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Thomas BLOOM & Norman LEVENBERG

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DISTRIBUTION OF NODES ON ALGEBRAIC CURVES IN $\mathbb{C}^N$

by T. BLOOM & N. LEVENBERG

Introduction.

In [GMS], the asymptotic behavior of "good" points for bivariate polynomial interpolation (in particular, Fekete points) was determined for sets in the plane consisting of a finite union of compact subsets of real algebraic curves, each having genus 0. To define this notion of "good" points, we generalize to the following situation. Let A be an *algebraic curve*; i.e., A is a pure one (complex)-dimensional irreducible algebraic subvariety in $\mathbb{C}^N$, $N > 1$. Note in particular we are assuming that A is connected. Now for each $d = 0, 1, 2, ...$, let $m_d = \dim P_d|_A$ be the dimension of the complex vector space of all holomorphic polynomials of degree at most d (this is P_d) restricted to A (this is $P_d|_A$). By a standard result in algebraic geometry, if D is the degree of A (D is the generic number of points of intersection of A with an affine complex hyperplane), then for d sufficiently large, $m_d = dD + c$ for some integer c. Let $K \subset A$ be compact and nonpolar in A; i.e., $K \cap A^0$ is nonpolar as a subset of the complex manifold A^0 consisting of the regular points of A. Now for each $d = 1, 2, ...$, choose m_d points $\{A_{dj}\}_{j=1,...,m_d}$ in K such that the fundamental Lagrange interpolating polynomials (FLIP's) $l_j^{(d)}$, $j = 1, ..., m_d$ of degree d associated to $A_{d1}, ..., A_{dm_d}$ exist; i.e., each $l_j^{(d)}$ is a nonconstant polynomial of degree

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at most d and $l_j^{(d)}(A_{dk}) = \delta_{jk}$. We call

$$\Lambda_d := \|\sum_{j=1}^{m_d} |l_j^{(d)}|\|_K$$

the d–th Lebesgue constant of the array $\{A_{dj}\}$; this is the operator norm of $L_d : C(K) \to P_d \subset C(K)$ on the Banach space $C(K)$ of continuous, complex-valued functions on K equipped with the supremum norm $\|f\|_K = \max_{z\in K} |f(z)|$, where

$$L_d f(z) := \sum_{j=1}^{m_d} f(A_{dj}) l_j^{(d)}(z)$$

is the Lagrange interpolating polynomial for f, $\{A_{dj}\}_{j=1,\ldots,m_d}$. Note that if we take a basis $e_1, ..., e_{m_d}$ of $P_d|_A$, and points $A_{d1}, ..., A_{dm_d} \in K$, and we let

$$V(A_{d1}, ..., A_{dm_d}) := \det[e_j(A_{dk})]_{j,k=1,\ldots,m_d}$$

be the generalized Vandermonde (VDM) determinant of these points, then, assuming $V(A_{d1}, ..., A_{dm_d}) \neq 0$, we can write

$$l_j^{(d)}(z) = \frac{V(A_{d1}, ..., z, ..., A_{dm_d})}{V(A_{d1}, ..., A_{dm_d})}. \tag{0.1}$$

Definition 0.1. — *We call an array $\{A_{dj}\}_{j=1,\ldots,m_d;d=1,2,\ldots}$ good for K if*

$$\limsup_{d\to\infty} \Lambda_d^{1/d} \leqslant 1. \tag{0.2}$$

As an example of such an array, taking points $\{A_{dj}\}_{j=1,\ldots,m_d} \in K$ such that

$$V(A_{d1}, ..., A_{dm_d}) = \max_{x_1,\ldots,x_{m_d}\in K} V(x_1, ..., x_{m_d})$$

(d–th order *Fekete points* for K), we have $\Lambda_d \leqslant m_d$ and hence (0.2) holds. In [GMS], as mentioned earlier, the authors considered the situation where $K = \cup_{j=1}^m K_j$ was a finite union of compact subsets $K_j \subset A^{(j)}$ and each $A^{(j)}$ was a real algebraic curve of genus 0 in $\mathbb{R}^2$. They showed that there is a probability measure μ_K supported on K such that for any array in K satisfying (0.2), the discrete measures

$$\mu_d := \frac{1}{m_d}\sum_{j=1}^{m_d} \delta_{A_{dj}},\ d = 1, 2, ..., \tag{0.3}$$

converge weak-$*$ to μ_K. In case such a curve K is irreducible, it has a rational (real) parameterization which [GMS] utilized to pull back the problem on K to a *weighted potential theory* problem on the real line or on the unit circle.

Since an irreducible real algebraic curve in $\mathbb{R}^2$ has a rational parameterization precisely when it is of genus 0, the technique of weighted potential theory is not available in the higher genus setting. However, complexifying the situation allows one to use techniques of *pluripotential theory* in several complex variables; we discuss this subject in the next section. At the end of Section 1, we state our main result (Theorem 1.1), which includes the claim that *for any irreducible algebraic curve A in $\mathbb{C}^N$, and any nonpolar compact subset $K \subset A$, there is a probability measure μ_K supported on ∂K such that for any array in K satisfying (0.2), the discrete measures*

$$\mu_d := \frac{1}{m_d}\sum_{j=1}^{m_d} \delta_{A_{dj}},\ \ d = 1, 2, ...,$$

converge weak-$*$ to μ_K. In particular, condition (0.2) implies that the support of the (unique) weak-$*$ limit of $\{\mu_d\}$ is contained in ∂K. The proof of the main theorem follows in Section 2; and some final remarks and examples comprise Section 3.

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1. Pluripotential theory and results of Sadullaev.

A real-valued function u defined on a domain D in $\mathbb{C}^N$ is plurisubharmonic (psh) on D if it is uppersemicontinuous on D and the restriction of u to each complex line ℓ is subharmonic (shm) on (components of) $D \cap \ell$. A plurisubharmonic function u on $\mathbb{C}^N$ is said to be in the class $\mathcal{L}(\mathbb{C}^N)$ (*minimal growth* in $\mathbb{C}^N$) if $u(z) - \log|z| = 0(1)$ as $|z| \to \infty$. Here, $z = (z_1, ..., z_N) \in \mathbb{C}^N$ and $|z|^2 = |z_1|^2 + \cdots + |z_N|^2$. Given $E \subset \mathbb{C}^N$, the function

$$V_E(z) \equiv \sup\{u(z) : u \in \mathcal{L}(\mathbb{C}^N),\ u \leqslant 0 \,\mathrm{on}\, E\}$$

is called the *global* or *$\mathcal{L}$-extremal function* of E; either $V_E^*(z) \equiv \limsup_{\zeta\to z} V_E(\zeta) \equiv +\infty$ or $V_E^* \in \mathcal{L}(\mathbb{C}^N)$; the former case occurs precisely when E is pluripolar; i.e., $E \subset \{z : u(z) = -\infty\}$ for some psh function u.

