ANNALES

DE

L'INSTITUT FOURIER

Maria ROGINSKAYA & Michaël WOJCIECHOWSKI

On summability of measures with thin spectra

Tome 54, n° 2 (2004), p. 413-430.

<http://aif.cedram.org/item?id=AIF_2004__54_2_413_0>

Ann. Inst. Fourier, Grenoble
54, 2 (2004), 413–430

ON SUMMABILITY OF MEASURES WITH THIN SPECTRA

by Maria ROGINSKAYA & Michał WOJCIECHOWSKI

1. Introduction.

According to the general uncertainty principle a distribution (a measure in our case) and its Fourier transform can not be both too concentrated. In particular, if the Fourier transform of a measure is supported on a set of a special form then it has no singular part. We call a set with this property a *Riesz set*. Many different sufficient conditions for Riesz sets are known - we refer to [M], [Sh], [A], [HJ], where the conditions for $\mathbb{T}^d$ are given - roughly speaking the set should be concentrated on a halfspace and it can not contain a line. Another sufficient condition (both for $\mathbb{R}^d$ and $\mathbb{T}^d$) is given in [R], where the set is required to be strongly antisymmetric. In the present paper we study phenomena which occur only in the non-compact setting. We give a new class of examples of Riesz sets on $\mathbb{R}^d$ which are both symmetric and also include a lot of lines.

In Section 2 we prove the following criterion inspired by the de Leeuw transference method, on which the examples of Riesz sets are based.

THEOREM 1. — *Suppose that $\alpha_j K \cap \mathbb{Z}^d$ is a Riesz set in $\mathbb{Z}^d$ for every $j = 1, 2, \dots$ for some $K \subset \mathbb{R}^d$, and a sequence $\alpha_j \to \infty$. Then K is a Riesz set in $\mathbb{R}^d$.*

Supported in part by KVA grant 11 29 3 551 and KBN grant 2 P03A 036 14.
Keywords: Rietz sets – Singular measures – Support of Fourier transform.
Math. classification: 42B10 – 42A55.

As a direct application of the above criterion we prove that so called f-poles are Riesz sets for every $f : \mathbb{R}_+ \to \mathbb{R}_+$ which decreases to 0. For any positive, decreasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$ we call a set *an f-pole* iff it is an image of the set $K_f = \{(x_1, x') \in \mathbb{R}^d : |x'| \leqslant f(|x_1|)\}$ under a linear transformation.

COROLLARY 1. — *For each function $f : \mathbb{R}_+ \to \mathbb{R}_+$ decreasing to 0, every f-pole is a Riesz set.*

We also give an example of a Riesz set in $\mathbb{R}^d$ ($d \geqslant 2$) whose interior contains all lines in one direction except for lines passing through a set of small $(d-1)$-dimensional Hausdorff measure, which disproves the conjecture, that a Riesz set in $\mathbb{R}^d$ can not contain a line, as it does in the $\mathbb{T}^d$ case.

The formulation of Theorem 1 is in the spirit of the criterion given by Meyer for compact group (cf. [M]). However, instead of using an argument of a topological nature, we transfer the results from tori to the Euclidean spaces.

In Section 3 we study special cases of Riesz sets for which the L^1-summability can be improved. It is easy to see that if the Fourier-Stieltjes transform of a measure $\mu \in M(\mathbb{R}^d)$ is supported on a set $K \subset \mathbb{R}^d$ of finite Lebesgue measure, then μ is a bounded continuous function and belongs to $L^p(\mathbb{R}^d)$ for $1 \leqslant p \leqslant \infty$. Moreover, $\|\mu\|_p \leqslant |K|^{\frac{p-1}{p}} \|\mu\|_M$. We consider the class of sets $K \subset \mathbb{R}^d$ such that the function assigning to $t \in \mathbb{R}$ the $(d-1)$-dimensional Lebesgue measure of the intersection of K with the hyperplane $\{x_1 = t\}$ is L^p-summable. We prove the following result.

THEOREM 2. — *Let $1 < p \leqslant 2$, $K \subset \mathbb{R}^d$ is a closed set and suppose that there exists $y \in \mathbb{R}^d$ such that the function*

$$h(t) = m_{d-1}\big(K \cap \{\xi :< y, \xi >= t\}\big),$$

where m_{d-1} is $(d-1)$-dimensional Hausdorff measure, belongs to $L^p(\mathbb{R})$. Then any finite measure with Fourier transform supported in K is locally $L^{p'}$-summable where $\frac{1}{p} + \frac{1}{p'} = 1$.

We also give in Section 3 several results about the sharpness of Theorem 2. Among them, we show that there exists a Riesz set K which is not a Hardy set, i.e. there exists a summable function with Fourier-Stieltjes transform supported on K, which does not belong to the class $H^1(\mathbb{R}^d)$.

In Section 4 we study conditions on the set of zeros of the Fourier-Stieltjes transform of a measure which imply that the measure is absolutely continuous. We call a sequence $\Lambda \subset \mathbb{R}^d$ a *co-Riesz sequence* iff every finite measure with Fourier-Stieltjes transform vanishing on Λ is absolutely continuous with respect to Lebesgue measure. We prove that the co-Riesz sequences exist.

THEOREM 3. — *No matter how slowly the sequence r_n tends to 0, there exists a co-Riesz sequence Λ such that* $\operatorname{dist}(\lambda_n, \Lambda\backslash\{\lambda_n\}) > r_n$.

On the other hand, the fact that the sequence of differences of a sequence Λ tends to 0, that doesn't guarantee, that the sequence Λ is a co-Riesz sequence. An example of such a sequence was provided to us by J.-P. Kahane.

Later in Section 4, we show that vanishing of the Fourier transform of a function on any sequence without limit points does not guarantee any additional summability of the function (compare with the Theorem 2). We also study some properties of co-Riesz sequences and formulate some problems.

In Section 5 we apply the method developed in the previous sections to co-Lebesgue sequences. A sequence $\Lambda \subset \mathbb{R}^d$ is *a co-Lebesgue* iff for every measure $\mu \in M(\mathbb{R}^d)$ with Fourier-Stieltjes transform vanishing on Λ, the Fourier-Stieltjes transforms of its singular and absolutely continuous parts also vanish on Λ. We establish a criterion for being co-Lebesgue and apply it to the sequences $(n^{1/k})_{n=1}^{\infty}$ $(k = 2, 3, \ldots)$ and $(\log n)_{n=1}^{\infty}$.

Notation. — We denote by $\mathbb{R}^d$ the d-dimensional Euclidean space with the scalar product $\langle\cdot,\cdot\rangle$ and Euclidean norm $|\cdot|$. By $\mathbb{T}^d$ we denote d-dimensional torus identified naturally with the unit cube in $\mathbb{R}^d$. All measures are supposed to be finite Borel measures. The space of finite Borel measures of bounded total variation on $\mathbb{R}^d$ is denoted by $M(\mathbb{R}^d)$. By $\|\cdot\|$ we denote the usual norm on this space, i.e. the total variation of a measure. We denote by μ_s the part of μ singular with respect to Lebesgue measure (cf. [HR, Chapt. III, Th. 14.22]). If a measure μ is absolutely continuous with respect to Lebesgue measure m_d, there exists $f \in L^1(\mathbb{R}^d)$ such that $d\mu = f\,dm_d$. In this case we write for shortness $\mu \in L^1(\mathbb{R}^d)$, i.e we identify the measure with f. The restriction of a measure μ to a Borel set Ω is denoted by $\mu|_\Omega$. By $\widehat{\mu}(\xi) = \int e^{-i2\pi\langle x,\xi\rangle}\,d\mu(x)$ we denote the Fourier-Stieltjes transform of the measure $\mu \in M(\mathbb{R}^d)$. For $A, B \subset \mathbb{R}^d$ by $A + B$ we denote the Minkowski sum $\{x + y : x \in A, y \in B\}$; rA denotes the set $\{ra \in \mathbb{R}^d : a \in A\}$ $(r \in \mathbb{R})$. By $\operatorname{dist}(x, A)$ we denote the distance between $x \in \mathbb{R}^d$ and the nonempty set $A \subset \mathbb{R}^d$. The symbol C (possibly with indexes) denotes a non-negative constant which can change in value from one occurrence to another.

