T^\sqcup$ and $d \colon T \to \operatorname{im} d$ is an isomorphism. Let $\{z_1, \ldots, z_h\}$ be a basis of H and let $\{t_1, \ldots, t_r\}$ be a basis of T. Then

$$\{z_1, \ldots, z_h, t_1, \ldots, t_r, dt_1, \ldots, dt_r\}$$

is a basis of A. Consider the Poincaré dual basis

$$\{z_1^*, \ldots, z_h^*, t_1^*, \ldots, t_r^*, (dt_1)^*, \ldots, (dt_r)^*\}.$$

Since $T \oplus \operatorname{im} d = H^\perp$ we have that $z_i^* \in H$.

Next we prove that $t_i^* = \pm d((dt_i)^*)$. Since

$$\begin{aligned}
0 &= \omega(d(t_i.(dt_j)^*)) \\
&= \omega((dt_i).(dt_j)^*) + (-1)^{\deg(t_i)}\omega(t_i.d((dt_j)^*)) \\
&= \langle dt_i, (dt_j)^*\rangle + (-1)^{\deg(t_i)}\langle t_i, d((dt_j)^*)\rangle
\end{aligned}$$

we have that

$$\langle t_i, -(-1)^{\deg(t_j)}d((dt_j)^*)\rangle = \delta_{ij}.$$

Since $\langle \operatorname{im} d, \operatorname{im} d\rangle = 0 = \langle H, \operatorname{im} d\rangle$ we have also

$$\langle dt_i, -(-1)^{\deg(t_j)}d((dt_j)^*)\rangle = 0$$
$$\langle z_i, -(-1)^{\deg(t_j)}d((dt_j)^*)\rangle = 0.$$

As a consequence we get that $t_i^* = -(-1)^{\deg(t_i)}d((dt_i)^*)$.

Therefore the diagonal class is

$$\Delta = \sum_{i=1}^{h}(-1)^{\deg(z_i)}z_i \otimes z_i^* + \sum_{i=1}^{r} -t_i \otimes d((dt_i)^*) + \sum_{i=1}^{r}(-1)^{\deg(dt_i)}dt_i \otimes (dt_i)^*.$$

A direct computation using the fact that $d(H) = 0$ gives $d(\Delta) = 0$. $\square$

5. Small CDGA-model of $F(M,2)$.

In this section we establish a small CDGA-model for $F(M,2)$ when M admits a connected Poincaré duality model (see Theorem 1.2 of the introduction).

In all the section we consider a connected differential Poincaré duality algebra (A,d) of formal dimension m. Let ω be an orientation of A and consider the associated diagonal class $\Delta \in (A \otimes A)^m$. Remember from Lemma 4.5 the map

$$\hat{\Delta} \colon s^{-m}A \to A \otimes A, \quad s^{-m}a \to \Delta.(1 \otimes a).$$

We establishes a series of lemmas to prepare the proof of Theorem 1.2.

LEMMA 5.1. — $\hat{\Delta}$ is a morphism of $A \otimes A$-dgmodules and its mapping cone

$$C(\hat{\Delta}) := A \otimes A \oplus_{\hat{\Delta}} ss^{-m}A$$

is an $A \otimes A$-dgmodule.

Proof. — By Lemma 4.5 we know that $\hat{\Delta}$ is a morphism of $A \otimes A$-modules. The fact that $\hat{\Delta}$ commutes with the differential is an immediate consequence of $d(\Delta) = 0$ (Proposition 4.11). Thus $\hat{\Delta}$ is a morphism of $A \otimes A$-dgmodules, hence $C(\hat{\Delta})$ is an $A \otimes A$-dgmodule.   □

We equip the mapping cone $C(\hat{\Delta})$ with a multiplication defined by, for $a, b, a', b', x, y \in A$,

$$(5.1) \quad \begin{cases} (a \otimes b).(a' \otimes b') & = (-1)^{|b||a'|}(aa' \otimes bb'), \\ (a \otimes b).(ss^{-m}x) & = (-1)^{(m-1)(|a|+|b|)}ss^{-m}(abx), \\ (ss^{-m}x).(a \otimes b) & = ss^{-m}(xab), \\ (ss^{-m}x).(ss^{-m}y) & = 0. \end{cases}$$

LEMMA 5.2. — The mapping cone $C(\hat{\Delta}) = A \otimes A \oplus_{\hat{\Delta}} ss^{-m}A$ equipped with the multiplication (5.1) is a CDGA.

