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REPRESENTATION THEORY OF NEVEU-SCHWARZ AND RAMOND ALGEBRAS II : FOCK MODULES

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Contents.

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1. Introduction.

In this paper, following [IK2], we continue our study on the $N = 1$ super Virasoro algebras. There, we have analyzed the structure of Verma modules, and here we will analyze the structure of Fock modules over the $N = 1$ super Virasoro algebras. In the case of the Virasoro algebra, the modules realized on the space of semi-infinite forms are investigated by B. Feigin and D. Fuchs [FeFu]. Motivated by their work, Tsuchiya and Kanie [TK] constructed the representations of the Virasoro algebra on bosonic Fock spaces via Boson-Fermion correspondence. Here, we first recall the Fock modules of the $N = 1$ super Virasoro algebras, and study their structures. As an application, we will construct the Bechi-Rouet-Stora-Tyutin (BRST for short) type resolutions announced in [IK1], in detail.

The main idea to study the structure of Fock modules is a generalization of the Jantzen filtration [Ja]. This filtration was used in [FeFu] to study the structure of Virasoro modules realized on the space of semi-infinite forms. Here, we reformulate their generalization and state some general properties of the construction (see §2.3). By our reformulation, in particular the duality stated in Proposition 2.3, we could even simplify the original arguments done for the Virasoro algebra. One of the technical difficulties here arises at the so-called super-symmetric point, and we could resolve this difficulties by a technique we have developed in our previous paper [IK2].

Concerning the BRST type resolutions, we use the detailed structure of Fock modules studied in §4 to construct BRST type complex. Our proof here is different from those given by [Fel] and [BP] for the Virasoro case in an essential point, viz., we do not use the so-called screening operators to construct the complex itself. We can prove the existence of the coboundary maps by an abstract manner, and we see that the explicit form of the coboundary maps is given by the screening operators under a weak assumption. Thus, in particular, our proof is new even for the Virasoro case.

We consider the supersymmetric point and non-supersymmetric points separately. The proof for the supersymmetric point uses our construction of the Jantzen filtration à la Feigin and Fuchs (see [FeFu] and §2.3). On the other hand the proof for the non-supersymmetric point is rather similar to the Virasoro case due to [Fel] and [BP] except for the point where it is not necessary to use the screening operators to construct the complex itself.

This paper is organized as follows. In §2.1, we will recall the definition of the $N = 1$ super Virasoro algebras and their Fock modules. §2.2 is a collection of the necessary facts that follow from the results stated in [IK2]. §2.3 is the core of this paper where we describe a generalization of the Jantzen filtration. In §3, we will study basic tools, such as screening operators and the determinant formulae. In §4, we first state the results on the BRST type resolutions and the proofs for the supersymmetric point and non-supersymmetric points are given in different subsections. In §A, we provide some data which will be used in the main body of the paper.

2. Preliminary.

In this section, we present our framework of the representation theory of the $N = 1$ super Virasoro algebras.

In §2.1, we introduce all of the objects considered in this article. §2.2 is devoted to a reformulation of the Jantzen filtration that fits to our arguments developed in the further sections.

2.1. Definitions.

Here, we recall the objects that will be considered in this article, the $N = 1$ super Virasoro algebras, Verma modules and Fock modules etc.

The Lie superalgebras we are going to consider are the following:

Definition 2.1. — *The $N = 1$* **super Virasoro algebras** Vir_ε *($\varepsilon = \frac{1}{2}, 0$) are the Lie superalgebras*

$$\mathrm{Vir}_\varepsilon := \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \bigoplus_{m\in\varepsilon+\mathbb{Z}} \mathbb{C}G_m \oplus \mathbb{C}c,$$

which satisfy the following commutation relations:

$$\begin{aligned}
&\deg L_n = \bar{0} \quad (n \in \mathbb{Z}), \quad \deg G_m = \bar{1} \quad (m \in \varepsilon + \mathbb{Z}), \quad \deg c = \bar{0},\\
&[L_m, L_n] = (m-n)L_{m+n} + \delta_{m+n,0}\frac{1}{12}(m^3 - m)c,\\
&[G_m, L_n] = \left(m - \frac{1}{2}n\right) G_{m+n},\\
&[G_m, G_n] = 2L_{m+n} + \delta_{m+n,0}\frac{1}{3}\left(m^2 - \frac{1}{4}\right) c,\\
&[\mathrm{Vir}_\varepsilon, c] = \{0\}.
\end{aligned}$$

$\mathrm{Vir}_{\frac{1}{2}}$ *and* Vir_0 *are called the* **Neveu-Schwarz** *and the* **Ramond** *algebras respectively. Furthermore,* Vir_ε *is* $\mathbb{Z}$*-graded by setting*

$$\mathfrak{h} := \mathbb{C}L_0 \oplus \mathbb{C}c$$

and

$$(\mathrm{Vir}_{\frac{1}{2}})_n := \begin{cases} \mathbb{C}L_{\frac{1}{2}n} & \text{if } n \in 2\mathbb{Z} \setminus \{0\},\\ \mathbb{C}G_{\frac{1}{2}n} & \text{if } n \in 2\mathbb{Z}+1,\\ \mathfrak{h} & \text{if } n = 0, \end{cases} \quad (\mathrm{Vir}_0)_n := \begin{cases} \mathbb{C}L_n \oplus \mathbb{C}G_n & \text{if } n \neq 0,\\ \mathfrak{h} \oplus \mathbb{C}G_0 & \text{if } n = 0. \end{cases}$$

By definition, Vir_ε satisfies the following decomposition:

$$\mathrm{Vir}_\varepsilon = \mathrm{Vir}_\varepsilon^{\bar{0}} \oplus \mathrm{Vir}_\varepsilon^{\bar{1}}, \qquad \mathrm{Vir}_\varepsilon^{\bar{0}} := \bigoplus_{n\in\mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}c, \quad \mathrm{Vir}_\varepsilon^{\bar{1}} := \bigoplus_{m\in\varepsilon+\mathbb{Z}} \mathbb{C}G_m.$$

Moreover, Vir_ε possesses the following triangular decomposition:

$$\mathrm{Vir}_\varepsilon = (\mathrm{Vir}_\varepsilon)_+ \oplus (\mathrm{Vir}_\varepsilon)_0 \oplus (\mathrm{Vir}_\varepsilon)_-, \qquad (\mathrm{Vir}_\varepsilon)_\pm := \bigoplus_{\pm n\in(1-\varepsilon)\mathbb{Z}_{>0}} (\mathrm{Vir}_\varepsilon)_n.$$

Below, we define the objects that will be treated in this article.

Namely, we introduce Fock-modules of Vir_ε.

As a preliminary step, we define the Heisenberg algebra $\mathcal{H}$. Let $\mathcal{H} := \bigoplus_{n\in\mathbb{Z}} \mathbb{C}a_n \oplus \mathbb{C}K_{\mathcal{H}}$ be the Lie algebra over the field $\mathbb{C}$ whose commutation relations are given by

$$\begin{gathered}
\deg a_n := \bar{0}, \quad \deg K_{\mathcal{H}} := \bar{0},\\
[a_m, a_n] := \delta_{m+n,0} m K_{\mathcal{H}}, \quad [\mathcal{H}, K_{\mathcal{H}}] = \{0\}.
\end{gathered}$$

If we put $\mathcal{H}_\pm := \bigoplus_{\pm n\in\mathbb{Z}_{>0}} \mathbb{C}a_n$ and $\mathcal{H}_0 := \mathbb{C}a_0 \oplus \mathbb{C}K_{\mathcal{H}}$, then we have a triangular decomposition

$$\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_0 \oplus \mathcal{H}_+.$$

Further we set $\mathcal{H}_\geqslant := \mathcal{H}_0 \oplus \mathcal{H}_+$, $\mathcal{H}_\leqslant := \mathcal{H}_- \oplus \mathcal{H}_0$.

For $\varepsilon = \frac{1}{2}, 0$, let $\mathcal{D}_\varepsilon := \bigoplus_{n\in\varepsilon+\mathbb{Z}} \mathbb{C}\varphi_n \oplus \mathbb{C}K_{\mathcal{D}}$ be the Lie superalgebra over the field $\mathbb{C}$ satisfying the following commutation relations:

$$\deg \varphi_n := \bar{1}, \quad \deg K_{\mathcal{D}} := \bar{0},$$
$$[\varphi_m, \varphi_n] := \delta_{m+n,0} K_{\mathcal{D}}, \quad [\mathcal{D}_\varepsilon, K_{\mathcal{D}}] = \{0\}.$$

If we set

$$\mathcal{D}_{\varepsilon;\pm} := \bigoplus_{\pm n\in\mathbb{Z}_{>0}-\varepsilon} \mathbb{C}\varphi_n, \quad \mathcal{D}_{\varepsilon;0} := \begin{cases} \mathbb{C}K_{\mathcal{D}} & \text{if } \varepsilon = \frac{1}{2}, \\ \mathbb{C}\varphi_0 \oplus \mathbb{C}K_{\mathcal{D}} & \text{if } \varepsilon = 0, \end{cases}$$

then we have a triangular decomposition

$$\mathcal{D}_\varepsilon = \mathcal{D}_{\varepsilon;-} \oplus \mathcal{D}_{\varepsilon;0} \oplus \mathcal{D}_{\varepsilon;+}.$$

We also set $\mathcal{D}_{\varepsilon;\geqslant} := \mathcal{D}_{\varepsilon;0} \oplus \mathcal{D}_{\varepsilon;+}$, $\mathcal{D}_{\varepsilon;\leqslant} := \mathcal{D}_{\varepsilon;-} \oplus \mathcal{D}_{\varepsilon;0}$.

Recall the so-called Fock modules of $\mathcal{H}$ and $\mathcal{D}_\varepsilon$.

For $\eta \in \mathbb{C}$, let $\mathbb{C}_\eta := \mathbb{C}1_\eta$ be the one-dimensional $\mathcal{H}_\geqslant$-module given by

1. $\deg \mathbf{1}_\eta = \bar{0}$.
2. $a_n.\mathbf{1}_\eta = \eta\delta_{n,0}\mathbf{1}_\eta$ $(n \in \mathbb{Z}_{\geqslant 0})$.
3. $K_{\mathcal{H}}.\mathbf{1}_\eta = \mathbf{1}_\eta$.

We consider the induced module

$$\mathcal{F}^{\eta;\varepsilon} := \operatorname{Ind}_{\mathcal{H}_\geqslant}^{\mathcal{H}} \mathbb{C}_\eta.$$

For $\varepsilon = \frac{1}{2}, 0$, let

$$\mathbb{C}_{\mathcal{D}}^\varepsilon := \begin{cases} \mathbb{C}\mathbf{1}^{\frac{1}{2}} & \text{if } \varepsilon = \frac{1}{2}, \\ \mathbb{C}\mathbf{1}^0 \oplus \mathbb{C}\varphi_0.\mathbf{1}^0 & \text{if } \varepsilon = 0, \end{cases}$$

be the $\mathcal{D}_{\varepsilon;\geqslant}$-module whose structure is defined by

1. $\deg \mathbf{1}^\varepsilon = \bar{0}$.
2. $\varphi_n.\mathbf{1}^\varepsilon = 0$ if $n > 0$.
3. $K_{\mathcal{D}}.\mathbf{1}^\varepsilon = \mathbf{1}^\varepsilon$.

Furthermore, we set

$$\mathcal{F}_{\mathcal{D}}^\varepsilon := \mathrm{Ind}_{\mathcal{D}_{\varepsilon;\geqslant}}^{\mathcal{D}_\varepsilon} \mathbb{C}_{\mathcal{D}}^\varepsilon.$$

We define the space $\mathcal{F}^{\eta:\varepsilon}$ on which Vir_ε acts.

Definition 2.2. — *For $\eta \in \mathbb{C}$, we set*

$$\mathcal{F}^{\eta:\varepsilon} := \mathcal{F}^\eta \otimes \mathcal{F}_{\mathcal{D}}^\varepsilon,$$

and call it a Fock module.

Setting

$$a(z) := \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \qquad \varphi^\varepsilon(z) := \sum_{n\in\mathbb{Z}+\varepsilon} \varphi_n z^{-n-\frac{1}{2}},$$

Vir_ε-module structures on the space $\mathcal{F}^{\eta:\varepsilon}$ can be described as follows:

Lemma 2.1. — *For $\varepsilon = \frac{1}{2}, 0$ and $\lambda, \kappa \in \mathbb{C}$, we set*

$$\begin{aligned} T_{\lambda,\kappa}^\varepsilon(z) &:= \frac{1}{2} {}^\circ_\circ a(z)^2 {}^\circ_\circ + (\lambda\partial_z + \kappa z^{-1})a(z) + \frac{1}{2}\kappa(\kappa - 2\lambda)z^{-2} \\ &\qquad + \frac{1}{2} {}^\circ_\circ (\partial_z\varphi^\varepsilon)(z)\varphi^\varepsilon(z) {}^\circ_\circ + \frac{3}{8} z^{-1} {}^\circ_\circ \varphi^\varepsilon(z)^2 {}^\circ_\circ, \\ G_{\lambda,\kappa}^\varepsilon(z) &:= a(z)\varphi^\varepsilon(z) + \kappa z^{-1}\varphi^\varepsilon(z) + 2\lambda\partial_z\varphi^\varepsilon(z), \end{aligned}$$

$$\sum_{n\in\mathbb{Z}} L_n^{\lambda,\kappa:\varepsilon} z^{-n-2} := T_{\lambda,\kappa}^\varepsilon(z), \qquad \sum_{m\in\varepsilon+\mathbb{Z}} G_m^{\lambda,\kappa:\varepsilon} z^{-m-\frac{3}{2}} := G_{\lambda,\kappa}^\varepsilon(z),$$

where we set

$${}^\circ_\circ a_k a_l {}^\circ_\circ := \begin{cases} a_k a_l & (k \leqslant l), \\ a_l a_k & (k > l), \end{cases} \qquad {}^\circ_\circ \varphi_k \varphi_l {}^\circ_\circ := \begin{cases} \varphi_k\varphi_l & (k \leqslant l), \\ -\varphi_l\varphi_k & (k > l). \end{cases}$$

Then we have

1. *Vir_ε acts on the space $\mathcal{F}^{\eta;\varepsilon}$ via*

$$L_n \mapsto L_n^{\lambda,\kappa:\varepsilon},\ G_m \mapsto G_m^{\lambda,\kappa:\varepsilon},\ c \mapsto z_\lambda \mathrm{id}_{\mathcal{F}^{\eta:\varepsilon}}$$

where $z_\lambda := \frac{3}{2}(1 - 8\lambda^2)$.

2.

$$\begin{aligned}
&L_0^{\lambda,\kappa:\varepsilon}.(1\otimes \mathbf{1}_\eta)\otimes(1\otimes \mathbf{1}^\varepsilon)=h_\lambda^{\eta+\kappa;\varepsilon}(1\otimes \mathbf{1}_\eta)\otimes(1\otimes \mathbf{1}^\varepsilon),\\
&G_0^{\lambda,\kappa:0}.(1\otimes \mathbf{1}_\eta)\otimes(1\otimes \varphi_0^i.\mathbf{1}^0)\\
&=\left(\frac{1}{2}\right)^i(\eta+\kappa-\lambda)(1\otimes \mathbf{1}_\eta)\otimes(1\otimes \varphi_0^{1-i}.\mathbf{1}^0)\quad (i=0,1),
\end{aligned}$$

where we set

$$h_\lambda^{\eta;\varepsilon}=\frac{1}{2}\eta(\eta-2\lambda)+\frac{1}{16}(1-2\varepsilon).$$

When we regard the space $\mathcal{F}^{\eta:\varepsilon}$ as a Vir_ε module via the above action, we denote it by $\mathcal{F}^{\eta:\varepsilon}_{\lambda,\kappa}$.

2.2. Some results on Verma modules.

In this subsection, we will summarize some results on Verma modules $M_\varepsilon(z,h)$ used in later sections that are not stated in [IK2] but are immediate consequences of the results. A result on singular vectors in pre-Verma module $N(z,h)$ will be also given.

The first result that we are going to state is the explicit form of submodules of Verma modules which belong to Class R^+. Let $p,q\in\mathbb{Z}_{>0}$ be integers satisfying $p-q\in 2\mathbb{Z}$ and $(\frac{p-q}{2},q)=1$, and fix the following central charge:

$$z:=\frac{15}{2}-3\left(\frac{p}{q}+\frac{q}{p}\right).$$

THEOREM 2.1.— *Let us fix $(r,s)\in K^+_{p,q}$.*

1. *Suppose that (r,s) does not belong to Case 5^+. Then, any proper submodule of $M_\varepsilon(z,h_{i;\varepsilon})$ is one of the following forms:*

(i) *If (r,s) belongs to Case 1^+ $(i\in\mathbb{Z})$, then we have*

$$M_\varepsilon(z,h_{j;\varepsilon}),\quad M_\varepsilon(z,h_{-j;\varepsilon}),\quad M_\varepsilon(z,h_{j;\varepsilon})+M_\varepsilon(z,h_{-j;\varepsilon})$$
$$(j\in\mathbb{Z}_{>0},\ j>|i|).$$

(ii) *If (r,s) belongs to Case 2^+ $(i\in\mathbb{Z}_{\geqslant 0})$, then we have*

$$M_\varepsilon(z,h_{j;\varepsilon})\qquad (j\in\mathbb{Z}_{>0},\ j>i).$$

(iii) *If (r,s) belongs to Case 3^+ $(i \in \mathbb{Z}_{\geqslant 0})$, then we have*

$$M_\varepsilon(z, h_{(-1)^{j-1}j;\varepsilon}) \qquad (j \in \mathbb{Z}_{>0},\ j > |i|).$$

(iv) *If (r,s) belongs to Case 4.1^+ $(i \in 2\mathbb{Z}_{\geqslant 0})$, then we have*

$$M_{\frac{1}{2}}(z, h_{j;\frac{1}{2}}) \qquad (j \in 2\mathbb{Z}_{>0},\ j > i).$$

(v) *If (r,s) belongs to Case 4.2^+ $(i \in 2\mathbb{Z}_{\leqslant 0})$, then we have*

$$M_\varepsilon(z, h_{j;\varepsilon}) \qquad (j \in 2\mathbb{Z}_{<0},\ j < i).$$

2. *Suppose that (r,s) belongs to Case 5^+ $(i \in \mathbb{Z}_{\geqslant 0})$.*

(i) *If $i = 0$, then any submodule of $M_0(z, h_{0;0})$ is of the form:*

$$\begin{aligned} \mathrm{Im}\varphi_j \quad \textit{where} \quad \varphi_j &: M_0(z, h_{j;0}) \longrightarrow M_0(z, h_{0;0}) \\ &: \ \textit{non-injective map } (j \in \mathbb{Z}_{>0}). \end{aligned}$$

(ii) *Any submodule of $\widetilde{M}(z, h_{0;0})$ is of the form:*

$$\begin{aligned} &\mathrm{Im}\varphi_j, \quad \Pi\,\mathrm{Im}\varphi_k, \quad \mathrm{Im}\varphi_j \oplus \Pi\,\mathrm{Im}\varphi_k, \quad M_0(z, h_{j;0}), \\ &(j,k \in \mathbb{Z}_{>0},\ j,k > i). \end{aligned}$$

(iii) *If $i > 0$, then any submodule of $M_0(z, h_{i;0})$ is of the form:*

$$\begin{aligned} &W^{(j)}_{2c_P+c_Q}, \quad W^{(k)}_{c_Q}, \quad W^{(j)}_{2c_P+c_Q} \oplus W^{(k)}_{c_Q}, \quad M_0(z, h_{j;0}), \\ &(j,k \in \mathbb{Z}_{>0},\ j,k > i), \end{aligned}$$

where $W^{(j)}_{2c_P+c_Q}, W^{(j)}_{c_Q}$ are submodules generated by an even singular vector $(PG_0 + Q).(1 \otimes 1_{z,h_{i;0}})$ of L_0-weight $h_{j;0}$ whose coefficients c_P, c_Q of P, Q expanded with respect to the basis $\mathcal{B}^{\bar{0}}_{h_{j,0}-h_{i,0}}$ defined in §3 of [IK2]

$$\begin{aligned} P &= c_P G_{-1} L_{-1}^{h_{j;0}-h_{i;0}-1} + \cdots, \\ Q &= c_Q L_{-1}^{h_{j;0}-h_{i,0}} + \cdots, \end{aligned}$$

satisfy $2c_P + c_Q = 0$ (resp. $c_Q = 0$).

The second result that we will use later is the multiplicities

$$[M_\varepsilon(z, h_{i;\varepsilon}) : L_\varepsilon(z, h_{j;\varepsilon})]$$

in the case when they belong to Class R^+. Let $K(\mathcal{O})$ be the Grothendieck group of the category $\mathcal{O}$, and we denote the element of $K(\mathcal{O})$ corresponding to an object V of $\mathcal{O}$ by $[V]$. By Theorem 5.2 and (17) of [IK2], we obtain the following formulae:

LEMMA 2.2. — *Let us fix $(r,s) \in K^+_{p,q}$.*

1. *If (r,s) belongs to Case 1^+, then we have*

$$[M_\varepsilon(z,h_{i;\varepsilon})] = [L_\varepsilon(z,h_{i;\varepsilon})] + \sum_{k\in\mathbb{Z},\ |k|>|i|} [L(z,h_{k;\varepsilon})] \qquad (i\in\mathbb{Z}).$$

2. *If (r,s) belongs to Case 2^+, then we have*

$$[M_\varepsilon(z,h_{i;\varepsilon})] = \sum_{k\geqslant i}[L_\varepsilon(z,h_{k;\varepsilon})] \qquad (i\in\mathbb{Z}_{\geqslant 0}).$$

3. *If (r,s) belongs to Case 3^+, then we have*

$$[M_\varepsilon(z,h_{(-1)^{i-1}i;\varepsilon})] = \sum_{k\geqslant i}[L_\varepsilon(z,h_{(-1)^{k-1}k;\varepsilon})] \qquad (i\in\mathbb{Z}_{\geqslant 0}).$$

4. *If (r,s) belongs to Case 4.1^+, then we have*

$$[M_\varepsilon(z,h_{2i;\frac{1}{2}})] = \sum_{k\geqslant i}[L_\varepsilon(z,h_{2k;\frac{1}{2}})] \qquad (i\in\mathbb{Z}_{\geqslant 0}).$$

5. *If (r,s) belongs to Case 4.2^+, then we have*

$$[M_\varepsilon(z,h_{2i;\varepsilon})] = \sum_{k\leqslant i}[L_\varepsilon(z,h_{2k;\varepsilon})] \qquad (i\in\mathbb{Z}_{\leqslant 0}).$$

6. *If (r,s) belongs to Case 5^+, then we have*

$$[M_0(z,h_{0;0})] = [L_0(z,h_{0;0})] + \sum_{k>0}[L_0(z,h_{k;0})],$$

$$[M_0(z,h_{i;0})] = [L_0(z,h_{i;0})] + 2\sum_{k>i}[L_0(z,h_{k;0})] \qquad (i\in\mathbb{Z}_{>0}).$$

Let $\mathcal{C}$ be the category $\mathcal{C}^{\mathbb{Z}_2}_{(\mathfrak{g},\mathfrak{h})}$ introduced in §2 of [IK2]. The third result that we will use later is the extension:

LEMMA 2.3. — *Suppose that $(r,s) \in K^+_{p,q}$ belongs either to Case 1^+ or to Case 5^+. For $m,n \in \mathbb{Z}$ and $\sigma,\tau \in \mathbb{Z}_2$, we have*

$$\mathrm{Ext}^1_{\mathcal{C}}(L_\varepsilon(z,h_{m;\varepsilon};\sigma),L_\varepsilon(z,h_{n;\varepsilon};\tau))$$

$$\cong \begin{cases} \mathbb{C}^2 & ||m|-|n||=1 \ \wedge \ \varepsilon=0 \ \wedge \ (r,s)=(\frac{q}{2},\frac{p}{2}) \ \wedge \ mn\neq 0, \\ \mathbb{C} & ||m|-|n||=1 \ \wedge \begin{cases} \varepsilon=0 \ \wedge & \begin{cases} (r,s)\neq(\frac{q}{2},\frac{p}{2}), \\ (r,s)=(\frac{q}{2},\frac{p}{2}) \quad \wedge \ mn=0, \end{cases} \\ \varepsilon=\frac{1}{2} \ \wedge & 2(h_{m;\varepsilon}-h_{n;\varepsilon}).\bar{1}=\sigma-\tau, \end{cases} \\ \{0\} & \text{otherwise.} \end{cases}$$