2. Symmetric Riesz sets.

We begin with the proof of Theorem 1.

Proof of Theorem 1. — Suppose that K is not a Riesz set. Then there exists $\mu \in M(\mathbb{R}^d)$ such that $\operatorname{supp} \widehat{\mu} \subset K$ and $\mu_s \neq 0$. Let us choose an integer j such that $|\mu_s|(\alpha_j I^d) > \frac{2}{3}\|\mu_s\|$ (here $I^d = \{x \in \mathbb{R}^d : -\frac{1}{2} < x_k \leqslant \frac{1}{2}\}$). Let $\nu \in M(\mathbb{T}^d)$ be the measure defined by $\nu(E) = \mu(\alpha_j E + \alpha_j \mathbb{Z}^d)$ for $E \subset \mathbb{T}^d$. It is easy to see that $\widehat{\nu}(\xi) = \widehat{\mu}(\frac{\xi}{\alpha_j})$ for every $\xi \in \mathbb{Z}^d$. Since $\operatorname{supp} \widehat{\mu} \subset K$, the Fourier-Stieltjes transform of ν vanishes outside a Riesz subset of $\mathbb{Z}^d$. Hence $\nu_s = 0$. But $\nu_s(E) = \sum_{\xi \in \mathbb{Z}^d} \mu_s(\alpha_j E + \alpha_j \xi)$ and therefore

$$\|\nu_s\| \geqslant \|(\mu|\alpha_j I^d)_s\| - \|(\mu|\mathbb{R}^d \setminus \alpha_j I^d)_s\| > \frac{1}{3}\|\mu_s\| > 0.$$

This contradiction completes the proof. □

Corollary 1 is a direct consequence of Theorem 1. Linear transformation preserve Riesz sets, and one always can shift the f-pole in such a way that it does not contain any line with rational points. For this shifted f-pole the sequence $\alpha_j = j$ is the sequence required in Theorem 1.

Example 1. — Given $\varepsilon > 0$ there exists a closed symmetric (with respect to the origin) subset E of the hyperplane $L = \{x_1 = 0\} \subset \mathbb{R}^d$ and a Riesz set $K \subset \mathbb{R}^d$ such that $m_L(L \setminus E) < \varepsilon$ and $\mathbb{R} \times E = \{(x_1, x') : (0, x') \in E\} \subset \operatorname{Int} K$.

Let $A \subset L$ be an open symmetric set of measure $m_L(A) < \varepsilon$ containing all the rational points in L and $A_1 \subset A_2 \subset \ldots \subset A$ be a sequence of open symmetric sets such that $\frac{1}{n}\mathbb{Z}^d \cap L \subset A_n$ and $\overline{A}_n \subset A$ for $n = 1, 2, \ldots$. Then we put $K = \mathbb{R}^d \setminus \bigcup F_n$ where

$$F_n = \left\{(x_1, x') \in \mathbb{R}^d : |x_1| > \frac{n}{1 + |x'|} - 1 \quad \text{and} \quad (0, x') \in A_n\right\}.$$

We put $E = L \setminus A$. Clearly $nK \cap \mathbb{Z}^d \subset \{|x_1| \leqslant \frac{n^3}{n + |x'|} - n\}$ which is a finite subset of $\mathbb{Z}^d$. Hence, by Theorem 1, K is a Riesz set. The remaining property, $\mathbb{R} \times E \subset \operatorname{Int} K$, is obvious. □

It might happen (however we do not know it) that a strengthened form of Corollary 1 is valid: every L^1 function with the Fourier transform supported by an f-pole is better than L^1 integrable (e.g., belongs to some fixed Orlicz space), and this could be the reason for being a Riesz set. Theorem 2 being applied to f-poles supports this conjecture. The next result shows however that this possible improvement cannot be uniform for all functions f.

Let Φ be a Young function which defines the Orlicz norm on $\mathbb{R}^d$; we denote the corresponding Orlicz space by $L^\Phi(\mathbb{R}^d)$ (cf. [RR]). We say that a function f belongs to $L^\Phi_{\mathrm{loc}}(\mathbb{R}^d)$ iff for every $x \in \mathbb{R}^d$ there exists a neighbourhood U such that $f \cdot \chi_U \in L^\Phi(\mathbb{R}^d)$.

PROPOSITION 2. — *Let the Young function Φ be such that $L^1(\mathbb{R}^d) \not\subset L^\Phi(\mathbb{R}^d)$. Then there exists $f : \mathbb{R}_+ \to \mathbb{R}_+$ and a function $F \in L^1(\mathbb{R}^d)$ with the Fourier-Stieltjes transform supported on the f-pole K_f, such that $F \notin L^\Phi_{\mathrm{loc}}(\mathbb{R}^d)$.*

Proof. — Let $\psi \in C^\infty(\mathbb{R}^d)$ be a positive function such that $\|\psi\|_1 = 1$ and its Fourier transform $\widehat{\psi}$ is positive and supported on the unit cube I^d. We can get such a function as the square of an L^1 function with smooth positive Fourier transform supported on $\frac{1}{2}I^d$. Clearly we have $\psi(x) > \sigma > 0$ for a fixed small enough constant σ for $x \in rI^d$ for some $r > 0$. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be a function decreasing to 0 (to be fixed later). For $n = 1, 2, \dots$ we define ψ_n by

$$\widehat{\psi_n}(x_1, x') = \widehat{\psi}\Big(\frac{x_1}{2^n}, \frac{(d-1)x'}{f(2^{n-1})}\Big),$$

where $x = (x_1, x')$ with $x_1 \in \mathbb{R}$, $x' \in \mathbb{R}^{d-1}$.

Note that

1) $\operatorname{supp} \widehat{\psi_n} \subset K_f$

2) $\psi_n \geqslant 0$;

3) $\psi_n > 2^n(d-1)^{-(d-1)} f^{d-1}(2^{n-1})\sigma$ on $E_n = 2^{-n} rI \times \big(\frac{(d-1)r}{f(2^{n-1})}\big) I^{d-1}$;

4) $\|\psi_n\|_1 = 1$.

Then we put

$$F = \sum_{j=1}^{\infty} \frac{1}{j^2} \psi_{n_j},$$

where the increasing sequence of integers (n_j) will be fixed later. We are going to show that if f is chosen properly then $\int_{\varepsilon I^d} \Phi(\alpha|F|) = \infty$ for every

$\varepsilon > 0$ and $\alpha > 0$. Put $\phi(t) = t^{-1}\Phi(t)$. Since $L^1(\mathbb{R}^d) \not\subset L^\Phi(\mathbb{R}^d)$, we have $\phi(t) \to \infty$ as $t \to \infty$. Since Φ is superadditive, we have

$$\int_{\varepsilon I^d} \Phi(\alpha|F|) \geqslant \sum_j \int_{\varepsilon I^d} \Phi\Big(\frac{\alpha\psi_{n_j}}{j^2}\Big).$$

