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MODULI SPACES OF DECOMPOSABLE MORPHISMS OF SHEAVES AND QUOTIENTS BY NON-REDUCTIVE GROUPS

by J.-M. DRÉZET and G. TRAUTMANN

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1. Introduction.

Let X be a projective algebraic variety over the field of complex numbers. Given two coherent sheaves $\mathcal{E}, \mathcal{F}$ on X the algebraic group $G = \mathrm{Aut}(\mathcal{E}) \times \mathrm{Aut}(\mathcal{F})$ acts naturally on the affine space $W = \mathrm{Hom}(\mathcal{E}, \mathcal{F})$ by $(g, h) \cdot w = h \circ w \circ g^{-1}$. If two morphisms are in the same G-orbit then they have isomorphic cokernels and kernels. Therefore it is natural to ask for good quotients of such actions in the sense of geometric invariant theory.

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1.1. Morphisms of type (r,s).

In general $\mathcal{E}$ and $\mathcal{F}$ will be decomposable such that G is not reductive. More specifically let $\mathcal{E}$ and $\mathcal{F}$ be direct sums

$$\mathcal{E} = \bigoplus_{1\le i\le r} M_i \otimes \mathcal{E}_i \quad \text{and} \quad \mathcal{F} = \bigoplus_{1\le \ell\le s} N_\ell \otimes \mathcal{F}_\ell,$$

where M_i and N_ℓ are finite dimensional vector spaces and $\mathcal{E}_i$, $\mathcal{F}_\ell$ are simple sheaves, i.e. their only endomorphisms are the homotheties, and such that $\mathrm{Hom}(\mathcal{E}_i,\mathcal{E}_j) = 0$ for $i > j$ and $\mathrm{Hom}(\mathcal{F}_\ell,\mathcal{F}_m) = 0$ for $\ell > m$. In this case we call homomorphisms $\mathcal{E} \to \mathcal{F}$ of type (r,s). Then the groups $\mathrm{Aut}(\mathcal{E})$ and $\mathrm{Aut}(\mathcal{F})$ can be viewed as groups of matrices of the following type. The group $\mathrm{Aut}(\mathcal{E})$, say, is the group of matrices

$$\begin{pmatrix} g_1 & 0 & \cdots & 0 \\ u_{2,1} & g_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ u_{r,1} & \cdots & u_{r,r-1} & g_r \end{pmatrix}$$

where $g_i \in \mathrm{GL}(M_i)$ and $u_{j,i} \in \mathrm{Hom}(M_i, M_j \otimes \mathrm{Hom}(\mathcal{E}_\imath,\mathcal{E}_j))$.

In the literature on moduli of vector bundles and coherent sheaves many quotients of spaces $\mathbb{P}\,\mathrm{Hom}(\mathcal{E},\mathcal{F})$ of type $(1,1)$ by the reductive group $\mathrm{Aut}(\mathcal{E}) \times \mathrm{Aut}(\mathcal{F})$ have been investigated, see for example [6], [14], [15], [20], [26]. The moduli spaces described in this way are the simplest ones, and this allows to test in these cases some conjectures that are expected to be true on more general moduli spaces of sheaves (cf. [7], [36]). We think that the moduli spaces of morphisms of type (r,s) will be as useful to treat other less simple moduli problems of sheaves. In fact, if one wants to use the spaces $\mathrm{Hom}(\mathcal{E},\mathcal{F})$ as parameter spaces for moduli spaces of sheaves, which are as close as possible to the moduli spaces, the higher types (r,s) are unavoidable.

The homomorphisms in a Beilinson complex of a bundle on projective n-space, for example, have in general arbitrary type (r,s) depending on the dimensions of the cohomology spaces of the bundle. In several papers, see [25], [30] for example, semi-stable sheaves or ideal sheaves of subschemes of projective spaces, are represented as quotients of injective morphisms of type (r,s), and one should expect that the moduli spaces of such sheaves are isomorphic to a good quotient of an open subset of the corresponding space of homomorphisms. In some cases of type $(2,1)$ this has been verified for semi-stable sheaves on $\mathbb{P}_2$ in [8].

In case of type (r, s) there are good and projective quotients if one restricts the action to the reductive subgroup

$$G_{\mathrm{red}} = \prod \mathrm{GL}(M_i) \times \prod \mathrm{GL}(N_\ell).$$

This has been shown recently by A. King in [21]. The quotient problem for $\mathrm{Hom}(\mathcal{E}, \mathcal{F})$ of type (r, s) with respect to the full group $\mathrm{Aut}(\mathcal{E}) \times \mathrm{Aut}(\mathcal{F})$ is however the generic one and indispensable.

Unfortunately the by now standard geometric invariant theory (GIT) doesn't provide a direct answer for these quotient problems in case $\mathrm{Aut}(\mathcal{E}) \times \mathrm{Aut}(\mathcal{F})$ is not reductive. There are several papers dealing with the action of an arbitrary algebraic group like [16], [17], [3], [4] and older ones, but their results are insufficient for the above problem. The conditions of [16] are close to what we need, but they don't allow a concrete description of the set of semi-stable points in our case and they don't guarantee good or projective quotients, see Remark 4.1.2.

1.2. The main idea.

Our procedure is very close to standard GIT and we finally reduce the problem of the quotient to the one of a reductive group action. We introduce polarizations $\Lambda \in \mathbb{Q}^{r+s}$ of tuples of rational numbers for the action of G on the affine space in analogy to the ones of A. King in [21], which are refinements of the polarizations by ample line bundles on the projective space $\mathbb{P}W$, and then introduce open sets $W^s(G, \Lambda) \subset W^{ss}(G, \Lambda)$ of stable and semi-stable points depending on Λ and study the quotient problem for these open subsets. There are chambers in $\mathbb{Q}^{r+s}$ such that the polarizations in one chamber define the same open set, in accordance with the chamber structure in Neron-Severi spaces of polarizations in the reductive case, see for example I. Dolgachev-Y. Hu [5] and M. Thaddeus [35]. However, in contrast to the reductive case, good quotients $W^{ss}(G, \Lambda)//G$ don't exist for all polarizations, see 4.2. As a main achievement we are providing numerical conditions on the polarizations, depending on the dimensions of the spaces M_i and N_ℓ, under which such quotients exist. The main step for that is to embed the group actions $G \times W \to W$ into an action $\mathbf{G} \times \mathbf{W} \to \mathbf{W}$ of a reductive group $\mathbf{G}$ and to compare the open sets $W^{ss}(G, \Lambda)$ and $\mathbf{W}^{ss}(\mathbf{G}, \widetilde{\Lambda})$, where $\widetilde{\Lambda}$ is a polarization for the $\mathbf{G}$-action associated to Λ.

1.3. Construction of quotients by non reductive groups.

To be more precise, a polarization Λ is a tuple $(\lambda_1, \ldots, \lambda_r, \mu_1, \ldots, \mu_s)$

of positive rational numbers, called weights of the factors $M_i \otimes \mathcal{E}_i$ and $N_\ell \otimes \mathcal{F}_\ell$ respectively, which satisfy $\sum \lambda_i m_i = \sum \mu_\ell n_\ell = 1$, where m_i, n_ℓ denote the dimensions of the spaces of the same name. We use then the numerical criterion of A. King [21], as definition for semi-stability with respect to the reductive group G_{red}. An element $w \in W$ is $(G_{\mathrm{red}}, \Lambda)$-stable if for any proper choice of subspaces $M_i' \subset M_i$, $N_\ell' \subset N_\ell$ such that w maps $\bigoplus(M_i' \otimes \mathcal{E}_i)$ into $\bigoplus(N_\ell' \otimes \mathcal{F}_\ell)$, we have $\sum \lambda_i m_i' < \sum \mu_\ell n_\ell'$, or semi-stable if equality is allowed. Let $W^s(G_{\mathrm{red}}, \Lambda) \subset W^{ss}(G_{\mathrm{red}}, \Lambda)$ denote the set of stable and semi-stable points so defined. If $H \subset G$ is the unipotent radical of G, which is generated by the homomorphisms $\mathcal{E}_i \to \mathcal{E}_j$ and $\mathcal{F}_\ell \to \mathcal{F}_m$ for $i < j$ and $\ell < m$, we say that w is (G, Λ)-(semi-)stable if $h \cdot w$ is $(G_{\mathrm{red}}, \Lambda)$-(semi-)stable for any $h \in H$, see 4.1. We thus have open subsets $W^{ss}(G, \Lambda) \subset W^{ss}(G_{\mathrm{red}}, \Lambda)$ and $W^s(G, \Lambda) \subset W^{ss}(G_{\mathrm{red}}, \Lambda)$.

The main result of our paper is that there are sufficient numerical and effective bounds for the polarizations Λ such that $W^{ss}(G, \Lambda)$ admits a good and even projective quotient $W^{ss}(G, \Lambda)//G$ and that in addition $W^s(G, \Lambda)$ admits a geometric quotient, which is smooth and quasi-projective, see Proposition 6.1.1 and the results 7.2.2, 7.5.3, and Section 8.

The definitions of good and geometric quotients are recalled in 6.1. By using correspondences between spaces of morphisms, called *mutations*, it is possible to deduce from our results other polarizations such that there exist a good projective quotient (see [10], [12]).

All this is achieved by embedding the action $G \times W \to W$ into an action $\mathbf{G} \times \mathbf{W} \to \mathbf{W}$ of a reductive group and then imposing conditions for the equality $W^{ss}(G, \Lambda) = W \cap \mathbf{W}^{ss}(\mathbf{G}, \widetilde{\Lambda})$, where $\widetilde{\Lambda}$ is the associated polarization. The quotient is then the quotient of the saturated subvariety $\mathbf{G}W^{ss}(G, \Lambda) \subset \mathbf{W}^{ss}(\mathbf{G}, \widetilde{\Lambda})$. The quotient will be projective if $\overline{\mathbf{G} \cdot W} \setminus \mathbf{G} \cdot W$ doesn't meet $\mathbf{W}^{ss}(\mathbf{G}, \widetilde{\Lambda})$. Also for this, numerical conditions can be found in Section 8.

The idea of embedding the non-reductive action $G \times W \to W$ into the action $\mathbf{G} \times \mathbf{W} \to \mathbf{W}$ is simply to replace the $\mathcal{E}_i$ by $\mathcal{E}_1$ using the evaluation maps $\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_i) \otimes \mathcal{E}_1 \to \mathcal{E}_i$. It is explained in 5.1 and 5.1.1 that this is the outcome when we start to replace the sheaves $\mathcal{E}_i$ step by step and similarly for the sheaves $\mathcal{F}_\ell$. Since we have to deal everywhere with the dimensions of the vector spaces $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j)$ and $\mathrm{Hom}(\mathcal{F}_\ell, \mathcal{F}_m)$ which form the components of the unipotent group H, we have translated the whole setup into an abstract multilinear setting and related actions by technical reasons. This gives more general results although we have only applications in the theory of sheaves. The reader should always keep in mind the motivation in 5.1.

