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FUNCTIONAL MODELS AND ASYMPTOTICALLY ORTHONORMAL SEQUENCES

by I. CHALENDAR, E. FRICAIN & D. TIMOTIN

1. Introduction.

A canonical orthonormal basis in the Hilbert space $L^2(-\pi,\pi)$ is formed by the exponentials $\exp int$, $n \in \mathbb{Z}$. Starting with the works of Paley-Wiener ([12] and Levinson ([8]), a whole direction of research has investigated other families of exponentials, looking for properties as completeness, minimality, or being an unconditional basis. In this context, functional models have been used in [7], together with some other tools from operator theory on a Hilbert space. The model spaces are subspaces of the Hardy space H^2, invariant under the adjoints of multiplications; their theory is connected to dilation theory for contractions on Hilbert spaces (see [14], [9]). The approach has been proved fruitful; it has allowed the recapture of all the classical results and has lead to many generalizations.

We are interested in investigating, along the line of research from [7], the case when the basis is asymptotically close to an orthogonal one (see definition below). This is a particular case of unconditional basis, where more rigidity is required, but the conclusions obtained are usually more precise. A basic result appears in Volberg's paper [15], where it is shown that the usual Carleson condition for an interpolation set can be adapted to obtain a characterization of asymptotically orthonormal sequences of reproducing kernels; further developments can be found in [3]. We intend to

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provide a comprehensive treatment of this subject, emphasizing the parallel with unconditional bases.

The plan of the paper is the following. The next two sections contain preliminary material. The case of reproducing kernels for the whole Hardy space is treated in Section 4; we give an equivalent form of Volberg's condition and prove some related results. Section 5 investigates the relevance of Volberg's condition for model spaces; the main results are Theorem 5.2 and Corollary 5.6, which allow the characterization of asymptotically orthonormal sequences of reproducing kernels. Perturbation results are obtained in Section 6. In the last two sections we discuss the important case of exponentials, as well as some other examples.

2. Preliminaries.

For most of the definitions and facts below, one can use [9] as a main reference.

Let $\mathcal{H}$ be a complex Hilbert space. A sequence $(x_n)_{n\geqslant 1} \subset \mathcal{H}$ is called:

- *complete* if $\operatorname{Span}\{x_n : n \geqslant 1\} = \mathcal{H}$;
- *minimal* if for all $n \geqslant 1$, $x_n \notin \operatorname{Span}\{x_m : m \neq n\}$;
- *Riesz* if there are positive constants c, C verifying, for all finite complex sequences $(a_n)_{n\geqslant 1}$,

$$c\sum_{n\geqslant 1}|a_n|^2 \leqslant \Big\|\sum_{n\geqslant 1} a_n x_n\Big\|^2 \leqslant C\sum_{n\geqslant 1}|a_n|^2. \tag{2.1}$$

A Riesz sequence is minimal, but the converse is in general not true.

The *Gram matrix* of the sequence $(x_n)_{n\geqslant 1}$ is $\Gamma = (\langle x_n, x_m\rangle)_{n,m\geqslant 1}$. Riesz sequences are characterized by the fact that Γ defines an invertible operator on ℓ^2.

The basic Hilbert space in which our objects live is the Hardy space H^2 of the open unit disc $\mathbb{D}$; this is the Hilbert space of analytic functions $f(z) = \sum_{n\geqslant 0} a_n z^n$ defined in the unit disc $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, such that $\sum_{n\geqslant 0}|a_n|^2 < \infty$. Alternately, it can be identified with a closed subspace of the Lebesgue space $L^2(\mathbb{T})$ on the unit circle, by associating to each analytic function its radial limit. The algebra of bounded analytic functions on $\mathbb{D}$ is denoted by H^∞. Any $\phi \in H^\infty$ acts as a multiplication operator on H^2, that we will denote by T_ϕ.

Evaluations at points $\lambda \in \mathbb{D}$ are bounded functionals on H^2 and the corresponding reproducing kernel is $k_\lambda(z) = \frac{1}{1-\bar{\lambda}z}$; thus, $f(\lambda) = \langle f, k_\lambda \rangle$. If $\phi \in H^\infty$, then k_λ is an eigenvector for T_ϕ^*, and $T_\phi^* k_\lambda = \overline{\phi(\lambda)} k_\lambda$. By normalizing k_λ we obtain $h_\lambda = \frac{k_\lambda}{\|k_\lambda\|} = \sqrt{1-|\lambda|^2} k_\lambda$.

Suppose now Θ is an inner function. We define the corresponding *model space* by the formula $K_\Theta = H^2 \ominus \Theta H^2$; the orthogonal projection onto K_Θ is denoted by P_Θ. In K_Θ the reproducing kernel for a point $\lambda \in \mathbb{D}$ is the function

$$k_\lambda^\Theta(z) = \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{1 - \bar{\lambda}z} \tag{2.2}$$

and the normalized reproducing kernel

$$h_\lambda^\Theta(z) = \sqrt{\frac{1-|\lambda|^2}{1-|\Theta(\lambda)|^2}}\, k_\lambda^\Theta(z). \tag{2.3}$$

Note that, according to (2.2), we have the orthogonal decomposition

$$k_\lambda = k_\lambda^\Theta + \Theta\overline{\Theta(\lambda)} k_\lambda. \tag{2.4}$$

Suppose $\Lambda = (\lambda_n)_{n\geqslant 1}$ is a *Blaschke sequence* of distinct points in $\mathbb{D}$ (which means that $\sum_{n\geqslant 1} 1 - |\lambda_n| < \infty$). As usual, we denote by $B = B_\Lambda = \prod_{n\geqslant 1} b_{\lambda_n}$ the associated Blaschke product, and $B_n = B/b_{\lambda_n}$; here $b_{\lambda_n}(z) = \frac{|\lambda_n|}{\lambda_n} \frac{\lambda_n - z}{1-\overline{\lambda_n}z}$. As B is an inner function, we may consider the model space K_B; it is well known that $(h_{\lambda_n})_{n\geqslant 1}$ is a complete minimal system in K_B. It is a Riesz basis if and only if it satisfies the Carleson condition

$$\delta(\Lambda) = \inf_{n\geqslant 1} |B_n(\lambda_n)| > 0;$$

we will write in this case $\Lambda \in (C)$ and say that Λ is a *Carleson sequence*. Also, the sequence Λ is called *separated* if $\inf_{n\neq m} |b_{\lambda_n}(\lambda_m)| > 0$.

In connection with Blaschke products, we will have the opportunity to use the following two formulas. If $\lambda, \mu \in \mathbb{D}$, then

$$\left|\frac{\lambda - \mu}{1-\bar{\lambda}\mu}\right|^2 = 1 - \frac{(1-|\lambda|^2)(1-|\mu|^2)}{|1-\bar{\lambda}\mu|^2}; \tag{2.5}$$

the denominator in the right hand side is given by

$$|1-\bar{\lambda}\mu|^2 = (1-|\lambda\mu|)^2 + 4|\lambda\mu|\sin^2\frac{\theta}{2}, \tag{2.6}$$

where $\theta \in (-\pi, \pi]$ is the argument of $\bar{\lambda}\mu$.

The following two lemmas are proved in [7], II.

LEMMA 2.1. — *If $(h_{\lambda_n}^{\Theta})_{n\geqslant 1}$ is a minimal but not complete sequence in K_Θ, then, for all $\mu \in \mathbb{D}\setminus\Lambda$, $\{h_\mu^\Theta, h_{\lambda_n}^\Theta : n \geqslant 1\}$ is still a minimal sequence.*

LEMMA 2.2. — *If Θ_1, Θ_2 are two inner functions, then* $\operatorname{dist}(\bar{\Theta}_1\Theta_2, H^\infty) = \|P_{\Theta_1}T_{\Theta_2}|K_{\Theta_1}\|$, *and this quantity is strictly smaller than 1 if and only if $P_{\Theta_2}|K_{\Theta_1}$ is an isomorphism onto its image.*

We end the preliminaries with a lemma pertaining to Riesz sequences of normalized reproducing kernels.