Proof. — It is straightforward to check that this multiplication endows $C(\hat{\Delta})$ with a structure of graded commutative algebra. Moreover this multiplication extends the one defined by the $A \otimes A$-module structure.

We need only to verify the Leibnitz rule for the differential d on $C(\hat{\Delta})$. Since this mapping cone is an $A \otimes A$-dgmodule, it is enough to check that, for $x, y \in A$, we have

$$d((ss^{-m}x).(ss^{-m}y)) = (d(ss^{-m}x)).(ss^{-m}y) + (-1)^{|ss^{-m}x|}(ss^{-m}x).d(ss^{-m}y).$$

The left hand side of this equation is zero. We develop the right hand side:

$$(d(ss^{-m}x)).(ss^{-m}y) + (-1)^{|ss^{-m}x|}(ss^{-m}x).d(ss^{-m}y)$$
$$= \hat{\Delta}(s^{-m}x).(ss^{-m}y) + \pm(ss^{-m}dx).(ss^{-m}y) + (-1)^{|ss^{-m}x|}(ss^{-m}x).\hat{\Delta}(s^{-m}y)$$
$$\qquad + \pm(ss^{-m}x).(ss^{-m}dy)$$
$$= \hat{\Delta}(s^{-m}x).(ss^{-m}y) + (-1)^{|ss^{-m}x|}(ss^{-m}x).\hat{\Delta}(s^{-m}y)$$
$$= \Delta.(1 \otimes x).(ss^{-m}y) + (-1)^{|ss^{-m}x|}(ss^{-m}x).\Delta.(1 \otimes y).$$

Using the graded commutativity of the multiplication and the $A \otimes A$-module structure on $ss^{-m}A$, the last expression becomes

$$(-1)^{(m-1) \cdot |x|} \Delta.(ss^{-m}xy) - (-1)^{(m-1) \cdot |x|} \Delta.(ss^{-m}xy) = 0.$$

This proves the Leibnitz rule. □

LEMMA 5.3. — *The ideal* $(\Delta) := \Delta.(A \otimes A)$ *generated by* Δ *in* $A \otimes A$ *is a differential ideal and the quotient* $(A \otimes A)/(\Delta)$ *is a CDGA.*

Proof. — By Proposition 4.11, $d(\Delta) = 0$, hence the ideal (Δ) is differential. This implies immediately that the quotient $(A \otimes A)/(\Delta)$ inherits a CDGA structure. □

LEMMA 5.4. — *The map* $\hat{\Delta}$ *induces an isomorphism*

$$\hat{\Delta} \colon s^{-m}A \xrightarrow{\cong} \mathrm{im}(\hat{\Delta}) = (\Delta).$$

Proof. — It is clear from the definition of $\hat{\Delta}$ that $\mathrm{im}(\hat{\Delta}) = (\Delta)$. Hence we need only to prove that $\hat{\Delta}$ is injective.

Let ω be the orientation class of the Poincaré duality algebra A. Since A is connected, $A^m \cong \mathbb{Q}$ and there exists a unique element $\mu \in A^m$ such that $\omega(\mu) = 1$. The definition of the diagonal class implies that $\Delta.(1 \otimes \mu) = \mu \otimes \mu$.

Let a be a non-zero element of A^i. By Poincaré duality there exists an element $b \in A^{m-i}$ such that $a.b = \mu$. We have

$$(\hat{\Delta}(s^{-m}a)).(1 \otimes b) = \Delta.(1 \otimes a).(1 \otimes b) = \Delta.(1 \otimes \mu) = \mu \otimes \mu \neq 0.$$

Therefore $\hat{\Delta}(s^{-m}a) \neq 0$, which proves that $\hat{\Delta}$ is injective. □

We extend the canonical projection

$$\pi \colon A \otimes A \to (A \otimes A)/(\Delta)$$

into a map

$$\hat{\pi} \colon A \otimes A \oplus_{\hat{\Delta}} ss^{-m}A \to (A \otimes A)/(\Delta)$$

by setting $\hat{\pi}(ss^{-m}A) = 0$.