The results obtained in the simplest case (morphisms of type (2,1) or (1,2)) are stated in 1.5. They are characteristic for the general case in which only the conditions are more complicated.

1.4. Remark on finite generatedness.

One would expect that the quotients of W could be obtained by first forming the quotient W/H with respect to the unipotent radical H and then in a second step a quotient of W/H by $G/H \cong G_{\text{red}}$. However, the actions of unipotent groups behave generally very badly, [19], and we are not able to prove that the algebra $\mathbb{C}[W/H]$ is finitely generated. This would be an essential step in a direct construction of the quotient. Of course, the main difficulty also in this paper arises from the presence of the group H. The counterexample of M. Nagata [28] also shows that the finite generatedness depends on the dimensions of the problem. So from a philosophical point of view we are determining bounds for the dimensions involved under which we can expect local affine G-invariant coordinate rings which are finitely generated, and thus to obtain good quasi-projective quotients, even so the bounds might not be the best. The simple examples 4.2, 4.3 show that a good quotient $W^{ss}(G,\Lambda)//G$ might not exist if the conditions are not fulfilled.

1.5. Morphisms of type (2,1).

In this case the homomorphisms of sheaves are of the type

$$m_1\mathcal{E}_1 \oplus m_2\mathcal{E}_2 \longrightarrow n_1\mathcal{F}_1,$$

where we use the notation $m\mathcal{E}$ for $\mathbb{C}^m \otimes \mathcal{E}$. For this type a polarization is given by a pair (λ_1, λ_2) of positive rational numbers such that $\lambda_1 m_1 + \lambda_2 m_2 = 1$. It is determined by the rational number $t = m_2\lambda_2$ which lies in $[0,1]$. Writing $W^{ss}(t)$ for W^{ss} and $W^s(t)$ for W^s for the moment, our results depend on constants $c(k)$ defined as follows: Let

$$\tau\colon \operatorname{Hom}(\mathcal{E}_1,\mathcal{F}_1)^* \otimes \operatorname{Hom}(\mathcal{E}_1,\mathcal{E}_2) \longrightarrow \operatorname{Hom}(\mathcal{E}_2,\mathcal{F}_1)^*$$

be the linear map induced by the composition map

$$\operatorname{Hom}(\mathcal{E}_2,\mathcal{F}_1) \otimes \operatorname{Hom}(\mathcal{E}_1,\mathcal{E}_2) \longrightarrow \operatorname{Hom}(\mathcal{E}_1,\mathcal{F}_1), \quad \text{and}$$

$$\tau_k = \tau \otimes I_{\mathbb{C}^k} : \operatorname{Hom}(\mathcal{E}_1,\mathcal{F}_1)^* \otimes (\operatorname{Hom}(\mathcal{E}_1,\mathcal{E}_2) \otimes \mathbb{C}^k) \longrightarrow \operatorname{Hom}(\mathcal{E}_2,\mathcal{F}_1)^* \otimes \mathbb{C}^k.$$

Let $\mathcal{K}$ be the set of proper linear subspaces $K \subset \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2) \otimes \mathbb{C}^k$ such that for every proper linear subspace $F \subset \mathbb{C}^k$, K is not contained in $\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2) \otimes F$. Let

$$c(k) = \sup_{K \in \mathcal{K}} \Big(\frac{\mathrm{codim}(\tau_k(\mathrm{Hom}(\mathcal{E}_1, \mathcal{F}_1)^* \otimes K)}{\mathrm{codim}(K)} \Big).$$

1.5.1. THEOREM. — *There exist a good projective quotient $W^{ss}(t)//G$ and a geometric quotient $W^s(t)/G$ if*

$$t > \frac{m_2 \dim(\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2))}{\dim(\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2)) + m_1} \quad \text{and} \quad t > \dim\big(\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2)\big) \frac{c(m_2) m_2}{n_1} \cdot$$

In the case of morphisms $m_1\mathcal{O}(-2) \oplus m_2\mathcal{O}(-1) \to n_1\mathcal{O}$ on projective spaces the constants have been computed in [12] and we obtain the more explicit result:

1.5.2. THEOREM. — *Let $n \geq 2$ be an integer. There exist a good projective quotient $W^{ss}(t)//G$ and a geometric quotient $W^s(t)/G$ in the case of morphisms $m_1\mathcal{O}(-2) \oplus m_2\mathcal{O}(-1) \to n_1\mathcal{O}$ on the projective space $\mathbb{P}_n$ if*

$$\begin{aligned} t &> \frac{(n+1)m_2}{(n+1)m_2 + m_1}, \\ t &> \frac{(n+1)m_2^2(m_2-1)}{2n_1(m_2(n+1)-1)} && \textit{if} \quad 2 \leq m_2 \leq n+1, \\ t &> \frac{(n+1)^2 m_2}{2(n+2)n_1} && \textit{if} \quad m_2 > n+1. \end{aligned}$$

1.6. Construction of fine moduli spaces of torsion free sheaves.

In Section 10 we construct smooth projective fine moduli spaces of torsion free coherent sheaves on $\mathbb{P}_n$ using morphisms

$$(*) \qquad \mathcal{O}(-2) \otimes \mathbb{C}^2 \longrightarrow \mathcal{O}(-1) \oplus (\mathcal{O} \otimes \mathbb{C}^k),$$

(for $\frac{1}{2}(n+1)(n+2) < k < (n+1)^2$). More precisely we prove that for all polarizations, semi-stable morphisms are injective outside a closed subvariety of codimension ≥ 2, hence their cokernels are torsion free sheaves. A generic morphism is injective and its cokernel is locally free. In this case we can construct

$$q = \frac{(n+1)(n+2)}{2} - \Big\lceil \frac{n+k+1}{2} \Big\rceil$$

distinct smooth projective moduli spaces $\mathbb{M}_1, \ldots, \mathbb{M}_q$ of such morphisms, of dimension $2(n-1)+k((n+1)^2-k)$. Moreover, all the $\mathbb{M}_i$ are birational to each other. For $1 \leq i \leq q$, we construct a coherent sheaf $\mathcal{E}_i$ on $\mathbb{M}_i \times \mathbb{P}_n$, flat over $\mathbb{M}_i$, such that for every closed point $z \in \mathbb{M}_i$, $\mathcal{E}_{iz}$ is isomorphic to the cokernel of the morphism (*) corresponding to z. We prove that $\mathbb{M}_i$ is a fine moduli space of torsion free sheaves with universal sheaf $\mathcal{E}_i$. In particular, this means that for every closed point $z \in \mathbb{M}_i$, the Kodaïra-Spencer map

$$T_z\mathbb{M}_i \longrightarrow \operatorname{Ext}^1(\mathcal{E}_{iz}, \mathcal{E}_{iz})$$

is bijective, and for any two distinct closed points $z_1, z_2 \in \mathbb{M}_i$, the sheaves $\mathcal{E}_{iz_1}$, $\mathcal{E}_{iz_2}$ are not isomorphic.

1.7. Open problems.

Even in the simplest case of morphisms of type (2,1) we do not know what all the polarizations are for which a good quotient $W^{ss}//G$ exists. More generally it would be interesting to find all the saturated open subsets U of W such that a good quotient (quasiprojective or not) $U//G$ exists, or all the open subsets U such that a geometric quotient U/G exists. The corresponding problem for reductive groups has been studied in [27], 1.12, 1.13, and in [1], [2].

1.8. Organization of the paper.

In Section 2 we describe our problem in terms of multilinear algebra.

In Section 3 we recall results of A. King [21]. The reductive group actions considered in this paper, the action of G_{red} on W and that of $\mathbf{G}$ on $\mathbf{W}$, are particular cases of [21]. We also discuss the relation of Λ-(semi)-stability in W with that in the projective space $\mathbb{P}W$. But we cannot work solely on the projective niveau, because the embedding $W \subset \mathbf{W}$ is not linear.

After defining G-(semi-)stability for the non-reductive group in Section 4 we describe the embedding in Section 5 and introduce the associated polarizations. Section 6 contains the step of constructing the quotient $W^{ss}(G,\Lambda)//G$ using the GIT-quotient $\mathbf{W}^{ss}(\mathbf{G},\widetilde{\Lambda})//\mathbf{G}$ of A. King.

Sections 7 and 8 are the hard parts of the paper. Here the conditions of the weights which define good polarizations are derived. It seems that the constants appearing in these estimates had not been considered before.

In Section 9 we are investigating a few examples in order to test the strength of the bounds. Here we restrict ourselves to small type

$(2,1),(2,2),(3,1)$ in order to avoid long computations of the constants which give the bounds for the polarizations. What we discover in varying the polarizations are flips between the moduli spaces, as one has to expect from the general results on the variation of linearizations of group actions, cf. [32], [5], [35]. In Example 9.2 we have a very simple effect of a flip, but in Example 9.5 the chambers of the polarizations look already very complicated.

In Section 10 we define new fine moduli spaces of torsion free sheaves using our moduli spaces of morphisms.

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2. The moduli problem for decomposable homomorphisms.

