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Kayo MASUDA & Masayoshi MIYANISHI

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THE ADDITIVE GROUP ACTIONS ON $\mathbb{Q}$-HOMOLOGY PLANES

by K. MASUDA* and M. MIYANISHI†

Introduction.

A $\mathbb{Q}$-homology plane is, by definition, a smooth algebraic surface X defined over the complex field $\mathbb{C}$ such that $H_i(X;\mathbb{Q}) = (0)$ for every $i > 0$ [12]. It is known that X is affine and rational [7]. If there is a nontrivial action of the additive group scheme G_a on X, the orbits will form the fibers of an $\mathbb{A}^1$-fibration $\rho : X \to \mathbb{A}^1$. Hence X has log Kodaira dimension $\overline{\kappa}(X) = -\infty$. Write $R = \Gamma(X, \mathcal{O}_X)$. Then there is a well-known bijective correspondence between the set of G_a-actions on X and the set of locally nilpotent derivations on R (cf. [10]). The correspondence is given by assigning to a locally nilpotent derivation δ on R an algebra homomorphism $\varphi : R \to R \otimes_{\mathbb{C}} \mathbb{C}[t]$ giving rise to the coaction

$$\varphi(a) = \sum_{n=0}^{\infty} \frac{1}{n!} \delta^n(a) t^n.$$

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The set of invariant elements of R under the given G_a-action is obtained as $\operatorname{Ker}\delta$ consisting of elements annihilated by δ. Then $\operatorname{Ker}\delta$ is isomorphic to a polynomial ring in one variable and the base curve of the $\mathbb{A}^1$-fibration which is isomorphic to $\mathbb{A}^1$ is obtained as the spectrum of $\operatorname{Ker}\delta$ (cf. [10]).

The Makar-Limanov invariant $\mathrm{ML}(X)$ for X is then introduced by Kaliman and Makar-Limanov [8] as the set $\bigcap_\delta \operatorname{Ker}\delta$, where δ ranges over all possible locally nilpotent derivations of R. Then it is shown that $\mathrm{ML}(X)$ for a $\mathbb{Q}$-homology plane X is the coordinate ring R, a polynomial ring in one variable $\mathbb{C}[x]$ or $\mathbb{C}$. We are particularly interested in such $\mathbb{Q}$-homology planes X that the Makar-Limanov invariant $\mathrm{ML}(X)$ is equal to $\mathbb{C}$. We shall consider two algebraically independent G_a-actions σ, σ' and define the intertwining number $\iota(\sigma, \sigma')$ associated with these G_a-actions. It is then shown that the intertwining number is actually a multiple of m^2, where $m = |H_1(X;\mathbb{Z})|$. We define a *minimal* pair $\{\sigma, \sigma'\}$ of algebraically independent G_a-actions as such with $\iota(\sigma, \sigma') = m^2$.

Recently, Bandman and Makar-Limanov [1] considered a problem of characterizing in terms of the boundary divisors the smooth affine rational surfaces with trivial Makar-Limanov invariants. They succeeded in obtaining a characterization in the case where the surfaces are embedded into $\mathbb{A}^3$ as hypersurfaces. Furthermore, the hypersurfaces are defined by the equations of the form $xy = p(z)$ with respect to a suitable system of coordinates $\{x, y, z\}$, where $p(z)$ is a polynomial in z such that $p(z) = 0$ has distinct roots.

In the present article, we shall show that a $\mathbb{Q}$-homology plane with trivial Makar-Limanov invariant has a Bandman-Makar-Limanov hypersurface as the universal covering (Theorem 3.1). More precisely, if X is a $\mathbb{Q}$-homology plane with trivial Makar-Limanov invariant and with $m = |H_1(X;\mathbb{Z})|$ then X is a quotient of the hypersurface $xy = z^m - 1$ under a suitable, free $\mathbb{Z}/m\mathbb{Z}$-action (Theorem 3.4). The possibilities of the existence of non-minimal pairs of G_a-actions on $\mathbb{Q}$-homology planes are also observed (cf. Section 4). The final section 5 deals with étale endomorphisms of $\mathbb{Q}$-homology planes.

1. Intertwining number.

Let X be a smooth affine surface defined over the ground field k, which we assume mostly to be the complex field $\mathbb{C}$. We *assume* always that X is rational and $\Gamma(X, \mathcal{O}_X)^* = k^*$. The *Makar-Limanov invariant* $\mathrm{ML}\,(X)$ is defined as the intersection

$$\mathrm{ML}\,(X) = \bigcap_{\delta} \mathrm{Ker}\,\delta,$$

where δ runs over all locally nilpotent derivations δ on the coordinate ring $R = \Gamma(X, \mathcal{O}_X)$, where δ corresponds in a bijective way to an algebraic G_a-action σ on X. Then it is known that $\mathrm{Ker}\,\delta = k[t]$ a polynomial ring in one variable for any locally nilpotent derivation δ.

We begin with the following result.

LEMMA 1.1. — *We have one of the following three cases:*

(1) $\mathrm{ML}\,(X) = R$ *and there are no nontrivial G_a-actions on X. In particular,* $\overline{\kappa}(X) \geqslant 0$ *provided* $\mathrm{Pic}\,(X) \otimes \mathbb{Q} = (0)$.

(2) $\mathrm{ML}\,(X) = k[t]$, *and any two locally nilpotent derivations δ, δ' on R are conjugate to each other in the sense that $a\delta = a'\delta'$ for nonzero elements $a, a' \in \mathrm{ML}\,(X)$. The surface X has a unique $\mathbb{A}^1$-fibration defined by the inclusion* $\mathrm{ML}\,(X) \hookrightarrow R$.

(3) $\mathrm{ML}\,(X) = k$, *and there are two non-conjugate locally nilpotent derivations on R.*

Proof. — Our proof consists of several steps.

(I) Note that there exists an $\mathbb{A}^1$-fibration on X with the affine line as the base curve if and only if there exists an algebraic G_a-action on X. In fact, if there exists a nontrivial G_a-action σ, let δ be the corresponding locally nilpotent derivation. Let $R_0 = \mathrm{Ker}\,\delta$. Then R_0 is a normal rational algebra of dimension one with $R_0^* = k^*$. Hence $R_0 = k[t]$. The G_a-action σ gives rise to an $\mathbb{A}^1$-fibration with the base curve $\mathrm{Spec}\,R_0$. In particular, $\overline{\kappa}(X) = -\infty$. Conversely, if X has an $\mathbb{A}^1$-fibration $\rho : X \to B \cong \mathbb{A}^1$, write $B = \mathrm{Spec}\,k[t]$ and $X = \mathrm{Spec}\,R$. Then there exists an element $a \in k[t]$ such that $\rho^{-1}(U) \cong U \times \mathbb{A}^1$, where $U = \mathrm{Spec}\,k[t, a^{-1}]$. Hence $R[a^{-1}] = k[t, a^{-1}][\xi]$, where we can take ξ to be an element of R. Consider a derivation $\delta = a^N \frac{\partial}{\partial \xi}$ with $N > 0$. This is a locally nilpotent derivation on $k[t, a^{-1}][\xi]$. Since R is finitely generated over k, it follows that $\delta(R) \subseteq R$

if $N \gg 0$. Then δ defines a G_a-action σ and the associated $\mathbb{A}^1$-fibration consisting of σ-orbits is the given $\mathbb{A}^1$-fibration ρ. We note that any $\mathbb{A}^1$-fibration $\rho : X \to B$ has the base curve B isomorphic to $\mathbb{A}^1$ provided $\mathrm{Pic}\,(X) \otimes \mathbb{Q} = (0)$. In fact, B is isomorphic to $\mathbb{A}^1$ or $\mathbb{P}^1$ because X is rational. If $B \cong \mathbb{P}^1$, then $\mathrm{Pic}\,(X) \otimes \mathbb{Q} \neq (0)$. So, $B \cong \mathbb{A}^1$. Hence, if $\overline{\kappa}(X) = -\infty$, then there is an $\mathbb{A}^1$-fibration on X with the affine line as the base curve. Here we note that when we speak of an $\mathbb{A}^1$-fibration $\rho : X \to B$ it means that general fibers are isomorphic to the affine lines, while singular fibers may not be irreducible or reduced.

(II) Suppose that δ and δ' are locally nilpotent derivations on R. Then $\mathrm{Ker}\,\delta = k[t]$ and $\mathrm{Ker}\,\delta' = k[u]$. If t and u are algebraically independent over k, we have $k[t] \cap k[u] = k$. In this case, we say that δ and δ' (or the corresponding G_a-actions σ and σ') are *algebraically independent* over k. Then $\mathrm{ML}\,(X) = k$.

(III) Suppose that u is algebraic over $k(t)$. Then there exists an algebraic equation

$$a_0(t)u^n + a_1(t)u^{n-1} + \cdots + a_{n-1}(t)u + a_n(t) = 0, \tag{1}$$

where $a_i(t) \in k[t]$, and we may assume that (1) is minimal. Since $\mathrm{Ker}\,\delta = k[t]$, we have

$$\left\{na_0(t)u^{n-1} + (n-1)a_1(t)u^{n-2} + \cdots + a_{n-1}(t)\right\}\delta(u) = 0. \tag{2}$$

Since (1) is minimal, $na_0(t)u^{n-1} + \cdots + a_{n-1}(t) \neq 0$. This implies that $\delta(u) = 0$. Hence $k[u] \subseteq k[t]$, and t is then algebraic over $k(u)$. By the same reasoning as above, we infer that $k[t] \subseteq k[u]$. So, $k[t] = k[u]$. The $\mathbb{A}^1$-fibrations associated with σ and σ' coincide with the morphism $X \to \mathbb{A}^1$ defined by the inclusion $k[t] = k[u] \hookrightarrow R$. By (I) above, $R[a^{-1}] = k[t, a^{-1}][\xi] = k[u, a^{-1}][\xi]$ for $a \in k[t]$ and an element $\xi \in R$ which is algebraically independent over $k(t)$. Then $a_1\delta = b_1\frac{\partial}{\partial\xi}$ and $a_2\delta' = b_2\frac{\partial}{\partial\xi}$ for $a_1, a_2, b_1, b_2 \in k[t]$. By adjusting the coefficients, we have $a\delta = a'\delta'$ for some nonzero elements $a, a' \in k[t]$. Namely, δ and δ' are conjugate to each other. These observations yield the assertions (2) and (3). □

We consider the case where $\mathrm{ML}\,(X) = k$. In this case, there are two G_a-actions σ, σ' which are algebraically independent over k. We have the following result.

LEMMA 1.2. — *Let σ, σ' be algebraically independent G_a-actions as above. Let $\rho : X \to B$ and $\rho' : X \to B'$ be the $\mathbb{A}^1$-fibrations associated* *with σ and σ', respectively. Let T and T' be arbitrary fibers of ρ and ρ', respectively. Define the intersection number $(T \cdot T')$ by*

$$(T \cdot T') = \sum_{Q \in T \cap T'} i(T, T'; Q),$$

where $i(T, T'; Q)$ is the local intersection multiplicity. Then $(T \cdot T')$ is independent of the choice of T and T', and the intersection of T and T' are transverse and normal at each point $Q \in T \cap T'$ provided T and T' are general fibers of ρ and ρ'.

Proof. — There exists a smooth compactification V of X such that the $\mathbb{A}^1$-fibrations ρ and ρ' extend to the $\mathbb{P}^1$-fibrations $p : V \to \overline{B}$ and $p' : V \to \overline{B}'$. Since B and B' are isomorphic to $\mathbb{A}^1$, it follows that $\overline{B}$ and $\overline{B}'$ are isomorphic to $\mathbb{P}^1$. Consider the $\mathbb{A}^1$-fibration ρ. Let $\{P_\infty\} = \overline{B} - B$ and let $F_\infty = p^*(P_\infty)$. Let T_1, T_2 be fibers of ρ and let T' be an irreducible curve on X such that $T' \cong \mathbb{A}^1$ and $\rho \mid_{T'}: T' \to B$ is dominant. Let $\overline{T'}$ be the closure of T' on V. Then $\overline{T'}$ meets the fiber F_∞ in one point which is a one-place point. Except for this point, $\overline{T'}$ does not meet the boundary components $V - X$ because $T' \cong \mathbb{A}^1$. This implies that $(p^{-1}(\rho(T_1)) \cdot \overline{T'}) = \sum_{Q \in T_1 \cap T'} i(T_1, T'; Q)$ and $(p^{-1}(\rho(T_2)) \cdot \overline{T'}) = \sum_{Q \in T_2 \cap T'} i(T_2, T'; Q)$, which we set $(T_1 \cdot T')$ and $(T_2 \cdot T')$, respectively. Since $(p^{-1}(\rho(T_1)) \cdot \overline{T'}) = (p^{-1}(\rho(T_2)) \cdot \overline{T'})$, we have $(T_1 \cdot T') = (T_2 \cdot T')$. Hence $(T_1 \cdot T')$ is independent of the choice of T_1. Note that any fiber of ρ is of the form $\sum_i m_i C_i$, where $C_i \cong \mathbb{A}^1$. Take T_1 to be a general fiber and let $T_2 = \sum_i m_i C_i$. Let T_1', T_2' be fibers of ρ', where T_1' is a general fiber and $T_2' = \sum_j n_j D_j$ with $D_j \cong \mathbb{A}^1$. Then we have

$$\begin{aligned}(T_1 \cdot T_1') &= \left(\sum_i m_i C_i \cdot T_1'\right) = \sum_i m_i (C_i \cdot T_1') \\ &= \sum_i m_i (C_i \cdot T_2') \quad = \sum_{i,j} m_i n_j (C_i \cdot D_j) = (T_2 \cdot T_2').\end{aligned}$$

Let T and T' be the general fibers of ρ and ρ', respectively and let $\overline{T}$ and $\overline{T}'$ be the closures of T and T'. Consider the restriction $p_{\overline{T}'} : \overline{T}' \to \overline{B}$ of p. Since $\overline{T}'$ has only one place outside of X, which must dominate the point of the fiber F_∞ of p, the restriction $\rho_{T'} : T' \to B$ is a finite morphism. Then $\rho_{T'}$ is unramified over an open set W of B. This means that the intersection of T' and a fiber $\rho^{-1}(Q)$ with $Q \in W$ is transversal and consists of the same number of points. □

We call the above intersection number $(T{\cdot}T')$ the *intertwining number* of σ and σ', and denote it by $\iota(\sigma,\sigma')$. Choose a general point $P \in X$ and let T (resp. T') be the σ-orbit (resp. σ'-orbit) passing through P. Define a morphism $\Phi_P : \mathbb{A}^2 \to X$ by $\Phi_P(g,g') = \sigma(g)\sigma'(g')P$, where $(g,g') \in \mathbb{A}^2 \cong G_a \times G_a$. Then we have the following result.

LEMMA 1.3. — *The morphism Φ_P has degree $\iota(\sigma,\sigma')$.*

Proof. — For $(g,g') = (0,0)$, we have $\Phi_P(0,0) = P$. With the above notations, any point of $T \cap T'$ is written as $\sigma(g_i)(P) = \sigma'(g_i')(P), 1 \leqslant i \leqslant n$, where $n = |T \cap T'| = \iota(\sigma,\sigma')$. Conversely, $\Phi_P^{-1}(P)$ consists of the (g,g') such that $\sigma(g)\sigma'(g')P = P$, i.e., $\sigma(g^{-1})P = \sigma'(g')P$.

