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ZETA FUNCTIONS FOR THE RIEMANN ZEROS

by André VOROS

1. Introduction.

This work proposes to investigate certain meromorphic functions defined by Dirichlet series over the nontrivial zeros $\{\rho\}$ of the Riemann zeta function $\zeta(s)$, and to thoroughly compile their explicit analytical features. If the Riemann zeros are listed in pairs as usual,

$$(1) \qquad \{\rho = \tfrac{1}{2} \pm \mathrm{i}\tau_k\}_{k=1,2,\ldots}, \qquad \{\mathrm{Re}\ \tau_k\} \text{ positive and non-decreasing,}$$

then the Dirichlet series to be mainly studied read as

$$(2) \qquad \mathscr{Z}(\sigma, v) \stackrel{\mathrm{def}}{=} \sum_{k=1}^{\infty} ({\tau_k}^2 + v)^{-\sigma}, \qquad \mathrm{Re}\ \sigma > \tfrac{1}{2}, \quad v > -{\tau_1}^2,$$

extended to meromorphic functions of $\sigma \in \mathbb{C}$, and parametrized by v — with emphasis on two cases, $v = \frac{1}{4}$ and especially $v = 0$. Their analysis will simultaneously yield some results for the variant series

$$(3) \quad \mathfrak{Z}(\sigma, a) \stackrel{\mathrm{def}}{=} \sum_{k=1}^{\infty} (\tau_k + a)^{-2\sigma}, \qquad \mathrm{Re}\ \sigma > \tfrac{1}{2} \quad (\text{and, e.g., } |\arg a| \leqslant \pi/2).$$

Those Zeta functions are "secondary": arising from the nontrivial zeros of a classic zeta function (here, $\zeta(s)$); and "generalized": they

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admit an auxiliary shift parameter just like the Hurwitz zeta function ($\zeta(s,a) \stackrel{\text{def}}{=} \sum_0^\infty (n+a)^{-s}$). Such "$\zeta$-Zeta" functions have occasionally appeared in the literature, but mostly through particular cases or under very specific aspects. On the other hand, their *abundance of general explicit properties* seems to have been largely ignored, although it can be revealed by quite *elementary* means (as we will do). And with regard to the Riemann zeros, reputed to be highly elusive quantities, those properties constitute additional explicit information: this is enough to motivate a more comprehensive treatment (and bibliography) of the subject. The present work just aims to do that, in a self-contained and very concrete way, as a kind of "All you ever wanted to know about ζ-Zeta functions..." handbook (without prejudice to the usefulness of any single result by itself).

We begin by developing the background and our motivations in greater detail.

First, if a *Selberg zeta function* is used in place of $\zeta(s)$ from the start (assuming the simplest setting of a compact hyperbolic surface X here), then the $\{{\tau_k}^2+\frac{1}{4}\}$ become the eigenvalues of the (positive) Laplacian on X, and the Zeta functions (2) turn into *bona fide* spectral (Minakshisundaram–Pleijel) zeta functions, for which numerous explicit results have indeed been displayed (with the help of Selberg trace formulae: cf. [32] for $v \geqslant \frac{1}{4}$, [35] for $v = \frac{1}{4}$, [7], [38] for $v = 0$).

Some transposition of those results to the Riemann case can then be expected, in view of the formal analogies between the trace formulae for Selberg zeta functions on the one hand, and the Weil "explicit formula" for $\zeta(s)$ on the other hand [19]. Indeed, a few symmetric functions over the Riemann zeros that resemble spectral functions have been well described, mainly $V(t) \stackrel{\text{def}}{=} \sum_\rho \mathrm{e}^{\rho t}$ ([10], [18], [23] chap.II). Zeta functions like (2) have also been considered, but almost solely to establish their meromorphic continuation to the whole σ-plane — apart from the earliest occurrence we found: a mention by Guinand [17] (see also [8]) of the series $\sum_k \tau_k^{-s}$ ($\equiv \mathcal{Z}(s/2, v=0)$) on one side of a functional relation (equation (79) below) arising as an instance of a generalized Poisson summation formula. Later, Delsarte introduced that function again (as $\phi(s)$ in [12]) to describe its poles qualitatively, displaying (only) its principal polar part at $s=1$, as $(2\pi)^{-1}/(s-1)^2$; Kurokawa [24] made the same study at $v = \frac{1}{4}$, not only for $\zeta(s)$ but also for Dedekind zeta functions and Selberg zeta functions for $\mathrm{PSL}_2(\mathbb{Z})$ [or congruence subgroups] (then, Zeta functions like (2) occur within the *parabolic* components) ; and Matiyasevitch [27] discussed the special values $\theta_n \equiv 2\,\mathscr{Z}(n, \frac{1}{4})$, $(n \in \mathbb{N}^*)$. Extensions in the style of the Lerch zeta function have also been studied ([16], [23] chap.VI).

Independently, Deninger [13] and Schröter–Soulé [34] considered a *different* Hurwitz-type family (we keep their factor $(2\pi)^s$ just to avoid multiple notations),

$$(4)\qquad \xi(s,x) \stackrel{\text{def}}{=} (2\pi)^s \sum_\rho (x-\rho)^{-s} \qquad (\text{Re } s > 1),$$

mainly to evaluate $\partial_s \xi(s,x)_{s=0}$ (as equation (101) below); earlier, Matsuoka [29], then Lehmer [26] had focused upon the sums

$$(5)\qquad \mathscr{Z}_n \stackrel{\text{def}}{=} \sum_\rho \rho^{-n}, \quad n \in \mathbb{N}^* \quad [\equiv (2\pi)^{-n}\xi(n,1), \text{ for } n \neq 1].$$

Here we easily recover $\xi(s,x)$ from the *other* Zeta function (3) (but not the reverse), as

$$(6)\qquad \xi\Big(s, \frac{1}{2}+y\Big) \equiv (2\pi)^s \Big[\mathrm{e}^{\mathrm{i}\pi s/2}\, \mathfrak{Z}\Big(\frac{1}{2}s, \mathrm{i}y\Big) + \mathrm{e}^{-\mathrm{i}\pi s/2}\, \mathfrak{Z}\Big(\frac{1}{2}s, -\mathrm{i}y\Big)\Big].$$

The present work proposes a broader, and unified, description for all those ζ-Zeta functions, with a wealth of explicit results comparable to usual spectral zeta functions [38].

Tools for the purpose could also be borrowed from spectral theory (trace formulae, etc.), but the objects under scrutiny are more singular here (the Zeta functions for the Riemann zeros manifest double poles, vs simple poles in the Selberg case); this then imposes so many adaptations upon the classic procedures that a self-contained treatment of the Riemann case alone is actually simpler. Even then, we cannot get maximally explicit outputs for all cases at once (e.g., Weil's «explicit formula» *diverges* for $v \leqslant \frac{1}{4}$), and our analysis has to develop gradually.

So, we begin (Section 2) by setting up a minimal abstract framework, sufficient to handle $\mathscr{Z}(\sigma, v)$ (with a permanent distinction between properties in the half-planes $\{\text{Re } \sigma < 1\}$ and $\{\text{Re } \sigma > \frac{1}{2}\}$ respectively). We next obtain a first batch of explicit results for the case $v = \frac{1}{4}$ (Section 3), then for general values of v (Section 4). Specializing to the case $v = 0$ in Section 5, we reach an almost explicit meromorphic continuation formula for $\mathscr{Z}(\sigma, 0)$ into the half-plane $\{\text{Re } \sigma < \frac{1}{2}\}$, which immediately implies many more properties of that function, and is generalizable to L-series and other number-theoretical zeta functions. In Section 6 we exploit the latter results to sharpen the descriptions of both Hurwitz-type functions $\mathcal{Z}(\sigma, v)$ and $\mathfrak{Z}(\sigma, a)$. Section 7 provides a summary of the results; essentially, a Table collects the main formulae for $v = 0$ and $\frac{1}{4}$, also referring to the main text like an index. (Which text can in some sense be viewed, and simply used, as a set of "notes" for this Table !) Finally, Appendices A and B treat some subsidiary issues: a meromorphic continuation method for the Mellin transforms of Section 2, and certain numerical aspects.

For convenience, we recall the needed elementary results and notations [1], [15], [9]:

$$B_n : \text{Bernoulli numbers } \Big(B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ldots;\quad B_{2m+1} = 0 \text{ for } m = 1, 2, \ldots\Big)$$

$(B_n(x)$: Bernoulli polynomials)

$\gamma =$ Euler's constant;

$$\text{Stieltjes constants : } \gamma_n \stackrel{\text{def}}{=} \lim_{M\to\infty} \left\{ \sum_{m=1}^{M} \frac{(\log m)^n}{m} - \frac{(\log M)^{n+1}}{n+1} \right\}$$

$(\gamma_0 = \gamma \approx 0.5772156649,\ \gamma_1 \approx -0.0728158455,\ \gamma_2 \approx -0.0096903632, \ldots)$

$$E_n : \text{Euler numbers } (E_0 = 1,\ E_2 = -1,\ E_4 = 5, \ldots;\quad E_{2m+1} = 0 \text{ for } m = 0, 1, \ldots). \tag{7}$$

[E_n is also a standard symbol for quantized energy levels, a concept often invoked purely rhetorically about the Riemann zeros; let us then insist that our (present) work is totally decoupled from such a viewpoint, whereas it sees the Euler numbers as truly essential !]

Concerning the Riemann zeta function $\zeta(s) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} k^{-s}$ [36], [11], [14], we will need its special values,

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1} \quad (n = 0, 1, \ldots); \qquad \zeta(2m) = \frac{(2\pi)^{2m}}{2(2m)!} |B_{2m}| \quad (m = 1, 2, \ldots); \tag{8}$$

$$\zeta(0) = -\tfrac{1}{2} \qquad \text{and} \qquad \zeta'(0) = -\tfrac{1}{2} \log 2\pi, \tag{9}$$

and its functional equation in the form

$$\Xi(s) \stackrel{\text{def}}{=} \frac{\zeta(s)}{F(s)} \equiv \frac{\zeta(1-s)}{F(1-s)}, \qquad F(s) \stackrel{\text{def}}{=} \frac{\pi^{s/2}}{s(s-1)\,\Gamma(s/2)}, \tag{10}$$

where $\Xi(s)$ is an *entire* function, which is *even* under the symmetry $s \leftrightarrow (1-s)$ (this expresses the functional equation), and only keeps the nontrivial zeros of $\zeta(s)$.

In order to analyze $\zeta(s)$, we will be *forced* to invoke a particular Dirichlet L-series as well (associated with the Dirichlet character χ_4 [1], [15], [11])

$$\beta(s) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} (-1)^{k+1}(2k-1)^{-s} \quad [= L(\chi_4, s)]; \tag{11}$$

$\beta(s)$ extends to an entire function, having the special values

$$\beta(-m) = \frac{E_m}{2}, \qquad \beta(2m+1) = \frac{(\pi/2)^{2m+1}}{2(2m)!}|E_{2m}| \qquad (m = 0, 1, \ldots); \tag{12}$$

$$\beta(0) = \frac{1}{2} \quad \text{and} \quad \beta'(0) = -\frac{3}{2}\log 2 - \log \pi + 2\log\left(\frac{1}{4}\right), \tag{13}$$

and a functional equation expressible as

$$\Xi_{\chi_4}(s) \stackrel{\text{def}}{=} \left(\frac{4}{\pi}\right)^{(s+1)/2} \Gamma\left(\frac{s+1}{2}\right)\beta(s) \equiv \Xi_{\chi_4}(1-s) \tag{14}$$

where the function Ξ_{χ_4} is entire and only keeps the nontrivial zeros of $\beta(s)$.

2. General Delta and Zeta functions of order < 1.

2.1. Admissible sequences and Delta functions.

Throughout this work, a numerical sequence $\{x_k\}$ (systematically labeled by *positive* integers $k = 1, 2, \ldots$) will be called *admissible of order* $\mu_0 < 1$ if

(i) $0 < x_1 \leqslant x_2 \leqslant \cdots, \quad x_k \uparrow +\infty$;

(or: complex $x_k \to \infty$ with $|\arg x_k|$ sufficiently bounded [31], [20], to provide an unconditionally valid framework "in case of need");

(ii) $\sum_k |x_k|^{-1} < \infty$, making the following Weierstrass product *converge*:

$$\Delta(z \mid \{x_k\}) \stackrel{\text{def}}{=} \prod_k (1 + z/x_k) \qquad (\forall z \in \mathbb{C}); \tag{15}$$

it then defines an *entire* "Delta" function $\Delta(z)$ (we omit the argument $\{x_k\}$ except when an ambiguity may result);

(iii) for $z \to \infty$, $\log \Delta(z) = o(|z|^{\mu_0+\delta})$ $\forall \delta > 0$, and it admits a complete *uniform asymptotic expansion* in some sector $\{|\arg z| < \theta\}$, of a form governed by some strictly decreasing sequence of real exponents $\{\mu_n\}$, as

$$\log \Delta(z) \sim \sum_{n=0}^{\infty} (\tilde{a}_{\mu_n} \log z + a_{\mu_n}) z^{\mu_n} \qquad (z \to \infty) \tag{16}$$
$$\text{with } \mu_0 > \mu_1 > \cdots, \quad \mu_n \downarrow -\infty, \quad \text{and } 0 < \mu_0 < 1$$

("generalized Stirling expansion", by extension from the case $x_k = k$ [22]); such a uniform expansion is repeatedly differentiable in z.