Thus, using properties 1) - 4), we get that for j such that $n_j > j^2(\alpha\sigma)^{-1}(d-1)^{d-1}$, $\varepsilon f(2^{n_j-1}) < (d-1)r$ and $2^{n_j} > \frac{r}{\varepsilon}$

$$\begin{aligned}\int_{\varepsilon I^d} \Phi\Big(\frac{\alpha}{j^2}\psi_{n_j}\Big) &\geqslant \int_{\varepsilon I^d \cap E_{n_j}} \Phi\Big(\frac{\alpha}{j^2}2^{n_j}(d-1)^{-(d-1)}f^{d-1}(2^{n_j-1})\sigma\Big)\\
&\geqslant 2^{-d}\varepsilon^{d-1}\frac{r}{2^{n_j}}\Phi\Big(\frac{\alpha}{j^2}2^{n_j}(d-1)^{-(d-1)}f^{d-1}(2^{n_j-1})\sigma\Big)\\
&= 2^{-d}\varepsilon^{d-1}r\frac{\alpha}{j^2}\sigma(d-1)^{-(d-1)}f^{d-1}(2^{n_j-1})\\
&\quad \phi\Big(\frac{\alpha}{j^2}2^{n_j}(d-1)^{-(d-1)}f^{d-1}(2^{n_j-1})\sigma\Big).\end{aligned}$$

Put now

$$f(t) = \begin{cases} \max\big(\frac{1}{\log_2 t}, \phi^{-\frac{1}{d}}(\frac{t}{\log_2^d t})\big) & \text{for } t > 2^d,\\ \max\big(\frac{1}{d}, \phi^{-\frac{1}{d}}(\frac{2^d}{d^d})\big) & \text{for } t < 2^d.\end{cases}$$

Choose the sequence (n_j) such that $\phi(2^{n_j}n_j^{-d}) > j^d$ and $n_j > j^3$. Then, using the above estimation and the definition of f, we get for the large values of j

$$\begin{aligned}\int_{\varepsilon I^d} \Phi\Big(\frac{\alpha}{j^2}\psi_{n_j}\Big) &\geqslant 2^{-d}\varepsilon^{d-1}(d-1)^{-(d-1)}r\frac{\alpha}{j^2}\sigma f^{d-1}(2^{n_j})\phi(2^{n_j}n_j^{-d})\\
&\geqslant 2^{-d}\varepsilon^{d-1}(d-1)^{-(d-1)}r\alpha\sigma j^{-2}\phi^{\frac{1}{d}}(2^{n_j}n_j^{-d})\\
&\geqslant 2^{-d}\varepsilon^{d-1}(d-1)^{-(d-1)}r\alpha\sigma\cdot j^{-1}.\end{aligned}$$

Hence the integral $\int_{\varepsilon I^d}\Phi(\alpha|F|)$ is estimated from below by a tail of the divergent series. □

Using now the well known fact that $H^1_+(\mathbb{R}^d) \subset (L\log L)_{\mathrm{loc}}(\mathbb{R}^d)$ (cf. [St, Chap. III.5.3]) and that the constructed function F is positive, we get as a corollary that on $\mathbb{R}^d$ the class of Riesz sets is slightly larger that the class of Hardy sets:

COROLLARY 2. — *There exists an f-pole $K_f \subset \mathbb{R}^d$ which is not a Hardy set, i.e. there exists $F \in L^1(\mathbb{R}^d)$ with Fourier transform supported by K_f such that $F \notin H^1(\mathbb{R}^d)$.*

3. Proof of Theorem 2.

We can assume that $y = (1,0,\dots,0)$. Let $\mu \in M(\mathbb{R}^d)$ satisfy $\operatorname{supp}\widehat{\mu} \subset K$. For $t > 0$ we put $\mu_t = \mu * P_t$, where $\{P_t\}_{t>0}$ are Poisson kernels. Clearly $\mu_t \in L^1(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d)$ and $\|\mu_t\|_1 \leqslant \|\mu\|$. It is also clear that $\operatorname{supp}\widehat{\mu}_t \subset K$ and $\widehat{\mu}_t \in L^1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$. We have

$$\begin{aligned}\mu_t(x) &= \int_{\mathbb{R}^d} \widehat{\mu}_t(\xi) e^{2\pi i\langle (x_1,x'),\xi\rangle}\, d\xi \\ &= \int_{-\infty}^{\infty} \mathrm{P}(\widehat{\mu}_t(\xi)e^{2\pi i\langle x',\xi'\rangle})(\xi_1) e^{2\pi i x_1\xi_1}\, d\xi_1 \\ &= (\mathrm{P}(\widehat{\mu}_t(\xi_1,\cdot)e^{2\pi i\langle x',\cdot\rangle}))^\wedge(-x_1),\end{aligned}$$

where

$$\mathrm{P}(f)(s) = \int_{\{x_1=s\}} f(x_1,x')\, dm_{d-1}(x').$$

Since $|\widehat{\mu}_t(\xi)e^{2\pi i\langle x',\xi'\rangle}| \leqslant \|\mu_t\|_1 \leqslant \|\mu\|$, and $m_{d-1}(\{x\in K : x_1 = s\}) = h(s)$, we get

$$\|\mathrm{P}(\widehat{\mu}_t(\xi_1,\cdot)e^{2\pi i\langle x',\cdot\rangle})\|_{L^p(d\xi_1)} \leqslant \|\mu\|\cdot\|h\|_p.$$

Hence, if $\frac{1}{p}+\frac{1}{p'} = 1$, by the Hausdorff-Young inequality,

$$\|(\mathrm{P}(\widehat{\mu}_t(\xi_1,\cdot)e^{2\pi i\langle x',\cdot\rangle}))^\wedge\|_{p'} \leqslant \|\mu\|\cdot\|h\|_p.$$

Thus $\|\mu_t(\cdot,x')\|_{p'} \leqslant \|\mu\|\cdot\|h\|_p$ for every $x' \in \mathbb{R}^{d-1}$.

Let $y = (y_1,y') \in \mathbb{R}^d$ and $U = \mathbb{R}\times\Omega$ be an open neighborhood of y such that $\Omega \in \mathbb{R}^{d-1}$ is an open neighborhood of y' with finite $(d-1)$-dimensional Lebesgue measure. Then

$$\begin{aligned}\int_U |\mu_t|^{p'}\, dm_d &= \int_\Omega\int_{\mathbb{R}} |\mu_t(x_1,x')|^{p'}\, dx_1 dx' \\ &\leqslant m_{d-1}(\Omega)\cdot\|\mu\|^{p'}\|h\|_p^{p'}.\end{aligned}$$

Hence there exists $C > 0$ such that for $t > 0$,

$$\|\mu_t\|_{L^{p'}(U)} \leqslant C.$$

By assumption $(\mu_t)_{|U} \to \mu_{|U}$ in the $*$-weak topology. Since $\|(\mu_t)_{|U}\|_{p'}$ is bounded for $t > 0$, we get that $\mu_{|U} \in L^{p'}(U)$. □

The f-pole with $f(t) = \min\{1, t^{-\frac{q-1}{q(d-1)}}\}$ is called a q-pole.

Corollary 3. — *Let $2 \leqslant p < \infty$. If the support of the Fourier transform of a measure μ is contained in a finite union of q-poles, where $q > p$, then $\mu \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$.*

Corollary 3 gives another proof that q-poles are Riesz sets for $q > 2$. However, by applying Theorem 2, one can construct Riesz sets which do not seem to be treated by Theorem 1.

Example 2. — Let $K \in \mathbb{R}^d$ be any q-pole ($q > 2$) which does not contain a line orthogonal to the first coordinate. Let $K_n = K \cap \{n \leqslant x_1 \leqslant n+1\}$. If $(r_n)_{n=-\infty}^{\infty} \subset \mathbb{R}^d$ is any sequence with bounded first coordinate, then the set $\bigcup_{n\in\mathbb{Z}}(K_n + r_n)$ satisfies the assumption of Theorem 2 for $p > q'$.