Let $\mathcal{E} = \bigoplus \mathcal{E}_i \otimes M_i$ and $\mathcal{F} = \bigoplus \mathcal{F}_\ell \otimes N_\ell$ be semi-simple sheaves as in the introduction. In order to describe the action of $G = \mathrm{Aut}(\mathcal{E}) \times \mathrm{Aut}(\mathcal{F})$ on $W = \mathrm{Hom}(\mathcal{E}, \mathcal{F})$ in greater detail we use the abbreviations

$$H_{\ell i} = \mathrm{Hom}(\mathcal{E}_i, \mathcal{F}_\ell), \quad A_{ji} = \mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j), \quad B_{m\ell} = \mathrm{Hom}(\mathcal{F}_\ell, \mathcal{F}_m),$$

such that we are given the natural pairings

$$\begin{aligned}
H_{\ell j} \otimes A_{ji} &\longrightarrow H_{\ell i} && \text{for } i \leq j,\\
A_{kj} \otimes A_{ji} &\longrightarrow A_{ki} && \text{for } i \leq j \leq k,\\
B_{m\ell} \otimes H_{\ell i} &\longrightarrow H_{mi} && \text{for } \ell \leq m,\\
B_{nm} \otimes B_{m\ell} &\longrightarrow B_{n\ell} && \text{for } \ell \leq m \leq n.
\end{aligned}$$

The group G consists now of pairs (g, h) of matrices

$$g = \begin{pmatrix} g_1 & 0 & \cdots & 0 \\ u_{2,1} & g_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ u_{r,1} & \cdots & u_{r,r-1} & g_r \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} h_1 & 0 & \cdots & 0 \\ v_{2,1} & h_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ v_{s,1} & \cdots & v_{s,s-1} & h_s \end{pmatrix}$$

with diagonal elements $g_i \in \mathrm{GL}(M_i), h_\ell \in \mathrm{GL}(N_\ell)$ and

$$u_{j,i} \in \mathrm{Hom}(M_i, M_j \otimes A_{ji}), \quad v_{m,\ell} \in \mathrm{Hom}(N_\ell, N_m \otimes B_{m\ell}).$$

Similarly a homomorphism $w \in \mathrm{Hom}(\mathcal{E}, \mathcal{F})$ is represented by a matrix $w = (\varphi_{\ell i})$ of homomorphisms

$$\varphi_{\ell i} \in \mathrm{Hom}(M_i, N_\ell \otimes H_{\ell i}) = \mathrm{Hom}(H^*_{\ell i} \otimes M_i, N_\ell).$$

Using the natural pairings, the left action $(g,h).w = hwg^{-1}$ of G on W is described by the matrix product

$$\begin{pmatrix} h_1 & 0 & \cdots & 0 \\ v_{2,1} & h_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ v_{s,1} & \cdots & v_{s,s-1} & h_s \end{pmatrix} \circ \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1r} \\ \vdots & & \vdots \\ \varphi_{s1} & \cdots & \varphi_{sr} \end{pmatrix} \circ \begin{pmatrix} g_1 & 0 & \cdots & 0 \\ u_{2,1} & g_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ u_{r,1} & \cdots & u_{r,r-1} & g_r \end{pmatrix}^{-1}$$

where the compositions $v_{m,\ell} \circ \varphi_{\ell i}$ and $\varphi_{\ell j} \circ u_{j,i}$ are compositions as sheaf homomorphisms but can also be interpreted as compositions induced by the pairings of the vector spaces above. Thus the group G, the space W and the action are already determined by the vector spaces A_{ji}, $B_{m\ell}$, $H_{\ell i}$ and the pairings between them. Therefore, in the following we define G, W and the actions $G \times W \to W$ by abstractly given vector spaces and pairings. The resulting statements can then be applied to systems of sheaves by specifying the spaces as spaces of homomorphisms as above.

2.1. The abstract setting.

Let r, s be positive integers and let for $1 \le i \le j \le r$, $1 \le \ell \le m \le s$ finite dimensional vector spaces $A_{ji}, B_{m\ell}, H_{\ell i}$ be given, where we assume that $A_{ii} = \mathbb{C}$ and $B_{\ell\ell} = \mathbb{C}$. Moreover we suppose that we are given linear maps, called *compositions*,

$$\begin{aligned} H_{\ell j} \otimes A_{ji} &\longrightarrow H_{\ell i} && \text{for } 1 \le i \le j \le r,\ 1 \le \ell \le s, \\ A_{kj} \otimes A_{ji} &\longrightarrow A_{ki} && \text{for } 1 \le i \le j \le k \le r, \\ B_{m\ell} \otimes H_{\ell i} &\longrightarrow H_{mi} && \text{for } 1 \le i \le r,\ 1 \le \ell \le m \le s, \\ B_{nm} \otimes B_{m\ell} &\longrightarrow B_{n\ell} && \text{for } 1 \le \ell \le m \le n \le s. \end{aligned}$$

We assume that all these maps and the induced maps

$$H^*_{\ell i} \otimes A_{ji} \longrightarrow H^*_{\ell j} \quad \text{and} \quad H^*_{mi} \otimes B_{m\ell} \longrightarrow H^*_{\ell i}$$

are ***surjective***. This is the case when all the spaces are spaces of sheaf homomorphisms as above for which the sheaves $\mathcal{E}_i$ and $\mathcal{F}_\ell$ are line bundles on a projective space or each of them is a bundle $\Omega^p(p)$.

We may and do assume that these pairings are the identities if $i = j$, $\ell = m$, etc. Finally, we suppose that these maps verify the natural associative properties of compositions. This means that the induced diagrams

$$\begin{array}{ccc} A_{kj} \otimes A_{ji} \otimes A_{ih} & \longrightarrow & A_{ki} \otimes A_{ih} \\ \downarrow & & \downarrow \\ A_{kj} \otimes A_{jh} & \longrightarrow & A_{kh} \end{array} \qquad \begin{array}{ccc} B_{on} \otimes B_{nm} \otimes B_{m\ell} & \longrightarrow & B_{om} \otimes B_{n\ell} \\ \downarrow & & \downarrow \\ B_{on} \otimes B_{n\ell} & \longrightarrow & B_{o\ell} \end{array}$$

$$\begin{array}{ccc} H_{\ell k} \otimes A_{kj} \otimes A_{ji} & \longrightarrow & H_{\ell j} \otimes A_{ji} \\ \downarrow & & \downarrow \\ H_{\ell k} \otimes A_{ki} & \longrightarrow & H_{\ell i} \end{array} \qquad \begin{array}{ccc} B_{nm} \otimes B_{m\ell} \otimes H_{\ell i} & \longrightarrow & B_{n\ell} \otimes H_{\ell i} \\ \downarrow & & \downarrow \\ B_{nm} \otimes H_{mi} & \longrightarrow & H_{ni} \end{array}$$

$$\begin{array}{ccc} B_{m\ell} \otimes H_{\ell j} \otimes A_{ji} & \longrightarrow & H_{mj} \otimes A_{ji} \\ \downarrow & & \downarrow \\ B_{m\ell} \otimes H_{\ell i} & \longrightarrow & H_{mi} \end{array}$$

are commutative for all possible combinations of indices.

In our setup we also let finite dimensional vector spaces M_i for $1 \leq i \leq r$ and N_ℓ for $1 \leq \ell \leq s$ be given and we consider finally the vector space

$$W = \bigoplus_{i,\ell} \mathrm{Hom}(M_i, N_\ell \otimes H_{\ell i}) = \bigoplus_{i,\ell} \mathrm{Hom}(H_{\ell i}^* \otimes M_i, N_\ell)$$

where summation is over $1 \leq i \leq r$ and $1 \leq \ell \leq s$. This is the space of homomorphisms in the abstract setting. The group G and its action on W are now also given in the abstract setting as follows.

2.2. The group G.

We define G as a product $G_L \times G_R$ of two groups where the left group G_L replaces $\mathrm{Aut}(\mathcal{E})$ and the right group G_R replaces $\mathrm{Aut}(\mathcal{F})$ in our motivation. Let G_L be the set of matrices

$$\begin{pmatrix} g_1 & 0 & \cdots & 0 \\ u_{2,1} & g_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ u_{r,1} & \cdots & u_{r,r-1} & g_r \end{pmatrix}$$

with $g_i \in \mathrm{GL}(M_i)$ and $u_{ji} \in \mathrm{Hom}(M_i, M_j \otimes A_{ji}) = \mathrm{Hom}(A_{ji}^* \otimes M_i, M_j)$. The group law in G_L is now defined as matrix multiplication where we define the compositions $u_{kj} * u_{ji}$ naturally according to the given pairings as the composition

$$M_i \xrightarrow{u_{ji}} M_j \otimes A_{ji} \xrightarrow{u_{kj} \otimes \mathrm{id}} M_k \otimes A_{kj} \otimes A_{ji} \xrightarrow{\mathrm{id} \otimes \mathrm{comp}} M_k \otimes A_{ki}.$$

Explicitly, if g has the entries g_i, u_{ji} and g' has the entries g_i', u_{ji}' then the product

$$g'' = g'g$$

in G_L is defined as the matrix with the entries $g_i'' = g_i' \circ g_i$ in the diagonal and

$$u_{ki}'' = u_{ki}' \circ g_i + \sum_{i<j<k} u_{kj}' * u_{ji} + (g_k' \otimes \mathrm{id}) \circ u_{ki}$$

for $1 \le i < k \le r$. The verification that this defines a group structure on G_L is now straightforward.

As a set G_L is the product of all the $\mathrm{GL}(M_i)$ and all $\mathrm{Hom}(M_i, M_j \otimes A_{ji})$ for $i < j$ and thus has the structure of an affine variety. Since multiplication is composed by a system of bilinear maps it is a morphism of affine varieties. Hence G_L is naturally endowed with the structure of an algebraic group. The group G_R is now defined in the same way by replacing the spaces M_i and A_{ji} by N_ℓ and $B_{m\ell}$. Finally $G = G_L \times G_R$ is defined as an algebraic group.

2.3. The action of G and W.

We will define a left action of G_R and a right action of G_L on W such that the action of G on W can be defined by $(g, h) \cdot w = h \cdot w \cdot g^{-1}$. Both actions are defined as matrix products as described above in the case of sheaf homomorphisms using the abstract compositions as in the definition of the group law.

If w has the entries $\varphi_{\ell i} \in \mathrm{Hom}(H_{\ell i}^* \otimes M_i, N_\ell)$ and $g \in G_L$ has the entries g_i and u_{ij} then $w \cdot g$ is defined as the matrix product

$$\begin{pmatrix} \varphi_{11}' & \cdots & \varphi_{1r}' \\ \vdots & & \vdots \\ \varphi_{s1}' & \cdots & \varphi_{sr} \end{pmatrix} = \begin{pmatrix} \varphi_{11} & \cdots & \varphi_{1r} \\ \vdots & & \vdots \\ \varphi_{s1} & \cdots & \varphi_{sr} \end{pmatrix} \begin{pmatrix} g_1 & 0 & \cdots & 0 \\ u_{2,1} & g_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ u_{r,1} & \cdots & u_{r,r-1} & g_r \end{pmatrix}$$

with

$$\varphi'_{\ell i} = \varphi_{\ell i} \circ g_i + \sum_{i<j} \varphi_{\ell j} * u_{ji} \quad (\text{if } i = r \text{ the last sum is } 0),$$

where $\varphi_{\ell j} * u_{ji}$ is the composition

$$M_i \longrightarrow M_j \otimes A_{ji} \longrightarrow N_\ell \otimes H_{\ell j} \otimes A_{ji} \longrightarrow N_\ell \otimes H_{\ell i}$$

or dually the composition

$$H^*_{\ell i} \otimes M_i \longrightarrow H^*_{\ell j} \otimes A^*_{ji} \otimes M_i \longrightarrow H^*_{\ell j} \otimes M_j \longrightarrow N_\ell.$$

The left action of G_R is defined in the same way. In the next two sections we give an analysis of stability and semi-stability for the action of G and its natural reductive subgroup G_{red}. In the reductive case this is due to A. King.