Let Q be a general point of X. Then $\Phi_P^{-1}(Q)$ consists of the $(g,g') \in \mathbb{A}^2$ such that $\sigma(g)\sigma'(g')P = Q$, i.e., $\sigma(g^{-1})Q = \sigma'(g')P$. Suppose $\sigma(g_1)\sigma'(g_1')P = \sigma(g)\sigma'(g')P$. Then we have

$$\sigma'(g_1')P = \sigma(g_1^{-1}g)\sigma'(g')P \in \sigma(G_a)(\sigma'(g')P) \cap \sigma'(G_a)P.$$

This implies that $\Phi_P^{-1}(Q)$ corresponds bijectively to the set of intersection points of the σ-orbit $\sigma(G_a)(\sigma'(g')P)$ and the σ'-orbit $\sigma'(G_a)P$. So, $\Phi_P^{-1}(Q)$ consists of $\iota(\sigma,\sigma')$ points. □

As an immediate consequence of Lemma 1.3, we have:

COROLLARY 1.4. — *With the notations and assumptions, $\pi_1(X)$ is a finite group of order less than or equal to $\iota(\sigma,\sigma')$.*

Let σ,σ' be algebraically independent G_a-actions on X and let δ,δ' be the corresponding locally nilpotent derivations on R. We can interpret the intertwining number $\iota(\sigma,\sigma')$ in terms of δ,δ'. Write $\operatorname{Ker}\delta = k[t]$ and $\operatorname{Ker}\delta' = k[t']$ for two elements t,t' of R which are algebraically independent over k. Then we have:

LEMMA 1.5. — *With the notations as above, the following equalities hold:*

$$\begin{aligned}\iota(\sigma,\sigma') &= \min\{n \mid \delta^n(t') = 0\} - 1\\ &= \min\{n \mid \delta'^n(t) = 0\} - 1.\end{aligned}$$

Proof. — By [10], there exist $a \in \operatorname{Ker}\delta$ and $\xi \in R$ such that $R[a^{-1}] = k[t,a^{-1}][\xi]$. Then t' is written as

$$t' = c_0\xi^N + c_1\xi^{N-1} + \cdots + c_N,$$

where $c_i \in k[t, a^{-1}]$ and $c_0 \neq 0$. We may assume, after replacing t' by $t' + \lambda$ with $\lambda \in k$, that $t' = 0$ defines a general σ'-orbit T'. Similarly, we can take $\mu \in k$ so that $c_i(\mu)$ is defined for $0 \leqslant i \leqslant N$, $c_0(\mu) \neq 0$ and the curve $t = \mu$ is a general σ-orbit T. Then the intersection number $(T \cdot T')$ is equal to the number of roots of the equation

$$c_0(\mu)\xi^N + c_1(\mu)\xi^{N-1} + \cdots + c_N(\mu) = 0,$$

where each root is counted with multiplicity. Namely $(T \cdot T') = N$. On the other hand, since δ is equivalent to the derivation $\partial/\partial\xi$, it follows that $N = \min\{n \mid \delta^n(t') = 0\} - 1$. So, we have the assertion. □

2. $\mathbb{Q}$-homology planes and the Makar-Limanov invariants.

In this section, X denotes a $\mathbb{Q}$-homology plane, that is, a smooth algebraic surface defined over the complex field such that $H_i(X; \mathbb{Q}) = (0)$ for every $i > 0$. In particular, X is affine and rational [7]. Furthermore, $\pi_1(X) \cong H_1(X; \mathbb{Z}) \cong \mathrm{Pic}\,(X)$. We consider the existence of G_a-actions on X and the structure of X when X has enough G_a-actions.

We recall the following result [12, Th.1.2].

LEMMA 2.1.— *Let X be a $\mathbb{Q}$-homology plane with an $\mathbb{A}^1$-fibration $\rho : X \to B$. Then every fiber $\rho^{-1}(P)$ is irreducible and $\rho^{-1}(P)_{\mathrm{red}}$ is isomorphic to $\mathbb{A}^1$. Let $m_1A_1, \ldots, m_nA_n$ exhaust all multiple fibers with $A_i \cong \mathbb{A}^1$. Then $H_1(X; \mathbb{Z}) \cong \prod_{i=1}^n \mathbb{Z}/m_i\mathbb{Z}$.*

With the hypothesis of Lemma 2.1, X is isomorphic to the affine plane $\mathbb{A}^2$ if $H_1(X; \mathbb{Z}) = 0$. Since we are interested in $\mathbb{Q}$-homology planes which are not isomorphic to $\mathbb{A}^2$, we assume in the subsequent arguments that $H_1(X; \mathbb{Z}) \neq 0$.

If ρ has a unique multiple fiber mA, then the universal covering Y of X is constructed as follows. Let $P = \rho(A)$ and let $C \to B(\cong \mathbb{A}^1)$ be a finite covering of degree m totally ramifying over P and the point at infinity P_∞. Let Y be the normalization of $X \times_B C$ and let $\pi : Y \to X$ be a composite of the normalization morphism $\nu : Y \to X \times_B C$ and the first projection $X \times_B C \to X$. Then π is a Galois covering with Galois group $\mathbb{Z}/m\mathbb{Z}$, and $\pi^*(A) = L_1 + \cdots + L_m$. Furthermore, Y has an $\mathbb{A}^1$-fibration $\widetilde{\rho} : Y \to C$ which is a composite of ν and the second projection $X \times_B C \to C$, and $\widetilde{\rho}^*(Q) = L_1 + \cdots + L_m$, where Q is a unique point of C lying over P. Since the other fibers of $\widetilde{\rho}$ are reduced and irreducible, an open set $Y - \bigcup_{i\neq 1} L_i$ is isomorphic to $\mathbb{A}^2$. Hence Y is simply connected. So, $\pi : Y \to X$ is a universal covering of X.

We need the following result.

LEMMA 2.2.— *Let $X = \operatorname{Spec} R$ be an affine variety defined over k and let $f : Y \to X$ be an étale finite morphism. Suppose that there exists a G_a-action σ on X. Then σ lifts up uniquely to a G_a-action $\widetilde{\sigma}$ on the variety Y.*

Proof.— Let δ be the locally nilpotent derivation associated with σ. Let $R_0 = \operatorname{Ker}\delta$. Then $R[a^{-1}] = R_0[a^{-1}][\xi]$ for some element $a \in R_0$, and δ is conjugate to $\partial/\partial\xi$, i.e., $a_0\delta = a_1 \frac{\partial}{\partial\xi}$ for nonzero elements $a_0, a_1 \in R_0$. Let $S = \Gamma(Y, \mathcal{O}_Y)$. Then the derivation δ extends *uniquely* to a derivation $\widetilde{\delta}$ on S because $\operatorname{Der}_k(S,S) \cong \operatorname{Der}_k(R,R) \otimes_R S$, which follows from the hypothesis that S is finite and étale over R. On the other hand, δ extends uniquely to a derivation δ on the function field $Q(R)$ and to a derivation on $Q(S)$ which must coincide with the extension of $\widetilde{\delta}$ on $Q(S)$. Since $f : Y \to X$ is étale and finite and since $D(a) \cong \operatorname{Spec} R_0[a^{-1}] \times \mathbb{A}^1$, it follows that $f^{-1}(D(a)) \cong \operatorname{Spec} S_0 \times \mathbb{A}^1$, where $f\,|_{f^{-1}(D(a))}$ is induced by an étale finite morphism $f_0 : \operatorname{Spec} S_0 \to \operatorname{Spec} R_0[a^{-1}]$ via the fiber product $f = f_0 \times \mathbb{A}^1$. Hence $S[a^{-1}] = S_0[\xi]$. Then the derivation $\widehat{\delta} = \frac{a_1}{a_0}\frac{\partial}{\partial\xi}$ is a derivation on $Q(S)$ which is zero on $Q(S_0)$. Since $\widehat{\delta}$ is clearly an extension of δ on $Q(S)$, the uniqueness of the extension implies that $\widehat{\delta} = \widetilde{\delta}$. In particular, $\widehat{\delta}$ is zero on S_0. This implies that $\widehat{\delta}$ is a locally nilpotent derivation on S, and $\widetilde{\delta}$ defines a G_a-action $\widetilde{\sigma}$ on Y which extends σ on X. □

The existence of two algebraically independent G_a-actions on a $\mathbb{Q}$-homology plane gives a strong restriction on the structure of X. Namely we have:

LEMMA 2.3.— *Let X be a $\mathbb{Q}$-homology plane with algebraically independent G_a-actions σ, σ'. Then each of the $\mathbb{A}^1$-fibrations $\rho : X \to B$ and $\rho' : X \to B'$ associated respectively with σ and σ' has a unique multiple fiber of multiplicity m, where $m = |H_1(X;\mathbb{Z})|$. Furthermore, $\iota(\sigma,\sigma')$ is a multiple of m^2.*

Proof.— Consider the $\mathbb{A}^1$-fibration $\rho : X \to B$. Let $m_1A_1, \ldots, m_nA_n$ exhaust all multiple fibers of ρ. Then there is a Galois covering $\pi : C \to \overline{B}$ which ramifies over the points $P_1 = \rho(A_1), \ldots, P_n = \rho(A_n)$ and P_∞ with respective multiplicities $m_1, \ldots, m_n$ and m_∞, where $\overline{B}$ is the smooth compactification of B and $\{P_\infty\} = \overline{B} - B$. By [3] and [5], such a covering exists for a suitable choice of $m_\infty > 1$ provided $n \geqslant 1$. The genus g of C is computed by the Riemann-Hurwitz formula

$$\begin{aligned} 2g - 2 &= -2d + \sum_{i=1}^{n} \frac{d}{m_i}(m_i - 1) + \frac{d}{m_\infty}(m_\infty - 1) \\ &= d\left\{(n-1) - \left(\frac{1}{m_1} + \cdots + \frac{1}{m_n} + \frac{1}{m_\infty}\right)\right\}, \end{aligned}$$

where d is the degree of the morphism π. Hence $g \geqslant 1$ if and only if

$$n - 1 \geqslant \frac{1}{m_1} + \cdots + \frac{1}{m_n} + \frac{1}{m_\infty}.$$

Since $m_i \geqslant 2$ $(1 \leqslant i \leqslant n)$ and $m_\infty \geqslant 2$, it follows that $g = 0$ only if $n - 1 < (n+1)/2$, i.e., $n \leqslant 2$. If $n = 2$, then $g = 0$ only if

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_\infty} > 1.$$

If $n = 1$, then $g = 0$ always. The above observation implies that we can choose $\{m_1, \ldots, m_n, m_\infty\}$ to make the genus $g > 0$ unless one of the following cases takes place:

(1) $n = 1$

(2) $\{m_1, m_2\} = \{2, 2\}$.

Suppose we can take C to have genus $g \geqslant 1$. Let $C_0 = C - \pi^{-1}(P_\infty)$. Let Y be the normalization of the fiber product $X \times_B C_0$ and let $f : Y \to X$ be the composite of the normalization morphism and the projection $X \times_B C_0 \to X$. Then f is a finite étale morphism. Hence the $\mathbb{A}^1$-fibration ρ lifts up to the $\mathbb{A}^1$-fibration $\widetilde{\rho} : Y \to C_0$. Let T' be a general orbit of the G_a-action σ'. Then $f^{-1}(T')$ splits into a disjoint union of the affine lines $\widetilde{T}'_1, \ldots, \widetilde{T}'_d$, where $d = \deg \pi$. Since T' is transversal to ρ, each of $\widetilde{T}'_1, \ldots, \widetilde{T}'_d$ is transversal to the $\mathbb{A}^1$-fibration $\widetilde{\rho}$. Then $\widetilde{\rho} : \widetilde{T}'_j \to C_0$ is dominant. Since the genus of C is positive by the assumption, this is a contradiction.

In the case (2) above, we have $H_1(X; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. By Lemma 2.1, the $\mathbb{A}^1$-fibration ρ' then has also two multiple fibers of multiplicity two. Let $2A_1, 2A_2$ be the multiple fibers of ρ and let $2A'_1, 2A'_2$ be the multiple fibers of ρ'. Since $\iota(\sigma,\sigma') = (2A_1, 2A'_1) = 4(A_1, A'_1)$, write $\iota(\sigma,\sigma') = 4d$. Consider the restriction $\rho'_1 : A'_1 \to B$ of ρ onto A'_1. Since A'_1 has only one place point lying over the point $P_\infty := \overline{B} - B$, the Riemann-Hurwitz formula applied to ρ'_1, which has degree $2d$, yields

$$\begin{aligned}-2 &= -4d + (2d-1) + \{\text{contributions from ramifying points over } B\}\\ &\geqslant -4d + (2d-1) + d + d,\end{aligned}$$

which is a contradiction, where we obtain the above inequality by counting the ramifications at the intersection points of A'_1 with A_1 and A_2. This implies that the case (2) does not occur.

In the case (1), let mA_1 (resp. mA'_1) be a unique multiple fiber of ρ (resp. ρ'), where $m = m_1$. Then $\iota(\sigma,\sigma') = (mA_1, mA'_1) = m^2(A_1, A'_1)$. Hence $\iota(\sigma,\sigma')$ is a multiple of m^2. □

Let X be a $\mathbb{Q}$-homology plane with two algebraically independent G_a-actions σ, σ'. *Suppose that* $|H_1(X;\mathbb{Z})| = m > 1$. Embed X into a smooth projective surface V in such a way that the following conditions are satisfied:

(1) There exists a $\mathbb{P}^1$-fibration $p : V \to \overline{B}$ which restricts to the $\mathbb{A}^1$-fibration $\rho : X \to B$ associated with σ, where $\overline{B}$ is isomorphic to $\mathbb{P}^1$.

(2) The boundary divisor $D := V - X$ is a divisor with simple normal crossings.

(3) The divisor D is written as $D = F_\infty + S + G$, where F_∞ is a smooth fiber of p lying over the point $P_\infty = \overline{B} - B$, S is a cross-section of p and G together with the closure $\overline{A}_0$ of a unique multiple fiber mA_0 of ρ supports a fiber of p lying over the point $P_0 := \rho(A_0)$.

(4) The connected component G contains no (-1) components.

We consider the linear pencil Λ' on V generated by the closures of σ'-orbits. Then we have the following result.

LEMMA 2.4. — *We may furthermore assume that the following conditions are satisfied:*

(5) *Λ' has a unique base point Q on F_∞, which is different from the point $Q_0 = S \cap F_\infty$.*

(6) $(S^2) = -1$.