LEMMA 5.5. — *The map* $\hat{\pi} \colon A \otimes A \oplus_{\hat{\Delta}} ss^{-m}A \to (A \otimes A)/(\Delta)$ *defined above is a CDGA quasi-isomorphism.*

Proof. — It is straightforward to check that $\hat{\pi}$ is a CDGA morphism. Since $\hat{\Delta}$ is injective, we have a short exact sequence

$$0 \longrightarrow s^{-m}A \xrightarrow{\hat{\Delta}} A \otimes A \xrightarrow{\mathrm{pr}} (A \otimes A)/\mathrm{im}(\hat{\Delta}) \longrightarrow 0.$$

The five lemma applied to the long exact sequences in homology implies that the map

$$A \otimes A \oplus_{\hat{\Delta}} ss^{-m}A \xrightarrow{\mathrm{pr} \oplus 0} (A \otimes A)/\mathrm{im}(\hat{\Delta})$$

is a quasi-isomorphism ([9], Lemma 2.14). By Lemma 5.4 the map $\hat{\pi}$ can be identified with $\mathrm{pr} \oplus 0$. $\qquad\square$

We are now ready for the proof of the main result of this section:

THEOREM 5.6. — *Let M be a connected closed manifold orientable of dimension m such that $H^1(M;\mathbb{Q}) = H^2(M;\mathbb{Q}) = 0$. Let (A,d) be a CDGA-model of M such that A is a connected Poincaré duality algebra of formal dimension m and let $\Delta \in (A \otimes A)^m$ be the diagonal class. Then the ideal $(\Delta) = \Delta.(A \otimes A)$ is a differential ideal in $A \otimes A$ and the quotient CDGA*

$$\frac{A \otimes A}{(\Delta)}$$

is a CDGA model of $F(M,2)$.

Proof. — We have proved in Lemma 5.3 that (Δ) is a differential ideal.

Since A is a connected Poincaré duality algebra of formal dimension m and since $H^m(A,d) = H^m(M;\mathbb{Q}) \neq 0$, Proposition 4.8 implies that (A,d,ω) is a differential Poincaré duality algebra in the sense of Definition 4.6 for some orientation $\omega \in \#A^m$.

Consider the multiplication $\phi\colon A \otimes A \to A, a \otimes b \mapsto a.b$ which is a CDGA-model of the diagonal $A_{PL}(\Delta)\colon A_{PL}(M \times M) \to A_{PL}(M)$. Set $n = 2m$. The morphism ϕ induces an obvious $A \otimes A$-dgmodule structure on A, hence on $s^{-n}\#A$. By Proposition 4.7 (i) $s^{-n}\#A = s^{-m}(s^{-m}\#A)$ is isomorphic to $s^{-m}A$ as an A-dgmodule, therefore also as an $A \otimes A$-dgmodule.

By Lemma 5.1 the map $\hat{\Delta}\colon s^{-m}A \to A \otimes A$ is of $A \otimes A$-dgmodules. Moreover it induces an isomorphism in H^n because $\hat{\Delta}(s^{-m}\mu) = \mu \otimes \mu$ where $\mu \in A^m$ is the element such that $\omega(\mu) = 1$. Thus $\hat{\Delta}$ is a shriek map in the sense of Definition 3.1 in [9].

Let I_0 be a complement of the cocycles in $(A \otimes A)^{n-3}$ and set $I = I_0 \oplus (A \otimes A)^{>n-3}$. Let K_0 be a complement of the cocycles in $(s^{-m}A)^{n-2}$ and set $K = K_0 \oplus (s^{-m}A)^{>n-2}$. Then Theorem 6.2 of [9] (see also the proof of Theorem 1.3 in the present paper) implies that the CDGA

$$(5.2) \qquad \frac{A \otimes A \oplus_{\hat{\Delta}} ss^{-m}A}{I \oplus sK}$$

is a CDGA model of $F(M,2)$.