2.4. Canonical subgroups of G.

We let $H_L \subset G_L$ and $H_R \subset G_R$ be the maximal normal unipotent subgroups of G_L and G_R defined by the condition that all $g_i = \mathrm{id}_{M_i}$ and all $h_\ell = \mathrm{id}_{N_\ell}$. Then $H = H_L \times H_R$ is a maximal normal unipotent subgroup of G. Similarly we consider the reductive subgroups $G_{L,\mathrm{red}}$ and $G_{R,\mathrm{red}}$ of G_L and G_R defined by the conditions $u_{ji} = 0$ and $v_{m\ell} = 0$ for all indices. Then $G_{\mathrm{red}} = G_{L,\mathrm{red}} \times G_{R,\mathrm{red}}$ is a reductive subgroup of G and it is easy to see that $G/H \cong G_{\mathrm{red}}$. The restricted action of G_{red} is much simpler and reduces to the natural actions of $\mathrm{GL}(M_i)$ on M_i and $\mathrm{GL}(N_\ell)$ on N_ℓ

3. Actions of reductive groups.

3.1. Results of A. King.

Let Q be a finite set, $\Gamma \subset Q \times Q$ a subset such that the union of the images of the two projections of Γ is Q. For each $\alpha \in Q$, let m_α be a positive integer, M_α a vector space of dimension m_α and for each $(\alpha, \beta) \in \Gamma$, let $V_{\alpha\beta}$ be a finite dimensional nonzero vector space. Let

$$W_0 = \bigoplus_{(\alpha,\beta)\in\Gamma} \mathrm{Hom}(M_\alpha \otimes V_{\alpha\beta}, M_\beta).$$

On W_0 we have the following action of the reductive group:

$$G_0 = \prod_{\alpha\in Q} \mathrm{GL}(M_\alpha)$$

arising naturally in this situation. If $(f_{\beta\alpha}) \in W_0$ and $(g_\alpha) \in G_0$, then

$$(g_\alpha) \cdot (f_{\beta\alpha}) = \big(g_\beta \circ f_{\beta\alpha} \circ (g_\alpha \otimes \mathrm{id})^{-1}\big).$$

Let $(e_\alpha)_{\alpha \in Q}$ be a sequence of integers such that

$$\sum_{\alpha \in Q} e_\alpha m_\alpha = 0.$$

To this sequence is associated the character χ of G_0 defined by

$$\chi(g) = \prod_{\alpha \in Q} \det(g_\alpha)^{-e_\alpha}.$$

This character is trivial on the canonical subgroup of G_0 isomorphic to $\mathbb{C}^*$ (for every $\lambda \in \mathbb{C}^*$, the element (g_α) of G_0 corresponding to λ is such that $g_\alpha = \lambda \cdot \mathrm{id}$ for each α). This subgroup acts trivially on W_0. A point $x \in W_0$ is called *χ-semi-stable* if there exist an integer $n \geq 1$ and a polynomial $f \in \mathbb{C}[W_0]$ which is χ^n-invariant and such that $f(x) \neq 0$ (f is called χ^n-invariant if for every $w \in W_0$ and $g \in G_0$ we have $f(gw) = \chi^n(g)f(w)$). The point x is called *χ-stable* if moreover

- $\dim(G_0 x) = \dim(G_0/\mathbb{C}^*)$ and
- the action of G_0 on $\{w \in W_0, f(w) \neq 0\}$ is closed.

A. King proves in [21] the following results:

1) A point $x = (f_{\beta\alpha}) \in W_0$ is χ-semi-stable (resp. χ-stable) if and only if for each family $(M'_\alpha), \alpha \in Q$, of subspaces $M'_\alpha \subset M_\alpha$ which is neither the trivial family (0) nor the given family (M_α) and which satisfies

$$f_{\beta\alpha}(M'_\alpha \otimes V_{\alpha\beta}) \subset M'_\beta$$

for each $(\alpha, \beta) \in \Gamma$, we have

$$\sum_{\alpha \in Q} e_\alpha \dim(M'_\alpha) \leq 0 \quad (\text{resp. } < 0).$$

2) Let W_0^{ss} (resp. W_0^s) be the open subset of W_0 consisting of semi-stable (resp stable) points. Then there exist a good quotient

$$\pi : W_0^{ss} \longrightarrow M$$

by $G_0/\mathbb{C}^*$ which is a projective variety.

3) The restriction of this quotient

$$W_0^s \longrightarrow M^s = \pi(W_0^s)$$

is a geometric quotient and M^s is smooth.

3.2. Polarizations.

The (semi-)stable points of W_0 remain the same if we replace (e_α) by (ce_α), c being a positive integer. So the notion of (semi-)stability is fully described by the reduced parameters (e_α/t), where

$$t = \sum_{\alpha\in Q, e_\alpha>0} e_\alpha m_\alpha.$$

So we can define the *polarization* of the action of G_0 on W_0 by any sequence $(c_\alpha)_{\alpha\in Q}$ of nonzero rational numbers such that

$$\sum_{\alpha\in Q} c_\alpha m_\alpha = 0, \qquad \sum_{\alpha\in Q, c_\alpha>0} c_\alpha m_\alpha = 1.$$

By multiplying this sequence by the smallest common denominator of the c_α we obtain a sequence (e_α) of integers and the corresponding character of G_0. Therefore the loci of stable and semi-stable points of W_0 with respect to G_0 and a polarization $\Lambda_0 = (c_\alpha)$ are well defined and denoted by

$$W_0^s(G_0,\Lambda_0) \quad \text{and} \quad W_0^{ss}(G_0,\Lambda_0).$$

3.3. Conditions imposed by the non-emptiness of the quotient.

If W_0^s is not empty, the e_α must satisfy some conditions. We will derive this only in the three situations which occur in this paper. Polarizations satisfying these necessary conditions will be called *proper*. The first is that of the action of G_{red} in 2.4 and the second is that of $\mathbf{G}$ and $\mathbf{W}$ in Section 5, and the third is the case in between occurring in 7.4.2.

3.3.1. *First case.* — Let r, s be positive integers. We take

$$Q = \{\alpha_1 \ldots, \alpha_r, \beta_1, \ldots, \beta_s\}, \quad \Gamma = \{\alpha_1 \ldots, \alpha_r\} \times \{\beta_1, \ldots, \beta_s\}.$$

This is the case of morphisms of type (r, s). For $1 \le i \le r$, let $M'_{\alpha_i} = M_{\alpha_i}$ if $e_{\alpha_i} > 0$, and $\{0\}$ otherwise, and for $1 \le \ell \le s$, let $M'_{\beta_\ell} = M_{\beta_\ell}$. Then if one e_{α_i} is not positive, we have

$$\sum_{\alpha\in Q} e_\alpha \dim(M'_\alpha) \ge 0$$

and $(M'_\alpha) \ne (M_\alpha)$, so in this case no point of W_0 is stable. So we obtain, if W_0^s is non-empty, the conditions

$$e_{\alpha_i} > 0, \text{ for any } i, \quad \text{and} \quad e_{\beta_\ell} < 0, \text{ for any } \ell.$$

A proper polarization is in this case a sequence

$$(\lambda_1, \dots, \lambda_r, -\mu_1, \dots, -\mu_s)$$

of rational numbers such that the λ_i and the μ_ℓ are positive and satisfy

$$\sum_{1 \le i \le r} \lambda_i m_{\alpha_i} = \sum_{1 \le \ell \le s} \mu_\ell m_{\beta_\ell} = 1.$$

3.3.2. *Second case.* — This case appears when we use a bigger reductive group to define the quotient (this is the case of $\mathbf{W}$ later on). Let r, s be positive integers. Here we take

$$\begin{aligned} Q &= \{\alpha_1 \dots, \alpha_r, \beta_1, \dots, \beta_s\}, \\ \Gamma &= \big\{(\alpha_i, \alpha_{i-1}),\ 2 \le i \le r,\ (\alpha_1, \beta_s), (\beta_\ell, \beta_{\ell-1}),\ 2 \le \ell \le s\big\}. \end{aligned}$$

Then the necessary conditions for W_0^s to be non-empty are:

$$\sum_{i \le j \le r} e_{\alpha_j} m_{\alpha_j} > 0 \quad \text{for any } i, \text{ and } \sum_{1 \le \ell \le m} e_{\beta_\ell} m_{\beta_\ell} < 0 \quad \text{for any } m.$$

To derive the first set of conditions we consider for any i the family (M'_γ) for which $M'_{\alpha_j} = 0$ if $i \le j \le r$ and $M'_\gamma = M_\gamma$ for all other $\gamma \in Q$. Then $f_{\alpha\beta}(M'_\alpha \otimes V_{\alpha\beta}) \subset M'_\beta$ for any $f \in W_0$ and any $(\alpha, \beta) \in \Gamma$. If f is stable we obtain

$$-\sum_{i \le j \le r} e_{\alpha_j} m_{\alpha_j} = \sum_{\gamma \in Q} e_\gamma \dim(M'_\gamma) < 0.$$

Moreover, if the family (M'_γ) is defined by $M'_{\alpha_j} = 0$ for $1 \le j \le r, M'_{\beta_\ell} = 0$ if $m < \ell \le s$ and $M'_\gamma = M_\gamma$ else, we obtain directly

$$\sum_{1 \le \ell \le m} e_{\beta_\ell} m_{\beta_\ell} = \sum_{\gamma \in Q} e_\gamma \dim(M'_\gamma) < 0.$$

A proper polarization in this case is then a sequence

$$(\rho_1, \dots, \rho_r, -\sigma_1, \dots, -\sigma_s)$$

of rational numbers satisfying

$$\sum_{1 \le i \le r} \rho_i m_{\alpha_i} = \sum_{1 \le \ell \le s} \sigma_\ell m_{\beta_\ell} = 1$$

and

$$\sum_{i \le j \le r} \rho_j m_{\alpha_j} > 0 \quad \text{for any } i \quad \text{and} \quad \sum_{1 \le \ell \le m} \sigma_\ell m_{\beta_\ell} > 0 \quad \text{for any } m.$$

We could also drop the normalization condition.