Then, the Dirichlet series

$$Z(\sigma \mid \{x_k\}) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} x_k^{-\sigma} \qquad (\text{convergent for } \operatorname{Re} \sigma > \mu_0) \tag{17}$$

defines the *Zeta function of* $\{x_k\}$, holomorphic in that half-plane. The point $\sigma = 1$ has to lie in the latter by assumption (ii), which imposes $\mu_0 < 1$; then $(\log \Delta)$ can moreover be expanded term by term in equation (15) to yield the Taylor series

$$\log \Delta(z) \equiv \sum_{m=1}^{\infty} (-1)^{m-1} \frac{Z(m)}{m} z^m \qquad (\text{convergent for } |z| < x_1). \tag{18}$$

Motivations: the idea here is to assume certain properties for an entire function $\Delta(z)$ of order $\mu_0 < 1$, so as to generate a function $Z(\sigma)$ meromorphic in all of $\mathbb{C}$ with poles at the μ_n, their maximum order r being 2 here (as dictated by the specific form of equation (16)). Such $\Delta(z)$ are very special instances of zeta-regularized infinite products, for which much more general frameworks exist (e.g., [22], [23], [20]). However, singularities and essential complications definitely increase each time r or the integer part $[\mu_0]$ can become larger (chiefly, the formalism leaves $([\mu_0] + 1)$ meaningful integration constants undetermined). Efficiency then commands to minimize both latter parameters (subject to $r \geqslant 1$ and $\mu_0 > 0$); specially, $\mu_0 < 1$ is simpler to handle than any $\mu_0 \geqslant 1$. In this respect, spectral zeta functions frequently have $\mu_0 \geqslant 1$ (e.g., for Laplacians on compact Riemannian manifolds, $\mu_0 = \frac{1}{2} \times$[dimension]) but only simple poles, hence the simplifying assumption $r = 1$ is suitable for them [37], [31]; by contrast, the present functions $\mathcal{Z}(\sigma, v)$ will accept the lower value $\mu_0 = \frac{1}{2}$ but will need $r = 2$. Another difference is that for eigenvalue spectra, a "partition function" $\sum_k \mathrm{e}^{-t x_k}$ is a natural starting point; in the Riemann case, that type of function ($V(t) = \sum_\rho e^{\rho t}$) exhibits a more remote and contrived structure [10], while Delta-type functions are easily accessible (by simple alterations of equation (10)). Thus, Riemann zeros and standard eigenvalue spectra have several *mutually singular* features, making their unified handling rather cumbersome.

2.2. Meromorphic continuation of Zeta functions.

The bounds $\log \Delta(z) = O(z^1)$ for $z \to 0$, and $O(z^{\mu_0})$ for $z \to +\infty$, with $\mu_0 < 1$, imply that the following Mellin transform of $\log \Delta(z)$:

$$(19) \qquad I(\sigma) \stackrel{\text{def}}{=} \int_0^{+\infty} \log \Delta(z) z^{-\sigma-1} \mathrm{d}z \qquad (\mu_0 < \mathrm{Re}\, \sigma < 1)$$

converges to a holomorphic function of σ in the stated vertical strip.

Then, by standard arguments (see App.A, and [21], [37], [6]), $I(\sigma)$ extends to a meromorphic function on either side of that strip:

a) by virtue of the expansion (16) for $z \to \infty$, $I(\sigma)$ extends to all $\mathrm{Re}\, \sigma < 1$, with

$$(20) \qquad \text{(at most) double poles at } \sigma = \mu_n, \qquad \text{polar parts} = \frac{\tilde{a}_{\mu_n}}{(\sigma - \mu_n)^2} + \frac{a_{\mu_n}}{(\sigma - \mu_n)};$$

b) just because $\log \Delta(z)$ has a Taylor series at $z = 0$, $I(\sigma)$ extends to all $\mathrm{Re}\, \sigma > \mu_0$, with

$$(21) \qquad \text{(at most) simple poles at } \sigma = m \in \mathbb{N}^*, \qquad \text{of residues } -(\log \Delta)^{(m)}(0)/m!\ .$$

Now, for $\mu_0 < \mathrm{Re}\, \sigma < 1$, the integral (19) can also be done term by term after inserting the expansion (15), giving

$$(22) \qquad \frac{\sigma \sin \pi\sigma}{\pi} I(\sigma) \equiv \sum_{k=1}^{\infty} x_k^{-\sigma} \equiv Z(\sigma).$$

The meromorphic extension of $I(\sigma)$ then entails that of $Z(\sigma)$ to the whole σ-plane as well, i.e., to the half-planes $\{\mathrm{Re}\, \sigma < 1\}$ by *a*), resp. $\{\mathrm{Re}\, \sigma > \mu_0\}$ by *b*) independently, so that

a') any *non-integer* pole ($\mu_n \notin \mathbb{Z}$) of $I(\sigma)$ generates at most a double pole for $Z(\sigma)$, with

$$\text{(23)}\quad Z(\mu_n+\varepsilon) = \Big[\frac{\mu_n \sin \pi\mu_n}{\pi}\tilde a_{\mu_n}\Big]\frac{1}{\varepsilon^2} + \Big[\frac{\mu_n \sin \pi\mu_n}{\pi}a_{\mu_n} + \Big(\frac{\sin \pi\mu_n}{\pi} + \mu_n \cos \pi\mu_n\Big)\tilde a_{\mu_n}\Big]\frac{1}{\varepsilon}\ \text{[+regular part]};$$

a'') any *negative integer* pole ($\mu_n = -m$) of $I(\sigma)$ generates at most a simple pole for $Z(\sigma)$, through partial cancellation with the zeros of $(\sin \pi\sigma)$, with

$$(m\in\mathbb{N}^*):\ Z(-m+\varepsilon) = (-1)^m\Big[-\frac{m\tilde a_{-m}}{\varepsilon} + (\tilde a_{-m} - m a_{-m})\Big] + O(\varepsilon)$$

$$\text{(24)}\quad \text{i.e., residue: } \operatorname{Res}_{\sigma=-m} Z(\sigma) = (-1)^{m+1} m\tilde a_{-m}\ ;$$

$$\text{(25)}\quad \text{finite part: } \operatorname{FP}_{\sigma=-m} Z(\sigma) = (-1)^m(\tilde a_{-m} - m a_{-m})$$

(we call equation (25) "trace identities" by extension from spectral theory, specially when $\tilde a_{-m} = 0$, in which case explicit finite values for the $Z(-m)$ result);

a''') any pole at $\sigma = 0$ of $I(\sigma)$ gets fully cancelled by the double zero of $(\sigma \sin \pi\sigma)$, giving

$$\text{(26)}\quad Z(\varepsilon) = \tilde a_0 + a_0\varepsilon + O(\varepsilon^2) \quad\Longrightarrow\quad Z(0) = \tilde a_0, \quad Z'(0) = a_0.$$

b') each *positive integer* pole of $I(\sigma)$ ($\sigma = m \in \mathbb{N}^*$, from equation (21)) gets cancelled by a zero of $(\sin \pi\sigma)$, giving

$$\text{(27)}\quad Z(m) = (-1)^{m-1}(\log \Delta)^{(m)}(0)/(m-1)!$$

but this last output just duplicates with the previous Taylor formula (18).

Ultimately, all the poles of $Z(\sigma)$ lie in a *single* decreasing sequence $\{\mu_n\}_{n\in\mathbb{N}}$, and have maximum order $r = 2$ under the specific assumption (16).

3. Delta function based at $s = 0$, and the Zeta function $Z(\sigma) \stackrel{\text{def}}{=} \mathcal{Z}(\sigma, v = \frac{1}{4})$.

We now apply the previous framework to the Riemann zeros upon just slight changes with regard to the usual factorization of $\zeta(s)$ around $s = 0$.

3.1. Basic facts and notations.

The most convenient starting point is the entire function (10), which keeps precisely the non-trivial zeros $\{\rho\}$ of $\zeta(s)$, and has the familiar Hadamard product representation [14]

$$(28)\quad \Xi(s) = \mathrm{e}^{Bs} \prod_\rho (1-s/\rho)\, \mathrm{e}^{s/\rho}, \qquad B \stackrel{\text{def}}{=} [\log \Xi]'(0) = \log 2\sqrt{\pi} - 1 - \tfrac{1}{2}\gamma\,.$$

Zeros are henceforth grouped in pairs as $\{\rho = \frac{1}{2} \pm \mathrm{i}\tau_k\}_{k=1,2,\dots}$ according to equation (1). Their corresponding counting function, $\mathcal{N}(T) \stackrel{\text{def}}{=} \operatorname{card}\{\tau_k \mid \operatorname{Re}\tau_k < T\}$, follows a well-known estimate $\overline{\mathcal{N}}(T)$, as [36], [14]

$$(29)\qquad \mathcal{N}(T) \sim \overline{\mathcal{N}}(T) \stackrel{\text{def}}{=} \frac{T}{2\pi}\Big[\log\frac{T}{2\pi} - 1\Big] \qquad (T \to +\infty).$$

Thus, the sequence of zeros itself could be admissible of order 1 at best; fortunately, a transformed sequence $\{x_k\}$ and its Zeta function will immediately arise:

$$(30)\qquad x_k \stackrel{\text{def}}{=} \tfrac{1}{4} + {\tau_k}^2, \qquad \text{and} \qquad Z(\sigma) \stackrel{\text{def}}{=} Z(\sigma \mid \{x_k\}) = \sum_k {x_k}^{-\sigma}$$

(the latter series will converge for $\operatorname{Re}\sigma > \frac{1}{2}$, again by the estimate (29)).

Indeed, once the zeros have been reordered in pairs, it first follows that $\mathscr{Z}_1 \stackrel{\text{def}}{=} \sum_\rho \rho^{-1} = \sum_k {x_k}^{-1} = Z(1)$ (convergent sums), and then, $\Xi(s) \equiv \mathrm{e}^{[B+Z(1)]s} \prod_k [1+s(s-1)/x_k]$; the parity property $\Xi(s) = \Xi(1-s)$ thereupon imposes $Z(1) = -B$, hence

$$(31)\qquad Z(1) = \mathscr{Z}_1 = -[\log \Xi]'(0) = -\log 2\sqrt{\pi} + 1 + \tfrac{1}{2}\gamma,$$

a classic result ([11], ch. 12; [14], Section 3.8). All in all, the product formulae and functional equation boil down to

$$(32)\quad \frac{\zeta(s)}{F(s)} \equiv \Delta(\lambda \mid \{x_k\}) \equiv \frac{\zeta(1-s)}{F(1-s)} \qquad \Big[F(s) \stackrel{\text{def}}{=} \frac{\pi^{s/2}}{s(s-1)\,\Gamma(s/2)}\Big],$$

$$(33)\qquad \text{where} \qquad \Delta(\lambda \mid \{x_k\}) \equiv \Xi(s) \ \text{ with } \ \lambda \stackrel{\text{def}}{=} s(s-1):$$

i.e., $\Xi(s)$ has been rewritten as an infinite product $\Delta(\lambda \mid \{x_k\})$ which is *manifestly even* (under $s \longleftrightarrow (1-s)$), and will qualify as a Delta function of order $\mu_0 = \frac{1}{2}$ *in the variable* $z = \lambda$, a much simpler situation than $\mu_0 = 1$ (naively suggested by equation (28)). We now derive the ensuing properties of $Z(\sigma)$ (as compiled in Section 7, Table 1).