Proof. — First, we note that for $z,h,h' \in \mathbb{C}$ and $\sigma,\tau \in \mathbb{Z}_2$, one can compute $\mathrm{Ext}^1_{\mathcal{C}}(M_\varepsilon(z,h;\sigma),L_\varepsilon(z,h';\tau))$ as a direct consequence of Proposition 2.1 and Theorem 5.1 in [IK2] if either $\varepsilon\neq 0$ or $h\neq\frac{1}{24}z$ is satisfied. If $\varepsilon=0$ and $h=\frac{1}{24}z$, then by the short exact sequence

$$0 \longrightarrow M_0\left(z,\frac{1}{24}z;\bar{1}-\sigma\right) \longrightarrow N\left(z,\frac{1}{24}z;\sigma\right) \longrightarrow M_0\left(z,\frac{1}{24}z;\sigma\right) \longrightarrow 0,$$

we have the next exact sequence

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{C}}\left(N\left(z,\frac{1}{24}z;\sigma\right),L_0(z,h';\tau)\right) \\ \longrightarrow\ &\mathrm{Hom}_{\mathcal{C}}\left(M_0\left(z,\frac{1}{24}z;\bar{1}-\sigma\right),L_0(z,h';\tau)\right) \\ \longrightarrow\ &\mathrm{Ext}^1_{\mathcal{C}}\left(M_0\left(z,\frac{1}{24}z;\sigma\right),L_0(z,h';\tau)\right) \\ \longrightarrow\ &\mathrm{Ext}^1_{\mathcal{C}}\left(N\left(z,\frac{1}{24}z;\sigma\right),L_0(z,h';\tau)\right). \end{aligned}$$

By Proposition 2.1 and Theorem 5.1 in [IK2], we get

$$\mathrm{Ext}^1_{\mathcal{C}}\left(N\left(z,\frac{1}{24}z;\sigma\right),L_0(z,h';\tau)\right)\cong\{0\},$$

from which it follows that

$$\mathrm{Ext}^1_{\mathcal{C}}\left(M_0\left(z,\frac{1}{24}z;\sigma\right),L_0(z,h';\tau)\right)\cong\begin{cases}\mathbb{C} & h'=\frac{1}{24}z \ \wedge \ \tau\neq\sigma, \\ \{0\} & \text{otherwise.}\end{cases}$$

Second, by the short exact sequence

$$0 \longrightarrow M_\varepsilon(z,h_{m;\varepsilon};\sigma)(1) \longrightarrow M_\varepsilon(z,h_{m;\varepsilon};\sigma) \longrightarrow L_\varepsilon(z,h_{m;\varepsilon};\sigma) \longrightarrow 0,$$

we get the following long exact sequence:

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{C}}(M_\varepsilon(z,h_{m;\varepsilon};\sigma), L_\varepsilon(z,h_{n;\varepsilon};\tau)) \\ \longrightarrow\ &\mathrm{Hom}_{\mathcal{C}}(M_\varepsilon(z,h_{m;\varepsilon};\sigma)(1), L_\varepsilon(z,h_{n;\varepsilon};\tau)) \\ \longrightarrow\ &\mathrm{Ext}^1_{\mathcal{C}}(L_\varepsilon(z,h_{m;\varepsilon};\sigma), L_\varepsilon(z,h_{n;\varepsilon};\tau)) \\ \longrightarrow\ &\mathrm{Ext}^1_{\mathcal{C}}(M_\varepsilon(z,h_{m;\varepsilon};\sigma), L_\varepsilon(z,h_{n;\varepsilon};\tau)) \\ \longrightarrow\ &\mathrm{Ext}^1_{\mathcal{C}}(M_\varepsilon(z,h_{m;\varepsilon};\sigma)(1), L_\varepsilon(z,h_{n;\varepsilon};\tau)). \end{aligned}$$

Now, suppose $||m|-|n|| \neq 1$. Then, by Theorem 4.1 in [IK2], we have

$$\mathrm{Hom}_{\mathcal{C}}(M_\varepsilon(z,h_{m;\varepsilon};\sigma)(1), L_\varepsilon(z,h_{n;\varepsilon};\tau)) \cong \{0\}.$$

Moreover, if at least one of the conditions $\varepsilon = \frac{1}{2}$, $(r,s) \neq (\frac{q}{2},\frac{p}{2})$, $(m,n) \neq (0,0)$ and $\tau = \sigma$ is satisfied, we have

$$\mathrm{Ext}^1_{\mathcal{C}}(M_\varepsilon(z,h_{m;\varepsilon};\sigma), L_\varepsilon(z,h_{n;\varepsilon};\tau)) \cong \{0\}.$$

Otherwise, we have an inclusion

$$\mathrm{Ext}^1_{\mathcal{C}}(L_0(z,h_{0;0};\sigma), L_0(z,h_{0;0};\tau)) \hookrightarrow \mathrm{Ext}^1_{\mathcal{C}}(M_0(z,h_{0;0};\sigma), L_0(z,h_{0;0};\tau)) \cong \mathbb{C}.$$

Assume that $\mathrm{Ext}^1_{\mathcal{C}}(L_0(z,h_{0;0};\sigma), L_0(z,h_{0;0};\tau)) \neq \{0\}$, and let E_L be the corresponding non-trivial extension. Then, there exist a non-trivial extension E_M which corresponds to a non-zero element of $\mathrm{Ext}^1_{\mathcal{C}}(M_0(z,h_{0;0};\sigma), L_0(z,h_{0;0};\tau))$ and a surjective morphism $\pi : E_M \longrightarrow E_L$. But, this is impossible since E_M is the co-kernel of the map

$$M_0(z,h_{0;0};\tau)(1) \hookrightarrow N(z,h_{0;0};\sigma),$$

there is no submodule of E_M whose character is the same as the character of $\mathrm{Ker}\pi$ by Proposition 6.1 in [IK2]. Thus, we get the result in this case. Next, if we have $|m|-|n| = -1$, we get

$$\mathrm{Hom}_{\mathcal{C}}(M_\varepsilon(z,h_{m;\varepsilon};\sigma), L_\varepsilon(z,h_{n;\varepsilon};\tau)) \cong \{0\}$$

and

$$\mathrm{Ext}^1_{\mathcal{C}}(M_\varepsilon(z,h_{m;\varepsilon};\sigma), L_\varepsilon(z,h_{n;\varepsilon};\tau)) \cong \{0\}$$

from which it follows that

$$\mathrm{Ext}^1_{\mathcal{C}}(L_\varepsilon(z,h_{m;\varepsilon};\sigma),L_\varepsilon(z,h_{n;\varepsilon};\tau))\cong \mathrm{Hom}_{\mathcal{C}}(M_\varepsilon(z,h_{m;\varepsilon};\sigma)(1),L_\varepsilon(z,h_{n;\varepsilon};\tau)).$$

Hence, the result follows from Theorem 4.1 in [IK2].

Next, if we have $|m|-|n|=1$, it follows from Theorem 4.1 in [IK2] that

$$\mathrm{Hom}_{\mathcal{C}}(M_\varepsilon(z,h_{m;\varepsilon};\sigma)(1),L_\varepsilon(z,h_{n;\varepsilon};\tau))\cong\{0\}.$$

Let $\sigma_\pm\in\mathbb{Z}_2$ be the elements satisfying $2(h_{\pm(|m|+1);\varepsilon}-h_{m;\varepsilon}).\bar{1}=\sigma_\pm-\sigma$. Again by Theorem 4.1 in [IK2], we have the following short exact sequence:

$$\begin{aligned}0 &\longrightarrow M_\varepsilon(z,h_{m;\varepsilon};\sigma)(2)\longrightarrow M_\varepsilon(z,h_{|m|+1;\varepsilon};\sigma_+)\oplus M_\varepsilon(z,h_{-(|m|+1);\varepsilon};\sigma_-)\\ &\longrightarrow M_\varepsilon(z,h_{m;\varepsilon};\sigma)(1)\longrightarrow 0,\end{aligned}$$

from which we get the following exact sequence:

$$\begin{aligned}&\mathrm{Hom}_{\mathcal{C}}(M_\varepsilon(z,h_{m;\varepsilon};\sigma)(2),L_\varepsilon(z,h_{n;\varepsilon};\tau))\\ \longrightarrow\ &\mathrm{Ext}^1_{\mathcal{C}}(M_\varepsilon(z,h_{m;\varepsilon};\sigma)(1),L_\varepsilon(z,h_{n;\varepsilon};\tau))\\ \longrightarrow\ &\mathrm{Ext}^1_{\mathcal{C}}(M_\varepsilon(z,h_{|m|+1;\varepsilon};\sigma_+)\oplus M_\varepsilon(z,h_{-(|m|+1);\varepsilon};\sigma_-),L_\varepsilon(z,h_{n;\varepsilon};\tau)).\end{aligned}$$

Thus, by assumption, we get

$$\mathrm{Ext}^1_{\mathcal{C}}(M_\varepsilon(z,h_{m;\varepsilon};\sigma)(1),L_\varepsilon(z,h_{n;\varepsilon};\tau))\cong\{0\}$$

which implies

$$\mathrm{Ext}^1_{\mathcal{C}}(L_\varepsilon(z,h_{m;\varepsilon};\sigma),L_\varepsilon(z,h_{n;\varepsilon};\tau))\cong \mathrm{Ext}^1_{\mathcal{C}}(M_\varepsilon(z,h_{m;\varepsilon};\sigma),L_\varepsilon(z,h_{n;\varepsilon};\tau)).\ \square$$

Finally, we state a result on singular vectors of $N(z,h)$. It can be easily seen by Proposition 3.2 in [IK2] and its proof that an even (resp. an odd) singular vector of $N(z,h)$ can be parametrized by a certain two dimensional vector space. Here, we specify a one dimensional subspace which in fact parametrizes an even (resp. odd) singular vector at non-supersymmetric points, i.e., except for the case when it belongs to Case $5^\pm$.

Recall that for $\varepsilon\in\{\frac{1}{2},0\}$, $\alpha,\beta\in\mathbb{Z}_{>0}$ satisfying $\alpha-\beta\in 1-2\varepsilon+2\mathbb{Z}$ and $t\in\mathbb{C}^*$, we defined complex numbers $z(t)$ and $h_{\alpha,\beta;\varepsilon}(t)$ by

$$\begin{aligned}z(t)&:=\frac{15}{2}-3(t+t^{-1}),\\ h_{\alpha,\beta;\varepsilon}(t)&:=\frac{1}{8}(\alpha^2-1)t-\frac{1}{4}(\alpha\beta-1)+\frac{1}{8}(\beta^2-1)t^{-1}+\frac{1}{16}(1-2\varepsilon).\end{aligned}\tag{1}$$

Remark that we have the following symmetry:

$$h_{\beta,\alpha;0}(t^{-1}) = h_{\alpha,\beta;0}(t). \tag{2}$$

As one can see from the determinant formulae (see, e.g., Lemma 6.1 in [IK2]), the pre-Verma module $N(z(t), h_{\alpha,\beta;0}(t))$ contains even and odd singular vectors of level $\frac{1}{2}\alpha\beta$. Let us describe the even singular vector in this case. Let $|z,h\rangle \in N(z,h)$ be an even highest weight vector:

$$L_0.|z,h\rangle = h|z,h\rangle, \quad c.|z,h\rangle = z|z,h\rangle.$$

Let $w_{\alpha,\beta} := (X_{\alpha,\beta}G_0 + Y_{\alpha,\beta}).|z(t), h_{\alpha,\beta;0}(t)\rangle \in N(z(t), h_{\alpha,\beta;0}(t))$ be an even singular vector of level $\frac{1}{2}\alpha\beta$ which is regular in $t^{\pm 1}$. Expanding $w_{\alpha,\beta}$ with respect to the basis $\mathcal{B}^{\bar{0}}_{\frac{1}{2}\alpha\beta}$ defined in §6 of [IK2],

$$X_{\alpha,\beta} = c^X_{\alpha,\beta} G_{-1} L_{-1}^{\frac{1}{2}\alpha\beta - 1} + \cdots, \quad Y_{\alpha,\beta} = c^Y_{\alpha,\beta} L_{-1}^{\frac{1}{2}\alpha\beta} + \cdots,$$

we see that the coefficients $c^X_{\alpha,\beta}, c^Y_{\alpha,\beta}$ satisfy the following quadratic relation:

$$((\alpha t - \beta)c^X_{\alpha,\beta} + \alpha t c^Y_{\alpha,\beta})((\alpha - \beta t^{-1})c^X_{\alpha,\beta} - \beta t^{-1} c^Y_{\alpha,\beta}) = 0 \tag{3}$$

by Lemma 3.3 in [IK2]. Indeed, we can say more about the coefficients $c^X_{\alpha,\beta}, c^Y_{\alpha,\beta}$ as follows:

PROPOSITION 2.1. — *Suppose that $(z(t), h_{\alpha,\beta;0}(t))$ does not belong to Case $5^{\pm}$. Then, the coefficients $c^X_{\alpha,\beta}, c^Y_{\alpha,\beta}$ satisfy*

$$\begin{cases} (\alpha t - \beta)c^X_{\alpha,\beta} + \alpha t c^Y_{\alpha,\beta} = 0 & \alpha \equiv 0 \ (2), \\ (\alpha - \beta t^{-1})c^X_{\alpha,\beta} - \beta t^{-1} c^Y_{\alpha,\beta} = 0 & \alpha \equiv 1 \ (2). \end{cases}$$

Proof. — By the symmetry (2), we may assume $\alpha > \beta$ without loss of generality.

We prove this proposition by induction on the level $n := \frac{1}{2}\alpha\beta$.

The first step $(\alpha,\beta) = (2,1)$ can be checked by direct computation. Assume that we could prove the statement up to level $n-1$. To prove the statement for level n, we use the embedding diagram (Figure 2 in [IK2]) for Case $2^+, 3^+, 4.2^+$. Since we already know that the coefficients $c^X_{\alpha,\beta}, c^Y_{\alpha,\beta}$ satisfy the relation (3), we have only to check the statement at a special value.

For each $(\alpha,\beta)\in(\mathbb{Z}_{>0})^2$ satisfying $\alpha\beta=2n$ and $\alpha-\beta\in -1+2\mathbb{Z}_{>0}$, there exist $t\in\mathbb{C}^*$, $(\alpha',\beta'),(\alpha'',\beta'')\in(\mathbb{Z}_{>0})^2$ satisfying

$$\begin{aligned}
&\alpha'-\beta',\alpha''-\beta''\in -1+2\mathbb{Z}_{>0},\\
&h_{\alpha',\beta';0}(t)=h_{\alpha,\beta}(t),\quad h_{\alpha'',\beta'';0}(t)=h_{\alpha',\beta';0}(t)+\frac{1}{2}\alpha'\beta',\\
&\alpha'\beta'+\alpha''\beta''=\alpha\beta.
\end{aligned}$$

(The choice of such $t,(\alpha',\beta'),(\alpha'',\beta'')$ will be given in Appendix A.2.) These conditions imply the existence of the following commutative diagram:

$$\begin{tikzcd}
 & N(z(t),h_{\alpha',\beta';0}(t)+\frac{1}{2}\alpha'\beta') \arrow[d, equal] & \\
N(z(t),h_{\alpha'',\beta'';0}(t)+\frac{1}{2}\alpha''\beta'') \arrow[r, tail] \arrow[d, equal] & N(z(t),h_{\alpha'',\beta'';0}(t)) \arrow[r, tail] & N(z(t),h_{\alpha',\beta';0}(t)) \arrow[d, equal] \\
N(z,h_{\alpha,\beta;0}(t)+\frac{1}{2}\alpha\beta) \arrow[rr, tail] & & N(z,h_{\alpha,\beta;0}(t))
\end{tikzcd}$$

Figure 1. Splitting of singular vectors

This commutative diagram ensures the following relations:

$$\begin{aligned}
c^X_{\alpha,\beta}&=2c^X_{\alpha'',\beta''}c^X_{\alpha',\beta'}+c^X_{\alpha'',\beta''}c^Y_{\alpha',\beta'}+c^X_{\alpha',\beta'}c^Y_{\alpha'',\beta''},\\
c^Y_{\alpha,\beta}&=c^Y_{\alpha',\beta'}c^Y_{\alpha'',\beta''}.
\end{aligned}$$

Thus, the statement follows from these formulae and the induction hypothesis. □

2.3. Jantzen filtration à la Feigin & Fuchs.

In this subsection, we will formulate a generalization of the Jantzen filtration [Ja] à la Feigin and Fuchs [FeFu]. Here, we assume that our ground field K is of any characteristic.

Let S be an algebraic variety, and $\mathbb{V},\mathbb{W}$ be vector bundles over S of the same rank, say r. Suppose that a morphism of the vector bundles $f:\mathbb{V}\longrightarrow\mathbb{W}$ is given.

We denote the sheaf of sections of $\mathbb{V}$, $\mathbb{W}$ by $\mathcal{V}$ (resp. $\mathcal{W}$).

Now, let us fix a regular point $P \in S$ and a curve $C \subset S$ containing the point P as a regular point. We denote the restrictions of $\mathbb{V}, \mathbb{W}, \mathcal{V}, \mathcal{W}$ and f to the fixed curve C by $\mathbb{V}_C, \mathbb{W}_C, \mathcal{V}_C, \mathcal{W}_C$ and f_C respectively. Moreover, the restriction of them to the point P are also denoted by $\mathbb{V}_P, \mathbb{W}_P, \mathcal{V}_{C,P}, \mathcal{W}_{C,P}$ and $f_{C,P}$ respectively. Here, we assume the following:

Ass The rank of $\mathrm{Im} f_{C,P}$ is of full, i.e., r.

We regard the map $f_{C,P}$ as a morphism $\mathcal{V}_{C,P} \longrightarrow \mathcal{W}_{C,P}$. Since the map $f_{C,P}$ and f_P are the same morphism, regarded as a morphism between the fibres $\mathbb{V}_P \longrightarrow \mathbb{W}_P$, we denote it by f_P in this case. Let $\mathcal{O}_C$ be the structure sheaf of C. Then, by assumption, $\mathcal{O}_{C,P}$ is a discrete valuation ring with its unique maximal ideal $\mathfrak{m}_P = (t)$ with a uniformizing element $t \in \mathcal{O}_{C,P}$.

Under the above setting, we can formulate the Jantzen filtration à la Feigin & Fuchs as follows:

DEFINITION-PROPOSITION 2.1. — *For $n \in \mathbb{Z}_{\geqslant 0}$, we define an $\mathcal{O}_{C,P}$-submodule $\mathcal{V}_{C,P}(n)$ of $\mathcal{V}_{C,P}$ and a K-vector subspace $\mathbb{V}_{C,P}(n)$ of $\mathbb{V}_P$ as follows:*

$$\mathcal{V}_{C,P}(n) := \{u \in \mathcal{V}_{C,P} | f_{C,P}(u) \in \mathfrak{m}_P^n \mathcal{W}_{C,P}\},$$
$$\mathbb{V}_{C,P}(n) := \{u(P) \in \mathbb{V}_P | u \in \mathcal{V}_{C,P}(n)\}.$$

Similarly, we define an $\mathcal{O}_{C,P}$-submodule $\mathcal{IK}_{C,P}(n)$ of $\mathcal{W}_{C,P}$ and a K-vector subspace $\mathbb{IK}_{C,P}(n)$ of $\mathbb{W}_P$ by

$$\mathcal{IK}_{C,P}(n) := t^{-n} \left(\mathfrak{m}_P^n \mathcal{W}_{C,P} \cap \mathrm{Im} f_{C,P}\right),$$
$$\mathbb{IK}_{C,P}(n) := \{u(P) \in \mathbb{W}_P | u \in \mathcal{IK}_{C,P}(n)\}.$$

We set $\mathbb{IK}_{C,P}(0) := \mathrm{Im} f_{C,P}$. Then, the quotient space $\mathbb{W}_{C,P}(n)$ of $\mathbb{W}_P$ is defined by

$$\mathbb{W}_{C,P}(n) := \mathbb{W}_P / \mathbb{IK}_{C,P}(n-1).$$

Let π_n be the canonical projection $\mathbb{W}_P \twoheadrightarrow \mathbb{W}_{C,P}(n)$. We define the 'n-th. derivative of f'

$$f_{C,P}^{(n)} : \mathbb{V}_{C,P}(n) \longrightarrow \mathbb{W}_{C,P}(n)$$

by

$$f_{C,P}^{(n)}(u) := \pi_n((t^{-n} f_{C,P}(\tilde{u}))(P)),$$

where $\tilde{u}$ is a 'lift' of $u \in \mathbb{V}_{C,P}(n)$, i.e., $\tilde{u}$ is an element of $\mathcal{V}_{C,P}(n)$ satisfying $\tilde{u}(P) = u$.

Then, we have the following:

1. *The map $f^{(n)}$ is well-defined, i.e.,*

$$((t^{-n}f_{C,P})(\tilde{u}))(P) \in \mathbb{IK}_{C,P}(n-1) \quad ({}^{\forall}\tilde{u} \in \mathcal{V}_{C,P}(n) \cap \mathfrak{m}_P\mathcal{V}_{C,P}).$$

2. *We have the following filtration:*

$$\mathbb{V}_P =: \mathbb{V}_{C,P}(0) \supset \mathbb{V}_{C,P}(1) \supset \mathbb{V}_{C,P}(2) \supset \cdots, \qquad \bigcap_{n=1}^{\infty} \mathbb{V}_{C,P}(n) = \{0\}.$$

3. *We have the following co-filtration:*

$$\mathbb{W}_P =: \mathbb{W}_{C,P}(0) \twoheadrightarrow \mathbb{W}_{C,P}(1) \twoheadrightarrow \mathbb{W}_{C,P}(2) \twoheadrightarrow \cdots.$$

We call the above filtration $\{\mathbb{V}_{C,P}(n)\}$ of $\mathbb{V}_P$ the Jantzen filtration of $(\mathbb{V}_P, \mathbb{W}_P; f; C)$, and the above co-filtration $\{\mathbb{W}_{C,P}(n)\}$ of $\mathbb{W}_P$ the Jantzen co-filtration of $(\mathbb{V}_P, \mathbb{W}_P; f; C)$.

We remark that a choice of uniformizing element is rather inessential to define the higher derivatives $f_{C,P}^{(n)}$, i.e., they are defined up to a scalar multiplication.

This (co-)filtration enjoys more properties:

Proposition 2.2. — *For any $n \in \mathbb{Z}_{>0}$, we have*

1. $\mathbb{V}_{C,P}(n) = \mathrm{Ker} f_{C,P}^{(n-1)}$,
2. $\mathbb{W}_{C,P}(n) = \mathrm{Coker}\, f_{C,P}^{(n-1)}$.

This proposition suggests the following duality between the Jantzen filtration and the co-filtration. Namely, let $\mathbb{V}^\vee, \mathbb{W}^\vee$ be the dual vector bundles to $\mathbb{V}$ (resp. $\mathbb{W}$), and let

$${}^t f : \mathbb{W}^\vee \longrightarrow \mathbb{V}^\vee$$

be the transpose of f (to be precise, ${}^t f$ is the transpose of the morphism f at each fibre). Then it defines the Jantzen filtration $\{\mathbb{W}_{C,P}^\vee(n)\}$ and the co-filtration $\{\mathbb{V}_{C,P}^\vee(n)\}$ of $(\mathbb{W}_P^\vee, \mathbb{V}_P^\vee; {}^t f; C)$. By the duality, we mean the following proposition:

Proposition 2.3. — *For any $n \in \mathbb{Z}_{>0}$, we have*

$$\mathbb{W}_{C,P}^\vee(n) \cong \mathbb{W}_{C,P}(n)^*, \qquad \mathbb{V}_{C,P}^\vee(n) \cong \mathbb{V}_{C,P}(n)^*,$$

where $*$ signifies the dual as K-vector space.

The proof of the above three propositions is rather straightforward, and we will omit it here.

Now, for each $s \in C$, let $U \subset C$ be an open neighborhood of s over which the vector bundles $\mathbb{V}_C$ and $\mathbb{W}_C$ become trivial. Let $\{m_i\}_{1 \leqslant i \leqslant r} \subset \Gamma(U, \mathcal{V}_C)$, $\{n_i\}_{1 \leqslant i \leqslant r} \subset \Gamma(U, \mathcal{W}_C)$ be $\Gamma(U, \mathcal{O}_C)$-free basis of $\Gamma(U, \mathcal{V}_C)$ (resp. $\Gamma(U, \mathcal{W}_C)$). We define $\det f_{C,s} \in \mathcal{O}_{C,s}$ by

$$(\det f_{C,s}) n_1 \wedge n_2 \wedge \cdots \wedge n_r := f_{C,s}(m_1) \wedge f_{C,s}(m_2) \wedge \cdots \wedge f_{C,s}(m_r).$$

We remark that $\det f_{C,s}$ is well-defined up to a multiplication by the units $\mathcal{O}_{C,s}^{\times}$.

The next statement follows by an argument similar to the case of the original Jantzen filtration [Ja], and is called the character sum formula. Let ν_P be the valuation of $(\mathcal{O}_{C,P}, \mathfrak{m}_P)$ satisfying $\nu_P(\mathfrak{m}_P \setminus \mathfrak{m}_P^2) = \{1\} \subset \mathbb{Z}$.

