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Fabien NAPOLITANO

Cohomology rings of spaces of generic bipolynomials and extended affine Weyl groups of serie A

COHOMOLOGY RINGS OF SPACES OF GENERIC BIPOLYNOMIALS AND EXTENDED AFFINE WEYL GROUPS OF SERIE A

by Fabien NAPOLITANO

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Introduction.

A polynomial is a holomorphic mapping of a sphere onto a sphere such that some point on the target sphere has only one preimage. In his famous paper on topological invariants of algebraic functions [Arn70a], Arnol'd described the connections between braid groups, algebraic functions and spaces of polynomials without multiple roots and he proved that the cohomology rings of spaces of polynomials without double roots stabilize when the degree of the polynomials tends to infinity. In the present paper we similarly study the cohomology rings of the spaces of bipolynomials without multiple roots. The study of bipolynomials has been initiated by Goryunov [Gor81] and continued by Arnol'd [Arn96] in connection with combinatorics of graphs. A bipolynomial of bidegree (k, ℓ) is the restriction of a polynomial in two variables

$$h(z, w) = w^k + a_1 w^{k-1} + \cdots + a_{k-1} w + a_k + a_{k+1} z + \cdots + a_{k+\ell-1} z^{\ell-1} + z^\ell$$

to the "hyperbola" $wz = \gamma$. The space of bipolynomials of bidegree (k, ℓ) is a complex affine space of dimension $k + \ell$ with coordinates $(a_1, \ldots, a_{k+\ell-1}, \gamma)$ in $\mathbb{C}^{p+q}$.

Remark 1. — If $\gamma = 0$ then $h(z, w)$ is a polynomial on a pair of intersecting lines in $\mathbb{C}^2$. If $\gamma \neq 0$ then $h(z, w)$ is a Laurent polynomial in the variable w (resp. z) with a pole of order k (resp. ℓ) at infinity and a pole of order ℓ (resp. k) at zero.

A generic bipolynomial of bidegree (k, ℓ) has $k + \ell$ distinct roots.

DEFINITION 1. — *The discriminant of the space $P^{k,\ell}$ of bipolynomials of bidegree (k, ℓ) is the hypersurface $\Delta^{k,\ell}$ of bipolynomials having less than $k + \ell$ distinct roots. The extended discriminant $\widetilde{\Delta}^{k,\ell}$ is the union of the discriminant with the hyperplane $\gamma = 0$ of improper bipolynomials.*

The study of spaces of generic bipolynomials leads to many interesting connections with rational functions, Brieskorn braid groups and generalized braid groups associated to extended affine Weyl groups. Here are a few examples:

1) The hypersurface $\Delta^{k,\ell}$ is the complement to the branching manifold of the rational function with two poles of order (k, ℓ) given by the equations

$$w^k + a_1 w^{k-1} + \cdots + a_{k-1} w + a_k + a_{k+1} z + \cdots + a_{k+\ell-1} z^{\ell-1} + z^\ell = 0,$$
$$wz = \gamma$$

(a root (w, z) is regarded as a function of the coefficients $(a_1, \ldots, a_{k+\ell-1}, \gamma)$). The complement to the discriminant is the base of a $(k+\ell)$-sheeted covering. In particular the cohomology classes of spaces of generic bipolynomials provide characteristic classes of rational functions with two poles (a similar connection between generic polynomial and algebraic functions is described in [Arn70b]).

2) A bipolynomial of bidegree $(k, 0)$ is a polynomial of degree k in the variable w. In particular the discriminant of the space $P^{k,0}$ coincides with the discriminant of the space of polynomials of degree k. Hence $\Delta^{k,0}$ is a hypersurface in $P^{k,0}$ and the complement to $\Delta^{k,0}$ is a Eilenberg-MacLane space $K(\pi, 1)$, where π is the braid group on k strings [Arn70a] $(\pi_1(P^{k,0} - \Delta^{k,0}) = \mathrm{Br}(k), \pi_i(P^{k,0} - \Delta^{k,0}) = 0$ for $i > 1)$. The cohomology classes of polynomials without multiple roots have been used by Arnol'd in [Arn70b] to prove that algebraic functions of a given number of variables cannot be obtained by superposition of algebraic functions of fewer variables (the results of Arnol'd have been later improved by Lin [Lin72], [Lin76] using other methods).