Proof. — Let $\overline{T}'$ be the closure of a general σ'-orbit T'. If $\overline{T}' \cap F_\infty = \emptyset$, then the $\mathbb{A}^1$-fibrations ρ, ρ' associated respectively with σ, σ' coincide with each other, which is impossible. Thence it follows that $\overline{T}' \cap F_\infty \neq \emptyset$. Suppose that Λ' has no base points. Since $\overline{T}'$ has a single one-place point on F_∞, this implies that F_∞ is a cross-section of Λ'. This implies that $\iota(\sigma, \sigma') = 1$, which is impossible because $\iota(\sigma, \sigma')$ is a multiple of m^2 by Lemma 2.3 and $m > 1$ by the hypothesis. So, Λ' has a unique one-place base point Q on F_∞. Suppose that $Q = Q_0$. Then blow up the point Q_0 to obtain an exceptional (-1) curve E and the proper transform E' of F_∞ with $(E'^2) = -1$. Then contract E' to obtain a smooth projective surface V'. We call this process of obtaining V' from V the *elementary transformation* with center Q_0. By this process we have a new compactification $X \hookrightarrow V'$ which satisfies the same conditions (1) $\sim$ (4) as above. By applying the elementary transformations with center Q_0 several times, the proper transform of Λ' will have no base points on the proper transform of S. We may assume that this situation is already realized on the surface V at the beginning.

Then the components of $S + G$ are contained in one and the same member M_0 of Λ'. Since these components are untouched until the base points of Λ' are eliminated, it follows that $(S^2) \leqslant -1$. Suppose that $(S^2) \leqslant -2$. Let μ be the multiplicity of $\overline{T}'$ at the point Q. Let $\iota(\sigma, \sigma') = m^2 d$. Suppose $\mu = m^2 d$. Blow up the point Q. Let E be the exceptional curve and let F'_∞ be the proper transform of F_∞. Then E is a component of the member M'_0 of the proper transform of Λ' corresponding to M_0. Otherwise, E is a cross-section and $m^2 d = \mu = 1$, which is impossible. By contracting F'_∞, we obtain a new compactification of X with the same property but with (S^2) increased by 1. Hence we may assume that $m^2 d > \mu$. Then $(S^2) = -1$. For otherwise, the member M_0 of Λ' containing $S + G$ will have no (-1) components when the base points of Λ' are eliminated and the last (-1) curve arising from the elimination process gives rise to a cross-section. This is impossible. □

Lemma 2.4 has the following consequence (cf. [11]).

THEOREM 2.5. — *With the notations as in Lemma 2.4, the dual graph of G is a linear chain. In particular, if C is a projective plane curve defined by an equation $X_0 X_1^{m-1} = X_2^m$ with $m > 2$, then the surface $X := \mathbb{P}^2 - C$ has a unique G_a-action up to equivalence which is associated with the pencil generated by C and $m\ell_0$, where ℓ_0 is the line $X_1 = 0$.*

Proof. — Let $\varphi : \widetilde{V} \to V$ be the shortest sequence of blowing-ups to eliminate the base points of the pencil Λ' and let $\widetilde{\Lambda}'$ be the proper transform of Λ' by φ. Let $\widetilde{M_0}$ be the member of $\widetilde{\Lambda}'$ containing $S+G$, where we denote the proper transforms of S, G by the same symbols. Then S is a unique (-1) curve in $\widetilde{M_0}$ because $m^2d > \mu$ with the notations in the proof of Lemma 2.4. One can obtain a smooth member by a sequence of blowing-downs which starts with the contraction of S. If the dual graph of G contains a branch point, then there appears in the course of the above sequence of blowing-downs a (-1) component meeting three or more components, one of which might be replaced by the cross-section. Hence the dual graph of G must be a linear chain. The second assertion is a straightforward consequence if one notices that a smooth compactification V of X satisfying the conditions (1) $\sim$ (6) as listed above is obtained by blowing up the point $(1,0,0)$ and its infinitely near points and that the dual graph of D is then as given in [11, Figure 1, p. 456], where $r = m > 2$ and $n = 1$. Hence the dual graph of the component G is not linear. □

Another consequence of Lemma 2.4 (and also Theorem 2.5) is the following result.

THEOREM 2.6.— *Let X be a $\mathbb{Q}$-homology plane with $H_1(X;\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Suppose that X has two algebraically independent G_a-actions. Then X is isomorphic to $\mathbb{P}^2 - C$, where C is a smooth conic.*

Proof.— With the notations in Lemma 2.4, we consider the fiber F_0 which restricts on X a unique multiple fiber $2A$. The fiber F_0 is supported by $\overline{A} + G$ and $\overline{A}$ is a unique (-1) component. By Theorem 2.5, the dual graph of G is a linear chain. Then it is readily verified that G consists of three irreducible components $G_1 + G_2 + G_3$ which are all (-2) curves. Furthermore, $\overline{A}$ meets the component G_2, and we may assume that G_1 meets the cross-section S of the $\mathbb{P}^1$-fibration $p : V \to \overline{B}$. Now contract $S + G_1 + G_2 + G_3$. Then we obtain a projective plane $\mathbb{P}^2$ and the proper transforms of $F_\infty, \overline{A}$ become respectively a smooth conic C and a line tangent to the conic. Hence X is isomorphic to $\mathbb{P}^2 - C$. □

We assume that the conditions (1) $\sim$ (6) are satisfied when we consider a projective embedding $X \hookrightarrow V$. A pair (σ, σ') of two algebraically independent G_a-actions on a $\mathbb{Q}$-homology plane X is *minimal* if $\iota(\sigma,\sigma') = m^2$, where $m = |H_1(X;\mathbb{Z})|$. The following result, which is essentially contained in [2, 1.10, 1.11], guarantees the existence of a minimal pair of G_a-actions in the case $m = 2$.

LEMMA 2.7.— *Let C be a smooth conic on $\mathbb{P}^2$ and let $X = \mathbb{P}^2 - C$. Then the following assertions hold:*

(1) *X is a $\mathbb{Q}$-homology plane with $m = 2$.*

(2) *Let Q be a point on C and let ℓ_Q be the tangent line of C at Q. Let Λ_Q be the linear pencil spanned by C and $2\ell_Q$. Then the pencil Λ_Q defines an $\mathbb{A}^1$-fibration $\rho_Q : X \to \mathbb{A}^1$, and hence the conjugate class of G_a-actions σ_Q on X.*

(3) *If Q, Q' are distinct points on C, then $\sigma_Q, \sigma_{Q'}$ are algebraically independent. Furthermore, $\iota(\sigma_Q, \sigma_{Q'}) = 4$. Hence $(\sigma_Q, \sigma_{Q'})$ is a minimal pair.*

If X is isomorphic to $\mathbb{P}^2 - C$ as above, the universal covering Y of X is the complement of the diagonal Δ in $\mathbb{P}^1 \times \mathbb{P}^1$, which is a hypersurface $xy = z^2 - 1$ in $\mathbb{A}^3$. The lift $\widetilde{\sigma}_Q$ of σ_Q onto Y is associated with a pencil $\Lambda_{\widetilde{Q}}$ spanned by Δ and $\ell_{\widetilde{Q}} + M_{\widetilde{Q}}$, where $\widetilde{Q}$ is a point of Δ lying over the point Q of C and where $\ell_{\widetilde{Q}}$ and $M_{\widetilde{Q}}$ are respectively the fiber and section passing through the point $\widetilde{Q}$. We have the following result.

Lemma 2.8. — *With the above notations, express $\widetilde{Q} \in \Delta$ as $\{(1, a), (1, a)\}$ with $a \in k$. Then the locally nilpotent derivation associated with $\widetilde{\sigma}_Q$ is conjugate to δ_a defined by*

$$\delta_a(x) = 2(z - ax),\ \delta_a(y) = 2a(y - az),\ \delta_a(z) = y - a^2 x.$$

Furthermore, $\operatorname{Ker} \delta_a = k[u]$ *with $u = y - 2az + a^2 x$.*

Proof. — It is straightforward to show that δ_a is locally nilpotent and $u \in \operatorname{Ker} \delta_a$. By substituting y by $u + 2az - a^2 x$ in the equation $xy = z^2 - 1$, we have $xu = (z - ax)^2 - 1$. Hence it follows that $\operatorname{Ker} \delta_a = k[u]$. In order to see that δ_a is associated with the pencil $\Lambda_{\widetilde{Q}}$, set $X_0 = x_0 x_1, X_1 = x_0 y_1, X_2 = x_1 y_0$ and $X_3 = x_1 y_1$, where (x_0, x_1) (resp. (y_0, y_1)) is a system of homogeneous coordinates on $\mathbb{P}^1$ (resp. a copy of $\mathbb{P}^1$). Let $U = X_1 - X_2$, where the diagonal Δ of $\mathbb{P}^1 \times \mathbb{P}^1$ is defined by $U = 0$. Note that $\mathbb{P}^1 \times \mathbb{P}^1$ is defined by $X_0 X_3 = X_1 X_2 = X_2(X_2 + U)$ as a quadric hypersurface in $\mathbb{P}^3$. Set $x = 2X_0/U, y = 2X_3/U$ and $z = 2X_2/U + 1$. Then $Y := \mathbb{P}^1 \times \mathbb{P}^1 - \Delta$ is a hypersurface in $\mathbb{P}^3 - \{U = 0\} \cong \mathbb{A}^3$ defined by $xy = z^2 - 1$. Note that $\ell_{\widetilde{Q}} + M_{\widetilde{Q}}$ is defined by $(x_1 - ax_0)(y_1 - ay_0) = 0$, which is written as $y - 2az + a^2 x = 0$ on Y. Hence the $\mathbb{A}^1$-fibration induced by the pencil $\Lambda_{\widetilde{Q}}$ is given by the inclusion $k[u] \hookrightarrow \Gamma(Y, \mathcal{O}_Y)$. □

In order to show the existence of a minimal pair of G_a-actions on a $\mathbb{Q}$-homology plane, we shall consider a hypersurface $xy = p(z)$ in $\mathbb{A}^3$

which is treated in [1] as a smooth affine hypersurface in $\mathbb{A}^3$ with trivial Makar-Limanov invariant.

LEMMA 2.9. — *Let Y be a hypersurface $xy = p(z)$ in $\mathbb{A}^3$, where $p(z)$ is a polynomial of degree $m > 1$ in z with distinct linear factors and let $R = \Gamma(Y, \mathcal{O}_Y)$. Then the following assertions hold.*

(1) *Define a derivation $\widetilde{\delta}$ (resp. $\widetilde{\delta}'$) on R by $\widetilde{\delta}(x) = 0$, $\widetilde{\delta}(y) = p'(z)$ and $\widetilde{\delta}(z) = x$ (resp. $\widetilde{\delta}'(y) = 0, \widetilde{\delta}'(x) = p'(z)$ and $\widetilde{\delta}'(z) = y$). Then $\widetilde{\delta}$ and $\widetilde{\delta}'$ are locally nilpotent derivations. Hence they define G_a-actions $\widetilde{\sigma}$ and $\widetilde{\sigma}'$ on Y which are algebraically independent.*

(2) *The intertwining number $\iota(\widetilde{\sigma}, \widetilde{\sigma}')$ is equal to m.*

(3) *Write $p(z) = a\prod_{i=1}^{m}(z - \alpha_i)$, and let L_i (resp. M_i) be the curve on Y defined by $x = z - \alpha_i = 0$ (resp. $y = z - \alpha_i = 0$). Then the L_i and the M_j are isomorphic to $\mathbb{A}^1$, and $(L_i \cdot M_i) = 1$ and $(L_i \cdot M_j) = 0$ if $i \neq j$.*

(4) *The Picard group $\mathrm{Pic}\,(Y)$ is a free group of rank $m - 1$ generated by the classes $[L_1], \ldots, [L_m]$ (or $[M_1], \ldots, [M_m]$) with the relations*

$$[L_1] + \cdots + [L_m] \sim 0 \quad \text{and} \quad [L_i] \sim -[M_i]$$

for $1 \leqslant i \leqslant m$.

Proof. — The first and the third assertions are verified in a straightforward fashion. To prove the second assertion, note that $\mathrm{Ker}\,\widetilde{\delta} = k[x]$ and $\mathrm{Ker}\,\widetilde{\delta}' = k[y]$. Then apply Lemma 1.5 to show that $\iota(\widetilde{\sigma}, \widetilde{\sigma}') = m$. In order to verify the fourth assertion, consider the $\mathbb{A}^1$-fibrations $\widetilde{\rho}$ and $\widetilde{\rho}'$ on Y defined by $\widetilde{\delta}$ and $\widetilde{\delta}'$, respectively. □

Let $Y(m)$ be a hypersurface $xy = z^m - 1$ in $\mathbb{A}^3$ for $m > 1$. Since $Y(m) - \bigcup_{i \neq 1} L_i$ is an open set of $Y(m)$ isomorphic to $\mathbb{A}^2$, it follows that $Y(m)$ is simply connected. Let ζ be a primitive m-th root of the unity. We have the following result.

THEOREM 2.10. — *Consider an action of a cyclic group $\mathbb{Z}/m\mathbb{Z}$ on $Y(m)$ defined by $x \mapsto \zeta x, y \mapsto \zeta^{-1}y$ and $z \mapsto \zeta^j z$ for $0 < j < m$ with $\gcd(j, m) = 1$. We denote by $Y(m, j)$ the hypersurface $Y(m)$ with this action τ_j of $\mathbb{Z}/m\mathbb{Z}$. Then the following assertions hold:*

(1) *The $\mathbb{Z}/m\mathbb{Z}$-action τ_j is free. Let $X(m, j)$ be the quotient of $Y(m, j)$ under this action of $\mathbb{Z}/m\mathbb{Z}$. Then $X(m, j)$ is a smooth affine surface with $Y(m, j)$ as its universal covering.*

(2) *Let* $\widetilde{\delta}_j = x^{j-1}\widetilde{\delta}$ *and* $\widetilde{\delta}'_j = y^{j-1}\widetilde{\delta}'$. *Then* $\widetilde{\delta}_j$ *and* $\widetilde{\delta}'_j$ *are locally nilpotent derivations on* $R := \Gamma(Y(m), \mathcal{O}_{Y(m)})$ *such that* $\widetilde{\delta}_j$ *and* $\widetilde{\delta}'_j$ *are algebraically independent and commute with the* $\mathbb{Z}/m\mathbb{Z}$*-action* τ_j, *i.e.,* $\tau_j \cdot \widetilde{\delta}_j = \widetilde{\delta}_j \cdot \tau_j$ *and* $\tau_j \cdot \widetilde{\delta}'_j = \widetilde{\delta}'_j \cdot \tau_j$. *Hence* $\widetilde{\delta}_j$ *and* $\widetilde{\delta}'_j$ *induce locally nilpotent derivations* δ_j *and* δ'_j *on* $R(m,j)$ *such that* δ_j *and* δ'_j *are algebraically independent, where* $R(m,j)$ *is the invariant subring of* R *under the action* τ_j *of* $\mathbb{Z}/m\mathbb{Z}$ *and hence the coordinate ring of* $X(m,j)$.

(3) $X(m,j)$ *is a* $\mathbb{Q}$*-homology plane with two algebraically independent* G_a*-actions* σ_j *and* σ'_j *associated respectively with* δ_j *and* δ'_j. *Furthermore,* $m = |H_1(X(m,j); \mathbb{Z})|$.