LEMMA 2.4.

$$\nu_P(\det f_{C,P}) = \sum_{n=1}^{\infty} \dim_K \mathbb{V}_{C,P}(n).$$

3. Fock modules I: basic properties.

In this section, we will study some basic properties of Fock modules.

3.1. Screening operators.

Let us first recall isomorphisms among Fock modules.

LEMMA 3.1 (cf. [IK2]). — *For $\lambda, \eta, \kappa \in \mathbb{C}$ and $\varepsilon \in \{0, \frac{1}{2}\}$, we have*

1. $\mathcal{F}_{\lambda,\kappa}^{\eta;\varepsilon} \cong \mathcal{F}_{\lambda,0}^{\eta+\kappa;\varepsilon}$,
2. $\mathcal{F}_{\lambda,\kappa}^{\eta,\varepsilon} \cong \mathcal{F}_{-\lambda,-\kappa}^{-\eta;\varepsilon}$,
3. $(\mathcal{F}_{\lambda,\kappa}^{\eta;\varepsilon})^c \cong \mathcal{F}_{-\lambda,\kappa-2\lambda}^{\eta;\varepsilon}$.

In particular, it follows that

$$(\mathcal{F}_{\lambda}^{\eta;\varepsilon})^c \cong \mathcal{F}_{\lambda}^{2\lambda-\eta;\varepsilon}.$$

Proof. — The first isomorphism is induced by the automorphism of the Lie algebra $\mathcal{H}$ defined by

$$a_n \longmapsto a_n + \kappa\delta_{n,0}K_{\mathcal{H}}, \qquad K_{\mathcal{H}} \longmapsto K_{\mathcal{H}}.$$

The second isomorphism is induced by the automorphism of the Lie algebra $\mathcal{H}$ defined by

$$a_n \longmapsto -a_n, \qquad K_{\mathcal{H}} \longmapsto K_{\mathcal{H}},$$

and the automorphism of the Lie superalgebra $\mathcal{D}_\varepsilon$ defined by

$$\varphi_k \longmapsto -\varphi_k, \qquad K_{\mathcal{D}} \longmapsto K_{\mathcal{D}}.$$

The third isomorphism is induced by the isomorphism $(\mathcal{F}^{\eta;\varepsilon})^c \cong \mathcal{F}^{\eta;\varepsilon}$ as $(\mathcal{H} \oplus \mathcal{D}_\varepsilon)$-module, where the anti-involutions $\sigma_{\mathcal{H}}, \sigma_{\mathcal{D}_\varepsilon}$ of $U(\mathcal{H})$ and $U(\mathcal{D}_\varepsilon)$ respectively are defined by

$$\begin{aligned} &\sigma_{\mathcal{H}}(a_n) := a_{-n}, \qquad &&\sigma_{\mathcal{H}}(K_{\mathcal{H}}) := K_{\mathcal{H}}, \\ &\sigma_{\mathcal{D}_\varepsilon}(\varphi_k) := \varphi_{-k}, \qquad &&\sigma_{\mathcal{D}_\varepsilon}(K_{\mathcal{D}_\varepsilon}) := K_{\mathcal{D}_\varepsilon}. \end{aligned}$$

□

By this lemma, it is enough to study the properties of Fock module $\mathcal{F}^{\eta;\varepsilon}_\lambda$ instead of seemingly general $\mathcal{F}^{\eta;\varepsilon}_{\lambda,\kappa}$.

Our arguments in the rest of this subsection follow [TK], where they considered the Virasoro algebra. For $\mu \in \mathbb{C}$ and an indeterminate ζ, we define the operators $e^{\mu q}, \zeta^{\mu a_0} \in \mathrm{End}_{\mathbb{C}}\left(\bigoplus_{\eta\in\mathbb{C}} \mathcal{F}^{\eta;\varepsilon}\right)$ by

$$\begin{aligned} e^{\mu q}.(1\otimes \mathbf{1}_\eta)\otimes(1\otimes\mathbf{1}^\varepsilon) &:= (1\otimes\mathbf{1}_{\eta+\mu})\otimes(1\otimes\mathbf{1}^\varepsilon), \\ \zeta^{\mu a_0}.(1\otimes \mathbf{1}_\eta)\otimes(1\otimes\mathbf{1}^\varepsilon) &:= \zeta^{\mu\eta}(1\otimes\mathbf{1}_{\eta})\otimes(1\otimes\mathbf{1}^\varepsilon). \end{aligned}$$

The operators $X_\mu(\zeta)$ and $S^\varepsilon_\mu(\zeta)$ are defined by

$$\begin{aligned} X_\mu(\zeta) &:= \exp\left(\mu\sum_{k>0}\frac{a_{-k}}{k}\zeta^k\right)\exp\left(-\mu\sum_{k>0}\frac{a_k}{k}\zeta^{-k}\right)e^{\mu q}\zeta^{\mu a_0}, \\ S^\varepsilon_\mu(\zeta) &:= X_\mu(\zeta)\varphi^\varepsilon(\zeta). \end{aligned}$$

The commutation relations between $S^\varepsilon_\mu(\zeta)$ and Vir_ε can be computed by the direct computation.

LEMMA 3.2. — *For $n \in \mathbb{Z}$ and $m \in \varepsilon + \mathbb{Z}$, we have*

$$[L_n, S_\mu^\varepsilon(\zeta)] = \zeta^n \left\{ \zeta \frac{\partial}{\partial \zeta} + \frac{1}{2}(\mu^2 - 2\mu\lambda + 1)(n+1) \right\} S_\mu^\varepsilon(\zeta),$$
$$[G_m, S_\mu^\varepsilon(\zeta)]_+ = \zeta^{m-\frac{1}{2}} \left\{ \mu^{-1} \zeta \frac{\partial}{\partial \zeta} + (\mu - 2\lambda)(m + \frac{1}{2}) \right\} X_\mu(\zeta),$$

where $[\ ,\ ]_+$ signifies the anti-commutator.

For $a \in \mathbb{Z}_{>1}$, the composition of the operators $X_\mu(\zeta_i)$ $(i = 1, \cdots, a)$ is given by

$$\begin{aligned} X_\mu(\zeta_1) \cdots X_\mu(\zeta_a) = & \prod_{1 \leqslant i < j \leqslant a} (\zeta_i - \zeta_j)^{\mu^2} \\ & \times \exp\left(\mu \sum_{k>0} \frac{a_{-k}}{k} \sum_{i=1}^a \zeta_i^k \right) \exp\left(-\mu \sum_{k>0} \frac{a_k}{k} \sum_{i=1}^a \zeta_i^{-k} \right) \\ & \times e^{a\mu q} \left(\prod_{i=1}^a \zeta_i \right)^{\mu a_0} . \end{aligned}$$

We remark that the left hand side of this formula converges on $\{(\zeta_1, \cdots, \zeta_a) | \, |\zeta_1| > |\zeta_2| > \cdots > |\zeta_a| > 0\}$, and the right hand side provides us its analytical continuation. Motivated by this formula and Lemma 3.2, for $1 \leqslant j \leqslant a$ we set

$$K_\varepsilon(\mu; \zeta_1, \cdots, \zeta_a) := S_\mu^\varepsilon(\zeta_1) \cdots S_\mu^\varepsilon(\zeta_a) \left(\prod_{i=1}^a \zeta_i \right)^{-\mu a_0 - \frac{1}{2}(a-1)(\mu^2+1)},$$

$$K_\varepsilon^{(j)}(\mu; \zeta_1, \cdots, \zeta_a) := S_\mu^\varepsilon(\zeta_1) \cdots X_\mu(\zeta_j) \cdots S_\mu^\varepsilon(\zeta_a) \left(\prod_{i=1}^a \zeta_i \right)^{-\mu a_0 - \frac{1}{2}(a-1)(\mu^2+1)},$$

where in the second formula only the j-th factor is replaced by $X_\mu(\zeta_j)$. Then, Lemma 3.2 implies the following commutation relations:

LEMMA 3.3. — *For $n \in \mathbb{Z}$ and $m \in \varepsilon + \mathbb{Z}$, we have*

$$\begin{aligned} [L_n, K_\varepsilon(\mu; \zeta_1, \cdots, \zeta_a)] = & \sum_{i=1}^a \zeta_i^n \left\{ \zeta_i \frac{\partial}{\partial \zeta_i} + \frac{1}{2}(\mu^2 - 2\mu\lambda + 1)(n+1) \right. \\ & \left. + \mu a_0 - \frac{1}{2}(a+1)\mu^2 + \frac{1}{2}(a-1) \right\} K_\varepsilon(\mu; \zeta_1, \cdots, \zeta_a), \end{aligned}$$

$$[G_m, K_\varepsilon(\mu;\zeta_1,\cdots,\zeta_a)]_+ = \sum_{i=1}^{a}(-1)^{i-1}\zeta_i^{m-\frac{1}{2}}\left\{\mu^{-1}\zeta_i\frac{\partial}{\partial\zeta_i} + (\mu-2\lambda)\left(m+\frac{1}{2}\right)\right.$$
$$\left. +a_0 - \frac{1}{2}(a+1)\mu + \frac{1}{2}(a-1)\mu^{-1}\right\} K_\varepsilon^{(i)}(\mu;\zeta_1,\cdots,\zeta_a).$$

Next, we will look at the Fourier components of $K_\varepsilon(\mu;\zeta_1,\cdots,\zeta_a)$. For $a \in \mathbb{Z}_{>0}$, set

$$M_a := \{(\zeta_1,\cdots,\zeta_a) \in (\mathbb{C}^*)^a | \zeta_i \neq \zeta_j \ (i \neq j)\}.$$

Let $\mathcal{S}_\mu$ be the local system with coefficient in $\mathbb{C}$ associated to the monodromy group of the multi-valued function

$$\prod_{1\leqslant i<j\leqslant a}(\zeta_i-\zeta_j)^{\mu^2+1}\prod_{i=1}^{a}\zeta_i^{-\frac{1}{2}(a-1)(\mu^2+1)},$$

and $\mathcal{S}_\mu^\vee$ be its dual. For each cycle $\Gamma \in H_a(M_a, \mathcal{S}_\mu^\vee)$ and half integers $l_i \in (\frac{1}{2}-\varepsilon)+\mathbb{Z}$ $(1 \leqslant i \leqslant a)$, we set

$$\mathcal{O}_\varepsilon(\mu;\Gamma;l_1,\cdots,l_a) := \int_\Gamma K_\varepsilon(\mu;\zeta_1,\cdots,\zeta_a)\prod_{i=1}^{a}\zeta_i^{-l_i-1}d\zeta_i.$$

By Lemma 3.3 and integration by parts, we get

Proposition 3.1. — *If λ, η and l_i's $(1 \leqslant i \leqslant a)$ satisfy $\lambda = \frac{1}{2}(\mu - \mu^{-1})$, $\eta - \lambda = -\frac{1}{2}a\mu - b\mu^{-1}$ and $l_i = b - \frac{1}{2}a$ for some $\mu \in \mathbb{C}^*$, then we have*

$$\mathcal{O}_\varepsilon(\mu;\Gamma;l_1,\cdots,l_a) \in \mathrm{Hom}^{a\bar{1}}_{\mathrm{Vir}_\varepsilon}(\mathcal{F}_\lambda^{\eta;\varepsilon}, \mathcal{F}_\lambda^{\eta+a\mu;\varepsilon}).$$

Here and after, we fix the relation between λ and μ as in Proposition 3.1.

Definition 3.1. — *The operator*

$$S_\varepsilon(\mu;\Gamma;a,b) := \mathcal{O}_\varepsilon\left(\mu;\Gamma;\underbrace{b-\frac{1}{2}a,\cdots b-\frac{1}{2}a}_{a}\right)$$

is called the screening operator associated to the cycle $\Gamma \in H_a(M_a, \mathcal{S}_\mu^\vee)$.

We will consider when the operator $S_\varepsilon(\mu;\Gamma;a,b)$ is non-trivial. We may assume that $a>1$, since the case $a=1$ is trivial. By looking at the contribution from the fermions

$$\begin{cases} \varphi_\varepsilon \cdots \varphi_{(a-1)+\varepsilon} & b \geqslant 0, \\ \varphi_{-(a-1)-\varepsilon} \cdots \varphi_{-\varepsilon} & b<0, \end{cases}$$

it turns out that the non-triviality of the operator $S_\varepsilon(\mu;\Gamma;a,b)$ can be observed by the non-triviality of

$$\int_\Gamma \prod_{1\leqslant i<j\leqslant a} (\zeta_i-\zeta_j)^{\mu^2+1} \prod_{i=1}^{a} \zeta_i^{-\frac{1}{2}(a-1)(\mu^2+1)} \prod_{i=1}^{a} \frac{d\zeta_i}{\zeta_i}. \tag{4}$$

If $\frac{1}{2}(\mu^2+1)\in\mathbb{Z}$, then the local system $\mathcal{S}_\mu^\vee$ becomes trivial, and we may take the residue around $\zeta_1=\cdots=\zeta_a=0$ which is non-trivial. Thus, we may assume that $\frac{1}{2}(\mu^2+1)\notin\mathbb{Z}$. Set

$$Y_{a-1}:=\{(z_1,\cdots,z_{a-1})\in(\mathbb{C}^*)^{a-1} | z_i\neq z_j \ (i\neq j),\ z_i\neq 1\}.$$

Then, by the change of variables

$$\zeta_1:=\zeta, \qquad \zeta_i:=\zeta z_{i-1} \ \ (1<i\leqslant a),$$

the above integrand becomes

$$\prod_{1\leqslant i<j<a} (z_i-z_j)^{\mu^2+1} \prod_{1\leqslant i<a} (1-z_i)^{\mu^2+1} z_i^{-\frac{1}{2}(a-1)(\mu^2+1)} \prod_{1\leqslant i<a} \frac{dz_i}{z_i} \times \frac{d\zeta}{\zeta},$$

which is a multi-valued function on Y_{a-1} and define the local system $\widetilde{\mathcal{S}}_\mu$ with coefficients in $\mathbb{C}$ associated to the monodromy group of the above integrand. We denote the dual of $\widetilde{\mathcal{S}}_\mu$ by $\widetilde{\mathcal{S}}_\mu^\vee$. Then, Lemma 3.9 in [TK] implies

$$H_a(M_a,\mathcal{S}_\mu^\vee)\cong H_{a-1}(Y_{a-1},\widetilde{\mathcal{S}}_\mu^\vee)\otimes H_1(\mathbb{C}^*,\mathbb{C}).$$

This means that for a cycle $\Gamma\in H_a(M_a,\mathcal{S}_\mu^\vee)$, there exist $\Gamma_1\in H_{a-1}(Y_{a-1},\widetilde{\mathcal{S}}_\mu^\vee)$ and $\Gamma_2\in H_1(\mathbb{C}^*,\mathbb{C})$ such that the integration (4) is equal to

$$\int_{\Gamma_1} \prod_{1\leqslant i<j<a} (z_i-z_j)^{\mu^2+1} \prod_{1\leqslant i<a} (1-z_i)^{\mu^2+1} z_i^{-\frac{1}{2}(a-1)(\mu^2+1)} \prod_{1\leqslant i<a} \frac{dz_i}{z_i} \int_{\Gamma_2} \frac{d\zeta}{\zeta}.$$

Thus, we have reduced the problem to the non-triviality of the first factor. Set

$$\Omega_a := \left\{ \mu \in \mathbb{C} \,\middle|\, \frac{1}{2}d(d+1)\mu^2 \notin \mathbb{Z},\ \frac{1}{2}d(d-a)(\mu^2+1) \notin \mathbb{Z} \quad 0 < d < a \right\}.$$

Then, Proposition 4.2 in [TK] and [Sel] ensure the following:

PROPOSITION 3.2. — *There exist cycles $\Gamma_\mu \in H_{a-1}(Y_{a-1}, \widetilde{\mathcal{S}}_\mu^\vee)$ defined on Ω_a such that*

1. *Γ_μ is holomorphic on Ω_a.*
2. *The following formula holds:*

$$\int_{\Gamma_\mu} \prod_{1 \leqslant i < j < a} (z_i - z_j)^{\mu^2+1} \prod_{1 \leqslant i < a} (1-z_i)^{\mu^2+1} z_i^{-\frac{1}{2}(a-1)(\mu^2+1)} \prod_{1 \leqslant i < a} \frac{dz_i}{z_i}$$
$$= \frac{(-\pi)^{a-1}\Gamma\left(\frac{1}{2}a(\mu^2+1)+1\right)}{(a-1)!\Gamma\left(\frac{1}{2}(\mu^2+1)+1\right)^a \prod_{1 \leqslant i < a} \sin\frac{1}{2}i(\mu^2+1)\pi},$$

where $\Gamma(s)$ in the formula is the Euler Gamma function. Take an integer $a \in \mathbb{Z}_{>0}$, a half integer $b \in \frac{1}{2}\mathbb{Z}$ satisfying $b - \frac{1}{2}a \in (\frac{1}{2} - \varepsilon) + \mathbb{Z}$ and a complex number $\mu \in \Omega_a$. Take a cycle $\Gamma_1 = \Gamma_\mu \in H_{a-1}(Y_{a-1}, \widetilde{\mathcal{S}}_\mu^\vee)$, and set $\Gamma = \Gamma_\mu \times \Gamma'$ where Γ' is a generator of $H_1(\mathbb{C}^*, \mathbb{C})$. Under this situation, we have

THEOREM 3.1. — *The screening operator*

$$S_\varepsilon(\mu; \Gamma; a, b) : \mathcal{F}_\lambda^{\lambda - \frac{1}{2}a\mu - b\mu^{-1};\varepsilon} \longrightarrow \mathcal{F}_\lambda^{\lambda + \frac{1}{2}a\mu - b\mu^{-1};\varepsilon}$$

is non-trivial, i.e.,

1. *for $b \geqslant 0$, the image of $(1 \otimes \mathbf{1}_{\lambda - \frac{1}{2}a\mu - b\mu^{-1}}) \otimes (\varphi_{-(a-1)-\varepsilon} \cdots \varphi_{-\varepsilon} \otimes \mathbf{1}^\varepsilon)$ is a non-zero vector in $\mathcal{F}_\lambda^{\lambda + \frac{1}{2}a\mu - b\mu^{-1};\varepsilon}$, and*
2. *for $b < 0$, there exists a vector in $\mathcal{F}_\lambda^{\lambda - \frac{1}{2}a\mu - b\mu^{-1};\varepsilon}$ whose image is $(1 \otimes \mathbf{1}_{\lambda + \frac{1}{2}a\mu - b\mu^{-1}}) \otimes (\varphi_{-(a-1)-\varepsilon} \cdots \varphi_{-\varepsilon} \otimes \mathbf{1}^\varepsilon)$.*

One can prove this theorem by a way similar to those given in [TK].

3.2. Determinant formulae.

For $(z,h)\in\mathbb{C}^2$, set

$$\widetilde{M}_\varepsilon(z,h) := \begin{cases} M_{\frac{1}{2}}(z,h) & \varepsilon=\frac{1}{2}, \\ \widetilde{M}(z,h) & \varepsilon=0. \end{cases}$$

In this subsection, we will compute the determinants of two maps $\widetilde{M}_\varepsilon(z_\lambda,h_\lambda^{\eta;\varepsilon}) \longrightarrow \mathcal{F}_\lambda^{\eta;\varepsilon}$ and $\mathcal{F}_\lambda^{\eta;\varepsilon} \longrightarrow \widetilde{M}_\varepsilon(z_\lambda,h_\lambda^{\eta;\varepsilon})^c$, whose composition is the map $\widetilde{S}_{z_\lambda,h_\lambda^{\eta;\varepsilon}} : \widetilde{M}_\varepsilon(z_\lambda,h_\lambda^{\eta;\varepsilon}) \longrightarrow \widetilde{M}_\varepsilon(z_\lambda,h_\lambda^{\eta;\varepsilon})^c$ defined by

$$1\otimes\mathbf{1}^\tau_{z_\lambda,h_\lambda^{\eta;\varepsilon}} \longmapsto (1\otimes\mathbf{1}^\tau_{z_\lambda,h_\lambda^{\eta;\varepsilon}})^*, \qquad (1\otimes\mathbf{1}^\tau_{z_\lambda,h_\lambda^{\eta;\varepsilon}})^*(1\otimes\mathbf{1}^\sigma_{z_\lambda,h_\lambda^{\eta;\varepsilon}}) := (-1)^\tau\delta_{\sigma,\tau}$$

for $\sigma,\tau\in\mathbb{Z}_2$. Clearly, the modules $\widetilde{M}_\varepsilon(z,h), \widetilde{M}_\varepsilon(z,h)^c$ and the map $\widetilde{S}_{z,h}$ are $\mathbb{Z}\times\mathbb{Z}_2$-graded for $(z,h)\in\mathbb{C}^2$. Thus, for $n\in(1-\varepsilon)\mathbb{Z}$ and $\tau\in\mathbb{Z}_2$, we set

$$\begin{aligned}
\widetilde{M}_\varepsilon(z,h)^\tau_n &:= \{u\in\widetilde{M}_\varepsilon(z,h) | L_0.u=(h+n)u, \quad \deg u=\tau\}, \\
\left(\widetilde{M}_\varepsilon(z,h)^c\right)^\tau_n &:= \{u\in\widetilde{M}_\varepsilon(z,h)^c | L_0.u=(h+n)u, \quad \deg u=\tau\}, \\
(\widetilde{S}_{z,h})^\tau_n &:= \widetilde{S}_{z,h}\Big|_{\widetilde{M}_\varepsilon(z,h)^\tau_n}.
\end{aligned}$$

Fixing basis of $\widetilde{M}_\varepsilon(z,h)^\tau_n$, $\left(\widetilde{M}_\varepsilon(z,h)^c\right)^\tau_n$, we define the determinant of the map $(\widetilde{S}_{z,h})^\tau_n$ which is denoted by $\widetilde{\det}_\varepsilon(z,h)^\tau_n$. This determinant can be easily computed, and the result looks as follows:

Lemma 3.4.— *For $n\in(1-\varepsilon)\mathbb{Z}_{>0}$ and $\tau\in\mathbb{Z}_2$, we have*

$$\widetilde{\det}_\varepsilon(z,h)^\tau_n \propto \prod_{\substack{\alpha,\beta\in\mathbb{Z}_{\geqslant 0},\\ 1\leqslant\alpha\beta\leqslant 2n,\\ \alpha-\beta\in 1-2\varepsilon+2\mathbb{Z}_{\geqslant 0}}} \Phi_{\alpha,\beta;\varepsilon}(z,h)^{p_\varepsilon\left(n-\frac{1}{2}\alpha\beta\right)}.$$

Next, we introduce our basic tools to study Fock modules.

Definition 3.2.— *For $(\lambda,\eta)\in\mathbb{C}^2$ and $\varepsilon\in\{0,\frac{1}{2}\}$, we define two Vir_ε-module maps $\Gamma_{\lambda,\eta;\varepsilon}$, $\mathrm{L}^{\lambda,\eta;\varepsilon}$ as follows:*

$$\begin{aligned}
\Gamma_{\lambda,\eta;\varepsilon} :& \widetilde{M}_\varepsilon(z_\lambda,h_\lambda^{\eta;\varepsilon}) \longrightarrow \mathcal{F}_\lambda^{\eta;\varepsilon}, \\
& 1\otimes\mathbf{1}^{\bar{0}}_{z_\lambda,h_\lambda^{\eta;\varepsilon}} \longmapsto (1\otimes\mathbf{1}_\eta)\otimes(1\otimes\mathbf{1}^\varepsilon),
\end{aligned}$$

in particular, we have

$$\Gamma_{\lambda,\eta;0}(1\otimes \mathbf{1}^{\bar{1}}_{z_\lambda,h_\lambda^{\eta;0}}) = \varphi_0(1\otimes \mathbf{1}_\eta)\otimes(1\otimes \mathbf{1}^0).$$

$$\mathrm{L}^{\lambda,\eta;\varepsilon}:\mathcal{F}_\lambda^{\eta;\varepsilon} \cong (\mathcal{F}_\lambda^{2\lambda-\eta;\varepsilon})^c \xrightarrow{{}^t\Gamma_{\lambda,2\lambda-\eta;\varepsilon}} \widetilde{M}_\varepsilon(z_\lambda, h_\lambda^{2\lambda-\eta;\varepsilon})^c = \widetilde{M}_\varepsilon(z_\lambda, h_\lambda^{\eta;\varepsilon})^c.$$

Note that the Fock module $\mathcal{F}_\lambda^{\eta;\varepsilon}$ is $\mathbb{Z}\times\mathbb{Z}_2$-graded, and the maps $\Gamma_{\lambda,\eta;\varepsilon}, \mathrm{L}^{\lambda,\eta;\varepsilon}$ are also $\mathbb{Z}\times\mathbb{Z}_2$-graded. Hence, for $n\in(1-\varepsilon)\mathbb{Z}_{\geqslant 0}$ and $\tau\in\mathbb{Z}_2$, we set

$$(\mathcal{F}_\lambda^{\eta;\varepsilon})_n^\tau := \{u\in\mathcal{F}_\lambda^{\eta;\varepsilon} | L_0.u = (h_\lambda^{\eta;\varepsilon}+n)u, \quad \deg u = \tau\},$$

$$(\Gamma_{\lambda,\eta;\varepsilon})_n^\tau := (\Gamma_{\lambda,\eta;\varepsilon})|_{\widetilde{M}_\varepsilon(z_\lambda,h_\lambda^{\eta;\varepsilon})_n^\tau} : \widetilde{M}_\varepsilon(z_\lambda, h_\lambda^{\eta;\varepsilon})_n^\tau \longrightarrow (\mathcal{F}_\lambda^{\eta;\varepsilon})_n^\tau,$$

$$(\mathrm{L}^{\lambda,\eta;\varepsilon})_n^\tau := (\mathrm{L}^{\lambda,\eta;\varepsilon})|_{(\mathcal{F}_\lambda^{\eta;\varepsilon})_n^\tau} : (\mathcal{F}_\lambda^{\eta;\varepsilon})_n^\tau \longrightarrow \left(\widetilde{M}_\varepsilon(z_\lambda, h_\lambda^{\eta;\varepsilon})^c\right)_n^\tau.$$

We denote the determinants of the maps $(\Gamma_{\lambda,\eta;\varepsilon})_n^\tau$ and $(\mathrm{L}^{\lambda,\eta;\varepsilon})_n^\tau$ by $\det(\Gamma_{\lambda,\eta;\varepsilon})_n^\tau$ and $\det(\mathrm{L}^{\lambda,\eta;\varepsilon})_n^\tau$ respectively.