3.3.3. *Third case.* — This case is a combination of the first and second case. It appears in the proof of the equivalence of semi-stability in 7.3. Here Q is the same as in the previous cases and

$$\Gamma = \{(\alpha_i, \alpha_{i-1}),\ 2 \le i \le r,\ (\alpha_1, \beta_\ell),\ 1 \le \ell \le s\}.$$

Now the necessary conditions for W_0^s to be non-empty are:

$$\sum_{i \le j \le r} e_{\alpha_j} m_{\alpha_j} > 0 \quad \text{for any } i, \text{ and } e_{\beta_\ell} < 0 \text{ for any } \ell.$$

The first condition follows as in the second case when we consider the family (M'_γ) with $M'_{\alpha_j} = 0$ for $i \le j \le r$ and $M'_\gamma = M_\gamma$ for all other $\gamma \in Q$. The second condition follows when all M'_γ are zero except $M'_{\beta_\ell} = M_{\beta_\ell}$ for one ℓ. Again a proper polarization in this case is a sequence $(\rho_1, \ldots, \rho_r, -\mu_1, \ldots, -\mu_\ell)$ with

$$\sum_{1 \le i \le r} \rho_i m_{\alpha_i} = \sum_{1 \le \ell \le s} \mu_\ell m_{\beta_\ell} = 1$$

and

$$\sum_{i \le j \le r} \rho_j m_{\alpha_j} > 0 \quad \text{for any } i \quad \text{and } \mu_\ell > 0 \text{ for any } \ell.$$

3.4. The action of G_0 on $\mathbb{P}(W_0)$.

We suppose that we are in one of the first two preceding cases and that there exist stable points in W_0. Let P be a nonzero homogeneous polynomial, χ^n-invariant for some positive integer n. The χ^n-invariance implies that P has degree $n \cdot t$ where in case 1 (action of G_{red} on W)

$$t = \sum_{1 \le i \le r} e_{\alpha_i} m_{\alpha_i},$$

and in case 2 (action of $\mathbf{G}$ on $\mathbf{W}$)

$$t = \sum_{1 \le i \le r} i e_{\alpha_i} m_{\alpha_i} - \sum_{1 \le \ell \le s} (s - \ell) e_{\beta_\ell} m_{\beta_\ell}.$$

To see this let $\lambda \in \mathbb{C}^*$ and let g be given by $g_{\alpha_i} = \lambda^{-1}\,\mathrm{id}$ and $g_{\beta_\ell} = \mathrm{id}$ in the first case and by $g_{\alpha_i} = \lambda^{-i}\,\mathrm{id}$ and $g_{\beta_\ell} = \lambda^{\ell - s}\,\mathrm{id}$ in the second case. Then $gx = \lambda x$ and $\chi^n(g) = \lambda^{nt}$ in both cases, such that $P(\lambda x) = \lambda^{nt} P(x)$.

Now we will see that there exist a G_0-line bundle $\mathcal{L}$ on $\mathbb{P}(W_0)$ such that the set W_0^{ss} of semi-stable points is exactly the set of points over $\mathbb{P}(W_0)^{ss}(G_0, \mathcal{L})$, which is the set of semi-stable points in the sense of Geometric Invariant Theory corresponding to

$$\mathcal{L} = \mathcal{O}_{\mathbb{P}(W_0)}(t),$$

cf. [27], [29], [31]. Here the action of G_0 on $\mathcal{L}$ is the natural action multiplied by χ. More precisely, the action of G_0 on W_0 induces an action of this group on $S^t W_0$ and on $S^t W_0^*$ by

$$(g \cdot F)(w) = F(g^{-1}w)$$

for all $g \in G_0$, $w \in W_0$ and $F \in S^t W_0^*$, viewed as an homogeneous polynomial of degree t on W_0. The line bundle space L of $\mathcal{L}$ is acted on by G_0 in the same way: if $\xi \in L_{\langle w \rangle}$ then $g \cdot \xi \in L_{\langle gw \rangle}$ is the form on $\langle gw \rangle^{*\otimes t} = L_{\langle gw \rangle}$ given by $(g \cdot \xi)(y) = \xi(g^{-1}y)$. We modify now the action of G_0 on L (resp. $S^t W_0^*$) by multiplying with $\chi(g)$:

$$g * \xi = \chi(g) g \cdot \xi \quad \text{for} \quad \xi \in L_{\langle w \rangle}, \text{ or}$$
$$g * F = \chi(g) g \cdot F \quad \text{for} \quad F \in H^0(\mathbb{P}(W_0), \mathcal{L}) = S^t W_0^*.$$

Now $P \in H^0(\mathbb{P}(W_0), \mathcal{L}^{\otimes n})$ is an invariant section if and only if P is a homogeneous polynomial of degree tn which satisfies

$$P(gw) = \chi^n(g) P(w).$$

From the definition of semi-stable points in W_0 and $\mathbb{P}(W_0)$ with respect to the modified G_0-structure on $\mathcal{L} = \mathcal{O}_{\mathbb{P}(W_0)}(t)$, we get immediately

3.4.1. Lemma. — *Assume that $W_0^s(G_0, \Lambda_0) \neq \emptyset$ and let t be defined as above in the two cases of W_0. Then the set $W_0^{ss}(G_0, \Lambda_0)$ is the cone of the set $\mathbb{P}(W_0)^{ss}(G_0, \mathcal{O}_{\mathbb{P}(W_0)}(t))$ as defined in G.I.T.*

There are two definitions of stable points in $\mathbb{P}(W_0)$, the classical one, given in [27], [29], and a more recent one, given in [31]. If we take D. Mumford's definition, the cone of the set of stable points in $\mathbb{P}(W_0)$ does not coincide with W_0^s because every point of $\mathbb{P}(W_0)$ has a stabilizer of positive dimension. In fact there is a subgroup of $G_0/\mathbb{C}^*$ of positive dimension which acts trivially on $\mathbb{P}(W_0)$. In the first case for example such a group is given by $g_{\alpha_i} = \lambda\,\mathrm{id}$ and $g_{\beta_\ell} = \mu\,\mathrm{id}$ with $\lambda, \mu \in \mathbb{C}^*$. If we want to keep the coincidence between the sets of stable points for one and the same group, we would have to consider the action of a smaller reductive group in order to eliminate additional stabilizers. We will do this in 3.5 only in the first case. If we take the definition of V.L. Popov and E.G. Vinberg, then we obtain that the set $W_0^s(G_0, \Lambda_0)$ is exactly the cone of the set $\mathbb{P}(W_0)^s(G_0, \mathcal{O}_{\mathbb{P}(W_0)}(t))$.

3.5. The group G'.

Let G and W be as in Section 2 and let $\Lambda = (\lambda_1, \dots, \lambda_r, -\mu_1, \dots, -\mu_s)$ be a proper polarization as in 3.3.1 for the action of G_{red} on W. It is then convenient to use the subgroup G'_{red} of G_{red} consisting of elements $((g_i), (h_\ell))$ satisfying

$$\prod_{1 \le i \le r} \det(g_i)^{a_{i1}} = \prod_{1 \le \ell \le s} \det(h_\ell)^{b_{s\ell}} = 1,$$

where $a_{ji} = \dim(A_{ji})$ and $b_{m\ell} = \dim(B_{m\ell})$.

We consider the action of G'_{red} on $\mathcal{L}$ induced by the modified χ-action of G_{red}. Now the set $W^s(G_{\mathrm{red}}, \Lambda)$ of χ-stable points of W is exactly the cone over the locus $\mathbb{P}(W)^s(G'_{\mathrm{red}}, \mathcal{L})$ of stable points of $\mathbb{P}(W)$ in the sense of Geometric Invariant Theory.

4. Semi-stability in the non-reductive case.

Let G and W be as in Section 2. A character χ on G_{red} as in King's setup can be extended to a character of G. Also the modified action of G_{red} on $\mathcal{L}$ can be extended to an action of G. Let G' be the subgroup of G defined by the same equations as for G'_{red}. It contains H and G'_{red}, and we have $G'/H \simeq G'_{\mathrm{red}}$.

In the case of the action of G_{red} on W a proper polarization is given by a sequence $\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_s$ of positive rational numbers such that

$$\sum_{1 \le i \le r} \lambda_i m_i = \sum_{1 \le \ell \le s} \mu_\ell n_\ell = 1.$$

More precisely, the polarization is exactly the sequence

$$(\lambda_1, \dots, \lambda_r, -\mu_1, \dots, -\mu_s).$$

The parameter λ_i (resp. μ_ℓ) will be called the weight of the vector space M_i (resp. N_ℓ). We see that the dimension of the set of possible proper polarizations is $r+s-2$. Let t denote the smallest common denominator of the numbers λ_i and μ_ℓ and χ the character of G_{red} defined by the sequence of integers $(-t\lambda_1,\dots,-t\lambda_r,t\mu_1,\dots,t\mu_s)$. Let

$$\mathcal{L}=\mathcal{O}_{\mathbb{P}(W)}(t) \quad \text{with} \quad t=\sum_{1\le i\le r} m_i t\lambda_i.$$

As we have seen, if we consider the modified action of G_{red} on $\mathcal{L}$, then the χ-semi-stable points of W are exactly those over the semi-stable points of $\mathbb{P}(W)$ in the sense of Geometric Invariant Theory with respect to the action of $G_{\mathrm{red}}/\mathbb{C}^*$ on $\mathcal{L}$. The χ^{tn}-invariant polynomials are the G_{red}-invariant sections of $\mathcal{L}^n$.

We are now going to define a notion of (semi-)stability for the points of W with respect to the given action of the non-reductive group G. Let $H\subset G$ be the above unipotent group, see also 2.4.

4.1. Definition. — *A point $w\in W$ is called G-semi-stable (resp. G-stable) with respect to the (proper) polarization*

$$\Lambda=(\lambda_1,\dots,\lambda_r,-\mu_1,\dots,-\mu_s)$$

if every point of Hw is G_{red}-semi-stable (resp. G_{red}-stable) with respect to this polarization. We denote these sets by $W^{ss}(G,\Lambda)$ (resp. $W^s(G,\Lambda)$).

For many of the quotient problems for the spaces of homomorphisms between $\bigoplus m_i\mathcal{E}_i$ and $\bigoplus n_j\mathcal{F}_j$ and their cokernel sheaves this is a fruitful notion. In 4.2 we investigate an example with an explicit description of the open sets $W^s(G,\Lambda)\subset W^s(G_{\mathrm{red}},\Lambda)$. This example also shows that the existence of a good quotient depends on the choice of the polarization.

4.1.1. *Situation for type* $(2,1)$. — In the case of morphisms of type $(2,1)$ we have $\mu_1=1/n_1$ and the polarization is completely described by the single parameter $t=m_2\lambda_2$. We must have $0<t<1$.

A polarization such that there exist integers m_1', m_2', n_1', with $0<n_1'<n_1$, $0\le m_i'\le m_i$, such that $m_1'n_1-m_1n_1'$, $m_2'n_1-m_2n_1'$ are not both 0, and that

$$\lambda_1 m_1'+\lambda_2 m_2'=\frac{n_1'}{n_1}$$

is called *singular*. There are only finitely many singular polarizations, corresponding to the values $0 < t_1 < t_2 < \cdots < t_p < 1$ of t. Let $t_0 = 0$, $t_{p+1} = 1$. If Λ, Λ' are polarizations corresponding to parameters t, t' such that for some $i \in \{0, \cdots, p\}$ we have $t_i < t, t' < t_{i+1}$, then

$$W^{ss}(G,\Lambda) = W^{ss}(G,\Lambda') \quad \text{and} \quad W^{s}(G,\Lambda) = W^{s}(G,\Lambda').$$

Hence there are exactly $2p+1$ notions of G-(semi-)stability in this case. Moreover, if m_1, m_2 and n_1 are relatively prime, and Λ is a non singular polarization, we have $W^{ss}(G,\Lambda) = W^{s}(G,\Lambda)$.