If K is a compact subset of $\mathbb{C}^N$, the $\mathcal{L}$-extremal function can be obtained using polynomials:

$$V_K(z) \equiv \max\Big\{0, \sup\Big\{\frac{1}{\deg p}\log\frac{|p(z)|}{||p||_K} : p \text{ is a polynomial, } \deg p > 0\Big\}\Big\}$$

(Theorem 5.1.7 [K]). Note that $V_K = V_{\hat{K}}$ where

$$\hat{K} = \{z : |p(z)| \leqslant ||p||_K, \quad p \text{ is a polynomial}\}$$

is the *polynomial hull* of K. For $K \subset \mathbb{C}$, $K = \hat{K}$ is equivalent to $\mathbb{C} \setminus K$ being connected; however, it is not necessarily the case for K a compact subset of a general algebraic curve A of $\mathbb{C}^N$ that $K = \hat{K}$ is equivalent to $A \setminus K$ being connected (note that in this setting $\hat{K} \subset A$). As an elementary example, take

$$A = \{(z_1, z_2) \in \mathbb{C}^2 : z_1 z_2 = 1\}$$

and $K = \{(z_1, z_2) \in A : |z_1| = 1\}$; here $\hat{K} = K$ but $A \setminus K$ consists of the two components $\{(z_1, z_2) \in A : |z_1| > 1\}$ and $\{(z_1, z_2) \in A : |z_2| > 1\}$. We will return to this example later.

Let A be a pure k-dimensional analytic subvariety in $\mathbb{C}^N$. Following Sadullaev, if we let A^0 be the set of regular points of A, then a real-valued function u on A is (weakly) *plurisubharmonic* on A if it is plurisubharmonic on the complex manifold A^0 and locally bounded above on A. There is a stronger notion of plurisubharmonicity on analytic varieties: a real-valued function u on A is (strongly) *plurisubharmonic* on A if u is locally the restriction to A of an ambient psh function (on a domain in $\mathbb{C}^N$). A good discussion of these notions can be found in Section 1 of [D]; we present (without proofs) the main results and some examples. As a simple (reducible) example, on the variety $A = \{(z_1, z_2) : z_1 z_2 = 0\}$, the function u defined as $u(z_1, 0) = 1$ and $u(0, z_2) = 0$ (if $z_2 \neq 0$) is weakly psh on A. Clearly u is not strongly psh since there is no psh function U defined on a neighborhood N of $(0,0)$ in $\mathbb{C}^2$ which agrees with u on $N \cap A$. In general, however, if u is weakly psh and continuous on A, then u is strongly psh on A. More generally, if A is *locally irreducible*, i.e., if each point of A has a neighborhood N in $\mathbb{C}^N$ such that $A \cap N$ is irreducible in N, then given any weakly psh function u on A, the function

$$u^*(z) := \limsup_{\zeta \to z,\ \zeta \in A} u(\zeta)$$

is strongly psh on A and $u = u^*$ a.e. on A; indeed, $u = u^*$ q.e. on A; i.e., off of a (possibly empty) pluripolar set. We remark that "irreducible" and "locally irreducible" are different notions; e.g., the variety $A = \{(z_1, z_2) :$

$z_1^2 + z_1^3 = z_2^2\}$ is irreducible (in $\mathbb{C}^2$) but is not locally irreducible at the origin.

We will always take A to be a pure k-dimensional *irreducible* analytic subvariety in $\mathbb{C}^N$; thus the regular points A^0 form a connected complex submanifold of $\mathbb{C}^N \setminus (A \setminus A^0)$. We now recall the fundamental result in [Sa1], giving a criterion for such a subvariety to be *algebraic.*

THEOREM [Sa1]. — *A is* **algebraic** *if and only if $V_K \in L^\infty_{\mathrm{loc}}(A)$ for some (and hence for each) nonpluripolar compact set K in A.*

Note that the *global* regularization $V_K^*(z) = \limsup_{\zeta\to z} V_K(\zeta) \equiv +\infty$ since A and hence K is pluripolar in $\mathbb{C}^N$. Thus when we write V_K^* in this paper, we refer to the regularization of V_K along A; i.e.,

$$V_K^*(z) := \limsup_{\zeta\to z,\ \zeta\in A} V_K(\zeta).$$

We now restrict our attention to the algebraic curve case; i.e., $k = 1$. If A is algebraic, by [Ru] we can choose a basis for $\mathbb{C}^N$ so that

$$A \subset \{(z', z_N) \in \mathbb{C}^N : |z'|^2 < C(1 + |z_N|^2)\},\ z' = (z_1, ..., z_{N-1}), \tag{1.1}$$

for some $C > 0$. It can then be shown ([Sa1], p. 497) that for the "disk"

$$K \equiv \{z \in A : |z_N| \leqslant 1\},$$

we have $V_K(z) = \max[0, \log |z_N|]$ for $z \in A$, yielding the "only if" direction. With respect to the notions of (pluri-) subharmonicity described earlier, if A is of degree D, we can assume that $\pi : A \subset \mathbb{C}^N \to \mathbb{C}$ via $\pi(z) = z_N$ is a $D-$sheeted covering map of A over $\mathbb{C} \setminus V$ where V is a finite set. Thus,

$$A = \{(z', z_N) = (s_j(z_N), z_N) : 1 \leqslant j \leqslant D\}$$

where the $s_j(z_N)$ are distinct for $z_N \notin V$. Roughly speaking, a weakly subharmonic function u on A can be given as D shm functions $u_1, ..., u_D$ on each "branch" $\{(z', z_N) = (s_j(z_N), z_N) : z_N \notin V\}$; if these functions coincide as they approach the branch points (when $z_N \in V$), then u is strongly shm on A.

As a concrete example of the theorem, in $\mathbb{C}^2$, if $Q(z_1, z_2)$ is an irreducible polynomial so that $A := \{(z_1, z_2) \in \mathbb{C}^2 : Q(z_1, z_2) = 0\}$ is an irreducible algebraic curve, and we let

$$K = A \cap \mathbb{R}^2 = \{(z_1, z_2) \in A : \Im z_1 = \Im z_2 = 0\},$$

then provided $K \cap A^0$ is a *real* 1-dimensional submanifold of $\mathbb{R}^2$, it is not polar in A (see [BLMT], Lemma 1.7). Thus V_K is locally bounded on A

(moreover, if K is smooth, then V_K is Lipshitz on A near K, as in the case of a real interval away from the endpoints). We will need the following fact from [Sa1] (Corollary 3.3).

PROPOSITION [Sa1]. — *If A is an irreducible algebraic curve, then for any nonpolar compact set $K \subset A$, V_K is a harmonic function on $A^0 \setminus K$.*

In the next section, we will always work on an irreducible algebraic curve $A \subset \mathbb{C}^N$ with coordinates chosen to satisfy (1.1); and "shm" will refer to "weakly shm." Let

$$\mathcal{L}(A) := \{u \,\text{shm in}\, A : u(z) - \log|z| = 0(1) \,\text{as}\, |z| \to \infty,\ z \in A\}.$$

If $K \subset A$ is compact, for $z \in A$ we have ([Sa1], Proposition 3.4)

$$V_K(z) = \sup\{u(z) : u \in \mathcal{L}(A),\, u \leqslant 0 \,\text{on}\, K\}.$$

If K is nonpolar in A, the function

$$V_K^*(z) := \limsup_{\zeta \to z,\, \zeta \in A} V_K(\zeta)$$

is shm on A; indeed $V_K^* \in \mathcal{L}(A)$.

We identify a positive measure μ on A with the (positive) $(1,1)$-current $\mu \cdot \beta|_A$ where

$$\beta = \Big(\frac{1}{4}\Big) dd^c(|z_1|^2 + \cdots + |z_N|^2) = \frac{i}{2} dz_1 \wedge d\overline{z}_1 + \ldots + \frac{i}{2} dz_N \wedge d\overline{z}_N.$$

Here $d = \partial + \overline{\partial}$ and $d^c = i(\overline{\partial} - \partial)$. The operator dd^c is considered on the (one-dimensional) complex manifold A^0; for a shm function u on A, $dd^c u$ is a positive measure on A^0 and is extended by zero on $A \setminus A^0$ (see [Be] for details). Recall that $m_d = \dim P_d|_A = dD + c$ for d sufficiently large where D is the degree of A.