LEMMA 2.3. — *Let $(\lambda_n)_{n\geqslant 1}$ be a Blaschke sequence of distinct points in $\mathbb{D}$, B the corresponding Blaschke product, and Θ an inner function. Suppose that $(h_{\lambda_n}^\Theta)_{n\geqslant 1}$ is a Riesz sequence in K_Θ, and denote by c, C the corresponding constants appearing in (2.1). Then*

$$\|P_B T_\Theta | K_B\| \leqslant \sqrt{\frac{C}{c}} \sup_{n\geqslant 1} |\Theta(\lambda_n)|.$$

Proof. — The subspace K_B is spanned by the eigenvectors h_{λ_n}, $n \geqslant 1$ of T_Θ^*, and $T_\Theta^* h_{\lambda_n} = \overline{\Theta(\lambda_n)} h_{\lambda_n}$. Take a sum (with a finite number of nonzero terms) $\sum_{n\geqslant 1} a_n h_{\lambda_n}$; we have

$$\Big\|\sum_{n\geqslant 1} a_n h_{\lambda_n}\Big\|^2 \geqslant c \sum_{n\geqslant 1} |a_n|^2$$

and

$$\begin{aligned}\Big\|T_\Theta^*\Big(\sum_{n\geqslant 1} a_n h_{\lambda_n}\Big)\Big\|^2 &= \Big\|\sum_{n\geqslant 1} \overline{\Theta(\lambda_n)} a_n h_{\lambda_n}\Big\|^2 \leqslant C \sum_{n\geqslant 1} |\Theta(\lambda_n)|^2 |a_n|^2 \\ &\leqslant C(\sup_{n\geqslant 1} |\Theta(\lambda_n)|)^2 \sum_{n\geqslant 1} |a_n|^2,\end{aligned}$$

whence

$$\Big\|T_\Theta^*\Big(\sum_{n\geqslant 1} a_n h_{\lambda_n}\Big)\Big\|^2 \leqslant \frac{(\sup_{n\geqslant 1} |\Theta(\lambda_n)|)^2 C}{c} \Big\|\sum_{n\geqslant 1} a_n h_{\lambda_n}\Big\|^2.$$

Since $(P_B T_\Theta | K_B)^* = T_\Theta^* | K_B$, the lemma is proved. □

3. Asymptotically orthonormal bases.

We will say that $(x_n)_{n\geqslant 1}$ is an *asymptotically orthonormal sequence* in $\mathcal{H}$ (abbreviated AOS) if there exists $N_0 \in \mathbb{N}$, such that for all $N \geqslant N_0$, there are constants $c_N, C_N > 0$ verifying

$$c_N \sum_{n\geqslant N} |a_n|^2 \leqslant \Big\|\sum_{n\geqslant N} a_n x_n\Big\|^2 \leqslant C_N \sum_{n\geqslant N} |a_n|^2, \tag{3.1}$$

and $\lim_{N\to\infty} c_N = 1 = \lim_{N\to\infty} C_N$.

If $N_0 = 1$, then one says that $(x_n)_{n\geqslant 1}$ is an *asymptotically orthonormal basic sequence* (abbreviated AOB). Obviously this is equivalent to $(x_n)_{n\geqslant 1}$ being an AOS as well as a Riesz sequence.

The following simple lemma is a basic tool.

LEMMA 3.1. — *If $(x_n)_{n\geqslant 1} \subset \mathcal{H}$, then $(x_n)_{n\geqslant 1}$ is an AOB if and only if it is minimal and an AOS.*

Proof. — If $(x_n)_{n\geqslant 1}$ is an AOB, then it is a Riesz sequence, and therefore minimal. Conversely, if $(x_n)_{n\geqslant 1}$ is an AOS, then $(x_n)_{n\geqslant N_0}$ is a Riesz sequence for some N_0. Now minimality ensures that we can add the first finite number of vectors and still preserve this property. □

As in the case of Riesz sequences, several equivalent characterizations are available for AOB's, as shown in the next proposition ([3], Section 3).

PROPOSITION 3.2. — *Let $(x_n)_{n\geqslant 1}$ be a sequence in $\mathcal{H}$. The following are equivalent:*

(i) *$(x_n)_{n\geqslant 1}$ is an AOB;*

(ii) *there exist a separable Hilbert space $\mathcal{K}$, an orthonormal basis $(e_n)_{n\geqslant 1} \subset \mathcal{K}$ and $U, K : \mathcal{K} \to \mathcal{H}$, U unitary, K compact, $U + K$ left invertible, such that $(U+K)(e_n) = x_n$;*

(iii) *the Gram matrix Γ associated to $(x_n)_{n\geqslant 1}$ defines a bounded invertible operator of the form $I + K$, with K compact.*

One can obtain complete AOB's by slightly perturbing orthonormal bases; this fact is made precise in the following lemma.

LEMMA 3.3. — *Let $\mathcal{H}$ be a Hilbert space, $(x_n)_{n\geqslant 1}$ an orthonormal basis in $\mathcal{H}$, and $(x'_n)_{n\geqslant 1}$ a sequence in $\mathcal{H}$, such that $\sum_{n\geqslant 1} \|x_n - x'_n\|^2 < 1$. Then $(x'_n)_{n\geqslant 1}$ is a complete AOB in $\mathcal{H}$.*

Proof. — Consider the operator $\Phi : \mathcal{H} \to \mathcal{H}$, defined by $\Phi(x_n) = x'_n$. The condition in the statement implies that $I_{\mathcal{H}} - \Phi$ is Hilbert-Schmidt, of norm strictly smaller than 1. Thus Φ is of the form unitary plus compact and invertible; from Proposition 3.2 it follows that $(x'_n)_{n\geqslant 1}$ is an AOB. On the other hand, since Φ is invertible, $(x_n)_{n\geqslant 1}$ complete implies $(x'_n)_{n\geqslant 1}$ complete. □

We should note the following way to obtain AOB's.

LEMMA 3.4. — *Let $(x_n)_{n\geqslant 1}$ be a normalized sequence in $\mathcal{H}$, tending weakly to 0. There exists a subsequence $(x_{k_n})_{n\geqslant 1}$ which is an AOB.*

Proof. — Choose recursively the sequence (x_{k_n}) by requiring that $\sum_{m<n} |\langle x_{k_n}, x_{k_m}\rangle|^2 \leqslant \frac{1}{2^{n+2}}$. Then $\sum_{m\neq n} |\langle x_{k_n}, x_{k_m}\rangle|^2 \leqslant 1/2$, whence, if Γ' is the Gram matrix associated to (x_{k_n}), then $\Gamma' - I$ has Hilbert-Schmidt norm smaller than 1/2. Applying Proposition 3.2 to Γ' implies that $(x_{k_n})_{n\geqslant 1}$ is an AOB. □

In particular, a Riesz sequence tends weakly to 0, and thus it contains AOB's as subsequences.

4. Reproducing kernels and AOB's.

Suppose $\Lambda = (\lambda_n)_{n\geqslant 1}$ is a Blaschke sequence of distinct points in $\mathbb{D}$. Since the reproducing kernels $(k_{\lambda_n})_{n\geqslant 1}$ are complete and minimal in K_B, if $(h_{\lambda_n})_{n\geqslant 1}$ is an AOS, then it is also a complete AOB in K_B. Such sequences are characterized by the following theorem of Volberg ([15]).

THEOREM A. — *The sequence $(h_{\lambda_n})_{n\geqslant 1}$ is a complete AOB in K_B if and only if*

$$\lim_{n\to\infty} |B_n(\lambda_n)| = 1. \tag{4.1}$$

Blaschke sequences that satisfy (4.1) have already appeared in literature (see, for instance, [5], [4], [11], [13]). In particular, it follows from results in [13] that, among Carleson sequences, they are characterized by the possibility of free interpolation with functions in $H^\infty \cap VMO$. We will adopt the terminology in [13] and call a sequence $(\lambda_n)_{n\geqslant 1}$ that satisfies (4.1) a *thin interpolating sequence* (or just *thin sequence*); we will write $(\lambda_n) \in (\vartheta)$. Thus thin interpolating sequences correspond to AOS of normalized reproducing kernels.

A different characterization can be stated by using the Gram matrix.

PROPOSITION 4.1. — *If $\Lambda = (\lambda_n)_{n\geqslant 1}$ is a Blaschke sequence of distinct points in $\mathbb{D}$, then the following are equivalent:*

(i) *$(h_{\lambda_n})_{n\geqslant 1}$ is a complete AOB in K_B;*

(ii) $(\Gamma - I)e_n \to 0$.

Proof. — (i)$\Rightarrow$(ii). By Proposition 3.2, (iii), it follows that $\Gamma = I+K$ with K compact; since $Ke_n \to 0$, $(\Gamma - I)e_n \to 0$.

(ii)$\Rightarrow$(i). By hypothesis $(\Gamma - I)e_n \to 0$. But

$$\|(\Gamma - I)e_n\|^2 = \sum_{p\neq n, p\geqslant 1} |\Gamma_{n,p}|^2 = \sum_{p\neq n, p\geqslant 1} \frac{(1-|\lambda_n|^2)(1-|\lambda_p|^2)}{|1-\bar{\lambda}_n\lambda_p|^2} = \sum_{p\neq n, p\geqslant 1} (1-|b_{\lambda_p}(\lambda_n)|^2) \tag{4.2}$$

(we have used 2.5)).