The explicit formulae of $\det(\Gamma_{\lambda,\eta;\varepsilon})_n^\tau$ and $\det(\mathrm{L}^{\lambda,\eta;\varepsilon})_n^\tau$ can be described as follows. For $\lambda\in\mathbb{C}$, set $\lambda_\pm := \lambda\pm\sqrt{\lambda^2+1}$.

Theorem 3.2. — *For $(\lambda,\eta)\in\mathbb{C}^2$, $\varepsilon\in\{0,\frac{1}{2}\}$, $n\in(1-\varepsilon)\mathbb{Z}_{\geqslant 0}$ and $\tau\in\mathbb{Z}_2$, we have*

$$\det(\Gamma_{\lambda,\eta;\varepsilon})_n^\tau \propto \prod_{\substack{r,s\in\mathbb{Z}_{>0},\\ 1\leqslant rs\leqslant 2n,\\ r-s\in(1-2\varepsilon)+2\mathbb{Z}}} \left\{(\eta-\lambda)+\frac{1}{2}(r\lambda_+ + s\lambda_-)\right\}^{p_\varepsilon(n-\frac{1}{2}rs)},$$

$$\det(\mathrm{L}^{\lambda,\eta;\varepsilon})_n^\tau \propto \prod_{\substack{r,s\in\mathbb{Z}_{>0},\\ 1\leqslant rs\leqslant 2n,\\ r-s\in(1-2\varepsilon)+2\mathbb{Z}}} \left\{(\eta-\lambda)-\frac{1}{2}(r\lambda_+ + s\lambda_-)\right\}^{p_\varepsilon(n-\frac{1}{2}rs)}.$$

Proof. — The proof of this theorem is based on the results of § 3.1. In fact, one can easily show that the left hand sides are divisible by the right hand sides. But since we have $\mathrm{L}^{\lambda,\eta;\varepsilon}\circ\Gamma_{\lambda,\eta;\varepsilon} = \widetilde{S}_{z_\lambda,h_\lambda^{\eta;\varepsilon}}$, we have

$$\widetilde{\det}_\varepsilon(z_\lambda, h_\lambda^{\eta;\varepsilon})_n^\tau = \det(\mathrm{L}^{\lambda,\eta;\varepsilon})_n^\tau \times \det(\Gamma_{\lambda,\eta;\varepsilon})_n^\tau \tag{5}$$

by a suitable choice of basis. Hence, we conclude that the left hand sides coincide with the right hand sides by Lemma 3.4.

The details are left to the reader. □

4. Fock modules II: structure theorem.

In this section, we will study the detailed structure of the Fock modules $\mathcal{F}_\lambda^{\eta;\varepsilon}$.

4.1. Classification of weights.

In this subsection, we will classify the pair (λ, η) of parameters which specify the module $\mathcal{F}_\lambda^{\eta;\varepsilon}$.

First, for $T \in \mathbb{C}^*$ and $(\alpha, \beta) \in \mathbb{Z}^2$, we set

$$\lambda(T) := \frac{1}{2}(T - T^{-1}), \qquad \eta_{\alpha,\beta}(T) := \frac{1}{2}(\alpha T - \beta T^{-1}).$$

These $\lambda(T)$,$\eta_{\alpha,\beta}(T)$ and $z(t), h_{\alpha,\beta;\varepsilon}(t)$ recalled in §2.2 are related by

$$z_{\lambda(T)} = z(T^2), \qquad h_{\lambda(T)}^{\lambda(T)\pm\eta_{\alpha,\beta}(T);\varepsilon} = h_{\alpha,\beta;\varepsilon}(T^2). \tag{6}$$

Thus, by abuse of notation, we say that a pair (λ, η) belongs to Class $*$ if the corresponding weight $(z_\lambda, h_\lambda^{\eta;\varepsilon})$ does.

If the pair (λ, η) belongs to Class V, we have nothing to do.

If the pair (λ, η) belongs to Class I, then there exist $(\alpha, \beta) \in (\mathbb{Z}_{>0})^2$ and $T \in \mathbb{C}^*$ satisfying

$$\alpha - \beta \in 1 - 2\varepsilon + 2\mathbb{Z}, \quad T^2 \notin \mathbb{Q}, \quad \lambda = \lambda(T), \quad \eta = \lambda(T) \pm \eta_{\alpha,\beta}(T).$$

If the pair (λ, η) belongs to Class $\mathrm{R}^\pm$, then there exist $p, q \in \mathbb{Z}_{>0}$ satisfying $p - q \in 2\mathbb{Z}$, $\left(\frac{p-q}{2}, q\right) = 1$ and

$$\lambda = \lambda\left(\omega_\pm\sqrt{\frac{p}{q}}\right),$$

where we set

$$\omega_+ := 1, \qquad \omega_- := (-1)^{\frac{1}{2}}.$$

First, if the pair (λ, η) belongs to Class R^-, then by Theorem 3.2, it turns out that both of the maps $\Gamma_{\lambda,\eta;\varepsilon}$ and $\Gamma^{\lambda,\eta;\varepsilon}$ can never degenerate at the same time. This means, in this case, the Fock module $\mathcal{F}_\lambda^{\eta;\varepsilon}$ is isomorphic to either the module $\widetilde{M}_\varepsilon(z_\lambda, h_\lambda^{\eta;\varepsilon})$ or its contragredient dual $\widetilde{M}_\varepsilon(z_\lambda, h_\lambda^{\eta;\varepsilon})^c$.

Thus, the structure of the Fock module $\mathcal{F}_\lambda^{\eta;\varepsilon}$ has already been investigated in §4 of [IK2]. Hence, we will classify the only pair (λ,η) that belongs to Class R^+ below.

Set

$$K_{p,q} := \left\{ (r,s) \in \mathbb{Z}^2 \,\middle|\, \begin{array}{l} 0 \leqslant r \leqslant q \\ 0 \leqslant s \leqslant p \end{array} \right\}.$$

For each $(r,s) \in K_{p,q}$, we set

$$\eta(r,s;i) := \lambda\left(\sqrt{\frac{p}{q}}\right) + \begin{cases} \eta_{(i-1)q+r,-s}\left(\sqrt{\frac{p}{q}}\right) & \mathrm{i} \equiv 1 \bmod 2, \\ \eta_{iq+r,s}\left(\sqrt{\frac{p}{q}}\right) & \mathrm{i} \equiv 0 \bmod 2. \end{cases}$$

According to the degeneracy of the weights $\eta(r,s;i)$, we regroup $K_{p,q}$ into four groups as follows:

TABLE 1. — Classification of $K_{p,q}$.

Group ♠	$0 < r < q \;\wedge\; 0 < s < p$ $(r,s) \neq (\frac{q}{2}, \frac{p}{2})$
Group ♡	$r = \frac{q}{2} \;\wedge\; s = \frac{p}{2}$
Group ♣	$(r \equiv 0\ (q) \;\wedge\; s \not\equiv 0\ (p)) \vee (r \not\equiv 0\ (q) \;\wedge\; s \equiv 0\ (p))$
Group ♢	$r \equiv 0\ (q) \;\wedge\; s \equiv 0\ (p)$

Remark 4.1. — The degeneration of $\eta(r,s;i)$ are summarized in the following table:

Group ♣	$r \equiv 0\ (q)$	$\eta(0,s;i) = \eta(q,p-s;i-1)$
	$s \equiv 0\ (p)$	$\eta(r,0;i+1) = \eta(r,0;i) \quad i \equiv 0\ (2)$ $\eta(r,p;i+1) = \eta(r,p;i) \quad i \equiv 1\ (2)$
Group ♢	$\forall s$	$\eta(0,s;i) = \eta(q,p-s;i-1)$
	$\forall r$	$\eta(r,0;i+1) = \eta(r,0;i) \quad i \equiv 0\ (2)$ $\eta(r,p;i+1) = \eta(r,p;i) \quad i \equiv 1\ (2)$

Thus, we may assume that the range of i in $\eta(r,s;i)$ for each case is given by the following table:

Group	$\spadesuit$	$\heartsuit$	$\clubsuit$	$\diamondsuit$
Range	$\mathbb{Z}$	$\mathbb{Z}$	$2\mathbb{Z}$	$2\mathbb{Z}$

Remark 4.2. — The automorphism of $K_{p,q}$ defined by

$$(r,s) \longmapsto (q-r, p-s)$$

has the following meaning. It can be checked directly that we have

$$\eta(q-r,p-s;i) - \lambda\left(\sqrt{\frac{p}{q}}\right) = -\left(\eta(r,s;-i) - \lambda\left(\sqrt{\frac{p}{q}}\right)\right),$$

which implies that

$$h_\lambda^{\eta(q-r,p-s;i);\varepsilon} = h_\lambda^{\eta(r,s;-i);\varepsilon}.$$

This, in particular for $i=0$, is nothing but the symmetry of the Kac table, and at the level of the Fock modules, we have

$$\mathcal{F}_\lambda^{\eta(q-r,p-s;i);\varepsilon} \cong (\mathcal{F}_\lambda^{\eta(r,s;-i);\varepsilon})^c$$

by Lemma 3.1. This observation simplifies the arguments given in § 4.3.

Remark 4.3. — Setting

$$h(r,s;i)_\varepsilon := h_\lambda^{\eta(r,s;i);\varepsilon}$$

for $(r,s) \in K_{p,q}$ and $i \in \mathbb{Z}$, the L_0-weights $h(r,s;i)_\varepsilon$ and $h_{i;\varepsilon}$ defined in [IK2] are related as follows. Set

$$\sigma : K_{p,q} \longrightarrow K_{p,q}^+, \quad \{(r,s)\} \longmapsto \{(r,s),(q-r,p-s)\} \cap K_{p,q}^+.$$

Then, for $(r,s) \in K_{p,q}$, the weights $\{h(r,s;i)_\varepsilon\}$ and the weights $\{h_{i;\varepsilon}\}$ for $\sigma(r,s) \in K^+_{p,q}$ are related, and their explicit relations are given by the following table:

Group	$(r,s) \in K_{p,q}$	$h(r,s;i)_\varepsilon$	$\sigma(r,s)$
♠	$rp+sq<pq$	$h_{i;\varepsilon}$	(r,s)
	$rp+sq>pq$	$h_{-i;\varepsilon}$	$(q-r,p-s)$
♡	$(\frac{q}{2},\frac{p}{2})$	$h_{i;0} \quad i \geqslant 0$ $h_{-i;0} \quad i \leqslant 0$	$(\frac{q}{2},\frac{p}{2})$
♣	$r \equiv 0\ (q)$	$h_{i-(1-\frac{r}{q});\varepsilon} \quad i \geqslant 1-\frac{r}{q}$ $h_{-i-\frac{r}{q};\varepsilon} \quad i < 1-\frac{r}{q}$	$(q,p-s) \quad r=0$ $(q,s) \quad r=q$
	$s \equiv 0\ (p)$	$h_{\mathrm{sgn}(sq-rp)(i-\frac{s}{p});\varepsilon} \quad i \geqslant \frac{s}{p}$ $h_{\mathrm{sgn}(sq-rp)(i+(1-\frac{s}{p}));\varepsilon} \quad i < \frac{s}{p}$	$(q-r,p) \quad s=0$ $(r,p) \quad s=p$
♢	$rp-sq \neq 0$	$h_{-i+2\frac{s}{p};\varepsilon} \quad i \geqslant \frac{s}{p}$ $h_{i+2\frac{r}{q};\varepsilon} \quad i < \frac{s}{p}$	$(0,p)$
	$rp-sq=0$	$h_{i;\frac{1}{2}} \quad i \geqslant 0$ $h_{-i;\frac{1}{2}} \quad i \leqslant 0$	(q,p)

The list of the lattice points on the line $l^\sigma_{r,s;i}$ $(\sigma \in \{\pm\})$ defined by the factors of determinant

$$l^\sigma_{r,s;i} : \left(\eta(r,s;i) - \lambda\left(\sqrt{\frac{p}{q}}\right)\right) - \frac{\sigma}{2\sqrt{pq}}(p\alpha - q\beta) = 0$$

in the first quadrant of the (α,β)-plane for Class R^+ is given in §A.1.

4.2. Structure of Fock modules: simple cases.

In this subsection, we will set up the necessary tools to study the structure of the Fock modules $\mathcal{F}^{\eta;\varepsilon}_\lambda$. In particular, we will also study the case when (λ,η) does not belong to Class R^+.

For each $\varepsilon \in \{0, \frac{1}{2}\}$, let $\mathbb{M}, \mathbb{F}, \mathbb{M}^c$ be the trivial vector bundles on $D = \mathbb{C}^2$ whose fibres at a point $(\lambda,\eta) \in D$ are $\widetilde{M}_\varepsilon(z_\lambda, h^{\eta;\varepsilon}_\lambda), \mathcal{F}^{\eta;\varepsilon}_\lambda$ and $\widetilde{M}_\varepsilon(z_\lambda, h^{\eta;\varepsilon}_\lambda)^c$ respectively. Moreover, let Γ, L be the morphisms of bundles

$$\begin{array}{ccccc} \mathbb{M} & \xrightarrow{\Gamma} & \mathbb{F} & \xrightarrow{\mathrm{L}} & \mathbb{M}^c \\ & \searrow & \downarrow & \swarrow & \\ & & D & & \end{array}$$

whose restriction to the fibre over a point $(\lambda, \eta) \in D$ are $\Gamma_{\lambda,\eta;\varepsilon}$ and $\mathrm{L}^{\lambda,\eta;\varepsilon}$ respectively. We define the morphism of vector bundles $S : \mathbb{M} \longrightarrow \mathbb{M}^c$ by $S := \mathrm{L} \circ \Gamma$. Since each fibre of $\mathbb{M}, \mathbb{F}, \mathbb{M}^c$ is a graded vector space with finite dimensional graded subspaces and the morphisms preserve this grading, we can apply the method developed in § 2.3 and define the Jantzen filtration and the co-filtration in a natural way. For each point $P := (\lambda_0, \eta_0) \in D$, let C_P be the line defined by

$$\lambda - \eta - (\lambda_0 - \eta_0) = 0.$$

Let $\{\widetilde{M}_\varepsilon(z_{\lambda_0}, h_{\lambda_0}^{\eta_0;\varepsilon})(n)\}_{n\in\mathbb{Z}_{\geqslant 0}}$, $\{\widetilde{M}_\varepsilon(z_{\lambda_0}, h_{\lambda_0}^{\eta_0;\varepsilon})(n]\}_{n\in\mathbb{Z}_{\geqslant 0}}$ and $\{\mathcal{F}_{\lambda_0}^{\eta_0;\varepsilon}[n)\}_{n\in\mathbb{Z}_{\geqslant 0}}$ be the Jantzen filtration of the quadruples $\{\mathbb{M}_P, \mathbb{M}_P^c; S, C_P\}$ (resp. $\{\mathbb{M}_P, \mathbb{F}_P; \Gamma, C_P\}$ and $\{\mathbb{F}_P, \mathbb{M}_P^c; \mathrm{L}, C_P\}$), and let $\{\widetilde{M}_\varepsilon(z_{\lambda_0}, h_{\lambda_0}^{\eta_0;\varepsilon})^c(n)\}_{n\in\mathbb{Z}_{\geqslant 0}}$, $\{\mathcal{F}_{\lambda_0}^{\eta_0;\varepsilon}(n]\}_{n\in\mathbb{Z}_{\geqslant 0}}$ and $\{\widetilde{M}_\varepsilon(z_{\lambda_0}, h_{\lambda_0}^{\eta_0;\varepsilon})^c[n)\}_{n\in\mathbb{Z}_{\geqslant 0}}$ be the Jantzen co-filtration of the quadruples $\{\mathbb{M}_P, \mathbb{M}_P^c; S, C_P\}$, $\{\mathbb{M}_P, \mathbb{F}_P; \Gamma, C_P\}$ and $\{\mathbb{F}_P, \mathbb{M}_P^c; \mathrm{L}, C_P\}$ respectively. We remark that since the maps $\widetilde{S}_{z_\lambda, h_\lambda^{\eta;\varepsilon}}, \Gamma_{\lambda,\eta;\varepsilon}$ and $\mathrm{L}^{\lambda,\eta;\varepsilon}$ are Vir_ε-module morphisms, it follows that the Jantzen filtrations and the co-filtrations defined above are sequences of Vir_ε-modules.

For $(\lambda, \eta) \in \mathbb{C}^2$ and each $n \in \mathbb{Z}_{\geqslant 0}$, set

$$\begin{aligned} &\mathrm{Pr}^{(n)} : \widetilde{M}_\varepsilon(z_\lambda, h_\lambda^{\eta;\varepsilon})^c \twoheadrightarrow \widetilde{M}_\varepsilon(z_\lambda, h_\lambda^{\eta;\varepsilon})^c(n), \\ &\mathrm{Pr}^{(n]} : \mathcal{F}_\lambda^{\eta;\varepsilon} \twoheadrightarrow \mathcal{F}_\lambda^{\eta;\varepsilon}(n], \\ &\mathrm{Pr}^{[n)} : \widetilde{M}_\varepsilon(z_\lambda, h_\lambda^{\eta;\varepsilon})^c \twoheadrightarrow \widetilde{M}_\varepsilon(z_\lambda, h_\lambda^{\eta;\varepsilon})^c[n). \end{aligned}$$

Fixing a uniformizing element t of $\mathcal{O}_{C_P,P}$, the n-th. derivative of $\widetilde{S}_{z_\lambda, h_\lambda^{\eta,\varepsilon}}$, $\Gamma_{\lambda,\eta;\varepsilon}$ and $\mathrm{L}^{\lambda,\eta;\varepsilon}$ are defined, and we denote them by $S^{(n)}_{z_\lambda, h_\lambda^{\eta;\varepsilon}}, \Gamma^{(n]}_{\lambda,\eta;\varepsilon}$ and $\mathrm{L}^{\lambda,\eta;\varepsilon}_{[n)}$ respectively.

LEMMA 4.1 ([FeFu]). — *Let $k, l \in \mathbb{Z}_{\geqslant 0}$, and assume there exists a vector*

$$\begin{aligned} w \in &\left\{\widetilde{M}_\varepsilon(z,h)(k+l) \setminus \widetilde{M}_\varepsilon(z,h)(k+l+1)\right\} \\ &\cap \left\{\widetilde{M}_\varepsilon(z,h)(k] \setminus \widetilde{M}_\varepsilon(z,h)(k+1]\right\}, \end{aligned}$$

where we set $(z,h)=(z_\lambda,h_\lambda^{\eta;\varepsilon})$. *Then, there exist vectors* $w^f\in\mathcal{F}_\lambda^{\eta;\varepsilon}[l)\setminus\mathcal{F}_\lambda^{\eta;\varepsilon}[l+1)$ *and* $w^c\in\widetilde{M}_\varepsilon(z,h)^c\setminus\{0\}$ *satisfying*

1. $\mathrm{Pr}^{(k]}(w^f)=\Gamma^{(k]}_{\lambda,\eta;\varepsilon}(w)$,

2. $\mathrm{Pr}^{(k+l)}(w^c)=S^{(k+l)}_{z,h}(w)$ *and* $\mathrm{Pr}^{[l)}(w^c)=\mathrm{L}^{\lambda,\eta;\varepsilon}_{[l)}(w^f)$.

Proof. — Set $P:=(\lambda,\eta)\in D$ and $(z,h):=(z_\lambda,h_\lambda^{\eta;\varepsilon})$. Let $\mathcal{M}_{C_P,P},\mathcal{F}_{C_P,P}$ and $\mathcal{M}^c_{C_P,P}$ be the stalk of the sheaves of sections of $\mathbb{M}_{C_P}$ (resp. $\mathbb{F}_{C_P}$ and $\mathbb{M}^c_{C_P}$) at the point P. For each $n\in\mathbb{Z}_{>0}$, we set

$$\begin{aligned}\mathcal{M}_{C_P,P}(n)&:=\left\{u\in\mathcal{M}_{C_P,P}\,\middle|\,S_{C_P,P}(u)\in\mathfrak{m}_P^n\mathcal{M}^c_{C_P,P}\right\},\\ \mathcal{M}_{C_P,P}(n]&:=\left\{u\in\mathcal{M}_{C_P,P}\,\middle|\,\Gamma_{C_P,P}(u)\in\mathfrak{m}_P^n\mathcal{F}_{C_P,P}\right\},\\ \mathcal{F}_{C_P,P}[n)&:=\left\{u\in\mathcal{F}_{C_P,P}\,\middle|\,\mathrm{L}^{C_P,P}(u)\in\mathfrak{m}_P^n\mathcal{M}^c_{C_P,P}\right\}.\end{aligned}$$

Then, since we have

$$\begin{aligned}&\left\{u(P)\,\middle|\,\begin{matrix}u\in\mathcal{M}_{C_P,P}(k+l)\setminus\mathcal{M}_{C_P,P}(k+l+1),\\ u\in\mathcal{M}_{C_P,P}(k]\setminus\mathcal{M}_{C_P,P}(k+1].\end{matrix}\right\}\\ =&\left\{\widetilde{M}_\varepsilon(z,h)(k+l)\setminus\widetilde{M}_\varepsilon(z,h)(k+l+1)\right\}\cap\left\{\widetilde{M}_\varepsilon(z,h)(k]\setminus\widetilde{M}_\varepsilon(z,h)(k+1]\right\}\end{aligned}$$

by definition, we can take

$$u\in\{\mathcal{M}_{C_P,P}(k+l)\setminus\mathcal{M}_{C_P,P}(k+l+1)\}\cap\{\mathcal{M}_{C_P,P}(k]\setminus\mathcal{M}_{C_P,P}(k+1]\}$$

satisfying $u(P)=w$. Now, setting

$$w^f:=((t^{-k}\Gamma_{C_P,P})(u))(P),\qquad w^c:=((t^{-(k+l)}S_{C_P,P})(u))(P),$$

where t is the fixed uniformizing element of $\mathcal{O}_{C_P,P}$, it is easy to check that these w^f,w^c satisfy the properties in the lemma. $\square$

Let us study the structure of the Fock modules $\mathcal{F}_\lambda^{\eta;\varepsilon}$ in each case. First, if a point $(\lambda,\eta)\in D$ belongs to Class V, then the module $\mathcal{F}_\lambda^{\eta;\varepsilon}$ is irreducible. Hence, we have

LEMMA 4.2.

$$\mathcal{F}_\lambda^{\eta;\varepsilon}\cong\widetilde{M}_\varepsilon(z_\lambda,h_\lambda^{\eta;\varepsilon})\cong\widetilde{M}_\varepsilon(z_\lambda,h_\lambda^{\eta;\varepsilon})^c.$$

Second, if a point $(\lambda,\eta)\in D$ belongs to either Class I or Class R^-, then only one of $\Gamma_{\lambda,\eta;\varepsilon}$ and $\mathrm{L}^{\lambda,\eta;\varepsilon}$ can vanish. To be precise, take $T\in\mathbb{C}^*$ satisfying $\lambda=\lambda(T)$ and $(\alpha,\beta)\in(\mathbb{Z}_{>0})^2$ satisfying $\eta\in\{\lambda(T)\pm\eta_{\alpha,\beta}(T)\}$. Then, by the determinant formulae (Theorem 3.2), we obtain the following lemma:

LEMMA 4.3. — *Under the above setting,*

1. *if* $\eta=\lambda(T)+\eta_{\alpha,\beta}(T)$, *then we have*

$$\mathcal{F}_\lambda^{\eta;\varepsilon}\cong\widetilde{M}_\varepsilon(z_\lambda,h_\lambda^{\eta;\varepsilon}),$$

2. *if* $\eta=\lambda(T)-\eta_{\alpha,\beta}(T)$, *then we have*

$$\mathcal{F}_\lambda^{\eta;\varepsilon}\cong\widetilde{M}_\varepsilon(z_\lambda,h_\lambda^{\eta;\varepsilon})^c.$$

4.3. Structure of Fock modules: Class R^+.

In this subsection, we study the structure of the Fock modules $\mathcal{F}_\lambda^{\eta;\varepsilon}$ in the case when (λ,η) belongs to Class R^+ in detail.