Corollary 3 shows that every q-pole is a "local" Λ_p for every $q > p \geqslant 2$. The next remark shows that (a) $\Lambda_{q,\mathrm{loc}} \not\subset \Lambda_p$ for $2 < q < \infty$ and $1 < p < \infty$, and (b) $\Lambda_{p,\mathrm{loc}} \neq \Lambda_{q,\mathrm{loc}}$ for $p, q \geqslant 2$ and $p \neq q$.

PROPOSITION 3. —

a) *Let $1 < q < \infty$. There exists a function $F \in L^1(\mathbb{R}^d)$ with the Fourier transform supported on a q-pole, such that $F \notin L^p(\mathbb{R}^d)$ for any $1 < p < \infty$.*

b) *Let $1 < q < \infty$. There exists a function F with the Fourier transform supported on a q-pole, such that $F \notin L^p_{\mathrm{loc}}(\mathbb{R}^d)$ for every $p > q$.*

Proof. — We use the function F constructed in the proof of Proposition 2. We let $f(t) = \min(1, t^{-\frac{q-1}{q(d-1)}})$ and $n_j = j$. Then

$$\begin{aligned}\|F\|_p &\geqslant j^{-2}\|\psi_j\|_p \\ &\geqslant j^{-2}\big(|E_{n_j}| \cdot (2^j(d-1)^{-(d-1)}f^{d-1}(2^{j-1})\sigma)^p\big)^{1/p} \\ &= j^{-2}r^{d/p}\sigma\big(2^j(d-1)^{-(d-1)}f^{d-1}(2^{j-1})\big)^{\frac{p-1}{p}} \\ &= j^{-2}r^{d/p}\sigma 2^{\frac{p-1}{p}}(d-1)^{-(d-1)\frac{p-1}{p}}2^{(j-1)(\frac{1}{q}\cdot\frac{p-1}{p})} \to \infty\end{aligned}$$

as $j \to \infty$. This proves part (a). For part (b) we have for j such that $\varepsilon f(2^{j-1}) < (d-1)r$,

$$\begin{aligned}\int_{\varepsilon I^d} |F|^p &\geqslant j^{-2}\int_{\varepsilon I^d}|\psi_j|^p \\ &\geqslant j^{-2}|\varepsilon I^d \cap E_j| \cdot \big(2^j(d-1)^{-(d-1)}f^{d-1}(2^{j-1})\sigma\big)^p \\ &= j^{-2}\varepsilon^{d-1}r\sigma^p 2^{p-1}(d-1)^{-p(d-1)}2^{(j-1)(\frac{p}{q}-1)} \to \infty\end{aligned}$$

as $j \to \infty$ for any fixed $\varepsilon > 0$. □

4. co-Riesz sequences on $\mathbb{R}$.

We call $\Lambda = (\lambda_n)_{n=1}^{\infty} \subset \mathbb{R}^d$ *a co-Riesz sequence* iff every measure $\mu \in M(\mathbb{R}^d)$ such that $\widehat{\mu}(\lambda) = 0$ for $\lambda \in \Lambda$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.

Though a number of results in this section make the impression that every sequence $\Lambda = (\lambda_n)_{n=1}^{\infty} \subset \mathbb{R}$ such that $\lim_{|x|\to\infty} \operatorname{dist}(\Lambda, x) = 0$ (resp. $\lim_{x\to\infty} \operatorname{dist}(\Lambda, x) = 0$, which, in case when Λ is monotone, is equivalent to $\lim_{n\to\infty} |\lambda_n - \lambda_{n+1}| = 0$) should be a co-Riesz sequence, this is not true. That is shown by Example 3 below, which was provided by J.-P. Kahane to the previous version of this manuscript (and appears here with his kind permission). The further study of co-Riesz sequences in connection with Helson sets is provided in the forthcoming paper [W].

On the other hand we can show that a number particular sequences from this class are indeed co-Riesz. This is in the case when Theorem 1 could be applied. Note that, despite the fact that Theorem 1 is formulated for Riesz sets, which by definition are closed, it remains valid in this setting – in the proof of Theorem 1 we only use the values of the Fourier transform at the points from some special (countable) set.

Proof of Theorem 3. — Without loss of generality we can assume that (r_j) is a non-increasing sequence consisting of powers of 2. Moreover we can assume that $\sum r_j = \infty$. Then we put $\lambda_n = \sum_{j=0}^{n} r_j$ for $n = 1, 2, \ldots$. It is easy to check that for every $n = 1, 2, \ldots$ the intersection $2^n(\mathbb{R} \setminus \Lambda) \cap \mathbb{Z}$ is a set contained in a halfline (i.e. bounded from above). Hence it follows from the theorem of F. and M. Riesz (cf. [HJ, 1.1.3, p.13]) and Theorem 1 that Λ is a co-Riesz sequence. □

Remark. — An obvious modification of the proof of Theorem 3 gives its analogue for several variables. Namely, one can prove that for every sequence (r_n) decreasing to 0 there exists a co-Riesz sequence $\Lambda = (\lambda_n) \subset \mathbb{R}^d$ such that $\operatorname{dist}(\lambda_n, \Lambda \setminus \{\lambda_n\}) > r_n$ for $n = 1, 2, \ldots$.

Contrasting to the example given by the Theorem 3 is the following example.

Example 3. — There exists a non co-Riesz set Λ, which is yet "thick" at infinity, i.e., $\lim_{x\to\infty} d(x, \Lambda) = 0$.

Proof. — Let us start from a sequence of Riesz products

$$\rho_n(x) = \prod_{j=1}^{\infty}(1 + c_{j,n} \cos 2\pi 4^j x),$$

where

(1) $0 < c_{j,n} < 1$, $\sum_{j=1}^{\infty} c_{j,n}^2 = \infty$,

and

(2) $\lim_{j\to\infty} \frac{c_{j,n+1}}{c_{j,n}} = \infty$ for each n.

We can e.g. take $c_{j,n} = \frac{1}{\sqrt[n]{j}}$ for $j > 2^n$, $c_{j,n} = \frac{1}{2}$ otherwise. For the Riesz products the condition (1) implies that ρ_n is a singular probability measure supported by $[-\frac{1}{2}, \frac{1}{2}]$, whose Fourier-Stieltjes transform is carried by the set $\Omega = \{\sum_{j=0}^{J} \varepsilon_j 4^j, \varepsilon_j = 0, \pm 1, J \geqslant 0\} \subset \mathbb{Z}$. We identify ρ_n with a 1-periodic measure on the real line and consider $\sigma_n = \varphi\rho_n$, so that $\widehat{\sigma_n} = \widehat{\varphi} * \widehat{\rho_n}$, where $\varphi \in C^\infty(\mathbb{R})$, $\operatorname{supp}\widehat{\varphi} = [-\frac{1}{2}, \frac{1}{2}]$, $\varphi \geqslant 0$, $\widehat{\varphi} \geqslant 0$, and $\widehat{\varphi} > 0$ on $(-\frac{1}{2}, \frac{1}{2})$. Note that $\operatorname{supp}\widehat{\sigma}_n = \Omega + [-\frac{1}{2}, \frac{1}{2}]$.

Finally, we set

(3) $\tau_n = \frac{\sigma_n}{\|\sigma_n\|} 2^{-n}$; $\mu_n = \tau_n * (\frac{\delta_{a_n} + \delta_{-a_n}}{2})$, and $\mu = \sum_{n=1}^{\infty} \mu_n$ where $\{a_n\}$ is a very rapidly increasing sequence of integers (to be chosen later). Observe that $\widehat{\mu_n}(\xi) = \widehat{\tau_n}(\xi)\cos(a_n\xi)$. Moreover, as all the measures in the sum $\mu = \sum_{j=1}^{\infty} \mu_j$ are positive, the measure μ is a singular probability measure: $\|\mu\| = \sum_{n=1}^{\infty} \|\tau_n\| = \sum_{n=1}^{\infty} 2^{-n} = 1$.