In particular, there is some N such that for $n \geqslant N$ we have $\|(\Gamma - I)e_n\|^2 < 1/2$, and therefore $|b_{\lambda_p}(\lambda_n)|^2 > 1/2$ if p or n are larger than N. Since the points λ_n are distinct, the whole sequence Λ is separated, and there exists $\varepsilon > 0$, such that $|b_{\lambda_p}(\lambda_n)| \geqslant \varepsilon$ for all $n \neq p$. Therefore $1 - |b_{\lambda_p}(\lambda_n)|^2 \geqslant -c\log|b_{\lambda_p}(\lambda_n)| \geqslant 0$ for some $c > 0$. It follows that

$$\|(\Gamma - I)e_n\|^2 \geqslant -c\log|B_n(\lambda_n)|$$

whence $|B_n(\lambda_n)| \to 1$; by Theorem A it follows that $(h_{\lambda_n})_{n\geqslant 1}$ is a complete AOB in K_B. □

It is well known that Carleson sequences $\Lambda = (\lambda_n)_{n\geqslant 1}$ can also be characterized by the fact that they are separated and the measure $\sum(1-|\lambda_n|)\delta_{\lambda_n}$ is a Carleson measure. A similar characterization can be obtained for thin interpolating sequences, as suggested by Lemma 7.1 in [13]. We need some notations: for any $z \in \mathbb{D}$, I_z will be the interval $I_z \subset \mathbb{T}$ of center $\frac{z}{|z|}$ and length $1-|z|$. For $I \subset \mathbb{T}$, $S_I = \{z \in \mathbb{D} : z/|z| \in I \text{ and } |z| \geqslant 1-|I|\}$; while, for $C > 0$, CI is the interval with the same center and length $C|I|$.

PROPOSITION 4.2. — *Suppose $\Lambda = (\lambda_n)_{n\geqslant 1}$ is a Blaschke sequence. The following are equivalent:*

(i) *Λ is a thin interpolating sequence;*

(ii) *for any $A \geqslant 1$,*

$$\lim_{n\to\infty} \frac{1}{|I_{\lambda_n}|} \sum_{\substack{p\neq n \\ \lambda_p \in S_{AI_{\lambda_n}}}} (1-|\lambda_p|) = 0. \tag{4.3}$$

Proof. — (i)$\Rightarrow$(ii). Fix $A \geqslant 1$; from (2.6) it follows easily that there is some constant $a > 0$ such that, if $z \in S_{AI_{\lambda_n}}$, then $|1-\bar{\lambda}_n z|^2 \leqslant a(1-|\lambda_n|)^2$.

Therefore, if $\lambda_p \in S_{AI_{\lambda_n}}$, then $\frac{(1-|\lambda_n|^2)(1-|\lambda_p|^2)}{|1-\bar\lambda_n\lambda_p|^2} \geqslant \frac{1-|\lambda_p|}{a(1-|\lambda_n|)}$. Consequently, (4.3) follows from Proposition 4.1, (ii).

(ii)$\Rightarrow$(i). We show first that $\sum(1-|\lambda_n|)\delta_{\lambda_n}$ is a Carleson measure. From (4.3) it follows that we may suppose (by deleting a finite number of terms, if necessary) that for all λ_n we have

$$\sum_{\substack{p\neq n\\ \lambda_p\in S_{5I_{\lambda_n}}}} (1-|\lambda_p|) \leqslant |I_{\lambda_n}|$$

and therefore

$$\sum_{\lambda_p\in S_{5I_{\lambda_n}}} (1-|\lambda_p|) \leqslant 2|I_{\lambda_n}|. \tag{4.4}$$

Fix the interval $I \subset \mathbb{T}$, and define $\sigma = \{n\in\mathbb{N} : \lambda_n \in S_I\}$. If $n\in\sigma$, then $I_{\lambda_n} \subset 2I$. By Vitali's covering lemma (see for instance [6], V.17), there is $\sigma' \subset \sigma$, such that the intervals I_{λ_n} are disjoint for $n\in\sigma'$, while $\bigcup_{n\in\sigma} I_{\lambda_n} \subset \bigcup_{n\in\sigma'} 5I_{\lambda_n}$; the last inclusion implies that $\{\lambda_n\in S_I\} \subset \bigcup_{n\in\sigma'} S_{5I_{\lambda_n}}$. Then, using (4.4), it follows that

$$\sum_{\lambda_n\in S_I}(1-|\lambda_n|) \leqslant \sum_{n\in\sigma'}\sum_{\lambda_p\in S_{5I_{\lambda_n}}}(1-|\lambda_p|) \leqslant 2\sum_{n\in\sigma'}|I_{\lambda_n}| \leqslant 4m(I);$$

thus $\sum(1-|\lambda_n|)\delta_{\lambda_n}$ is indeed a Carleson measure.

Fix $0<\varepsilon<1$, $A\geqslant 1$, and choose $n\in\mathbb{N}$, such that

$$\sum_{\substack{p\neq n\\ \lambda_p\in S_{AI_{\lambda_n}}}} (1-|\lambda_p|) \leqslant \varepsilon(1-|\lambda_n|). \tag{4.5}$$

We write

$$\sum_{p\neq n}\frac{(1-|\lambda_n|^2)(1-|\lambda_p|^2)}{|1-\bar\lambda_n\lambda_p|^2} = \sum_{\substack{p\neq n\\ \lambda_p\in S_{AI_{\lambda_n}}}}\frac{(1-|\lambda_n|^2)(1-|\lambda_p|^2)}{|1-\bar\lambda_n\lambda_p|^2} + \sum_{\lambda_p\notin S_{AI_{\lambda_n}}}\frac{(1-|\lambda_n|^2)(1-|\lambda_p|^2)}{|1-\bar\lambda_n\lambda_p|^2}.$$

Since $|1-\bar\lambda_n\lambda_p| \geqslant 1-|\lambda_n|$, it follows from (4.5) that the first sum is bounded by 4ε.

The second sum can be written as

$$\sum_{k=0}^{\infty}\sum_{\lambda_p\in S_{2^{k+1}AI_{\lambda_n}}\setminus S_{2^kAI_{\lambda_n}}}\frac{(1-|\lambda_n|^2)(1-|\lambda_p|^2)}{|1-\bar\lambda_n\lambda_p|^2}. \tag{4.6}$$

By (2.6), there is some constant $C > 0$ such that, for $z \notin S_{2^k A I_{\lambda_n}}$ one has $\frac{(1-|\lambda_n|^2)}{|1-\bar\lambda_n z|^2} \leqslant \frac{C}{2^{2k}A^2(1-|\lambda_n|)}$. On the other hand, since $\sum(1-|\lambda_n|)\delta_{\lambda_n}$ is a Carleson measure, there exists $C' > 0$, such that $\sum_{\lambda_p \in S_{2^{k+1}AI_{\lambda_n}}}(1 - |\lambda_p|^2) \leqslant C'2^{k+1}A(1-|\lambda_n|)$. It follows then that (4.6) is bounded by $4CC'/A$. On the whole, we obtain

$$\sum_{p\neq n, p\geqslant 1} \frac{(1-|\lambda_n|^2)(1-|\lambda_p|^2)}{|1-\bar\lambda_n\lambda_p|^2} \leqslant 4(\varepsilon + CC'/A).$$

This can be made as small as possible by choosing ε and A; by Proposition 4.1 it follows that Λ is a thin interpolating sequence. □

As a consequence, we mention the following two results that help to clarify the geometry of thin sequences; they are suggested by corresponding results related to Carleson sequences (see [9], VII.3).

PROPOSITION 4.3. — (i) *Suppose* $\Lambda = (\lambda_n)_{n\geqslant 1}$ *is an increasing sequence in* $\mathbb{D}$ *such that* $\lim_{n\to\infty}|\lambda_n| = 1$. *If*

$$\gamma := \lim_{k\to\infty} \frac{1-|\lambda_{k+1}|}{1-|\lambda_k|} = 0,$$

then Λ *is a thin interpolating sequence. If, moreover,* $\Lambda \subset [0,1)$, *then* Λ *is a thin sequence if and only if* $\gamma = 0$.

(ii) *Suppose* $(r_n)_{n\geqslant 1}$ *is a sequence of distinct positive numbers,* $0 < r_n < 1$, *such that* $\sum_{n\geqslant 1}(1-r_n) < \infty$. *Then there exist* $t_n \geqslant 0$ *such that* $(r_n e^{it_n})_{n\geqslant 1}$ *is a thin interpolating sequence.*

Proof. — (i) Fix $0 < \varepsilon < 1, A \geqslant 1$, and choose N such that for all $n \geqslant N$ we have

$$\frac{1-|\lambda_n|}{1-|\lambda_{n-1}|} < \varepsilon/A.$$

It follows that, if $n \geqslant N$ and $k < n$, then $\lambda_k \notin S_{AI_n}$.