Comparing the semi-trivial CDGA structure ([9], Definition 2.20) on $(A \otimes A \oplus_{\hat{\Delta}} ss^{-m}A)/(I \oplus sK)$ with the multiplication (5.1), it is clear that the projection

$$A \otimes A \oplus_{\hat{\Delta}} ss^{-m}A \to \frac{A \otimes A \oplus_{\hat{\Delta}} ss^{-m}A}{I \oplus sK}$$

is a CDGA map. Moreover it is a quasi-isomorphism by [9] Lemma 6.7.

Thus $A \otimes A \oplus_{\hat{\Delta}} ss^{-m}A$ is a CDGA model of $F(M,2)$. By Lemma 5.5 the same is true for $(A \otimes A)/(\Delta)$. □

Remark 5.7. — The proof above shows that the CDGA $A \otimes A \oplus_{\hat{\Delta}} ss^{-m}A$ is also a model of $F(M,2)$. When M is formal then we can take $(A,d) = (H^*(M;\mathbb{Q}),0)$. In that case the CDGA model $H^*(M;\mathbb{Q}) \otimes H^*(M;\mathbb{Q}) \oplus_{\hat{\Delta}} ss^{-m}H^*(M;\mathbb{Q})$ is exactly the Kriz model for $F(M,2)$ when M is a smooth compact projective variety [7]. In a forthcoming paper [10], we will build for $k \geqslant 2$ a CDGA analog of the Kriz model for F(M,k) but with a differential Poincaré duality algebra (A,d) replacing the cohomology algebra. Moreover this CDGA will be shown to be weakly equivalent as a *dgmodule* to $A_{PL}(F(M,k))$.

6. Formality.

In this last section we develop a few examples of manifolds admitting a differential Poincaré duality CDGA model and we discuss the formality of configuration spaces.

Recall that a space X is called *formal* if $(H^*(X;\mathbb{Q}),0)$ is a CDGA-model of X. If M is a formal closed orientable manifold then $(H^*(M;\mathbb{Q}),0)$ is a Poincaré duality CDGA-model of X. Therefore an immediate application of Theorem 1.2 gives the following

COROLLARY 6.1. — *If M is a closed connected formal manifold of dimension m such that $H^1(M;\mathbb{Q}) = H^2(M;\mathbb{Q}) = 0$ then $F(M,2)$ is a formal space and admits as a CDGA-model its cohomology algebra*

$$H^*(F(M,2);\mathbb{Q}) \cong (H^*(M;\mathbb{Q}) \otimes H^*(M;\mathbb{Q}))/(\Delta)$$

where Δ is the diagonal class.

There are also many non formal manifolds that admit Poincaré duality CDGA-models. For example let X be a finite simply connected CW-complex of cohomological dimension q. By truncating a connected CDGA model of X by a suitable ideal we can construct a CDGA model (B, d_B) for X that is connected and such that $B^{>q} = 0$. Let $m \geqslant 2q + 1$ and consider a thickening T of X of dimension $m + 1$, that is T is a compact $(m+1)$-manifold (with boundary) of the homotopy type of X. By [8] Theorem 6, the CDGA $(A,d) := (B \oplus s^{-m}\#B, d_B \oplus s^{-m}d_B^{\#})$ is a model of the boundary $M := \partial T$, and A is a connected Poincaré duality algebra of formal dimension m. Thus M admits a Poincaré duality CDGA model but it is not formal when X is not formal.

Another family of examples of manifolds admitting a Poincaré duality CDGA model are given by products $M = N \times S^{m-r}$ where N is a closed manifold of dimension $r < m/2$ (see [8] middle of p. 158 for the details).

A third family of examples is given by odd-sphere bundles over a manifold admitting a Poincaré duality CDGA model like in the following example.