(4) *We have* $\iota(\sigma_j, \sigma'_j) = m^2$. *Hence the pair* (σ_j, σ'_j) *is minimal.*

(5) *If* $j \neq j'$, *there are no isomorphisms* $\theta : X(m,j) \to X(m,j')$ *such that* $\theta^*(x^m) = x^m$ *or* $\theta^*(y^m) = y^m$.

Proof. — The first and second assertions are verified in a straightforward fashion. We prove the assertion (3). It is clear that $\mathbb{Z}/m\mathbb{Z}$ acts transitively via τ_j on the subset $\{[L_1], \ldots, [L_m]\}$ of $\operatorname{Pic} Y(m)$. Since $[L_1] + \cdots + [L_m] \sim 0$ and $\operatorname{Pic} X(m,j) \otimes \mathbb{Q}$ is the invariant subspace of $\operatorname{Pic} Y(m) \otimes \mathbb{Q}$ under the $\mathbb{Z}/m\mathbb{Z}$-action, it follows that $\operatorname{Pic} X(m,j) \otimes \mathbb{Q} = (0)$. On the other hand, since $X(m,j)$ is a rational surface with logarithmic Kodaira dimension $-\infty$ and $\Gamma(\mathcal{O}_{X(m,j)})^* = k^*$, we know that $X(m,j)$ is a $\mathbb{Q}$-homology plane (cf. [12]). Since $X(m,j)$ has two algebraically independent G_a-actions σ_j and σ'_j, any $\mathbb{A}^1$-fibration $\rho : X(m,j) \to B$, for example, the $\mathbb{A}^1$-fibration $\rho_j : X(m,j) \to B$ associated with σ_j, has at most one multiple fiber (cf. Proof of Lemma 2.3). The construction of the universal covering of $X(m,j)$ described after Lemma 2.1 and Lemma 2.3 implies that there is a unique multiple fiber of multiplicity m. Hence $m = |H_1(X(m,j); \mathbb{Z})|$.

In order to prove the assertion (4), let $\pi : Y(m,j) \to X(m,j)$ be the quotient morphism. Let T (resp. T') be a general orbit of the G_a-action σ_j (resp. σ'_j). Then $\pi^*(T) = T_1 + \cdots + T_m$ and $\pi^*(T') = T'_1 + \cdots + T'_m$, where the T_i (resp. the T'_i) are the general orbits of the G_a-action $\widetilde{\sigma}_j$ (resp. $\widetilde{\sigma}'_j$) on $Y(m,j)$ associated with $\widetilde{\delta}_j$ (resp. $\widetilde{\delta}'_j$). It is then clear that $\iota(\widetilde{\sigma}_j, \widetilde{\sigma}'_j) = \iota(\widetilde{\sigma}, \widetilde{\sigma}') = m$. Since $\iota(\sigma_j, \sigma'_j) = (T \cdot T')$ and since

$$m(T \cdot T') = (\pi^*(T) \cdot \pi^*(T')) = \sum_{i,\ell=1}^{m} (T_i \cdot T'_\ell) = \sum_{i,\ell=1}^{m} \iota(\widetilde{\sigma}, \widetilde{\sigma}') = m^3,$$

we know that $\iota(\sigma_j, \sigma'_j) = m^2$. Hence (σ_j, σ'_j) is a minimal pair.

Finally, we prove the assertion (5). Consider the derivation δ_j as a vector field on $X(m,j)$. Then δ_j is non-vanishing along the fibers of $\rho_j : X(m,j) \to B$ except for the fiber over the point P_0 of B which is defined by $\xi = 0$, where $\xi = x^m$ and $B = \operatorname{Spec} k[\xi]$. In fact, if $\rho_j^*(P_0) = mA$, we claim that δ_j vanishes along A to the order $j+1$. To show this claim, take an integer $0 < i < m$ so that $ij \equiv 1 \pmod{m}$. Then x/z^i is a rational function on $X(m,j)$ because it is invariant under the $\mathbb{Z}/m\mathbb{Z}$-action τ_j. Furthermore, it is regular near the fiber mA because $z \neq 0$ on $\pi^*(mA)$. Since $\xi = (z^m)^i(x/z^i)^m$, the curve A is locally defined by $x/z^i = 0$. Then we compute as follows:

$$\begin{aligned}\delta_j\left(\frac{x}{z^i}\right) &= \widetilde{\delta}_j\left(\frac{x}{z^i}\right) = x^j\widetilde{\delta}\left(z^{-i}\right) = (-i)\frac{x^{j+1}}{z^{i+1}}\\ &= \left(\frac{x}{z^i}\right)^{j+1}\cdot z^{\alpha m},\end{aligned}$$

where $ij = \alpha m + 1$. Thus the claim is proved. On the other hand, if δ and γ are locally nilpotent derivations giving rise to the same $\mathbb{A}^1$-fibration ρ_j on $X(m,j)$, then $a\delta = b\gamma$ with $a, b \in \operatorname{Ker}\delta = \operatorname{Ker}\gamma$ (cf. Lemma 1.1). Suppose that there is an isomorphism $\theta : X(m,j) \to X(m,j')$ such that $\theta(x^m) = x^m$, i.e., $\rho_{j'} \cdot \theta = \rho_j$. Then δ_j and $\delta_{j'}$ are considered to give the same $\mathbb{A}^1$-fibrations $\rho_j : X(m,j) \to B = \operatorname{Spec} k[x^m]$. By the above remark, we have $a\delta_j = b\delta_{j'}$ with $a, b \in k[\xi] = \operatorname{Ker}\delta_j = \operatorname{Ker}\delta_{j'}$, where $\xi = x^m$. Since δ_j and $\delta_{j'}$ are non-vanishing along the fibers of ρ_j except for mA, we have $a = c\xi^\ell$ and $b = d\xi^n$ with $c, d \in k^*$ and $\ell, n \geqslant 0$. Since δ_j (resp. δ'_j) vanishes along A to the order $j+1$ (resp. $j'+1$), it follows that $m\ell + j + 1 = mn + j' + 1$. Since $0 < j, j' < m$, we have $\ell = n$ and $j = j'$. This is a contradiction. □

3. $\mathbb{Q}$-homology planes whose Makar-Limanov invariants are trivial.

In this section, we shall prove that the $\mathbb{Q}$-homology planes with minimal pairs of G_a-actions are exhausted up to isomorphisms by the surfaces $X(m,j)$ observed in the previous section, where $0 < j < m$ and $\gcd(j,m) = 1$. We shall begin with a remark made by a doctoral student Adrien Dubouloz of the Université de Grenoble, which gives a relation between the $\mathbb{Q}$-homology planes with trivial Makar-Limanov invariants and the hypersurfaces $xy = p(z)$ in [1]. We here note that, in a setting similar to Theorem 3.1, an explicit local construction of obtaining a surface X with $\mathbb{C}^+$-action as the quotient of a surface Y with $\mathbb{C}^+$-action and $\mathbb{Z}/m\mathbb{Z}$-action has been initiated in [4, Example 1.6].

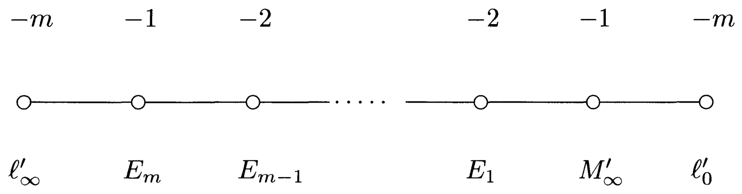
THEOREM 3.1. — *Let X be a $\mathbb{Q}$-homology plane with trivial Makar-Limanov invariant and let $\rho : X \to B$ be an $\mathbb{A}^1$-fibration with a unique multiple fiber mA of multiplicity $m > 1$. Let $B' \to B$ be a cyclic Galois covering of order m ramifying totally over the point $P_0 = \rho(A)$ and let Y be the normalization of the fiber product $X \times_B B'$. Then Y is isomorphic to a hypersurface $xy = p(z)$, where $p(z)$ is a polynomial of degree m in z with distinct linear factors. The given $\mathbb{Q}$-homology plane X is regained as the quotient of Y with respect to a $\mathbb{Z}/m\mathbb{Z}$-action.*

Proof. — We shall give a rough sketch of the proof, leaving the details to a paper by A. Dubouloz. We use the projective embedding $X \hookrightarrow V$ considered before and in Lemma 2.4. In particular, the fiber F_0 of $p : V \to \overline{B}$ over the point P_0 is supported by $G+\overline{A}$, where the dual graph of G is a linear chain and $\overline{A}$ is the closure of A in V. Let G_1 be the irreducible component of G such that $(G_1 \cdot \overline{A}) = 1$. Let $\sigma : \overline{B}' \to \overline{B}$ be a cyclic Galois covering of order m ramifying totally over the points P_0 and $P_\infty = p(F_\infty)$. Let W' be the normalization of V in the function field of Y and let $\tau' : W' \to V$ be the normalization morphism. Then the branch locus of τ' contains F_∞ and is contained in the sum $F_\infty + G$. Hence W' has a $\mathbb{P}^1$-fibration $q' : W' \to \overline{B}'$. The singularity of W' are at most cyclic quotient singularities which arise from the intersection points of the branch locus and lie on the fiber ${q'}^{-1}(P'_0)$, where P'_0 is the point of $\overline{B}'$ lying over P_0. Let $\nu : W \to W'$ be the minimal resolution of the singular points of W' and let $\tau = \tau' \cdot \nu : W \to V$. Then there is an induced $\mathbb{P}^1$-fibration $q : W \to \overline{B}'$, which satisfies $\sigma \cdot q = p \cdot \tau$. Remind that the component A splits into a disjoint union of m affine lines $L_1, \ldots, L_m$. This implies that the component G_1 is not contained in the branch locus of τ' and hence τ. Let H_1 be the irreducible component of $q^{-1}(P'_0)$ lying over G_1. Then $\tau |_{H_1}: H_1 \to G_1$ is a cyclic covering of order m, and there are m irreducible components $\overline{L}_1, \ldots, \overline{L}_m$ of $q^{-1}(P'_0)$ such that $(H_1 \cdot \overline{L}_i) = 1$ and $\overline{L}_i \cap Y = L_i$ for $1 \leqslant i \leqslant m$. Since $\overline{L}_1, \ldots, \overline{L}_m$ are reduced in $q^{-1}(P'_0)$, the multiplicity of H_1 in $q^{-1}(P'_0)$ is accordingly equal to 1. So, we can contract all the components of $q^{-1}(P'_0)$ except for H_1 and $\overline{L}_1, \ldots, \overline{L}_m$. Let $\widetilde{W}$ be the surface thus obtained from W. Then $\widetilde{W}$ has a $\mathbb{P}^1$-fibration $\widetilde{q} : \widetilde{W} \to \overline{B}'$ and Y is embedded into $\widetilde{W}$ as an open set, and the boundary divisor $\widetilde{D} := \widetilde{W} - Y$ consists of the cross-section $\widetilde{S}$ of $\widetilde{q}$, the fiber $\widetilde{F}_\infty$ lying above the point at infinity P'_∞, and the component $\widetilde{H}_1$ of the fiber $\widetilde{F}_0 = \widetilde{H}_1 + \sum_{i=1}^m \widetilde{L}_i$, where P'_∞ is a unique point of $\overline{B}'$ lying above P_∞, $\widetilde{S}$ is the inverse image of S and $\widetilde{H}_1, \widetilde{L}_1, \ldots, \widetilde{L}_m$ are respectively the proper transforms of $H_1, \overline{L}_1, \ldots, \overline{L}_m$. Then it is straightforward to see that the canonical divisor K_Y, that is to say, the restriction of $K_{\widetilde{W}}$ onto Y, is trivial.

On the other hand, since all the G_a-actions on X lifts up to Y by Lemma 2.2, Y is a smooth affine surface with trivial Makar-Limanov invariant. Hence, by [1, Lemma 4], Y is isomorphic to a hypersurface $xy = p(z)$ with $\deg p(z) = m$. □

Let Y be as above a hypersurface $xy = p(z)$ in $\mathbb{A}^3$, where we may write $p(z) = \prod_{i=1}^{m}(z - \alpha_i)$ with $\alpha_i \neq \alpha_j$ whenever $i \neq j$. We shall consider a smooth compactification of the hypersurface Y and how to construct it.

Example 3.2. — Let W_0 be a rational surface isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. We denote by ℓ and M the respective fibers of two projections from W_0 to $\mathbb{P}^1$. By fixing one projection, we call ℓ a fiber and M a section. Fix two fibers ℓ_0, ℓ_∞ and $m+1$ sections $M_1, \ldots, M_m, M_\infty$, where $m \geqslant 2$. Let $Q_i := \ell_0 \cap M_i$ for $1 \leqslant i \leqslant m$ and $Q_\infty := \ell_\infty \cap M_\infty$. Consider a linear system $\Lambda = |\ell + mM| - (Q_1 + \cdots + Q_m + mQ_\infty)$, which consists of curves linearly equivalent to $\ell + mM$ and passing through the points $Q_1, \ldots, Q_m$ simply and the point Q_∞ m times. Since $\dim|\ell + mM| = 2m+1$, it follows that Λ is a linear pencil and that the curves $\ell_\infty + M_1 + \cdots + M_m$ and $\ell_0 + mM_\infty$ are members of Λ. Let $\tau : W \to W_0$ be a composite of blowing-ups with centers $Q_1, \ldots, Q_m, Q_\infty$ and $m-1$ infinitely near points $Q_\infty^{(1)}, \ldots, Q_\infty^{(m-1)}$ of Q_∞, where $Q_\infty^{(i)}$ lies on the proper transform of ℓ_∞ and $Q_\infty^{(i)}$ is infinitely near to $Q_\infty^{(i-1)}$ for $1 \leqslant i < m$ with $Q_\infty^{(0)} = Q_\infty$. Let $L_i := \tau^{-1}(Q_i)$ for $1 \leqslant i \leqslant m$, let M_i denote the proper transform $\tau'(M_i)$ by the abuse of the notations, and let $\tau^{-1}(Q_\infty) = E_1 + E_2 + \cdots + E_m$. Then, with the proper transforms $\ell'_0 = \tau'(\ell_0), \ell'_\infty = \tau'(\ell_\infty)$ and $M'_\infty = \tau'(M_\infty)$, the curves $E_1, \ldots, E_m$ constitute a linear chain of rational curves whose dual graph is given as follows:



Note that if $m = 2$ the contraction of $E_2, \ell'_\infty, M'_\infty$ and ℓ'_0 brings W to a surface isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ with the proper transform of E_1 as the diagonal. Set $Z = W - (\ell'_0 + M'_\infty + E_1 + \cdots + E_m + \ell'_\infty)$. Then Z has two $\mathbb{A}^1$-fibrations, one of which is given by the pencil $|\ell|$ on W_0 and has a reducible fiber $L_1 + \cdots + L_m$ and another one of which is given by the pencil Λ on W_0 and contains a reducible fiber $M_1 + \cdots + M_m$, where we denote the intersections of $L_1 \cap Z, \ldots, L_m \cap Z$ and $M_1 \cap Z, \ldots, M_m \cap Z$ by the same letters $L_1, \ldots, L_m$ and $M_1, \ldots, M_m$ by the abuse of the notations.

The following result shows that the hypersurface Y in Theorem 3.1 is constructed in a way as described in the above example.