On the other hand, if $k > n$, $1-|\lambda_k| \leqslant (\varepsilon/A)^{k-n}(1-|\lambda_n|)$, and $\sum_{k>n}(1-|\lambda_k|) \leqslant \frac{\varepsilon/A}{1-\varepsilon/A}(1-|\lambda_n|)$. Therefore

$$\sum_{\substack{k\neq n\\ \lambda_k\in S_{AI_{\lambda_n}}}} (1-|\lambda_k|) \leqslant \frac{\varepsilon/A}{1-\varepsilon/A}(1-|\lambda_n|);$$

it follows by Proposition 4.2 that Λ is a thin sequence.

If $\Lambda \in (0,1)$, then

$$1 - \frac{1-\lambda_{k+1}}{1-\lambda_k} = \frac{\lambda_{k+1}-\lambda_k}{1-\lambda_k} \geqslant \frac{\lambda_{k+1}-\lambda_k}{1-\lambda_k\lambda_{k+1}} = b_{\lambda_k}(\lambda_{k+1}) \geqslant |B_k(\lambda_k)|.$$

Therefore, if $\lim_{k\to\infty}|B_k(\lambda_k)|=1$, then $\lim_{k\to\infty}\frac{1-\lambda_{k+1}}{1-\lambda_k}=0$.

(ii) We may suppose r_n is increasing. Choose numbers $b_n>0$, such that $\sum_{n\geqslant 1} b_n<\infty$, and $(1-r_n)/b_{n+1}\to 0$. Since the thin interpolating property is not changed by adding a finite number of distinct points, we may suppose that $\sum_{n\geqslant 1} b_n<\pi/2$. We will then define $t_n=\sum_{k=1}^n b_k$, and $\lambda_n=r_n e^{it_n}$.

If $A>0$ is given, then the condition $(1-r_n)/b_{n+1}\to 0$ implies that, for n sufficiently large, $S_{AI_{\lambda_n}}\cap\Lambda=\{\lambda_n\}$. Then (4.3) is trivially verified, whence Λ is a thin sequence. □

It should be mentioned that (ii) in the above proposition has already been noticed in [11].

We end this section by quoting a stability result from [3], Section 3, where it has been proved that thin sequences are stable with respect to "small" perturbations.

THEOREM B. — *Let $\Lambda=(\lambda_n)_{n\geqslant 1}$, $\Lambda'=(\lambda_n')_{n\geqslant 1}$ be two sequences in $\mathbb{D}$. If $\sup_{n\geqslant 1}|b_{\lambda_n}(\lambda_n')|<1$, then $\Lambda\in(\vartheta)$ if and only if $\Lambda'\in(\vartheta)$.*

5. Projection onto a model space.

Suppose now that Θ is an inner function, while Λ is a Blaschke sequence of distinct points in $\mathbb{D}$. We are interested in the AOB property for the corresponding sequences of normalized reproducing kernels $(h^\Theta_{\lambda_n})_{n\geqslant 1}$, as defined by (2.3). It turns out that Volberg's condition (4.1) is necessary also in this context, as is shown by the next result ([3], Section 3). Below we will give a simpler proof.

PROPOSITION 5.1. — *If $(h^\Theta_{\lambda_n})_{n\geqslant 1}$ is an AOS, then $(\lambda_n)_{n\geqslant 1}$ is a thin interpolating sequence.*

Proof. — By applying formula (2.5), we have

$$\begin{aligned}|\Gamma^\Theta_{n,p}|^2&=|\Gamma_{n,p}|^2\frac{|1-\overline{\Theta(\lambda_n)}\Theta(\lambda_p)|^2}{(1-|\Theta(\lambda_n)|^2)(1-|\Theta(\lambda_p)|^2)}\\&=\frac{|\Gamma_{n,p}|^2}{1-|b_{\Theta(\lambda_n)}(\Theta(\lambda_p))|^2}\geqslant|\Gamma_{n,p}|^2.\end{aligned}$$

Since Proposition 3.2, (iii), implies $\|(\Gamma^\Theta - I)e_n\|^2 = \sum_{p\neq n}|\Gamma^\Theta_{n,p}|^2 \to 0$, it follows that $\|(\Gamma - I)e_n\|^2 = \sum_{p\neq n}|\Gamma_{n,p}|^2 \to 0$. Proposition 4.1 implies then that $(h_{\lambda_n})_{n\geqslant 1}$ is an AOB in K_B. $\square$

There is no hope to obtain, without supplementary conditions, a converse to Proposition 5.1. Indeed, suppose (λ_n) is a thin sequence, converging nontangentially to a point $\zeta \in \mathbb{T}$, while Θ can be analytically extended on an neighborhood of ζ. It follows from Theorem C in Section 8 below that $(h^\Theta_{\lambda_n})$ is in this case norm convergent, and thus cannot be even a Riesz sequence.

We will therefore try to obtain partial converses to Proposition 5.1. It is then natural, in view of the theory of Riesz bases developed in [7], to work under the supplementary condition $\sup_{n\geqslant 1}|\Theta(\lambda_n)| < 1$. (Note that in the previous example we have $|\Theta(\lambda_n)| \to |\Theta(\zeta)| = 1$.)

THEOREM 5.2. — *Suppose* $\sup_{n\geqslant 1}|\Theta(\lambda_n)| < 1$. *If* $(\lambda_n)_{n\geqslant 1}$ *is a thin interpolating sequence, then either*

(i) $(h^\Theta_{\lambda_n})_{n\geqslant 1}$ *is an AOB,*

or

(ii) *there exists* $p \geqslant 2$ *such that* $(h^\Theta_{\lambda_n})_{n\geqslant p}$ *is a complete AOB in* K_Θ.

Proof. — The condition on $(h_{\lambda_n})_{n\geqslant 1}$ implies the existence of positive constants $(c_N)_{N\geqslant N_0}$, $(C_N)_{N\geqslant N_0}$, tending to 1, such that

$$c_N \sum_{n\geqslant N} |a_n|^2 \leqslant \Big\| \sum_{n\geqslant N} a_n h_{\lambda_n} \Big\|^2 \leqslant C_N \sum_{n\geqslant N} |a_n|^2. \tag{5.1}$$

According to (2.4), we have, applying (5.1),

$$\begin{aligned}
\Big\| \sum_{n\geqslant N} a_n h^\Theta_{\lambda_n} \Big\|^2 &= \Big\| \sum_{n\geqslant N} \frac{a_n}{\sqrt{1-|\Theta(\lambda_n)|^2}} h_{\lambda_n} \Big\|^2 - \Big\| \sum_{n\geqslant N} \frac{a_n\overline{\Theta(\lambda_n)}}{\sqrt{1-|\Theta(\lambda_n)|^2}} h_{\lambda_n} \Big\|^2 \\
&\leqslant C_N \sum_{n\geqslant N} \frac{|a_n|^2}{1-|\Theta(\lambda_n)|^2} - c_N \sum_{n\geqslant N} \frac{|a_n|^2|\Theta(\lambda_n)|^2}{1-|\Theta(\lambda_n)|^2} \\
&= C_N \sum_{n\geqslant N} |a_n|^2 + (C_N - c_N) \sum_{n\geqslant N} \frac{|a_n|^2|\Theta(\lambda_n)|^2}{1-|\Theta(\lambda_n)|^2} \\
&\leqslant C_N \sum_{n\geqslant N} |a_n|^2 + (C_N - c_N) \sup_n \frac{|\Theta(\lambda_n)|^2}{1-|\Theta(\lambda_n)|^2} \sum_{n\geqslant N} |a_n|^2.
\end{aligned}$$

Since $C_N \to 1$, $C_N - c_n \to 0$, while $\sup_n \frac{|\Theta(\lambda_n)|^2}{1-|\Theta(\lambda_n)|^2} < \infty$, we can find constants $C'_N \to 1$, such that

$$\Big\| \sum_{n \geqslant N} a_n h^{\Theta}_{\lambda_n} \Big\|^2 \leqslant C'_N \sum_{n \geqslant N} |a_n|^2.$$

A similar argument shows the existence of $c'_N \to 1$, such that

$$\Big\| \sum_{n \geqslant N} a_n h^{\Theta}_{\lambda_n} \Big\|^2 \geqslant c'_N \sum_{n \geqslant N} |a_n|^2.$$

It follows that $(h^{\Theta}_{\lambda_n})_{n \geqslant 1}$ is an AOS; hence there exists $m \geqslant 1$ such that $(h^{\Theta}_{\lambda_n})_{n \geqslant m}$ is an AOB.

Let p be the smallest positive integer with the property that $(h^{\Theta}_{\lambda_n})_{n \geqslant p}$ is an AOB. If $p = 1$ we are in case (i) of the statement. Otherwise, Lemmas 3.1 and 2.1 imply that $(h^{\Theta}_{\lambda_n})_{n \geqslant p}$ is complete in K_Θ. The theorem is thus proved. □

It should be mentioned that a weaker version of Theorem 5.2 appears in Lemma 3.9 in [3].