In the general case of morphisms of type (r,s), it is not difficult to see that there are only finitely many notions of G-(semi-)stability.

4.1.2. *Remark.* — In [16] semi-stability is defined as follows: A point $w \in W$ is semi-stable if there exist a positive integer k and a G'-invariant section s of $\mathcal{L}^k$ such that $s(w) \neq 0$ (there is also a condition on the action of H). It is clear that a semi-stable point in the sense of Fauntleroy is also G-semi-stable with respect to $(\lambda_1, \ldots, \lambda_r, -\mu_1, \ldots, -\mu_s)$. It is proved in [16] that there exist a *categorical quotient* of the open subset of semi-stable points in the sense of [16], but it is not clear that all G-semi-stable points are semi-stable. Moreover, in the general situation of [16] there is no way to impose conditions which would imply that the categorical quotient is a good quotient or even projective. Using Definition 4.1 we are able to derive a criterion for the existence of a good and projective quotient of W under the action of G.

4.2. Existence and non-existence of good quotients, an example.

We show here that we cannot expect that a good quotient $W^{ss}(\Lambda, G)//G$ will exist for any polarization Λ.

We consider morphisms $2\mathcal{O}(-2) \to \mathcal{O}(-1) \oplus \mathcal{O}$ on $\mathbb{P}_2$. There are three notions of G-(semi-)stability in this case, two corresponding to non singular polarizations. For one of the non singular polarizations the quotient $W^s(\Lambda, G)/G$ exists and for the other we prove the inexistence of a good quotient $W^s(\Lambda, G)//G$.

Let V be a complex vector space of dimension 3, and $\mathbb{P}_2 = \mathbb{P}V$. Let

$$W = \operatorname{Hom}\big(2\mathcal{O}(-2), \mathcal{O}(-1) \oplus \mathcal{O}\big)$$

on $\mathbb{P}_2$. A polarization for the action of G on W is a triple $(\frac{1}{2}, -\mu_1, -\mu_2)$ with positive numbers μ_1, μ_2 satisfying $\mu_1+\mu_2 = 1$. As in 4.1.1 such a polarization depends only on μ_1. There is only one singular polarization, corresponding to $\mu_1 = \frac{1}{2}$. Hence if we consider only non singular polarizations there are only two notions of G-(semi-)stability, the first one corresponding to polarizations such that $\mu_1 > \frac{1}{2}$ and the second to polarizations such that $\mu_1 < \frac{1}{2}$. In both cases semi-stable points are already stable. We are going to show that in the first case $W^s(G,\Lambda)$ has a geometric quotient which is projective and smooth and that in the second case $W^s(G,\Lambda)$ doesn't even admit a good quotient.

The elements $x \in W$ and $g \in G$ are written as matrices

$$x = \begin{pmatrix} z_1 & z_2 \\ q_1 & q_2 \end{pmatrix} \quad \text{and} \quad g = \left(\sigma, \begin{pmatrix} \alpha & 0 \\ z & \beta \end{pmatrix}\right)$$

where $z_1, z_2 \in V^*$, $q_1, q_2 \in S^2V^*$, $\sigma \in \mathrm{GL}(2)$, $\alpha, \beta \in \mathbb{C}^*$ and $z \in V^*$.

4.2.1. *The case $\mu_1 > \frac{1}{2}$.* — In this case $W^s(G,\Lambda)$ has a geometric quotient which is the universal cubic $Z \subset \mathbb{P}V \times \mathbb{P}S^3V^*$ of the Hilbert scheme of plane cubic curves in $\mathbb{P}_2 = \mathbb{P}V$. The quotient map is given by

$$x \longmapsto \big(\langle z_1 \wedge z_2\rangle, \langle z_1q_2 - z_2q_1\rangle\big).$$

Remark. — If $\mu_1 > \frac{3}{4}$, then $\mu_1 > 3\mu_2$ and the conditions of 1.5.1 (in the dual case $(1,2)$) for a good and projective quotient to exist in this case are satisfied.

The proof is done in several steps.

1) *Claim 1*: Let $x \in W$ be as above. Then

(i) $x \in W^s(G_{\mathrm{red}},\Lambda)$ if and only if $z_1 \wedge z_2 \neq 0$ in Λ^2V^* and q_1, q_2 are not both zero.

(ii) $x \in W^s(G,\Lambda)$ if and only if $z_1 \wedge z_2 \neq 0$ and $\det(x) = z_1q_2 - z_2q_1 \neq 0$ in S^3V^*.

Proof of Claim 1. — (i) follows easily from the criterion (1) in 3.1. As for (ii) let $x \in W^s(G_{\mathrm{red}},\Lambda)$ with $\det(x) \neq 0$. Then $\det(h\cdot x) = \det(x) \neq 0$ for any $h = \left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right)$ which implies that also $h \cdot x \in W^s(G_{\mathrm{red}},\Lambda)$. Let conversely $x \in W^s(G,\Lambda)$. Then $\det(x) \neq 0$ because otherwise there is a linear form $z \in V^*$ with $q_1 = zz_1$ and $q_2 = zz_2$ and with $h = \left(\begin{smallmatrix} 1 & 0 \\ -z & 1 \end{smallmatrix}\right)$ the element $h \cdot x$ is the matrix $\left(\begin{smallmatrix} z_1 & z_2 \\ 0 & 0 \end{smallmatrix}\right)$ which is not in $W^s(G_{\mathrm{red}},\Lambda)$.

2) By the result of A. King in 3.1, (3), there is a geometric quotient $W^s(G_{\mathrm{red}},\Lambda)/G_{\mathrm{red}}$ which is smooth and projective.

Claim 2: $W^s(G_{\mathrm{red}}, \Lambda)/G_{\mathrm{red}} \cong \mathbb{P}(Q^* \otimes S^2V^*)$.

Here $Q^* = \Omega^1(1)$ is the dual of the tautological quotient bundle over $\mathbb{P}V$. (The dimension of this quotient variety is 13 while $\dim W = 18$ and $\dim G_{\mathrm{red}}/\mathbb{C}^* = 5$.)

To verify Claim 2 we consider the map

$$x = \begin{pmatrix} z_1 & z_2 \\ q_1 & q_2 \end{pmatrix} \xmapsto{\ \alpha\ } \big(\langle z_1 \wedge z_2\rangle, \langle z_1 \otimes q_2 - z_2 \otimes q_1\rangle\big)$$

from $W^s(G_{\mathrm{red}}, \Lambda)$ to $\mathbb{P}V \times \mathbb{P}(V^* \otimes S^2V^*) \subset \mathbb{P}(V^* \otimes S^2V^* \otimes \mathcal{O}_{\mathbb{P}V})$ where we identify $\mathbb{P}\Lambda^2V^*$ with $\mathbb{P}V$ via $\langle z_1 \wedge z_2\rangle \leftrightarrow \langle a\rangle$, $z_1(a) = z_2(a) = 0$. Then each $\alpha(x) \in \mathbb{P}(Q^*_{\langle a\rangle} \otimes S^2V^*)$ because $Q^*_{\langle a\rangle} \subset V^*$ is the subspace of forms vanishing in $\langle a\rangle$. It follows immediately that α is a morphism

$$W^s(G_{\mathrm{red}}, \Lambda) \longrightarrow \mathbb{P}(Q^* \otimes S^2V^*)$$

which is surjective and G_{red}-equivariant. It induces a morphism of the geometric quotient to $\mathbb{P}(Q^* \otimes S^2V^*)$ which is even bijective. Since both, the quotient and the target are smooth, this is an isomorphism.

3) Since $Q^* \subset V^* \otimes \mathcal{O}_{\mathbb{P}V}$ we have an induced homomorphism $Q^* \otimes S^2V^* \to S^3V^* \otimes \mathcal{O}_{\mathbb{P}V}$. It is the middle part of the canonical exact sequence

$$0 \to \Lambda^2Q^* \otimes V^* \longrightarrow Q^* \otimes S^2V^* \longrightarrow S^3V^* \otimes \mathcal{O}_{\mathbb{P}V} \xrightarrow{\ \mathrm{ev}\ } \mathcal{O}_{\mathbb{P}V}(3) \to 0$$

of vector bundles on $\mathbb{P}V$. Let $\mathcal{Z}$ be the kernel of ev. From the left part of the sequence we obtain the affine bundle

$$\mathbb{P}(Q^* \otimes S^2V^*) \setminus \mathbb{P}(\Lambda^2Q^* \otimes V^*) \xrightarrow{\ \beta\ } \mathbb{P}(\mathcal{Z}) \subset \mathbb{P}V \times \mathbb{P}S^3V^*.$$

Here $\mathbb{P}(\mathcal{Z}) = Z$ is nothing but the universal cubic and the fibres of β are isomorphic to V^*.

Claim 3: $W^s(G, \Lambda) \subset W^s(G_{\mathrm{red}}, \Lambda)$ is the inverse image of

$$\mathbb{P}(Q^* \otimes S^2V^*) \setminus \mathbb{P}(\Lambda^2Q^* \otimes V^*)$$

under α and $\alpha_{|W^s(G,\Lambda)}$ is a geometric quotient with respect to G_{red}.

Proof of Claim 3. — $z_1 \otimes q_2 - z_2 \otimes q_1$ belongs to $\Lambda^2Q^*_{\langle a\rangle} \otimes V^*$ if and only if $z_1q_2 - z_2q_1 = 0$, see (ii) of Claim 1.

4) Let now $\pi = \beta \circ \alpha$ be the morphism $W^s(G,\Lambda) \to Z$, given by $x \mapsto (\langle a\rangle, \langle z_1q_2 - z_2q_1\rangle)$, where $z_1(a) = z_2(a) = 0$. It is obviously G-equivariant and its fibres coincide with the G-orbits. Since α is a geometric quotient and β is an affine bundle, then π is also a geometric quotient.