THEOREM 1.1. — *Let A be an irreducible algebraic curve in $\mathbb{C}^N$ and let $K \subset A$ be nonpolar. Given an array $\{A_{dj}\} \subset K$ satisfying (0.2), the probability measures*

$$\mu_d := \frac{1}{m_d} \sum_{j=1}^{m_d} \delta_{A_{dj}},\ d = 1, 2, \ldots,$$

converge weak-$*$ to $\mu_K := \frac{1}{2\pi D}\, dd^c V_K^*$, and $\mathrm{supp}(\mu_K) \subset \partial K$.

2. Proof of Theorem 1.1.

We begin with a sequence of lemmas.

LEMMA 2.1. — *Let $p(z)$ be a polynomial and let $a \in A^0$ be an isolated simple zero (relative to A) of p. Then in a sufficiently small coordinate disk $B := B(a,r) \subset A^0$ ($B(a,r)$ is the image of a disk $\Delta \subset \mathbb{C}$ under a one-to-one holomorphic map ψ), $dd^c(\log|p(z)|)|_B = 2\pi\delta_a\beta|_B$.*

Proof. — Via ψ, we can identify B with Δ and $\log|p|$ with $\log|p(\psi)|$. We may assume $\psi(0) = a$. The condition that $a \in A^0$ be an isolated simple zero (relative to A) of p may then be written as $p(\psi(t)) = t \cdot f(t)$ where f is a holomorphic, nonvanishing function on a neighborhood of 0. Then

$$\Delta_t \log|p(\psi(t))| = \Delta_t(\log|t|) + \Delta_t \log|f(t)|.$$

But since f is nonvanishing near $t = 0$, $\log|f(t)|$ is harmonic and the result follows. □

Remark. — A similar argument shows the following: let $p(z)$, $q(z)$ be polynomials and let $a \in A^0$ be an isolated simple zero (relative to A) of p and of q. Then in a sufficiently small coordinate disk $B \subset A^0$,

$$dd^c(\max[\log|p(z)|, \log|q(z)|])|_B = 2\pi\delta_a\beta|_B + \mu|_B$$

where μ is a positive measure with $\mu(\{a\}) = 0$. The only difference is that, writing $p(\psi(t)) = t \cdot f(t)$ and $q(\psi(t)) = t \cdot g(t)$ where f and g are holomorphic, nonvanishing functions in a neighborhood of 0,

$$\Delta_t\big(\max[\log|p(\psi(t))|, \log|q(\psi(t))|]\big) = \Delta_t(\log|t|) + \Delta_t(\max[\log|f(t)|, \log|g(t)|]).$$

Now since f and g are nonvanishing near $t = 0$, $\max[\log|f(t)|, \log|g(t)|]$ is bounded and hence $\Delta_t(\max[\log|f(t)|, \log|g(t)|])$ is a positive measure which can put no mass at the origin.

The next result is stated and proved in [H] (Theorem 3.2.12) in the case of shm functions on domains in $\mathbb{R}^N$. The proof is valid for shm functions on domains in algebraic curves A since such functions u are locally integrable with respect to area measure $\beta|_A$ and since $dd^c u$ is a positive measure.

LEMMA 2.2. — *Let $\{u_j\}$ be a sequence of* shm *functions on a domain $\Omega \subset A$ which are locally uniformly bounded above on Ω. Then* *either* $u_j \to -\infty$ *locally uniformly in* Ω *or else there exists a subsequence* $\{u_{j_k}\} \subset \{u_j\}$ *which converges in* $L^1_{\text{loc}}(\Omega)$ *to a* shm *function* u.

LEMMA 2.3. — *Let* $G \subset A$ *be a domain and let* v *be harmonic on* G*; i.e.,* $v|_{G\cap A^0}$ *is harmonic; equivalently,* $dd^c v = 0$ *on* G. *Let* $\{u_n\}$ *be a sequence of* shm *functions on* G *satisfying*

$$u_n(z) \leqslant v(z),\ n = 1, 2, ..., \text{ for all } z \in G \cap A^0.$$

Suppose $u_n \to u$ *in* $L^1_{\text{loc}}(G)$ *where* u *is* shm *in* G *and that there exists* **one** *point* $z_0 \in G \cap A^0$ *at which*

$$\lim_{n\to\infty} u_n(z_0) = v(z_0).$$

Then $u(z) = v(z)$ *for all* $z \in G \cap A^0$.

Proof. — Fix a coordinate disk $B := B(z_0, r)$ with $B(z_0, r) \subset G \cap A^0$. As in the proof of Lemma 2.1, we identify B with a disk $\Delta \subset \mathbb{C}$ and v, u_n, u restricted to B with the corresponding functions on Δ. By the sub-mean-value property,

$$u_n(z_0) \leqslant \frac{1}{\pi r^2}\int_B u_n(z)dm(z),\ n = 1, 2, ...$$

Since $u_n \to u$ in $L^1_{\text{loc}}(G)$, letting $n \to \infty$ we obtain

$$v(z_0) \leqslant \frac{1}{\pi r^2}\int_B u(z)dm(z).$$

On the other hand, by the mean-value property,

$$v(z_0) = \frac{1}{\pi r^2}\int_B v(z)dm(z),$$

so that

$$\int_B v(z)dm(z) \leqslant \int_B u(z)dm(z).$$

But $u_n \to u$ in $L^1_{\text{loc}}(G)$ and $u_n(z) \leqslant v(z)$, $n = 1, 2, ...$, for $z \in G$ and hence on B; so that $u \leqslant v$ a.e. on B. The above inequality thus implies that $u = v$ a.e. on B. Since u, v are shm, $u = v$ on all of B. Repeating the argument replacing z_0 by other points of B, and using connectedness of G yields the result. □

Proof of Theorem 1.1. — Note that since $m_d = dD + c$ for d sufficiently large where c is independent of d, $\mu_d \to \mu_K$ (see (0.3)) if and only if $\tilde{\mu}_d \to \mu_K$ where $\tilde{\mu}_d := \frac{1}{dD}\sum_{j=1}^{m_d} \delta_{A_{dj}}$. For the remainder of the proof, we always take d large so that $m_d = dD + c$.

Now each $l_j^{(d)}$, $j = 1, \dots, m_d$, is a polynomial of degree d which vanishes at the m_d points $\{A_{dk}\}_{k=1,\dots,m_d}$ except for A_{dj}. By Bezout's theorem, each $l_j^{(d)}$ has at most dD zeros on A. From the relation $m_d = dD + c$, we see that if we choose any $j = j(d) \in \{1, \dots, m_d\}$, the sequence of measures

$$\nu_d := \frac{1}{2\pi D} dd^c\Big(\frac{1}{d}\log|l_j^{(d)}|\Big)$$

has the same weak$-*$ limits as the sequence $\{\mu_d\}$. We record this observation.

Remark. — If a subsequence $\{\mu_{d_k}\}$ of $\{\mu_d\}$ converges weak-$*$ to a probability measure ν on K, then so does the corresponding subsequence of $\frac{1}{2\pi D} dd^c(\frac{1}{d}\log|l_j^{(d)}|)$ where we may choose, for each d_k, any choice of $j = j(d_k) \in \{0, 1, \dots, m_{d_k}\}$.