3.2. Properties of $Z(\sigma)$ for $\operatorname{Re}\sigma < 1$.

As basic initial result, the sequence $\{x_k \stackrel{\text{def}}{=} \frac{1}{4} + \tau_k{}^2\}$ *is admissible of order* $\frac{1}{2}$: it obviously fulfills assumptions (i)–(ii); the function $\Xi(s)$ is entire of order 1 in s, hence $\frac{1}{2}$ in λ; finally, a large-λ expansion of the form (16) for $\log\Delta(\lambda \mid \{x_k\})$ is easily obtained as follows. In equation (32), for $s \to \infty$ ($|\arg s| < \pi/2$), $-\log F(s)$ can be replaced by its complete Stirling expansion and $\log\zeta(s) = O(s^{-\infty})$ can be deleted, giving

$$\begin{aligned}(34)\quad \log\Delta(\lambda) \sim \log\lambda - \frac{\log\pi}{2}s + \frac{s}{2}(\log s - \log 2 - 1)\\ -\frac{1}{2}\log\frac{s}{2} + \frac{1}{2}\log 2\pi + \sum_{m\geqslant 1}\frac{B_{2m}2^{2m-1}}{2m(2m-1)}s^{-2m+1};\end{aligned}$$

whereupon s is to be substituted by the relevant solution branch of $s(s-1) = \lambda$, namely

$$(35)\qquad s = \frac{1}{2} + \sqrt{\lambda + \frac{1}{4}} \sim \lambda^{1/2} + \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2^{-2n}\,\Gamma(3/2)}{n!\,\Gamma(-n+3/2)}\lambda^{1/2-n}.$$

The resulting λ-expansion then has all the required properties, with the exponents $\{\mu_n = \frac{1}{2}(1-n)\}$ — giving the order $\mu_0 = \frac{1}{2}$ — and leading coefficients

$$(36)\quad \tilde{a}_{1/2} = \frac{1}{4} \qquad\qquad a_{1/2} = -\frac{1}{2}(1 + \log 2\pi)$$

$$(37)\quad \tilde{a}_0 = \frac{7}{8} \qquad\qquad a_0 = \frac{1}{4}\log 8\pi$$

$$(38)\quad \tilde{a}_{-1/2} = \frac{1}{32} \qquad\qquad a_{-1/2} = -\frac{1}{16}\log 2\pi - \frac{1}{48}, \quad \textit{etc.}$$

This expansion can be computed to any power μ_n in principle; still, a reduced general formula for a_{μ_n} looks inaccessible this way; by contrast, all the $\tilde{a}_{\mu_n}$ with $\mu_n \neq 0$ arise from the single substitution of equation (35) into the prefactor $\frac{s}{2}$ of $\log s$ ($\sim \frac{1}{2}\log\lambda$) in equation (34), giving

$$(39)\quad \tilde{a}_{1/2-m} = \frac{2^{-2m-2}\,\Gamma(3/2)}{m!\,\Gamma(-m+3/2)}, \qquad \tilde{a}_{-1-m} = 0, \qquad m = 0, 1, 2, \ldots.$$

Now, by Section 2.2a'–a'''), a large-λ expansion for $\log\Delta(\lambda)$ translates into explicit properties of $Z(\sigma)$ for $\mathrm{Re}\,\sigma < 1$. Here, $Z(\sigma)$ gets a double pole at each *half*-integer $\frac{1}{2} - m$ ($m \in \mathbb{N}$), with principal polar term $\frac{(-1)^m}{\pi}(\frac{1}{2} - m)\tilde{a}_{1/2-m}/(\sigma + m - \frac{1}{2})^2$, and is regular elsewhere; the leading pole $\sigma = \frac{1}{2}$ has

$$\text{(40)} \qquad \text{full polar part } = \frac{1}{8\pi}\frac{1}{(\sigma-\frac{1}{2})^2} - \frac{\log 2\pi}{4\pi}\frac{1}{\sigma - \frac{1}{2}}.$$

At $\sigma = 0$, equations (37) and (26) deliver *two* explicit values,

$$\text{(41)} \qquad Z(0) = \tilde{a}_0 = \frac{7}{8}, \qquad\qquad Z'(0) = a_0 = \frac{1}{4}\log 8\pi \ (\approx 0.806042857);$$

the latter makes the *Stirling constant* (the value $\exp - [Z'(0)]$, or regularized determinant) also explicitly known for this sequence $\{x_k\}$.

Further quantities, tied to the yet unspecified general coefficients a_{μ_n}, will also acquire explicit closed forms: the polar terms of order $1/(\sigma+m-\frac{1}{2})$ and the finite values $Z(-m)$ for all $m \in \mathbb{N}$, see Table 1; but we will need a more indirect approach (Section 6.1).

3.3. Properties of $Z(\sigma)$ for $\mathrm{Re}\,\sigma > \frac{1}{2}$.

Now by Section 2.2b'), $Z(\sigma)$ is holomorphic in the half-plane $\mathrm{Re}\,\sigma > \frac{1}{2}$, and the values $Z(n)$ for $n = 1, 2, \dots$ lie in the Taylor series of $\log\Delta(\lambda)$ at $\lambda = 0$, which can be specified here through equation (32).

We first expand $\log\Delta(\lambda)$ in powers of s, where $\lambda = s(s-1)$. Equation (28) directly implies

$$\text{(42)} \qquad \log\Delta(\lambda) \equiv \log\Xi(s) = -\sum_{n=1}^{\infty}\frac{\mathscr{Z}_n}{n}s^n \qquad \Big(\mathscr{Z}_n \equiv \sum_\rho \rho^{-n}\Big);$$

then, from equation (32), $\log[\zeta(s)/F(s)]$ and $\log[\zeta(1-s)/F(1-s)]$ (expressing the functional equation) respectively yield the two Taylor series

$$-\log\sqrt{\pi}\ s + \log\Gamma(1+s/2) + \log(1-s) + \log[-2\zeta(s)]$$

$$\text{(43)} \quad = -\log\sqrt{\pi}\,s + \Big[-\frac{\gamma}{2}s + \sum_{n=2}^{\infty}\frac{(-1)^n\zeta(n)}{2^n n}s^n\Big] - \sum_{n=1}^{\infty}\frac{1}{n}s^n + \sum_{n=1}^{\infty}\frac{(\log|\zeta|)^{(n)}(0)}{n!}s^n,$$

$$\log(1-s) + \Big[\frac{\log\pi}{2}(s-1) + \log\Gamma\Big(\frac{1-s}{2}\Big)\Big] + \log[-s\zeta(1-s)]$$

$$\text{(44)} \quad = \Big(\frac{\gamma}{2} + \log 2\sqrt{\pi} - 1\Big)s + \sum_{n=2}^{\infty}\frac{(1-2^{-n})\zeta(n) - 1}{n}s^n - \sum_{n=1}^{\infty}\frac{\gamma_{n-1}^{\mathrm{c}}}{(n-1)!}s^n.$$

The γ_n^{c} in the last line are *cumulants* for the Stieltjes constants γ_n of equation (7) [1] (cf. also the η_n in ref. [39], Sec. 4), i.e.,

$$(45)\qquad \log[-s\zeta(1-s)] \equiv -\sum_{n=1}^{\infty} \frac{\gamma_{n-1}^{\mathrm{c}}}{(n-1)!} s^n \qquad vs \qquad -s\zeta(1-s) \equiv 1 - \sum_{n=1}^{\infty} \frac{\gamma_{n-1}}{(n-1)!} s^n$$

$$\Big(\gamma_0^{\mathrm{c}} = \gamma_0 = \gamma, \quad \gamma_1^{\mathrm{c}} = \gamma_1 + \frac{1}{2}\gamma^2 \approx 0.260362078, \quad \gamma_2^{\mathrm{c}} = \gamma_2 + 2\gamma\gamma_1 + \frac{2}{3}\gamma^3 \approx 0.034459088, \dots\Big).$$

The identification of the three series (42)–(44) at each order s^n now yields a countable sequence of 3-term identities: the first one just restores the result $\mathscr{Z}_1 = -B$ as in equation (31); then, the subsequent ones likewise express the higher $\mathscr{Z}_n$ in *two* ways,

$$(46)\qquad \mathscr{Z}_n = 1 - (-1)^n 2^{-n}\zeta(n) - \frac{(\log|\zeta|)^{(n)}(0)}{(n-1)!} = 1 - (1-2^{-n})\zeta(n) + \frac{n}{(n-1)!}\gamma_{n-1}^{\mathrm{c}}, \quad n = 2,3,\dots$$

That short argument subsumes several earlier results. The rightmost and center expressions in equation (46) amount to formulae for $\mathscr{Z}_n$ by Matsuoka [29] and Lehmer ([26], equation (12)) respectively; the implied relations between the derivatives $\zeta^{(n)}(0)$ and the Stieltjes constants γ_n were also discussed in [2], [9], together with Euler–Maclaurin formulae for the $\zeta^{(n)}(0)$ which parallel the specification of the γ_n in equation (7).

As for the values $Z(m)$ themselves, they are given by equation (27) now using $\lambda = s(s-1)$ as expansion variable, i.e.,

$$(47)\quad Z(m) = \frac{(-1)^{m-1}}{(m-1)!}\left[\left(\frac{1}{2s-1}\frac{\mathrm{d}}{\mathrm{d}s}\right)^m \log \Xi(s)\right]_{s=0 \text{ or } 1} \quad (m = 1,2,\dots).$$

But alternatively, $Z(m) \equiv \mathscr{Z}_m + \big[$a *finite* linear combination of the $\{\mathscr{Z}_n\}_{n=1,\dots,m-1}\big]$, and vice-versa: as shown below,

$$(48)\qquad \frac{\mathscr{Z}_n}{n} = \sum_{0\leqslant \ell \leqslant n/2} (-1)^\ell \binom{n-\ell}{\ell} \frac{Z(n-\ell)}{n-\ell} \iff Z(m) = \sum_{\ell=0}^{m-1} \binom{m+\ell-1}{m-1} \mathscr{Z}_{m-\ell}$$

($Z(1) = \mathscr{Z}_1$, $Z(2) = \mathscr{Z}_2 + 2\mathscr{Z}_1, \ldots$). The $\mathscr{Z}_n$ being already known from equations (31) and (46), $Z(m)$ then reduces to an explicit affine combination (over the rationals) of $B = [\log \Xi]'(0)$, $\zeta(n)$, and either $(\log|\zeta|)^{(n)}(0)$ or $\gamma^{\mathrm{c}}_{n-1}$ for $1 < n \leqslant m$ (see Table 1), as stated earlier by Matiyasevich [27]. (More recently, $Z(1)$ and $Z(2)$ also got revived in studies of the distribution of primes [33], [25]).

Since ref. [27] uses equation (48) for $Z(m)$ without mentioning any proof, we sketch one. First, if $x \stackrel{\text{def}}{=} \rho(1-\rho)$, the expansion of the identity $\log\big[(1-s/\rho)(1-s/(1-\rho))\big] \equiv \log\big[1-s(1-s)/x\big]$ in powers of s yields $[\rho^{-n}+(1-\rho)^{-n}]/n \equiv \sum_{0\leqslant\ell\leqslant n/2}(-1)^\ell\binom{n-\ell}{\ell}x^{-n+\ell}/(n-\ell)$ for $n=1,2,\ldots$. By recursion, this must invert in the form $x^{-m} \equiv \sum_{n=1}^m V_{m,n}[\rho^{-n}+(1-\rho)^{-n}]$; then, $\sum_{n=1}^m V_{m,n}\rho^{-n}$ has to be the singular part in the Laurent series of x^{-m} around $\rho = 0$, resulting in $V_{m,n} = \binom{2m-n-1}{m-1}$. Now, summing both sets of identities over the Riemann zeros $\{\rho\}$ yields the stated decompositions (48). (We stress that their finite character is specific to the $Z(m)$ as opposed to all other values $\mathcal{Z}(m,v)$, $v \neq \frac14$).

Note added in proof.— For completeness, we quote two other sets of identities for the sums $\mathscr{Z}_n$ [40]:

$$2\,\mathscr{Z}_k = -\sum_{l=k+1}^{\infty}\binom{\ell-1}{k-1}\mathscr{Z}_\ell \qquad \text{for each odd } k \geqslant 1$$

(a countable sequence of «sum rules», easy but unreported, wich allow to eliminate any *finite* subset of odd values); and the connection to *Li's coefficients* (cf. [39], thm 2), $\lambda_n \stackrel{\text{def}}{=} \sum_\rho[1-(1-1/\rho)^n]$ [wich allow to recast the Riemann Hypothesis as $\lambda_n > 0$ $(\forall n)$]:

$$\lambda_n = \sum_{j=1}^{n}(-1)^{j+1}\binom{n}{j}\mathscr{Z}_j \iff \mathscr{Z}_n = \sum_{j=1}^{n}(-1)^{j+1}\binom{n}{j}\lambda_j \ (n=1,2,\ldots).$$

4. Generalized Zeta functions $\mathcal{Z}(\sigma, v) \equiv Z(\sigma \mid \{\tau_k{}^2 + v\})$.

We begin to discuss the Hurwitz-like generalizations of the preceding case obtained by shifting the *squared* parameters $\tau_k{}^2$. The obviously allowed translations ($x_k \mapsto x_k + v''$, $v'' > -x_1$ in the real case) preserve the notion of admissible sequences (with their values of μ_0, r); that validates the earlier definition (2), as $\mathcal{Z}(\sigma, v) = Z(\sigma \mid \{\tau_k{}^2 + v\})$.

The corresponding transformation of Delta functions (as Hadamard products of order < 1) only involves an explicit constant denominator to preserve their specific normalization $\Delta(0) = 1$, as

$$(49) \quad \Delta(\lambda \mid \{{\tau_k}^2 + v\}) \equiv \Delta(\lambda + v - v' \mid \{{\tau_k}^2 + v'\}) / \, \Delta(v - v' \mid \{{\tau_k}^2 + v'\}).$$

Alternatively, we might have opted to normalize Delta functions as *zeta-regularized* infinite products, i.e.,

$$(50) \qquad \Delta_{\mathrm{zr}}(\lambda) \stackrel{\mathrm{def}}{=} \exp\left[-\partial_\sigma Z(\sigma \mid \{{\tau_k}^2 + \lambda\})\right]_{\sigma=0},$$

which are fully translation-covariant, but at the same time less explicit. The overall benefit of this normalization is then dubious within the restricted scope of this work; but here, it explains a dichotomy between algebraic and transcendental properties of Zeta functions, which roughly follows our overall division between $\{\mathrm{Re}\,\sigma < 1\}$ and $\{\mathrm{Re}\,\sigma > \mu_0\}$ properties, but not quite.