Corollary 5.3. — *Suppose that* $\sup_{n \geqslant 1} |\Theta(\lambda_n)| < 1$ *and* Λ *is a thin interpolating sequence. Then* $(h^{\Theta}_{\lambda_n})_{n \geqslant 1}$ *is an AOB if and only if it is minimal.*

Case (ii) in Theorem 5.2 corresponds to $(h^{\Theta}_{\lambda_n})_{n \geqslant 1}$ not minimal; an example can be obtained by taking Θ to be a proper inner divisor of B. Minimality of sequences of reproducing kernels has been investigated in [1]; using Theorem 4.7 therein, we obtain the following characterization.

Corollary 5.4. — *Suppose that* $\sup_{n \geqslant 1} |\Theta(\lambda_n)| < 1$. *Then* $(h^{\Theta}_{\lambda_n})_{n \geqslant 1}$ *is an AOB if and only if* $(\lambda_n)_{n \geqslant 1}$ *is a thin interpolating sequence and there exists* $f \in H^\infty$, $f \neq 0$, *such that* $\|\Theta + Bf\|_\infty \leqslant 1$.

It is instructive to compare Corollary 5.4 with a result in [9], VIII.6, where it is proved that under the hypothesis $\sup_{n \geqslant 1} |\Theta(\lambda_n)| < 1$, $(h^{\Theta}_{\lambda_n})_{n \geqslant 1}$ is a Riesz sequence if and only if $\Lambda \in (C)$ and $\operatorname{dist}(\Theta \bar{B}, H^\infty) < 1$. This last condition is obviously stronger than the last requirement of Corollary 5.4. On the other hand, the thin interpolating condition is much more restrictive than Carleson's.

We can say more in case Θ is not a Blaschke product. The next result adapts an argument in [7], Theorem 3.2.

Proposition 5.5. — *Let* Θ *be an inner function with a nontrivial singular part, and suppose* $\sup_{n \geqslant 1} |\Theta(\lambda_n)| < 1$. *If the sequence* $(h^{\Theta}_{\lambda_n})_{n \geqslant 1}$ *is an AOB in* K_Θ, *then its span has infinite codimension.*

Proof. — Suppose that $\sup_{n\geqslant 1}|\Theta(\lambda_n)| = \eta < 1$. We shall write $\Theta = \beta S$, with β a Blaschke product and S singular, nonconstant. Let us also denote $B^{(N)} = \prod_{n\geqslant N} b_{\lambda_n}$.

By Proposition 5.1, $(h_{\lambda_n})_{n\geqslant 1}$ is an AOB (and in particular a Riesz sequence). If c_N, C_N are the constants in (3.1), then applying Lemma 2.3 to Θ and $B^{(N)}$ it follows that $\|P_{B^{(N)}}T_\Theta|K_{B^{(N)}}\| \leqslant (C_N/c_N)^{1/2}\eta$. Since $C_N/c_N \to 1$, we may find $N \in \mathbb{N}$, such that $\|P_{B^{(N)}}T_\Theta|K_{B^{(N)}}\| < 1$, which, according to Lemma 2.2, implies that $P_\Theta|K_{B^{(N)}}$ is an isomorphism on its image.

Now, if we define $\Theta' = \beta S^{1/2}$, Θ' is also an inner function, and

$$|\Theta'(\lambda_n)| \leqslant |\beta(\lambda_n)|^{1/2}|S(\lambda_n)|^{1/2} = |\Theta(\lambda_n)|^{1/2} \leqslant \eta^{1/2}.$$

If we apply the same argument to Θ', it follows that we can find $N \in \mathbb{N}$, such that both $P_\Theta|K_{B^{(N)}}$ and $P_{\Theta'}|K_{B^{(N)}}$ are isomorphisms on their images.

But we have

$$P_{\Theta'}|K_{B^{(N)}} = (P_{\Theta'}|K_\Theta)(P_\Theta|K_{B^{(N)}}).$$

The operator on the left is one-to-one, while the image of $(P_\Theta|K_{B^{(N)}})$ is closed. Therefore this image cannot intersect $\mathrm{Ker}(P_{\Theta'}|K_\Theta)$, which is infinite dimensional. But the image of $(P_\Theta|K_{B^{(N)}})$ is the space spanned by $h^\Theta_{\lambda_n}$ for $n \geqslant N$; it follows that the space spanned by all the $h^\Theta_{\lambda_n}$ $(n \geqslant 1)$ also has infinite codimension. □

In this case one can improve Corollary 5.4.

COROLLARY 5.6. — *Suppose that Θ has a nontrivial singular part and* $\sup_{n\geqslant 1}|\Theta(\lambda_n)| < 1$. *The following assertions are equivalent:*

(i) Λ *is a thin interpolating sequence;*

(ii) $(h^\Theta_{\lambda_n})_{n\geqslant 1}$ *is an AOB.*

Moreover, in this case, $\mathrm{Span}\{h^\Theta_{\lambda_n} : n \geqslant 1\}$ *has infinite codimension in* K_Θ.

Proof. — If Λ is a thin sequence, Proposition 5.5 shows that we are in Case (i) of Theorem 5.2; consequently $(h^\Theta_{\lambda_n})_{n\geqslant 1}$ is an AOB. The converse is contained in Proposition 5.1. □

6. Stability of AOB's.

We will next study the stability of AOB's with respect to small perturbations.

THEOREM 6.1. — *Suppose that* $\sup_{n\geqslant 1}|\Theta(\lambda_n)|<1$ *and* $(h^\Theta_{\lambda_n})_{n\geqslant 1}$ *is an AOB. If* $\Lambda'=(\lambda'_n)_{n\geqslant 1}$ *is a sequence of distinct points in* $\mathbb{D}$ *that satisfies*

$$\limsup_{n\to\infty}|b_{\lambda_n}(\lambda'_n)|<\frac{1-\operatorname{dist}(\Theta\bar B,H^\infty)}{1+\operatorname{dist}(\Theta\bar B,H^\infty)},\tag{6.1}$$

then $(h^\Theta_{\lambda'_n})_{n\geqslant 1}$ *is an AOB.*

Proof. — Fix $N\geqslant 1$, and define

$$\gamma_n=\begin{cases}\lambda_n & \text{if } n<N,\\ \lambda'_n & \text{if } n\geqslant N,\end{cases}$$

and Φ the Blaschke product associated to $(\gamma_n)_{n\geqslant 1}$. Proposition 5.1 implies that Λ is a thin sequence, whence, by Theorem B, Λ' and $(\gamma_n)_{n\geqslant 1}$ are both thin sequences. If $g,h\in H^\infty$, then the equality $\Theta\bar\Phi-gh=\Theta\bar B(B\bar\Phi-g)+(\Theta\bar B-h)g$ implies

$$\|\Theta\bar\Phi-gh\|_\infty\leqslant\|B\bar\Phi-g\|_\infty+\|(\Theta\bar B-h)g\|_\infty,$$

which shows that

$$\operatorname{dist}(\Theta\bar\Phi,H^\infty)\leqslant\operatorname{dist}(\Theta\bar B,H^\infty)+(1+\operatorname{dist}(\Theta\bar B,H^\infty))\operatorname{dist}(B\bar\Phi,H^\infty).$$

Now, if $B^{(N)}=\prod_{n\geqslant N}b_{\lambda_n}$, $\Phi^{(N)}=\prod_{n\geqslant N}b_{\lambda'_n}$, then $B\bar\Phi=B^{(N)}\overline{\Phi^{(N)}}$. Suppose C_N and c_N are the constants associated to Λ' as in 3.1, while $\varepsilon_N=\sup_{n>N}|b_{\lambda_n}(\lambda'_n)|$; one has then obviously $\sup_{n\geqslant N}|B^{(N)}(\lambda'_n)|\leqslant\varepsilon_N$. Applying Lemmas 2.2 and 2.3, it follows that

$$\begin{aligned}\operatorname{dist}(B\bar\Phi,H^\infty)=\operatorname{dist}(B^{(N)}\overline{\Phi^{(N)}},H^\infty)&=\|P_{\Phi^{(N)}}T_{B^{(N)}}|K_{\Phi^{(N)}}\|\\&\leqslant\varepsilon_N(C_N/c_N)^{1/2}.\end{aligned}$$

Consequently,

$$\operatorname{dist}(\Theta\bar\Phi,H^\infty)\leqslant\operatorname{dist}(\Theta\bar B,H^\infty)+(1+\operatorname{dist}(\Theta\bar B,H^\infty))\varepsilon_N(C_N/c_N)^{1/2}.$$

The hypothesis implies that, for N sufficiently large, $\varepsilon_N(C_N/c_N)^{1/2}<\frac{1-\operatorname{dist}(\Theta\bar B,H^\infty)}{1+\operatorname{dist}(\Theta\bar B,H^\infty)}$ and therefore $\operatorname{dist}(\Theta\bar\Phi,H^\infty)<1$. There exists thus $f\in H^\infty$, $f\neq 0$, such that $\|\Theta-\Phi f\|_\infty<1$, and therefore $\sup_{n\geqslant 1}|\Theta(\gamma_n)|\leqslant\operatorname{dist}(\Theta\bar\Phi,H^\infty)<1$. It follows by Corollary 5.4 that $(h^\Theta_{\gamma_n})_{n\geqslant 1}$ is an AOB. Applying repeatedly Lemma 2.1, we obtain that $(h^\Theta_{\lambda'_n})_{n\geqslant 1}$ is an AOB. $\square$

In the particular case where Θ has a nontrivial singular part, we can improve the stability constant in Theorem 6.1.