LEMMA 3.3.— *Let Y be a hypersurface $xy = p(z)$ as above. Assume that $m \geqslant 2$ and $p(0) \neq 0$. The hypersurface Y is then isomorphic to Z as constructed as above with suitably chosen points $Q_1, \ldots, Q_m$ and Q_∞.*

Proof.— Let $\rho_x : Y \to B_x \cong \mathbb{A}^1$ and $\rho_y : Y \to B_y \cong \mathbb{A}^1$ be respectively the $\mathbb{A}^1$-fibrations parametrized by x and y. So, the generic fiber of ρ_x (resp. ρ_y) is defined by $y = x^{-1}p(z)$ (resp. $x = y^{-1}p(z)$). Furthermore, let $L_1 + \cdots + L_m$ (resp. $M_1 + \cdots + M_m$) be a unique reducible reduced fiber of ρ_x (resp. ρ_y). We may assume that $L_i + M_i$ is defined by $z - \alpha_i = 0$ for $1 \leqslant i \leqslant m$, where $p(z) = \prod_{i=1}^{m}(z - \alpha_i)$ with $\alpha_i \neq \alpha_j$ whenever $i \neq j$.

Consider a smooth compactification W' of Y such that ρ_x extends to a $\mathbb{P}^1$-fibration $\pi_x : W' \to \overline{B}_x \cong \mathbb{P}^1$. We may assume that ρ_y extends to a $\mathbb{P}^1$-fibration $\pi_y : W' \to \overline{B}_y \cong \mathbb{P}^1$. We denote the closures of the L_i and the M_j on W' by the same letters. The boundary $D' := W' - Y$ consists of $\Gamma_0 - (L_1 + \cdots + L_m), M_\infty$ and Γ_∞, where Γ_0 and Γ_∞ are fibers of π_x and M_∞ is a section of π_x. Note that $M_1, \ldots, M_m$ are mutually disjoint cross-sections of π_x. Similarly, $L_1, \ldots, L_m$ are mutually disjoint cross-sections of π_y. Note that the fibers of π_y except for $\pi_y^{-1}(P_\infty)$ with $(P_\infty) = \overline{B}_y - B_y$ do not intersect the components of $\Gamma_0 - (L_1 + \cdots + L_m)$. Hence we may contract all smoothly contractible components of $\Gamma - (L_1 + \cdots + L_m)$.

We claim that we can take W' in such a way that $\Gamma_0 - (L_1 + \cdots + L_m)$ is an irreducible component L_0 satisfying $(L_0 \cdot M_\infty) = 1, ({L_0}^2) = -m$ and $({L_i}^2) = -1$ for $1 \leqslant i \leqslant m$. In fact, let $\overline{Y}$ be the projective closure of Y in $\mathbb{P}^3$, where $\mathbb{A}^3$ is naturally embedded into $\mathbb{P}^3$ as the complement of a hyperplane. Then $\overline{Y}$ is defined by an equation

$$XYU^{m-2} = P(Z, U),$$

where $x = X/U, y = Y/U, z = Z/U$ and $P(Z, U)$ is a homogeneous polynomial in Z, U of degree m with $p(z) = P(z, 1)$. Consider a fiber A_α of the $\mathbb{A}^1$-fibration ρ_x for $x = \alpha \in k^*$. The curve A_α has a parametric representation

$$x = \alpha, \quad y = \alpha^{-1}p(t) \quad \text{and} \quad z = t.$$

Let $x' = X/Y, z' = Z/Y$ and $u' = U/Y$. Then, in an open set $D_+(Y)$ of $\mathbb{P}^3$, the hypersurface $\overline{Y}$ is defined by $x'u'^{m-2} = P(z', u')$, which has singularity along the curve $z' = u' = 0$ if $m \geqslant 3$. The curve A_α has a parametric representation

$$x' = \frac{\alpha^2\tau^m}{P(1,\tau)}, \quad z' = \frac{\alpha\tau^{m-1}}{P(1,\tau)}, \quad u' = \frac{\alpha\tau^m}{P(1,\tau)},$$

where $\tau = t^{-1}$. Let $x'' = x'/z'$ and $u'' = u'/z'$. Then the proper transform of $\overline{Y}$ is defined in $\operatorname{Spec} k[z', x'', u'']$ by the equation

$$x''u''^{m-2} = z'P(1, u'')$$

which is a smooth surface. The proper transform $\overline{A}_\alpha$ of the closure of the curve A_α has a parametric representation

$$x'' = \alpha\tau, \quad u'' = \tau, \quad z' = \frac{\alpha\tau^{m-1}}{P(1,\tau)}.$$

Hence $\overline{A}_\alpha$ is a smooth curve with tangent direction $x'' = \alpha u''$. The fiber A_0 of ρ_x for $x = 0$ corresponds to a reducible curve $\overline{A}_0$ which consists of the curve $L_0 = \{x'' = 0\}$ and the irreducible components $L_1, \ldots, L_m$ of $P(1, u'') = 0$. Hence the blowing-up of the point $(x'' = 0, u'' = 0)$ produces a $\mathbb{P}^1$-fibration π_x which extends the $\mathbb{A}^1$-fibration ρ_x and for which $L_0 + L_1 + \cdots + L_m$ is a fiber. Thus we have shown our claim.

By a similar observation, we may assume that the fiber of π_y containing $M_1 + \cdots + M_m$ has as an extra component a unique irreducible reduced component M_0 with $(M_0 \cdot M_i) = 1$ for $1 \leqslant i \leqslant m$. Note that M_0 is a component of the fiber Γ_∞ of π_x. Let $q : W' \to \overline{W}$ be the contractions of $L_1, \ldots, L_m$ and the components of Γ_∞ except for M_0 such that $\pi_x \cdot q^{-1} : \overline{W} \to \overline{B}_x$ is a relatively minimal $\mathbb{P}^1$-fibration. Then $\overline{W}$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ because the respective images $\overline{M}_1, \cdots, \overline{M}_m, \overline{M}_\infty$ of $M_1, \cdots, M_m, M_\infty$ on $\overline{W}$ are mutually disjoint cross-sections. The linear pencil Λ consisting of the images of the fibers of π_y is of the form $|\ell + mM| - (Q_1 + \cdots + Q_m + mQ_\infty)$ as described in the above example. □

We note that the hypothesis $p(0) \neq 0$ is easily realized by replacing z by $z - c$ with some $c \in k$. The following result will determine the form of $p(z)$ when the hypersurface Y is obtained as the universal covering of a $\mathbb{Q}$-homology plane with trivial Makar-Limanov invariant.

THEOREM 3.4. — *Let X be a $\mathbb{Q}$-homology plane with trivial Makar-Limanov invariant. Suppose that $m \geqslant 2$ for $m = |H_1(X; \mathbb{Z})|$. Then the* *universal covering of X is isomorphic to the hypersurface $xy = z^m - 1$ in $\mathbb{A}^3$, and X is isomorphic to $X(m,j)$ constructed in Theorem 2.9 for some $0 < j < m$ with $\gcd(j,m) = 1$.*

The proof of Theorem 3.4 consists of the following two lemmas.

LEMMA 3.5. — *With the notations and assumptions of Theorem 3.4, suppose that X has a pair (σ, σ') of G_a-actions such that $\iota(\sigma,\sigma') = m^2$ with $m \geqslant 2$. Then the assertion of Theorem 3.4 holds true.*

Proof. — We use the smooth compactification V of X as constructed before and in Lemma 2.4. As explained after Lemma 2.1, the universal covering Y of X is obtained as the normalization of $X \times_B B'$, where B' is a cyclic covering of degree m totally ramifying over the point $P_0 := \rho(A)$ and the point at infinity P_∞. We employ the notations of the proof of Theorem 3.1. Then the Galois group $\mathbb{Z}/m\mathbb{Z}$ acts regularly on W' as well as on W, where W' is the normalization of V in the function field of Y and W is the minimal resolution of W'. The $\mathbb{P}^1$-fibration $q : W \to \overline{B}'$ is $\mathbb{Z}/m\mathbb{Z}$-equivariant, and the divisors $q^{-1}(P'_0)$, $q^{-1}(P'_\infty)$ and $\tau^{-1}(S)$, which lie on the boundary $W - Y$, are $\mathbb{Z}/m\mathbb{Z}$-stable, where P'_0 and P'_∞ are the points of $\overline{B}'$ lying over the points P_0 and P_∞ respectively. Furthermore, the contraction, say μ, of all the components of $q^{-1}(P'_0)$ except for H_1 and $\overline{L}_1, \ldots, \overline{L}_m$ is $\mathbb{Z}/m\mathbb{Z}$-equivariant, and $\mathbb{Z}/m\mathbb{Z}$ stabilizes H_1 and permutes transitively the set $\{\overline{L}_1, \ldots, \overline{L}_m\}$. Thus $\mathbb{Z}/m\mathbb{Z}$ acts regularly on the surface $\widetilde{W}$ obtained by the contraction μ and the $\mathbb{P}^1$-fibration $\widetilde{q} : \widetilde{W} \to \overline{B}'$ is $\mathbb{Z}/m\mathbb{Z}$-equivariant. Furthermore, $\mathbb{Z}/m\mathbb{Z}$ stabilizes $\widetilde{F}_\infty, \widetilde{S}$ and $\widetilde{H}_1$, and permutes transitively $\{\widetilde{L}_1, \ldots, \widetilde{L}_m\}$, where $\widetilde{F}_\infty$ is the fiber $\widetilde{q}^{-1}(P'_\infty)$, $\widetilde{S}$ is the image of $\tau^{-1}(S)$ and $\widetilde{H}_1, \widetilde{L}_1, \ldots, \widetilde{L}_m$ are the images of $H_1, \overline{L}_1, \ldots, \overline{L}_m$ on $\widetilde{W}$.

The surface $\widetilde{W}$ has a $\mathbb{P}^1$-fibration $\widetilde{q} : \widetilde{W} \to \overline{B}'$ for which $\widetilde{S}$ is a cross-section, $\widetilde{F}_\infty$ is a smooth fiber and $\widetilde{q}^{-1}(P'_0) = \widetilde{H} + \widetilde{L}_1 + \cdots + \widetilde{L}_m$. Note that there are at least two fixed points R_1, R_2 on $\widetilde{F}_\infty$, where we can take $R_1 = \widetilde{S} \cap \widetilde{F}_\infty$. By the elementary transformation with center at R_1 or R_2, which is $\mathbb{Z}/m\mathbb{Z}$-equivariant, we can decrease or increase the self-intersection number $(\widetilde{S}^2)$ by 1. So, applying the $\mathbb{Z}/m\mathbb{Z}$-equivariant elementary transformations several times if necessary, we may assume that $(\widetilde{S}^2) = -1$. Then we can contract $\widetilde{S}, \widetilde{L}_1, \ldots, \widetilde{L}_m$ without losing the regular $\mathbb{Z}/m\mathbb{Z}$-action to obtain the projective plane $\mathbb{P}^2$ so that the respective images ℓ_0, ℓ_∞ of $\widetilde{H}_1, \widetilde{F}_\infty$ are lines.

On the other hand, since (σ, σ') is a minimal pair, the $\mathbb{A}^1$-fibration ρ' on X associated with σ' has a unique multiple fiber mA', and the inverse image of A' on Y splits into a disjoint sum $M_1 + \cdots + M_m$ of the affine lines such that $(L_i \cdot M_i) = 1$ and $(L_i \cdot M_j) = 0$ if $i \neq j$. For $1 \leqslant j \leqslant m$, let $\overline{M}_j$ be the closure of M_j on W, and denote by γ_j the image of $\overline{M}_j$ on $\mathbb{P}^2$. Let $(Q_0) = \ell_0 \cap \ell_\infty$. Then γ_j meets $\ell_0 - (Q_0)$ in one point Q_j transversally and meet ℓ_∞ in one-place point Q, where the point Q is common for the curves $\gamma_1, \ldots, \gamma_m$ because otherwise $\gamma_1, \ldots, \gamma_m$ would be mutually disjoint from each other, which is impossible for the curves on $\mathbb{P}^2$. The $\mathbb{A}^1$-fibration ρ' on Y is produced from a linear pencil Λ on $\mathbb{P}^2$ for which $\gamma_1 + \cdots + \gamma_m$ is a member. We consider the two cases $Q \neq Q_0$ and $Q = Q_0$ separately.

Case $Q \neq Q_0$. It is then easy to see that $\gamma_1, \ldots, \gamma_m$ are lines and that the pencil Λ is spanned by $\gamma_1 + \cdots + \gamma_m$ and $\ell_0 + (m-1)\ell_\infty$. Choose a system of homogeneous coordinates (x_0, x_1, x_2) so that the points Q_0 and Q are written respectively as $(0,0,1)$ and $(0,1,0)$ and the line ℓ_0 is defined by $x_1 = 0$. Furthermore, since $\mathbb{Z}/m\mathbb{Z}$ acts transitively on the set of the points $\{Q_1, \ldots, Q_m\}$, we can adjust the coordinate x_1 so that the curve $\gamma_1 + \ldots + \gamma_m$ is defined by $x_2^m - x_0^m$. Then a general member of Λ is written as $\lambda x_0^{m-1} x_1 = x_2^m - x_0^m$, where λ is an inhomogeneous parameter of the pencil Λ. Set $x = x_1/x_0$ and $z = x_2/x_0$. Then we have a linear pencil $\{xy = z^m - 1\}$, where y is a parameter. If y moves over the elements of k, we know that the curves $xy = z^m - 1$ exhaust all the points of Y without overlappings. Hence Y itself is realized as a hypersurface $xy = z^m - 1$ in $\mathbb{A}^3$. The $\mathbb{Z}/m\mathbb{Z}$-action on $\mathbb{P}^2$ is given by $(x_0, x_1, x_2) \mapsto (x_0, \zeta x_1, \zeta^j x_2)$, where $0 < j < m$. Since $xy = z^m - 1$ is $\mathbb{Z}/m\mathbb{Z}$-invariant, the action on the coordinate y is given by $y \mapsto \zeta^{-1} y$.