3) A bipolynomial of bidegree $(k, 1)$ is the restriction of a polynomial in two variables

$$h(w, z) = w^k + a_1 w^{k-1} + \cdots + a_{k-1} w + a_k + z,$$

to the "hyperbola" $wz = \gamma$. If $\gamma \neq 0$ such a bipolynomial has a double root exactly when the polynomial in one variable $w \times h(w, \gamma/w) = w^{k+1} + a_1 w^k + \cdots + a_{k-1} w^2 + a_k w + \gamma$ has a double root. In particular the extended discriminant of $P^{k,1}$ and the discriminant of the boundary singularity B_k coincide (the discriminant of B_k is the subspace of polynomials $w^{k+1} + a_1 w^k + \cdots + a_{k-1} w^2 + a_k w + \gamma$ having either a double roots or a root in zero [Arn78]). In particular $\Delta^{k,1}$ is a hypersurface in $P^{k,1}$ and the complement to $\Delta^{k,1}$ is a Eilenberg-MacLane space $K(\pi, 1)$, where π is the generalized braid group on k strings $\mathrm{Br}(B_k)$ (for the definition of generalized braid groups see [Bri72]).

4) Dubrovin and Zhang [DZ98] proved that the complement to the extended discriminant of $P^{k,\ell}$ is homeomorphic to the space of regular orbits of the extended affine Weyl group $\widetilde{W}^{(k)}(A_\ell)$. In particular the complement to the extended discriminant is a Frobenius manifold.

5) Define the generalized braid group $\mathrm{Br}(k, \ell)$ as the Poincaré group of the space of regular orbits of the extended affine Weyl group $\widetilde{W}^{(k)}(A_\ell)$ (this definition is analogous to the definition of generalized braid groups of types B, C, D by Brieskorn [Bri72]). In [Nap98] we proved that the complement to the extended discriminant is an Eilenberg-MacLane space $K(\pi, 1)$. In particular the cohomology groups of the groups of generalized braids $\mathrm{Br}(k, \ell)$ coincide with the cohomology groups of the spaces $P^{k,\ell} - \widetilde{\Delta}^{k,\ell}$.

Remark 2. — Contrarily to the complement of the extended discriminant, the complement to the discriminant of $P^{k,\ell}$ is not a $K(\pi, 1)$ (see [Knö82] for the case $k = 2$, $\ell = 2$).

Our main result is the stabilization of the cohomology rings of the spaces $P^{k,\ell} - \widetilde{\Delta}^{k,\ell}$ and $P^{k,\ell} - \Delta^{k,\ell}$ as the bidegree grows. More precisely we prove the following theorem:

STABILISATION THEOREM. — *The cohomology groups of the spaces* $P^{k,\ell} - \Delta^{k,\ell}$ *and* $P^{k,\ell} - \widetilde{\Delta}^{k,\ell}$ *stabilize as k tends to infinity:*

$$H^i(P^{k,\ell} - \Delta^{k,\ell}) \approx H^i(P^{k+1,\ell} - \Delta^{k+1,\ell}) \quad \text{for } i < \left[\frac{1}{2}k\right],$$

$$H^i(P^{k,\ell} - \widetilde{\Delta}^{k,\ell}) \approx H^i(P^{k+1,\ell} - \widetilde{\Delta}^{k+1,\ell}) \quad \text{for } i < \left[\frac{1}{2}k\right],$$

where $\left[\frac{1}{2}k\right]$ denotes the integral part of $\frac{1}{2}k$. Moreover the stabilization of the cohomology groups are induced by natural embeddings $\phi^{k,\ell} : P^{k,\ell} \to P^{k+1,\ell}$ sending the complement of the discriminant (resp. extended discriminant) to the complement of the discriminant (resp. extended discriminant).

From this theorem it follows that for any given ℓ the cohomology rings of $P^{k,\ell} - \widetilde{\Delta}^{k,\ell}$ and $P^{k,\ell} - \Delta^{k,\ell}$ stabilize as k tends to infinity. Hence there exists a sequence of stable cohomology rings $H^*(P^{\infty,\ell} - \widetilde{\Delta}^{\infty,\ell})$ and $H^*(P^{\infty,\ell} - \Delta^{\infty,\ell})$. Moreover the theorem also implies that this sequence of stable cohomology rings stabilize as ℓ tends to infinity. Hence there exist two bistable cohomology rings $H^*(P^{\infty,\infty} - \widetilde{\Delta}^{\infty,\infty})$ and $H^*(P^{\infty,\infty} - \Delta^{\infty,\infty})$. Figure 1 represents the diagram of stabilization obtained. In Section 4 we prove an analog of Snaith splitting formula for the stable cohomology groups of $P^{\infty,\ell} - \Delta^{\infty,\ell}$.