PROPOSITION 6.2. — *Suppose that Θ has a nontrivial singular part, $\sup_{n\geqslant 1}|\Theta(\lambda_n)|<1$ and $(h^\Theta_{\lambda_n})_{n\geqslant 1}$ is an AOB. If $\Lambda'=(\lambda'_n)_{n\geqslant 1}$ is a sequence of distinct points in $\mathbb{D}$ that satisfies*

$$\limsup_{n\to\infty}|b_{\lambda_n}(\lambda'_n)|<\frac{1-\limsup_{n\to\infty}|\Theta(\lambda_n)|}{1+\limsup_{n\to\infty}|\Theta(\lambda_n)|},$$

then $(h^\Theta_{\lambda'_n})_{n\geqslant 1}$ is an AOB.

Proof. — By Theorem B, $\Lambda'\in(\vartheta)$. On the other hand,

$$\left|\frac{\Theta(\lambda'_n)-\Theta(\lambda_n)}{b_{\lambda_n}(\lambda'_n)}\right|\leqslant\left\|\frac{\Theta-\Theta(\lambda_n)}{b_{\lambda_n}}\right\|_\infty\leqslant 1+|\Theta(\lambda_n)|,$$

whence

$$|\Theta(\lambda'_n)|\leqslant|\Theta(\lambda_n)|+(1+|\Theta(\lambda_n)|)|b_{\lambda_n}(\lambda'_n)|.$$

Therefore $\sup_n|\Theta(\lambda'_n)|<1$; Corollary 5.6 implies that $(h^\Theta_{\lambda'_n})_{n\geqslant 1}$ is an AOB. □

It is also possible to complement these results by studying the completeness of the perturbed sequence. As concerns the effect of small perturbations on Riesz basis, the following theorem was proved in [3].

THEOREM ([3], 3.1). — *Suppose that $\sup_{n\geqslant 1}|\Theta(\lambda_n)|<1$. If $(h^\Theta_{\lambda_n})_{n\geqslant 1}$ is a Riesz basis in K_Θ, then there exists $\varepsilon=\varepsilon(\Theta,\Lambda)<1$ such that for all sequences $\Lambda'=(\lambda'_n)_{n\geqslant 1}$ in $\mathbb{D}$ satisfying $|b_{\lambda_n}(\lambda'_n)|\leqslant\varepsilon$, we have $(h^\Theta_{\lambda'_n})_{n\geqslant 1}$ is a Riesz basis in K_Θ.*

Combining this result with Theorem 6.1, we obtain the following consequence for complete AOB's.

COROLLARY 6.3. — *Suppose that $\sup_{n\geqslant 1}|\Theta(\lambda_n)|<1$. If $(h^\Theta_{\lambda_n})_{n\geqslant 1}$ is a complete AOB in K_Θ, then there exists $\varepsilon=\varepsilon(\Theta,\Lambda)<1$ such that for all sequences $\Lambda'=(\lambda'_n)_{n\geqslant 1}$ in $\mathbb{D}$ satisfying*

$$|b_{\lambda_n}(\lambda'_n)|\leqslant\varepsilon,$$

we have $(h^\Theta_{\lambda'_n})_{n\geqslant 1}$ is a complete AOB in K_Θ.

A few words are in order concerning the different stability constants appearing in this section. The analogue for Riesz sequences of Theorem 6.1

appears in [3], Theorem 3.3. The right hand side of (6.1) is replaced therein by

$$\frac{\delta(\Lambda)^6}{8}\,\frac{1-\operatorname{dist}(\Theta\bar{B},H^\infty)}{1+\operatorname{dist}(\Theta\bar{B},H^\infty)}.$$

For AOB's, one should have expected a similar result, with $\delta(\Lambda)$ replaced by 1; Theorem 6.1 is therefore a sensible improvement.

As concerns completeness, there exists also an explicit upper bound for the constant $\varepsilon(\Theta,\Lambda)$ which appears in Theorem 3.1 of [3]; namely, we must have

$$\varepsilon < \min\left\{\frac{\delta}{2}, \frac{1-\sup_{n\geqslant 1}|\Theta(\lambda_n)|}{2}\right\}$$

as well as

$$\frac{2\varepsilon}{\delta/2-\varepsilon}\|\Gamma\|^{1/2}\left(128\frac{1+\varepsilon}{1-\varepsilon}(1-6\log\delta)\frac{1+\varepsilon+\delta/2}{1-\varepsilon-\delta/2}\right)^{1/2} < 1,$$

where $\delta = \inf_{n\geqslant 1}|B_n(\lambda_n)|$ and Γ is the Gram matrix associated to $(h^\Theta_{\lambda_n})_{n\geqslant 1}$. One can see that this is much more complicated than the bound given by formula (6.1).

7. Bases of exponentials.

The study of bases of exponentials in $L^2(0,a)$ has provided the original motivation for the development of the functional model approach in [7]. It is therefore natural to discuss in more detail AOB's of exponentials. Some preliminaries are needed to translate the problem into the language of model spaces. Note also that, as is customary, the index set will now be $\mathbb{Z}$ rather than $\mathbb{N}^*$.

If $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, then we define $\phi : \mathbb{C}_+ \to \mathbb{D}$ by $\phi(z) = \frac{z-i}{z+i}$ (ϕ is a conformal map from $\mathbb{C}_+$ to $\mathbb{D}$). The operator

$$(Uf)(z) = \frac{1}{\pi(z+i)} f(\phi(z)) \tag{7.1}$$

maps H^2 unitarily onto $H^2(\mathbb{C}_+)$, the Hardy space of the upper half-plane. The corresponding transformation for functions in H^∞ is

$$f \mapsto f\circ\phi; \tag{7.2}$$

it maps inner functions in $\mathbb{D}$ into inner functions in $\mathbb{C}_+$. We have then $UK_\Theta = H^2(\mathbb{C}_+)\ominus(\Theta\circ\phi)H^2(\mathbb{C}_+)$, and $U(k^\Theta_\lambda)$ is the reproducing kernel for the point $\phi(\lambda)$.

The Blaschke factor corresponding to $\mu \in \mathbb{C}_+$ is

$$b_\mu^+(z) = \frac{z-\mu}{z-\bar{\mu}}$$

and the Blaschke product with zeros $(\mu_n)_{n\in\mathbb{Z}}$ is

$$B^+(z) = \prod_{n\in\mathbb{Z}} c_{\mu_n} b_{\mu_n}^+(z),$$

the coefficients c_{μ_n} being chosen as to make all terms positive in i. The thin interpolating condition, which we will denote by (ϑ_+), becomes

$$\lim_{|n|\to\infty} \prod_{m\neq n} |b_{\mu_m}^+(\mu_n)| = 1.$$

Let $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the Fourier transform. Then $\mathcal{F}U$ maps H^2 unitarily onto $L^2(0,\infty)$. If $\Theta_a(z) = e^{a\frac{z+1}{z-1}}$, then $\mathcal{F}U$ maps K_{Θ_a} unitarily onto $L^2(0,a)$; the normalized reproducing kernel $h_\lambda^{\Theta_a}$ ($\lambda \in \mathbb{D}$) is mapped into $\chi_\mu^a(t) = \kappa_a(\mu)e^{i\mu t}$, where $\mu = -\overline{\phi^{-1}(\lambda)}$, and $\kappa_a(\mu) = \left(\frac{2\operatorname{Im}\mu}{1-e^{-2a\operatorname{Im}\mu}}\right)^{1/2}$. We also have $\Theta_a(\lambda) = e^{ia\bar{\mu}}$.

The results from the previous sections concerning reproducing kernels can then be adapted to the case of exponentials $e^{i\mu t}$, with $\mu \in \mathbb{C}_+$; note that the relevant inner function Θ_a is singular. The next theorem deals, however, with a more general class of exponentials.

