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QUASI-ISOMETRIC VECTOR BUNDLES AND BOUNDED FACTORIZATION OF HOLOMORPHIC MATRICES

by B. BERNDTSSON & J.-P. ROSAY

1. Introduction.

Let E and F be two holomorphic hermitian vector bundles over a complex manifold. We say that E and F are *quasi-isometric* if there is a holomorphic bundle map

$$G : F \longrightarrow E$$

and a constant c such that

$$1/c|\xi|_F \leqslant |G\xi|_E \leqslant c|\xi|_F$$

for all vectors in F. The principal aim of this note is to give a sufficient condition for a bundle over the disk, Δ, to be quasi-isometric to the trivial bundle. We will also apply the theorem (or the method of its proof) to prove a version of Cartan's lemma on the factorization of holomorphic matrices with uniform bounds for the solution. It should be stressed that our method is strictly one-variable, and that in particular we do not know if a similar statement on uniform bounds in Cartan's lemma holds in higher dimensions.

It is a well-known fact that any holomorphic vector bundle over Δ is holomorphically trivial. Therefore there is always some bundle map from

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the trivial bundle to our bundle E and it can be thought of as giving a global frame for E. Our theorem then gives a criterion for when we can find a global frame such that the metric on E is equivalent to the trivial metric in this frame.

Let a vector in our bundle E be represented by a column vector $\xi = (\xi_1, \dots, \xi_n)^t$ with respect to a certain holomorphic frame. The metric on E is then given by a positive definite Hermitian matrix A, so that

$$|\xi|_E^2 = \xi^* A\xi = (\xi, \xi)_A.$$

The curvature, Θ, of the metric is given, in the same frame, as

$$\Theta = \Theta_A = -\frac{\partial}{\partial \bar{z}}\left[A^{-1}\frac{\partial A}{\partial z}\right].$$

If we make a holomorphic change of frame, so that $\xi = g\eta$, with g a holomorphic matrix valued function, the matrix A is transformed to g^*Ag, and the curvature is represented by

$$\Theta = \Theta_{g^*Ag} = g^{-1}\Theta_A g.$$

Hence the curvature is naturally defined as a linear operator from E to itself. If E is a line bundle and we write $A = e^{-\phi}$ we have

$$\Theta_A = \frac{\partial^2 \phi}{\partial z \partial \bar{z}}.$$

In general one verifies that Θ is a Hermitian operator so that

$$(\Theta_A \xi, \eta)_A = (\xi, \Theta_A \eta)_A.$$

We can now state our main result. Here, and in the sequel, we use the convention that

$$\Delta\psi = \frac{\partial^2 \psi}{\partial z \partial \bar{z}}.$$

THEOREM 1.1. — *Assume there is a bounded subharmonic function ψ such that*

$$|\langle \Theta\xi, \xi\rangle| \leqslant \Delta\psi |\xi|_E^2. \tag{1}$$

Then E is quasi-isometric to the trivial bundle with a constant $c \leqslant e^{2\|\psi\|_\infty}$.

Note that when E is a line bundle and $A = e^{\phi}$ then (1.1) just means that $|\Delta\phi| \leqslant \Delta\psi$ with ψ bounded. This implies that the Green potential of $\Delta\phi$, G, is bounded. Since $\phi = G - \log|h|^2$ with h holomorphic, we can make a holomorphic change of frame so that A becomes bounded. This proves the theorem in the line bundle case. In the general case we will replace the existence of a Green potential, *i.e.* the solvability of the Dirichlet problem, by a non-linear analog, the *Wiener-Masani theorem*.

In Section 2 we give the proof of Theorem 1.1. In Section 3 we discuss Cartan's lemma with uniform bounds. Our result here follows from Theorem 1.1, but we also give a direct proof. Although the direct proof is based on the same idea as Theorem 1.1 it avoids all references to bundles and curvature, so the reader who is principally interested in Cartan's lemma may go directly to that proof. In Section 4 we apply Theorem 1.1 to a known generalization of the Corona theorem: A bounded holomorphic map, f, from the disk to the space of linear maps from $\mathbb{C}$ to $\mathbb{C}^n$ can be completed to a bounded holomorphic map to the space of invertible linear maps from $\mathbb{C}^n$ to itself, provided $|f(z)\xi| \geqslant \delta|\xi|$ for some positive δ. In Section 5 we prove a uniform estimate for solutions of the non-linear $\bar{\partial}$-equation

$$(\bar{\partial}u)u^{-1} = f$$

where f is a matrix of $(0,1)$-forms. In the last section we finally discuss various proofs of the Wiener-Masani theorem.