On each particular interval $(\omega - \frac{1}{2}, \omega + \frac{1}{2})$, with $\omega = \sum_{j=1}^{J} \varepsilon_j 4^j \in \Omega$, $\varepsilon_j = 0, \pm 1$, we have $\widehat{\tau}_n / \widehat{\tau}_{n'} = \prod_{j=1}^{J} c_{j,n}^{|\varepsilon_j|} / \prod_{j=1}^{J} c_{j,n'}^{|\varepsilon_j|}$, and $\widehat{\tau}_n$ vanishes outside of these intervals. So, taking into account the condition (2), we can construct a sequence m_n such that $\sum_{j=1}^{n} \widehat{\tau}_j(\xi) \leqslant \widehat{\tau}_{n+1}(\xi)$ for all $|\xi| > m_n$.

Now we will chose inductively the sequence a_n and the sequence of sequences Λ_n such that

(4) $(\sum_{j=1}^{n} \mu_j)^\wedge(\lambda) = 0$ for all $\lambda \in \Lambda_n$;

(5) for any $\lambda \in \Lambda_n \cap [-m_n, m_n]$ the distance $\operatorname{dist}(\lambda, \Lambda_{n+1}) < 2^{-n}$;

and

(6) for all $|\xi| > m_n$, $\operatorname{dist}(\xi, \Lambda_{n+1}) < 2^{-n}$.

After such sequence Λ_n is constructed, we can take $\Lambda = \lim_{n\to\infty} \Lambda_n$ (in the sense that $\lambda \in \Lambda$ if $\operatorname{dist}(\lambda, \Lambda_n) \to 0$). The norm convergence of the sum $\sum_{j=0}^{\infty} \mu_j$ and (4) imply $\widehat{\mu}|_\Lambda \equiv 0$. And the conditions (5) and (6) give that $\operatorname{dist}(x, \Lambda) < 2^{-n+1}$, for all $x \in [-m_{n+1}, m_{n+1}] \setminus [-m_n, m_n]$, which completes the proof.

To construct the sequence Λ_n notice that outside of the intervals $(\omega - \frac{1}{2}, \omega + \frac{1}{2})$, with $\omega \in \Omega$, all the functions $\widehat{\tau}_n$ (and so $\widehat{\mu}_n$) vanishes, by the construction, and thus the only difficulty in the choice of Λ_n occurs on these intervals. Let s_n be the partial sum $\sum_{j=1}^{n} \mu_j$. Note that, $\widehat{s}_n = \tau_1 P$, where P is a trigonometric polynomial on each of the intervals under consideration. As any trigonometric polynomial has only finitely many roots on an interval, $\widehat{s}_n$ has finitely many roots on the interval $(\omega - \frac{1}{2}, \omega + \frac{1}{2})$ for each ω. Denote the set of those roots by $E_{n,\omega}$. It is clear, that $\Lambda_n \cap (\omega - \frac{1}{2}, \omega + \frac{1}{2}) \subset E_{n,\omega}$. As for any $\lambda \in E_{n,\omega}$ by the definition $\widehat{s}_n(\lambda) = 0$, for each $\lambda \in E_{n,\omega}$ we can choose an interval $I_\lambda \subset (\omega - \frac{1}{2}, \omega + \frac{1}{2})$ which contains λ and has length less than 2^{-n}, such that $|\widehat{s}_n| < \widehat{\tau}_n$ on I_λ. As there are only finitely many intervals $(\omega - \frac{1}{2}, \omega + \frac{1}{2})$ in the interval $[-m_n, m_n]$ and there are only finitely many points in each interval, there are only finitely many intervals I_λ, with $\lambda \in (\bigcup_\omega E_{n,\omega}) \cap [-m_n, m_n]$. Let r_n be the minimal length of such an interval. Then we choose $a_{n+1} = \max\{\frac{2\pi}{r_n}, 2^{n+1}\pi\}$. Let Λ_{n+1} on the intervals $(\omega - \frac{1}{2}, \omega + \frac{1}{2})$ be the sequence of zeros of the function $\sum_{j=1}^{n+1} \widehat{\mu_j}$. Notice that on each interval I_λ the function $\widehat{s}_n(\xi) + \widehat{\tau}_{n+1}(\xi)\cos(a_{n+1}\xi)$ takes a non-positive value at the point where $\cos(a_{n+1}\xi) = -1$ and a non-negative value at the point where $\cos(a_{n+1}\xi) = 1$, as $|\widehat{s}_n(\xi)| \leqslant \widehat{\tau}_{n+1}$. For the chosen value of a_{n+1} we can find all the range of values of $\cos(a_{n+1}\xi)$ on the interval I_λ. So there exists a root of the function $\sum_{j=1}^{n+1} \widehat{\mu_j}$ (and so an element of Λ_{n+1}) on each interval I_λ, for all $\lambda \in \big((\cup E_{n,\omega}) \cap [-m_n, m_n]\big)$. As the length of all the intervals I_λ is less than 2^{-n} this means that condition (5) is satisfied. Similar arguments show that condition (6) is satisfied as well. As condition (4) is satisfied by the construction, this complete the proof. □

It appears that there is no estimate on the growth of the distribution of the values of a function with the Fourier transform vanishing on a sequence Λ (unlike the result about f-poles).

PROPOSITION 4. — *For every sequence $\Lambda \subset \mathbb{R}^d$ with no limit points, and every Young function Φ such that $L^1(\mathbb{R}^d) \not\subset L^\Phi(\mathbb{R}^d)$, there exists $f \in L^1(\mathbb{R}^d) \setminus L^\Phi_{\mathrm{loc}}(\mathbb{R}^d)$ such that $\widehat{f}(\lambda) = 0$ for $\lambda \in \Lambda$.*

Proof. — Put $\phi(t) = t^{-1}\Phi(t)$. Since $L^1(\mathbb{R}^d) \neq L^\Phi(\mathbb{R}^d)$, $\phi(t) \to \infty$ as $t \to \infty$. Let ψ be the function from the proof of Proposition 2 which is, additionally, decreasing to 0 at infinity (this requirement is equivalent to smoothness of $\widehat{\psi}$). Let $x_0 = (1, 0, \dots, 0) \in \mathbb{R}^d$ and put for $m, k \in \mathbb{Z}$

$$f_{m,k}(x) = 2^{md}\psi(2^m x) - 2^{md}\psi(2^m x + 2^m k x_0).$$

Obviously $\widehat{f}_{m,k}(\xi) = (1 - e^{i2\pi k\langle\xi, x_0\rangle})\widehat{\psi}(\xi 2^{-m})$. Thus the Fourier transform of $f_{m,k}$ is supported on the cube $2^m I^d$, and $|\widehat{f}_{m,k}| = |1 - e^{i2\pi k\langle\xi,x_0\rangle}|$ for $\xi \in \mathbb{R}^d$. Since $\psi > 0$ and ψ is decreasing at infinity, for every $m \in \mathbb{Z}$ there exists $K = K(m) \in \mathbb{Z}$ such that $f_{m,k}$ is positive on the cube I^d and $f_{m,k}(x) > 2^{md}\sigma$ for $x \in r2^{-m}I^d$, for all $|k| > K(m)$. Since the set $\Lambda \cap 2^m I^d$ is finite, for every $\varepsilon > 0$ we can always find an (arbitrarily large) integer N such that $\operatorname{dist}(N\langle\lambda, x_0\rangle, \mathbb{Z}) < \varepsilon$ for $\lambda \in \Lambda \cap 2^m I^d$. Hence there exists integer k_m such that we have $\widehat{f}_{m,k_m}(\lambda) < \frac{1}{M}$ for $\lambda \in \Lambda \cap 2^m I^d$, where $M = \#(\Lambda \cap 2^m I^d)$. Clearly $\widehat{f}_{m,k_m}(\lambda) = 0$ for $\lambda \in \Lambda \setminus 2^m I^d$. Put

$$h_1 = \sum \frac{1}{n^2} f_{m_n,k_{m_n}},$$

where the numbers m_n $(n = 1, 2, \dots)$ are going to be chosen later. We prove that for every $\alpha > 0$ the function $\Phi(\alpha h_1)$ is not integrable on any fixed neighbourhood of the origin, say aI^d (note that we can assume that h_1 is positive in aI^d). Since Φ is a superadditive function,

$$\int_{aI^d} \Phi(\alpha h_1) \geqslant \sum \int_{aI^d} \Phi\Big(\frac{\alpha}{n^2} f_{m_n,k_{m_n}}\Big).$$