Covariance implies that if we translate $\lambda \mapsto (\lambda + v)$, the expansion of $\log \Delta_{\mathrm{zr}}(\lambda)$ around the *invariant* point $\lambda = +\infty$ can be recomputed to any order by straight substitution, yielding *explicit polynomials* in v as coefficients. When $\mu_0 < 1$ as here, then $\Delta_{\mathrm{zr}}(\lambda) \equiv \mathrm{e}^{-a_0}\, \Delta(\lambda)$ [37], hence the previous statement holds for the expansion (16) *minus its term of order* λ^0; i.e., all the shifted coefficients $\tilde{a}_{\mu_n}(v)$, $a_{\mu_n}(v)$ will be polynomial excepting $a_0(v)$. For $\log \Delta(\lambda \mid \{{\tau_k}^2 + v\})$ specifically, equations (36–39) (at $v = \frac{1}{4}$) imply that

$$(51) \quad \tilde{a}_{1/2},\ a_{1/2},\ \tilde{a}_0 \text{ stay constant} \qquad (\text{as well as } \tilde{a}_{-1} \equiv \tilde{a}_{-2} \equiv \cdots \equiv 0);$$

$$(52) \qquad \tilde{a}_{1/2-m}(v) \equiv \frac{\Gamma(3/2)}{\Gamma(-m+3/2)} \frac{v^m}{4m!}.$$

For functions like $\mathcal{Z}(\sigma, v)$, the consequences are that their polar parts and "trace identities" will depend polynomially on v; furthermore, a single (fixed-v) large-λ expansion, such as equation (34) for $\log \Delta(\lambda \mid \{{\tau_k}^2 + \frac{1}{4}\})$, suffices to express those v-dependences in full. Precisely here, by equation (51): $\mathcal{Z}(\sigma, v)$ keeps its rightmost ($\sigma = \frac{1}{2}$) full polar part *constant* (and given by equation (40)), as well as its value at 0 ($\mathcal{Z}(0, v) \equiv \frac{7}{8}$); all its other poles (of order 2, except at $v = 0$) keep fixed locations. (As a by-product, any difference function $[\mathcal{Z}(\sigma, v) - \mathcal{Z}(\sigma, v_0)]$ is holomorphic for $\mathrm{Re}\,\sigma > -\frac{1}{2}$). We can specify such polynomial formulae in the half-plane $\mathrm{Re}\,\sigma < \frac{1}{2}$ still further, but only later by a different path (Section 6).

By contrast, all formulae for $\mathcal{Z}(\sigma, v)$ in the half-plane $\{\mathrm{Re}\, \sigma > \frac{1}{2}\}$ refer to Taylor coefficients of $\log \Delta_{\mathrm{zr}}(v+\lambda)$ around v *finite*, which evolve *transcendentally* with v: here they will only express in terms of $\log |\zeta(s)|$ (or $\log \Xi(s)$, from equation (10)) and its derivatives at $s = \frac{1}{2} \pm v^{1/2}$. The first of those coefficients, $\log \Delta_{\mathrm{zr}}(v)$ itself ($= -a_0(v)$), actually yields a special value lying at $\sigma = 0$, by equation (50):

$$\partial_\sigma \mathcal{Z}(\sigma, v)_{\sigma=0} = a_0(v) = a_0\left(\frac{1}{4}\right) - \log \Delta\left(v - \frac{1}{4} \mid \left\{{\tau_k}^2 + \frac{1}{4}\right\}\right)$$

$$= \frac{1}{4}\log 8\pi - \log \Xi\left(\frac{1}{2} \pm v^{1/2}\right) \tag{53}$$

(a result which fully matches equation (101) below for the Hurwitz-type function ξ [13], [34]). Then, the Taylor coefficients of order $n \geqslant 1$ (identical for $\log \Delta(\lambda)$ and $\log \Delta_{\mathrm{zr}}(\lambda)$) yield

$$\mathcal{Z}(n, v) = \frac{(-1)^{n-1}}{(n-1)!} \frac{\mathrm{d}^n}{\mathrm{d}v^n} \log \Xi\left(\tfrac{1}{2} \pm v^{1/2}\right) \qquad (n = 1, 2, \ldots), \tag{54}$$

$$\textit{e.g.,} \qquad \mathcal{Z}(1, v) = \pm\frac{1}{2} v^{-1/2} (\log \Xi)'\left(s = \frac{1}{2} \pm v^{1/2}\right) \quad (v \neq 0),$$

$$\mathcal{Z}(1, 0) = \frac{1}{2} (\log \Xi)''\left(\frac{1}{2}\right)$$

(but we are in lack of more reduced closed forms for general (n, v)).

In summary, the polar parts of $\mathcal{Z}(\sigma, v)$ and the special values $\{\mathcal{Z}(-n, v)\}_{n \in \mathbb{N}}$ ("trace identities") have polynomial expressions in v; whereas $\{\mathcal{Z}(n, v)\ (n \neq 0)$ *plus* $\partial_\sigma \mathcal{Z}(0, v)$ *at* $n = 0\}$ are also special values, but only computable transcendentally. (This conclusion is moreover fully obeyed for typical spectral zeta functions.)

5. Delta function based at $s = \frac{1}{2}$, and the Zeta function $\mathcal{Z}(\sigma) \stackrel{\text{def}}{=} \mathcal{Z}(\sigma, v = 0)$.

An interesting option is now to shift the parameter v from its initial value $\frac{1}{4}$ in $Z(\sigma)$, to the most symmetrical value $v = 0$. By equation (49), the Hadamard product (33) becomes based at $s = \frac{1}{2}$ as

$$\Xi\left(\frac{1}{2} + t\right) \equiv \Xi\left(\frac{1}{2}\right) \Delta(t^2 \mid \{{\tau_k}^2\}) \qquad \text{with} \quad t \stackrel{\text{def}}{=} s - \frac{1}{2}, \quad t^2 = \lambda + \frac{1}{4}, \tag{55}$$

$$\text{(56)} \qquad \text{and} \quad \Xi\left(\frac{1}{2}\right) = -\pi^{-1/4}\,\Gamma\left(\frac{5}{4}\right)\zeta\left(\frac{1}{2}\right) \qquad (\approx 0.994241556)$$

(but very little is known about $\zeta(\frac{1}{2})$ [28] and we cannot make this constant factor any more explicit, contrary to the special case $v = \frac{1}{4}$ where that factor was $\Xi(0) = 1$).

The factorized representation (32) then transforms to

(57)

$$\Delta(t^2 \mid \{\tau_k{}^2\})\frac{\mathbf{D}(t)}{1-2t} = \zeta\left(\frac{1}{2}+t\right), \; \mathbf{D}(t) \stackrel{\text{def}}{=} \zeta\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{4}\right)\pi^{t/2}\Big/\Gamma\left(\frac{5}{4}+\frac{t}{2}\right).$$

(This Delta function is closest to the determinant of Riemann zeros used by Berry–Keating for other purposes [4].)

We accordingly switch to the Zeta function of the sequence $\{\tau_k{}^2\}$ [17], [12], or in short,

$$\text{(58)} \qquad \mathcal{Z}(\sigma) \stackrel{\text{def}}{=} \mathcal{Z}(\sigma, 0) = \sum_{k=1}^{\infty} \tau_k{}^{-2\sigma}, \qquad \operatorname{Re}\sigma > \frac{1}{2}.$$

Numerically, this new function looks almost indistinguishable from $Z(\sigma)$ (see Appendix B). (Also, by Section 4, $(Z - \mathcal{Z})(\sigma)$ extends holomorphically to $\operatorname{Re}\sigma > -\frac{1}{2}$.) By contrast, the *meromorphic continuation* of $\mathcal{Z}(\sigma)$ will prove to be distinctly simpler than that of $Z(\sigma)$, thanks to specially explicit representation formulae in the half-plane $\{\operatorname{Re}\sigma < \frac{1}{2}\}$. To obtain these, we now switch to a more powerful, special to $v = 0$, approach (whereas the earlier considerations would still describe $\mathcal{Z}(\sigma)$, but just to the same extent as $Z(\sigma)$).

5.1. The shifted spectrum of trivial zeros.

The factor $\mathbf{D}(t)$ in equation (57) has the structure of a *spectral determinant* built over the "spectrum" of trivial zeros *in the variable* $(-t)$, namely $\{\frac{1}{2} + 2k\}$ ($\mathbf{D}(t)$ is not exactly the zeta-regularized determinant, but again this will not matter here). That spectral interpretation can be extended to the factor $(1 - 2t)^{-1}$, by treating the pole $t = \frac{1}{2}$ (of $\zeta(\frac{1}{2} + t)$) as a "ghost eigenvalue" of multiplicity (-1). A major role of the spectrum of trivial zeros is to make $\log[\mathbf{D}(t)/(1 - 2t)]$ asymptotically cancel $\log\Delta(t^2 \mid \{\tau_k{}^2\})$ *to all orders* when $t \to \infty$ in $|\arg t| < \pi/2$, given that $\log\zeta(\frac{1}{2} + t)$ decreases exponentially there.

We therefore expect the *spectral zeta function of the trivial zeros* (of $\zeta(\frac{1}{2} - t)$) to play an important role; this "shadow zeta function of $\zeta(s)$"

(for short) involves *both $\zeta(s)$ itself and the partner function $\beta(s)$*, in the combination

$$(59)\qquad \mathbf{Z}(s) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \left(\frac{1}{2}+2k\right)^{-s} \equiv 2^s \left[\frac{1}{2}\Big((1-2^{-s})\zeta(s)+\beta(s)\Big)-1\right]$$

($= 2^{-s}\zeta(s,\frac{5}{4})$ in terms of the Hurwitz zeta function). $\mathbf{Z}(s)$ has a single simple pole at $s=1$, of residue $\frac{1}{2}$, and admits the special values

$$(60)\qquad \mathbf{Z}(-n) = -\frac{2^n}{n+1} B_{n+1}\left(\frac{1}{4}\right) - 2^{-n}$$
$$= \frac{1}{2}\left[(1-2^{-n})\frac{B_{n+1}}{n+1} + 2^{-n-1}E_n\right] - 2^{-n}, \quad n=0,1,\dots$$

$$(61)\quad \mathbf{Z}(n) = \frac{(-1)^n 2^{-n}}{(n-1)!}[\log\Gamma]^{(n)}\Big(\frac{5}{4}\Big), \qquad n=2,3,\dots$$
$$= \left.\begin{cases} \dfrac{1}{2}\left[(2^{2m}-1)\frac{(2\pi)^{2m}}{2(2m)!}|B_{2m}| + 2^{2m}\beta(2m)\right] - 2^{2m}, & n=2m \\ \dfrac{1}{2}\left[(2^{2m+1}-1)\zeta(2m+1) + \frac{\pi^{2m+1}}{2(2m)!}|E_{2m}|\right] - 2^{2m+1}, & n=2m+1 \end{cases}\right\} m=1,2,\dots$$

$$(62)\quad \mathbf{Z}(0) = -\frac{3}{4} \qquad \text{and} \qquad \mathbf{Z}'(0) = -\frac{7}{4}\log 2 - \frac{1}{2}\log\pi + \log\Gamma\left(\frac{1}{4}\right).$$

Remark. — Literally, the framework of Section 2 excludes the sequence of trivial zeros (of linear growth, and order $\mu_0 = 1$), but the truly relevant function here will be $\mathbf{Z}(2\sigma)$, as Zeta function of the modified sequence $\{(\frac{1}{2}+2k)^2\}$, which is admissible of order $\frac{1}{2}$ again.

5.2. Meromorphic continuation formulae for $\mathcal{Z}(\sigma)$.

We start from a slight variant of the representation (22) for $\mathcal{Z}(\sigma)$, obtained through an integration by parts upon the Mellin formula (19) (where $z \equiv t^2$, by equation (55)):

$$(63)\quad \mathcal{Z}(\sigma) = \frac{\sin\pi\sigma}{\pi} J(\sigma),$$
$$J(\sigma) \stackrel{\text{def}}{=} \int_0^{+\infty} (t^2)^{-\sigma} \mathrm{d}\,\log\Delta(t^2 \mid \{{\tau_k}^2\}) \quad \left(\frac{1}{2} < \operatorname{Re}\sigma < 1\right).$$

We next introduce a (regularized) *resolvent trace* for the spectrum of trivial zeros,

$$(64)\qquad \mathbf{R}(t) \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}t}\log \mathbf{D}(t) = \frac{1}{2}\left[\log\pi - \frac{\Gamma'}{\Gamma}\left(\frac{5}{4}+\frac{t}{2}\right)\right],$$

which has a simple pole of residue $+1$ at each trivial zero of $\zeta(\frac{1}{2}+t)$; a corresponding function for the pole ("ghost") at $t=\frac{1}{2}$ is

$$R_{\mathrm{g}}(t) = -1/\Big(t-\frac{1}{2}\Big) \qquad \text{(with residue } (-1)). \tag{65}$$

Then, upon insertion of the factorization formula (57), equation (63) yields

$$J(\sigma) \equiv \int_0^{+\infty} t^{-2\sigma}\Big[-\mathbf{R}(t) - R_{\mathrm{g}}(t) + \frac{\zeta'}{\zeta}\Big(\frac{1}{2}+t\Big)\Big]\mathrm{d}t. \tag{66}$$

Now a crucial feature of the case $v=0$ is that *this* integral is also a Mellin transform with respect to the argument appearing in the factorized form of $\zeta(s)$ (namely the variable t, in equation (57)). As a consequence, the contribution to J from $\mathbf{R}(t)$ (and also $R_{\mathrm{g}}(t)$) can be neatly extracted and evaluated, in closed and interpretable form. Because the factor in brackets in equation (66) is $O(t)$ at $t=0$ (due to the functional equation), $J(\sigma)$ can be split as

$$J(\sigma) = \mathbf{J}(\sigma) + J_{\mathrm{r}}(\sigma), \qquad \mathbf{J}(\sigma) \stackrel{\text{def}}{=} \int_0^{+\infty} t^{-2\sigma}[\mathbf{R}(0)-\mathbf{R}(t)]\mathrm{d}t \tag{67}$$

$$J_{\mathrm{r}}(\sigma) \stackrel{\text{def}}{=} \int_0^{+\infty} t^{-2\sigma}\Big[R_{\mathrm{g}}(0) - R_{\mathrm{g}}(t) + \frac{\zeta'}{\zeta}\Big(\frac{1}{2}+t\Big) - \frac{\zeta'}{\zeta}\Big(\frac{1}{2}\Big)\Big]\mathrm{d}t,$$

(*this* splitting preserves the convergence strip $\{\frac{1}{2} < \operatorname{Re}\sigma < 1\}$ for both resulting integrals).