The measure $dd^c V_K^*$ is supported on $\partial\hat{K}$ since $V_K = 0$ on $\hat{K}$ and V_K^* is a harmonic function on $A^0 \setminus K$ (recall also that $dd^c V_K^*$ puts no mass on the polar set $A \setminus A^0$).

We let $\phi_d(z) := \max_{j=1,\dots,m_d} \frac{1}{d}\log|l_j^{(d)}(z)|$.

Claim. — $\lim_{d\to\infty} \phi_d(z) = V_K(z)$ for all $z \in A$.

Proof of Claim. — This is where we use the hypothesis (0.2). The proof is essentially in [BBCL], 2.3. First of all,

$$\frac{|l_j^{(d)}(z)|}{\Lambda_d} \leqslant \frac{|l_j^{(d)}(z)|}{||l_j^{(d)}||_K} \leqslant e^{dV_K(z)}$$

so that

$$\phi_d(z) = \max_{j=1,\dots,m_d} \frac{1}{d}\log|l_j^{(d)}(z)| \leqslant \frac{1}{d}\log\Lambda_d + V_K(z);$$

and, using (0.2),

$$\limsup_{d\to\infty} \phi_d(z) \leqslant V_K(z).$$

On the other hand, for any polynomial p_d of degree at most d,

$$p_d(z) = \sum_{j=1}^{m_d} p_d(A_{dj}) l_j^{(d)}(z),\ z \in A,$$

so that, if $||p_d||_K \leqslant 1$,

$$|p_d(z)| \leqslant \sum_{j=1}^{m_d} |l_j^{(d)}(z)| \leqslant m_d e^{d\phi_d(z)}.$$

Hence

$$\sup\left\{\frac{1}{d}\log|p_d(z)| : \deg p_d \leqslant d,\ ||p_d||_K \leqslant 1\right\} \leqslant \frac{1}{d}\log m_d + \phi_d(z);$$

thus

$$V_K(z) = \lim_{d\to\infty}\left[\sup\left\{\frac{1}{d}\log|p_d(z)| : \deg p_d \leqslant d, ||p_d||_K \leqslant 1\right\}\right] \leqslant \liminf_{d\to\infty}\phi_d(z)$$

and the claim follows. □

Now we put all these ingredients together. Let $\{\mu_{d_k}\} \subset \{\mu_d\}$ be a subsequence which converges to a probability measure ν on K. We want to show $\nu = \frac{1}{2\pi D}dd^cV_K^*$.

Case I: Suppose $\mathrm{int}K = \emptyset$, *i.e., the relative interior of K (relative to A) is empty, and $G := A \setminus \partial K = A \setminus K$ is connected.* Note that K has empty interior, for example, in the case of a *real* curve K in A. Fix a point $a_0 \in G \cap A^0$ at which $V_K(a_0) = V_K^*(a_0)$. For each d, choose $j_d \in \{1, ..., m_d\}$ so that $\phi_d(a_0) = \frac{1}{d}\log|l_{j_d}^{(d)}(a_0)|$. By the claim,

$$\lim_{d\to\infty}\phi_d(a_0) = \lim_{d\to\infty}\frac{1}{d}\log|l_{j_d}^{(d)}(a_0)| = V_K(a_0);$$

and, since this limit exists and equals $V_K(a_0)$, the same is true for any subsequence. Thus we pick a subsequence $\{d_k\}$ of $\{d\}$ as above; i.e., such that $\mu_{d_k} \to \nu$. We will use the notation $u_{d_k}(z) := \frac{1}{d_k}\log|l_{j_{d_k}}^{(d_k)}(z)|$; these are shm functions in A. Since $u_{d_k} \leqslant \phi_{d_k} \leqslant \frac{1}{d_k}\log\Lambda_{d_k} + V_K$ in all of A, $\{u_{d_k}\}$ is a sequence of shm functions (indeed, strongly shm) which are locally uniformly bounded above on A. Since all but at most a bounded number b of the zeroes of each $l_{j_{d_k}}^{(d_k)}$ lie in K, with b independent of d, it is clear that $u_{d_k} \not\to -\infty$ in all of A and hence, by Lemma 2.2 applied to $\Omega = A$, we may choose a subsequence, which we again call $\{u_{d_k}\}$, which converges in $L^1_{\mathrm{loc}}(A)$ to a shm function u. Note that $u \leqslant V_K$ a.e. on A, so that $u \leqslant V_K^*$ on A, and, by the previous remark, $dd^cu = 2\pi D\nu$.

We need to make a minor adjustment to apply Lemma 2.3 on G (where $dd^cV_K^* = 0$): we don't quite have $u_{d_k} \leqslant V_K$ in G; only

$$u_{d_k} \leqslant \frac{1}{d_k}\log\Lambda_{d_k} + V_K.$$

Since $\frac{1}{d_k}\log\Lambda_{d_k} \to 0$ as $k \to \infty$, we can take

$$\tilde{u}_{d_k} := u_{d_k} - \frac{1}{d_k}\log\Lambda_{d_k} \leqslant V_K \text{ in } A;\ \lim_{k\to\infty}\tilde{u}_{d_k}(a_0) = V_K(a_0);$$

$$\text{and } \tilde{u}_{d_k} \to u \text{ in } L^1_{\mathrm{loc}}(A).$$

Applying Lemma 2.3 to $\{\tilde{u}_{d_k}\}$ on G, we conclude that $u = V_K^*$ in $G \cap A^0$. Note that

$$u \leqslant V_K^* \text{ in all of } A \tag{2.1}$$

since $\tilde{u}_{d_k} \leqslant V_K$ in all of A and $\tilde{u}_{d_k} \to u$ in $L^1_{\mathrm{loc}}(A)$ imply that $u \leqslant V_K$ a.e. on A; since u, V_K^* are shm on A, this yields (2.1).

Since $V_K = V_K^*$ for q.e. $z \in A$, we have

$$V_K^*(z) := \limsup_{\zeta \in A^0, \zeta \to z} V_K(\zeta) = \limsup_{\zeta \in G \cap A^0, \zeta \to z} V_K(\zeta) = 0 = V_K(z) \tag{2.2}$$

q.e. on ∂K. Note that since V_K^* is shm and $V_K \leqslant V_K^*$ on all of A,

$$\limsup_{\zeta \in G \cap A^0, \zeta \to z} V_K(\zeta) \leqslant \limsup_{\zeta \in G \cap A^0, \zeta \to z} V_K^*(\zeta) \leqslant \limsup_{\zeta \in A, \zeta \to z} V_K^*(\zeta) = V_K^*(z) = 0 \tag{2.3}$$

for $z \in \partial K$ satisfying (2.2).

We show that $u = V_K^*$ q.e. on ∂K. To see this, fix $z \in \partial K \cap A^0$ as in (2.2) so that, combining (2.2) and (2.3),

$$V_K^*(z) = V_K(z) = \limsup_{\zeta \in G \cap A^0,\ \zeta \to z} V_K(\zeta) = \limsup_{\zeta \in G \cap A^0,\ \zeta \to z} V_K^*(\zeta) = 0.$$

Since $u = V_K^*$ in $G \cap A^0$,

$$\limsup_{\zeta \in G \cap A^0,\ \zeta \to z} u(\zeta) = \limsup_{\zeta \in G \cap A^0,\ \zeta \to z} V_K^*(\zeta) = V_K^*(z) = 0.$$

But from (2.1), $u \leqslant V_K^*$ in all of A; hence, using the fact that at points $z \in A^0$, $\limsup_{\zeta \in A^0,\ \zeta \to z} u(\zeta) = u(z)$ (by shm of u),

$$V_K^*(z) = \limsup_{\zeta \in G \cap A^0,\ \zeta \to z} u(\zeta) \leqslant \limsup_{\zeta \in A^0,\ \zeta \to z} u(\zeta) = u(z) \leqslant V_K^*(z)$$

and equality holds throughout this string of inequalities. Since K has empty interior relative to A, $u = V_K^*$ q.e. on $K = \partial K$, and we have that $u = V_K^*$ q.e. on A^0. In particular, $dd^c u = 2\pi D\nu = dd^c V_K^*$.