If $r2^{-m_n} < a$ we have

$$\begin{aligned}\int_{aI^d} \Phi\Big(\frac{\alpha}{n^2} f_{m_n,k_{m_n}}\Big) &\geqslant \int_{r2^{-m_n}I^d} \Phi\Big(\frac{\alpha\sigma}{n^2} 2^{m_n d}\Big) \\ &= 2r\frac{\alpha\sigma}{n^2}\phi\Big(\frac{\alpha\sigma}{n^2}2^{m_n d}\Big).\end{aligned}$$

If m_n is chosen to satisfy $\phi(n^{-3}2^{m_n d}) > n$, then for n sufficiently large

$$\int_{aI^d} \Phi\Big(\frac{\alpha}{n^2} f_{m_n,k_{m_n}}\Big) \geqslant 2r\alpha\sigma n^{-1}.$$

Thus $h_1 \notin L^{\Phi}_{\mathrm{loc}}(\mathbb{R}^d)$. Let us consider now the functions g_{m_n} defined by

$$\widehat{g}_{m_n}(\xi) = \sum_{\lambda \in \Lambda \cap 2^{m_n} I^d} \widehat{f}_{m_n,k_{m_n}}(\lambda)\widehat{\psi}\Big(\sqrt{d}\,\frac{\xi - \lambda}{\tau_{m_n}}\Big),$$

where $\tau_m = \min\{1, |\lambda - \lambda'| : \quad \lambda, \lambda' \in \Lambda \cap (2^m + 1)I^d, \quad \lambda \neq \lambda'\}$. Then, obviously, $\widehat{g}_{m_n}(\lambda) = \widehat{f}_{m_n,k_{m_n}}(\lambda)$ for all $\lambda \in \Lambda$. Also

$$\|g_{m_n}\|_1 \leqslant \sum_{\lambda \in \Lambda \cap 2^{m_n} I^d} |\widehat{f}_{m_n,k_{m_n}}(\lambda)| < 1,$$

and

$$\begin{aligned}\|g_{m_n}\|_\infty &\leqslant \|\widehat{g}_{m_n}\|_1 \\ &\leqslant \sum_{\lambda \in \Lambda \cap 2^{m_n} I^d} |\widehat{f}_{m_n,k_{m_n}}(\lambda)| \tau_{m_n}^d \sqrt{d^{-d}} \|\widehat{\psi}\|_1 \\ &\leqslant \sqrt{d^{-d}}\|\widehat{\psi}\|_1.\end{aligned}$$

Thus the function $h_2 = \sum \frac{1}{n^2} g_{m_n}$ is bounded, summable, and $\widehat{h}_1(\lambda) = \widehat{h}_2(\lambda)$ for all $\lambda \in \Lambda$. Hence the function $f = h_1 - h_2 \in L^1(\mathbb{R}^d)$, its Fourier transform vanishes on Λ and f is still not in $L^{\Phi}_{\mathrm{loc}}(\mathbb{R}^d)$. □

Remark. — If $\Phi(x) = x\log(x+1)$, then one can modify the above construction to get a function which does not belong to $H^1(\mathbb{R})$.

Proof. — After some minor modification we can assume that h_2 is continuously differentiable. Indeed, during the construction, when we define numbers k_m, we replace the condition $\widehat{f}_{m,k_m}(\lambda) < \frac{1}{M}$ by the condition $\widehat{f}_{m,k_m}(\lambda) < \frac{1}{(2^m+1)M}$. Then we get the estimate on the gradient of g_{m_n} in the same way as the estimate for the sup norm of g_{m_n} in the proof above. Let χ be a smooth function supported on $2I^d$ such that $\chi \equiv 1$ on I^d. Since the function

$$h_3(x) = h_2(x)\chi(x) - h_2(x - 3x_0)\chi(x - 3x_0)$$

belongs to the space H^1, the function f belongs to H^1 iff $f + h_3$ does. The function $f + h_3$ is positive on the cube I^d, because it coincides there with h_1. Hence, by [St, Chapt. III.5.3], the restriction $(f+h_3)|_{I^d} = h_1|_{I^d}$ should agree with an $L\log L$-summable function on every compact subset of I^d, which is not the case. □

In the proof of Theorem 3 the arithmetic relations between elements of Λ were crucial, as we used the fact that all but finitely many elements of $\alpha_j^{-1}\mathbb{Z}$ belong to Λ. On the other hand, the set of non co-Riesz sequences is open in the following sense.

PROPOSITION 5. — *For any function f on $\mathbb{R}_+$, which decreases to 0, and any sequence $\Lambda = (\lambda_n) \subset \mathbb{R}$ which has no limit points, there exists a sequence of positive numbers (r_n) such that for every measure $\mu \in M(\mathbb{R})$, for which $\|\mu\| = 1$ and $|\mu|(\mathbb{R} \setminus [-r,r]) \leqslant f(r)$ and any sequence $\Lambda' = (\lambda'_n) \subset \mathbb{R}$ such that $|\lambda_n - \lambda'_n| < r_n$ $(n = 1, 2, \ldots)$, there exists a measure $\mu' \in M(\mathbb{R})$ such that $\widehat{\mu}(\lambda_n) = \widehat{\mu}'(\lambda'_n)$ for $n = 1, 2, \ldots$ and $\mu_s = \mu'_s$.*

Proof. — We need the following lemma.

Lemma 1. — Given $c \in (0,1)$, $r > 0$, $x \in \mathbb{R}$ and a measure $\mu \in M(\mathbb{R})$ as in the Proposition 5, there exists a measure $\nu = \nu(c,r,x) \in M(\mathbb{R})$ absolutely continuous with respect to Lebesgue measure, such that $\operatorname{supp}\widehat{\nu} \subset [x - 2r, x + 2r]$, $\widehat{\mu}(x) = (\mu - \nu)^{\wedge}(y)$ for every $y \in (x - r, x + r)$, and $\|\nu\| < C\big(c + f(\frac{c}{r})\big)$, where the constant C does not depend on c, r, x and μ.

We show first how Proposition 5 follows from the lemma. Let (r_n) and (c_n) be sequences of positive numbers such that the intervals $[\lambda_n - 2r_n, \lambda_n + 2r_n]$ are pairwise disjoint, $\sum c_n < \infty$ and $\sum f(\frac{c_n}{r_n}) < \infty$. Then $\mu' = \mu - \sum \nu(c_n, r_n, \lambda_n)$ is a finite measure which satisfies all the requirements.