Remark 3. — Some of the terms of this sequence of stable cohomology rings are well known. The cohomology ring $H^*(P^{\infty,0} - \widetilde{\Delta}^{\infty,0})$ is isomorphic to the cohomology ring of the space of two fold loops in the three dimensional sphere $H^*(\Omega^2(S^3))$ (May-Segal formula [May72], [Seg73]) and also to the cohomology ring of Artin's braid group on an infinite number of strings $H^*(\mathrm{Br}(\infty))$ (Arnol'd [Arn70a]). The cohomology ring $H^*(P^{\infty,1} - \widetilde{\Delta}^{\infty,1})$ is isomorphic to the cohomology ring $H^*(\Omega S^2 \times \Omega^2 S^3)$ (Fuchs [Fuc74]), where ΩS^2 is the space of loops in the two dimensional sphere, and also to the cohomology ring of the generalized braid group on an infinite number of strings $\mathrm{Br}(B_\infty)$ associated to the sequence of boundary singularities B_k (Goryunov [Gor78], [Gor82]).

Remark 4. — According to Examples 2 and 3 above and Remark 3, our theorem implies the stabilization of cohomology rings of braid groups of types A_k and B_k.

Remark 5. — The main results of Arnol'd paper [Arn70a] on the cohomology of the spaces of polynomials without multiple roots consist of three theorems: the stabilization, repetition and finiteness theorems. In the setting of bipolynomials only the stabilization theorem holds: the repetition and finiteness theorems do not hold already for the space $P^{k,1}$ (see Example 3 and [Gor82]).

Remark 6. — This theorem corrects the result announced in [Nap98] where the stability of the first $\left[\frac{1}{2}(k+\ell)\right]$ cohomology groups was asserted.

$$
\begin{array}{ccccccccc}
P^{0,0} & \longrightarrow \cdots \longrightarrow & P^{0,\ell} & \xrightarrow{[\frac{1}{2}\ell]} & P^{0,\ell+1} & \longrightarrow \cdots \longrightarrow & P^{0,\infty} \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\
P^{1,0} & \longrightarrow \cdots \longrightarrow & P^{1,\ell} & \xrightarrow{[\frac{1}{2}\ell]} & P^{1,\ell+1} & \longrightarrow \cdots \longrightarrow & P^{1,\infty} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
P^{k,0} & \longrightarrow \cdots \longrightarrow & P^{k,\ell} & \xrightarrow{[\frac{1}{2}\ell]} & P^{k,\ell+1} & \longrightarrow \cdots \longrightarrow & P^{k,\infty} \\
\downarrow{\scriptstyle [\frac{1}{2}k]} & & \downarrow{\scriptstyle [\frac{1}{2}k]} & & \downarrow{\scriptstyle [\frac{1}{2}k]} & & \downarrow{\scriptstyle [\frac{1}{2}k]} \\
P^{k+1,0} & \longrightarrow \cdots \longrightarrow & P^{k+1,\ell} & \xrightarrow{[\frac{1}{2}\ell]} & P^{k+1,\ell+1} & \longrightarrow \cdots \longrightarrow & P^{k+1,\infty} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
P^{\infty,0} & \longrightarrow \cdots \longrightarrow & P^{\infty,\ell} & \xrightarrow{[\frac{1}{2}\ell]} & P^{\infty,\ell+1} & \longrightarrow \cdots \longrightarrow & P^{\infty,\infty}
\end{array}
$$

Figure 1. *Stabilization of cohomology rings of spaces of generic bipolynomials. The number besides each arrow indicates the number of coinciding cohomology groups for the complement of the discriminant (resp. for the complement of the extended discriminant).*

The main tools in the proof of the stabilization theorem are Alexander duality, simplicial resolutions and Mayer-Vietoris exact sequence.