Example 6.2. — Let M be an S^5-bundle over $S^3 \times S^3$ with non-zero Euler class. Such a bundle exists because twice the universal Euler class $e \in H^6(BSO(6), \mathbb{Z})$ is in the image of Hurewicz and our bundle is classified by the obvious map $S^3 \times S^3 \to S^6 \xrightarrow{2e} BSO(6)$. Standard arguments in rational homotopy theory show that a Sullivan model of that manifold M is given by

$$A = (\wedge(x, y, z), dx = dy = 0, dz = xy)$$

with $\deg(x) = \deg(y) = 3$ and $\deg(z) = 5$. This is a Poincaré duality CDGA model of M.

We describe the model of $F(M,2)$ for this example. The diagonal class is

$$\Delta = 1 \otimes xyz - x \otimes yz - y \otimes xz - z \otimes xy + xy \otimes z + xz \otimes y + yz \otimes x - xyz \otimes 1.$$

A minimal Sullivan model of $(A \otimes A)/(\Delta)$, hence of $F(M,2)$, is of the form

$$\wedge(x_3, y_3, z_5, x_3', y_3', z_5', \text{ other generators of degrees } \geqslant 10).$$

From this we prove that $F(M,2)$ is not formal. Consider the cohomology classes $[x], [y] \in H^3(F(M,2); \mathbb{Q})$. Then

$$\langle [x], [y], [x] \rangle$$

is a non trivial Massey product because it contains the cohomology class $[xz]$ which does not belong to $[x].H^*(F(M,2); \mathbb{Q})$. According to [2] Section 4 this implies that $F(M,2)$ is not formal.

The end of the section is devoted to the proof of the fact that if M is a simply connected non-formal manifold then $F(M,k)$ is not formal either (Proposition 6.6). To prove this result we need the following three lemmas.

LEMMA 6.3. — Let M be a simply-connected closed manifold of dimension m and denote by $\overset{\circ}{M}$ the manifold with one point removed, that is $\overset{\circ}{M} := M \setminus \{*\}$. If $\overset{\circ}{M}$ is formal then so is M.

Proof. — The map of algebras $H^*(M; \mathbb{Q}) \overset{i^*}{\longrightarrow} H^*(\overset{\circ}{M}; \mathbb{Q})$ can be realized by a map between formal spaces that we denote by $i': \overset{\circ}{M'} \to M'$. Suppose that $\overset{\circ}{M}$ is formal. Then $\overset{\circ}{M} \simeq_{\mathbb{Q}} \overset{\circ}{M'}$ where "$\simeq_{\mathbb{Q}}$" means "has the same rational homotopy type". Since the cohomology algebras of M and M' agree and satisfy Poincaré duality, Theorem 1 of [14] implies that $M \simeq_{\mathbb{Q}} M'$. Therefore M is formal. $\qquad \square$

LEMMA 6.4. — A homotopy retract of a connected formal space is formal.

Proof. — Let X be a homotopy retract of a formal space Y. Let A and B be Sullivan models of X and Y respectively. By taking a CDGA model r of the retraction and a CDGA model i of the inclusion we get a diagram of CDGA's $A \overset{r}{\longrightarrow} B \overset{i}{\longrightarrow} A$ such that ir is homotopic to the identity. Since Y is formal there exists also a quasi-isomorphism $\beta: B \overset{\simeq}{\to} (H^*(B,d), 0)$. Then the composite $(H^*(i)\beta r): A \to (H^*(A), 0)$ is a CDGA morphism which induces the identity in homology. Thus X is formal. $\qquad \square$

LEMMA 6.5. — A skeleton of a formal connected CW-complex is formal.

Proof. — Let $f: S \hookrightarrow X$ be the inclusion of a k-skeleton in a formal CW-complex X. Set $K = \ker(H_k(f): H_k(S; \mathbb{Q}) \to H_k(X; \mathbb{Q}))$. Let A be