2. Proof of Theorem 1.1.

As mentioned in the introduction we may assume that E is already trivial and that the metric is given by a matrix of smooth functions A. We may also assume that A is smooth up to the boundary as long as we give bounds on the change of frame that we look for that depend only on the sup norm of ψ. The following theorem, [13], is the basis of our proof.

THEOREM 2.1 (Wiener-Masani). — *Let A_0 be a positive definite $N \times N$ matrix of smooth functions defined on the circle. Then there is a $N \times N$ matrix, h, of holomorphic functions in the disk, extending smoothly to the boundary, such that*

$$A_0 = h^* h$$

on the circle, and such that $g = h^{-1}$ is also holomorphic in the disk and also extends smoothly to the boundary. The matrix h is uniquely determined up to multiplication from the left by a constant unitary matrix.

In [13] this theorem is given without the statement about regularity on the boundary (and for a more general class of matrices A_0). The theorem as we have stated it can be found in [8]. See also the main theorem in [1] which deals with the more delicate case of metrics of Lorentz signature. In Section 4 we will discuss several different proofs of Theorem 2.1.

Now choose A_0 in the Wiener-Masani theorem to be the boundary values of our matrix A defining the metric on our bundle E. Choosing g as in the Theorem 2.1 we get

$$g^* A g = I,$$

(I being the identity matrix) on the boundary. We claim that

$$\widetilde{A} = g^* A g$$

is uniformly bounded from above and below in the disk. Thus

$$\xi \longrightarrow g\xi$$

gives a quasi-isometry from the trivial bundle to E and Theorem 1.1 follows. We will need the following (well-known) maximum principle for negatively curved bundles.

LEMMA 2.2. — *Let $A(z)$ be smooth and positive definite Hermitian for $z \in \bar{\Delta}$. Assume $A\Theta_A \leqslant 0$. Then*

$$z \longrightarrow |\xi|^2_{A(z)}$$

is, for any $\xi \in \mathbb{C}^N$, subharmonic in the disk. In particular, if

$$|\xi|_{A(z)} \leqslant |\xi|$$

for $z \in \partial\Delta$, then the same inequality holds for $z \in \Delta$.

Proof. — If $h(z)$ is a smooth $\mathbb{C}^N$-valued function, we let

$$\partial_A h = \frac{\partial h}{\partial z} + A^{-1}\frac{\partial A}{\partial z} h$$

be the holomorphic covariant derivative defined by the metric A. Note that if h is holomorphic,

$$\frac{\partial}{\partial \bar{z}} \partial_A h = -\Theta_A h.$$

Now

$$\frac{\partial}{\partial \bar{z}} (h(z), k(z))_A = (\partial h/\partial \bar{z}, k)_A + (h, \partial_A h)_A$$

and

$$\frac{\partial}{\partial z} (h(z), k(z))_A = (\partial_A h, k)_A + (h, \partial k/\partial \bar{z})_A.$$

Hence

$$\frac{\partial^2 |\xi|^2_A}{\partial z \partial \bar{z}} = \frac{\partial}{\partial z}(\xi, \partial_A \xi)_A = (\partial_A \xi, \partial_A \xi)_A - (\xi, \Theta_A \xi)_A \geqslant 0. \qquad \square$$

PROPOSITION 2.3. — *Let A satisfy*

$$|(\Theta_A\xi,\xi)_A| \leqslant \frac{\partial^2\psi}{\partial z\partial\bar{z}}(\xi,\xi)_A,$$

and assume

$$|\xi|^2_{A(z)} \leqslant |\xi|^2$$

for $z\in\partial\Delta$. Then

$$|\xi|^2_{A(z)} \leqslant e^{2\|\psi\|_\infty}|\xi|^2$$

in Δ. If

$$|\xi|^2_{A(z)} \geqslant |\xi|^2$$

for $z\in\partial\Delta$, then

$$|\xi|^2_{A(z)} \geqslant e^{-2\|\psi\|_\infty}|\xi|^2$$

in Δ.

Proof. — Consider a metric of the form $e^\phi A$. A direct computation shows that

$$\Theta_{e^\phi A} = \Theta_A - \frac{\partial^2\phi}{\partial z\partial\bar{z}}.$$

From this together with our assumption on A it follows that the metric $e^\psi A$ has negative curvature, and that the metric $e^{-\psi}A$ has positive curvature. The first part of the proposition therefore follows from Lemma 2.2 applied to $e^{\psi-\|\psi\|_\infty}$.