Alexander duality relates the cohomology classes of the complement to the discriminant to the Borel-Moore homology classes of the discriminant (the Borel-Moore homology is the homology of the one point compactification modulo the added point):

$$
H^i(P^{k,\ell} - \Delta^{k,\ell}) = \overline{H}_{2(k+\ell)-i-1}(\Delta^{k,\ell}) \quad \text{for } i > 0,
$$

where $\overline{H}_j$ denotes the j-th Borel-Moore homology group. In particular to prove the stabilization of the cohomology groups of the complement to the discriminant it suffices to prove the dual theorem for the Borel-Moore homology groups of the discriminant.

In general the discriminant is a space with many complicated self-intersections. To study its homology group we replace it by its

simplicial resolution. The *simplicial resolution* of the discriminant is a topological space homotopy equivalent to the discriminant but where all self-intersections are replaced by simpler normal forms. The simplicial resolution is endowed with a filtration corresponding to the stratification of the space of bipolynomials by the number of multiple roots. This stratification gives rise to a spectral sequence converging to the Borel-Moore homology of the discriminant. The construction of the simplicial resolution is based on analogous constructions of Vassiliev [Vas92a], [Vas92b], [Vas89].

Denote by $\Gamma^{k,\ell}$ the affine hypersurface $\gamma = 0$ in $P^{k,\ell}$. The extended discriminant of $P^{k,\ell}$ is the union of $\Gamma^{k,\ell}$ and the discriminant. Mayer-Vietoris exact sequence relates the Borel-Moore homology groups of the extended discriminant and the Borel-Moore homology groups of the discriminant. More precisely we have the following exact sequences for $i > 0$:

$$\overline{H}_i(\Delta^{k,\ell}\cap\Gamma^{k,\ell}) \longrightarrow \overline{H}_i(\Delta^{k,\ell})\oplus\overline{H}_i(\Gamma^{k,\ell}) \longrightarrow \overline{H}_i(\widetilde{\Delta}^{k,\ell}) \longrightarrow \overline{H}_{i-1}(\Delta^{k,\ell}\cap\Gamma^{k,\ell}).$$

By Alexander duality, this sequence induces a dual sequence linking the cohomology groups of the complement to $\Delta^{k,\ell}$ in $P^{k,\ell}$ and $\Gamma^{k,\ell}$ to the cohomology groups of the complement to $\widetilde{\Delta}^{k,\ell}$ in $P^{k,\ell}$. Hence the stabilization of the cohomology groups of the complement of the extended discriminant is a consequence of the stabilization of the cohomology groups of the complement to the discriminant in $P^{k,\ell}$ and in $\Gamma^{k,\ell}$ (the embedding $\phi^{k,\ell}$ induce morphisms of exact sequences).

1. Discriminants and bidegree shift.

To prove our main theorem we need to relate explicitly the spaces $P^{k,\ell} \setminus \Delta^{k,\ell}$ and $P^{k+1,\ell} \setminus \Delta^{k+1,\ell}$. This is done by considering neighborhood of the points $(w^k + z^\ell, \gamma = 0)$ and $(w^k(w + 1) + z^\ell, \gamma = 0)$.

LEMMA 1. — *The discriminant $\Delta^{k,\ell}$ is a quasi-homogeneous subspace of $P^{k,\ell}$.*

Proof. — Consider the family of maps $E_\lambda : P^{k,\ell} \to P^{k,\ell}$, $\lambda \in \mathbb{R}$, $\lambda > 0$, defined as follows: if $\gamma \neq 0$, the map E_λ multiplies all roots of the Laurent polynomial $h(w, \gamma/w)$ (resp. $h(\gamma/z, z)$) by λ^ℓ (resp. λ^k). If $\gamma = 0$, the map E_λ multiplies all roots of the bipolynomial $h(w, z)$ on the line $z = 0$ by λ^ℓ and all roots of $h(w, z)$ on the line $w = 0$ by λ^k. By definition E_λ preserves the multiplicities of all roots. In particular it sends the discriminant to

itself. In the coordinates $(a_1, \ldots, a_k, a_{k+1}, \ldots, a_{k+\ell-1}, \gamma)$ the map E_λ is given by

$$a_i \longmapsto \lambda^{\ell i} a_i \qquad \text{for} \quad 1 \leq i \leq k,$$

$$a_{k+i} \longmapsto \lambda^{k(\ell-i)} a_{k+i} \quad \text{for} \quad k+1 \leq k+i \leq k+\ell-1,$$

$$\gamma \longmapsto \lambda^{k+\ell} \gamma.$$

In particular the map E_λ is quasi-homogeneous. Hence the discriminant is a quasi-homogeneous subspace of $P^{k,\ell}$. $\square$