Let us fix $p,q\in\mathbb{Z}_{>0}$ satisfying

$$p-q\in 2\mathbb{Z},\quad \left(\frac{p-q}{2},q\right)=1,$$

and set

$$\lambda:=\lambda\left(\sqrt{\frac{p}{q}}\right),z:=z_\lambda.$$

As in [IK2], we use the character sum formulae to study the structure of Fock modules. By the duality stated in Proposition 2.3 and Remark 4.2, it is enough to compute the character sum formulae

$$\sum_{k>0}\mathrm{ch}\widetilde{M}_\varepsilon(z,h_{i;\varepsilon})(k].$$

The results are given as follows:

LEMMA 4.4. — *The sum*

$$\sum_{k>0}\mathrm{ch}\widetilde{M}_\varepsilon(z,h(r,s;i)_\varepsilon)(k]$$

is given by

1. *Group* ♠ & ♡: $(i \in \mathbb{Z})$,

$$\sum_{k>0} \mathrm{ch}\widetilde{M}_\varepsilon(z, h(r, s; -(|i| + 2k - 1))_\varepsilon),$$

2. *Group* ♣: $(i \in 2\mathbb{Z})$,

(i) $r \equiv 0\ (q)$:

$$\sum_{k>0} \mathrm{ch}\widetilde{M}_\varepsilon\left(z, h\left(r, s; \left|i + \frac{r}{q}\right| + \left(2k - \frac{r}{q}\right)\right)_\varepsilon\right),$$

(ii) $s \equiv 0\ (p)$:

$$\sum_{k>0} \mathrm{ch}\widetilde{M}_\varepsilon\left(z, h\left(r, s; -\left|i - \frac{s}{p}\right| - \left(2k - \frac{s}{p}\right)\right)_\varepsilon\right),$$

3. *Group* ♢: $(i \in 2\mathbb{Z})$,

(i) $i \neq 0$:

$$\sum_{k>0} \mathrm{ch}\widetilde{M}_\varepsilon(z, h(r, s; i + 2(\mathrm{sgn}\, i)k)_\varepsilon),$$

(ii) $i = 0 \ \wedge \ rp - sq \neq 0$:

$$\sum_{k>0} \mathrm{ch}\widetilde{M}_\varepsilon(z, h(r, s; 2(\mathrm{sgn}(rp - sq))k)_\varepsilon),$$

(iii) $i = 0 \ \wedge \ rp - sq = 0$:

$$\sum_{k>0} \mathrm{ch}\widetilde{M}_\varepsilon(z, h(r, s; \pm 2k)_\varepsilon).$$

This lemma is a simple consequence of Lemma 2.4, Theorem 3.2 and §A.1.

Now, we first analyze the structure of the Fock module $\mathcal{F}_\lambda^{\eta;\varepsilon}$ where (λ, η) belongs to either Group ♠ or Group ♡. In this case, we have the following lemma:

LEMMA 4.5 (Group ♠ & ♡). — *For $i \in \mathbb{Z}$ and $k \in \mathbb{Z}_{>0}$, we have*

$$\widetilde{M}_\varepsilon(z, h(r,s;i)_\varepsilon)(k] \cong \widetilde{M}_\varepsilon(z, h(r,s;-(|i|+2k-1))_\varepsilon).$$

Proof. — We first prove the case when (λ, η) belongs to Group ♠. We prove the statement by induction on k.

By Lemma 4.4, it follows that $\dim \widetilde{M}_\varepsilon(z, h(r,s;i)_\varepsilon)(1]_{h-h(r,s;i)_\varepsilon} = \{0\}$ for $h < h(r,s;-(|i|+1))_\varepsilon$ and $\widetilde{M}_\varepsilon(z, h(r,s;i)_\varepsilon)(1]_{h(r,s;-(|i|+1))_\varepsilon - h(r,s;i)_\varepsilon} \neq \{0\}$. Thus we conclude that

$$\left\{\widetilde{M}_\varepsilon(z, h(r,s;i)_\varepsilon)(1]_{h(r,s;-(|i|+1))_\varepsilon - h(r,s;i)_\varepsilon}\right\}^{(\mathrm{Vir}_\varepsilon)_+} \neq \{0\}.$$

On the other hand, Lemma 4.4 implies that

$$\left\{\widetilde{M}_\varepsilon(z, h(r,s;i)_\varepsilon)(1]_{h(r,s;|i|+1)_\varepsilon - h(r,s;i)_\varepsilon}\right\}^{(\mathrm{Vir}_\varepsilon)_+} = \{0\},$$

since we have $[\widetilde{M}_\varepsilon(z, h(r,s;i)_\varepsilon)(1] : L_\varepsilon(z, h(r,s;|i|+1)_\varepsilon)] = 0$. Thus, by Theorem 2.1, the statement for $k=1$ is proved. Assume that we could prove the statement up to $k-1$. Then, it follows from Lemma 4.4 and the hypothesis that

$$\sum_{l \geqslant k} \mathrm{ch}\, \widetilde{M}_\varepsilon(z, h(r,s;i)_\varepsilon)(l] = \sum_{l \geqslant k} \mathrm{ch}\, \widetilde{M}_\varepsilon(z, h(r,s;-(|i|+2l-1))_\varepsilon).$$

Thus, by an argument similar to the case of $k=1$, we can prove that the statement is also true for k, and we complete the induction.

Second, we prove the case when (λ, η) belongs to Group ♡. Again, we prove the statement by induction on k.

By Lemma 4.4, it follows that
$\dim \widetilde{M}_0\left(z, h(\frac{q}{2}, \frac{p}{2}; i)_0\right)(1]_{h-h(\frac{q}{2},\frac{p}{2};i)_0} = \{0\}$ for $h < h\left(\frac{q}{2}, \frac{p}{2}; -(|i|+1)\right)_0$

and

$$\widetilde{M}_\varepsilon\left(z, h(\frac{q}{2}, \frac{p}{2}; i)_0\right)(1]_{h(\frac{q}{2},\frac{p}{2};-(|i|+1))_0 - h(\frac{q}{2},\frac{p}{2};i)_0} \neq \{0\}.$$

Thus, we see that

$$\dim\left\{\widetilde{M}_0\left(z, h\left(\frac{q}{2}, \frac{p}{2}; i\right)_0\right)(1]^{\bar{0}}_{h(\frac{q}{2},\frac{p}{2};-(|i|+1))_0 - h(\frac{q}{2},\frac{p}{2};i)_0}\right\}^{(\mathrm{Vir}_0)_+} = 1. \tag{7}$$

First, we prove the statement in the case $i = 0$ and $k = 1$. Let $S \in U((\mathrm{Vir}_0)_-)^{\bar{0}}_{h(\frac{q}{2},\frac{p}{2};-1)_0-h(\frac{q}{2},\frac{p}{2};0)_0}\setminus\{0\}$ be an element such that $S(1\otimes \mathbf{1}^{\bar{0}}_{z,\frac{1}{24}z})$ is an even singular vector. Suppose $S.(1\otimes \mathbf{1}^{\bar{0}}_{z,\frac{1}{24}z}) \in \mathrm{Ker}\Gamma_{\lambda,\lambda;0}$. Then, since we have $S.(1\otimes \mathbf{1}_\lambda)\otimes(1\otimes \mathbf{1}^0) = 0$ by assumption, we see that S is an element of the left ideal of a certain completion of $U(\mathcal{H}\oplus\mathcal{D}_0)$ generated by $\mathcal{H}_+$, $\mathcal{D}_{0;+}$ and $a_0-\lambda$. But then, it is easy to see that $G_0S.(1\otimes \mathbf{1}^{\bar{1}}_{z,\frac{1}{24}z})$ is also an even (non-zero) singular vector, and since we have $\Gamma_{\lambda,\lambda;0}(1\otimes \mathbf{1}^{\bar{1}}_{z,\frac{1}{24}z}) = \varphi_0(1\otimes \mathbf{1}_\lambda)\otimes(1\otimes \mathbf{1}^0)$, it turns out that $G_0S.(1\otimes \mathbf{1}^{\bar{1}}_{z,\frac{1}{24}z}) \in \mathrm{Ker}\Gamma_{\lambda,\lambda;0}$. This is impossible, since we have

$$\widetilde{M_0}\left(z,\frac{1}{24}z\right) \cong M_0\left(z,\frac{1}{24}z\right)\oplus \Pi M_0\left(z,\frac{1}{24}z\right)$$

and (7). Thus, by Theorem 2.1, we conclude that the statement is true in this case.

Second, we prove the statement in the case $i \neq 0$ and $k = 1$. Let

$$X \in U((\mathrm{Vir}_0)_-)^{\bar{1}}_{h(\frac{q}{2},\frac{p}{2};-(|i|+1))_0-h(\frac{q}{2},\frac{p}{2};i)_0},$$

$$Y \in U((\mathrm{Vir}_0)_-)^{\bar{0}}_{h(\frac{q}{2},\frac{p}{2};-(|i|+1))_0-h(\frac{q}{2},\frac{p}{2};i)_0}$$

be elements such that $(XG_0+Y).(1\otimes \mathbf{1}^{\bar{0}}_{z,h(\frac{q}{2},\frac{p}{2};i)_0})$ is a non-zero even singular vector. Then, it is easy to see that $(\eta(\frac{q}{2},\frac{p}{2};i)-\lambda)X\varphi_0+Y$ is an element of the left ideal of a certain completion of $U(\mathcal{H}\oplus\mathcal{D}_0)$ generated by $\mathcal{H}_+$, $\mathcal{D}_{0;+}$ and $a_0-\eta(\frac{q}{2},\frac{p}{2};i)$. Then, it is also easy to check that the even singular vector $G_0(XG_0+Y)G_0.(1\otimes \mathbf{1}^{\bar{0}}_{z,h(\frac{q}{2},\frac{p}{2};i)_0})$ is also an element of $\mathrm{Ker}\Gamma_{\lambda,\eta(\frac{q}{2},\frac{p}{2};i);0}$, which implies that

$$(8)\quad G_0(XG_0+Y)G_0.(1\otimes \mathbf{1}^{\bar{0}}_{z,h(\frac{q}{2},\frac{p}{2};i)_0}) \propto (XG_0+Y).(1\otimes \mathbf{1}^{\bar{0}}_{z,h(\frac{q}{2},\frac{p}{2};i)_0}).$$

Expanding X and Y with respect to the basis $\mathcal{B}^{\bar{0}}_{h(\frac{q}{2},\frac{p}{2};-(|i|+1))_0-h(\frac{q}{2},\frac{p}{2};i)_0}$ as in [IK2]

$$X = c_X G_{-1}L_{-1}^{h(\frac{q}{2},\frac{p}{2};-(|i|+1))_0-h(\frac{q}{2},\frac{p}{2};i)_0-1}+\cdots,$$
$$Y = c_Y L_{-1}^{h(\frac{q}{2},\frac{p}{2};-(|i|+1))_0-h(\frac{q}{2},\frac{p}{2};i)_0}+\cdots,$$

and comparing the coefficients of $G_{-1}L_{-1}^{h(\frac{q}{2},\frac{p}{2};-(|i|+1))_0-h(\frac{q}{2},\frac{p}{2};i)_0-1}G_0$ and $L_{-1}^{h(\frac{q}{2},\frac{p}{2};-(|i|+1))_0-h(\frac{q}{2},\frac{p}{2};i)_0}$, we obtain

$$\begin{aligned}
&4\left(h\left(\frac{q}{2},\frac{p}{2};i\right)_0-\frac{1}{24}z\right)c_X^2+4\left(h\left(\frac{q}{2},\frac{p}{2};i\right)_0-\frac{1}{24}z\right)c_Xc_Y\\
&-\left(h\left(\frac{q}{2},\frac{p}{2};-(|i|+1)\right)_0-h\left(\frac{q}{2},\frac{p}{2};i\right)_0\right)c_Y^2=0\\
&\iff 4i^2c_X^2+4i^2c_Xc_Y-(2|i|+1)c_Y^2=0
\end{aligned}$$

by (8). This implies that $c_Y\neq 0$ and $2c_X+c_Y\neq 0$, and hence the statement is true in this case by Theorem 2.1. Assume that we could prove the statement up to $k-1$. Then, it follows from Lemma 4.4 and the hypothesis that

$$\sum_{l\geqslant k}\mathrm{ch}\widetilde{M}_0\left(z,h\left(\frac{q}{2},\frac{p}{2};i\right)_0\right)(l]=\sum_{l\geqslant k}\mathrm{ch}\widetilde{M}_0\left(z,h\left(\frac{q}{2},\frac{p}{2};-(|i|+2l-1)\right)_0\right).$$

Thus, we see that

$$\dim\left\{\widetilde{M}_0\left(z,h\left(\frac{q}{2},\frac{p}{2};i\right)_0\right)(k]^{\bar{0}}_{h(\frac{q}{2},\frac{p}{2};-(|i|+2k-1))_0-h(\frac{q}{2},\frac{p}{2};i)_0}\right\}^{(\mathrm{Vir}_0)_+}=1.$$

Paying attention to the fact that there exists a section $u\in\mathcal{M}_{C_P,P}(k]_{h(\frac{q}{2},\frac{p}{2};-(|i|+2k-1))_0-h(\frac{q}{2},\frac{p}{2};i)_0}\setminus\{0\}$ such that $u(P)$ is an element of $\{\widetilde{M}_0(z,h(\frac{q}{2},\frac{p}{2};i)_0)(k]^{\bar{0}}_{h(\frac{q}{2},\frac{p}{2};-(|i|+2k-1))_0-h(\frac{q}{2},\frac{p}{2};i)_0}\}^{(\mathrm{Vir}_0)_+}\setminus\{0\}$, we can prove that the statement is also true for k by similar arguments. Therefore, we have completed the proof. □

First, suppose that (λ,η) belongs to Group ♠. For $k\in\mathbb{Z}\setminus\{0\}$, let $w_k\in\widetilde{M}_\varepsilon(z,h(r,s;i)_\varepsilon)$ be a singular vector of L_0-weight being $h(r,s;(\mathrm{sgn}\,k)(|i|+|k|))_\varepsilon$. Second, when (λ,η) belongs to Group ♡, for $k\in 2\mathbb{Z}\setminus\{0\}$, we let $w_k\in\widetilde{M}_0(z,h(\frac{q}{2},\frac{p}{2};i)_0)$ be a singular vector of L_0-weight being $h(\frac{q}{2},\frac{p}{2};(\mathrm{sgn}\,k)(|i|+|k|))_0$ which is an image of a highest weight vector under an injective map (see Theorem 4.4 in [IK2]). For $k\in 1+2\mathbb{Z}_{<0}$, we let $w_k\in\widetilde{M}_0(z,h(\frac{q}{2},\frac{p}{2};i)_0)$ be a singular vector of L_0-weight being $h(\frac{q}{2},\frac{p}{2};-(|i|-k))_0$ which belongs to $\widetilde{M}_0(z,h(\frac{q}{2},\frac{p}{2};i)_0)(\frac{-k+1}{2}]$, and for $k\in -1+2\mathbb{Z}_{>0}$, we let $w_k\in\widetilde{M}_0(z,h(\frac{q}{2},\frac{p}{2};i)_0)$ be a singular vector of L_0-weight being $h(\frac{q}{2},\frac{p}{2};|i|+k)_0$ which is not a scalar multiple of w_{-k} and is an image of a highest weight vector under an injective map.

Now, for $k\in\mathbb{Z}\setminus\{0\}$, we define a vector $w_k^f\in\mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon}$ as a vector that corresponds to $w_k\in\widetilde{M}_\varepsilon(z,h(r,s;i)_\varepsilon)$ in the sense of Lemma 4.1. Here we also set $w_0^f:=(1\otimes\mathbf{1}_\eta)\otimes(1\otimes\mathbf{1}^\varepsilon)$.

As an application of Lemma 4.5, we can illustrate the structure of the Fock module $\mathcal{F}_\lambda^{\eta;\varepsilon}$ in the case when it belongs to Group ♠ or ♡ as follows:

THEOREM 4.1 (Group ♠ & ♡). — *The structure of the Fock module $\mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon}$ $(i \in \mathbb{Z})$ can be described as follows:*

1. *For $k \in \mathbb{Z}_{>0}$, we have*

$$U(\mathrm{Vir}_\varepsilon).w^f_{2k-1} \cong L_\varepsilon(z, h(r,s;|i|+2k-1)_\varepsilon).$$

Thus, we set $\mathcal{G}_\lambda^{\eta(r,s;i);\varepsilon} := \mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon} / \oplus_{k\in\mathbb{Z}_{>0}} U(\mathrm{Vir}_\varepsilon).w^f_{2k-1}$, and let

$$\pi : \mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon} \twoheadrightarrow \mathcal{G}_\lambda^{\eta(r,s;i);\varepsilon}$$

be the canonical projection.

2. *For $l \in \mathbb{Z}$, we have*

$$U(\mathrm{Vir}_\varepsilon).\pi(w^f_{2l}) \cong \begin{cases} L_\varepsilon(z, h(r,s;(\operatorname{sgn} l)(|i|+2|l|))_\varepsilon) & l \neq 0, \\ L_\varepsilon(z, h(r,s;i)_\varepsilon) & l = 0. \end{cases}$$

Hence, we set $\overline{\mathcal{G}}_\lambda^{\eta(r,s;i);\varepsilon} := \mathcal{G}_\lambda^{\eta(r,s;i);\varepsilon} / \oplus_{l\in\mathbb{Z}} U(\mathrm{Vir}_\varepsilon).\pi(w^f_{2l})$, and let

$$\overline{\pi} : \mathcal{G}_\lambda^{\eta(r,s;i);\varepsilon} \twoheadrightarrow \overline{\mathcal{G}}_\lambda^{\eta(r,s;i);\varepsilon}$$

be the canonical projection.

3. *We have the following isomorphisms:*

$$\overline{\mathcal{G}}_\lambda^{\eta(r,s;i);\varepsilon} = \bigoplus_{k\in\mathbb{Z}_{>0}} U(\mathrm{Vir}_\varepsilon).\overline{\pi}\circ\pi(w^f_{-2k+1}),$$
$$U(\mathrm{Vir}_\varepsilon).\overline{\pi}\circ\pi(w^f_{-2k+1}) \cong L_\varepsilon(z, h(r,s;-(|i|+2k-1))_\varepsilon) \quad (k \in \mathbb{Z}_{>0}).$$

Remark 4.4. — Pictorially, the structure of the Fock module $\mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon}$ that belongs to either Group ♠ or Group ♡ can be illustrated as follows:

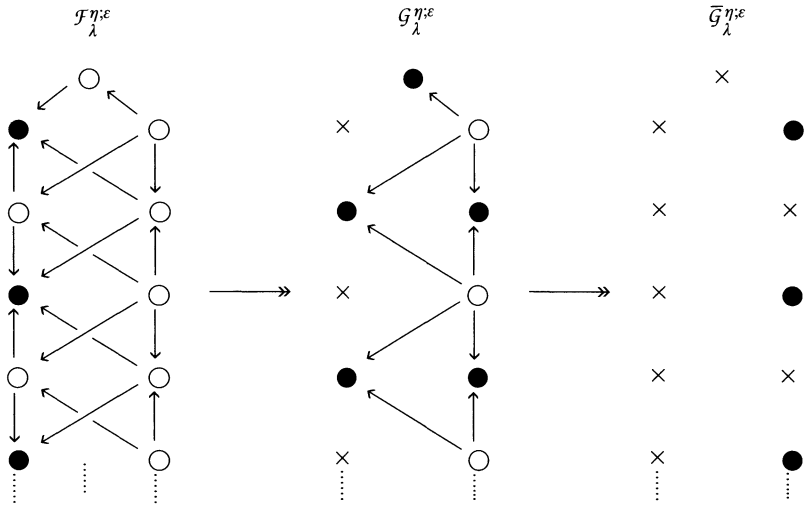


Figure 2. Group ♠ & ♡

Here, • and × denote a singular vector, and the zero vector in the indicated quotient respectively. The arrow

$$v \longrightarrow w$$

signifies the fact that the vector w lies in $U(\mathrm{Vir}_\varepsilon).v$ in an appropriate quotient module. The existence of these arrows is the direct consequences of Propositions 2.2, 2.3 and Lemma 4.5. Indeed, we have

$$\begin{aligned}\widetilde{M}_\varepsilon(z,h(r,s;i)_\varepsilon)(k] &\cong \widetilde{M}_\varepsilon(z,h(r,s;-(|i|+2k-1))_\varepsilon),\\ \widetilde{M}_\varepsilon(z,h(r,s;i)_\varepsilon)^c[k) &\cong \widetilde{M}_\varepsilon(z,h(r,s;|i|+2k-1)_\varepsilon)^c\end{aligned}$$

in this case.

Now, let us prove Theorem 4.1.

Proof. — By the fact that we mentioned in Remark 4.4 and Lemma 4.1, it is easy to see the following facts:

(A) The character of the modules $\mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon}[k)$ and $\mathrm{KerPr}^{(k]}$ for $k \in \mathbb{Z}_{>0}$ can be expressed as follows:

$$\begin{aligned}&\mathrm{ch}\mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon}[k) = \mathrm{ch}\widetilde{M}_\varepsilon(z,h(r,s;|i|+2k-1)_\varepsilon),\\ &\mathrm{chKerPr}^{(k]} = \mathrm{ch}\widetilde{M}_\varepsilon(z,h(r,s;i)_\varepsilon) - \mathrm{ch}\widetilde{M}_\varepsilon(z,h(r,s;-(|i|+2k-1))_\varepsilon).\end{aligned}$$

(B) For $k \in \mathbb{Z}_{>0}$ and $l \in \mathbb{Z}$, we have

$$w_l^f \in \mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon}[k) \iff l \leqslant -2k \ \vee \ l \geqslant 2k-1.$$

(C) For $k \in \mathbb{Z}_{>0}$ and $l \in \mathbb{Z}$, we have

$$w_l^f \in \mathrm{KerPr}^{(k]} \iff -2(k-1) \leqslant l \leqslant 2k-1.$$

Hence, to prove the first statement of the theorem, we have only to consider the modules

$$\mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon}[k) \cap \mathrm{KerPr}^{(k]} \quad (k \in \mathbb{Z}_{>0}).$$

In fact, it follows from (B) and (C) that for $k \in \mathbb{Z}_{>0}$ and $l \in \mathbb{Z}$,

$$w_l^f \in \mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon}[k) \cap \mathrm{KerPr}^{(k]} \iff l = 2k-1.$$

Moreover, (A) and (B) guarantee that the vector w_{2k-1}^f is a singular vector. Thus, the module

$$U(\mathrm{Vir}_\varepsilon).w_{2k-1}^f \subset \mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon}[k) \cap \mathrm{KerPr}^{(k]}$$

is a highest weight module with highest L_0-weight $h(r,s;|i|+2k-1)_\varepsilon$. We remark that this module is non-zero, and the following inequalities hold ($\forall h \in \mathbb{C}$):

$$\begin{aligned}
&[U(\mathrm{Vir}_\varepsilon).w_{2k-1}^f : L_\varepsilon(z,h)] \leqslant [\mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon}[k) : L_\varepsilon(z,h)],\\
&[U(\mathrm{Vir}_\varepsilon).w_{2k-1}^f : L_\varepsilon(z,h)] \leqslant [\mathrm{KerPr}^{(k]} : L_\varepsilon(z,h)].
\end{aligned}$$

Now, by Lemma 2.2 and (A), it turns out that

$$U(\mathrm{Vir}_\varepsilon).w_{2k-1}^f \cong L_\varepsilon(z, h(r,s;|i|+2k-1)_\varepsilon)$$

and the first statement is proved. In fact, we have proved the existence of an isomorphism:

$$\mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon}[k) \cap \mathrm{KerPr}^{(k]} \cong L_\varepsilon(z, h(r,s;|i|+2k-1)_\varepsilon).$$

Similarly, to prove the second statement, we have only to consider the modules

$$\mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon}[k) \cap \mathrm{KerPr}^{(k+1]} \quad (k \in \mathbb{Z}_{\geqslant 0}).$$

In fact, similar arguments show that

$$\mathrm{ch}\pi\{\mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon}[k)\cap \mathrm{KerPr}^{(k+1]}\} = \sum_{l\in\{\pm 2k\}} \mathrm{ch}L_\varepsilon(z, h(r,s;l)_\varepsilon).$$

Now, the second statement follows from Lemma 2.3.