Proof of Lemma 1. — Let $\psi \in L^1(\mathbb{R})$ be such that $\|\psi\|_1 < C$, $\|\psi'\|_\infty < C$, $|\psi'(x)| < \frac{C}{|x|^2}$, $\operatorname{supp}\widehat{\psi} \subset [-2,2]$ and $\widehat{\psi}(x) = 1$ for $x \in [-1,1]$. Put $\rho_R(x) = \sup_{y\in[x-R,x+R]} |\psi'(y)|$. It is easy to see that $\|\rho_R\|_1 < C_1 \max(1,R)$. We denote $\psi_t(x) = t\psi(tx)$. Without loss of generality we suppose that $x = 0$. We consider first the case $\widehat{\mu}(0) = 0$. Then we have $\|\mu * \psi_r\| \leqslant C_2(c + f(\frac{c}{r})$. Indeed, since $\widehat{\mu}(0) = 0$, we can represent $\mu = \mu^R + \mu_R$ where μ_R is supported on the interval $[-R,R]$, $\int_{\mathbb{R}} d\mu_R = 0$ and $\|\mu^R\| < 2f(R)$. Then we estimate the convolution separately for μ_R and μ^R

$$\|\mu^R * \psi_r\| \leqslant \|\mu^R\| \cdot \|\psi_r\|_1 \leqslant C \cdot f(R),$$

$$|(\mu_R * \psi_r)(x)| = \left| \int \psi_r(x - \ \cdot) - \psi_r(x)\, d\mu_R \right| \leqslant R \sup_{y\in[x-R,x+R]} |\psi_r'(y)| \cdot \|\mu_R\|.$$

Since $\sup_{y\in[x-R,x+R]} |\psi_r'(y)| = r^2 \rho_{rR}(rx)$ we get

$$\|\mu_R * \psi_r(x)\|_1 \leqslant CrR\max(1,rR)\|\mu\|.$$

Putting $R = \frac{c}{r}$ we get the desired estimation. Hence the measure $\nu = \mu * \psi_r$ satisfies the conditions of the lemma. If $\widehat{\mu}(0) \neq 0$ we put $\nu = (\mu - (\int d\mu)\delta_0) * \psi_r$. □

Remarks. — 1) It follows from the proof that if Λ satisfies the assumption of Proposition 5 and $\widehat{\mu}(\lambda) = 0$ for $\lambda \in \Lambda$ then there exists $\mu' \in M(\mathbb{R})$ such that $\mu_s = \mu_s'$ and $\widehat{\mu}'$ vanishes on some open set containing Λ.

2) Proposition 5 can be easily extended to a multidimensional case.

The next result shows that in the previous proposition the sequence (r_n) could not be chosen uniformly for all measures, without the decrease condition.

PROPOSITION 6. — *For every positive sequence (r_j) there exist sequences $\Lambda = (\lambda_n)$ and $\Lambda' = (\lambda_n')$ and $\mu \in M(\mathbb{R})$ such that $|\lambda_j - \lambda_j'| < r_j$ and there is no $\mu' \in M(\mathbb{R})$ such that $\widehat{\mu}(\lambda_n) = \widehat{\mu}'(\lambda_n')$ for $n = 1, 2, \ldots$.*

Proof. — We set $\Lambda' = \mathbb{Z}$. We index the sequences Λ and Λ' by integer numbers rather then natural ones. Let (a_n) be a decreasing sequence of positive numbers such that $\sum a_n < \infty$ and (k_n) be the sequence of positive integers such that $\sum k_n a_n^2 = \infty$. Put $b_n = a_j$ where j is the unique index such that $k_1 + \ldots + k_{j-1} < n \leqslant k_1 + \ldots + k_{j-1} + k_j$ (here we put $k_0 = 0$). Set

$$\lambda_m = \begin{cases} m & \text{if } m \neq 2^j,\ (j = 1, 2, \ldots) \\ m + \frac{1}{2}\omega_j^{-1} & \text{if } m = 2^n, \quad k_1 + \ldots + k_{j-1} < n \leqslant k_1 + \ldots + k_j, \end{cases}$$

where (ω_j) is a sequence of positive integers satisfying for $j = 1, 2, \ldots$

1) $(2\omega_j)|\omega_{j+1}$;

2) $\omega_j^{-1} < \min\{r_{2^n} : n < k_1 + \ldots + k_j\}$.

Let $\mu_n = (2i)^{-1}(\delta_{\omega_n} - \delta_{-\omega_n})$ and $\mu = \sum a_n \mu_n$ We have $\widehat{\mu}_n(t) = \sin \pi\omega_n t$. Hence $\widehat{\mu}_j(\lambda_{2^n})$ is positive for $n \leqslant k_1 + \ldots + k_{j-1}$, equals 1 for $k_1 + \ldots + k_{j-1} < n \leqslant k_1 + \ldots + k_j$ and vanishes for $k_1 + \ldots + k_j < n$. Thus $\widehat{\mu}(\lambda_{2^n}) > b_n$ for $n = 1, 2 \ldots$. Clearly $\widehat{\mu}(\lambda_n) = 0$ for $n \neq 2^j$, $(j = 1, 2, \ldots)$.

Suppose to the contrary that there exists a finite measure μ' such that $\widehat{\mu}'(j) = \widehat{\mu}(\lambda_j)$. By the de Leeuw transference theorem (cf. [deL], [StW, Chapt. VII, Th. 3.8]), there exists a bounded measure $\nu \in M(\mathbb{T})$ such that $\|\nu\|_{M(\mathbb{T})} \leqslant \|\mu'\|_{M(\mathbb{R})}$ and $\widehat{\nu}(n) = \widehat{\mu}'(n)$ for $n = 1, 2, \ldots$. But $\sum |\widehat{\nu}(n)|^2 = \sum b_j^2 = \infty$ which contradicts the fact that $A = \{2^k : k = 1, 2, \ldots\}$ is a Λ_2 set, i.e. $\nu \in L^2(\mathbb{T})$ and $\|\nu\|_2 \leqslant C\|\nu\|_M$ for every measure $\nu \in M(\mathbb{T})$ with the Fourier transform vanishing outside A. □

The above construction has one more application. We can use it to construct a sequence which does not allow co-balayage.

PROPOSITION 7. — *There exists a sequence $\Lambda = (\lambda_n)$ such that*

$$\inf_{\lambda \in \Lambda} \operatorname{dist}(\lambda, \Lambda \setminus \{\lambda\}) > 0,$$

and measure $\mu \in M(\mathbb{R})$ such that there is no measure $\mu' \in M(\mathbb{R})$ supported on a compact set such that $\widehat{\mu}(\lambda_n) = \widehat{\mu}'(\lambda_n)$ for $n = 1, 2, \ldots$.

Proof. — Let Λ and μ be the same as in the proof of Proposition 6 with one modification: the condition 2) on the sequence (ω_j) is replaced by another condition

2') $\omega_j^{-1} < j^{-1} a_j$.

Suppose that there exists $\mu' \in M(\mathbb{R})$ such that $\widehat{\mu}'(\lambda_n) = \widehat{\mu}(\lambda_n)$ for $n = 1, 2, \ldots$ and $\operatorname{supp} \mu' \subset [-T, T]$ for some $T > 0$. Then the derivative of

$\widehat{\mu}$ is bounded by $T \cdot \|\mu'\|_M$. Hence, for sufficiently large n, we have

$$\begin{aligned}|\widehat{\mu}'(2^n)| &\geqslant |\widehat{\mu}'(\lambda_{2^n})| - T \cdot \|\mu'\|_M \cdot |\lambda_{2^n} - 2^n| \\ &= |\widehat{\mu}'(\lambda_{2^n})| - T \cdot \|\mu'\|_M \cdot (2\omega_j)^{-1} \\ &> \frac{1}{2}|\widehat{\mu}'(\lambda_{2^n})| = \frac{1}{2}b_n.\end{aligned}$$

Thus $\sum |\widehat{\mu}'(n)|^2 = \infty$, and $\widehat{\mu'}$ vanish on $m \neq 2^n$. We finish proceeding as in the proof of Proposition 6. □

Remarks. — 1) Note that for $\Lambda = \mathbb{Z}$, the measure μ' with properties postulated by Proposition 7 exists, and it is supported by an interval of length 1. This is exactly what the de Leeuw theorem says:

2) We say that a sequence $\Lambda \subset \mathbb{R}$ has *de Leeuw property* iff for every measure $\mu \in M(\mathbb{R})$, there exists a measure $\mu' \in M(\mathbb{R})$ with compact support, such that $\widehat{\mu}'(\lambda) = \widehat{\mu}(\lambda)$ for every $\lambda \in \Lambda$. By the de Leeuw transference theorem, for every finite set $F \subset \mathbb{R}$ and $r \in \mathbb{R}$, any subset of the set $F + r\mathbb{Z}$ has the de Leeuw property. We do not know whether the converse is true.