$J_{\mathrm{r}}(\sigma)$ can be split still further, once its integration path has been rotated by a small angle: either $+\varepsilon$ or $-\varepsilon$, in order to bypass the poles of $R_{\mathrm{g}}(t)$ and of $\frac{\zeta'}{\zeta}\big(\frac{1}{2}+t\big)$ at $t=\frac{1}{2}$,

$$J_{\mathrm{r}}(\sigma) = J_{\mathrm{g}}^{\pm}(\sigma) + J_{\zeta}^{\pm}(\sigma), \qquad J_{\mathrm{g}}^{\pm}(\sigma) \stackrel{\text{def}}{=} \int_0^{+\mathrm{e}^{\pm\mathrm{i}\varepsilon}\infty} t^{-2\sigma}[R_{\mathrm{g}}(0)-R_{\mathrm{g}}(t)]\mathrm{d}t \tag{68}$$

$$J_{\zeta}^{\pm}(\sigma) \stackrel{\text{def}}{=} \int_0^{+\mathrm{e}^{\pm\mathrm{i}\varepsilon}\infty} t^{-2\sigma}\Big[\frac{\zeta'}{\zeta}\Big(\frac{1}{2}+t\Big) - \frac{\zeta'}{\zeta}\Big(\frac{1}{2}\Big)\Big]\mathrm{d}t.$$

Now, $\mathbf{J}(\sigma)$, $J_{\mathrm{g}}^{\pm}(\sigma)$ can be straightforwardly transformed into Hankel contour integrals and then *computed in closed form* (by the residue calculus), giving

$$\mathbf{J}(\sigma) = -\frac{\pi\,\mathbf{Z}(2\sigma)}{\sin 2\pi\sigma}, \qquad J_{\mathrm{g}}^{\pm}(\sigma) = \frac{\pi\, 2^{2\sigma}\,\mathrm{e}^{\mp 2\pi\mathrm{i}\sigma}}{\sin 2\pi\sigma}, \tag{69}$$

both of which are explicit functions, meromorphic in the whole plane; chiefly, $\mathbf{J}$ brings in the shadow zeta function (59).

Then, again upon back-and-forth integrations by parts, $J_\zeta^\pm(\sigma)$ continue to

$$(70)\quad J_\zeta^\pm(\sigma) \equiv \int_0^{+\,\mathrm{e}^{\pm \mathrm{i}\varepsilon}\infty} t^{-2\sigma} \frac{\zeta'}{\zeta}\Big(\frac{1}{2}+t\Big)\mathrm{d}t \quad \text{analytic for } -\infty < \mathrm{Re}\ \sigma < \frac{1}{2},$$

and (cf. equation (21)) these integrals admit meromorphic extensions to the whole plane, with

$$(71)\ \text{simple poles at } \sigma = \frac{n}{2}, \quad \text{of residues } -\frac{1}{2}\frac{(\log|\zeta|)^{(n)}(\frac{1}{2})}{(n-1)!}, \quad n=1,2,\ldots$$

(the difference $J_\zeta^+(\sigma) - J_\zeta^-(\sigma) \equiv 2\mathrm{i}\pi 2^{2\sigma}$ is entire).

All in all, we finally get two complex conjugate Mellin representations for $\mathcal{Z}(\sigma)$:

$$(72)\quad \mathcal{Z}(\sigma) = \frac{-\mathbf{Z}(2\sigma) + 2^{2\sigma}\,\mathrm{e}^{\mp 2\pi \mathrm{i}\sigma}}{2\cos\pi\sigma} + \frac{\sin\pi\sigma}{\pi}\int_0^{+\,\mathrm{e}^{\pm \mathrm{i}\varepsilon}\infty} t^{-2\sigma}\frac{\zeta'}{\zeta}\big(\tfrac{1}{2}+t\big)\mathrm{d}t;$$

and one real principal-value integral representation given by their half-sum,

$$(73)\quad \mathcal{Z}(\sigma) = \frac{-\mathbf{Z}(2\sigma) + 2^{2\sigma}\cos 2\pi\sigma}{2\cos\pi\sigma} + \frac{\sin\pi\sigma}{\pi}\,-\!\!\!\!\!\!\int_0^{+\infty} t^{-2\sigma}\frac{\zeta'}{\zeta}\big(\tfrac{1}{2}+t\big)\mathrm{d}t,$$

(each of the above converges in the full half-plane $\{\mathrm{Re}\ \sigma < \frac{1}{2}\}$).

Another real form can be obtained with a regular integrand, directly from equation (67):

$$(74)\quad \mathcal{Z}(\sigma) = -\frac{\mathbf{Z}(2\sigma)}{2\cos\pi\sigma} + \frac{\sin\pi\sigma}{\pi}\int_0^{+\infty} t^{-2\sigma}\Big[\frac{\zeta'}{\zeta}\Big(\frac{1}{2}+t\Big) + \frac{1}{t-\frac{1}{2}}\Big]\mathrm{d}t,$$

however this last integral only converges in the strip $\{0 < \mathrm{Re}\ \sigma < \frac{1}{2}\}$.

Remarks.

– As analytical extension formulae, equations (72–74) are precise *counterparts of the functional equation* for $\zeta(s)$; they also stand as more explicit forms of Guinand's functional relation for $\mathcal{Z}(\sigma)$ [17], as discussed below (equation (79)).

– A similar formula exists for the function $\xi(s,x)$ of equation (4) ([13], middle of p. 149), only requiring $\mathrm{Re}\, x > 1$ (which precisely avoids the problem raised above by the pole of $\zeta(s)$); in comparison, the present results correspond to the fixed value $x = \frac{1}{2}$, since $\mathcal{Z}(\sigma) \equiv (2\pi)^{-2\sigma}(2\cos\pi\sigma)^{-1}\xi(2\sigma,\frac{1}{2})$ by equation (6);

As we will elaborate next, analytical properties of $\mathcal{Z}(\sigma)$ in the half-plane $\{\mathrm{Re}\ \sigma < \frac{1}{2}\}$ are made totally straightforward by the Mellin formulae (72–74) (while $\mathcal{Z}(\sigma)$ is holomorphic in the half-plane $\{\mathrm{Re}\ \sigma > \frac{1}{2}\}$, where its defining series (58) converges). Detailed results are also recollected in fully reduced form in Section 7, Table 1.

5.3. Properties of $\mathcal{Z}(\sigma)$ for Re $\sigma < 1$.

A few leading properties of $\mathcal{Z}(\sigma)$ in the half-plane $\{\mathrm{Re}\ \sigma < 1\}$ emerge more easily as special cases from Section 4 (although they can be drawn from equation (72) as well):

– $\sigma = \frac{1}{2}$ is a *double pole*, with the *same full polar part* as for $Z(\sigma)$, equation (40);

– by specializing equations (51), (53),

$$(75)\quad \mathcal{Z}(0) = Z(0) = \frac{7}{8}; \qquad \mathcal{Z}'(0) = Z'(0) - \log \Xi\left(\frac{1}{2}\right)\ (\approx 0.811817944).$$

Otherwise, a Mellin representation like (72) gives a better global view of $\mathcal{Z}(\sigma)$ over the whole half-plane $\{\mathrm{Re}\ \sigma < \frac{1}{2}\}$. Indeed, its non-elementary part (the integral (70)) becomes *regular* there, hence can be *ignored* both for the polar analysis and (thanks to the $\sin \pi\sigma$ factor) for the "trace identities" at integer σ: all such information lies then in the *first term* alone, accessible by mere inspection. We thus obtain that

– $\mathcal{Z}(\sigma)$ only has *simple poles* at the negative *half*-integers $\sigma = \frac{1}{2} - m$, with residues

$$(76)\quad \mathcal{R}_m = \frac{(-1)^m}{2\pi}[\mathbf{Z}(1-2m) + 2^{1-2m}] \equiv \frac{(-1)^m}{8\pi m}(1 - 2^{1-2m})B_{2m}, \qquad m = 1, 2, \dots$$

(hence, only the leading pole $\sigma = \frac{1}{2}$ stays double);

– at the negative integers, the "*trace identities*" read as

$$(77)\quad \mathcal{Z}(-m) = \frac{(-1)^m}{2}[-\mathbf{Z}(-2m) + 2^{-2m}] \equiv (-1)^m 2^{-2m}(1 - \frac{1}{8}E_{2m}), \qquad m = 0, 1, \dots$$

(both formulae (76), (77) were fully reduced using equation (60)).

– a ($\sigma \to -\infty$) *asymptotic formula* follows for $J_\zeta^\pm(\sigma)$, from the term-by-term substitution of the Euler product for $\zeta(s)$ into the integrand of equation (72), giving

$$J_\zeta^\pm(\sigma) \sim -\Gamma(1-2\sigma)\sum_{n\geqslant 2}\Lambda(n)n^{-1/2}(\log n)^{2\sigma-1}, \quad \sigma\to-\infty \tag{78}$$

where as usual, $\Lambda(n) = \log p$ if $n = p^r$ for some prime p, else 0. (An asymptotic formula for $\mathscr{Z}(\sigma)$ itself then follows from equation (72) and the functional equations for $\zeta(s)$, $\beta(s)$.)

Remark. — In our notations, Guinand's functional relation for $\mathscr{Z}(\sigma)$ [17] reads as

$$\mathscr{Z}(\sigma) = -\frac{\mathbf{Z}(2\sigma)}{2\cos\pi\sigma} - \frac{Z_{\mathrm{p}}(1-2\sigma)}{2\,\Gamma(2\sigma)\cos\pi\sigma}, \tag{79}$$

where $Z_{\mathrm{p}}(1-2\sigma) \stackrel{\mathrm{def}}{=} \lim_{T\to+\infty}\Big\{\sum_{2\leqslant n<\mathrm{e}^T}\Lambda(n)n^{-1/2}(\log n)^{2\sigma-1} - \int_0^T \mathrm{e}^{x/2}\,x^{2\sigma-1}\mathrm{d}x\Big\}$ — subject to the Riemann Hypothesis [8] — clearly specifies a (real-valued) resummation of the *divergent* series in equation (78) (the asymptotic series for $-J_\zeta^\pm(\sigma)/\,\Gamma(1-2\sigma)$). Equation (79) was only asserted for $0 < \mathrm{Re}\ \sigma < \frac14$, with no clue as to the analytic structure of either $\mathscr{Z}(\sigma)$ or $Z_{\mathrm{p}}(1-2\sigma)$ elsewhere. The present formulae (72–74) are thus resummed versions of equation (79), with a definitely more explicit content.

5.4. Properties of $\mathscr{Z}(\sigma)$ for $\mathrm{Re}\ \sigma > \frac12$.

As stated before, $\mathscr{Z}(\sigma)$ is regular in the half-plane $\{\mathrm{Re}\ \sigma > \frac12\}$, where analytical results are identities directly obtainable by expanding the logarithm of the functional relation (57) in Taylor series around $t = 0$. Here we will extract those results from the Mellin representation (72), invoking the meromorphic properties of its integral term in the whole plane as given by equation (71).