a Sullivan model of X. The formality of X implies that there exists a quasi-isomorphism $\alpha\colon A \xrightarrow{\simeq} (H^*(X;\mathbb{Q}),0)$. The map $f\colon S \to X$ admits a minimal relative Sullivan model $\phi\colon A \to (A \otimes \wedge W, d)$. By minimality and since f is k-connected we have that $W = W^{\geq k}$ and $W^k \cap \ker d$ is isomorphic to K. Let C be a complement of $W^k \cap \ker d$ in W. The composite $H^*(f)\alpha\colon A \to (H^*(S,\mathbb{Q}),0)$ can be extended in an algebra map $\bar{\alpha}\colon A \otimes \wedge W \to H^*(S,\mathbb{Q})$ by sending $W^k \cap \ker d$ bijectively on K and C to 0. Since $H^{>k}(S;\mathbb{Q}) = 0$, $\bar{\alpha}$ commutes with the differential hence it is a CDGA morphism. Moreover it induces an isomorphism in homology. Therefore $(A\otimes\wedge W, d)$ is a CDGA-model of both $A_{PL}(S)$ and $(H^*(S,\mathbb{Q}),0)$, hence S is formal. $\qquad\square$

PROPOSITION 6.6. — Let M be a closed simply-connected manifold and let $k \geqslant 1$. If $F(M,k)$ is formal then M is formal.

Proof. — Set $m = \dim M$. The configuration space $F(M,k)$ is the complement in M^k of the fat diagonal which is of dimension $\leqslant (k-1)m$. A general position argument implies then that the inclusion $i\colon F(M,k) \hookrightarrow M^k$ is $(m-2)$-connected (that is $\pi_{m-2}(i)$ is surjective and $\pi_{<m-2}(i)$ is an isomorphism).

Set $\overset{\circ}{M}= M \setminus \{*\}$. The simple connectivity and Poincaré duality imply that the cohomological dimension of $\overset{\circ}{M}$ is $\leqslant m - 2$, hence $\overset{\circ}{M}$ has the homotopy type of a CW-complex of dimension $\leqslant m - 2$. Moreover the inclusion map $j\colon \overset{\circ}{M}\hookrightarrow M$ is $(m-1)$-connected and the same is true for the self-product map $j^{\times k}\colon (\overset{\circ}{M})^k \to M^k$. Consider the inclusion of the first factor $i_1\colon \overset{\circ}{M}\to (\overset{\circ}{M})^k$ and the projection on the first factor $p_1\colon (\overset{\circ}{M})^k \to\overset{\circ}{M}$.

The configuration space $F(M,k)$ can be realized by a CW-complex and let F^{m-2} be its $(m\text{–}2)$-skeleton. Then the inclusion map $f\colon F^{m-2}\hookrightarrow M^k$ is $(m-2)$-connected.

Collecting this data we get the following solid arrow diagram

$$
\begin{array}{ccccc}
F(M,k) & \xrightarrow{\;i\;} & M^k & & \\
\Big\uparrow{\scriptstyle f} & & \Big\uparrow{\scriptstyle j^{\times k}} & & \\
F^{m-2} & \dashrightarrow{\;g\;} & (\overset{\circ}{M})^k & \xrightarrow{\;p_1\;} & \overset{\circ}{M}\;. \\
& {\scriptstyle h}\nwarrow\diagdown & \Big\uparrow{\scriptstyle i_1} & \diagup\!\diagup & \\
& & \overset{\circ}{M} & &
\end{array}
$$

Since F^{m-2} is of dimension $m-2$ and $j^{\times k}$ is $(m-2)$-connected there exists a lift g making the top square homotopy commutative. Since f and i are $(m-2)$-connected and $j^{\times k}$ is $(m-1)$-connected, the lift g is $(m-2)$ connected.

Since $\overset{\circ}{M}$ has the homotopy type of a CW-complex of dimension $m-2$, we can lift i_1 along the $(m-2)$-connected map g into a map h making the bottom triangle homotopy commutative. Therefore $\overset{\circ}{M}$ is a homotopy retract of F^{m-2}.

Suppose that $F(M,k)$ is formal. Then by Lemma 6.5 its skeleton F^{m-2} is also formal. This implies by Lemma 6.4 that $\overset{\circ}{M}$ is formal and by Lemma 6.3 that M is formal. $\qquad\square$

Open problems. — We finish by a few open problems:

– Can we get rid of the 2-connectivity hypothesis? It is interesting to note that Levitt and Klein have also this connectivity restriction. This is related to the possibility of "knotting" the inclusion $\Delta \hookrightarrow M \times M$.