Remark. — If $\operatorname{int} K \neq \emptyset$ but we take points $\{A_{dj}\}$ which belong to ∂K, then $dd^c u$ has its support in ∂K. The functions u and V_K^* agree q.e. on $G \cup \partial K$; in particular, $u \geqslant 0$ q.e. on $G \cup \partial K$. Thus $\tilde{u} := \max[u, 0] = V_K^*$ q.e. on A and $dd^c \tilde{u} = dd^c u$ so the conclusion is valid in this situation as well. Note this applies to a "disk" $K \equiv \{z \in A : |z_N| \leqslant 1\}$ where

$$A \subset \{(z', z_N) \in \mathbb{C}^N : |z'|^2 < C(1 + |z_N|^2)\},\ z' = (z_1, \dots, z_{N-1}),$$

for some $C > 0$. Here, $V_K(z) = \max[0, \log|z_N|]$ for $z \in A$ and we thus have

$$\int_A dd^c V_K = \int_K dd^c V_K = 2\pi D$$

where D is the degree of A. This will be used in the next proposition.

PROPOSITION 2.4. — *Let A be an algebraic curve of degree D in $\mathbb{C}^N$. For any $u \in \mathcal{L}(A)$,*

$$\int_A dd^c u \leqslant 2\pi D.$$

Proof. — From the remark we have that

$$\int_A dd^c V_K = \int_K dd^c V_K = 2\pi D$$

where $K = \{z \in A : |z_N| \leqslant 1\}$. Now first suppose $u \in \mathcal{L}(A) \cap L^\infty_{loc}(A)$. We show that

$$\int_A dd^c u \leqslant \int_A dd^c V_K. \tag{2.4}$$

To see this, we follow an argument of Taylor [T]. Let $M = \sup_K u(z)$. Then $\tilde{u} := u - M \leqslant V_K$ in A. For $\epsilon > 0$ and $c > 0$, let $w(z) := (1+\epsilon)V_K(z) - c$. Set

$$G := \{z \in A : w(z) < \tilde{u}(z)\}.$$

Then $G \subset\subset A$ and by the comparison theorem (valid for locally bounded psh functions on relatively compact subsets of analytic sets (cf., [Be] or [Ze])),

$$\int_G dd^c \tilde{u} \leqslant \int_G dd^c w;$$

i.e.,

$$\int_G dd^c u \leqslant (1+\epsilon) \int_G dd^c V_K \leqslant (1+\epsilon) \int_A dd^c V_K.$$

The above inequality is valid for any $c > 0$; letting $c \to +\infty$, so that $G \to A$, we have

$$\int_A dd^c u = \int_{A \cap \{u > -\infty\}} dd^c u \leqslant (1+\epsilon) \int_A dd^c V_K.$$

Now let $\epsilon \to 0$ to get (2.4).

If $u \in \mathcal{L}(A)$ is not locally bounded, we can apply the above argument to $u_n(z) := \max[u(z), -n]$ to get, from (2.4),

$$\int_A dd^c u_n \leqslant \int_A dd^c V_K.$$

Now $\{u_n\}$ decrease pointwise to u so that $dd^c u_n \to dd^c u$ as measures ((1,1)−currents with measure coefficients); thus, for any $\psi \in C_0^\infty(A)$ (or just in $C_0(A)$) we have

$$\lim_{n\to\infty} \int_A \psi dd^c u_n = \int_A \psi dd^c u.$$

Taking $0 \leqslant \psi \leqslant 1$ with $\psi \equiv 1$ on $K_R := \{z \in A : |z_N| \leqslant R\}$, we see that

$$\int_{K_R} dd^c u \leqslant \int_A \psi dd^c u = \lim_{n\to\infty} \int_A \psi dd^c u_n \leqslant \limsup_{n\to\infty} \int_A dd^c u_n \leqslant \int_A dd^c V_K.$$

The above inequality is valid for any R; hence the result. □

Case II: Suppose $A \setminus \partial K$ consists of finitely many components. For simplicity in notation, we give the proof for $A \setminus \partial K = G_1 \cup G_2$ (two components). Recall that

$$\mu_d := \frac{1}{m_d} \sum_{j=1}^{m_d} \delta_{A_{dj}}; \tag{0.3}$$

$\{\mu_{d_k}\} \subset \{\mu_d\}$ is a subsequence which converges to a probability measure ν on K; and we want to show $\nu = \frac{1}{2\pi D} dd^c V_K^*$. Fix points $a_i \in G_i$ at which $V_K(a_i) = V_K^*(a_i)$. For each d, we first choose $j_d \in \{1, ..., m_d\}$ so that $\phi_d(a_1) = \frac{1}{d} \log |l_{j_d}^{(d)}(a_1)|$. By the claim,

$$\lim_{d\to\infty} \phi_d(a_1) = \lim_{d\to\infty} \frac{1}{d} \log |l_{j_d}^{(d)}(a_1)| = V_K(a_1);$$

and, since this limit exists and equals $V_K(a_1)$, the same is true for any subsequence. Thus we pick a subsequence $\{d_k\}$ of $\{d\}$ such that $\mu_{d_k} \to \nu$. We write $u_{d_k}(z) := \frac{1}{d_k} \log |l_{j_{d_k}}^{(d_k)}(z)|$; as before, $\{u_{d_k}\}$ is a sequence of shm functions which are locally uniformly bounded above on A and by Lemma 2.2 applied to $\Omega = A$, we may choose a subsequence, which we again call $\{u_{d_k}\}$, which converges in $L^1_{\text{loc}}(A)$ to a shm function u satisfying $u \leqslant V_K^*$ on A and $dd^c u = 2\pi D \nu$.

Again we adjust (to apply Lemma 2.3) by setting $\tilde{u}_{d_k} := u_{d_k} - \frac{1}{d_k} \log \Lambda_{d_k}$; then

$$\tilde{u}_{d_k} \leqslant V_K \text{ in } A; \ \lim_{k\to\infty} \tilde{u}_{d_k}(a_1) = V_K(a_1); \text{ and } \tilde{u}_{d_k} \to u \text{ in } L^1_{\text{loc}}(A).$$

Applying Lemma 2.3 to $\{\tilde{u}_{d_k}\}$ on G_1, we conclude that $u = V_K^*$ in $G_1 \cap A^0$. Note that

$$u \leqslant V_K^* \text{ in all of } A \tag{2.5}$$

since $\tilde{u}_{d_k} \leqslant V_K$ in all of A and $\tilde{u}_{d_k} \to u$ in $L^1_{\text{loc}}(A)$ imply that $u \leqslant V_K$ a.e. on A; since u, V_K^* are shm on A, this yields (2.5). Hence we have

$$u(z) \begin{cases} = V_K^*(z), & z \in G_1 \cap A^0; \\ \leqslant V_K^*(z), & z \in A \setminus G_1. \end{cases} \tag{2.6}$$