– For half-integer $\sigma = \frac12 + m$: the residues of the two summands in (72) have to cancel given that $\mathscr{Z}(\sigma)$ is analytic in the half-plane; this imposes

$$(\log|\zeta|)^{(2m+1)}\Big(\frac12\Big) = (2m)!\ [\mathbf{Z}(2m+1) + 2^{2m+1}], \quad m = 1,2,\dots \tag{80}$$
$$\Big(= -2^{-2m-1}(\log\Gamma)^{(2m+1)}\Big(\frac14\Big)\ \Big),$$

which simply amounts to $(\log \Xi)^{(2m+1)}(\frac{1}{2}) = 0$ (itself a consequence of the functional equation $\Xi(\frac{1}{2}+t) \equiv \Xi(\frac{1}{2}-t)$); that result further reduces, using equation (61), to the identity

$$(81)\quad (\log|\zeta|)^{(2m+1)}(\tfrac{1}{2}) = \tfrac{1}{2}(2m)!\,(2^{2m+1}-1)\zeta(2m+1) + \tfrac{1}{4}\pi^{2m+1}|E_{2m}|.$$

The case $m = 0$ is singular, but $(\log \Xi)'(\frac{1}{2}) = 0$ directly yields

$$(82)\quad \frac{\zeta'}{\zeta}(\tfrac{1}{2}) = \tfrac{1}{2}\Big[\log\pi - (\log\Gamma)'(\tfrac{1}{4})\Big] = \tfrac{1}{2}\log 8\pi + \frac{\pi}{4} + \frac{\gamma}{2} \quad (\approx 2.68609171).$$

– For integer $\sigma = m$, the pole of the integral is cancelled by the zero of $\sin\pi\sigma$, and the following explicit relation results,

$$(83)\quad 2(-1)^{m+1}\mathcal{Z}(m) - \frac{1}{(2m-1)!}(\log|\zeta|)^{(2m)}(\tfrac{1}{2}) = \mathbf{Z}(2m) - 2^{2m}, \qquad m = 1, 2, \ldots .$$

which can also be further reduced with the help of equation (61), see Table 1.

Unfortunately, we hardly know anything else about the values $(\log|\zeta|)^{(n)}(\frac{1}{2})$, $n = 0, 1, \ldots$ (cf. [28] for $\zeta(\frac{1}{2})$). To supplement the relation (81) with $\zeta(n)$ for n odd, we can only refer to other formulae for $\zeta(2m+1)$ (compiled in [5]), and to Euler–Maclaurin formulae for $\zeta^{(n)}(s)$ (valid at $s = \frac{1}{2}$) with related numerical data [2], [9], [5]. So, even at $v = 0$, the transcendental values $\partial_\sigma \mathcal{Z}(0, v)$ (equation (75)) and $\mathcal{Z}(m, v)$ currently remain more elusive than at the (exceptional) point $v = \frac{1}{4}$ (Section 3.3). Furthermore, we found no reference at all to those values (i.e., $\mathcal{Z}'(0)$, $\mathcal{Z}(m)$) in the literature.

5.5. Speculations and generalizations.

The results of Sections 5.2–4 for $\mathcal{Z}(\sigma)$ are similar to those yielded by the "sectorial" trace formula for the analogous spectral zeta function $\mathcal{Z}_X(\sigma)$ over a compact hyperbolic surface X [7], [38]. The present formulae for the Riemann case nevertheless show several distinctive features.

– As announced end of Section 2.1, the sequence $\{{\tau_k}^2\}$ and the analogous spectrum of the Laplacian on X have mutually singular features: the former has the parameter values $\mu_0 = \frac{1}{2}$, $r = 2$ ($\mathcal{Z}(\sigma)$ has its leading pole *double*, at $\sigma = \frac{1}{2}$), whereas the latter more precisely has $\mu_0 = 1$, $r = 1$

($\mathcal{Z}_X(\sigma)$ has all its poles *simple*, starting at $\sigma = 1$), hence this spectral analogy for the Riemann zeros holds only partially;

– in the continuation formulae (72–74), $\zeta(s)$ itself *reenters* as an *additive* component of the shadow zeta function $\mathbf{Z}(s)$. This is an altogether different incarnation of $\zeta(s)$ from its initial, *multiplicative* involvement, which remains in the integral term and indirectly through the zeros, in the left-hand side $\mathcal{Z}(\sigma)$. It is curious to find two such copies of $\zeta(s)$ to coexist in one formula, especially with the additive $\zeta(2\sigma)$ represented in its critical strip;

– however, those formulae relative to $\zeta(s)$ are not fully closed as they also invoke the other Dirichlet series $\beta(2\sigma)$ (as second additive component in the shadow zeta function $\mathbf{Z}(2\sigma)$). The question then arises whether $\beta(s)$ and other zeta functions can be handled on the same footing as $\zeta(s)$ (as in [24]), so we now outline a possible extension of equations (72–74).

We assume that $\tilde{\zeta}(s)$ is a Dirichlet zeta or L-series, having

– a single pole, at $s = 1$ and of order q (typically, $q = 0$ or 1);

– the asymptotic property $\log \tilde{\zeta}(s) = o(s^{-N})$ for all N ($s \to +\infty$);

– a functional equation of the form

$$\tilde{\Delta}(t^2)\frac{\tilde{\mathbf{D}}(t)}{(1-2t)^q} = \tilde{\zeta}\Big(\frac{1}{2}+t\Big), \quad \text{where} \tag{84}$$

– $\tilde{\Delta}(t^2)$ is an entire function of order < 1 *in the variable* t^2, and

– $\tilde{\mathbf{D}}(t)$ is an entire function with all of its zeros lying in the half-plane $\{\mathrm{Re}\, t < 0\}$.

Then the Zeta functions $\mathcal{Z}_{\tilde{\zeta}}(\sigma)$ (for the zeros of $\tilde{\Delta}$) and $\mathbf{Z}_{\tilde{\zeta}}(\sigma)$ (for the zeros of $\tilde{\mathbf{D}}$) are related by this formula corresponding to equation (74) (we omit the others),

$$\mathcal{Z}_{\tilde{\zeta}}(\sigma) = -\frac{\mathbf{Z}_{\tilde{\zeta}}(2\sigma)}{2\cos\pi\sigma} + \frac{\sin\pi\sigma}{\pi}\int_0^{+\infty} t^{-2\sigma}\Big[\frac{\tilde{\zeta}'}{\tilde{\zeta}}\Big(\frac{1}{2}+t\Big) + \frac{q}{t-\frac{1}{2}}\Big]\mathrm{d}t. \tag{85}$$

Apart from $\zeta(s)$ itself, with equation (74), the next independent example is $\beta(s)$. Its functional equation (14) has the form (84) with $q = 0$ (no pole) and $\tilde{\Delta}(t^2) = \Xi_{\chi_4}(\frac{1}{2}+t)$, $\tilde{\mathbf{D}}(t) = (\frac{\pi}{4})^{\frac{3}{4}+\frac{t}{2}} / \Gamma(\frac{3}{4}+\frac{t}{2})$; its spectrum of trivial zeros (for $\beta(\frac{1}{2}-t)$) is $\{-\frac{1}{2}+2k\}$ $(= \frac{3}{2}, \frac{7}{2}, \cdots)$, giving as shadow zeta function

$$\mathbf{Z}_\beta(\sigma) = \sum_{k=1}^{\infty}\Big(-\frac{1}{2}+2k\Big)^{-s} \equiv 2^s\Big[\frac{1}{2}\big((1-2^{-s})\zeta(s) - \beta(s)\big)\Big]. \tag{86}$$

Under $q = 0$, all Mellin representations (72–74) coalesce into the single regular form

(87)
$$\mathcal{Z}_\beta(\sigma) = -\frac{\mathbf{Z}_\beta(2\sigma)}{2\cos\pi\sigma} + \frac{\sin\pi\sigma}{\pi}\int_0^{+\infty} t^{-2\sigma}\frac{\beta'}{\beta}\Big(\frac{1}{2}+t\Big)\mathrm{d}t \qquad \Big(\mathrm{Re}\ \sigma < \frac{1}{2}\Big),$$

and all consequences previously drawn for $\mathcal{Z}(\sigma)$ have analogs for $\mathcal{Z}_\beta(\sigma)$.

Various such integral representations will naturally *add*, whenever the initial zeta functions combine nicely under multiplication. For instance, equations (74) and (87) add up to

$$(88)\quad (\mathcal{Z}_\zeta + \mathcal{Z}_\beta)(\sigma) = -\frac{2^{2\sigma}[(1-2^{-2\sigma})\zeta(2\sigma) - 1]}{2\cos\pi\sigma} + \frac{\sin\pi\sigma}{\pi}\int_0^{+\infty} t^{-2\sigma}\Big[\Big(\frac{\zeta'}{\zeta}+\frac{\beta'}{\beta}\Big)\Big(\frac{1}{2}+t\Big) + \frac{1}{t-\frac{1}{2}}\Big]\mathrm{d}t.$$

Here, the shadow zeta function purely invokes $\zeta(s)$; on the other hand, under the integral sign we now find $\big[\frac{(\beta\zeta)'}{\beta\zeta}\big]\big(\frac{1}{2}+t\big)$ so that the new multiplicative zeta function is $\beta(s)\zeta(s)$, also recognized as $\frac{1}{4}$ times $Z_4(s)$, the *zeta function of the ring of Gaussian integers* $\mathbb{Z}[\mathrm{i}]$ [6]. Hence equation (88) becomes

$$(89)\qquad \mathcal{Z}_{Z_4}(\sigma) = -\frac{2^{2\sigma}[(1-2^{-2\sigma})\zeta(2\sigma) - 1]}{2\cos\pi\sigma} + \frac{\sin\pi\sigma}{\pi}\int_0^{+\infty} t^{-2\sigma}\Big[\frac{Z_4{}'}{Z_4}\Big(\frac{1}{2}+t\Big) + \frac{1}{t-\frac{1}{2}}\Big]\mathrm{d}t;$$

thus, to isolate $\zeta(s)$, here in the additive position, we again had to allow a different zeta function elsewhere, this time $Z_4(s)$ in the multiplicative position.

Likewise, by subtracting equation (87) from (74) instead, we could get the shadow zeta function to be $\beta(s)$; then the counterpart of equations (81), (82) is a *fully explicit* identity,

$$(90)\quad (\log|\zeta|)^{(2m+1)}\Big(\frac{1}{2}\Big) - (\log\beta)^{(2m+1)}\Big(\frac{1}{2}\Big) = \frac{1}{2}\pi^{2m+1}|E_{2m}| + \delta_{m,0}\log 2, \quad m = 0, 1, \dots,$$

whereas each of the two left-hand-side terms separately needs $\zeta(2m+1)$ (or γ for $m = 0$).

6. More about the Hurwitz-type functions.

The purpose of this section is twofold. First, we analyze the Zeta functions $\mathscr{Z}(\sigma, v) = \sum_k ({\tau_k}^2 + v)^{-\sigma}$ more explicitly over the half-plane $\{\mathrm{Re}\ \sigma < 1\}$ than in Section 4, by exploiting the latest special properties of the function $\mathscr{Z}(\sigma, 0)$ (with new results even for the case $v = \frac{1}{4}$). Then, by the same approach, we (briefly) discuss the other Hurwitz-type Zeta functions $\mathfrak{Z}(\sigma, a)$ and $\xi(s, x)$, defined through equations (3) and (4) respectively.

6.1. Further properties of $\mathscr{Z}(\sigma, v)$ for $\mathrm{Re}\ \sigma < 1$.

To describe the Hurwitz-type function $\mathscr{Z}(\sigma, v)$ better, we now systematically expand it in terms of $\mathscr{Z}(\sigma)$, as

$$(91) \quad \mathscr{Z}(\sigma, v) = \sum_{k=0}^{\infty} ({\tau_k}^2)^{-\sigma} \Big(1 + \frac{v}{{\tau_k}^2}\Big)^{-\sigma} = \sum_{\ell=0}^{\infty} \frac{\Gamma(1-\sigma)}{\ell!\, \Gamma(1-\sigma-\ell)}\, \mathscr{Z}(\sigma+\ell)\, v^\ell \qquad (|v| < {\tau_1}^2).$$

Such an expansion can be formulated around any reference point v_0, but it will be specially useful for $v_0 = 0$ as above. For instance, coupled with equation (72) (say), it can express the meromorphic continuation of the general $\mathscr{Z}(\sigma, v)$ to $\{\mathrm{Re}\ \sigma < \frac{1}{2}\}$, while we lack an analog of equation (72) itself for any $v \neq 0$.

For the polar structure of $\mathscr{Z}(\sigma, v)$ at $\sigma = -m + \frac{1}{2}$, $m \in \mathbb{N}$, the series (91) reduces to

$$(92) \quad \mathscr{Z}\Big(-m + \frac{1}{2} + \varepsilon, v\Big) = \sum_{\ell=0}^{m} \frac{\Gamma(\frac{1}{2}+m-\varepsilon)}{\ell!\, \Gamma(\frac{1}{2}+m-\ell-\varepsilon)}\, \mathscr{Z}\Big(-m+\ell+\frac{1}{2}+\varepsilon\Big)\, v^\ell \quad [+\text{regular part for } \varepsilon \to 0];$$

then, importing the polar structure of $\mathscr{Z}(\sigma)$ from equation (76), we get

$$(93) \quad \mathscr{Z}\Big(-m + \frac{1}{2} + \varepsilon, v\Big) = \frac{1}{8\pi} \frac{\Gamma(m+\frac{1}{2})}{m!\, \Gamma(\frac{1}{2})} v^m\, \varepsilon^{-2} + \mathcal{R}_m(v)\, \varepsilon^{-1} + O(1) \qquad (\varepsilon \to 0),$$

just by brute-force polar expansion of the right-hand side of equation (92). Here, the polar part of order 2 at every $(-m + \frac{1}{2})$ is clearly induced by the only such part of $\mathcal{Z}(\sigma)$ (at $\sigma = \frac{1}{2}$), through the term with $\ell = m$ in equation (92); whereas the residue $\mathcal{R}_m(v)$ is built from all residues of $\mathcal{Z}(\sigma)$ at poles with $\sigma \geqslant -m + \frac{1}{2}$, as

$$(94) \qquad \mathcal{R}_m(v) = -\frac{\Gamma(m+\frac{1}{2})}{m!\,\Gamma(\frac{1}{2})}\Big[\frac{1}{4\pi}\sum_{j=1}^{m}\frac{1}{2j-1} + \frac{\log 2\pi}{4\pi}\Big]v^m + \sum_{j=1}^{m}\frac{\Gamma(\frac{1}{2}+m)}{(m-j)!\,\Gamma(\frac{1}{2}+j)}\mathcal{R}_j\, v^{m-j},$$

(the residues $\mathcal{R}_j$ of $\mathcal{Z}(\sigma)$ at $-j+\frac{1}{2}$ are known from equation (76)).