3) It is much easier to construct a sequence Λ without the de Leeuw property if we omit the condition $\inf_{\lambda\in\Lambda} \operatorname{dist}(\lambda, \Lambda \setminus \{\lambda\}) > 0$. Moreover, every sequence Λ which contains an increasing subsequence (x_n) such that $\lim x_n = \infty$ and $\lim(x_{2n} - x_{2n+1}) = 0$, has not de Leeuw property. Indeed, let $\nu \in L^1(\mathbb{R})$ be a measure with Fourier transform supported on the interval $[-1, 1]$ such that $\widehat{\nu}(0) = 1$ and let $\nu_r \in M(\mathbb{R})$ be defined by $\widehat{\nu}_r(t) = \widehat{\nu}(\frac{t}{r})$. Passing, if necessary, to a subsequence we can assume that $\sum r_n^{1/2} < \infty$ where $r_n = x_{2n+1} - x_{2n} < x_{2n} - x_{2n-1}$ for $n = 1, 2, \ldots$. Put $\mu = \sum r_n^{1/2} \nu_{r_n} e^{2\pi i x_{2n} t}$. Then we have $\|\mu\| < \|\nu\| \cdot \sum r_n^{1/2} < \infty$, $\widehat{\mu}(x_{2n}) = r_n^{1/2}$ and $\widehat{\mu}(x_{2n+1}) = 0$ for $n = 1, 2, \ldots$. Hence the supremum of the derivative of $\widehat{\mu}$ on the interval (x_{2n}, x_{2n+1}) is greater then $r_n^{-1/2}$. Therefore the derivative of $\widehat{\mu}$ is unbounded, which means that μ is not compactly supported.

5. co-Lebesgue sequences.

We call the sequence $\Lambda \subset \mathbb{R}^d$ a *co-Lebesgue* sequence iff for every measure $\mu \in M(\mathbb{R}^d)$ such that $\widehat{\mu}(\xi) = 0$ for $\xi \in \Lambda$, the singular part μ_s shares the same property, i.e. $\widehat{\mu}_s(\xi) = 0$ for $\xi \in \Lambda$. Clearly every co-Riesz sequence is co-Lebesgue. A slight modification of Theorem 1 allows to state the following criterion.

PROPOSITION 8. — *Assume that $\Lambda \subset \mathbb{R}^d$ has the following property. For every $\xi \in \Lambda$ there exists $\alpha \in \mathbb{R}$ such that $\mathbb{Z}^d \setminus \alpha\Lambda$ is a Riesz set, and $\alpha\xi \in \mathbb{Z}^d$. Then Λ is a co-Lebesgue sequence.*

Proof. — Let $\xi \in \Lambda$ and $\alpha \in \mathbb{R}$ be such that $\alpha\xi \in \mathbb{Z}^d$ and $\mathbb{Z}^d \setminus \alpha\Lambda$ is a Riesz set. Let $\nu \in M(\mathbb{T}^d)$ be the measure defined by $\nu(E) = \mu(\alpha E + \alpha\mathbb{Z}^d)$ for $E \subset \mathbb{T}^d$. Clearly $\nu_s(E) = \mu_s(\alpha E + \alpha\mathbb{Z}^d)$. It is easy to see that for every $k \in \mathbb{Z}^d$,

$$\widehat{\nu}(k) = \widehat{\mu}\Big(\frac{1}{\alpha}k\Big),$$

as well as

$$\widehat{\nu}_s(k) = \widehat{\mu}_s\Big(\frac{1}{\alpha}k\Big).$$

Since $\widehat{\mu}(\xi) = 0$ for $\xi \in \Lambda$, the Fourier transform of ν vanishes outside some Riesz subset of $\mathbb{Z}^d$. Hence, by the assumption, $\nu_s = 0$. Since $\alpha\xi \in \mathbb{Z}^d$, the above formula yields that $\mu_s(\xi) = \nu_s(\alpha\xi) = 0$. □

Examples 4.5. — 4) Let $k = 2, 3, \ldots$. Then the sequence $\Lambda_k = (n^{1/k})_{n=1}^{\infty} \subset \mathbb{R}$ is co-Lebesgue one. Indeed, let $a \in \Lambda_k$. Then $a^k \in \mathbb{Z}$. Therefore $j^k a^k \in \mathbb{Z}$ for $j = 1, 2, \ldots$. Hence $ja \in \Lambda_k$ for $j = 1, 2 \ldots$. Therefore $\frac{1}{a}\Lambda_k \cap \mathbb{Z} = \mathbb{Z}_+$, and, by F. and M. Riesz theorem, $\mathbb{Z} \setminus (\frac{1}{a}\Lambda_k)$ is a Riesz set.

5) Let $\Lambda_0 = (\log n)_{n=1}^{\infty}$. If $a = \log m \in \Lambda_0$ then $na = \log m^n \in \Lambda$ for $n = 1, 2, \ldots$ and hence, similarly as in Example 4, $\mathbb{Z} \setminus \frac{1}{a}\Lambda_0 = \mathbb{Z}_-$ is a Riesz set.

Remarks. — 1) In fact, Proposition 8 together with the above example give something more, namely if $\widehat{\mu}(\xi) = 0$ for $\xi \in \Lambda_k$ then $\widehat{\mu}_s(\xi) = 0$ for $\xi \in \Lambda_k \cup -\Lambda_k$.

2) We do not know whether Λ_k are co-Riesz sequences.

BIBLIOGRAPHY

[A] A.B. ALEKSANDROV, Essays on non locally convex Hardy classes, Lecture Notes in Math., Springer, 864 (1981), 1–89.

[deL] K. de LEEUW, On L_p multipliers, Annals of Math., 81 (1965), 364–379.

[E] R.E. Edwards, Fourier series, Holt, Rinehart and Winston, Inc., 1967.

[HJ] V.P. Havin and B. Jörike, The Uncertainty Principle in Harmonic Analysis, Springer-Verlag, Berlin, 1994.

[HR] E. Hewitt and K. Ross, Abstract Harmonic Analysis, Springer-Verlag, Berlin-Goettingen-Heidelberg, 1963.

[M] Y. Meyer, Spectres des mesures et mesures absolument continues, Studia Math., 30 (1968), 87–99.

[RR] M.M. Rao and Z.D. Ren, Theory of Orlicz spaces, Marcel Dekker Inc., New York, 1991.

[R] M. Roginskaya, Two analogues of M. and F. Riesz theorem, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 255 (1998), 164–176, 253.

[Sh] J. H. Shapiro, Subspaces of $L^p(G)$ spanned by characters:$0 < p < 1$, Israel J. Math., 29 (1978) 248–264.

[St] E.M. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, Princeton University Press, Princeton, NJ, 1993.

[StW] E.M. Stein and G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton Univ. Press, Princeton, 1971.

[W] M. Wojciechowski, On the roots of the Fourier transform of Singular measures, to appear.

Manuscrit reçu le 7 novembre 2002,
accepté le 9 mai 2003.

Maria ROGINSKAYA,
Department of Mathematics
Chalmers TH &Göteborg University
Eklandagatan 86
SE-41296 Göteborg (Sweden)
maria@math.chalmers.se

Michał WOJCIECHOWSKI,
Institute of Mathematics
Polish Academy of Sciences
Sniadeckich 8, I p.
00-950 Warszawa (Poland).
miwoj@impan.gov.pl