On the other hand one can verify that if the curvature of a metric A is positive, then the curvature of A^{-1} is negative. The second part of the proposition therefore follows from the lemma applied to $(e^{\|\psi\|_\infty-\psi}A)^{-1}$. □

Applying Proposition 2.3 to the metric $\widetilde{A}$ we get Theorem 1.1.

3. Cartan's lemma.

Let B be the ball in $\mathbb{C}^n$ and let $\{U_1,U_2\}$ be a covering of the ball with two well-separated open sets (by this we mean that the U_i's come from an open cover of the closed ball). Let g be an invertible $N\times N$ matrix of holomorphic functions defined on $U_1\cap U_2$. Cartan's lemma says that we can then find two invertible matrices of holomorphic functions, g_1 and g_2 defined on U_1 and U_2 respectively, such that $g = g_1g_2^{-1}$ on the intersection of U_1 and U_2.

Suppose now that in addition the given matrix of functions g and its inverse are uniformly bounded. Is it then possible to find g_1 and g_2 with the same property? The scalar case (when $N = 1$, *i.e.* the multiplicative Cousin problem) has been treated by E.L. Stout [11], [12]. Our next result answers that question affirmatively for any N when $n = 1$ so that we are dealing with the disk in $\mathbb{C}$. We will relax the condition that the sets be well-separated somewhat, and also treat coverings of the disk with more than two open sets.

To simplify some of the statements in the sequel we will for the remainder of this section call a bounded real-valued function χ in the disk "good" if there exists some bounded subharmonic function, ϕ, in the disk such that

$$|d\chi|^2 + |\Delta\chi| \leqslant \Delta\phi.$$

Notice that the class of good functions is closed under sums and products and also stable under composition with smooth functions on the real line.

THEOREM 3.1. — *Let $\Delta = \bigcup_1^N V_j$ be a finite covering of the disk with open sets. Assume there exists a subordinate partition of unity, (χ_i) consisting of good functions. Let (g_{ij}) be a collection of $N \times N$-matrices of bounded holomorphic functions defined on $V_i \cap V_j$ satisfying the cocycle conditions*

$$g_{ij} = g_{ji}^{-1}, \quad g_{ij}g_{jk}g_{ki} = I.$$

Then there exist $N \times N$-matrices of bounded holomorphic functions with bounded inverses, g_i, defined on V_i, such that

$$g_{ij} = g_i g_j^{-1}$$

on $V_i \cap V_j$.

For the proof we first need a preliminary lemma.

LEMMA 3.2. — *Let (V_i) be an open covering of the disk satisfying the condition in Theorem 2.3. Then there exists a refinement of the covering, (V_i'), which also has a subordinate partition of unity (χ_i') of good functions, and moreover is such that there are good functions χ_{ij} with*

$$\chi_{ij} = 1 \quad \text{on} \quad V_i' \cap V_j',$$
$$\chi_{ij} = 0 \quad \text{outside} \quad V_i \cap V_j.$$

Proof. — Put $V_i' = \{\chi_i > 1/3N\}$; clearly these sets form a covering of Δ. We first define smooth approximations, σ_i, of $\max(\chi_i, 1/2N) - 1/2N$

by

$$\sigma_i = f_N(\chi_i),$$

where $f_N(t)$ is a nonnegative smooth function in $\mathbb{R}$ that vanishes for $t \leqslant 1/2N$ and equals $t - 1/2N$ for $t \geqslant 1/N$. Then σ_i vanishes outside V_i' and at any point of Δ at least one σ_i is greater than $1/2N$, and we obtain our new partition of unity by letting $\chi_i' = \sigma_i / \sum \sigma_j$.

The functions $\chi_i\chi_j$ vanish outside $V_i \cap V_j$ and are greater than $(3N)^{-2}$ on $V_i' \cap V_j'$, so we can obtain χ_{ij} by composing $\chi_i\chi_j$ with a suitable function on the line. □

We can now return to the proof of Theorem 2.3.

Proof. — Choose a refinement (V_i') of the covering as in Lemma 3.2 and consider g_{ij} as the transition functions of a holomorphic vector bundle, E, of rank N over Δ. We will next define a metric on E that satisfies the condition of Theorem 1.1.

For this we start with the trivial metrics with respect to the local trivializations over V_j' and then patch them together with the partition of unity. The resulting metric is given by

$$A_i = \sum_j \chi_j' g_{ji}^* g_{ji}$$

in the trivialization over V_i'.