Remark. — For $m = 0$, the full polar part (40) at $\sigma = \frac{1}{2}$, independent of v, is recovered.

When $\sigma \in -\mathbb{N}$, the series (91) also terminates, as

$$(95) \qquad \mathcal{Z}(-m, v) \equiv \sum_{\ell=0}^{m}\binom{m}{\ell}\mathcal{Z}(-m+\ell)\, v^\ell \qquad (m \in \mathbb{N}),$$

so that explicit "trace identities" for general v derive from those for $v = 0$ (equation (77)). (For $v = \frac{1}{4}$, this result simplifies further, see Table 1.)

Remark. — In view of equations (23) and (25), the latter two results now imply a general-n formula for the coefficients $a_{(1-n)/2}(v)$ in the large-λ expansion (16) of $\log\Delta(\lambda \mid \{\tau_k{}^2 + v\})$. (Hitherto we had such a formula just at $v = 0$, not even at $v = \frac{1}{4}$, and knew only the *other* coefficients $\tilde{a}_{(1-n)/2}(v)$ for any v and n, by equations (51–52).)

Our initial emphasis on the special case $v = \frac{1}{4}$ might now seem misplaced: why didn't we operate at once from $v = 0$? In the first place, we saw the case $v = \frac{1}{4}$ arise more readily from the standard product representation of $\zeta(s)$. But mainly, the case $v = \frac{1}{4}$ also enjoys certain special properties, this time with the values $Z'(0)$ and $Z(n)$ (Section 3.3), and since these evolve from *transcendental* functions of v (Section 4), their expansions (91) around $v = 0$ are now *infinite*. So, each case $v = 0$ and $v = \frac{1}{4}$ has its own exceptional features, the former in the half-plane $\{\mathrm{Re}\,\sigma < 1\}$, and the latter for $\sigma \in \mathbb{N}$.

6.2. The Hurwitz-type functions $\mathfrak{Z}(\sigma, a)$ and $\xi(s, x)$.

The function $\mathfrak{Z}(\sigma, a)$ as defined by equation (3) is *a priori* more singular than $\mathcal{Z}(\sigma, a)$ (the sequence $\{\tau_k\}$ itself has $r = 2$ and $\mu_0 = 1$, which would require a formalism more elaborate than in Section 2). Fortunately, $\mathfrak{Z}(\sigma, a)$ can also be analyzed directly through its expansion around $\mathfrak{Z}(\sigma, 0) \equiv \mathcal{Z}(\sigma)$, by analogy with equation (91) (see also [41]):

$$(96) \qquad \mathfrak{Z}(\sigma, a) = \sum_{k=0}^{\infty} {\tau_k}^{-2\sigma} \Big(1 + \frac{a}{\tau_k}\Big)^{-2\sigma}$$
$$= \sum_{\ell=0}^{\infty} \frac{\Gamma(1-2\sigma)}{\ell!\,\Gamma(1-2\sigma-\ell)}\, \mathcal{Z}\left(\sigma + \frac{1}{2}\ell\right) a^{\ell} \quad (|a| < \tau_1).$$

This formula generates a pole for $\mathfrak{Z}(\sigma, a)$ now at every *half-integer* $\frac{1}{2}(1-n)$, $n \in \mathbb{N}$, according to

(97)
$$\mathfrak{Z}\Big(\frac{1}{2}(1-n)+\varepsilon, a\Big) = \sum_{\ell=0}^{n} \frac{\Gamma(n-2\varepsilon)}{\ell!\,\Gamma(n-\ell-2\varepsilon)}\, \mathcal{Z}\left(\frac{1}{2}(1-n+\ell)+\varepsilon\right) a^{\ell} \quad + O(\varepsilon).$$

Differences with equation (92) arise due to the factor $\Gamma(n-2\varepsilon)/\,\Gamma(n-\ell-2\varepsilon)$ vanishing whenever $\ell \geqslant n > 0$. Only the polar part at $\sigma = \frac{1}{2}$ remains the same as for $\mathcal{Z}(\sigma, v)$ (of order $r = 2$ and independent of a, given by equation (40)); all other poles $\frac{1}{2}(1-n)$ of $\mathfrak{Z}(\sigma, a)$ are now *simple*, of residues

$$(98) \qquad r_n(a) = -\frac{1}{4\pi n}\, a^n + \sum_{0<2m\leqslant n} \binom{n-1}{2m-1} \mathcal{R}_m\, a^{n-2m}, \quad n = 1, 2, \ldots$$

(again, $\mathcal{R}_m$ is the residue given by equation (76)). In addition, at $\sigma = 0$ the ε-expansion of equation (97) captures the *finite part* too:

$$(99) \qquad r_1(a) = \operatorname{Res}_{\sigma=0} \mathfrak{Z}(\sigma, a) = -\frac{a}{4\pi};$$
$$\text{finite part: } \operatorname{FP}_{\sigma=0} \mathfrak{Z}(\sigma, a) = \frac{7}{8} + \frac{\log 2\pi}{2\pi}\, a.$$

As for the function $\xi(s, x)$, if we express it by equation (6) in terms of $\mathfrak{Z}(\sigma, a)$, then we find this combination to be *less singular* overall: by mere substitution of equation (97), $\xi(s, x)$ shows a *simple* pole at $s = 1$, of residue $-\pi$ [34], and all other possible poles at $s = 1 - n$, $n = 1, 2, \ldots$ *cancel out*, resulting in the holomorphy of $\xi(s, x)$ for all $s \neq 1$ with the computable finite values ("trace identities")

$$(100) \quad \xi\Big(1-n, \frac{1}{2} + y\Big) = \frac{2}{(2\pi)^{n-1}}$$
$$\left[-\pi \frac{r_n(\mathrm{i}y)}{\mathrm{i}^n} + \sum_{0\leqslant 2m<n} (-1)^m \binom{n-1}{2m} \mathcal{Z}(-m) y^{n-2m-1}\right], \quad n = 1, 2, \ldots$$

(An alternative evaluation follows from Deninger's continuation formula for $\xi(s,x)$ ([13], middle of p. 149), as $\xi(1-n,\frac{1}{2}+y)=(2\pi)^{1-n}\big[(y+\frac{1}{2})^{n-1}+(y-\frac{1}{2})^{n-1}+2^{n-1}B_n(\frac{1}{4}+\frac{1}{2}y)\big]$, whose agreement with equation (100) can be verified.)

As for special values: first, $\partial_s\xi(s,x)_{s=0}$ is expressible as well, in terms of $\zeta(x)$ [13], [34]:

$$\text{(101)} \qquad \partial_s\xi(s,x)_{s=0} = \log 2^{1/2}(2\pi)^2 - \log\Xi(x)$$

$$\text{(102)} \iff \quad -\partial_s\Big[\sum_\rho (x-\rho)^{-s}\Big]_{s=0} = \log\Xi(x)+\frac{1}{4}(\log 2\pi)\,x-\frac{1}{4}\log 4\pi$$

(the equivalence of the two forms follows from equations (4) and (100) for $n=1$, i.e., $\xi(0,x)=\frac{1}{2}(x+3)$). Now, the exponentiated left-hand side of equation (102) precisely defines the zeta-regularized product $\tilde\Delta_{\rm zr}(x)$ built upon the sequence $\{\rho\}$ of Riemann zeros, while the right-hand side mainly involves $\Xi(x)$ of equation (28). So, equation (102) is converting a zeta-regularized product ($\tilde\Delta_{\rm zr}(x)$) to Hadamard product form. As an aside, we now verify that such a conversion formula is entirely *fixed by universal rules* for (complex) *admissible sequences*, specialized here to $r=1$ (as in [37], [31]) and $\mu_0=1$ — since the Zeta functions $\xi(s,x)$ have just a simple pole at $s=1$. Those rules yield these two prescriptions: $\log\tilde\Delta_{\rm zr}(x)\equiv\log\Xi(x)-(\alpha x+\beta)$, *and* the large-$x$ expansion of $\log\tilde\Delta_{\rm zr}(x)$ shall only retain *canonical* (or *standard*) terms, namely: $c_\mu x^\mu$ for $1>\mu\notin\mathbb{N}$, $c_1x(\log x-1)$, $c_0\log x$. Those conditions together fix (α,β) uniquely, and here, equations (33–34) for $\log\Xi(x)$ as input precisely lead to equation (102) as output.

Likewise, the special values $\xi(n,x)$, $n=1,2,\dots$ are expressible in terms of [the higher Laurent coefficients of] $\zeta(x)$, e.g., by applying residue calculus to Deninger's continuation formula ([13], p. 149).

Thus, a fair degree of structural parallelism finally shows up between the two Hurwitz-like families $\mathscr{Z}(s,v)$ and $\xi(s,x)$ [40].

7. Recapitulation of main results.

In way of conclusion, Table 1 collates the analytical results found for the two ζ-Zeta functions $Z(\sigma)$ $(=\mathscr{Z}(\sigma,v=\frac{1}{4}))$ and $\mathscr{Z}(\sigma)$ $(=\mathscr{Z}(\sigma,0))$. Furthermore, corresponding results for the general $\mathscr{Z}(\sigma,v)$ were derived in Section 4 (for Re $\sigma\geqslant 0$) and 6.1 (for Re $\sigma<1$), and partly extended to the functions $\mathfrak{Z}(\sigma,a)$ and $\xi(s,x)$ in Section 6.2.

The other novel results we have developed here concern $\mathcal{Z}(\sigma)$ in the half-plane $\{\mathrm{Re}\,\sigma < \frac{1}{2}\}$: the analytical continuation formulae (72–74), and the $\sigma \to -\infty$ asymptotic formula (78) as corollary. We also came across two elementary (but unfamiliar to us) formulae concerning $\zeta(s)$ itself: equations (81), (90).

Appendix B gives information on some numerical aspects based on our use of the 100,000 first Riemann zeros (made freely available on the Web by A.M. Odlyzko [30], to whom we express our gratitude). We also wish to thank C. Deninger, J.P. Keating, P. Lebœuf, V. Maillot, C. Soulé, and the Referee, for helpful references and comments.

σ	$Z(\sigma) = \sum_{k=1}^{\infty} ({\tau_k}^2 + \frac{1}{4})^{-\sigma} \quad [v = \frac{1}{4}]$	$\mathcal{Z}(\sigma) = \sum_{k=1}^{\infty} {\tau_k}^{-2\sigma} \quad [v = 0]$
$-m$	$(-1)^{m+1} 2^{-2m-3} \sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^{\ell} E_{2(m-\ell)}$ (95,77)	$(-1)^m 2^{-2m} (1 - \frac{1}{8} E_{2m})$ (77)
$-m + \frac{1}{2} + \varepsilon$	$[\frac{2^{-2m}}{8\pi} \frac{\Gamma(m+1/2)}{m!\,\Gamma(1/2)}]\, \varepsilon^{-2} + \mathcal{R}_m(\frac{1}{4})\, \varepsilon^{-1} + O(1)$ (93,94)	$[\frac{(-1)^m}{8\pi m} (1 - 2^{1-2m}) B_{2m}]\, \varepsilon^{-1} + O(1)$ (76)
$\vdots$	$\vdots$	$\vdots$
-1	$-1/16$	$-9/32$
$-\frac{1}{2} + \varepsilon$	$\frac{1}{64\pi}\, \varepsilon^{-2} - [\frac{3\log 2\pi + 4}{96\pi}]\, \varepsilon^{-1} + O(1)$	$-\frac{1}{96\pi}\, \varepsilon^{-1} + O(1)$
0	$7/8$ (41)	$7/8$ (75)
derivative at 0	$Z'(0) = \frac{1}{4} \log 8\pi$ (41)	$\mathcal{Z}'(0) = \log\big[2^{11/4} \pi^{1/2}\, \Gamma(\frac{1}{4})^{-1} \lvert\zeta(\frac{1}{2})\rvert^{-1}\big]$ (75)
$+\frac{1}{2} + \varepsilon$	$\frac{1}{8\pi}\, \varepsilon^{-2} - \frac{\log 2\pi}{4\pi}\, \varepsilon^{-1} + O(1)$ (40)	$\frac{1}{8\pi}\, \varepsilon^{-2} - \frac{\log 2\pi}{4\pi}\, \varepsilon^{-1} + O(1)$ (40,51)
$+1$	$-\frac{1}{2} \log 4\pi + 1 + \frac{1}{2}\gamma \equiv \mathscr{Z}_1$ (31)	$\frac{1}{2} (\log\lvert\zeta\rvert)''(\frac{1}{2}) + \frac{1}{8}\pi^2 + \beta(2) - 4$
$+2$	$\begin{Bmatrix} -\log 4\pi + 3 + \gamma - (\log\lvert\zeta\rvert)''(0) - \frac{1}{24}\pi^2 \\ -\log 4\pi + 3 + \gamma + 2\gamma_1 + \gamma^2 - \frac{1}{8}\pi^2 \end{Bmatrix}$	$-\frac{1}{12} (\log\lvert\zeta\rvert)^{(4)}(\frac{1}{2}) - \frac{1}{24}\pi^4 - 4\beta(4) + 16$
$\vdots$	$\vdots$	$\vdots$
m	$\sum_{\ell=0}^{m-1} \binom{m+\ell-1}{m-1} \mathscr{Z}_{m-\ell}$ (48)	$(-1)^m \Big\{ -\frac{1}{2(2m-1)!} (\log\lvert\zeta\rvert)^{(2m)}(\frac{1}{2}) - \frac{1}{4}\big[(2^{2m} - 1)\zeta(2m) + 2^{2m}\beta(2m)\big] + 2^{2m} \Big\}$ (83)