Now for each integer d_k, $k = 1, 2, ...$, choose $n_{d_k} \in \{1, ..., m_{d_k}\}$ such that $\phi_{d_k}(a_2) = \frac{1}{d_k} \log |l_{n_{d_k}}^{(d_k)}(a_2)|$. By the claim,

$$\lim_{k\to\infty} \phi_{d_k}(a_2) = \lim_{k\to\infty} \frac{1}{d_k} \log |l_{n_{d_k}}^{(d_k)}(a_2)| = V_K(a_2);$$

and, since this limit exists and equals $V_K(a_2)$, the same is true for any subsequence. Thus we pick a subsequence $\{\tilde{d}_k\}$ of $\{d_k\}$; note that $\mu_{\tilde{d}_k} \to \nu$. We write $v_{\tilde{d}_k}(z) := \frac{1}{\tilde{d}_k} \log |l_{n_{\tilde{d}_k}}^{(\tilde{d}_k)}(z)|$; these form a sequence of shm functions which are locally uniformly bounded above on A and by Lemma 2.2 applied to $\Omega = A$, we may choose a subsequence, which we again call $\{v_{\tilde{d}_k}\}$, which converges in $L^1_{\text{loc}}(A)$ to a shm function v. Note that $v \leqslant V_K$ a.e. on A, so that $v \leqslant V_K^*$ on A. By an earlier remark, the fact that $\{\mu_{d_k}\} \subset \{\mu_d\}$ is a subsequence which converges to the probability measure ν on K, and the fact that $\{\tilde{d}_k\}$ is a subsequence of $\{d_k\}$, we have $dd^c v = 2\pi D\nu$.

Define $\tilde{v}_{\tilde{d}_k} := v_{\tilde{d}_k} - \frac{1}{\tilde{d}_k} \log \Lambda_{\tilde{d}_k}$; then

$$\tilde{v}_{\tilde{d}_k} \leqslant V_K \text{ in } A; \ \lim_{k\to\infty} \tilde{v}_{\tilde{d}_k}(a_2) = V_K(a_2); \text{ and } \tilde{v}_{\tilde{d}_k} \to v \text{ in } L^1_{\text{loc}}(A).$$

Applying Lemma 2.3 to $\{\tilde{v}_{\tilde{d}_k}\}$ on G_2, we conclude that $v = V_K^*$ in $G_2 \cap A^0$. Note that

$$v \leqslant V_K^* \text{ in all of } A \tag{2.7}$$

since $\tilde{v}_{\tilde{d}_k} \leqslant V_K$ in all of A and $\tilde{v}_{\tilde{d}_k} \to v$ in $L^1_{\text{loc}}(A)$ imply that $v \leqslant V_K$ a.e. on A; since v, V_K^* are shm on A, this yields (2.7). Hence we have

$$v(z) \begin{cases} \leqslant V_K^*(z), & z \in A \setminus G_2; \\ = V_K^*(z), & z \in G_2 \cap A^0. \end{cases} \tag{2.8}$$

We put these sequences together in the following way: for $k = 1, 2, \ldots$, set

$$w_k(z) := \max\left[\tilde{u}_{\tilde{d}_k}(z), \tilde{v}_{\tilde{d}_k}(z)\right] = \max\left[\frac{1}{\tilde{d}_k} \log \left|l_{j_{\tilde{d}_k}}^{(\tilde{d}_k)}(z)\right|, \frac{1}{\tilde{d}_k} \log \left|l_{n_{\tilde{d}_k}}^{(\tilde{d}_k)}(z)\right|\right] - \frac{1}{\tilde{d}_k} \log \Lambda_{\tilde{d}_k}.$$

Thus w_k is defined from two FLIP's of the same degree $\tilde{d}_k$; and $w_k \leqslant V_K$ for all k. By taking a subsequence of $\{w_k\}$, if necessary, we may assume $w_k \to w$ in $L^1_{\text{loc}}(A)$, where $w \geqslant \max[u, v]$ (since $\tilde{u}_{\tilde{d}_k} \to u$ in $L^1_{\text{loc}}(A)$ and $\tilde{v}_{\tilde{d}_k} \to v$ in $L^1_{\text{loc}}(A)$); and, in particular, from (2.6) and (2.8),

$$w = V_K^* \text{ on } A^0 \setminus \partial K = A^0 \cap (G_1 \cup G_2).$$

We claim that $dd^c w = 2\pi D\nu$. Essentially this follows since $dd^c u = dd^c v = 2\pi D\nu$. Note first that $l_{j_{\tilde{d}_k}}^{(\tilde{d}_k)}$ and $l_{n_{\tilde{d}_k}}^{(\tilde{d}_k)}$ have $m_{\tilde{d}_k} - 2$ common zeros in K (all but 2 of the points $A_{\tilde{d}_k 1}, \ldots, A_{\tilde{d}_k m_{\tilde{d}_k}}$). By the remark after Lemma 2.1, Bezout's theorem, and the fact that $m_{\tilde{d}_k} = \tilde{d}_k D + c$, it follows that $dd^c w_k$

puts mass $\frac{2\pi}{\tilde{d}_k}$ at each of these $m_{\tilde{d}_k} - 2$ points except perhaps a fixed number, independent of k. By Proposition 2.4, the total mass of each measure $dd^c w_k$ is at most $2\pi D$. Since $1/m_{\tilde{d}_k} \to 0$ as $k \to \infty$, this implies that $\{dd^c w_k\}$ has a unique weak-* limit, namely $2\pi D\nu$.

Since $dd^c w = 2\pi D\nu$ and $w = V_K^*$ on $A^0 \setminus \partial K$, we complete the proof exactly as in Case I, obtaining next that $w = V_K^*$ q.e. on ∂K so that $w = V_K^*$ q.e. on A^0; hence $dd^c w = dd^c V_K^* = 2\pi D\nu$.

Case III. Suppose $A \setminus \partial K = \cup_{j=1}^{\infty} G_j$ has countably many components. By performing the procedure in Case II recursively, we obtain a sequence of subharmonic functions $\{w^{(n)}\}$ on A with the following properties:

1. $w^{(n)} \in \mathcal{L}(A)$ and
$$w^{(n)}(z) \begin{cases} \leqslant V_K^*(z), & z \in A \setminus \cup_{j=n+1}^{\infty} G_j; \\ = V_K^*(z), & z \in A^0 \cap (G_1 \cup \cdots \cup G_n); \end{cases}$$
2. $w^{(n)} \leqslant w^{(n+1)}$ on A (recall $w \geqslant \max[u, v] \geqslant u$ in the construction of Case II);
3. $dd^c w^{(n)} = 2\pi D\nu, \ n = 1, 2, \ldots.$

Thus the sequence $\{w^{(n)}\}$ is monotonically increasing and bounded above by V_K^*; hence
$$\lim_{n\to\infty} w^{(n)}(z) = w(z)$$
defines a function w with $w^* \in \mathcal{L}(A)$ and $w = w^* = V_K^*$ q.e. on $A \setminus \partial K$. By 3., $dd^c w^* = 2\pi D\nu$. Again we complete the proof exactly as in Case I. □

Remark. — Note the proof of the theorem yields the conclusion that the weak-* limit of the measures μ_d is supported on ∂K (since $\mathrm{supp}(dd^c V_K^*) \subset \partial K$) regardless of whether or not the array $\{A_{dj}\}$ satisfying (0.2) contains points in $\mathrm{int}\, K$.