By our assumptions on the g_{ij} the metric is equivalent to the trivial metrics defined by the local trivialization over any V_i' . The curvature of the metric, computed in any of the trivializations will be a sum of various terms. Such a term either contains the Laplacian of a function χ_j' and no derivative on any g_{ij} , or is of the form bD_1D_2 where b is bounded and the D_j's are first order derivatives of either a χ_j' or an entry of a g_{ij}.

Now put

$$\psi = \sum \chi_{ij}^2 |g_{ij}|^2,$$

where $|g|$ stands for the Hilbert-Schmidt norm of the matrix g. Then

$$\Delta\psi \geqslant 1/2 \sum \chi_{ij}^2 |g_{ij}'|^2 - \Delta\psi'$$

for some bounded subharmonic ψ' (since the χ_{ij} are "good"). Hence $2(\psi + \psi')$ is a bounded subharmonic function in the disk whose Laplacian dominates $|g_{ij}'|^2$ on any $V_i' \cap V_j'$. It follows that the curvature of E is bounded by the Laplacian of some bounded functions so Theorem 1.1 applies.

Let G be the quasi-isometry from the trivial bundle to E. Over each V_j', G is given by a holomorphic matrix g_j. Then $g_{ij}g_j = g_i$. Since G is a quasi-isometry and the metric on E is equivalent to the trivial metric over each trivialization it follows that g_j and g_j^{-1} are uniformly bounded. Finally, the equations $g_i = g_{ij}g_j$ define extensions of g_i to the union of the $V_i \cap V_j'$ and therefore to all of V_i. □

The hypothesis of Theorem 3.1 also covers some cases when the sets U_i are not well-separated. One such case is when the U_i's are bounded by simple smooth curves that meet at boundary points under positive angles. We next give an example where the boundary curves meet tangentially and the conclusion of Theorem 2.3 fails. (A somewhat less precise example can be found in [11].)

PROPOSITION 3.3. — *Let Ω be the disk of radius 1 and center i, and put $U_1 = \{z = x + iy \in \Omega \ ; \ x < y^2\}$, $U_2 = \{z = x + iy \in \Omega \ ; \ x > -y^2\}$. Let $g = e^{-1/z}$. Then g is a holomorphic function in $U_1 \cap U_2$, bounded from above and below, which can not be written $g = g_1 g_2^{-1}$ with g_i bounded and holomorphic in U_i.*

Proof. — It is clear that g is holomorphic and bounded from above and below on $U_1 \cap U_2$. If $g = g_1 g_2^{-1}$ then there is a holomorphic function in Ω, h, such that $g_1 = he^{-1/z}$ on U_1 and $g_2 = h$ on U_2. In particular h is bounded on all of Ω. On U_1, $\log |h| \leqslant x/|z|^2$. This contradicts the fact that the boundary values of $\log |h|$ are integrable. □

We next give a direct proof of Theorem 2.3 which does not rely on Theorem 1.1. We will treat the case of a covering with two well-separated open sets, and moreover, for ease of exposition, we will discuss principally the following special situation - hopefully it will be clear how the argument can be generalized: Assume that $\{z \in \Delta \ ; \ \mathrm{Re}\, z \leqslant 0\}$ is compactly included in U_1 and $\{z \in \Delta \ ; \ \mathrm{Re}\, z \geqslant 0\}$ is compactly included in U_2. As before we may assume that our matrix g extends holomorphically slightly over the disk. Consider first an arbitrary solution $g = \gamma_1 \gamma_2^{-1}$ to the factorization problem, where the γ_i's also extend slightly over the boundary. We look for a solution satisfying good estimates of the form $g_i = \gamma_i h$, where h is an invertible matrix of holomorphic functions in the disk. By the norm of a matrix valued function we understand in the sequel the Hilbert Schmidt norm so that $|g|^2 = \mathrm{trace}\, g^* g$.