The third statement can be proved by noting the fact that for $k \in \mathbb{Z}_{>0}$ and $l \in \mathbb{Z}$, we have

$$0 \neq \overline{\pi}\circ\pi(w_l^f) \in \overline{\pi}\circ\pi\{\mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon}[k-1)\cap \mathrm{KerPr}^{(k+1]}\} \iff l = -(2k-1).$$

□

Second, we analyze the case when (λ, η) belongs to group ♣. In this case, we have the following lemma which can be proved in a way similar to the case of Group ♠:

LEMMA 4.6 (Group ♣). — *For $i \in 2\mathbb{Z}$ and $k \in \mathbb{Z}_{>0}$, we have the following:*

1. $s \not\equiv 0\ (p)$:

(i) $i > 0 \ \vee\ [i = 0 \ \wedge\ r = q]$,

$$\widetilde{M}_\varepsilon(z, h(r,s;i)_\varepsilon)(k] \cong \widetilde{M}_\varepsilon(z, h(r,s;i+2k)_\varepsilon).$$

(ii) $i < 0 \ \vee\ [i = 0 \ \wedge\ r = 0]$,

$$\widetilde{M}_\varepsilon(z, h(r,s;i)_\varepsilon)(k] \cong \widetilde{M}_\varepsilon\left(z, h\left(r, s; -i + 2\left(k - \frac{r}{q}\right)\right)_\varepsilon\right).$$

2. $r \not\equiv 0\ (q)$:

(i) $i > 0 \ \vee\ [i = 0 \ \wedge\ s = 0]$,

$$\widetilde{M}_\varepsilon(z, h(r,s;i)_\varepsilon)(k] \cong \widetilde{M}_\varepsilon\left(z, h\left(r, s; -i - 2\left(k - \frac{s}{p}\right)\right)_\varepsilon\right).$$

(ii) $i < 0 \ \vee\ [i = 0 \ \wedge\ s = p]$,

$$\widetilde{M}_\varepsilon(z, h(r,s;i)_\varepsilon)(k] \cong \widetilde{M}_\varepsilon(z, h(r,s;i-2k)_\varepsilon).$$

As an application of this lemma, we can show the following structure theorem of the Fock module $\mathcal{F}_\lambda^\eta$ when (λ, η) belongs to Group ♣:

Theorem 4.2 (Group ♣). — *The structure of the Fock module $\mathcal{F}_\lambda^{\eta(r,s;i),\varepsilon}$ $(i \in 2\mathbb{Z})$ can be described as follows:*

1. $[r \equiv 0\ (q)\ \wedge [i > 0\ \vee [i = 0\ \wedge\ r = q]]]\ \vee\ [s \equiv 0\ (p)\ \wedge [i < 0\ \vee [i = 0\ \wedge\ s = p]]]$:

(i) For $k \in \mathbb{Z}_{>0}$, we have

$$\mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon}[k] \cap \mathrm{KerPr}^{(k]} \cong \begin{cases} L_\varepsilon\left(z, h\left(r, s; -i - 2k + 2\left(1 - \frac{r}{q}\right)\right)_\varepsilon\right) & r \equiv 0\ (q), \\ L_\varepsilon\left(z, h\left(r, s; -i + 2k - 2\left(1 - \frac{s}{p}\right)\right)_\varepsilon\right) & s \equiv 0\ (p). \end{cases}$$

Thus, we set $\mathcal{G}_\lambda^{\eta(r,s;i);\varepsilon} := \mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon} / \oplus_{k \in \mathbb{Z}_{>0}} \mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon}[k] \cap \mathrm{KerPr}^{(k]}$, and let

$$\pi : \mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon} \twoheadrightarrow \mathcal{G}_\lambda^{\eta(r,s;i);\varepsilon}$$

be the canonical projection.

(ii) We have

$$\mathcal{G}_\lambda^{\eta(r,s;i);\varepsilon} = \bigoplus_{l \in \mathbb{Z}_{\geqslant 0}} \pi\{\mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon}[l) \cap \mathrm{KerPr}^{(l+1]}\},$$

$$\pi\{\mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon}[l) \cap \mathrm{KerPr}^{(l+1]}\} \cong \begin{cases} L_\varepsilon(z, h(r, s; i + 2l)_\varepsilon) & r \equiv 0\ (q), \\ L_\varepsilon(z, h(r, s; i - 2l)_\varepsilon) & s \equiv 0\ (p). \end{cases}$$

2. $[r \equiv 0\ (q)\ \wedge [i < 0\ \vee [i = 0\ \wedge\ r = 0]]]\ \vee\ [s \equiv 0\ (p)\ \wedge [i > 0\ \vee [i = 0\ \wedge\ s = 0]]]$:

(i) For $k \in \mathbb{Z}_{\geqslant 0}$, we have

$$\mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon}[k) \cap \mathrm{KerPr}^{(k+1]} \cong \begin{cases} L_\varepsilon(z, h(r, s; i - 2k)_\varepsilon) & r \equiv 0\ (q), \\ L_\varepsilon(z, h(r, s; i + 2k)_\varepsilon) & s \equiv 0\ (p). \end{cases}$$

Thus, we set $\mathcal{G}_\lambda^{\eta(r,s;i);\varepsilon} := \mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon} / \oplus_{k \in \mathbb{Z}_{\geqslant 0}} \mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon}[k) \cap \mathrm{KerPr}^{(k+1]}$, and let

$$\pi : \mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon} \twoheadrightarrow \mathcal{G}_\lambda^{\eta(r,s;i);\varepsilon}$$

be the canonical projection.

(ii) We have

$$\mathcal{G}_\lambda^{\eta(r,s;i);\varepsilon} = \bigoplus_{l\in\mathbb{Z}_{>0}} \pi\{\mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon}[l-1)\cap \mathrm{KerPr}^{(l+1]}\},$$

$$\pi\{\mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon}[l-1)\cap \mathrm{KerPr}^{(l+1]}\}$$
$$\cong \begin{cases} L_\varepsilon\left(z, h\left(r,s;-i+2\left(l-\frac{r}{q}\right)\right)_\varepsilon\right) & r\equiv 0\ (q), \\ L_\varepsilon\left(z, h\left(r,s;-i-2\left(l+\frac{s}{p}\right)\right)_\varepsilon\right) & s\equiv 0\ (p). \end{cases}$$

One can prove this theorem by a similar way to the proof of Theorem 4.1, so we will omit its proof here.

Third, we analyze the case when (λ,η) belongs to group $\diamondsuit$. In this case, we have the following lemma which can be proved in a way similar to the case of Group $\spadesuit$:

LEMMA 4.7 (Group $\diamondsuit$). — *For $i\in 2\mathbb{Z}$ and $k\in\mathbb{Z}_{>0}$, we have the following:*

1. $i>0\ \vee\ [i=0\ \wedge\ rp-sq\geqslant 0]$*:*

$$\widetilde{M}_\varepsilon(z,h(r,s;i)_\varepsilon)(k]\cong \widetilde{M}_\varepsilon(z,h(r,s;i+2k)_\varepsilon).$$

2. $i<0\ \vee\ [i=0\ \wedge\ rp-sq\leqslant 0]$*:*

$$\widetilde{M}_\varepsilon(z,h(r,s;i)_\varepsilon)(k]\cong \widetilde{M}_\varepsilon(z,h(r,s;i-2k)_\varepsilon).$$

Remark 4.5. — According to our choice of the space of parameters (i.e, $\{(\lambda,\eta)\}$), one has to be careful when one study the Jantzen filtration $\{\widetilde{M}_\varepsilon(z,h(r,s;i)_\varepsilon)(k)\}_{k\in\mathbb{Z}_{>0}}$. In fact, in the case when (λ,η) belongs to Group $\diamondsuit$, one can check that the following hold:

1. $i>0\ \vee\ [i=0\ \wedge\ rp-sq\geqslant 0]$:

$$\widetilde{M}_\varepsilon(z,h(r,s;i)_\varepsilon)(k)\cong \widetilde{M}_\varepsilon\left(z, h\left(r,s;i+2\left\lceil\frac{k+1}{2}\right\rceil\right)_\varepsilon\right).$$

2. $i<0\ \vee\ [i=0\ \wedge\ rp-sq\leqslant 0]$:

$$\widetilde{M}_\varepsilon(z,h(r,s;i)_\varepsilon)(k)\cong \widetilde{M}_\varepsilon\left(z, h\left(r,s;i-2\left\lceil\frac{k+1}{2}\right\rceil\right)_\varepsilon\right).$$

This happens because of the ramification of the covering

$$\mathbb{C}^2 \longrightarrow \mathbb{C}^2, \quad (\lambda, \eta) \longmapsto (z_\lambda, h_\lambda^{\eta;\varepsilon}).$$

Note that this is the only subtle case, i.e., the case where the results require some modifications. (Compare this with Theorem 4.1 in [IK2].)

By Lemma 4.7 and Remark 4.5, the following theorem can be shown in a way similar to the proof of Theorem 4.1:

THEOREM 4.3 (Group $\diamondsuit$). — *The Fock module $\mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon}$ is semi-simple, and we have*

$$\mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon} \cong \bigoplus_{k \in \mathbb{Z}_{>0}} \mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon}[k-1) \cap \mathrm{KerPr}^{(k)},$$

$$\mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon}[k-1) \cap \mathrm{KerPr}^{(k)} \cong \begin{cases} L_\varepsilon(z, h(r,s;i+2k)_\varepsilon) & i > 0 \ \vee \ [i = 0 \ \wedge \ rp - sq \geqslant 0], \\ L_\varepsilon(z, h(r,s;i-2k)_\varepsilon) & i < 0 \ \vee \ [i = 0 \ \wedge \ rp - sq \leqslant 0]. \end{cases}$$

Remark 4.6. — The following diagrams illustrate the structure of the Fock modules in the case when (λ, η) belongs to Group $\clubsuit$ or Group $\diamondsuit$:

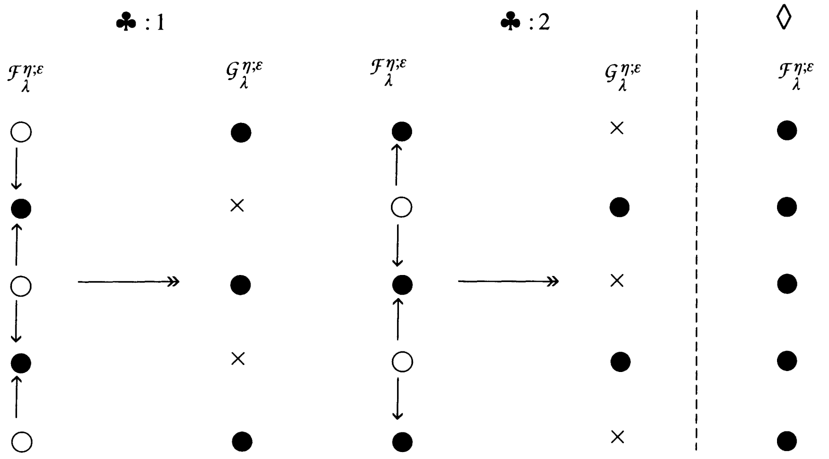


Figure 3. *Group* $\clubsuit$ & $\diamondsuit$

Here, $\clubsuit$: 1 (resp. $\clubsuit$: 2) signifies the classification in Theorem 4.2.

5. Bechi-Rouet-Stora-Tyutin resolution.

In this section, we construct Bechi-Rouet-Stora-Tyutin (BRST) resolution for the minimal cases.

5.1. Bechi-Rouet-Stora-Tyutin complex.

Let us fix $p,q \in \mathbb{Z}_{>1}$ satisfying $p-q \in 2\mathbb{Z}$ and $\left(\frac{p-q}{2}, q\right) = 1$. For $\alpha, \beta \in \mathbb{Z}$ and $(r,s) \in K_{p,q}$, we set

$$\mathcal{F}^{\varepsilon}_{\alpha,\beta} := \mathcal{F}^{\lambda(\sqrt{\frac{p}{q}})+\eta_{\alpha,\beta}(\sqrt{\frac{p}{q}});\varepsilon}_{\lambda(\sqrt{\frac{p}{q}})}.$$

Moreover, for $(z,h) \in \mathbb{C}^2$, we set

$$\widetilde{L}_{\varepsilon}(z,h) := \begin{cases} L_0(z,h;\bar{0}) \oplus L_0(z,h;\bar{1}) & \varepsilon = 0 \wedge h = \frac{1}{24}z, \\ L_{\varepsilon}(z,h). & \text{otherwise.} \end{cases}$$

Then, we have the following theorem:

THEOREM 5.1. — *Let us fix $(r,s) \in K_{p,q}$ satisfying $(0 < r < q, 0 < s < p)$. Take $k \in \mathbb{Z}_{\geqslant 0}$ and $j \in \{\pm k\}$.*

1. *We have the following complex:*

$$\begin{aligned} \mathcal{C} : \cdots \longrightarrow \mathcal{F}^{\varepsilon}_{\alpha_{-k-2},\beta_{-k-2}} \xrightarrow{d_{-2}} \mathcal{F}^{\varepsilon}_{\alpha_{-k-1},\beta_{-k-1}} \xrightarrow{d_{-1}} \mathcal{F}^{\varepsilon}_{\alpha_j,\beta_j} \\ \xrightarrow{d_0} \mathcal{F}^{\varepsilon}_{\alpha_{k+1},\beta_{k+1}} \xrightarrow{d_1} \mathcal{F}^{\varepsilon}_{\alpha_{k+2},\beta_{k+2}} \xrightarrow{d_2} \cdots, \end{aligned}$$

where α_l, β_l $(l \in \mathbb{Z})$ and the BRST differentials

$$d_l := S_{\varepsilon}(\mu; \Gamma_l; a_l, b_l)$$

are given by one of the following two cases:

(I) *For $l \in \mathbb{Z}$, we have*

$$\alpha_l := -lq + r, \qquad \beta_l := \begin{cases} s & l \equiv 0 \bmod 2, \\ p-s & l \equiv 1 \bmod 2. \end{cases}$$

In this case, we have

$$\mu = -\sqrt{\frac{q}{p}},$$

and the numbers a_l and b_l are given by

(i) *if* $l \in \mathbb{Z} \setminus \{-\frac{j+|j|}{2}\}$,

$$a_l := \frac{1+(-1)^{k+l}}{2}p - (-1)^{k+l}s, \quad b_l := \begin{cases} \frac{1}{2}(r-(-k+l+1)q) & l<0, \\ \frac{1}{2}(r-(k+l+1)q) & l \geqslant 0, \end{cases}$$

(ii) *if* $j=-k$,

$$a_0 := \left(k+\frac{1+(-1)^k}{2}\right)p - (-1)^k s, \quad b_0 := \frac{1}{2}(r-q),$$

(iii) *if* $j=k$,

$$a_{-1} := \left(k+\frac{1-(-1)^k}{2}\right)p + (-1)^k s, \qquad b_{-1} := \frac{1}{2}r,$$

provided that μ is an element of Ω_{a_l} for each $l \in \mathbb{Z}$.

(II) *For $l \in \mathbb{Z}$, we have*

$$\alpha_l := \begin{cases} r & l \equiv 0 \bmod 2, \\ q-r & l \equiv 1 \bmod 2, \end{cases} \qquad \beta_l := -lp+s.$$

In this case, we have

$$\mu = \sqrt{\frac{p}{q}},$$

and the numbers a_l and b_l are given by

(i) *if* $l \in \mathbb{Z} \setminus \{-\frac{j+|j|}{2}\}$,

$$a_l := \frac{1+(-1)^{k+l}}{2}q - (-1)^{k+l}r,$$
$$b_l := \begin{cases} \frac{1}{2}(s-(-k+l+1)p) & l<0, \\ \frac{1}{2}(s-(k+l+1)p) & l \geqslant 0, \end{cases}$$

(ii) *if* $j=-k$,

$$a_0 := \left(k+\frac{1+(-1)^k}{2}\right)q - (-1)^k r, \quad b_0 := \frac{1}{2}(s-p),$$

(iii) *if* $j = k$,

$$a_{-1} := \left(k + \frac{1-(-1)^k}{2}\right) q + (-1)^k r, \quad b_{-1} := \frac{1}{2} s,$$

provided that μ *is an element of* Ω_{a_l} *for each* $l \in \mathbb{Z}$.

Here, in both cases, the twisted cycles $\Gamma_l \in H_{a_l}(M_a, \mathcal{S}_\mu^\vee)$ *are so chosen that the co-boundary operators* d_l *are non-trivial morphisms.*

2. *The cohomologies of the complex* $\mathcal{C}$

$$\mathrm{H}^i(\mathcal{C}) := \mathrm{Ker}\ d_i / \mathrm{Im}\ d_{i-1}$$

can be described as follows:

(a) *For* $i = 0$, *we have*

$$\mathrm{H}^0(\mathcal{C}) \cong \begin{cases} \widetilde{L}_\varepsilon(z, h(r, s; -j)_\varepsilon) & \text{for (I)}, \\ \widetilde{L}_\varepsilon(z, h(r, s; (-1)^j j)_\varepsilon) & \text{for (II)}. \end{cases}$$

(b) *For* $i \neq 0$, *we have*

$$\mathrm{H}^i(\mathcal{C}) \cong \{0\}.$$

Remark 5.1. — As we will see in the next subsection, we do not use screening operators to construct the BRST complex. The existence of a non-trivial screening operators for any (r, s) satisfying $0 < r < q$, $0 < s < p$ is rather subtle, and it seems that there are some cases whose existence problem has not solved yet. The same situation also happens even for the ordinary Virasoro algebra.

5.2. Existence of the coboundary maps.

In this subsection, we prove the existence of a non-trivial morphism $\mathcal{C}_i \longrightarrow \mathcal{C}_{i+1}$ for all $i \in \mathbb{Z}$.

To be precise, we will prove the following proposition:

Proposition 5.1. — *Take* $k \in \mathbb{Z}_{\geqslant 0}$, *and let* $j \in \mathbb{Z}$ *be an integer satisfying* $j \in \{\pm k\}$. *We have*

$$\dim \mathrm{Hom}_{\mathrm{Vir}_\varepsilon}(\mathcal{F}^\varepsilon_{\alpha_{-k-1}, \beta_{-k-1}}, \mathcal{F}^\varepsilon_{\alpha_j, \beta_j}) = 1,$$
$$\dim \mathrm{Hom}_{\mathrm{Vir}_\varepsilon}(\mathcal{F}^\varepsilon_{\alpha_j, \beta_j}, \mathcal{F}^\varepsilon_{\alpha_{k+1}, \beta_{k+1}}) = 1.$$

Let us fix $(r, s) \in K_{p,q}$ satisfying $0 < r < q,\ 0 < s < p$. We first note that we have the following formulae:

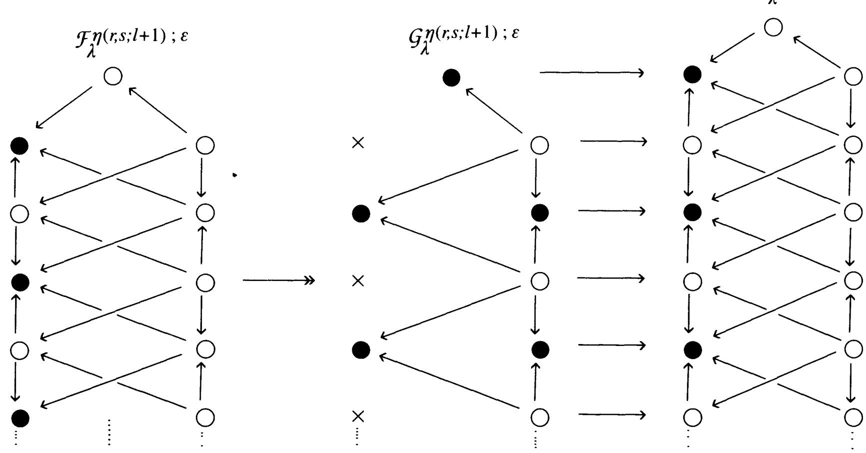
$$\lambda\left(\sqrt{\frac{p}{q}}\right) + \eta_{\alpha_l,\beta_l}\left(\sqrt{\frac{p}{q}}\right) = \begin{cases} \eta(r,s;-l) & \text{for (I)}, \\ \eta(r,s;l) & \text{for (II)} \wedge l \equiv 0 \bmod 2, \\ \eta(q-r,p-s;l) & \text{for (II)} \wedge l \equiv 1 \bmod 2. \end{cases}$$

Below, we will only study a non-trivial morphism

$$\mathcal{F}_\lambda^{\eta(r,s;l+1);\varepsilon} \longrightarrow \mathcal{F}_\lambda^{\eta(r,s;l);\varepsilon}$$

for $l \in \mathbb{Z}_{\geqslant 0}$, since the other cases can be treated by a similar manner.

For $i = l+1, l$ and $k \in \mathbb{Z}$, we denote the element of $\mathcal{F}_\lambda^{\eta(r,s;i);\varepsilon}$ introduced before Theorem 4.1 as w_k^f by $w_k^f(i)$. By Theorem 4.1, it follows that any morphism from $\mathcal{F}_\lambda^{\eta(r,s;l+1);\varepsilon}$ to $\mathcal{F}_\lambda^{\eta(r,s;l);\varepsilon}$ factors through $\mathcal{G}_\lambda^{\eta(r,s;l+1);\varepsilon}$, i.e., we have the following figure:



First, we will show that the arrows $\longrightarrow$ in the above diagram is in fact an isomorphism in the case when (r, s) belongs to Group ♠, i.e., $(r,s) \neq (\frac{q}{2}, \frac{p}{2})$. In this case, it follows from Theorem 4.1 that there is no singular vector in $\mathcal{F}_\lambda^{\eta(r,s;l);\varepsilon}$ whose L_0-weight is the same as that of $w_{-2k}^f(l+1)$ for $k \in \mathbb{Z}_{>0}$. Thus, in this case, any morphism

$$\mathcal{F}_\lambda^{\eta(r,s;l+1);\varepsilon} \longrightarrow \mathcal{F}_\lambda^{\eta(r,s;l);\varepsilon}$$

factors through $\mathcal{F}_\lambda^{\eta(r,s;l+1);\varepsilon}/\sum_{k>0} U(\mathrm{Vir}_\varepsilon).w^f_{(-1)^{k-1}k}(l+1)$. Let

$$\pi : \mathcal{F}_\lambda^{\eta(r,s;l+1);\varepsilon} \longrightarrow \mathcal{F}_\lambda^{\eta(r,s;l+1);\varepsilon}/\sum_{k>0} U(\mathrm{Vir}_\varepsilon).w^f_{(-1)^{k-1}k}(l+1)$$

be the canonical projection. For $j \in \mathbb{Z}_{\geqslant 0}$, we set

$$\mathcal{F}_j := \sum_{k=0}^{2j} U(\mathrm{Vir}_\varepsilon).w^f_{(-1)^{k-1}(k+1)}(l), \quad \mathcal{G}_j := \sum_{k=0}^{2j} U(\mathrm{Vir}_\varepsilon).\pi(w^f_{(-1)^{k-1}k}(l+1)).$$

Below, we will prove that $\mathcal{F}_j$ is isomorphic to $\mathcal{G}_j$ for each $j \in \mathbb{Z}_{\geqslant 0}$. Let V_j be one of $\mathcal{F}_j, \mathcal{G}_j$. Then, for $k \in \mathbb{Z}_{>0}$, it satisfies the following short exact sequences:

$$\begin{aligned} 0 &\longrightarrow V_{k-1} \longrightarrow V_k/L_\varepsilon(z, h(r,s;l+2k+1)_\varepsilon) \\ &\longrightarrow L_\varepsilon(z, h(r,s;-(l+2k))_\varepsilon) \longrightarrow 0, \end{aligned} \tag{9}$$

$$\begin{aligned} 0 &\longrightarrow L_\varepsilon(z, h(r,s;l+2k+1)_\varepsilon) \longrightarrow V_k \\ &\longrightarrow V_k/L_\varepsilon(z, h(r,s;l+2k+1)_\varepsilon) \longrightarrow 0. \end{aligned} \tag{10}$$

(Here and after, we omit denoting the dependency of the parity of the highest weight vectors to simplify the notations. Thus, in particular, we assume that the parity is chosen appropriately.) Notice that these exact sequences are both non-splitting. Therefore, it is enough to show that

$$\mathrm{Ext}^1_{\mathcal{C}}(L_\varepsilon(z, h(r,s;-(l+2j))_\varepsilon), V_{j-1}) \cong \mathbb{C}, \tag{11}$$

$$\mathrm{Ext}^1_{\mathcal{C}}(V_j/L_\varepsilon(z, h(r,s;l+2j+1)_\varepsilon), L_\varepsilon(z, h(r,s;l+2j+1)_\varepsilon)) \cong \mathbb{C}. \tag{12}$$

First, we will prove (11). We remark that (11) for $j = 1$ is a direct consequence of Lemma 2.3, since $V_0 = L_\varepsilon(z, h(r,s;l+1)_\varepsilon)$ by Theorem 4.1. From the long sequence of Ext associated to (10) for $k = j-1$, it is enough to show

$$\mathrm{Ext}^1_{\mathcal{C}}(L_\varepsilon(z, h(r,s;-(l+2j))_\varepsilon), V_{j-1}/L_\varepsilon(z, h(r,s;l+2j-1)_\varepsilon)) = \{0\},$$

since by definition, we have

$$\mathrm{Hom}_{\mathcal{C}}(L_\varepsilon(z, h(r,s;-(l+2j))_\varepsilon), V_{j-1}/L_\varepsilon(z, h(r,s;l+2j-1)_\varepsilon)) = \{0\}.$$

By the long sequence of Ext associated to (9) for $0<k<j$ and Lemma 2.3, we get

$$\begin{aligned}&\mathrm{Ext}^1_{\mathcal{C}}(L_\varepsilon(z,h(r,s;-(l+2j))_\varepsilon),V_k/L_\varepsilon(z,h(r,s;l+2k+1)_\varepsilon))\\ \cong&\mathrm{Ext}^1_{\mathcal{C}}(L_\varepsilon(z,h(r,s;-(l+2j))_\varepsilon),V_{k-1}).\end{aligned}\tag{13}$$

On the other hand, the long sequence of Ext associated to (10) for $0<k<j-1$ and Lemma 2.3 yields the following inclusion:

$$\begin{aligned}&\mathrm{Ext}^1_{\mathcal{C}}(L_\varepsilon(z,h(r,s;-(l+2j))_\varepsilon),V_k)\\ \hookrightarrow&\mathrm{Ext}^1_{\mathcal{C}}(L_\varepsilon(z,h(r,s;-(l+2j))_\varepsilon),V_k/L_\varepsilon(z,h(r,s;l+2k+1)_\varepsilon)).\end{aligned}\tag{14}$$

Now, (13), (14) together with Lemma 2.3 prove (11).