Case $Q = Q_0$. We work on the surface $\widetilde{W}$ instead of $\mathbb{P}^2$, where $\widetilde{W}$ is the Hirzebruch surface Σ_1 of degree 1 and $\widetilde{S}$ is the minimal section. Only for this case, we denote $\widetilde{S}$ and a general fiber of $\widetilde{q}$ by M and ℓ according to the customary usage of the notations. We denote the images of the $\overline{M}_j$ on $\widetilde{W}$ by C_j. Since C_j meets the fiber ℓ_0 at the point Q_j transversally, C_j is linearly equivalent to $n\ell + M$ for some $n \geqslant 1$. Hence C_j is smooth. If $n = 1$ then $C_j \cap M = \emptyset$ and we are reduced to the former case $Q \neq Q_0$. So, $n \geqslant 2$. Since C_j has only one place on the boundary $\widetilde{W} - Y$ and since $C_j \cap \widetilde{F}_\infty \neq \emptyset$, C_j passes through the point $\widetilde{Q} := \widetilde{F}_\infty \cap M$ and touches the section M with order $n-1$. Let $p_1 : W_1 \to \widetilde{W}$ be the composite of $n-1$ blowing-ups with centers at the infinitely near points of $\widetilde{Q}$ which lie on the proper transforms of M and let $E_1, \ldots, E_{n-1}$ be the irreducible exceptional curves of p_1. Then the curves $p_1'(\widetilde{F}_\infty), E_1, \ldots, E_{n-1}, p_1'(M)$ arranged in this order form a linear chain, and $(p_1'(\widetilde{F}_\infty)^2) = -1, ({E_i}^2) = -2$ for $1 \leqslant i \leqslant n-2$ and

$({E_{n-1}}^2) = -1$. Since $({C_j}^2) = 2n-1$, the proper transforms $p'_1(C_j)$ meet in one point of E_{n-1} which is different from $E_{n-2} \cap E_{n-1}$ and $E_{n-1} \cap p'_1(M)$. Let $p_2 : W_2 \to W_1$ be the composite of n blowing-ups by which the proper transforms $p'_2(p'_1(C_j))$ get separated from each other and let $F_1, \ldots, F_n$ be the irreducible exceptional curves. Then $F_1 + F_2 + \cdots + F_n$ is a linear chain sprouting from the proper transform $p'_2(E_{n-1})$ with $({F_i}^2) = -2$ for $1 \leqslant i \leqslant n-1$ and $({F_n}^2) = -1$. Note that $(p'_2(E_{n-1})^2) = -2$. We can then contract $p'_2(p'_1(\widetilde{F}_\infty)), p'_2(E_1), \ldots, p'_2(E_{n-1}), F_1, \ldots, F_{n-1}$ in this order. Let $p_3 : W_2 \to W_3$ be the contraction of these curves. By the abuse of the notations, denote the images of $p'_2(p'_1(M)), F_n, p'_2(p'_1(C_j))$ on W_3 by $M_\infty, \ell_\infty, M_j$ respectively. Since $({M_\infty}^2) = ({\ell_\infty}^2) = 0$, it follows that W_3 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. In fact, we regain the same picture as in Example 3.2 with the curves $M_1, \ldots, M_m$. Since we did not change anything on the open set Y, we may start with the situation treated in Example 3.2.

The proper transform Λ' of the pencil Λ on $\mathbb{P}^2$ becomes a linear pencil $|\ell_\infty + mM_\infty| - (Q_1 + \cdots + Q_m + mQ_\infty)$, where $Q_i = \ell_0 \cap M_j$ and $Q_\infty = \ell_\infty \cap M_\infty$. Eliminate the base points of the pencil Λ' by blowing up the point Q_∞ and its infinitely near points $Q_\infty^{(1)}, \ldots, Q_\infty^{(m-1)}$ which lies on the proper transform of ℓ_∞. The exceptional curves with the proper transforms $\ell'_\infty, M'_\infty, \ell'_0$ of $\ell_\infty, M_\infty, \ell_0$ form a linear chain as exhibited in Example 3.2. The proper transforms of $M_1, \ldots, M_m$ intersect ℓ'_∞. Now contract $E_m, E_{m-1}, \ldots, E_2, \ell'_\infty$ and M'_∞ in this order. The resulting surface is $\mathbb{P}^2$, and the proper transforms of E_1, ℓ'_0 and the M_j $(1 \leqslant j \leqslant m)$ fit to the previous case where $Q \neq Q_0$. So, we have settled this case as well. □

LEMMA 3.6. — *Let X be a $\mathbb{Q}$-homology plane with trivial Makar-Limanov invariant. Then there exists a minimal pair (σ, σ') of G_a-actions on X.*

Proof. — If $X \cong \mathbb{A}^2$ then the assertion holds obviously. So, we assume that $m = |H_1(X; \mathbb{Z})| \geqslant 2$. We fix a G_a-action σ and consider the associated $\mathbb{A}^1$-fibration $\rho : X \to B$. We employ the arguments in the proof of Lemma 3.5 up to the point where the surface $\widetilde{W}$ and the $\mathbb{P}^1$-fibration $\widetilde{q} : \widetilde{W} \to \overline{B}'$ are constructed. With the same notations there, we may assume, after performing $\mathbb{Z}/m\mathbb{Z}$-equivariant elementary transformations with center at R_1 or R_2, that $(\widetilde{S}^2) = 0$. Then $|\widetilde{S}|$ is a linear pencil and defines a $\mathbb{P}^1$-fibration $\Phi_{|\widetilde{S}|} : \widetilde{W} \to \mathbb{P}^1$. Then, by the count of rank Pic $(\widetilde{W})$, it follows that $\Phi_{|\widetilde{S}|}$ has exactly m degenerate fibers $\widetilde{L}_i + \widetilde{M}_i$ $(1 \leqslant i \leqslant m)$, where $\widetilde{M}_i$ is a (-1) curve with $(\widetilde{L}_i \cdot \widetilde{M}_i) = 1$. Since the Galois group $\mathbb{Z}/m\mathbb{Z}$ stabilizes

$\widetilde{S}$ and permutes the curves $\{\widetilde{L}_1, \ldots, \widetilde{L}_m\}$, it follows that it permutes the curves $\{\widetilde{M}_1, \ldots, \widetilde{M}_m\}$ as well.

Now contract $\widetilde{L}_1, \ldots, \widetilde{L}_m$ to obtain a surface $\overline{W}$, which is the Hirzebruch surface $\Sigma_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$. Denote the images of $\widetilde{S}, \widetilde{H}, \widetilde{M}_i, \widetilde{F}_\infty$ by $M_\infty, \ell_0, M_i, \ell_\infty$, respectively. Let $Q_\infty = M_\infty \cap \ell_\infty$ and $Q_i = \ell_0 \cap M_i$. Then $\ell_0 + mM_\infty$ and $\ell_\infty + M_1 + \cdots + M_m$ are $\mathbb{Z}/m\mathbb{Z}$-stable divisors. Hence the linear pencil $\Lambda = |\ell + mM| - (Q_1 + \cdots + Q_m + mQ_\infty)$ is closed under the $\mathbb{Z}/m\mathbb{Z}$-action (cf. Example 3.2). Then Λ induces a $\mathbb{Z}/m\mathbb{Z}$-stable $\mathbb{A}^1$-fibration $\widetilde{\rho}' : Y \to B_1'$, where $Y = \widetilde{W} - (\widetilde{H} + \widetilde{S} + \widetilde{F}_\infty)$ and $B_1' \cong \mathbb{A}^1$. So, $\widetilde{\rho}'$ induces a G_a-action $\widetilde{\sigma}'$ on Y, which descends down to a G_a-action σ' on X. It is then clear by the construction that (σ, σ') is a minimal pair of G_a-actions. □

4. Intertwining at infinity of the curves belonging to the two pencils.

Let X be a $\mathbb{Q}$-homology plane with two algebraically independent G_a-actions (σ, σ'). We consider a projective embedding $X \hookrightarrow V$ considered before and in Lemma 2.4 and observe how the curves belonging to the pencils Λ and Λ' intertwine each other at infinity, where Λ (resp. Λ') is the pencil associated to σ (resp. σ'). We shall employ the notations and assumptions in Lemma 2.4 and Theorem 2.5.

By Theorem 2.5, the dual graph of G is a linear chain. The linear pencil Λ' has a base point Q on F_∞ which is different from the point $S \cap F_\infty$. Let $\overline{T}'$ be a general member of Λ'. As in the proof of Lemma 2.4, we may assume that $\mu < m^2 d$, where $m^2 d = i(\overline{T}', F_\infty; Q)$ and $\mu = \operatorname{mult}_Q \overline{T}'$. The pencil contains a member $m\overline{A}'$, where mA' with $A' := \overline{A}' \cap X$ is a unique multiple fiber of the $\mathbb{A}^1$-fibration $\rho' : X \to B'$ which is induced by Λ'. Let $\mu' := \operatorname{mult}_Q \overline{A}'$. Let $\varphi : \widetilde{V} \to V$ be the shortest sequence of blowing-ups which eliminates the base points of Λ' and let $\widetilde{\Lambda}'$ be the proper transform of Λ' by φ. Let E be the last (-1) curve appearing in the process φ and write $\varphi^{-1}(Q) = \Gamma + E + \Delta$, where Γ (resp. Δ) is the connected component of $\varphi^{-1}(Q) - E$ which meets the proper transform $\widetilde{F}_\infty$ (resp. $\widetilde{A}'$) of F_∞ (resp. $\overline{A}'$). Theorem 2.5 applied to the σ'-action implies that the dual graph of Δ is a linear chain.

Lemma 4.1. — *The following assertions hold true:*

(1) $m\mu' \geqslant \mu$.

(2) *Suppose that* $m\mu' > \mu$. *Then the dual graph of* Γ *is either an emptyset or a linear chain. Furthermore,* $m\mu' - \mu = 1$.

(3) *Suppose that* $m\mu' = \mu$. *Then the dual graph of* Γ *has a branch point.*

Proof. — (1) This is clear because the multiplicity $\text{mult}_Q \overline{T}' = \mu$ is the minimum of the multiplicities which the members of Λ' take at the point Q.

(2) Let φ_1 be the first blowing-up in the process φ and let E_1 be the exceptional curve. Then we have

$$\varphi_1^*(m\overline{A}') = m\varphi_1'(\overline{A}') + m\mu' E_1$$
$$\varphi_1^*(\overline{T}') = \varphi_1'(\overline{T}') + \mu E_1.$$

Hence in the proper transform Λ_1' of Λ' by φ_1, the (-1) curve E_1 belongs to the member containing $\varphi_1'(\overline{A}')$. If the dual graph $\varphi^{-1}(Q) = \Gamma + E + \Delta$ has a branching point, the member $\widetilde{M}_0'$ of $\widetilde{\Lambda}'$ containing $S + G$ has to coincide with the member containing $\varphi'(\overline{A}')$, which is a contradiction. So, the dual graph of Γ is a linear chain. Under the assumption $m\mu' > \mu$, the proper transform of E_1 by $\varphi \cdot \varphi_1^{-1}$ is the end component of Δ. Since $\Delta + \varphi'(\overline{A}')$ is contractible to a smooth fiber of a $\mathbb{P}^1$-fibration, it follows that $m\mu' - \mu = 1$.

(3) With the above notation, E_1 belongs to the member $\widetilde{M}_0'$. Let $\psi : \widehat{V} \to V$ be the oscillating sequence of blowing-ups with the data (md, μ') (cf. [12]) and let E' be the last (-1) curve. Since the proper transforms of E_1 and F_∞ by φ are contained in the member $\widetilde{M}_0'$, all the exceptional curves of ψ are also contained in $\widetilde{M}_0'$. In order to eliminate the base points of Λ', we have therefore to blow up a point on E'. Hence the dual graph of Γ has a branch point which represent the proper transform of E'. □

LEMMA 4.2. — *The following assertions hold:*

(1) *Suppose* $\mu' = 1$ *and* $m\mu' > \mu$. *Then the pair* (σ, σ') *is minimal.*

(2) *Suppose* $\mu' \leqslant d$ *and* $m\mu' > \mu$. *Then* $\mu' = 1$.

Proof. — (1) By Lemma 4.1 and the hypothesis $\mu' = 1$, we have $\mu = m - 1$. Then the curve $\overline{A}'$ touches F_∞ with multiplicity md. Let $\psi : V' \to V$ be a sequence of md blowing-ups with centers Q and its infinitely near points lying on the proper transforms of F_∞. Let $E_1, \dots, E_{md}$ be the irreducible exceptional curves. Then $\psi'(F_\infty) + E_{md} + \cdots + E_1$ is a linear chain and $\psi'(\overline{A}')$ meets E_{md} transversally. Let M_0' (resp. M_1') be the member of $\psi'(\Lambda')$ containing $\psi'(F_\infty)$ (resp. $\psi'(\overline{A}')$). Then we have

$$M_0' = (m-1)\psi'(F_\infty) + \text{a divisor supported by } \psi'(S) + \psi^*(G)_{\text{red}}$$
$$M_1' = m\psi'(\overline{A}') + E_1 + 2E_2 + \cdots + mdE_{md}.$$

The general member $\psi'(\overline{T}')$ passes the point $Q' := \psi'(F_\infty) \cap E_{md}$ with

$$i(\psi'(F_\infty), \psi'(\overline{T}'); Q') = m^2d - (m-1)md = md,$$
$$i(\psi'(\overline{T}'), E_{md}; Q') = m - 1.$$

Let $\varphi : \widetilde{V} \to V$ be the sequence of blowing-ups as above which eliminates the base points of Λ'. Then the member $\widetilde{M_1}$ of $\varphi'(\Lambda')$ containing $\varphi'(\overline{A}')$ is a degenerate fiber of a $\mathbb{P}^1$-fibration which contains only one (-1) curve $\varphi'(\overline{A}')$. Since the coefficient of $\varphi'(\overline{A}')$ in $\widetilde{M_1}$ is m, it is the largest coefficient among those for the components of $\widetilde{M_1}$. This implies that $md \leqslant m$. Hence $d = 1$. So, the pair (σ, σ') is a minimal pair.

(2) Suppose on the contrary that $\mu' \geqslant 2$. Write

$$md = c_1\mu' + \mu_1', \quad 0 \leqslant \mu_1' < \mu'.$$

Then

$$m^2d = m(c_1\mu' + \mu_1') = c_1\mu + (c_1 + m\mu_1').$$

Since $\mu' \leqslant d$, we have $c_1 \geqslant m$. In the case $c_1 > m$, we abuse the notations to denote by $\psi : V' \to V$ a sequence of c_1 blowing-ups with center Q and its infinitely near points lying on F_∞. It produces the member M_1' of $\psi'(\Lambda')$ such that

$$M_1' = m\psi'(\overline{A}') + E_1 + 2E_2 + \cdots + c_1E_{c_1},$$

which leads to a contradiction as in the proof of the previous assertion. Consider the case $c_1 = m$. Suppose $\mu_1' > 0$. Then we have

$$i(\psi'(F_\infty), \psi'(\overline{A}); Q') = \mu_1',$$
$$i(\psi'(\overline{A}'), E_{c_1}; Q') = \mu',$$

where $Q' = \psi'(F_\infty) \cap E_{c_1}$. Then, after the base points of Λ' are removed by $\varphi : \widetilde{V} \to V$, $\varphi'(\overline{A}')$ does not meet any one of the proper transforms of $E_1, \ldots, E_{c_1}$. This implies that a component of the member $\widetilde{M_1}$ has coefficient greater than m, where $\widetilde{M_1}$ is a member of the proper transform $\varphi'(\Lambda')$ containing $\varphi'(\overline{A}')$. This is a contradiction. So, we must have $\mu_1' = 0$.