$$\left[\ \mathscr{Z}_n \equiv \begin{Bmatrix} 1 - (-1)^n 2^{-n} \zeta(n) - \dfrac{(\log\lvert\zeta\rvert)^{(n)}(0)}{(n-1)!} \\ 1 - (1 - 2^{-n})\zeta(n) + \dfrac{n}{(n-1)!}\, \gamma^{\mathrm{c}}_{n-1} \end{Bmatrix} (n = 2, 3, \ldots)\right] \quad (46)$$

TABLE 1. – Analytical results for ζ-Zeta functions of the Riemann zeros. Notations: see equations (1), (7), (11) [for $\beta(s)$], (45) [for γ^{c}_n]; $\mathscr{Z}_m \equiv \sum_\rho \rho^{-m}$; m stands for any *positive* integer, and $\varepsilon \to 0$. As an indexing tool, the superscripts refer to the relevant equation numbers in the main text. [In the very last formula (for $\mathcal{Z}(m)$), one can still use equation (8) for $\zeta(2m)$].

Appendix A. Meromorphic Mellin transforms.

We briefly recall the meromorphic continuation argument for a Mellin transform like equation (19), $I(\sigma) \overset{\text{def}}{=} \int_0^{+\infty} L(z) z^{-\sigma-1} \mathrm{d}z$, assuming the function $L(z)$ to be regular on $\mathbb{R}^+$ (for simplicity), with

$$(103) \qquad L(z) = O(z^{\nu_0}) \quad (z \to 0^+),$$

$$\text{and} \quad L(z) \sim \sum_{n=0}^{\infty} (\tilde{a}_{\mu_n} \log z + a_{\mu_n}) z^{\mu_n} \quad (z \to +\infty)$$

as in equation (16) (asymptotic estimates are repeatedly differentiable); and crucially, $\mu_0 < \nu_0$.

Sequential directed integrations by parts can be used (see [21], [37], [6] for details).

Step 1. — $I(\sigma)$ converges for $\mu_0 < \mathrm{Re}\ z < \nu_0$, and in that strip,

$$(104) \qquad I(\sigma) = \int_0^{+\infty} [L(z) z^{-\mu_0}]' \frac{z^{\mu_0 - \sigma}}{\sigma - \mu_0} \mathrm{d}z.$$

If $\tilde{a}_{\mu_0} = 0$, this suffices: the new integral actually converges for $\mu_1 < \mathrm{Re}\ z < \nu_0$ (thanks to $z^{\mu_0+1}[L(z) z^{-\mu_0}]' = O(z^{\mu_1} \log z)$ for $z \to \infty$), hence $I(\sigma)$ is manifestly meromorphic in that wider strip, with a simple pole at $\sigma = \mu_0$ of residue

$$(105) \qquad \int_0^{+\infty} [L(z) z^{-\mu_0}]' \, \mathrm{d}z = a_{\mu_0};$$

furthermore, in the complementary strip $\{\mu_1 < \mathrm{Re}\ z < \mu_0\}$, backward integration by parts now yields

$$(106) \qquad I(\sigma) = \int_0^{+\infty} [L(z) - a_{\mu_0} z^{\mu_0})] z^{-\sigma-1} \mathrm{d}z.$$

Then the whole argument can be restarted from here, to extend $I(\sigma)$ further (across $\{\mathrm{Re}\ \sigma = \mu_1\}$), and so on: the case $r = 1$ thus gets settled.

Step 2. — If $\tilde{a}_{\mu_0} \neq 0$, one more integration by parts upon equation (104) yields

$$(107) \qquad I(\sigma) = \int_0^{+\infty} [z[L(z) z^{-\mu_0}]']' \frac{z^{\mu_0 - \sigma}}{(\sigma - \mu_0)^2} \mathrm{d}z \qquad (\mu_{\hat{1}} < \mathrm{Re}\ z < \nu_0).$$

All previous arguments then carry over, but the pole is now *double*, with

$$(108) \qquad \text{principal polar coefficient:} \quad \int_0^{+\infty} [z[L(z) z^{-\mu_0}]']' \, \mathrm{d}z = \tilde{a}_{\mu_0},$$

and residue $-\int_0^{+\infty}[z[L(z)z^{-\mu_0}]']'\log z\,\mathrm{d}z$ (from the residue calculus); integrations by parts (backwards, and split) reduce the latter to

$$\text{(109)}\quad \text{residue:}\quad \int_0^1 [L(z)z^{-\mu_0}]'\,\mathrm{d}z + \int_1^{+\infty}\{z[L(z)z^{-\mu_0}]' - \tilde{a}_{\mu_0}\}\frac{1}{z}\,\mathrm{d}z = a_{\mu_0}.$$

The last two formulae thus generate equation (20) for the leading double pole, and so on for $r = 2$.

(More generally, if the factor of z^{μ_n} in the expansion (103) is a polynomial of degree p_n in $\log z$, then μ_n becomes a pole of order $(p_n + 1)$ for $I(\sigma)$ [21].)

As for meromorphic continuation in the other direction (across $\{\mathrm{Re}\,\sigma = \nu_0\}$), it works likewise if $L(z)$ admits a $z \to 0$ expansion: the previous arguments apply upon exchanging the bounds 0 and $+\infty$ under $\sigma \mapsto -\sigma$. E.g., in the regular case, $L(z)$ expands in an entire series at $z = 0$, and step 1 suffices (as in the main text, where $\nu_0 = 1$).

Appendix B. Numerical aspects.

We complete our analytical study by describing some *very heuristic* numerical work with $Z(\sigma)$ and $\mathcal{Z}(\sigma)$, mostly in the range $\{\sigma \geqslant 0\}$, and focusing on the simpler case of $\mathcal{Z}(\sigma)$. (The same ideas apply for any generalized Zeta function $\mathcal{Z}(\sigma, v)$ and for complex σ, but the formulae get more involved.) Here, the Riemann Hypothesis is *de facto* implied throughout (there being no numerical counter-example).

Numerically, $Z(\sigma)$ looks almost indistinguishable from $\mathcal{Z}(\sigma)$ for $\sigma \geqslant 0$, because ${\tau_k}^2 \gg \frac{1}{4}$ $(\forall k)$ (an empirical fact; already, ${\tau_1}^2 \approx 199.790455$). If we expand $Z(\sigma)$ in terms of $\mathcal{Z}(\sigma)$ according to equation (91), and make the roughest approximations, we get that $Z(\sigma) \approx \mathcal{Z}(\sigma)[1 - \sigma/(4{\tau_1}^2)]$: i.e., the very first correction term is only of relative size $\approx \sigma/800$. As other related numerical observations:

$$\text{(110)}\qquad 0 < \mathcal{Z}(\sigma) - Z(\sigma) < 0.0003 \text{ for all real } \sigma > 0;$$

$$\text{(111)}\qquad A \stackrel{\text{def}}{=} 4[Z'(0) - \mathcal{Z}'(0)] \approx -0.0231003495 \quad vs \quad B \equiv -Z(1) \approx -0.0230957090$$

(not only is A small, but moreover, $A \equiv 4\log\Xi(\frac{1}{2})$ by equation (75) and $B \equiv [\log\Xi]'(0)$ by equation (31), hence $|B - A| < 5 \times 10^{-6}$ reflects how *little* the function $\log \Xi(s)$ deviates from the parabolic shape $As(1-s)$ over the interval $[0,1]$).

We now focus on the numerical evaluation of $\mathcal{Z}(\sigma)$ itself. The defining series (58) converges more and more poorly as $\sigma \to \frac{1}{2}^+$ (with divergence setting in at $\sigma = \frac{1}{2}$). We then replace a far tail of that series ($k > K$) by an integral according to the integrated-density estimate (29), and formally obtain a kind of Euler–Maclaurin formula,

$$(112)\quad \mathcal{Z}(\sigma) = \lim_{K\to+\infty} S_K(\sigma), \quad S_K(\sigma) \stackrel{\text{def}}{=} \sum_1^{K-1} \tau_k{}^{-2\sigma} + \tfrac{1}{2}\tau_K{}^{-2\sigma} + \overline{R}_K(\sigma),$$

$$\overline{R}_K(\sigma) \stackrel{\text{def}}{=} \int_{\tau_K}^{+\infty} T^{-2\sigma} \mathrm{d}\overline{\mathcal{N}}(T) = \frac{1}{2\pi}\frac{\tau_K{}^{1-2\sigma}}{2\sigma-1}\left[\log\frac{\tau_K}{2\pi} + \frac{1}{2\sigma-1}\right]$$

(similar formulae can be written for $\mathcal{Z}'(\sigma)$, $\mathcal{Z}(\sigma,v)$, finite parts at $\sigma = \frac{1}{2}$, etc.).

σ	$Z(\sigma) = \sum_{k=1}^{\infty} (\tau_k{}^2 + \frac{1}{4})^{-\sigma}$ $[v = \frac{1}{4}]$	$\mathcal{Z}(\sigma) = \sum_{k=1}^{\infty} \tau_k{}^{-2\sigma}$ $[v = 0]$
-1	-0.0625*	-0.28125*
$-3/4$	1.69388	0.54319
$-1/4$	0.800805	0.785321
0	0.875*	0.875*
derivative at 0	*0.8060429*	*0.8118179*
$+1/4$	1.548829	1.549060
finite part at 1/2	*0.251546*	*0.251637*
$+3/4$	0.247730	0.247760
$+1$	0.0230957	0.0231050
$+3/2$	0.0007287	0.0007295
$+2$	0.0000371	0.0000372

TABLE 2. – Numerical values for ζ-Zeta functions of the Riemann zeros (*: exact values). Implied precision is expected to hold, but not guaranteed.

The approximate remainder term $\overline{R}_K(\sigma)$ balances the dominant *trend* of the partial sums $\sum_{k<K} \tau_k{}^{-2\sigma}$. It thereby accelerates the convergence of the partial sums in equation (112) for $\sigma > \frac{1}{2}$, while for $\sigma \leqslant \frac{1}{2}$ it counters their divergent trend, so that $S_K(\sigma)$ converges (as $K \to +\infty$) when $\sigma > 0$ ([17], p.116, last line). The next obstruction to convergence arises at $\sigma = 0$

but is of another type: $S_K(\sigma)$ displays *erratic fluctuations* in K (*roughly* of the order ${\tau_K}^{-2\sigma}(\log\log\tau_K)^{1/2}$, according to [30], equation (2.5.7)), and those numerically blow up indeed (as $K \to +\infty$) when $\sigma \leqslant 0$. Further convergence now requires to perform a damping of those fluctuations (as argued previously for "chaotic" spectra [3]). Here, a *Cesaro averaging* (defined by $\langle S\rangle_K \stackrel{\text{def}}{=} K^{-1}\sum_1^K S_{K'}$) *appears* to work well initially (results can be verified at $\sigma = 0$), but not very far down: already at $\sigma = -0.25$, the fluctuations of $\langle S\rangle_K(\sigma)$ itself retain a standard deviation $> 10^{-3}$ up to $K \approx 10^5$. So, instead of pursuing ever more severe (and unproven, after all) numerical regularizations as σ decreases below $\frac{1}{2}$, we advocate the switch to the continuation formulae (72–73) for numerical work as well. Thus, we first tested equation (73) against equation (112) for $\mathscr{Z}(+\frac{1}{4})$, then used it to evaluate $\mathscr{Z}(-\frac{1}{4})$, plus equation (91) with $v = \frac{1}{4}$ (3 terms sufficed) to obtain $Z(-\frac{1}{4})$.

Table 2 gives a summary of the numerical results we obtained. (We found no earlier analogs, except for the *other* special sums $\mathscr{Z}_n$ in [29], [26].)

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