Second, we will prove (12). From the long sequence of Ext associated to (9) for $k=j$, it is sufficient to show

$$\mathrm{Ext}^1_{\mathcal{C}}(V_{j-1},L_\varepsilon(z,h(r,s;l+2j+1)_\varepsilon))=\{0\},$$

since by definition, we have

$$\mathrm{Hom}_{\mathcal{C}}(V_{j-1},L_\varepsilon(z,h(r,s;l+2j+1)_\varepsilon))=\{0\}.$$

By the long sequence of Ext associated to (10) for $0<k<j$ and Lemma 2.3, we have

$$\begin{aligned}&\mathrm{Ext}^1_{\mathcal{C}}(V_k/L_\varepsilon(z,h(r,s;l+2k+1)_\varepsilon),L_\varepsilon(z,h(r,s;l+2j+1)_\varepsilon))\\ \cong&\mathrm{Ext}^1_{\mathcal{C}}(V_k,L_\varepsilon(z,h(r,s;l+2j+1)_\varepsilon)).\end{aligned}\tag{15}$$

On the other hand, the long sequence of Ext associated to (9) for $0<k<j$ and Lemma 2.3 yields the following inclusion:

$$\begin{aligned}&\mathrm{Ext}^1_{\mathcal{C}}(V_k/L_\varepsilon(z,h(r,s;l+2k+1)_\varepsilon),L_\varepsilon(z,h(r,s;l+2j+1)_\varepsilon))\\ \hookrightarrow&\mathrm{Ext}^1_{\mathcal{C}}(V_{k-1},L_\varepsilon(z,h(r,s;l+2j+1)_\varepsilon)).\end{aligned}\tag{16}$$

Now, (15), (16) together with Lemma 2.3 prove (12). Thus, to prove Proposition 5.1 for Group ♠, it is sufficient to prove

$$\mathrm{End}_{\mathrm{Vir}_\varepsilon}(\xrightarrow{\lim}\mathcal{G}_j)\cong\mathbb{C}.$$

We will prove this together with Theorem 5.1 for Group ♠, and will give it in the next subsection.

Second, we will prove that the arrows $\longrightarrow$ in the above figure is a non-trivial morphism, which is unique up to a scalar, in the case when (r,s) belongs to Group $\heartsuit$, i.e., $(r,s)=(\frac{q}{2},\frac{p}{2})$. In this case, let π be the canonical projection

$$\pi:\mathcal{F}_\lambda^{\eta(r,s,l+1);0}\longrightarrow\mathcal{G}_\lambda^{\eta(r,s,l+1);0}.$$

For $j\in\mathbb{Z}_{\geqslant 0}$, we set

$$\mathcal{F}_j:=\sum_{k=0}^{j}U(\mathrm{Vir}_0).w_{2k+1}^f(l)+\sum_{-j\leqslant k\leqslant j,\,k\neq 0}U(\mathrm{Vir}_0).w_{2k}^f(l),$$

$$\mathcal{G}_j:=\sum_{k=1}^{j}U(\mathrm{Vir}_0).\pi(w_{-(2k-1)}^f(l+1))+\sum_{-j\leqslant k\leqslant j}U(\mathrm{Vir}_0).\pi(w_{2k}^f(l+1)).$$

Note that we have the following short exact sequences:

$$\begin{aligned}0\longrightarrow\mathcal{G}_0\longrightarrow\mathcal{G}_k/L_0\left(z,h\left(\frac{q}{2},\frac{p}{2};l+3\right)_0\right)^{\oplus 2}\\ \longrightarrow L_0\left(z,h\left(\frac{q}{2},\frac{p}{2};-(l+2)\right)_0\right)\longrightarrow 0,\end{aligned}\tag{17}$$

$$\begin{aligned}0\longrightarrow\mathcal{F}_0\longrightarrow\mathcal{F}_k/L_0\left(z,h\left(\frac{q}{2},\frac{p}{2};l+3\right)_0\right)\\ \longrightarrow L_0\left(z,h\left(\frac{q}{2},\frac{p}{2};-(l+2)\right)_0\right)^{\oplus 2}\longrightarrow 0.\end{aligned}\tag{18}$$

We will show that $\dim\mathrm{Hom}_{\mathrm{Vir}_0}(\mathcal{G}_j,\mathcal{F}_j)=1$ and a non-trivial morphism in $\mathrm{Hom}_{\mathrm{Vir}_0}(\mathcal{G}_j,\mathcal{F}_j)$ extends to $\mathrm{Hom}_{\mathrm{Vir}_0}(\mathcal{G}_{j+1},\mathcal{F}_{j+1})$ by induction on j.

The first step, i.e., for $j=0$, the first assertion is trivial by definition. By Lemma 2.3, (17) and (18), it follows that there exists a unique, up to scalar, non-trivial (mono-)morphism

$$\bar{f}_0:\mathcal{G}_1/L_0\left(z,h\left(\frac{q}{2},\frac{p}{2};l+3\right)_0\right)^{\oplus 2}\longrightarrow\mathcal{F}_1/L_0\left(z,h\left(\frac{q}{2},\frac{p}{2};l+3\right)_0\right).$$

Here, we denote the domain of $\bar{f}_0$ and $\mathrm{Im}\bar{f}_0$ by $\overline{\mathcal{G}}_0$ and $\overline{\mathcal{F}}_0$, respectively. By Lemma 2.3, the long sequence of Ext associated to (17) and (18) yields

$$\begin{aligned}\mathrm{Ext}_{\mathcal{C}}^1\left(\overline{\mathcal{G}}_0,L_0\left(z,h\left(\frac{q}{2},\frac{p}{2};l+3\right)_0\right)\right)\cong\mathrm{Ext}_{\mathcal{C}}^1\left(\overline{\mathcal{F}}_0,L_0\left(z,h\left(\frac{q}{2},\frac{p}{2};l+3\right)_0\right)\right)\\ \cong\mathbb{C}^2,\end{aligned}$$

which implies the second assertion for $j = 0$. Hence, we have a unique non-trivial morphism $f_1 \in \mathrm{Hom}_{\mathrm{Vir}_0}(\mathcal{G}_1, \mathcal{F}_1)$. By construction, the first statement for $j = 1$ is also trivial. Set

$$\begin{aligned}
&\widetilde{\mathcal{G}}_1 := \mathcal{G}_1/\mathrm{Ker} f_1, && \widetilde{\mathcal{F}}_1 := \mathrm{Im} f_1,\\
&\mathcal{G}(1) := \mathcal{G}_\lambda^{\eta(\frac{q}{2},\frac{p}{2};l+1);0}/\mathrm{Ker} f_1, && \pi_1 : \mathcal{G}_\lambda^{\eta(\frac{q}{2},\frac{p}{2};l+1);0} \twoheadrightarrow \mathcal{G}(1).
\end{aligned}$$

Letting $\tilde{f}_1$ be an isomorphism $\widetilde{\mathcal{G}}_1 \cong \widetilde{\mathcal{F}}_1$, it is clear that the morphism f_1 factors as

$$f_1 = \tilde{f}_1 \circ \pi_1\Big|_{\mathcal{G}_1}.$$

Since Lemma 2.3 and the long sequence of Ext associated to the short exact sequences

$$0 \longrightarrow L_0\left(z, h\left(\frac{q}{2}, \frac{p}{2}; l+3\right)_0\right) \longrightarrow \widetilde{\mathcal{G}}_1 \longrightarrow \overline{\mathcal{G}}_0 \longrightarrow 0, \tag{19}$$

$$\begin{aligned}
0 &\longrightarrow L_0\left(z, h\left(\frac{q}{2}, \frac{p}{2}; l+1\right)_0\right) \longrightarrow \overline{\mathcal{G}}_0\\
&\longrightarrow L_0\left(z, h\left(\frac{q}{2}, \frac{p}{2}; l+2\right)_0\right) \longrightarrow 0,
\end{aligned} \tag{20}$$

yield

$$\begin{aligned}
\mathrm{Ext}^1_{\mathcal{C}}\left(L_0\left(z, h\left(\frac{q}{2}, \frac{p}{2}; l+4\right)_0\right), \widetilde{\mathcal{G}}_1\right) &\cong \mathrm{Ext}^1_{\mathcal{C}}\left(L_0\left(z, h\left(\frac{q}{2}, \frac{p}{2}; l+4\right)_0\right), \widetilde{\mathcal{F}}_1\right)\\
&\cong \mathbb{C}^2,
\end{aligned}$$

the morphism $\tilde{f}_1$ extends to a unique, up to a scalar, non-trivial (mono-)morphism

$$\bar{f}_1 : \pi_1(\mathcal{G}_2)/L_0\left(z, h\left(\frac{q}{2}, \frac{p}{2}; l+5\right)_0\right)^{\oplus 2} \longrightarrow \mathcal{F}_2/L_0\left(z, h\left(\frac{q}{2}, \frac{p}{2}; l+5\right)_0\right).$$

Let us denote the domain of $\bar{f}_1$ and $\mathrm{Im}\bar{f}_1$ by $\overline{\mathcal{G}}_1$ and $\overline{\mathcal{F}}_1$ respectively. Clearly, we have the next short exact sequence

$$0 \longrightarrow \widetilde{\mathcal{G}}_1 \longrightarrow \overline{\mathcal{G}}_1 \longrightarrow L_0\left(z, h\left(\frac{q}{2}, \frac{p}{2}; l+4\right)_0\right) \longrightarrow 0. \tag{21}$$

By Lemma 2.3 and the long sequences of Ext associated to the short sequences (19), (20) and (21), it follows that

$$\operatorname{Ext}^1_{\mathcal{C}}\left(\overline{\mathcal{G}}_1, L_0\left(z, h\left(\frac{q}{2}, \frac{p}{2}; l+5\right)_0\right)\right) \cong \operatorname{Ext}^1_{\mathcal{C}}\left(\overline{\mathcal{F}}_1, L_0\left(z, h\left(\frac{q}{2}, \frac{p}{2}; l+5\right)_0\right)\right) \\ \cong \mathbb{C}^2,$$

from which the second assertion for $j = 1$ follows.

Now, one can proceed a general step of the induction by a similar argument. We will leave the detail to the reader.

5.3. Proof for Group ♠.

Before proving Theorem 5.1, let us recall an analogue of Schur's lemma:

LEMMA 5.1. — *Let M be an object of the category $\mathcal{O}$. Suppose M is indecomposable and has a Jordan-Hölder series. Then, we have*

$$\operatorname{End}_{\mathrm{Vir}_\varepsilon}(M) \cong \mathbb{C}.$$

Proof. — The proof of this lemma is a consequence of the following statement that can be proved directly:

Let K, L, N be objects of the category $\mathcal{O}$. Suppose that K is indecomposable and $\operatorname{End}_{\mathrm{Vir}_\varepsilon}(K) \cong \mathbb{C}$, and that L is irreducible. If N is a non-trivial extension satisfying the following short exact sequence

$$0 \longrightarrow K \longrightarrow N \longrightarrow L \longrightarrow 0,$$

then we have

$$\operatorname{End}_{\mathrm{Vir}_\varepsilon}(N) \cong \mathbb{C}.$$

□

Now, let us turn to the proof of Theorem 5.1.

By Theorem 4.1, it follows that for $k \in \mathbb{Z}_{>0}$ satisfying $k \leqslant j$, there exists a unique submodule of $\mathcal{G}_k$which is isomorphic to $\mathcal{G}_j$. Moreover, since we have

$$\operatorname{End}_{\mathrm{Vir}_\varepsilon}(\mathcal{G}_j) \cong \mathbb{C}$$

by Lemma 5.1, it turns out that

$$\operatorname{Hom}_{\mathrm{Vir}_\varepsilon}(\mathcal{G}_j, \varinjlim \mathcal{G}_k) \cong \mathbb{C}.$$

Hence, we get

$$\begin{aligned}&\mathrm{End}_{\mathrm{Vir}_\varepsilon}(\varinjlim \mathcal{G}_j)\\ \cong\ &\varprojlim \mathrm{Hom}_{\mathrm{Vir}_\varepsilon}(\mathcal{G}_j, \varinjlim \mathcal{G}_k) \cong \mathbb{C}.\end{aligned}$$

Now, a concrete isomorphism from $\varinjlim \mathcal{G}_j$ to $\varinjlim \mathcal{F}_j$ is given by Theorem 3.1 under the assumption in Theorem 5.1.

5.4. Proof for Group ♡.

Let us first state a variant of Proposition 2.1 for quasi-Verma modules $\widetilde{M}(z,h)$. For $(\lambda,\eta)\in D$ (see §4.2 for the definitions), let $|\lambda,\eta;\tau\rangle$ $(\tau\in\mathbb{Z}_2)$ be generators of $\widetilde{M}(z_\lambda, h_\lambda^{\eta;0})$ satisfying

$$\begin{aligned}&L_0.|\lambda,\eta;\tau\rangle = h_\lambda^{\eta;0}|\lambda,\eta;\tau\rangle, \quad c.|\lambda,\eta;\tau\rangle = z_\lambda|\lambda,\eta;\tau\rangle, \qquad \deg|\lambda,\eta;\tau\rangle=\tau,\\ &G_0.|\lambda,\eta;\tau\rangle = \frac{1}{\sqrt{2}}(\eta-\lambda)|\lambda,\eta;\bar{1}-\tau\rangle.\end{aligned}$$

We may assume the following equalities:

$$|\lambda,\eta;\bar{0}\rangle = |\lambda,2\lambda-\eta;\bar{0}\rangle = |-\lambda,-\eta;\bar{0}\rangle = |-\lambda,\eta-2\lambda;\bar{0}\rangle$$

without loss of generality.

A basis of each weight subspace of $\widetilde{M}(z_\lambda,h_\lambda^{\eta;0})$ can be described as follows. For $j\in\mathbb{Z}$, $i_j\in\mathbb{Z}_{>0}$ and $\tau,\varepsilon,\varepsilon_j\in\mathbb{Z}_2$, set

$$x_j^\varepsilon := \begin{cases} L_j & \varepsilon=\bar{0},\\ G_j & \varepsilon=\bar{1},\end{cases} \qquad m^\tau_{(i_k,\varepsilon_k),\cdots,(i_1,\varepsilon_1)} := x_{-i_k}^{\varepsilon_k}\cdots x_{-i_1}^{\varepsilon_1}.|\lambda,\eta;\tau\rangle.$$

Then, for $\sigma\in\mathbb{Z}_2$ and $n\in\mathbb{Z}_{>0}$,

$$\mathcal{B}_n^\sigma := \left\{ m^\tau_{(i_k,\varepsilon_k),\cdots,(i_1,\varepsilon_1)} \;\middle|\; \begin{array}{c} 1\leqslant i_1\leqslant\cdots\leqslant i_k,\ \varepsilon_1,\cdots,\varepsilon_k\in\mathbb{Z}_2,\ \sum_{j=1}^k i_j=n\\ \sum_{j=1}^k\varepsilon_j+\tau=\sigma,\ \varepsilon_s=\bar{1}\implies i_s<i_{s+1}\end{array}\right\}$$

form a basis of $\widetilde{M}(z_\lambda,h_\lambda^{\eta;0})_n^\sigma$. For $T\in\mathbb{C}^*$, let

$$\begin{aligned}w_{\alpha,\beta}(T) &:= X_{\alpha,\beta}|\lambda(T),\lambda(T)+\eta_{\alpha,\beta}(T);\bar{1}\rangle + Y_{\alpha,\beta}|\lambda(T),\lambda(T)+\eta_{\alpha,\beta}(T);\bar{0}\rangle\\ &\in \widetilde{M}(z(T^2),h_{\alpha,\beta;0}(T^2))\end{aligned}$$

(see (6)) be an even singular vector of level $\frac{1}{2}\alpha\beta$ which is regular in $T^{\pm1}$. Expanding $w_{\alpha,\beta}$ with respect to the basis $\mathcal{B}^{\bar{0}}_{\frac{1}{2}\alpha\beta}$, set

$$X_{\alpha,\beta} = \tilde{c}^X_{\alpha,\beta}G_{-1}L_{-1}^{\frac{1}{2}\alpha\beta-1}+\cdots,\quad Y_{\alpha,\beta}=\tilde{c}^Y_{\alpha,\beta}L_{-1}^{\frac{1}{2}\alpha\beta}+\cdots.$$

The next proposition is a corollary of Proposition 2.1:

PROPOSITION 5.2. — *Suppose that (λ, η) does not belong to Group $\heartsuit$. Then, the coefficients $\tilde{c}^X_{\alpha,\beta}, \tilde{c}^Y_{\alpha,\beta}$ satisfy*

$$\begin{cases} 2\sqrt{2}\tilde{c}^X_{\alpha,\beta} + \alpha T\tilde{c}^Y_{\alpha,\beta} = 0 & \alpha \equiv 0\ (2), \\ 2\sqrt{2}\tilde{c}^X_{\alpha,\beta} - \beta T^{-1}\tilde{c}^Y_{\alpha,\beta} = 0 & \alpha \equiv 1\ (2). \end{cases}$$

Now, let us turn to the proof of Theorem 5.1 in the case when (r, s) belongs to Group $\heartsuit$, i.e., $(r, s) = (\frac{q}{2}, \frac{p}{2})$.

First, we construct the coboundary maps that commute with higher derivatives of Γ and L.

For $\sigma \in \{\pm\}$, $a \in \mathbb{Z}_{>0}$ and $b \in \frac{1}{2}\mathbb{Z}$ satisfying $b - \frac{1}{2}a \in \frac{1}{2} + \mathbb{Z}$, let $C^\sigma_{a,b}$ be two rational curves in D defined by

$$C^\sigma_{a,b} : (\eta - \lambda)^2 - (\sigma a + b)\lambda(\eta - \lambda) + \sigma ab\lambda^2 - \frac{1}{4}(\sigma a - b)^2 = 0,$$

and $\iota^\sigma_{a,b} : \mathbb{C}^* \hookrightarrow D$ be a morphism defined by

$$\iota^\sigma_{a,b} : \mathbb{C}^* \ni T \longmapsto \left(\lambda(T), \lambda(T) + \frac{1}{2}(\sigma aT - bT^{-1})\right) \in D.$$

It can be easily checked that $\iota^\sigma_{a,b}$ induce isomorphisms

$$\iota^\sigma_{a,b} : \mathbb{C}^* \cong \mathrm{Im}\iota^\sigma_{a,b} = C^\sigma_{a,b}$$

of algebraic varieties. Let $\mathbb{M}^\sigma_{a,b}, \mathbb{F}^\sigma_{a,b}, \mathbb{M}^{c;\sigma}_{a,b}$ be vector bundles over $\mathbb{C}^*$ whose fibre at a point $T \in \mathbb{C}^*$ is given by $(\mathbb{V}^\sigma_{a,b})_T := \mathbb{V}_{\iota^\sigma_{a,b}(T)}$ $(\mathbb{V} = \mathbb{M}, \mathbb{F}, \mathbb{M}^c)$, i.e., we set

$$\mathbb{M}^\sigma_{a,b} := (\iota^\sigma_{a,b})^* \left(\mathbb{M}\Big|_{C^\sigma_{a,b}}\right), \quad \mathbb{F}^\sigma_{a,b} := (\iota^\sigma_{a,b})^* \left(\mathbb{F}\Big|_{C^\sigma_{a,b}}\right),$$
$$\mathbb{M}^{c;\sigma}_{a,b} := (\iota^\sigma_{a,b})^* \left(\mathbb{M}^c\Big|_{C^\sigma_{a,b}}\right).$$

For $l \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geqslant 0}$, we define the morphisms of bundles over $\mathbb{C}^*$,

$$d_l : \mathbb{F}^-_{a_l,b_l} \longrightarrow \mathbb{F}^+_{a_l,b_l},$$

and

$$\iota_l : \begin{cases} \mathbb{M}^-_{a_l,b_l} \longrightarrow \mathbb{M}^+_{a_l,b_l} & l < 0, \\ \mathbb{M}^+_{a_l,b_l} \longrightarrow \mathbb{M}^-_{a_l,b_l} & l \geqslant 0, \end{cases} \qquad \pi_k : \mathbb{M}^{c;-}_{a_k,b_k} \longrightarrow \mathbb{M}^{c;+}_{a_k,b_k},$$

as follows. The morphism ι_l (resp. π_k) restricted to each fibre is a non-trivial embedding (resp. the canonical projection) of Vir_0-module. For $l \in \mathbb{Z}$, the restriction of d_l to each fibre is a non-trivial morphism which satisfies the following commutative diagram:

$$\begin{array}{ccc} & l<0 & \\ \mathbb{M}^-_{a_l,b_l} & \xrightarrow{\iota_l} & \mathbb{M}^+_{a_l,b_l} \\ {\scriptstyle\Gamma}\downarrow & & \downarrow{\scriptstyle\Gamma} \\ \mathbb{F}^-_{a_l,b_l} & \xrightarrow{d_l} & \mathbb{F}^+_{a_l,b_l} \end{array}, \qquad \begin{array}{ccc} & l\geqslant 0 & \\ \mathbb{F}^-_{a_l,b_l} & \xrightarrow{d_l} & \mathbb{F}^+_{a_l,b_l} \\ {\scriptstyle\mathrm{L}}\downarrow & & \downarrow{\scriptstyle\mathrm{L}} \\ \mathbb{M}^{c;-}_{a_l,b_l} & \xrightarrow{\pi_l} & \mathbb{M}^{c;+}_{a_l,b_l} \end{array}.$$

The existence of morphisms d_l are guaranteed by the fact that Γ and L are isomorphisms at the fibre over a general point.

Remark 5.2. — The existence of such morphisms d_l at a special point is given by Proposition 5.1 and its variant, and hence the global existence of d_l is guaranteed.

Second, we show that the long sequence in Theorem 5.1 is, in fact, a complex.