THEOREM 7.1. — *Let $(\mu_n)_{n\in\mathbb{Z}}, (\mu_n')_{n\in\mathbb{Z}}$ be two sequences of distinct complex numbers. If $(e^{i\mu_n t})_{n\in\mathbb{Z}}$ is a complete AOB in $L^2(0,1)$, and $\lim_{|n|\to\infty} |\mu_n - \mu_n'| = 0$, then $(e^{i\mu_n' t})_{n\in\mathbb{Z}}$ is a complete AOB in $L^2(0,1)$.*

Proof. — Fix $N \geqslant 1$, and define

$$\gamma_n = \begin{cases} \mu_n & \text{if } |n| \leqslant N, \\ \mu_n' & \text{if } |n| > N, \end{cases}$$

and V by $V(e^{i\mu_n t}) = e^{i\gamma_n t}$ for $n \in \mathbb{Z}$. For $(a_n)_{n\in\mathbb{Z}} \in \ell^2$, we have:

$$\begin{aligned}
\Big\|(V-I)\Big(\sum_{n\in\mathbb{Z}} a_n e^{i\mu_n t}\Big)\Big\| &= \|(V-I)(\sum_{|n|>N} a_n e^{i\mu_n t})\| \\
&= \|\sum_{|n|>N} a_n(e^{i\mu_n' t} - e^{i\mu_n t})\| \\
&= \|\sum_{|n|>N} a_n e^{i\mu_n t} \sum_{k\geqslant 1} \frac{(i(\mu_n'-\mu_n))^k t^k}{k!}\| \\
&\leqslant \sum_{k\geqslant 1} \frac{1}{k!} \|t^k \sum_{|n|>N} (i(\mu_n'-\mu_n))^k a_n e^{i\mu_n t}\| \\
&\leqslant C_N \sum_{k\geqslant 1} \frac{a^k}{k!} \Big(\sum_{|n|>N} |\mu_n'-\mu_n|^{2k}|a_n|^2\Big)^{1/2} \\
&\leqslant C_N(e^{a\sup_{|n|>N}|\mu_n'-\mu_n|}-1)\Big(\sum_{|n|>N}|a_n|^2\Big)^{1/2},
\end{aligned}$$

C_N being the constant in (3.1) corresponding to the AOB $(e^{i\mu_n t})_{n\in\mathbb{Z}}$. Since $\sup_{|n|>N}|\mu_n'-\mu_n| \to 0$ and $C_N \to 1$ for $N \to \infty$, it follows that if N is large enough, then $\|V-I\| < 1$ and thus V is invertible. If P_m is the orthogonal projection onto $\mathrm{Span}\{e^{i\mu_n t} : |n| \geqslant m\}$, similar computations for $m \geqslant N$ show then that $\|(V-I)P_m\| \to 0$, and therefore $V-I$ is compact. Proposition 3.2 shows that $(e^{i\gamma_n t})_{n\in\mathbb{Z}}$ is a complete AOB in $L^2(0,1)$.

Now the two sequences of complex numbers $(\mu_n')_{n\in\mathbb{Z}}$ and $(\gamma_n)_{n\in\mathbb{Z}}$ differ by a finite number of terms, and therefore $(e^{i\mu_n' t})_{n\in\mathbb{Z}}$ is an AOS. On the other hand, $\|e^{i\mu_n t}\| \to 1$ implies $\operatorname{Im}\mu_n \to 0$; thus $(\mu_n')_{n\in\mathbb{Z}}$ and $(\gamma_n)_{n\in\mathbb{Z}}$ are both contained in a strip, say $|\operatorname{Im} z| < A$. Multiplication by e^{-At} is an invertible operator on $L^2(0,1)$; thus $(e^{i(\gamma_n+iA)t})_{n\in\mathbb{Z}}$ is a Riesz basis in $L^2(0,1)$. An application of Lemma 2.1 implies that $(e^{i(\mu_n'+iA)t})_{n\in\mathbb{Z}}$ is also a Riesz basis, and therefore the same is true about $(e^{i\mu_n' t})_{n\in\mathbb{Z}}$; the proof is complete. □

In the case $\mu_n = 2\pi n$, one can compare Theorem 7.1 to Kadec's Theorem (see [7], I.5), which states, for real sequences (μ_n'), that Riesz bases are preserved under the requirement $|\mu_n'-2\pi n| < 1/4$. Such a uniform bound is not adequate for AOB's; indeed, since in $L^2(0,1)$,

$$\langle e^{2i\pi\mu_n t}, e^{2i\pi\mu_{n+1} t}\rangle = \frac{e^{2i\pi(\mu_n-\mu_{n+1})}-1}{2i\pi(\mu_n-\mu_{n+1})},$$

it follows that $\langle e^{2i\pi\mu_n t}, e^{2i\pi\mu_{n+1} t}\rangle \to 0$ implies $\mu_n-\mu_{n+1}-[\mu_n-\mu_{n+1}] \to 0$.

Suppose now that there is $\eta > 0$ such that $\operatorname{Im}\mu_n > \eta$ for all $n \in \mathbb{Z}$. In this case AOB of (normalized) exponentials in $L^2(0,a)$ $(a>0)$ are exactly characterized by the corresponding condition (ϑ_+).

PROPOSITION 7.2. — *If $\operatorname{Im}\mu_n > \eta > 0$ for all $n \in \mathbb{Z}$, then the following are equivalent:*

(i) *$(\mu_n)_{n\in\mathbb{Z}}$ is thin interpolating;*

(ii) *$(\chi^a_{\mu_n})_{n\in\mathbb{Z}}$ is an AOB in $L^2(0,a)$ for all $a > 0$;*

(iii) *$(\chi^a_{\mu_n})_{n\in\mathbb{Z}}$ is an AOB in $L^2(0,a)$ for some $a > 0$.*

Proof. — If we translate the problem in the disc, then the inner function Θ_a is singular, and if $\lambda_n = \phi(-\bar{\mu}_n)$, then $|\Theta_a(\lambda_n)| = e^{-a\operatorname{Im}\mu_n}$. Therefore $\sup_{n\in\mathbb{Z}}|\Theta_a(\lambda_n)| < 1$, and the results in the statement are a consequence of Corollary 5.6. □

One should remark that in this case the Volberg condition is independent of $a > 0$. This should be compared with the situation for Riesz sequences of exponentials (see, for instance, [10], D.5): in case $\inf_{n\in\mathbb{Z}}\operatorname{Im}\mu_n > -\infty$, if $(e^{i\mu_n t})_{n\in\mathbb{Z}}$ is a Riesz sequence in $L^2(0,a)$, then $(e^{i\mu_n t})_{n\in\mathbb{Z}}$ is a Riesz sequence in $L^2(0,a')$ for all $a' \geqslant a$, but usually not for $a' < a$.

Finally, a stability result can be obtained by translating Proposition 6.2.

COROLLARY 7.3. — *Suppose $\operatorname{Im}\mu_n > \eta > 0$ for all $n \in \mathbb{Z}$, and $(\chi^a_{\mu_n})_{n\in\mathbb{Z}}$ is an AOB in $L^2(0,a)$ for some $a > 0$. If $(\mu'_n)_{n\in\mathbb{Z}}$ is a sequence of distinct points in $\mathbb{C}_+$ that satisfies*

$$\limsup_{|n|\to\infty}\left|\frac{\mu_n-\mu'_n}{\bar{\mu}_n-\mu'_n}\right| < \frac{1-\limsup_{|n|\to\infty} e^{-a\operatorname{Im}\mu_n}}{1+\limsup_{|n|\to\infty} e^{-a\operatorname{Im}\mu_n}},$$

then $(\chi^a_{\mu'_n})_{n\in\mathbb{Z}}$ is an AOB in $L^2(0,a)$.

8. Examples.

As noticed in the previous section, bases of exponentials are related to a singular inner function Θ, with corresponding one-point supported measure. In this section we will give some examples related to other inner functions.

Since complete AOB's are asymptotically close to orthonormal bases, it is natural to try to obtain examples by perturbing orthonormal bases. If we take $\lambda, \lambda' \in \mathbb{D}$, then $\langle k_\lambda^\Theta, k_{\lambda'}^\Theta \rangle = \frac{1-\overline{\Theta(\lambda)}\Theta(\lambda')}{1-\bar{\lambda}\lambda'} \neq 0$, and thus the reproducing kernels themselves cannot be orthogonal. However, we may obtain orthogonal bases of reproducing kernels in case the evaluations on the boundary $\mathbb{T}$ of $\mathbb{D}$ are continuous; the precise statement appears in Theorem C below. Suppose $a_n \in \mathbb{D}$ are the zeros of the Blaschke factor of Θ, while σ is the positive singular measure on $\mathbb{T}$ corresponding to the singular factor of Θ. We define $E_\Theta \subset \mathbb{T}$ by the formula

$$E_\Theta = \left\{ \zeta \in \mathbb{T} \mid \sum_{k=1}^{\infty} \frac{1-|a_k|}{|\zeta - a_k|^2} + \int_0^{2\pi} \frac{d\sigma(t)}{|\zeta - e^{it}|^2} < \infty \right\}. \tag{8.1}$$

The following theorem appears in [2].