Note first that since $g = \gamma_1 \gamma_2^{-1}$ and g is uniformly bounded from above and below, the matrices $\gamma_1^*\gamma_1$ and $\gamma_2^*\gamma_2$ are comparable as positive matrices on $U_1 \cap U_2$. We claim that by the Wiener-Masani theorem we can choose h in such a way that $g_i^* g_i$ are uniformly bounded from above and below on U_i intersected with the boundary of the disk. To achieve this we first choose h so that $g_1^* g_1 = h^*\gamma_1^*\gamma_1 h$ is the identity matrix on the complement of U_2 and so that $g_2^* g_2 = h^*\gamma_2^*\gamma_2 h$ is the identity on the complement of U_1. This means that $(h^{-1})^* h^{-1} = \gamma_i^*\gamma_i$ on the respective pieces of the boundary. On the intersection of U_1, U_2 and the boundary we take $(h^{-1})^* h^{-1}$ equal to a suitable convex combination the $\gamma_i^*\gamma_i$'s. Since $\gamma_1^*\gamma_1$ and $\gamma_2^*\gamma_2$ are comparable as positive matrices on $U_1 \cap U_2$ it follows that $g_i^* g_i$ are uniformly bounded from above and below on the boundary of the disk intersected U_i. Now let c be a large constant and consider the function

$$v = \max(|g_1|^2 e^{-cx},\ |g_2|^2 e^{cx}),$$

on $U_1 \cap U_2$. Since by the argument above the matrices $g_i^* g_i$ are comparable as positive matrices on $U_1 \cap U_2$ it follows that for c big enough (only depending on the supremum norms of g, g^{-1} and the covering) $v = |g_1|^2 e^{-cx}$ near the boundary of U_2 and vice versa. Hence we can extend the definition of v to all of the disk by putting it equal to $|g_1|^2 e^{-cx}$ on the complement of U_2 and to $|g_2|^2 e^{cx}$ on the complement of U_1. The function v is then continuous, subharmonic, and uniformly bounded on the boundary. By the maximum principle v is bounded everywhere in the disk, so the norms of g_1 and g_2 are bounded. The same argument applies to the inverses of the g_i, so our second proof of Theorem 2.3 is complete.

Finally, we can not resist giving one last proof of Cartan's lemma with bounds. This proof, which is based on the reflection principle seems to work only in the case of a well-separated covering, and does not give explicit bounds easily. On the other hand, it is very short.

Consider a covering of the disk by the sets $\{\operatorname{Re} z > -\delta$ and $\{\operatorname{Re} z < \delta\}$. Let S be the part of the boundary of the disk where $-\delta < \operatorname{Re} z < \delta$. Choose a matrix-valued function, A_0, on the boundary of the disk which equals $g^* g$ on S and is bounded and positively definite everywhere. Write $A_0 = h^* h$ where h is a bounded holomorphic matrix with bounded inverse. (This choice of A_0 is not smooth, but there is a version of the Wiener-Masani theorem that applies.) It is now enough to factor $\gamma = gh^{-1}$. Since $\gamma^*\gamma$ is the identity matrix on S it follows from the Schwarz reflection principle that γ extends holomorphically across S. By the classical Cartan's lemma γ factors as $\gamma = \gamma_1\gamma_2^{-1}$ in a slightly larger disk, and the restriction of the γ_i's to our original disk will surely be bounded with bounded inverse.

4. Tolokonnikovs lemma.

Our next application is to the following (known) generalization of the Corona theorem (due to Tolokonnikov, see *e.g.* [9] for a proof, and [4] for generalizations).

THEOREM 4.1. — *Let $f^j = (f_1^j, \dots, f_N^j)^t$ for $j = 1, \dots, M \leqslant N$ be N-tuples of bounded holomorphic functions in the disk, and assume that $|\sum \xi_j f^j|^2 \geqslant \delta |\xi|^2$ for some $\delta > 0$. Then there are further N-tuples of bounded holomorphic functions f^j, $j = M+1, ...N$, such that the matrix*

$$F = (f^1, f^2, \dots, f^N)$$

is invertible with bounded inverse.

Proof. — Let first $M = 1$ and write $f^1 = f$. By the Corona theorem there are bounded holomorphic functions g_j such that $\sum g_j f_j = 1$.

Let γ be the map from the trivial bundle of rank N to the trivial line bundle defined by

$$\xi \longrightarrow \sum g_j \xi_j.$$

Let E be the kernel of γ and give E the metric as a subbundle of the trivial bundle. We claim that E satisfies the hypothesis of Theorem 1.1. To see this, cover the disk by the open sets U_j where $|g_j|^2 > \epsilon$. We must estimate the curvature of E on each of these sets and to be specific we choose U_1. On U_1 we can choose (e_k) where $e_k = (g_k, 0 \cdots - g_1, 0 \cdots)$ as a local holomorphic frame. With respect to this frame the metric is given by $((e_j, e_k)) = (g_j \bar{g}_k) + |g_1|^2 I$. The curvature of this matrix is a sum with bounded coefficients of products of two first order derivatives of the g_j. The hypothesis of Theorem 1.1 is therefore satisfied with $\psi = C|g|^2$.