3. Final remarks.

Slightly generalizing the situation in [GMS], we will say that a compact set $K \subset \mathbb{C}^N$ has an *asymptotic interpolation measure* (an AIM) if there exists a probability measure μ_K on K such that for any good array for K (Definition 0.1), the discrete measures
$$\mu_d := \frac{1}{m_d} \sum_{j=1}^{m_d} \delta_{A_{dj}}, \ d = 1, 2, \ldots,$$

converge weak-$*$ to μ_K. Here, $m_d = \dim P_d|_K$; we do not (a priori) require K to lie on an algebraic variety. The situation of [GMS] is to consider certain compact subsets K of $\mathbb{R}^2$; this is a special case of the $N = 2$ setting (considering $\mathbb{R}^2$ as the set of real points in $\mathbb{C}^2$). In this higher-dimensional setting (real or complex), essentially nothing is known about AIM's outside of the results of the [GMS] paper. In this vein, we mention an interesting fact proved in the real setting in [GMS]; the proof goes through with no changes to the complex setting.

PROPOSITION 3.1 (Theorem 3.2, [GMS]). — *Let $A = \cup_{j=1}^m A^{(j)}$ be an algebraic subvariety of $\mathbb{C}^N$ consisting of distinct algebraic curves $A^{(j)}$, $j = 1, ..., m$ of degrees D_j. Let $K_j \subset A^{(j)}$, $j = 1, ..., m$, be compact subsets, and let $K := \cup_{j=1}^m K_j$. If each K_j has an AIM μ_{K_j}, then K has an AIM μ_K; moreover, in this case,*

$$\mu_K = \sum_{j=1}^m \frac{D_j}{D} \mu_{K_j} \text{ where } D = \sum_{j=1}^m D_j.$$

Thus our Theorem 1.1 can be extended to the situation of a finite union of nonpolar compact pieces of algebraic curves. We next go through an interesting, albeit elementary, example. We return to the algebraic set $A = \{(z_1, z_2) \in \mathbb{C}^2 : z_1 z_2 = 1\}$ and the compact subset $K = \{(e^{i\theta}, e^{-i\theta}) \in A : \theta \in [0, 2\pi]\}$.

Claim. — $V_K(z_1, z_2)|_A = \max[\log^+ |z_1|, \log^+ |z_2|]$.

Proof of Claim. — Note if $S^1 := \{t = e^{i\theta} \in \mathbb{C} : \theta \in [0, 2\pi]\}$ and $g(w) = (w, 1/w)$, then $g(S^1) = K$ and $g(\mathbb{C} \setminus \{0\}) = A$. From Sadullaev [Sa1], we know that

$$V_K(z_1, z_2)$$
$$= \max\Big\{0, \sup\Big\{\frac{1}{\deg p} \log \frac{|p(z_1, z_2)|}{||p||_K} : p \text{ is a polynomial, } \deg p > 0\Big\}\Big\}$$

at points $(z_1, z_2) \in A$. Now for a polynomial $p(z_1, z_2)$, we have $p(g(w)) = p(w, 1/w)$ so that

$$||p||_K = \max_{e^{i\theta} \in S^1} |p(e^{i\theta}, e^{-i\theta})|.$$

But if p is of degree d, say, then

$$q(w) := w^d p(g(w)) = w^d p(w, 1/w)$$

is a polynomial of degree at most $2d$ in w; hence

$$||q||_{S^1} = \max_{e^{i\theta} \in S^1} |p(e^{i\theta}, e^{-i\theta})| = ||p||_K.$$

Thus, by the classical univariate Bernstein-Walsh inequality, if $|w| > 1$, and we write $g(w) = (z_1, z_2)$, then

$$|q(w)| = |w^d p(w, 1/w)| = |z_1|^d \cdot |p(z_1, z_2)| \leqslant |w|^{2d} ||q||_{S^1} = |w|^{2d} ||p||_K$$
$$= |z_1|^{2d} ||p||_K$$

so that

$$|p(z_1, z_2)| \leqslant |z_1|^d ||p||_K.$$

In particular, $|p(z_1, z_2)| \leqslant |z_1|^d$ if $||p||_K = 1$ and we conclude that $V_K(z_1, z_2) \leqslant \log |z_1|$ if $|z_1| > 1$. Similarly, we get that $V_K(z_1, z_2) \leqslant \log |z_2|$ if $|z_2| > 1$. On the other hand, clearly

$$\max[\log^+ |z_1|, \log^+ |z_2|] \leqslant V_K(z_1, z_2)$$

on all of A and the claim is proved. □

If we consider $S^1 = \{t \in \mathbb{C} : |t| = 1\}$ as a subset of $\mathbb{R}^2 \subset \mathbb{C}^2$, then the space of polynomials of degree at most d in two variables restricted to S^1 are the trigonometric polynomials

$$T_d := \text{span}\{1, t, \bar{t}, t^2, \bar{t}^2, ..., t^d, \bar{t}^d\}.$$

We label these monomials $e_j(t),\ j = 1, ..., 2d + 1$.

With $K = \{(z_1, z_2) \in \mathbb{C}^2 : z_1 z_2 = 1,\ |z_1| = 1\}$, the mapping $g(t) = (t, \bar{t}) = (t, 1/t)$ is a one-to-one map of S^1 onto K; moreover, the space T_d is transformed into

$$P_d|_K = \text{span}\{1, z_1, z_2, z_1^2, z_2^2, ..., z_1^d, z_2^d\}$$

which is the vector space of all polynomials of degree at most d restricted to K. We label these monomials $\tilde{e}_j(z_1, z_2)$; note that $\tilde{e}_j(g(t)) = e_j(t),\ j = 1, ..., 2d + 1$.

In particular, *any choice of* $2d + 1$ *points* $p_1, ..., p_{2d+1}$ *in* K, *where* $p_j = (z_1^{(j)}, z_2^{(j)}) = (z_1^{(j)}, 1/z_1^{(j)})$, *corresponds to a set of* $2d + 1$ *points* $t_1, ..., t_{2d+1}$ *on* S^1 *via* $p_j = g(t_j)$. Therefore

$$\det[e_j(t_k)]_{j,k=1,...,2d+1} = \det[\tilde{e}_j(p_k)]_{j,k=1,...,2d+1}.$$

Thus, not only do Fekete points of order d (equally spaced points on S^1) get transformed under g to Fekete points of order d on K, but for **any** set of $2d + 1$ points $t_1, ..., t_{2d+1}$ on S^1 with

$$\det[e_j(t_k)]_{j,k=1,...,2d+1} \neq 0$$

we have

$$\det[\tilde{e}_j(p_k)]_{j,k=1,...,2d+1} \neq 0$$

and, moreover, the corresponding FLIP's on S^1 and K "coincide":

$$\ell_j^{(d)}(t) = \tilde{\ell}_j^{(d)}(z_1, z_2),\ t \in S^1,\ g(t) = (z_1, z_2) \in K$$

since the ratio of the VDM's coincides; cf., (0.1). In particular, the Lebesgue constants for the arrays on S^1 and on K are equal.

We next show that $dd^cV_K = 2g_*(d\theta)$. To see this, note we have

$$dd^cV_K = g_*(\Delta_t(V_K \circ g))$$

where Δ_t is the (usual) Laplacian on $\mathbb{C}$. Now $V_K \circ g(t) = \max[0, \log|t|, \log 1/|t|] = \log^+|t| + \log^+(1/|t|)$. Since

$$\log^+|t| - \log^+(1/|t|) = \log|t|$$

which is harmonic for $t \neq 0$, we have $\Delta_t \log^+(1/|t|) = \Delta_t \log^+|t| = d\theta$ and hence $\Delta_t(V_K \circ g(t)) = 2d\theta$.