Remark. — The variety Z is isomorphic to the moduli space $M = M_{\mathbb{P}_2}(3m+1)$ of stable coherent sheaves on $\mathbb{P}_2$ with Hilbert polynomial $\chi\mathcal{F}(m) = 3m + 1$. This had been verified by J. Le Potier in [24]. The space $W^s(G,\Lambda)$ is a natural parametrization of M because any $\mathcal{F} \in M$ can be presented in an extension sequence $0 \to \mathcal{O}_C \to \mathcal{F} \to \mathbb{C}_p \to 0$ where C is the cubic curve supporting $\mathcal{F}$ and $p \in C$, and then $\mathcal{F}$ has a resolution

$$0 \to 2\mathcal{O}(-2) \xrightarrow{x} \mathcal{O}(-1) \oplus \mathcal{O} \longrightarrow \mathcal{F} \to 0.$$

This resolution is the Beilinson resolution as can easily be verified. Moreover, x is (G,Λ)-stable if and only if $\mathcal{F}$ is stable. (If p is a smooth point of C, then $\mathcal{F}$ is the line bundle $\mathcal{O}_C(p)$ and if p is a singular point of C, then $\mathcal{F}$ is the unique Cohen-Macaulay module on C with the given polynomial.) There is an obvious universal family $\mathcal{F}$ on $W^s(G,\Lambda) \times_H \mathbb{P}V$ which defines a G-equivariant morphism $W^s(G,\Lambda) \to M$ and then a bijective morphism $Z \to M$, which by smoothness, is an isomorphism. One knows that M carries a universal family $\mathcal{E}$. This family can be obtained as the non-trivial extension

$$0 \to \mathcal{O}_{Z\times_H Z} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_\Delta \to 0,$$

where $H = \mathbb{P}S^3V^*$ and $Z \times_H Z \subset Z \times \mathbb{P}V$, or can be obtained as the descent of the family $\mathcal{F}$. More details can be found in [18].

4.2.2 *The case $\mu_1 < \frac{1}{2}$.* — We suppose now that the polarization Λ is such that $\mu_1 < \frac{1}{2}$. In this case an element x of W is G-stable if and only if z_1, z_2 are not both zero, and if for every $z \in V^*$, $q_1 - zz_1$ and $q_2 - zz_2$ are linearly independent.

4.2.3. Proposition. — *For this polarization there does not exist a good quotient $W^s(G,\Lambda)//G$.*

Proof. — Let z_1 be a non-zero element of V^*, let $q \in S^2V^* \backslash z_1V^*$, and let $x \in W$ be the matrix

$$\begin{pmatrix} z_1 & 0 \\ q & z_1^2 \end{pmatrix}.$$

Then x is stable.

CLAIM. — *The orbit Gx is closed and if $y \in W^s(G,\Lambda)$ is such that $\overline{Gy}$ meets Gx, then $y \in Gx$.*

Before proving the claim, we will show that it implies Proposition 4.2.3. The stabilizer of a generic point in $W^s(G,\Lambda)$ is isomorphic to $\mathbb{C}^*$: it consists of pairs of homotheties (λ,λ). It follows that if $M = W^s(G,\Lambda)//G$ exists, then all the fibers of the quotient morphism $\pi : W^s(G,\Lambda) \to M$ are of dimension at least $\dim(G) - 1$. Now suppose that the claim is true. Then this implies that $\pi^{-1}(\pi(x)) = Gx$. But the stabilizer G_x of x has dimension 2: it consists of pairs

$$\left(\begin{pmatrix} \alpha & 0 \\ \beta & \alpha \end{pmatrix}, \begin{pmatrix} \alpha & 0 \\ \beta z_1 & \alpha \end{pmatrix}\right)$$

with $\alpha \in \mathbb{C}^*$, $\beta \in \mathbb{C}$, and hence has dimension 2. It follows that $\dim(\pi^{-1}(\pi(x))) < \dim(G) - 1$, a contradiction.

Proof of the claim. — Let $y \in W^s(G,\Lambda)$ such that $x \in \overline{Gy}$. Let

$$y = \begin{pmatrix} z & z_2 \\ q_1 & q_2 \end{pmatrix}.$$

Then z_1 is contained in the vector space spanned by z and z_2. Hence by replacing y with an element of Gy we can assume that $z = z_1$ and that $z_2 = 0$ if z_2 is a multiple of z_1.

According to Lemma 4.2.4 there exist a smooth irreducible curve C, $x_0 \in C$, and a morphism

$$\theta : C \backslash \{x_0\} \longrightarrow G$$

such that

$$\overline{\theta} : C \backslash \{x_0\} \longrightarrow W, \quad t \longmapsto \theta(t)y$$

can be extended to $\overline{\theta} : C \to W$, with $\overline{\theta}(x_0) = x$. We can write, for $t \in C \backslash \{x_0\}$,

$$\overline{\theta}(t) = \begin{pmatrix} a(t)z_1 + b(t)z_2 & c(t)z_1 + d(t)z_2 \\ q_1(t) & q_2(t) \end{pmatrix}$$

with

$$q_1(t) = \lambda(t)\big(a(t)q_1 + b(t)q_2 + u(t)z_1\big), \tag{1}$$

$$q_2(t) = \lambda(t)\big(c(t)q_1 + d(t)q_2 + u(t)z_2\big), \tag{2}$$

where λ, a, b, c, d are morphisms $C\backslash\{x_0\} \to \mathbb{C}$ and $u: C\backslash\{x_0\} \to V^*$. The morphisms λ, a, b, c, d can be extended to morphisms $C \to \mathbb{P}_1 = \mathbb{C} \cup \{0, \infty\}$, denoted by $\overline{\lambda}$, $\overline{a}$, $\overline{b}$, $\overline{c}$, $\overline{d}$ respectively, and u extends to $\overline{u}: C \to \mathbb{P}(V^* \oplus \mathbb{C})$. Now we use the fact that $\overline{\theta}$ is defined at x_0. The first consequence is that $\overline{a}(x_0) = 1$, $\overline{c}(x_0) = 0$, and if $z_2 \neq 0$ then $\overline{b}$ and $\overline{d}$ also vanish at x_0. The second is that the morphisms $q_1, q_2: C\backslash\{x_0\} \to S^2V^*$ can be extended to $\overline{q}_1, \overline{q}_2: C \to S^2V^*$, and we have $\overline{q}_1(x_0) = q$, $\overline{q}_2(x_0) = z_1^2$.

We will now consider three cases: $\overline{\lambda}(x_0) = 0$, $\overline{\lambda}(x_0) = \infty$, $\overline{\lambda}(x_0) \in \mathbb{C}^*$.

Suppose that $\overline{\lambda}(x_0) = 0$. If $z_2 \neq 0$, then (1) implies that $\overline{q}_1(x_0) = q$ is a multiple of z_1, but this is not true. If $z_2 = 0$ then (2) implies that q_2 is a multiple of z_1^2 and (1) implies then that q is also a multiple of z_1, which is not true. Hence we cannot have $\overline{\lambda}(x_0) = 0$.

Suppose that $\overline{\lambda}(x_0) = \infty$. If $z_2 \neq 0$, then (1) implies that

$$\mu: C\backslash\{x_0\} \longrightarrow S^2V^*, \quad t \longmapsto a(t)q_1 + b(t)q_2 + u(t)z_1$$

and

$$\eta: C\backslash\{x_0\} \longrightarrow S^2V^*, \quad t \longmapsto c(t)q_1 + d(t)q_2 + u(t)z_1$$

extend to morphisms $C \to S^2V^*$ which vanish at x_0. It follows from the fact that $\mu(x_0) = 0$ that $u = \overline{u}(x_0) \in V^*$, and that $q_1 = -uz_1$. Since $q_1 \neq 0$ (by G-stability of y), we have $u \neq 0$. But since $\overline{c}(x_0) = \overline{d}(x_0) = 0$, this contradicts the fact that $\eta(x_0) = 0$.

If $z_2 = 0$ then we deduce from the fact that $\mu(x_0) = 0$ that $q_1 \in \langle q_2, V^*z_1\rangle$, which contradicts the G-stability of y.

It follows that we have $\delta = \overline{\lambda}(x_0) \in \mathbb{C}^*$. If $z_2 \neq 0$, using the fact that $\overline{a}(x_0) = 1$ and $\overline{b}(x_0) = \overline{c}(x_0) = \overline{d}(x_0) = 0$ we see that $u = \overline{u}(x_0) \in V^*$ and that $z_1^2 = \delta u z_2$, which contradicts the fact that $z_1 \wedge z_2 \neq 0$.

Hence we have $z_2 = 0$. It follows from (2) that $\overline{d}(x_0) \in \mathbb{C}^*$ and that $z_1^2 = \delta\overline{d}(x_0)q_2$. By (1) we see that

$$\epsilon: C\backslash\{x_0\} \longrightarrow S^2V^*, \quad t \longmapsto b(t)q_2 + u(t)z_1$$

extends to C and that

$$\epsilon(x_0) = \frac{1}{\delta}q - q_1.$$

We have, if $t \neq x_0$

$$\epsilon(t) = z_1\Big(\frac{b(t)}{\delta\overline{d}(x_0)}z_1 + u(t)\Big).$$

It follows that $\epsilon(x_0)$ is a multiple of z_1: $\epsilon(x_0) = z_1 v$. We have then

$$q_1 = \frac{1}{\delta} q - z_1 v \quad \text{and} \quad y = \begin{pmatrix} z_1 & 0 \\ q_1 & q_2 \end{pmatrix} = \begin{pmatrix} z_1 & 0 \\ \frac{1}{\delta} q - z_1 v & \frac{1}{\delta \bar{d}(x_0)} z_1^2 \end{pmatrix} \in Gx$$

as claimed.

It remains to show that Gx is closed. This can be proved easily by computing the stabilizers of all the points in $W^s(G, \Lambda)$. We see then that G_x has the maximal possible dimension, hence Gx is closed. □

We now give a proof of the lemma used in the preceeding proposition:

4.2.4. LEMMA. — *Let W be a finite dimensional vector space, G a linear algebraic group acting algebraically on W, $y \in W$ and $x \in \overline{Gy} \backslash Gy$. Then there exist a smooth curve C, $x_0 \in C$ and a morphism*

$$\theta : C \backslash \{x_0\} \longrightarrow G$$

such that the morphism

$$\bar{\theta} : C \backslash \{x_0\} \longrightarrow W, \quad t \longmapsto \theta(t) y$$

extends to $\bar{\theta} : C \to W$ and that $\bar{\theta}(x_0) = x$.