By Theorem 1.1, E is quasi-isometric to the trivial bundle of rank $N-1$. Let $f^2, \dots, f^N$ be the sections of E that are images of the standard basis of the trivial bundle of rank $N-1$. Then the sections $f, f^2, \dots, f^N$ are uniformly linearly independent, which means that F is invertible with bounded inverse.

The case of $M > 1$ is proved by induction, so assume we have already treated the case of fewer N-tuples. By the first part of the proof we can find bounded holomorphic N-tuples $p^2, ...p^N$ such that the matrix $G = (f^1, p^2, ...p^N)$ is bounded with bounded inverse. Then

$$G^{-1}(f^1, ...f^M) = (\tilde{f}^1, ...\tilde{f}^M)$$

is an $N \times M$ matrix whose first column, $\tilde{f}^1$, equals $(1,0...0)^t$. Then the $(N-1)$-tuples $(k^2,...k^M)$, where k^j is $\tilde{f}^j$ with the first entry removed, satisfy the hypothesis of the theorem. By the induction hypothesis they can be completed with an $(N-1)\times(N-M-1)$ matrix H, so that $(k^2,...k^M, H)$ is holomorphic and bounded with bounded inverse. Let H_0 be H augmented with a top-row consisting of zeros. Then $F = G(\tilde{f}^1,...\tilde{f}^M, H_0)$ is the matrix we seek. □

5. A non-linear $\bar{\partial}$-equation.

Let f be an $N \times N$-matrix of $(0,1)$-forms in the disk. We will study the equation

$$(\bar{\partial}u)u^{-1} = f, \tag{5.1}$$

and in particular look for solutions u such that both u and u^{-1} are uniformly bounded in the disk. In the scalar case, *i.e.*, when $N = 1$ it is not hard to see that such a solution exists if and only if the equation

$$i\partial\bar{\partial}v = \operatorname{Re}\partial f$$

has a bounded solution. Our result in the higher dimensional case is the following .

THEOREM 5.1. — *Suppose there is a bounded subharmonic function, ϕ in the disk such that*

$$|f^* \wedge f + f \wedge f^* + \partial f - (\partial f)^*| \leqslant \Delta\phi. \tag{5.2}$$

Then the equation (5.1) *has a solution such that both u and its inverse are uniformly bounded.*

Theorem 5.1 is formally quite similar to the theorem of T. Wolff, stating that the scalar $\bar{\partial}$-equation has a bounded solution if

$$|f|^2 + |\partial f| \leqslant \Delta\phi$$

for some bounded ϕ. Wolff's theorem was devised to give a simple proof of the Corona theorem, see [7] for this and [2] for the formulation of the theorem that we have given here.

Notice however that in the scalar case (or more generally the case when our matrix f in (5.1) is diagonal) the matrices f and f^* anticommute under the wedge product so the condition (5.2) is actually a condition only on ∂f. Theorem 5.1 nevertheless seems quite related to the circle of ideas around Wolff's theorem as the following example shows.

Let f be a 2×2 matrix with all entries equal to 0 except the entry in the upper right corner where the entry is F. The inequality (5.2) is then equivalent to the hypothesis in Wolff's theorem. Suppose this condition is satisfied and let u be a solution to (5.1) bounded from above and below. Let (a, b) be the first row of u, and let (c, d) be the second row. Then (5.1) is equivalent to saying that c and d are holomorphic and that $\bar\partial a = cF$ and $\bar\partial b = dF$. Since u is uniformly bounded from below and above it follows that $|c| + |d| \geqslant \delta > 0$ in the disk. By the Corona theorem, there exist bounded holomorphic functions h and k such that $hc + kd = 1$. Then $\bar\partial(ha + kb) = F$ and we find a bounded solution to the scalar (and linear) $\bar\partial$-problem.

Admittedly this is not too impressive as the proof uses the Corona theorem – the main application of Wolff's result! Maybe there is some other way of obtaining all of Wolff's theorem from Theorem 5.1, but it seems more likely that Theorem 5.1 is really simpler. Our discussion then shows that modulo this simpler fact, Wolff's theorem and the Corona theorem are "equivalent".

We now turn to the proof of Theorem 5.1.