Added in proof: by the recent work of Longoni and Salvatore (to appear in Topology) we know that 0-connected is not enough.

– Is it possible to build a CDGA model for $F(M,k)$ when $k \geqslant 3$? This problem has been solved by Fulton-McPherson and Kriz for smooth projective varieties but the problem for non formal spaces is still very open. In a paper in preparation ([10]) we will show how to build a dgmodule model (over the CDGA $A^{\otimes k}$) out of a Poincaré duality CDGA model (A,d) of M. Moreover this model has a natural structure of CDGA and it generalizes the Kriz model. However it seems to be delicate to show that it is indeed a CDGA model of $F(M,k)$. The difficulty is also related to knotting problems (or to put it in other words, to the fact that the codimension of the fat diagonal in M^k is too low.)

– Is it true that any CDGA whose cohomology satisfies Poincaré duality is weakly equivalent to a differential Poincaré duality algebra? (This is an old problem in the folklore of Rational Homotopy Theory.)

BIBLIOGRAPHY

[1]   F. COHEN, L. TAYLOR, Computations of Gelfand-Fuks cohomology, the cohomology of function spaces, and the cohomology of configuration spaces, Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), I, 106–143, Lecture Notes in Math., 657, Springer, Berlin (1978).

[2]   P. DELIGNE, P. GRIFFITHS, J. MORGAN, D. SULLIVAN, Real homotopy theory of
       Kähler manifolds, Invent. Math., 29 (1975), 245–274.

[3]   Y. FÉLIX, S. HALPERIN et J.-C. THOMAS, Rational homotopy theory, Graduate
       Text in Mathematics, vol. 210, Springer-Verlag (2001).

[4]   W. FULTON and R. MAC PHERSON, A compactification of configuration spaces,
       Annals of Math., 139 (1994), 183–225.

[5]   J. KLEIN, Poincaré duality embeddings and fiberwise homotopy theory, Topology,
       38 (1999), 597–620.

[6]   J. KLEIN, Poincaré duality embeddings and fiberwise homotopy theory, II., Quart.
       Jour. Math. Oxford, 53 (2002), 319–335.

[7]   I. KRIZ, On the rational homotopy type of configuration spaces, Annals of Math.,
       139 (1994), 227–237.

[8]   P. LAMBRECHTS, Cochain model for thickenings and its application to rational
       LS-category, Manuscripta Math., 103 (2000), 143–160.

[9]   P. LAMBRECHTS and D. STANLEY, Algebraic models of the complement of a
       subpolyhedron in a closed manifold, submitted, available on:
       http://gauss.math.ucl.ac.be/~topalg/preprints/preprint-topalgUCL.html.

[10]   P. LAMBRECHTS and D. STANLEY, DGmodule models for the configuration space
       of k points, in preparation.

[11]   N. LEVITT, Spaces of arcs and configuration spaces of manifolds, Topology, 34
       (1995), 217–230.

[12]   J. MILNOR and D. HUSEMOLLER, Symmetric bilinear forms, Ergebnisse der
       Mathematik und ihrer Grenzgebiete, Band 73, Springer-Verlag, (1973).

[13]   B. TOTARO, Configuration spaces of algebraic varieties, Topology, 35, n° 4 (1996),
       1057–1067.

[14]   J. STASHEFF, Rational Poincaré duality spaces, Illinois J. Math., 27 (1983), 104–
       109.

[15]   D. SULLIVAN, Infinitesimal computations in topology, Inst. Hautes Études Sci.
       Publ. Math., n° 47 (1977), 269–331.

                    Pascal LAMBRECHTS,
                    Université de Louvain
                    Institut Mathématique
                    2, chemin du Cyclotron
                    B-1348 Louvain-la-Neuve (Belgique).
                    lambrechts@math.ucl.ac.be

                    Don STANLEY,
                    University of Ottawa
                    Department of Mathematics and Statistics
                    585 King Edward Avenue
                    Ottawa, ON K1N 6N5 (Canada).
                    stanley@math.uregina.ca