Fix $\mu \in \{-\sqrt{\frac{q}{p}}, \sqrt{\frac{p}{q}}\}$. For $l \in \mathbb{Z}_{<0}$, let $\{\widetilde{M}_{l;\sigma}(n]\}_{n\in\mathbb{Z}_{>0}}$ (resp. $\{\mathcal{F}_{l;\sigma}(n]\}_{n\in\mathbb{Z}_{>0}}$) be the Jantzen filtration (resp. the Jantzen co-filtration) associated to the quadruple $(\mathbb{M}_{\iota^\sigma_{a_l,b_l}(\mu)}, \mathbb{F}_{\iota^\sigma_{a_l,b_l}(\mu)}; \Gamma; C^\sigma_{a_l,b_l})$. For $l \in \mathbb{Z}_{\geqslant 0}$, let $\{\widetilde{M}'_{l;\sigma}(n]\}_{n\in\mathbb{Z}_{>0}}$ (resp. $\{\mathcal{F}'_{l;\sigma}(n]\}_{n\in\mathbb{Z}_{>0}}$) be the Jantzen filtration (resp. the Jantzen co-filtration) associated to the quadruple $(\mathrm{Ker}\Gamma_{\iota^\sigma_{a_l,b_l}(\mu)}, \mathrm{Coker}\,\Gamma_{\iota^\sigma_{a_l,b_l}(\mu)}; \Gamma'; C^\sigma_{a_l,b_l})$, where Γ' is the first derivative of Γ in the sense of Definition-Proposition 2.1. For $n \in \mathbb{Z}_{>0}$, we set

$$\widetilde{M}_{l;\sigma}(n] := \begin{cases} \mathrm{Ker}\Gamma & n=1, \\ \widetilde{M}'_{l;\sigma}(n-1] & n>1, \end{cases} \qquad \mathcal{F}_{l;\sigma}(n] := \begin{cases} \mathrm{Coker}\,\Gamma & n=1, \\ \mathcal{F}'_{l;\sigma}(n-1] & n>1. \end{cases}$$

As an application of Theorem 4.1, it follows from the genericness of the curves $C^\sigma_{a_l,b_l}$ that the following lemma holds:

LEMMA 5.2. — *The Jantzen filtrations* $\{\widetilde{M}_{l;-}(n]\}_{n\in\mathbb{Z}_{>0}}$, $\{\widetilde{M}_{l-1;+}(n]\}_{n\in\mathbb{Z}_{>0}}$ *coincide with those obtained in Lemma 4.5. Thus, in particular, the Jantzen co-filtrations* $\{\mathcal{F}_{l;-}(n]\}_{n\in\mathbb{Z}_{>0}}$, $\{\mathcal{F}_{l-1;+}(n]\}_{n\in\mathbb{Z}_{>0}}$ *are the same.*

Here and henceforth, we use the symbols

$$\widetilde{M}_l(n] := \widetilde{M}_{l;-}(n], \quad \mathcal{F}_l(n] := \mathcal{F}_{l;-}(n] \qquad (n \in \mathbb{Z}_{>0}).$$

For $l \in \mathbb{Z}$ and $n \in \mathbb{Z}_{>0}$, let

$$\mathrm{Pr}_l^{(n]} : \mathcal{F}_l \twoheadrightarrow \mathcal{F}_l(n]$$

be the canonical projection. By construction, we have the following key-lemma:

LEMMA 5.3. — *For $n \in \mathbb{Z}_{>0}$, we have the following:*

1. $l \in \mathbb{Z}_{<0}$*:*

(i) $\iota_l(\widetilde{M}_l(n]) \subset \widetilde{M}_{l+1}(n]$.

(ii) $d_l(\mathrm{Ker}\mathrm{Pr}_l^{(n]}) \subset \mathrm{Ker}\mathrm{Pr}_{l+1}^{(n]}$. *Hence d_l induces a morphism $\mathcal{F}_l(n] \longrightarrow \mathcal{F}_{l+1}(n]$ which will be denoted by the same symbol d_l.*

(iii) *The following diagram is commutative:*

$$\begin{array}{ccc} \widetilde{M}_l(n] & \xrightarrow{\iota_l} & \widetilde{M}_{l+1}(n] \\ {\scriptstyle \Gamma^{(n]}}\downarrow & & \downarrow{\scriptstyle \Gamma^{(n]}} \\ \mathcal{F}_l(n] & \xrightarrow[d_l]{} & \mathcal{F}_{l+1}(n]. \end{array}$$

2. $l \in \mathbb{Z}_{\geqslant 0}$*:*

(i) $\widetilde{M}_l(n] \subset \iota_l(\widetilde{M}_{l+1}(n-1])$.

(ii) $d_l(\mathrm{Ker}\mathrm{Pr}_l^{(n]}) \subset \mathrm{Ker}\mathrm{Pr}_{l+1}^{(n-1]}$. *Hence d_l induces a morphism $\mathcal{F}_l(n] \longrightarrow \mathcal{F}_{l+1}(n-1]$ which will be denoted by the same symbol d_l.*

(iii) *The following diagram is commutative:*

$$\begin{array}{ccc} \widetilde{M}_l(n] & \xrightarrow{\iota_l^{-1}} & \widetilde{M}_{l+1}(n-1] \\ {\scriptstyle \Gamma^{(n]}}\downarrow & & \downarrow{\scriptstyle \Gamma^{(n-1]}} \\ \mathcal{F}_l(n] & \xrightarrow{d_l} & \mathcal{F}_{l+1}(n-1]. \end{array}$$

Thus, by this lemma, ι_l induces the following morphisms:

$$\bar{\iota}_l^{(n)} : \widetilde{M}_l(n]/\widetilde{M}_l(n+1] \longrightarrow \widetilde{M}_{l+1}(n]/\widetilde{M}_{l+1}(n+1] \qquad (l<0,\ n \geqslant 0),$$
$$(\bar{\iota}_l^{(n)})^{-1} : \widetilde{M}_l(n]/\widetilde{M}_l(n+1] \longrightarrow \widetilde{M}_{l+1}(n-1]/\widetilde{M}_{l+1}(n] \qquad (l \geqslant 0,\ n>0).$$

By Proposition 5.2, one can prove the following lemma:

LEMMA 5.4. — *For $l \in \mathbb{Z}$, we have*

1. $l < -1,\ n \geqslant 0;\quad \bar{\iota}_{l+1}^{(n)} \circ \bar{\iota}_l^{(n)} = 0.$
2. $l \geqslant 0,\ n > 1;\quad (\bar{\iota}_{l+1}^{(n-1)})^{-1} \circ (\bar{\iota}_l^{(n)})^{-1} = 0.$
3. $n > 0;\quad (\bar{\iota}_0^{(n)})^{-1} \circ \bar{\iota}_{-1}^{(n)} = 0.$

Now, we show that the long sequence in Theorem 5.1 is a complex. For $l \in \mathbb{Z}$, we prove that

$$d_{l+1} \circ d_l(\mathrm{KerPr}_l^{(n]}) = \{0\} \qquad (n \in \mathbb{Z}_{>0}) \tag{22}$$

by induction on n. The first step, i.e., $n = 1$ case is trivial by Theorem 4.1. Suppose we could prove (22) up to n. By the commutative diagrams

$$l < 0$$

$$\begin{array}{ccc}
\widetilde{M}_l(n]/\widetilde{M}_l(n+1] & \xrightarrow{\iota_l} & \widetilde{M}_{l+1}(n]/\widetilde{M}_{l+1}(n+1] \\
\Gamma^{(n]}\big\downarrow & & \big\downarrow\Gamma^{(n]} \\
\mathrm{KerPr}_l^{(n+1]}/\mathrm{KerPr}_l^{(n]} & \xrightarrow{d_l} & \mathrm{KerPr}_{l+1}^{(n+1]}/\mathrm{KerPr}_{l+1}^{(n]},
\end{array}$$

$$l \geqslant 0$$

$$\begin{array}{ccc}
\widetilde{M}_l(n]/\widetilde{M}_l(n+1] & \xrightarrow{\iota_l^{-1}} & \widetilde{M}_{l+1}(n-1]/\widetilde{M}_{l+1}(n] \\
\Gamma^{(n-1]}\big\downarrow & & \big\downarrow\Gamma^{(n-1]} \\
\mathrm{KerPr}_l^{(n+1]}/\mathrm{KerPr}_l^{(n]} & \xrightarrow{\pi_l} & \mathrm{KerPr}_{l+1}^{(n]}/\mathrm{KerPr}_{l+1}^{(n-1]},
\end{array}$$

and Lemma 5.4, it follows that

$$d_{l+1} \circ d_l|_{\mathrm{KerPr}_l^{(n+1]}/\mathrm{KerPr}_l^{(n]}} = 0$$

which implies

$$d_{l+1} \circ d_l\left(\mathrm{KerPr}_l^{(n+1]}\right) \subset \begin{cases} \mathrm{KerPr}_{l+2}^{(n]} & l < -1, \\ \mathrm{KerPr}_{l+2}^{(n-1]} & l = -1, \\ \mathrm{KerPr}_{l+2}^{(n-2]} & l \geqslant 0. \end{cases}$$

By Lemma 2.2 and Lemma 4.5, it turns out that there does not exist $(z, h) \in \mathbb{C}^2$ satisfying

$$[\mathrm{KerPr}_l^{(n+1]}/\mathrm{KerPr}_l^{(n]} : L_0(z,h)] \neq 0 \ \wedge \ [\mathrm{KerPr}_{l+2}^{(n_l]} : L_0(z,h)] \neq 0,$$

where the positive integer n_l is defined by

$$n_l := \begin{cases} n & l < -1, \\ n-1 & l = -1, \\ n-2 & l \geqslant 0. \end{cases}$$

Thus, induction hypothesis (22) up to n implies (22) for $n+1$. Therefore, we have proved that the long sequences in Theorem 5.1 are complexes.

Next, we compute the cohomology of the complex $\mathcal{C}$ in Theorem 5.1.

By Theorem 4.1, Lemma 5.3 and the definition of coboundary morphisms d_l, one can prove the next lemma:

LEMMA 5.5. — *Let $l \in \mathbb{Z}_{>0}$ be a positive integer.*

1. *The complex*

$$\cdots \to \mathrm{KerPr}_{-l}^{(1]} \xrightarrow{d_{-l}} \mathrm{KerPr}_{-l+1}^{(1]} \xrightarrow{d_{-l+1}} \cdots \to \mathrm{KerPr}_0^{(1]} \to 0$$

is quasi-isomorphic to the complex

$$\cdots \to 0 \to 0 \to \cdots \to 0 \to \tilde{L}_0\left(z, h\left(\frac{q}{2}, \frac{p}{2}; j\right)_0\right) \to 0.$$

2. *For any $n \in \mathbb{Z}_{>0}$, the following long sequence is exact:*

$$\begin{aligned} \cdots \longrightarrow \mathcal{Q}_{-l,n} \longrightarrow \mathcal{Q}_{-l+1,n} \longrightarrow \cdots \mathcal{Q}_{-1,n} \longrightarrow \mathcal{Q}_{0,n} \\ \longrightarrow \mathcal{Q}_{1,n-1} \longrightarrow \cdots \longrightarrow \mathcal{Q}_{n-1,1} \longrightarrow \mathcal{Q}_{n,0} \longrightarrow 0, \end{aligned}$$

where we set

$$\mathcal{Q}_{l,n} := \mathrm{KerPr}_l^{(n+1]}/\mathrm{KerPr}_l^{(n]}.$$

Set

$$\mathcal{K}_{l,n} := \mathrm{KerPr}_l^{(n]},$$

and consider the following commutative diagram for $n \in \mathbb{Z}_{>0}$:

$$\begin{array}{ccccccccccccccc}
\cdots \longrightarrow & \mathcal{K}_{-l,n} & \longrightarrow & \mathcal{K}_{-l+1,n} & \cdots \longrightarrow & \mathcal{K}_{0,n} & \longrightarrow & \mathcal{K}_{1,n-1} & \cdots \longrightarrow & \mathcal{K}_{n-1,1} & \longrightarrow & 0 & & \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & \mathcal{K}_{-l,n+1} & \longrightarrow & \mathcal{K}_{-l+1,n+1} & \cdots \longrightarrow & \mathcal{K}_{0,n+1} & \longrightarrow & \mathcal{K}_{1,n} & \cdots \longrightarrow & \mathcal{K}_{n-1,2} & \longrightarrow & \mathcal{K}_{n,1} & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \longrightarrow & \mathcal{Q}_{-l,n} & \longrightarrow & \mathcal{Q}_{-l+1,n} & \cdots \longrightarrow & \mathcal{Q}_{0,n} & \longrightarrow & \mathcal{Q}_{1,n-1} & \cdots \longrightarrow & \mathcal{Q}_{n-1,1} & \longrightarrow & \mathcal{Q}_{n,0} & \longrightarrow & 0.
\end{array}$$

Since the long sequence in the last row is acyclic by 2 of Lemma 5.5, it turns out that the complexes in the first row and the second row are quasi-isomorphic. Namely, for $n \in \mathbb{Z}_{>0}$, if we define the complex $\mathcal{C}^{(n)}$ by

$$\mathcal{C}^{(n)} : \cdots \to \mathcal{K}_{-l,n} \to \mathcal{K}_{-l+1,n} \to \cdots \to \mathcal{K}_{0,n} \to \mathcal{K}_{1,n-1} \to \cdots \to \mathcal{K}_{n-1,1} \to 0,$$

then $\mathcal{C}^{(n)}$ and $\mathcal{C}^{(n+1)}$ are quasi-isomorphic. Moreover, since we have

$$\varinjlim \mathcal{C}^{(n)} = \mathcal{C},$$

Part 2 of Theorem 5.1 follows from 1 of Lemma 5.5.

A. Data.

In this section, we provide some numerical data used in the main body of this article.

A.1. Lattice points (α_k, β_k) on the line $l^\sigma_{r,s;i}$.

In this subsection, we will supplement some data used in §4.3. In particular, the data for Class R^+ will be provided.

Let p, q be positive integers satisfying $p - q \in 2\mathbb{Z}$ and $(\frac{p-q}{2}, q) = 1$. Fix

$$\lambda = \lambda\left(\sqrt{\frac{p}{q}}\right), \quad \lambda_+ := \sqrt{\frac{p}{q}}, \quad \lambda_- := -\sqrt{\frac{q}{p}}.$$

For each $\sigma \in \{\pm\}$, we will arrange the lattice points $\{(\alpha_k, \beta_k)\}_{k \in \mathbb{Z}_{>0}}$ of the line $l^\sigma_{r,s;i}$ in the first quadrant of the (α, β)-plane so that they satisfy

$$0 < \alpha_1\beta_1 < \alpha_2\beta_2 < \cdots.$$

Then, one can see that the following relation always holds:

$$(\alpha_k, \beta_k) = (\alpha_1, \beta_1) + (k-1)(q,p).$$

Thus, we will list (α_1, β_1) for each $\sigma \in \{\pm\}$. Moreover, we will also list $j \in \mathbb{Z}$ satisfying

$$h_\lambda^{\eta(r,s;i);\varepsilon} + \frac{1}{2}\alpha_k\beta_k = h_\lambda^{\eta(r,s;j);\varepsilon}.$$

Group ♠ & ♡ $\quad (r,s) \in \left\{ (a,b) \in \mathbb{Z}^2 \,\middle|\, \begin{matrix} 0 < a < q \\ 0 < b < p \end{matrix} \right\}, \quad (i \in \mathbb{Z})$:

$$j = \sigma(|i| + 2k - 1),$$

1. For $\sigma = +$;

$$(\alpha_1, \beta_1) = \begin{cases} (iq + r, s) & i \equiv 0\ (2)\ \wedge\ i \geqslant 0, \\ (r, -ip + s) & i \equiv 0\ (2)\ \wedge\ i < 0, \\ (iq + r, p - s) & i \equiv 1\ (2)\ \wedge\ i > 0, \\ (r, -ip + p - s) & i \equiv 1\ (2)\ \wedge\ i < 0. \end{cases}$$

2. For $\sigma = -$;

$$(\alpha_1, \beta_1) = \begin{cases} (q - r, ip + p - s) & i \equiv 0\ (2)\ \wedge\ i \geqslant 0, \\ (-iq + q - r, p - s) & i \equiv 0\ (2)\ \wedge\ i < 0, \\ (q - r, ip + s) & i \equiv 1\ (2)\ \wedge\ i > 0, \\ (-iq + q - r, s) & i \equiv 1\ (2)\ \wedge\ i < 0. \end{cases}$$

Group ♣ $\quad (i \in 2\mathbb{Z})$:

1. $r \equiv 0\ (q), \quad \sigma = +$;

$$(\alpha_1, \beta_1) = \begin{cases} (iq + r, s) & i > 0, \\ (q, -ip + s + \frac{p}{q}(q - r)) & i \leqslant 0, \end{cases}$$

$$j = -\left| i - \left(1 - \frac{r}{q}\right) \right| - \left(2k - \left(1 - \frac{r}{q}\right)\right).$$

2. $r \equiv 0\ (q), \quad \sigma = -;$

$$(\alpha_1, \beta_1) = \begin{cases} (q, (i+1)p - s + \frac{p}{q} r) & i \geqslant 0, \\ (-iq + q - r, p - s) & i < 0, \end{cases}$$

$$j = \left| i + \frac{r}{q} \right| + \left(2k - \frac{r}{q} \right).$$

3. $s \equiv 0\ (p), \quad \sigma = +;$

$$(\alpha_1, \beta_1) = \begin{cases} (iq + r + \frac{q}{p}(p - s), p) & i \geqslant 0, \\ (r, -ip + s) & i < 0, \end{cases}$$

$$j = \left| i + \left(1 - \frac{s}{p} \right) \right| + \left(2k - \left(1 - \frac{s}{p} \right) \right).$$

4. $s \equiv 0\ (p), \quad \sigma = -;$

$$(\alpha_1, \beta_1) = \begin{cases} (q - r, ip + p - s) & i > 0, \\ (-iq + q - r + \frac{q}{p} s, p) & i \leqslant 0, \end{cases}$$

$$j = -\left| i - \frac{s}{p} \right| - \left(2k - \frac{s}{p} \right).$$

Group $\diamondsuit \quad (i \in 2\mathbb{Z})$:

$$i \neq 0: \quad j = i + 2(\operatorname{sgn} i)k,$$

$$i = 0: \quad j = \begin{cases} \pm 2k & rp - sq = 0, \\ 2(\operatorname{sgn}(rp - sq))k & rp - sq \neq 0, \end{cases}$$

1. For $\sigma = +;$

$$(\alpha_1, \beta_1) = \begin{cases} ((i+1)q + \frac{1}{p}(rp - sq), p) & i > 0, \\ (q + \frac{|rp-sq|}{pq} r, p + \frac{|rp-sq|}{pq} s) & i = 0, \\ (q, -(i-1)p - \frac{1}{q}(rp - sq)) & i < 0. \end{cases}$$

2. For $\sigma = -;$

$$(\alpha_1, \beta_1) = \begin{cases} (q, (i+1)p + \frac{1}{q}(rp - sq)) & i > 0, \\ (q + \frac{|rp-sq|}{pq}(q - r), p + \frac{|rp-sq|}{pq}(p - s)) & i = 0, \\ (-(i-1)q - \frac{1}{p}(rp - sq), p) & i < 0. \end{cases}$$

A.2. Embedding pattern.

In this subsection, we will provide the data to prove the existence of the commutative diagram (Figure 1) in §2.2.

To be precise, we have to specify the value $p, q \in \mathbb{Z}_{>1}$ and $r \in \mathbb{Z}_{>0}$ (resp. $s \in \mathbb{Z}_{>0}$) to use the Embedding diagram in [IK2]. Thus, for each $(\alpha, \beta) \in (\mathbb{Z}_{>0})^2$, we will present the following data:

1. The case where $(z(t), h_{\alpha,\beta;0}(t))$ belongs to.

2. The values p, q (t is related to these number by $t = \frac{p}{q}$) and r (resp. s).

3. The commutative diagram together with the pairs $(\alpha', \beta'), (\alpha'', \beta'')$.

Here, we symbolize the commutative diagram (Figure 1) as

$$h_{i;0} \longleftarrow (\alpha', \beta') - h_{j;0} \longleftarrow (\alpha'', \beta'') - h_{k;0},$$

where $i, j, k \in \mathbb{Z}$ satisfy

$$h_{i;0} = h_{\alpha,\beta}(t), \quad h_{j;0} = h_{\alpha',\beta'}(t) + \frac{1}{2}\alpha'\beta', \quad h_{k;0} = h_{\alpha'',\beta''}(t) + \frac{1}{2}\alpha''\beta''.$$

Let $\alpha, \beta \in \mathbb{Z}_{>0}$ be positive integers satisfying $\alpha - \beta \in -1 + 2\mathbb{Z}_{>0}$.

1. $\beta \equiv 1\ (2)\ \wedge\ \beta \geqslant 5\ \wedge\ \alpha \neq \beta + 1$:
   (i) Case 3^+.
   (ii) $p = \beta,\ q = \beta + 2^k,\ r = 2\beta + 2^{k+1} - \alpha$,
   $(k \in \mathbb{Z}_{>0} \quad \text{s.t.} \quad \beta + 2^k < \alpha < 2(\beta + 2^k))$.
   (iii)

$$h_{0;0} \longleftarrow (2\beta + 2^{k+1} - \alpha, \beta) - h_{1;0} \longleftarrow (\alpha - \beta - 2^k, 2\beta) - h_{-2;0}.$$

2. $\beta \equiv 0\ (2)\ \wedge\ \beta \geqslant 4\ \wedge\ \alpha \neq \beta + 1$:
   (i) Case 3^+.
   (ii) $p = 2,\ q = 2^k,\ r = (\frac{1}{2}\beta + 1)q - \alpha$,
   $(k \in \mathbb{Z}_{>1} \quad \text{s.t.} \quad 2^{k-1}(\frac{1}{2}\beta + 1) < \alpha < 2^k(\frac{1}{2}\beta + 1))$.
   (iii)

$$h_{0;0} \longleftarrow \left(2^k\left(\frac{1}{2}\beta + 1\right) - \alpha, \beta\right) - h_{1;0} \longleftarrow (\alpha - 2^k, \beta + 2) - h_{-\beta;0}.$$

3. $\alpha = \beta + 1 \ \wedge \ \beta \geqslant 6$:
   (i) Case 3^+.
   (ii) $p = \beta, \ q = \beta - 2, \ r = \beta - 5$.
   (iii)

$$h_{0;0} \longleftarrow (\beta - 5, \beta) \longrightarrow h_{1;0} \longleftarrow (3, 2\beta) \longrightarrow h_{-2;0}.$$

4. $(\alpha, \beta) = (6, 5)$:
   (i) Case 2^+.
   (ii) $p = 4, \ q = 6, \ s = 3$.
   (iii)

$$h_{0;0} \longleftarrow (6, 3) \longrightarrow h_{1;0} \longleftarrow (12, 1) \longrightarrow h_{2;0}.$$

5. $(\alpha, \beta) = (5, 4)$:
   (i) Case 2^+.
   (ii) $p = 3, \ q = 5, \ s = 2$.
   (iii)

$$h_{0;0} \longleftarrow (5, 2) \longrightarrow h_{1;0} \longleftarrow (10, 1) \longrightarrow h_{2;0}.$$

6. $\alpha \equiv 0 \ (4) \ \wedge \ \beta = 3$:
   (i) Case 2^+.
   (ii) $p = 2, \ q = \alpha, \ s = 1$.
   (iii)

$$h_{0;0} \longleftarrow (\alpha, 1) \longrightarrow h_{1;0} \longleftarrow (2\alpha, 1) \longrightarrow h_{2;0}.$$

7. $\alpha \equiv 2 \ (4) \ \wedge \ \beta = 3$:
   (i) Case 3^+.
   (ii)

$$p = 3, \quad q = \frac{1}{2}(r + \alpha), \qquad r = \begin{cases} 8 & \alpha \equiv 2 \ (12), \\ 4 & \alpha \equiv 6, 10 \ (12). \end{cases}$$

   (iii)

$$h_{0;0} \longleftarrow (r, 3) \longrightarrow h_{1;0} \longleftarrow \left(\frac{\alpha - r}{2}, 6\right) \longrightarrow h_{-2;0}.$$

8. $\alpha \geqslant 5 \ \wedge \ \beta = 2$:

(i) Case 3^+.

(ii) $p = 2,\ q = \frac{1}{2}(r + \alpha), \quad r \in \{1,3,5,7\}$ s.t. $r \equiv -\alpha\ (8)$.

(iii)

$$h_{0;0} \longleftarrow (r,2) \longrightarrow h_{1;0} \longleftarrow \left(\frac{\alpha - r}{2}, 4\right) \longrightarrow h_{-2;0}.$$

9. $\alpha \geqslant 8 \ \wedge \ \beta = 1$:

(i) Case 3^+.

(ii) $p = 1,\ q = \frac{1}{2}(r + \alpha), \quad r \in \{2,4\}$ s.t. $r \equiv -\alpha + 2\ (4)$.

(iii)

$$h_{0;0} \longleftarrow (r,1) \longrightarrow h_{1;0} \longleftarrow \left(\frac{\alpha - r}{2}, 2\right) \longrightarrow h_{-2;0}.$$

10. $(\alpha, \beta) = (3,2)$:

(i) Case 4.2^+.

(ii) $p = q = 1$.

(iii)

$$h_{0;0} \longleftarrow (2,1) \longrightarrow h_{2;0} \longleftarrow (4,1) \longrightarrow h_{4;0}.$$

11. $(\alpha, \beta) = (6,1)$:

(i) Case 3^+.

(ii) $p = 1,\ q = 5,\ r = 4$.

(iii)

$$h_{0;0} \longleftarrow (4,1) \longrightarrow h_{1;0} \longleftarrow (1,2) \longrightarrow h_{-2;0}.$$

12. $(\alpha, \beta) = (4,1)$:

(i) Case 3^+.

(ii) $p = 1,\ q = 3,\ r = 2$.

(iii)

$$h_{0;0} \longleftarrow (2,1) \longrightarrow h_{1;0} \longleftarrow (1,2) \longrightarrow h_{-2;0}.$$

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