Proof. — There always exists some solution, u, to (5.2) if we leave aside the question of estimates. That this is possible locally means that any complex structure on a vector bundle over a Riemann surface is integrable and is discussed *e.g.* in [6]. If u and v are two local solutions it is easily verified that $u = vh$, where h is an invertible holomorphic matrix. A collection of local solutions to (5.2) on an open cover of the disk therefore defines a collection of holomorphic transition function on the overlaps which in turn defines a holomorphic vector bundle over the disk. Since this bundle is trivial we can piece together our local solutions to a global one.

We look for a solution $v = uh$ where h is an invertible matrix of holomorphic functions in the disk, and is chosen so that v satisfies the conditions of the theorem. To say that v is bounded from above and below is equivalent to saying that $v^*v = h^*u^*uh$ is bounded from above and below. If we let $A = u^*u$ and consider A as an Hermitian metric on the trivial vector bundle over Δ, finding h amounts to finding a holomorphic frame with respect to which the metric is equivalent to the trivial metric.

This is precisely the question discussed in Theorem 1.1, and we know that it is possible if the curvature of the metric A satisfies condition (1.1).

We claim that (1.1) is actually equivalent to (5.2), which then completes the proof. There are two ways of verifying the claim, and in both cases we leave the details to the reader. The first way is by brute calculation, using the definition of the curvature of A in the introduction.

The second, more conceptual way, is to think of

$$D\xi = d\xi + f^*\xi - f\xi = d\xi + \theta\xi$$

as a unitary connection on the trivial bundle (which is then compatible with its trivial metric). A solution of (5.1) gives us a change of frame such that D becomes holomorphic (*i.e.*, has vanishing $(0,1)$-part) in this frame. In the new holomorphic frame the metric is given by the matrix $A = u^*u$ and D is the unique holomorphic connection compatible with that metric. The curvature of the connection D, $D^2 = d\theta + \theta \wedge \theta$, is therefore equal to the curvature of the metric A and this means precisely that conditions (5.2) and (1.1) are equivalent.

□

6. The Wiener-Masani theorem.

The Wiener-Masani theorem was stated and proved in [13] in the context of stationary time series. There the authors only assume integrability of the logarithm of the determinant of A_0 and no regularity questions are discussed. It is instead proven that the logarithm of the modulus of the determinant of h is also integrable. For the proof of Theorem 1.1 we certainly do need at least continuity up to the boundary of the matrices h and h^{-1}. The theorem as we have stated it can be found in *e.g.* [8] (appendix) together with a neat proof based on linearization for data close to the identity matrix, and a reduction to this case by means of rational approximation. We shall now take the opportunity to discuss a few other proofs of the Wiener-Masani theorem which show its connection to several different areas of mathematics.

The first possibility is to reduce the Wiener-Masani theorem to the Birkhoff factorization theorem [5], [10]. This theorem says that any smooth invertible matrix valued function M on the circle can be smoothly factored

$$M = g_- D g_+$$

where g_+, g_+^{-1} and $\bar{g}_-, \bar{g}_-^{-1}$ extend holomorphically to the disk, and D is a diagonal matrix with entries z^{k_j}. We claim that if M is positive definite D

must be the unit matrix. To see this, note first that the winding number of the determinant of M, which is zero, equals that of D, so if all the "partial indices" k_j are not zero at least one of them is negative. We can then find a holomorphic vector f, vanishing at the origin such that Dg_+f is constant. Then Mf extends antiholomorphically to the disk, so f^*Mf also extends antiholomorphically and moreover vanishes at the origin, contradicting the fact that it is strictly positive on the boundary. Next, note that as $M = M^*$, $g_-^{-1}g_+^* = g_+g_-^{*-1} = c$ extends both holomorphically and antiholomorphically, so it is a constant. One checks that $c > 0$, and puts $\gamma_- = g_-c^{1/2}$, $\gamma_+ = c^{-1/2}g_+$. Then $M = \gamma_-\gamma_+$ and $\gamma_- = \gamma_+^*$ so we are done.

Alternatively we can view the Wiener-Masani theorem as a boundary problem for a nonlinear elliptic PDE. A positive definite matrix A can be factored $A = h^*h$ if and only if the curvature Θ_A vanishes, and the factor h will then have the same regularity as A if we measure regularity in terms of Hölder classes with non-integer exponents. By linearization one can solve the equation $\Theta_A = 0$, with $A = A_0$ on $\partial\Delta$ if A_0 is sufficiently close to the identity. This also means that the set of boundary data for which the problem is solvable is open, since any solvable boundary data can be transformed to the identity by the map $A_0 \to g^*A_0g$. An appropriate a priori estimate then shows that the boundary value problem is solvable for any positive definite boundary data. See also [1] where the more difficult question of Hermitian boundary values with Lorentz signature is discussed.