Define $V^{k,\ell}(\epsilon)$, $\epsilon > 0$, as the set of bipolynomials of bidegree (k, ℓ) whose roots in $\mathbb{C} \times \mathbb{C}$ have modulus less than ϵ. Lemma 1 implies:

COROLLARY 1. — *The space $\Delta^{k,\ell}$ is homeomorphic to its intersection with $V^{k,\ell}(\epsilon)$ for any $\epsilon > 0$.*

Define $U^{k,\ell}(\epsilon)$, $\epsilon > 0$, as the set of bipolynomials of bidegree (k, ℓ) having $k+\ell-1$ roots in $\mathbb{C} \times \mathbb{C}$ with modulus less than ϵ and one root differing from $(-1, 0)$ by less than ϵ. The set $U^{k,\ell}(\epsilon)$ is an open neighborhood of the bipolynomial $h(w, z) = w^{k-1}(w + 1) + z^\ell$ on the hyperbola $wz = 0$.

Let $\frac{1}{2} > \epsilon > 0$. Consider the map $\phi^{k,\ell} : V^{k,\ell}(\epsilon) \to U^{k+1,\ell}(\epsilon)$ sending the bipolynomial $h(w, z)$ on the hyperbola $wz = \gamma$ to the bipolynomial $(w+1)h(w, z)$ on the hyperbola $wz = \gamma$. This map respects the multiplicities of all roots. In particular it sends the discriminant to the discriminant. Hence by Corollary 1, we get:

LEMMA 2. — *The map $\phi^{k,\ell}$ induces a morphism of cohomology rings:*

$$H^*(P^{k+1,\ell} - \Delta^{k+1,\ell}) \longrightarrow H^*(P^{k,\ell} - \Delta^{k,\ell}).$$

2. Simplicial resolution of the discriminant.

The discriminant $\Delta^{k,\ell}$ has in general many complicated self-intersections. The simplicial resolution of $\Delta^{k,\ell}$ is a topological space Resolution($\Delta^{k,\ell}$) homotopy equivalent to $\Delta^{k,\ell}$ but where all self-intersections are replaced by simpler normal forms.

Let $h \in \Delta^{k,\ell}$ be a bipolynomial on the hyperbola $wz = \gamma$. The set of multiple roots of h is a finite set of points $\text{Root}(h) = \{(w_1, z_1), \ldots, (w_j, z_j)\}$

lying on the hyperbola $wz = \gamma$. For a generic point of the discriminant
Root(h) contains only one element. If the cardinal of Root(h) is $j > 1$ then
h belongs to the j-fold self-intersection of the discriminant. The key idea
of the simplicial resolution is to replace h by a $(j-1)$-dimensional simplex
Simplex(h) such that each vertex of this simplex corresponds to one of the
branches of the discriminant intersecting in h (see Figure 2).

Consider a continuous map S from $\mathbb{C} \times \mathbb{C}$ to an affine space $\mathbb{R}^N$ of
large dimension, such that the images of $k + \ell$ distinct points of $\mathbb{C} \times \mathbb{C}$ by S
do not lie in an affine subspace of dimension $k + \ell - 2$ (the images of these
points generate a simplex of dimension $k + \ell - 1$). To the bipolynomial h,
associate the simplex Simplex(h) generated by the images of the multiple
roots of h under the map S. The simplicial resolution of $\Delta^{k,\ell}$ is the union
of all simplices $h \times$ Simplex(h) in $P^{k,\ell} \times \mathbb{R}^N$ over all bipolynomials h in the
discriminant of $P^{k,\ell}$.

REMARK 7. — The simplicial resolutions of $\Delta^{k,\ell}$ and $\Delta^{k+1,\ell}$ are
compatible with the embedding $\phi^{k,\ell} : P^{k,\ell} \to P^{k+1,\ell}$: this embedding
induces an embedding of the simplicial resolutions.