We remark that since condition (0.2) for an array on S^1 implies that the corresponding probability measures converge weak-* to $d\theta/2\pi$, the above argument shows directly that if an array $\{A_{dj}\}_{j=1,\ldots,2d+1;d=1,2,\ldots} \subset K = \{(z_1, z_2) \in \mathbb{C}^2 : z_1z_2 = 1,\ |z_1| = 1\}$ satisfies

$$\limsup_{d\to\infty} \Lambda_d^{1/d} \leqslant 1,$$

then $\mu_d = \frac{1}{2d+1}\sum_{j=1}^{2d+1} \delta_{A_{dj}} \to \frac{1}{4\pi} dd^cV_K$ weak-*.

We conclude with an open question.

Q. *Is every* $u \in \mathcal{L}(A)$ *the restriction to* A *of a (global) function* $u \in \mathcal{L}(\mathbb{C}^N)$?

If $A^0 = A$, so that A is a complex submanifold of $\mathbb{C}^N$, it is well-known that psh functions on A extend to all of $\mathbb{C}^N$; we include the following elementary proof of Sadullaev [Sa2] since it is not readily available in the literature. We thank E. Poletsky for translating the contents of [Sa2].

THEOREM 3.2 [Sa2]. — *Let* M *be a complex submanifold of* $\mathbb{C}^N$. *Then any* psh *function* u *on* M *has a global extension to* $\mathbb{C}^N$; *i.e., there exists* w psh *in* $\mathbb{C}^N$ *such that* $w|_M = u$.

The proof hinges on two facts, the first of which is decidedly non-elementary.

1. *If* M *is a complex submanifold of* $\mathbb{C}^N$, *then there exist a neighborhood* V *of* M *in* $\mathbb{C}^N$ *and a holomorphic retraction* $r : V \to M$ [DG].

2. *If $D \subset \mathbb{C}^N$ is pseudoconvex and ϕ is any real-valued function which is locally bounded above on D, then there exists u* psh *in D with $u \geqslant \phi$ in D.*

Proof of 2: Let v be a psh exhaustion function for D and define, for r real,

$$q(r) := \sup\{\phi(z) : v(z) \leqslant r\}.$$

Then q is locally bounded above on $\mathbb{R}$. We can find a convex, increasing function $f(r)$ such that $f(r) \geqslant q(r)$; composing with v, $u(z) := (f \circ v)(z) \geqslant \phi(z)$. □

Proof of Theorem 3.2. — Let V be a neighborhood of M in $\mathbb{C}^N$ as in 1; i.e., such that there exists a holomorphic retraction $r : V \to M$. Now let u be psh on M and define $\tilde{u}(z) := u(r(z))$. Then $\tilde{u}$ is psh on V and $\tilde{u} = u$ on M. Next let D be a pseudoconvex domain with $M \subset D \subset \overline{D} \subset V$.

Since M is a complex submanifold of $\mathbb{C}^N$, we can find global holomorphic functions $\phi_1, ..., \phi_k$ such that

$$M = \{z \in \mathbb{C}^N : \phi_1(z) = \cdots = \phi_k(z) = 0\}$$

(cf. [Kr], Theorem 7.2.4). Define $\rho(z) := \log\left[|\phi_1(z)|^2 + \cdots + |\phi_k(z)|^2\right]$. Note that $-\rho$ is locally bounded above on ∂D; thus, we can find a real-valued function ϕ in $\mathbb{C}^N$ such that ϕ is locally bounded above on $\mathbb{C}^N$ and $\phi = \tilde{u} - \rho$ on ∂D. By 2, we can find v psh in $\mathbb{C}^N$ with $v \geqslant \phi$ on all of $\mathbb{C}^N$.

Finally, define

$$w(z) := \begin{cases} \max\left[\tilde{u}(z), v(z) + \rho(z)\right], & z \in D; \\ v(z) + \rho(z), & z \in \mathbb{C}^N \setminus D. \end{cases}$$

Clearly w is psh in all of $\mathbb{C}^N$ and, since $\rho(z) = -\infty$ for $z \in M$, $w(z) = \tilde{u}(z) = u(z)$ for $z \in M$. □

Using Theorem 3.2, we can almost answer the open question for the case when $A^0 = A$.

PROPOSITION 3.3. — *For any $u \in \mathcal{L}(A)$ and any $\epsilon > 0$, there exists a* psh *function U in $\mathbb{C}^N$ such that ¿$U = u$ on A and*

$$U(z) \leqslant \epsilon + (1 + \epsilon) \log(1 + |z|) \text{ on } \mathbb{C}^N. \tag{3.1}$$

Proof. — From Theorem 3.2, there exists $\tilde{u}$ psh in $\mathbb{C}^N$ with $\tilde{u} = u$ on A. Fix $\epsilon' < \epsilon$. By (3.1), If $\tilde{u}(z) < \log^+ |z| + \epsilon' \log(1 + |z|)$ on $\mathbb{C}^N$, we are done. If not,

$$\Theta := \{z \in \mathbb{C}^N : \tilde{u}(z) < \log^+ |z| + \epsilon' \log(1 + |z|)\}$$

is an open neighborhood of A. Let $A = \{z : Q_1(z) = ... = Q_{N-1}(z) = 0\}$ and let $v(z) = \frac{1}{2m}\log[|Q_1(z)|^2 + \cdots + |Q_{N-1}(z)|^2]$ where $m = \max_{i=1,\dots,N-1} \deg Q_i$. Take $M = M(\epsilon')$ large so that

$$\Theta' := \{z \in \mathbb{C}^N : v(z) < -M\} \subset\subset \Theta.$$

Then for any $\eta > 0$, the function $\phi(z) := \eta[v(z) + M]$ satisfies

a. $\phi(z) = -\infty$ on A;

b. $\phi(z) > 0$ outside $\overline{\Theta'}$;

c. $\phi(z) \leqslant \eta \log(1 + |z|) + \eta(C + M)$ in $\mathbb{C}^N$ where $C = C(A)$.

We define $U(z)$ as follows:

$$U(z) = \begin{cases} \max[\tilde{u}(z), \log^+|z| + \epsilon' \log(1 + |z|) + \phi(z)], & z \in \overline{\Theta'}; \\ \log^+|z| + \epsilon' \log(1 + |z|) + \phi(z), & z \notin \overline{\Theta'}. \end{cases}$$

Since $\Theta' \subset\subset \Theta$, using b, we have

$$\tilde{u} < \log^+|z| + \epsilon' \log(1 + |z|) \leqslant \log^+|z| + \epsilon' \log(1 + |z|) + \phi(z)$$

near $\partial\Theta'$ so that U is a well-defined global psh function; moreover, by a., $U(z) = \tilde{u}(z)$ on A. Condition c. gives a growth estimate of

$$U(z) \leqslant (1 + \epsilon' + \eta)\log(1 + |z|) + \eta(C + M)$$

on $\mathbb{C}^N$. Since $\epsilon' < \epsilon$ is fixed, we can choose $\eta > 0$ sufficiently small so that $\epsilon' + \eta < \epsilon$ and so that $\eta(C + M) < \epsilon$. The result is proved. □

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Thomas BLOOM,
University of Toronto
Department of Mathematics
Toronto, Ontario M5S 3G3 (Canada).
bloom@math.toronto.edu

Norman LEVENBERG,
University of Auckland
Department of Mathematics
Private Bag 92019
Auckland (New-Zealand).
levenber@math.auckland.ac.nz