Finally, we give a proof of the Wiener-Masani theorem on the lines of the proof of Tolokonnikov's lemma in [9] (which in turn follows the ideas in Beurling's description of invariant subspaces, [3]).

Let H^2 be the Hardy space of the disk and consider $F = (H^2)^N$ – the product of N copies of H^2. We have two natural scalar products on F. The first one, $(\cdot\,,\cdot)$, is the one inherited from the standard product on H^2. The second one, to be denoted $\langle\cdot\,,\cdot\rangle$ is defined by the matrix valued function A_0 on the circle: If $h = (h_1, \ldots, h_N)$ is an element in F

$$\langle h, h\rangle = 1/2\pi \int \sum (A_0)_{jk}h_j\bar{h}_k d\theta.$$

By the hypothesis on A_0 the two induced norms are equivalent.

Consider now the space F_0 of functions in F that vanish at the origin. Clearly F_0 has codimension N in F. Let G be the orthogonal complement of F_0 with respect to the scalar product $\langle\cdot\,,\cdot\rangle$. Let g_i, where i ranges from 1 to N be an orthonormal basis for G. The g_i's are vector-valued functions and we think of them as column vectors. Let g be the matrix $(g_1,\dots,g_N)$. We then have:

PROPOSITION 6.1. — *The matrix g is an invertible matrix of holomorphic functions in the disk that is bounded and has a bounded inverse. g is smooth up to the boundary (if A_0 is smooth) and*

$$g^* A_0 g$$

is the identity matrix on the boundary.

Proof. — By the construction, g_i is orthogonal to g_k if k is different from i, and orthogonal to $z^m g_k$ for any k if $m > 0$. Moreover g_i has norm 1. This implies that $g_i^* A_0 g_k = \delta_{ik}$ almost everywhere on the circle. In other words

$$g^* A_0 g = I$$

almost everywhere on the circle. Since A_0 is bounded from above and below and the g_i lie in H^2 this implies in particular that all the components of g lie in H^∞. We claim that g is invertible with bounded inverse.

To see this, let $\|h\|_{A_0}$ denote the norm of an element of F with respect to the metric defined by A_0 and let $\|h\|$ be the standard norm. Note that if h is in F, then $\|gh\|_{A_0} = \|h\|$ which shows in particular that the map $T : h \to gh$ from F to itself is injective with closed range. To prove that it is also surjective it suffices to prove that its range is dense. Suppose h is a non-zero element of F that is orthogonal to the range of T. Write $h = z^k h_0$, where $k \geqslant 0$ and $h_0(0) \neq 0$. Then, if $\gamma \in G$ (recall G is the orthogonal complement to F_0 under the A_0-scalar product) $z^k\gamma$ belongs to the range, so

$$\langle h_0, \gamma\rangle = \langle h, z^k\gamma\rangle = 0.$$

Hence h_0 lies in F_0, so $h_0(0) = 0$ contrary to assumption. In conclusion T is a one-to-one map from F to itself. Therefore there is a matrix of H^2 functions, H such that $gH = I$. Since g^{-1} lies in L^∞ on the boundary it follows that $H = g^{-1}$ has entries in H^∞.

It only remains to prove that g is smooth up to the boundary. For this, note that $A_0 g = g^{-1*}$ a. e on the boundary. The smoothness therefore follows from the next lemma, which is a particular case of a well-known reflection principle for smooth functions.

□

LEMMA 6.2. — *Suppose* $(h_1, \ldots, h_N)$ *is a vector of functions in* H^2 *and that* $(a_1, \ldots, a_N)$ *is a vector of smooth functions. Suppose moreover that* $\sum a_i h_i = \bar{f}$ *where* f *is in* H^2. *Then* f *is smooth.*

Proof. — Consider a Fourier coefficient $c(j)$ with negative index of a product ah where a is smooth and h belongs to H^2. Since h has vanishing Fourier coefficients with negative index and the coefficients of a, say $a(k)$, decay faster than polynomially we get

$$|c_j|^2 \leqslant \|h\|^2 \sum_{k \leqslant j} |a(k)|^2 \leqslant C_d |j|^{-d}$$

for any $d > 0$. It follows that the Fourier coefficients with negative indices of $\bar{f}$ decay faster than any polynomial. Since $\bar{f}$ has vanishing Fourier coefficients with positive indices, f is smooth. □

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