By construction the following lemma is straightforward:

LEMMA 3. — *The projection map* $\pi : P^{k,\ell} \times \mathbb{R}^N \to P^{k,\ell}$ *induces a
homotopy equivalence between* Resolution($\Delta^{k,\ell}$) *and* $\Delta^{k,\ell}$.
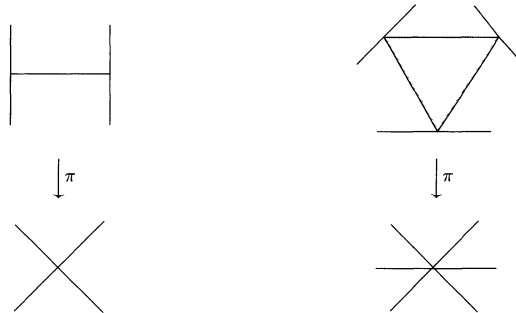

Figure 2. *Simplicial resolutions of two-fold and three-fold self-intersections*

3. The homological spectral sequence.

The simplicial resolution of $\Delta^{k,\ell}$ is endowed with a natural filtration
associated to the stratification of the space of bipolynomials by the

number of their multiple roots. The p-th term of this filtration is the space $F_p(k, \ell)$ defined as the union of all $(j - 1)$-dimensional simplices, $j \le p$, $h \times \text{Simplex}((w_1, z_1), \dots, (w_j, z_j))$ such that $\{(w_1, z_1), \dots, (w_j, z_j)\}$ is a subset of $\text{Root}(h)$. A bipolynomial of bidegree $k + \ell$ cannot have more than $\frac{1}{2}(k + \ell)$ multiple roots. Hence $\text{Resolution}(\Delta^{k,\ell}) = F_{[\frac{1}{2}(k+\ell)]}$ and we have the following sequence of inclusions:

$$\text{Resolution}(\Delta^{k,\ell}) = F_{[\frac{1}{2}(k+\ell)]} \supset F_{[\frac{1}{2}(k+\ell)]-1} \supset \cdots F_p \supset F_{p-1} \supset \cdots \supset F_1$$

(when there is no ambiguity on the bidegree (k, ℓ), we denote $F_p(k, \ell)$ simply by F_p).

Remark 8. — The filtration of $\text{Resolution}(\Delta^{k,\ell})$ is compatible with the map $\phi^{k,\ell}$: $\phi^{k,\ell}$ induces a continuous map sending the p-th term of the filtration of $\text{Resolution}(\Delta^{k,\ell})$ to the p-th term of the filtration of $\text{Resolution}(\Delta^{k+1,\ell})$.

The calculation of the Borel-Moore homology groups of $\Delta^{k,\ell}$ reduces to the calculation of the spectral sequence associated to this filtration. The first term of this spectral sequence is given by

$$E_{p,q}^1(k, \ell) = \overline{H}_{p+q}\big(F_p(k, \ell) - F_{p-1}(k, \ell)\big).$$

The homology groups of the spaces $F_p(k, \ell) - F_{p-1}(k, \ell)$ are strongly connected to the homology groups of the configuration spaces of roots $B^\ell(p)$ defined as follows. The space $B^\ell(p)$ is the set of all configuration $\{(w_1, z_1), \dots, (w_p, z_p)\}$ of p distinct points in $\mathbb{C} \times \mathbb{C}$ such that $w_1 z_1 = w_2 z_2 = \cdots = w_p z_p = \gamma$ and if $\gamma = 0$ at most $[\frac{1}{2}\ell]$ points lie on the line $w = 0$ (as in the introduction $[x]$ denotes the integer part of x). The following lemma is straightforward:

LEMMA 4. — The set of multiple roots of a bipolynomial belonging to the p-fold self-intersection of $\Delta^{k,\ell}$ is a configuration ζ in $B^\ell(p)$. Moreover if $p \le [\frac{1}{2}k]$, then for any configuration ζ in $B^\ell(p)$ the set of bipolynomials having multiple roots at the points of ζ is an affine subspace of $P^{k,\ell}$ of real codimension $4p + 2$.

LEMMA 5. — If $p \le [\frac{1}{2}k]$ the space $F_p(k, \ell) - F_{p-1}(k, \ell)$ is a fiber bundle above $B^\ell(p)$ whose fibers are open cells of dimension $2(k + \ell) - 3p - 3$.