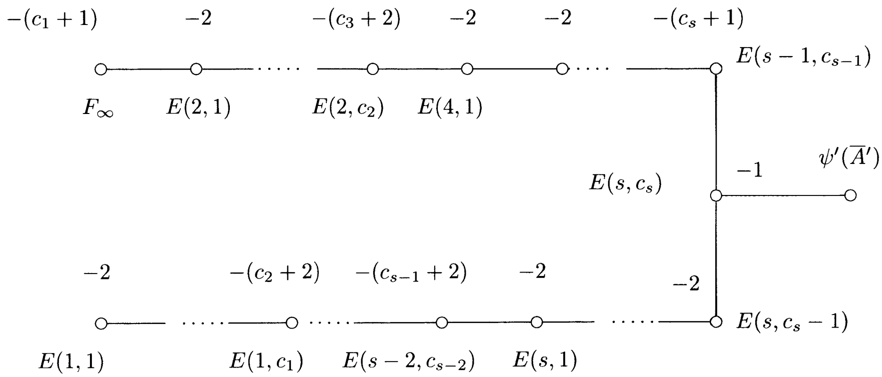
Then $c_1 = m$ and $\mu' = d$. Since $\mu' \geqslant 2$, $\psi'(\overline{A}')$ meets E_m in a single point with multiplicity μ', and this point is untouched in the further process of eliminating the base points of Λ'. This is a contradiction. □

We continue the analysis of the case $m\mu' > \mu$ and keep the same notations as above. In particular, we abuse the notations M_0' and M_1' to denote respectively the members of Λ' such that $\operatorname{Supp} M_0' = F_\infty + S + G$ and $M_1' = m\overline{A}'$, while $\overline{T}'$ denotes a general member of Λ'. Let $\varphi : \widetilde{V} \to V$ be the shortest sequence of blowing-ups with centers at the base point Q of Λ' and its infinitely near points such that the proper transform $\widetilde{\Lambda}'$ of Λ' has no base points. We denote by $\widetilde{M_0'}$ and $\widetilde{M_1'}$ the members of $\widetilde{\Lambda}'$ corresponding to M_0' and M_1' respectively. Let $\varphi^{-1}(Q) = \Gamma + E + \Delta$ as before, where $\Gamma \cap \varphi'(F_\infty) \neq \emptyset$ and $\Delta \cap \varphi'(\overline{A}') \neq \emptyset$. *We assume that* $m\mu' > \mu$. Then Γ is a linear chain and $m\mu' - \mu = 1$ by Lemma 4.1.

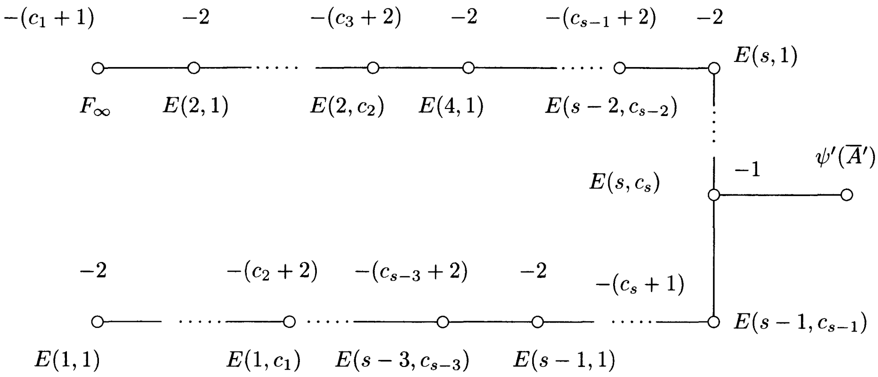
By the Euclidean algorithm with respect to md and μ', we introduce the integers c_i, μ_i' for $1 \leqslant i \leqslant s$ as follows:

$$
\begin{array}{lll}
md & = c_1\mu' + \mu_2', & 0 < \mu_2' < \mu' \\
\mu_1' & = c_2\mu_2' + \mu_3', & 0 < \mu_3' < \mu_2' \\
& \quad \cdots\cdots & \\
\mu_{s-2}' & = c_{s-1}\mu_{s-1}' + \mu_s', & 0 < \mu_s' < \mu_{s-1}' \\
\mu_{s-1}' & = c_s\mu_s', & c_s \geqslant 2,
\end{array}
$$

where we set $\mu_1' = \mu'$. Let $\psi : \widehat{V} \to V$ be an oscillating sequence of blowing-ups with respect to the data (md, μ') (cf. [12]). Then we have the following exceptional dual graph of $\psi^{-1}(Q)$. See also [10] for similar dual graphes and relevant explanations.



Case s is odd



Case s is even

LEMMA 4.3. — *The following assertions hold true:*

(1) $\psi'(\overline{A}')$ *meets the component* $E(s, c_s)$ *in one point transversally and does not meet any other components of* $\psi^{-1}(Q)$. *In particular,* $\mu'_s = 1$.

(2) *The components located on the lower side of* $E(s, c_s)$, *i.e.,* $E(1, 1), \ldots,$ $E(s, 1), \ldots, E(s, c_s - 1)$ *if s is odd and* $E(1, 1), \ldots, E(s-1, c_{s-1})$ *if s is even, are contained in the member* $\widehat{M'_1}$ *of* $\psi'(\Lambda')$ *corresponding to* M'_1 *of* Λ'.

(3) $\psi'(\overline{T}')$ *passes through the point* $E(s, c_s) \cap E(s - 1, c_{s-1})$ *if s is odd and the point* $E(s, c_s) \cap E(s, c_s - 1)$ *if s is even.*

(4) *The components located on the upper side of* $E(s, c_s)$ *are contained in the member* $\widehat{M'_0}$ *of* $\psi'(\Lambda')$, *where* $\widehat{M'_0}$ *corresponds to* M'_0 *of* Λ'.

Proof. — Let $\widehat{M'_0}$ and $\widehat{M'_1}$ be respectively the members of the proper transform $\psi'(\Lambda')$ of Λ' such that $\widehat{M'_0}$ (resp. $\widehat{M'_1}$) contains $\psi'(F_\infty)$ (resp. $\psi'(\overline{A}')$). Since every member of $\psi'(\Lambda')$ is connected, $\widehat{M'_1}$ contains a connected linear chain $\psi'(\overline{A}') + E(s, c_s) + \cdots + E(1, 1)$, which contains the lower half of the whole chain. We note that $\psi'(\overline{A}')$ meets $E(s, c_s)$ in one point with multiplicity μ'_s which is different from the points of $E(s, c_s)$ where $E(s, c_s)$ meets the other components $E(i, j)$'s.

The member $\widehat{M'_0}$ contains some connected part of the linear chain $E(2, 1) + \cdots + E(s - 1, c_{s-1})$ if s is odd (resp. $E(2, 1) + \cdots + E(s, c_s - 1)$ if s is even). We claim that $\widehat{M'_0}$ contains all of this linear chain and hence the point $E(s - 1, c_{s-1}) \cap E(s, c_s)$ (resp. $E(s, c_s - 1) \cap E(s, c_s)$) is the base point of $\psi'(\Lambda')$ if s is odd (resp. if s is even). Suppose on the contrary that the rightmost component E of $\widehat{M'_0}$ is not $E(s - 1, c_{s-1})$ (resp. $E(s, c_s - 1)$)

if s is odd (resp. if s is even). Then, from the mid-stage of ψ onward when E was the last (-1) curve, the general member $\overline{T}'$ (or precisely, its proper transform) keeps meeting the component E. Namely, the process φ is branched at this stage and should constitute of the blowing-ups with centers at the intersection point of E and $\overline{T}'$ and its infinitely near points. This implies that the component $\varphi'(\overline{A}')$ in the corresponding member $\widetilde{M}_1'$ of $\varphi'(\Lambda')$ has a singular point or meets two other components in a point. This is a contradiction. Hence our claim is ascertained. Furthermore, the point $Q_1 = E(s-1, c_{s-1}) \cap E(s, c_s)$ if s is odd (resp. $Q_1 = E(s, c_s - 1) \cap E(s, c_s)$ if s is even) is a base point of the pencil $\psi'(\Lambda')$.

Now the process φ is a sequence of blowing-ups with centers Q_1 and its infinitely near points. Let $\psi_1 = \psi^{-1} \cdot \varphi : \widetilde{V} \to \widehat{V}$ be the necessary process of eliminating the base points of $\psi'(\Lambda')$. Since $Q_1 \neq \psi'(\overline{A}') \cap E(s, c_s)$, it follows that $\mu_s' = 1$ because the proper transforms of $\psi'(\overline{A}')$ and $E(s, c_s)$ in $\widetilde{M}_1'$ meet each other transversally. All other assertions of Lemma 4.3 follow from these observations. □

Now let $\psi_1^{-1}(Q_1) = \Gamma_1 + E_1 + \Delta_1$, where E_1 is the last (-1) curve and Γ_1 (resp. Δ_1) is contained in $\widetilde{M}_0'$ (resp. $\widetilde{M}_1'$). Then

$$\Delta_1 + \varphi'(\overline{A}') + \psi_1'(E(s, c_s) + \cdots + E(1,1))$$

is contracted to a smooth $\mathbb{P}^1$-fiber, and the dual graph of Δ_1 (hence Γ_1) is therefore uniquely determined. In fact, the dual graph of Δ_1 coincides with the dual graph $F_\infty + E(2,1) + \cdots + E(s-1, c_{s-1})$ if s is odd (resp. $F_\infty + E(2,1) + \cdots + E(s, c_s - 1)$ if s is even).

We shall determine the multiplicity of $\psi_1'(E(s, c_s))$ as a component of a degenerate $\mathbb{P}^1$-fiber supported by $\Delta_1 + \varphi'(\overline{A}') + \psi_1'(E(s, c_s) + \cdots + E(1,1))$. For this purpose, identify Δ_1 with $F_\infty + E(2,1) + \cdots + E(s-1, c_{s-1})$ (resp. $F_\infty + E(2,1) + \cdots + E(s, c_s - 1)$) if s is odd (resp. if s is even), and let $\mu(i,j)$ be the multiplicity of $E(i,j)$ for $1 \leqslant i \leqslant s$ and $1 \leqslant j \leqslant c_i$, where $\mu(1,1) = 1$ and the multiplicity of F_∞ is 1. Then we have the following relations:

$$\begin{aligned}
\mu(1,j) &= j, && 1 \leqslant j \leqslant c_1 \\
\mu(2,j) &= 1 + j\mu(1, c_1), && 1 \leqslant j \leqslant c_2 \\
\mu(3,j) &= \mu(1, c_1) + j\mu(2, c_2), && 1 \leqslant j \leqslant c_3 \\
&\cdots\cdots \\
\mu(t,j) &= \mu(t-2, c_{t-2}) + j\mu(t-1, c_{t-1}), && 1 \leqslant j \leqslant c_t \\
&\cdots\cdots \\
\mu(s,j) &= \mu(s-2, c_{s-2}) + j\mu(s-1, c_{s-1}), && 1 \leqslant j \leqslant c_s.
\end{aligned}$$

Thence we have

$$\frac{\mu(s,c_s)}{\mu(s-1,c_{s-1})} = c_s + \cfrac{1}{c_{s-1} + \cfrac{1}{c_{s-2} + \cfrac{1}{\ddots + \cfrac{1}{c_1}}}} = [c_s, c_{s-1}, \dots, c_1],$$

while $md/\mu' = [c_1, \dots, c_s]$. Note that $\mu'_s = 1$ implies $\gcd(md, \mu') = 1$. Then it follows that $\mu(s, c_s) = md$. Meanwhile, the multiplicity of $\varphi'(\overline{A}')$ (and hence the one of $\psi'_1(E(s,c_s))$) is m. So, we conclude that $d = 1$ and that the pair (σ, σ') is minimal. Hence we proved the following result.

THEOREM 4.4. — *Suppose that $m\mu' > \mu$. Then the pair (σ, σ') is minimal.*

Continuing the previous arguments, we shall explain the elimination process $\varphi : \widetilde{V} \to V$ of the base points of the pencil Λ' in the case $m\mu' = \mu$. Let $\varphi_1 : V_1 \to V$ be the oscillating sequence of blowing-ups with center Q and data (md, μ'). With the observations before Lemma 4.3 taken into account, the proper transform $\varphi'_1(\Lambda')$ has a base point Q_1 on the last exceptional curve $E_1 := E(s, c_s)$, which does not lie on any other components of $\varphi_1^{-1}(Q)$. Note that the following assertions hold:

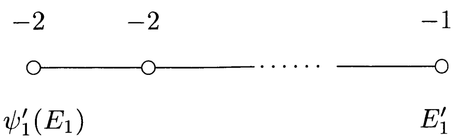
(1) Every component of $\varphi_1^{-1}(Q)$ belongs to the member $M'_0(1)$ of $\varphi'_1(\Lambda')$ which corresponds to the member M'_0 of Λ'.

(2) Write $\varphi_1^{-1}(Q) = \Gamma_1 + E_1 + \Delta_1$, where Γ_1 and Δ_1 are the connected components of $\varphi_1^{-1}(Q) - E_1$ such that $\Gamma_1 \cap \varphi'_1(F_\infty) \neq \emptyset$ and $\Delta_1 \cap \varphi'_1(F_\infty) = \emptyset$. Then $\varphi'(G + S + F_\infty) + \Gamma_1$ contracts to a smooth point.

(3) The general member $\varphi'_1(\overline{T}')$ of $\varphi'_1(\Lambda')$ satisfies

$$i(E_1, \varphi'_1(\overline{T}'); Q_1) = \operatorname{mult}_{Q_1} \varphi'_1(\overline{T}') = \mu_s = m\mu'_s.$$

Let $\psi_1 : V'_1 \to V_1$ be a sequence of blowing-ups such that $\psi^{-1}(Q_1)$ has the dual graph



where the proper transform $\Lambda_1' := (\varphi_1\psi_1)'(\Lambda')$ has a base point Q_1' lying only on the last (-1) curve E_1' and not on the other components, and where

$$m\mu_s' = i(E_1', (\varphi_1\psi_1)'(\overline{T}'); Q_1') > \mu^{(2)} := \operatorname{mult}_{Q_1'}(\varphi_1\psi_1)'(\overline{T}').$$

We note that $m(\varphi_1\psi_1)'(\overline{A}')$ is the member of Λ_1' and hence passes through the point Q_1' with

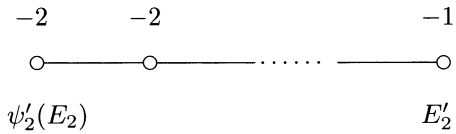
$$\mu_s' = i(E_1', (\varphi_1\psi_1)'(\overline{A}'); Q_1') \geqslant \mu'^{(2)} := \operatorname{mult}_{Q_1'}(\varphi_1\psi_1)'(\overline{A}').$$

Here $m\mu'^{(2)} \geqslant \mu^{(2)}$.

Suppose $\mu^{(2)} = m\mu'^{(2)}$. The next process is similar to the sequence φ_1 above. We let $\varphi_2 : V_2 \to V_1'$ be the oscillating sequence of blowing-ups with center Q_1' and data $(\mu_s', \mu'^{(2)})$. Let E_2 be the last (-1) curve of φ_2. Then the pencil $(\varphi_1\psi_1\varphi_2)'(\Lambda')$ has a base point Q_2 on E_2 not lying on any other components of $\varphi_2^{-1}(Q_1')$. Write $(\psi_1\varphi_2)^{-1}(Q_1) = \Gamma_2 + E_2 + \Delta_2$, where Γ_2 and Δ_2 are the connected components of $(\psi_1\varphi_2)^{-1}(Q_1) - E_2$ such that $\Gamma_2 \cap (\psi_1\varphi_2)'(E_1) \neq \emptyset$.