THEOREM C. — (i) *If $\zeta \in E_\Theta$, then Θ has a nontangential limit $\Theta(\zeta)$ in ζ, of modulus 1. The function $k_\zeta^\Theta(z) := \frac{1-\overline{\Theta(\zeta)}\Theta(z)}{1-\bar{\zeta}z}$ belongs to K_Θ, and $k_\lambda^\Theta \to k_\zeta^\Theta$ if $\lambda \to \zeta$ nontangentially. Moreover, any function $f \in K_\Theta$ has a nontangential limit $f(\zeta)$ in ζ, and $f(\zeta) = \langle f, k_\zeta^\Theta \rangle$.*

(ii) *If $\mathbb{T} \setminus E_\Theta$ is at most countable, then there exists a sequence $\zeta_n \in E_\Theta$ such that $\left(\frac{k_{\zeta_n}^\Theta}{\|k_{\zeta_n}^\Theta\|} \right)$ is an orthonormal basis of K_Θ.*

We may therefore obtain a large class of examples of complete AOB's in K_Θ formed by reproducing kernels.

COROLLARY 8.1. — *If $\mathbb{T} \setminus E_\Theta$ is at most countable, then there exist sequences $(\lambda_n)_{n\geqslant 1}$ in $\mathbb{D}$ such that $(h_{\lambda_n}^\Theta)_{n\geqslant 1}$ is a complete AOB in K_Θ.*

Proof. — By Theorem C, (ii), take a sequence $\zeta_n \in E_\Theta$ such that $\left(\frac{k_{\zeta_n}^\Theta}{\|k_{\zeta_n}^\Theta\|} \right)$ is an orthonormal basis of K_Θ. By (i) of the same proposition, it obviously follows that if $\zeta \in E_\Theta$, and $\lambda \to \zeta$ nontangentially, then $h_{\lambda_n}^\Theta \to \frac{k_\zeta^\Theta}{\|k_\zeta^\Theta\|}$. Choose then $\lambda_n \in \mathbb{D}$, $\lambda_n/\|\lambda_n\| = \zeta_n$, such that $\sum_{n\geqslant 1} \left\| h_{\lambda_n}^\Theta - \frac{k_{\zeta_n}^\Theta}{\|k_{\zeta_n}^\Theta\|} \right\|^2 < 1$. The required conclusion follows then by applying Lemma 3.3. □

One should note that the choice of λ_n can obviously be made such that $|\Theta(\lambda_n) - \Theta(\zeta_n)| \to 0$; it follows then that $|\Theta(\lambda_n)| \to 1$, and therefore we are not in the context of the results in Section 5.

In case $\Theta = \Theta_a$, $E_{\Theta_a} = \mathbb{T} \setminus \{1\}$, and thus obviously satisfies the hypotheses of Theorem C and Corollary 8.1. Actually, Clark's paper [2] indeed has the bases of exponentials as a starting point. A different type of example, adapted from [7], shows that complete AOB's can appear in a case when $E_\Theta = \emptyset$.

Take first a sequence of positive integers q_n, $n \geqslant 1$, such that $q_{n+1} - q_n \to \infty$. Choose then another sequence of positive integers p_n, $n \geqslant 1$, subject to the conditions

$$\sum_{n\geqslant 1} \frac{p_n}{2^{q_n}} < \infty \tag{8.2}$$

$$\sum_{n\geqslant 1} \frac{p_n \log p_n}{2^{q_n}} = \infty. \tag{8.3}$$

Choose p_n equidistant points on the circle centered in the origin and having radius $1 - \frac{1}{2^{q_n}}$; the union of all these points (for $n \geqslant 1$) will be denoted by Λ. We will also denote $r_n = 1 - \frac{1}{2^{q_n}}$.

We have $\sum_{\lambda\in\Lambda}(1 - |\lambda|) = \sum_{n\geqslant 1} p_n \frac{1}{2^{q_n}} < \infty$; thus Λ satisfies the Blaschke condition and we may form the corresponding product B. Take $A > 0$; for sufficiently large n, if $\lambda \in \Lambda$ has absolute value $1 - \frac{1}{2^{q_n}}$, then $(S_{AI_\lambda} \cap \Lambda) \setminus \{\lambda\}$ contains only points on the circles of radii strictly larger than $|\lambda|$. On each of these circles, the number of these points is of order $p_k \times A|I_\lambda| = p_k A(1 - |\lambda|)$. Therefore

$$\frac{1}{1 - |\lambda|} \sum_{\substack{\mu\neq\lambda \\ \mu\in S_{AI_\lambda}}} (1 - |\mu|)$$

can be estimated by $A \sum_{k\geqslant n+1} \frac{p_k}{2^{q_k}}$, and thus tends to 0 by (8.2). Therefore Λ is a thin sequence by Proposition 4.2.

On the other hand, $E_B = \emptyset$. Actually, as in [7], more can be proved, namely that $\sum_{\lambda\in\Lambda} \frac{1-|\lambda|}{|\zeta-\lambda|} = \infty$ for all $\zeta \in \mathbb{T}$. Indeed, for $\zeta \in \mathbb{T}$, we have

$$\sum_{\lambda\in\Lambda} \frac{1 - |\lambda|}{|\zeta - \lambda|} = \sum_{n\geqslant 1} \frac{1}{2^{q_n}} \sum_{|\lambda|=r_n} \frac{1}{|\zeta - \lambda|}.$$

For each fixed n, if $|\lambda| = r_n$, then, with the possible exception of two points, $|\zeta - \lambda|$ is comparable to $|r_n\zeta - \lambda|$. The other points λ on this circle are at distances to ζ comparable to $j \cdot \frac{2\pi}{p_n}$, with $j = 1, 2, \ldots, p_n - 2$. Therefore

$$\sum_{|\lambda|=r_n} \frac{1}{|\zeta - \lambda|} \geqslant C \sum_{1}^{p_n-2} \frac{1}{\frac{j}{p_n}} \leqslant C p_n \log p_n.$$

Then

$$\sum_{\lambda\in\Lambda}\frac{1-|\lambda|}{|\zeta-\lambda|}\geqslant C\sum_{n\geqslant 1}\frac{1}{2^{q_n}}p_n\log p_n=\infty$$

as required.

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BIBLIOGRAPHY

[1] I. Boricheva, Geometric properties of projections of reproducing kernels on z^*-invariant subspaces of H^2, Journal of Functional Analysis, 161 (1999), 397–417.

[2] D. Clark, One dimensional perturbations of restricted shifts, J. Analyse Math., 25 (1972), 169–191.

[3] E. Fricain, Bases of reproducing kernels in model spaces, J. Operator Theory, 46 (2001), 517–543.

[4] P. Gorkin, H.-M. Lingenberg, and R. Mortini, Homeomorphic disks in the spectrum of H^∞, Indiana Univ. Math. J., 39 (1990), 961–983.

[5] H. Hedenmalm, Thin interpolating sequences and three algebras of bounded functions, Proc. Amer. Math. Soc., 99 (1987), 489–495.

[6] E. Hewitt, and K. Stromberg, Real and Abstract Analysis, Springer-Verlag, Berlin-Heidelberg-New York, 1969.

[7] S.V. Hruščev, N.K. Nikolski, and B.S. Pavlov, Unconditional bases of exponentials and of reproducing kernels, in Complex Analysis and Spectral Theory, V.P. Havin and N.K. Nikolski, Eds., Lecture Notes in Mathematics. Springer-Verlag, Berlin Heidelberg New-York, 1981, 214–335.

[8] N. Levinson, Gap and Density Theorems, Amer. Math. Soc. Colloquium Publ., New-York, 26 (1940).

[9] N.K. Nikolski, Treatise on the Shift Operator, Springer-Verlag, Berlin, 1986. Grundlehren der mathematischen Wissenschafte, vol. 273.

[10] N.K. Nikolski, Operators, Functions and Systems: An Easy Reading, Volume 2: Model Operators and Systems, American Mathematical Society, 2002, Mathematical Surveys and Monographs, 93.

[11] N.K. Nikolski, and A.L. Volberg, Tangential and approximate free interpolation, Analysis and partial differential equations, C. Sadosky, Ed., Lecture Notes in Pure and Applied. Math. Dekker, New-York, 1990, 277–299.

[12] R.E. Paley, and N. Wiener, Fourier Transforms in the Complex Domain, vol. 19, Amer. Math. Soc. Colloquium Publ., Providence, 1934.

[13] C. Sundberg, and T.H. Wolff, Interpolating sequences for QA_B, Trans. Amer. Math. Soc., 276 (1983), 551–581.

[14] B. Sz-Nagy, and C. Foias, Harmonic Analysis of Operators on Hilbert Spaces, North-Holland Publishing Co., Amsterdam-London, 1970.

[15] A.L. VOLBERG, Two remarks concerning the theorem of S. Axler, S.-Y. A. Chang and D. Sarason, J. Operator Theory, 7 (1982), 209–218.

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