Proof. — Given a configuration $\zeta = \{(w_1, z_1), \ldots, (w_p, z_p)\}$ in $B^\ell(p)$, denote by Interior(ζ) the interior of the simplex Simplex(ζ) spanned by the images of the points of ζ by the map S. The space $F_p(k, \ell) - F_{p-1}(k, \ell)$ is the union of all open simplices $h \times$ Interior$((w_1, z_1), \ldots, (w_p, z_p))$ such that $\{(w_1, z_1), \ldots, (w_p, z_p)\}$ is a subset of Root(h). By definition of S given a point ξ in Interior(ζ), ζ is the only configuration of p points such that ξ belongs to Interior(ζ). Hence to any point in Interior(ζ) we can associate unambiguously the configuration ζ. Hence there exists a continuous projection map $F_p(k, \ell) - F_{p-1}(k, \ell) \to B^\ell(p)$. By Lemma 4, if $p \leq [\frac{1}{2} k]$ this map is onto and the fiber of this map above any configuration ζ is the product of an open $p - 1$-dimensional simplex by an affine subspace of $P^{k,\ell}$ of codimension $4p + 2$.   □

Consider the sheaf of coefficients $\pm\mathbb{Z}$ on $B^\ell(p)$ changing sign along loops inducing an odd permutation of $(w_1, z_1), \ldots, (w_p, z_p)$ (this is the orientation sheaf of the bundle $F_p(k, \ell) - F_{p-1}(k, \ell) \to B^\ell(p)$). By Thom isomorphism theorem and Lemma 5:

$$(1) \quad \overline{H}_i(F_p(k, \ell) - F_{p-1}(k, \ell)) \approx \overline{H}_{i+3p+3-2(k+\ell)}(B^\ell(p), \pm\mathbb{Z}) \text{ for } p \leq \left[\frac{1}{2} k\right].$$

The following lemma is an easy consequence of the construction of the simplicial resolution:

LEMMA 8. — *For any non negative integer p, the dimension of the space $F_p(k, \ell) - F_{p-1}(k, \ell)$ is bounded:*

$$\dim\big(F_p(k, \ell) - F_{p-1}(k, \ell)\big) \leq 2(k + \ell) - p - 1.$$

By Lemma 6 we have

$$(2) \qquad \overline{H}_i\big(F_p(k, \ell) - F_{p-1}(k, \ell)\big) \approx 0 \quad \text{for } i > 2(k + \ell) - p - 1.$$

4. The formal cohomological spectral sequence.

Consider the formal cohomological spectral sequence $E_r^{p,q}(k, \ell)$ defined by

$$(3) \qquad\qquad E_1^{p,q}(k, \ell) = E_{-p, 2(k+\ell)-q-1}^1(k, \ell).$$

By Alexander duality this spectral sequence converges to the cohomology groups of the space $P^{k,\ell} - \Delta^{k,\ell}$.

Remark 9. — Since the map $\phi^{k,\ell}$ is compatible with the filtrations of the resolutions of $\Delta^{k,\ell}$ and $\Delta^{k+1,\ell}$, it induces a morphism of cohomological spectral sequences.

By Equation (1), the cohomology groups $E_1^{p,q}(k,\ell)$ and $E_1^{p,q}(k+1,\ell)$ are isomorphic for $-[\frac{1}{2}k] \leq p \leq 0$. Moreover the isomorphism between these groups is induced by the map $\phi^{k,\ell}$. By Equation (2) the groups $E_1^{p,q}(k,\ell)$ and $E_1^{p,q}(k+1,\ell)$ are all zero for $2(k+\ell) - p - q - 1 > 2(k+\ell) + p - 1$. (In particular the spectral sequence $E_r^{p,q}$ is convergent.) Hence the differentials from the unstable area do not occur in the stable area for $p + q < [\frac{1}{2}k]$ (see Figure 3). This implies that, for any $r > 0$, the groups $E_r^{p,q}(k,\ell)$ and $E_r^{p,q}(k+1,\ell)$ such that $p+q < [\frac{1}{2}k]$ coincide. Since the morphism between the groups $E_r^{p,q}(k,\ell)$ and $E_r^{p,q}(k+1,\ell)$ is induced by $\phi^{k,\ell}$ this implies the stabilization theorem for the complement of the discriminant:

THEOREM 1. — *For any integer $\ell \geq 0$, the sequence of embeddings $\phi^{k,\ell}$, $k \geq 0$, induces stabilization of the cohomology rings of the spaces $P^{k,\ell} - \Delta^{k,\ell}$ as k tends to infinity:*

$$H^i(P^{k+1,\ell} - \Delta^{k+1,\ell}) \approx H^i(P^{k,\ell} - \Delta^{k,\ell}) \quad \text{for } i < \left[\tfrac{1}{2}k\right].$$
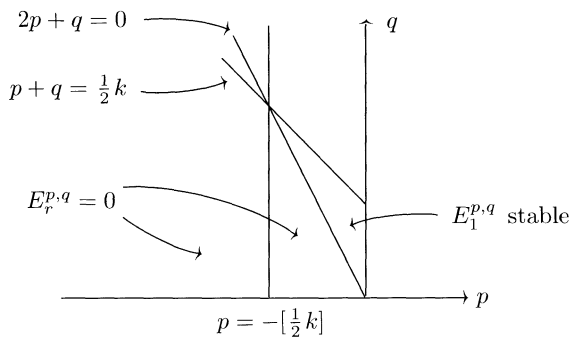
Figure 3. Stable terms in the spectral sequence $E^{p,q}(k,\ell)$ (the differentials δ^r go from $E_r^{p,q}$ to $E_r^{p+r,q-r+1}$)

Since the spectral sequence $E_r^{p,q}(k,\ell)$ degenerates at the first term for $p + q < [\frac{1}{2}k]$, we obtain the following splitting formula for $i < [\frac{1}{2}k]$:

$$(4) \quad H^i(P^{k,\ell} - \Delta^{k,\ell}) = \bigoplus_{p=0}^{\infty} \overline{H}_{3p-i+2}\big(B^\ell(p), \pm\mathbb{Z}\big) = \bigoplus_{p=0}^{\infty} H^{i-p}\big(B^\ell(p), \pm\mathbb{Z}\big).$$

THEOREM 2. — *The stable cohomology groups of the space* $P^{\infty,\ell} - \Delta^{\infty,\ell}$ *are given by the following splitting formula:*

$$H^i(P^{\infty,\ell} - \Delta^{\infty,\ell}) = \bigoplus_{p=0}^{\infty} H^{i-p}\big(B^\ell(p),\pm\mathbb{Z}\big) \quad \text{for } i \geq 0.$$

Remark 10. — In the case $\ell = 0$ this theorem and May-Segal formula imply Snaith splitting formula for the cohomology of the 2-fold loop space $\Omega^2 S^3$.

Proof. — Just let k tends to infinity in Formula (4).   □

We conjecture that this formula also stabilizes and that the following bi-stable splitting formula holds:

$$H^i(P^{\infty,\infty} - \Delta^{\infty,\infty}) = \bigoplus_{p=0}^{\infty} H^{i-p}\big(B^\infty(p),\pm\mathbb{Z}\big) \quad \text{for } i \geq 0.$$

The constructions carried over for the complement of the discriminant in $P^{k,\ell}$ can be carried over *mutatis mutandis* for the complement of the discriminant in $\Gamma^{k,\ell}$ (in this case, instead of the configuration spaces $B^\ell(p)$, we have to consider the configuration spaces $B_0^\ell(p)$ of p distinct points on the pair of lines $w = 0$, $z = 0$ such that at most $[\frac{1}{2}\ell]$ points lie on the line $w = 0$). From this follows the stabilization of the cohomology rings of the spaces $\Gamma^{k,\ell} \setminus \Delta^{k,\ell}$:

THEOREM 3. — *For any integer* $\ell \geq 0$, *the sequence of embeddings* $\phi^{k,\ell}$, $k \geq 0$, *induces stabilization of the cohomology rings of the spaces* $\Gamma^{k,\ell} - \Delta^{k,\ell}$ *as k tends to infinity:*

$$H^i(\Gamma^{k+1,\ell} - \Delta^{k+1,\ell}) \approx H^i(\Gamma^{k,\ell} - \Delta^{k,\ell}) \quad \text{for } i < \left[\tfrac{1}{2}k\right] + 2.$$

By Alexander duality and Mayer-Vietoris exact sequence we obtain the stabilization of the cohomology rings of the complement to the extended discriminant.

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Fabien NAPOLITANO,
Université Paris IX-Dauphine
CEREMADE – UMR CNRS 7534
Place du Maréchal DeLattre De Tassigny
75775 Paris cedex 16 (France).
napolita@ceremade.dauphine.fr