(4) Then $(\psi_1\varphi_2)'\,(\varphi_1'(G + S + F_\infty) + \Gamma_1 + E_1 + \Delta_1) + \Gamma_2$ contracts to a smooth point.

After a possible sequence of blowing-ups $\psi_2 : V_2' \to V_2$ like ψ_1 whose dual graph is a (-2) sequence



the proper transform $\Lambda_2' := (\varphi_2\psi_2)'(\Lambda_1')$ has a base point Q_2' lying only on the last (-1) curve E_2' and not lying on the other components. Furthermore,

$$i(E_2', (\varphi_1\psi_1\varphi_2\psi_2)'(\overline{T}'); Q_2') > \mu^{(3)} = \operatorname{mult}_{Q_2'}((\varphi_1\psi_1\varphi_2\psi_2)'(\overline{T}')).$$

We note that $m(\varphi_1\psi_1\varphi_2\psi_2)'(\overline{A}')$ is the member of Λ_2' and passes through the point Q_2' with

$$i(E_2', (\varphi_1\psi_1\varphi_2\psi_2)'(\overline{A}'); Q_2') \geqslant \mu'^{(3)} = \operatorname{mult}_{Q_2'}((\varphi_1\psi_1\varphi_2\psi_2)'(\overline{A}')),$$

where $m\mu'^{(3)} \geqslant \mu^{(3)}$.

After this process repeated several times, we reach to the t-th stage where $m\mu'^{(t)} > \mu^{(t)}$. As in Lemma 4.1, it then follows that $m\mu'^{(t)} - \mu^{(t)} = 1$.

As in the proof of Lemma 4.3 and the subsequent arguments, the oscillating sequence of blowing-ups with center Q'_{t-1} and data $(i(E'_{t-1}, \widehat{T}'; Q'_{t-1}), \mu^{(t)})$ eliminates the base points of the pencil Λ'_{t-1}, where $\widehat{T}'$ is the proper transform of $\overline{T}'$. Hence $V_t = \widetilde{V}$. Let E_t be the last (-1) curve of φ_t and write $(\psi_{t-1}\varphi_t)'(Q'_{t-1}) = \Gamma_t + E_t + \Delta_t$ as above, where Γ_t is connected to the proper transform of F_∞. Then we have:

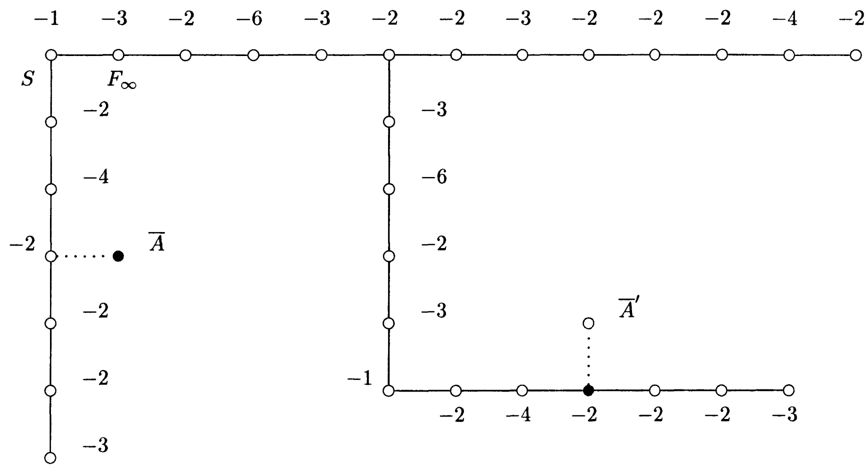
(5) All the components lying on the left side of E_t, i.e., the connected component containing Γ_t and the proper transform of $G + S + F_\infty$ contract to a smooth $\mathbb{P}^1$-fiber.

(6) Δ_t together with the proper transform of $\overline{A}'$ contracts to a smooth $\mathbb{P}^1$-fiber. In fact, the component of Δ_t where $\overline{A}'$ meets is the proper transform of the (-1) curve which appears as the last exceptional curve of the oscillating sequence of blowing-ups with center Q'_{t-1} and data $(i(E'_{t-1}, \widehat{A}'; Q'_{t-1}), \mu'^{(t)})$, where $\widehat{A}'$ is the proper transform of $\overline{A}'$ on V'_{t-1}.

(7) The same argument as the one leading to Theorem 4.4 shows that $(i(E'_{t-1}, \widehat{A}'; Q'_{t-1}), \mu'^{(t)}) = m$.

We do not know if such a pencil Λ' exists as satisfying all the above conditions. But the following example shows that the dual graph of exceptional curves of $\varphi : \widetilde{V} \to V$ together with the proper transform of $G + S + F_\infty$ is realizable.

Example 4.5. — Let $m = 7, d = 76, \mu' = 31, \mu = m\mu', s = 5, \mu'_s = 7, t = 1, \mu^{(1)} = 27, \mu'^{(1)} = 4$. The dual graph is given as follows:



5. Étale endomorphisms of $\mathbb{Q}$-homology planes.

In [6], the generalized Jacobian conjecture for $\mathbb{Q}$-homology planes is considered. It is shown that any étale endomorphism of a $\mathbb{Q}$-homology plane X is an automorphism if one of the following conditions is satisfied:

(1) $\overline{\kappa}(X) = 2$ or 1.

(2) $\overline{\kappa}(X) = -\infty$ and X has an $\mathbb{A}^1$-fibration $\rho : X \to B$ with at least two multiple fibers.

In this section, we rectify some of the arguments in [6]. We recall the following two lemmas (cf. [6, Lemma 6.1] and [6, 11, Lemma 3.1]).

LEMMA 5.1.— *Let $\rho : X \to B$ be an $\mathbb{A}^1$-fibration on a $\mathbb{Q}$-homology plane. Suppose that ρ has at least two singular fibers. Let $g : \mathbb{A}^1 \to X$ be a non-constant morphism. Then the image of g is a fiber of ρ.*

LEMMA 5.2.— *For $i = 1, 2$, let $\rho_i : X_i \to B_i$ be $\mathbb{A}^1$-fibrations on $\mathbb{Q}$-homology planes. Let $\phi : X_1 \to X_2$ and $\beta : B_1 \to B_2$ be dominant morphisms such that $\rho_2 \cdot \phi = \beta \cdot \rho_1$. Let $m\Gamma$ be an irreducible fiber of ρ_2 lying over a point $p \in B_2$ with $m \geqslant 1$ and Γ reduced, and let $q \in B_1$ be a point such that $\beta(q) = p$. Suppose $\rho_1^*(q) = \ell\Delta$, where Δ is reduced and irreducible and ℓ is its multiplicity. Suppose furthermore that ϕ is an étale morphism. If the ramification index of β at q is e then $\ell e = m$. In particular, if $m = 1$ then $\ell = e = 1$.*

Applying these lemmas, we shall show the following result.

LEMMA 5.3.— *Let X be a $\mathbb{Q}$-homology plane with an $\mathbb{A}^1$-fibration $\rho : X \to B$. Let $m_1A_1, \ldots, m_nA_n$ exhaust all multiple fibers of ρ. Let $\phi : X \to X$ be an étale endomorphism. Then the following assertions hold:*

(1) If $n \geqslant 2$, then there exists an endomorphism β of B such that $\rho \cdot \phi = \beta \cdot \rho$.

(2) The above endomorphism β is an automorphism provided $n \geqslant 3$ or $n = 2$ and $\{m_1, m_2\} \neq \{2, 2\}$.

Proof.— The first assertion is an immediate consequence of Lemma 5.1. So, we consider the second assertion. We employ the arguments in [9, Lemmas 3.1 and 3.3]. Note that $\beta : B \to B$ is a finite morphism because B is the affine line. By Lemma 5.2, the set $\{p_1, \ldots, p_n\}$ is mapped to itself by β, where $p_i = \rho(A_i)$. Suppose, furthermore, that the points $q_1, \ldots, q_s$, none of which belongs to $\{p_1, \dots, p_n\}$, are mapped to $\{p_1, \dots, p_n\}$. Then, by Lemma 5.2, the ramification index of β at q_j, say e_j, is larger than 1. In fact, if $\beta(q_j) = p_i$ then $e_j = m_i$.

Since β induces an étale finite morphism

$$\beta : B - \{p_1, \dots, p_n, q_1, \dots, q_s\} \longrightarrow B - \{p_1, \dots, p_n\},$$

the comparison of the Euler numbers gives rise to an equality

$$1 - (n + s) = d(1 - n), \tag{1}$$

where $d = \deg \beta$. On the other hand, by summing up the ramification indices, we have an inequality

$$2s + n \leqslant dn. \tag{2}$$

So, by combining (1) and (2) together, we have an inequality

$$2(d - 1)(n - 1) = 2s \leqslant (d - 1)n. \tag{3}$$

Suppose $d > 1$. Then $n \leqslant 2$. Hence, if $n \geqslant 3$ then $d = 1$ and β is an automorphism. Suppose that $d > 1$ and $n = 2$. Then the equality occurs in (3), and hence the equality occurs in (2). Namely, the ramification index e_j at q_j is two for all j, and $s = d - 1$. Since $d > 1$ implies $s > 0$, we may assume that q_1 is mapped to p_1. Then $m_1 = 2$. Suppose $d \geqslant 3$. Then $2s = 2(d - 1) > d$. Hence one of the q_j is mapped to $p_2, \dots, p_n$, say p_2. Hence $m_2 = 2$. In this case, after a suitable change of indices, one of the following two cases is possible:

(1) $s = s_1 + s_2 = d - 1$, and $q_1, \dots, q_{s_1}, p_1$ (or p_2) (resp. $q_{s_1+1}, \dots, q_s, p_2$ (or p_1)) are mapped to p_1 (resp. p_2).

(2) $s = s_1 + s_2, d = 2s_1 = 2s_2 + 2$, and $q_1, \dots, q_{s_1}$ (resp. $q_{s_1+1}, \dots, q_s, p_1$, p_2) are mapped to p_1 (resp. p_2).

Finally, suppose that $d = n = 2$ and $s = 1$. Then we may assume that $\beta(q_1) = p_1$ and $\beta(p_1) = \beta(p_2) = p_2$. Then $m_2 = 2$ as well by Lemma 4.2. So, if $\{m_1, m_2\} \neq \{2, 2\}$, then $d = 1$ and β is an automorphism. □

As a consequence of Lemma 5.3, we can prove the following result, which rectifies Theorem 6.1 in [6].

Theorem 5.4. — *Let X be a $\mathbb{Q}$-homology plane with an $\mathbb{A}^1$-fibration $\rho : X \to B$. Let $m_1 A_1, \dots, m_n A_n$ exhaust all multiple fibers of ρ. Suppose* *that either $n \geqslant 3$ or $n = 2$ and $\{m_1, m_2\} \neq \{2, 2\}$. Then any étale endomorphism $\phi : X \to X$ is an automorphism.*

Proof. — By Lemma 5.3, there exists an automorphism β of B such that $\rho \cdot \phi = \beta \cdot \rho$. Since β is an automorphism, Lemma 5.2 implies that β induces a permutation of the finite set $\{p_1, \ldots, p_n\}$. By replacing β by its suitable iteration β^r, we may assume that β induces the identity on $\{p_1, \ldots, p_n\}$. Since $n \geqslant 2$ and β (or rather an induced automorphism of the smooth compactification $\overline{B}$ of B) fixes the point at infinity p_∞. Hence β is then the identity automorphism.

Let $K = k(B)$ be the function field of B and let X_K be the generic fiber of ρ. Then X_K is isomorphic to the affine line over K, and ϕ induces an étale endomorphism ϕ_K of X_K. Since ϕ_K is then finite, ϕ_K is an automorphism. Hence ϕ is birational. Then Zariski's Main Theorem implies that ϕ is an open immersion. Note that $\mathrm{Pic}\,(X)_{\mathbb{Q}} = 0$ and $\Gamma(\mathcal{O}_X)^* = \mathbb{C}^*$. Suppose that $X \neq \phi(X)$. Then $X - \phi(X)$ has pure codimension one. Since $\mathrm{Pic}\,(X)_{\mathbb{Q}} = 0$, there exists a regular function h on X such that the zero locus $(h)_0$ of h is supported by $X - \phi(X)$. Then $\phi^*(h)$ is a non-constant invertible function on X, which contradicts the property $\Gamma(\mathcal{O}_X)^* = \mathbb{C}^*$. So, ϕ is an automorphism. □

In the case $\{m_1, m_2\} = \{2, 2\}, d = n = 2$ and $s = 1$, there exists the following counter-example to the generalized Jacobian conjecture.

Example 5.5. — Let $V_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Let M_0 be a cross-section and let $\ell_0, \ell_1, \ell_\infty$ be distinct three fibers with respect to the second projection $\pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. Let $\varphi : V \to V_0$ be q sequence of blowing-ups with centers at $\ell_0 \cap M_0, \ell_1 \cap M_0$ and their infinitely near points such that $\varphi^*(\ell_0) = \ell_0' + E_1 + 2E_2 + 2E_3$ and $\varphi^*(\ell_1) = \ell_1' + F_1 + 2F_2 + 2F_3$, where $({\ell_0'}^2) = ({\ell_1'}^2) = ({E_i}^2) = ({F_i}^2) = -2$ for $i = 1, 2$ and $({E_3}^2) = ({F_3}^2) = -1$. Let

$$X := V - (\ell_\infty + M_0' + \ell_0' + \ell_1' + E_1' + F_1' + E_2' + F_2').$$

Hence X has an $\mathbb{A}^1$-fibration $\rho : X \to B$ with two multiple fibers $2E_3 \cap X, 2F_3 \cap X$ of multiplicity 2. Then X has a degree two, non-finite étale endomorphism.

In fact, let $\sigma : B' \to B$ be a degree two covering ramifying over the point at infinity p_∞ and p_0, where $p_0 = \rho(E_3 \cap X)$. Let $\widetilde{X}$ be the normalization of $X \times_B B'$, let $\tau : \widetilde{X} \to X$ be the composite of the normalization morphism and the first projection $X \times_B B' \to X$ and let

$\widetilde{\rho} : \widetilde{X} \to B'$ be the $\mathbb{A}^1$-fibration induced naturally by ρ. Then $\widetilde{\rho}^*(q_0)$ is a disjoint sum $G_1 + G_2$ of two affine lines and $\tau : \widetilde{X} \to X$ is a finite étale morphism, where q_0 is a point of B' lying over p_0. Then $\widetilde{X} - G_1 \cong \widetilde{X} - G_2 \cong X$, and $\tau|_{\widetilde{X}-G_1}$ and $\tau|_{\widetilde{X}-G_2}$ induce a non-finite étale endomorphism of X.

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Kayo MASUDA,
Himeji Institute of Technology
Mathematical Science II
2167 Shosha, Himeji 671-2201 (Japan).
kayo@sci.himeji-tech.ac.jp

Masayoshi MIYANISHI,
Osaka University
Department of Mathematics
Graduate School of Science
1-1 Machikaneyama-cho, Toyonaka 560 (Japan).
miyanisi@math.sci.osaka-u.ac.jp