Proof. — Let $n = \dim(W)$, $d = \dim(Gy)$. The generic $(n-d+1)$-dimensional affine subspace $F \subset W$ through x meets $\overline{Gy}$ on a curve, and meets $\overline{Gy} \backslash Gy$ in a finite number of points. Hence we can find a curve $X \subset \overline{Gy}$ that meets $\overline{Gy} \backslash Gy$ only at x. Taking the normalization of X and substracting a finite number of points or unnecessary components if needed, we obtain a morphism $\alpha : Z \to \overline{Gy}$ (where Z is a smooth curve) and a point $z_0 \in Z$ such that $\alpha(z_0) = x$ and $\alpha(Z \backslash \{z_0\}) \subset Gy$. Consider now the restriction of α

$$Z \backslash \{z_0\} \longrightarrow Gy \simeq G/G_y.$$

There exist a smooth curve Z' and an etale surjective morphism $\phi : Z' \to Z \backslash \{z_0\}$ such that the principal G_y-bundle $\phi^* \alpha^* G$ on Z' is locally trivial. By considering completions $\overline{Z}', \overline{Z}$ of Z', Z and an extension of ϕ to a morphism $\overline{Z}' \to \overline{Z}$ we obtain a smooth curve Y, $y_0 \in Y$ and a morphism $\beta : Y \to Z$ such that $\beta(y_0) = z_0$ and that the principal G_y-bundle $\Gamma = \beta^* \alpha^* G$ is defined on $Y \backslash \{y_0\}$ and locally trivial. Let $U \subset Y$ be a nonempty open subset such that we have a G_y-isomorphism

$$\gamma : \Gamma_{|U} \simeq U \times G_y.$$

Then we can take $C = U \cup \{y_0\}$, $x_0 = y_0$, and for $t \in C \backslash \{x_0\} = U$, we have

$$\theta(t) = \psi\big(\gamma^{-1}(t,e)\big),$$

where ψ is the canonical morphism $\Gamma \to G$. □

4.3. More general counterexamples of inexistence of geometric quotients.

Let W be the space of homomorphisms

$$\mathcal{O}(-2) \oplus \mathcal{O}(-1) \longrightarrow \mathbb{C}^{2n} \otimes \mathcal{O}(1)$$

over $\mathbb{P}_n$ and let the homomorphism $\phi_0 \in W$ be given by the matrix

$$\begin{pmatrix} z_0^2 z_1 & z_1^2 \\ \vdots & \vdots \\ z_0^2 z_n & z_n^2 \\ z_0 z_1^2 & 0 \\ \vdots & \vdots \\ z_0 z_n^2 & 0 \end{pmatrix}$$

where the z_ν are homogeneous coordinates. The stabilizer of ϕ_0 contains $\mathbb{C}^*$ and the pairs

$$\begin{pmatrix} 1 & 0 \\ az_0 & 1 \end{pmatrix}, \quad \begin{pmatrix} I_n & -aI_n \\ 0 & I_n \end{pmatrix}$$

in $\mathrm{Aut}(\mathcal{O}(-2) \oplus \mathcal{O}(-1)) \times \mathrm{GL}(\mathbb{C}^{2n})$ and thus has dimension ≥ 2. If $\Lambda = (\lambda_1, \lambda_2, -\mu_1)$ is a polarization with $0 < \lambda_1, 0 < \lambda_2 < \frac{1}{2}$, then it is easy to see that ϕ_0 is Λ-stable in the sense of 4.1. For example $(m_1', m_2', n') = (0, 1, n)$ is the dimension vector of a ϕ_0-invariant choice of subspaces with $\lambda_1 m_1' + \lambda_2 m_2' - \mu_1 n' = \lambda_2 - \frac{1}{2} < 0$. There are however stable homomorphisms $\phi \in W$ with stabilizer $\mathbb{C}^*$. Therefore $W^s(G,\Lambda)/G$ can never admit the structure of a geometric quotient. We will see in 7.2.2 that a sufficient condition for that in the case of this W is $\lambda_2 > (n+1)\lambda_1$ or $\lambda_2 > (n+1)/(n+2)$ because $\lambda_1 + \lambda_2 = 1$.

5. Embedding into a reductive group action.

We will construct an algebraic reductive group $\mathbf{G}$, a finite dimensional vector space $\mathbf{W}$ on which $\mathbf{G}$ acts algebraically, and an injective morphism

$$\zeta : W \longrightarrow \mathbf{W}$$

compatible with a morphism of groups

$$\theta : G \longrightarrow \mathbf{G}.$$

The traces of $\mathbf{G}$-orbits on $\zeta(W)$ will be exactly the G-orbits. The space $\mathbf{W}$ is of the same type as those studied in 3.1. We will associate naturally to any polarization of the action of G on W a character χ of $\mathbf{G}/\mathbb{C}^*$, i.e. a polarization of the action of $\mathbf{G}$ on $\mathbf{W}$. We will prove that in certain cases a point w of W is G-(semi-)stable with respect to the given polarization if and only if $\zeta(w)$ is χ-(semi-)stable with respect to the associated polarization. The existence of a good and projective quotient of the open set of G-semi-stable points will follow from this.

5.1. Motivation in terms of sheaves.

The idea for the embedding of W into a space $\mathbf{W}$ with a reductive group action is to replace the sheaves $\mathcal{E}_i$ in $\mathcal{E} = \bigoplus(\mathcal{E}_i \otimes M_i)$ by $\mathcal{E}_1 \otimes \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_i)$ and dually the sheaves $\mathcal{F}_\ell$ in $\mathcal{F} = \bigoplus(\mathcal{F}_\ell \otimes N_\ell)$ by $\mathcal{F}_s \otimes \mathrm{Hom}(\mathcal{F}_\ell, \mathcal{F}_s)^*$ and then to consider the induced composed homomorphisms $\gamma(\Phi)$ for $\Phi \in \mathrm{Hom}(\mathcal{E}, \mathcal{F}) = W$:

$$\mathcal{E}_1 \otimes \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}) \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_s \otimes \mathrm{Hom}(\mathcal{F}, \mathcal{F}_s)^*$$

in the bigger space $\widetilde{W}$ of all homomorphisms between $\mathcal{E}_1 \otimes \mathrm{Hom}(\mathcal{E}_1, \mathcal{E})$ and $\mathcal{F}_s \otimes \mathrm{Hom}(\mathcal{F}, \mathcal{F}_s)^*$. This space is naturally acted on by the reductive group

$$\widetilde{G} = \mathrm{GL}\big(\mathrm{Hom}(\mathcal{E}_1, \mathcal{E})\big) \times \mathrm{GL}\big(\mathrm{Hom}(\mathcal{F}, \mathcal{F}_s)^*\big).$$

However it is not suitable enough for our purpose by two reasons. It does not allow enough polarizations as in Section 3 for direct sums in order to have consistency of (semi-)stability and, secondly the group actions $G \times W \to W$ and $\widetilde{G} \times \widetilde{W} \to \widetilde{W}$ don't have consistent orbits. Both insufficiencies are however eliminated when we consider the following enlargement of $\widetilde{W}$. We set

$$P_i = \mathrm{Hom}(\mathcal{E}_i, \mathcal{E}) \quad \text{and} \quad Q_\ell = \mathrm{Hom}(\mathcal{F}, \mathcal{F}_\ell)^*,$$

and introduce the auxiliary spaces

$$\mathbf{W}_L = \bigoplus_{1<i\leq r} \mathrm{Hom}\big(P_i \otimes \mathrm{Hom}(\mathcal{E}_{i-1}, \mathcal{E}_i), P_{i-1}\big),$$

$$\mathbf{W}_R = \bigoplus_{1\leq l<s} \mathrm{Hom}\big(Q_{\ell+1} \otimes \mathrm{Hom}(\mathcal{F}_\ell, \mathcal{F}_{\ell+1}), Q_\ell\big),$$

and define

$$\mathbf{W} = \mathbf{W}_L \oplus \mathrm{Hom}(\mathcal{E}_1 \otimes P_1, \mathcal{F}_s \otimes Q_s) \oplus \mathbf{W}_R.$$

There are distinguished elements

$$(\xi_2, \cdots, \xi_r) \in \mathbf{W}_L, \quad (\eta_1, \cdots, \eta_{s-1}) \in \mathbf{W}_R$$

whose components are the natural composition maps. The embedding of W into $\mathbf{W}$ will be defined as the affine map

$$W \xrightarrow{\zeta} \mathbf{W}, \quad \Phi \longmapsto \big((\xi_2, \cdots, \xi_r), \gamma(\Phi), (\eta_1, \cdots, \eta_{s-1})\big),$$

where $\gamma(\Phi)$ is the above composition for a given $\Phi \in W$. The components of $\mathbf{W}_L$ and $\mathbf{W}_R$ will guarantee a compatible action of a reductive group and at the same time the possibility of choosing enough polarizations for this action.

5.1.1. *Remark.* — One might hope to be able to do induction on r and/or s by simply replacing $M_{r-1} \otimes \mathcal{E}_{r-1} \oplus M_r \otimes \mathcal{E}_r$ by $(M_{r-1} \oplus M_r \otimes \mathrm{Hom}(\mathcal{E}_{r-1}, \mathcal{E}_r)) \otimes \mathcal{E}_{r-1}$ and keeping the other $\mathcal{E}_i$ for $i < r-1$. But then we drop the information about the homomorphisms $\mathcal{E}_i \to \mathcal{E}_r$. Therefore we are lead to replace all $\mathcal{E}_i, i \geq 2$, by $\mathcal{E}_1$ at a time, i.e. by

$$P_1 \otimes \mathcal{E}_1 = (M_1 \oplus M_2 \otimes A_{21} \oplus \cdots \oplus M_r \otimes A_{r1}) \otimes \mathcal{E}_1,$$

where $A_{ji} = \mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j)$. Moreover, in order to keep the information of the homomorphisms $\mathcal{E}_i \to \mathcal{E}_j$ for $2 \leq i \leq j$ we consider also the spaces

$$P_i = M_i \oplus M_{i+1} \otimes A_{i+1,j} \oplus \cdots \oplus M_r \otimes A_{ri}$$

together with the maps $P_i \otimes A_{i,i-1} \to P_{i-1}$ in the following. The reader may convince himself that only because of this the actions of the original group is compatible with the action of the bigger reductive group. It is a beautiful outcome that then we are able to compare the semi-stability with respect to related polarizations in Section 7.

5.2. The abstract definition of W.

The above motivating definition of the space $\mathbf{W}$ can immediately be turned into the following final definition using the spaces $H_{\ell i}, A_{ji}$ and $B_{m\ell}$

and the pairings between them. For any possible i and ℓ we introduce the spaces

$$P_i = \bigoplus_{i\le j\le r} M_j \otimes A_{ji} \quad \text{and} \quad Q_\ell = \bigoplus_{1\le m\le \ell} N_m \otimes B^*_{\ell m},$$

and we denote by p_i and q_ℓ their dimensions. For $1 < i$ and $\ell < s$ we let

$$P_i \otimes A_{i,i-1} \xrightarrow{\xi_i} P_{i-1} \quad \text{and} \quad Q_{\ell+1} \otimes B_{\ell+1,\ell} \xrightarrow{\eta_\ell} Q_\ell$$

be the canonical morphisms, defined as follows. On the component $M_j \otimes A_{ji}$ of P_i, the map ξ_i is the map

$$(M_j \otimes A_{ji}) \otimes A_{i,i-1} \longrightarrow M_j \otimes A_{j,i-1}$$

induced by the composition map of the spaces A. The map η_ℓ is defined in the same way. As in 5.1 we set

$$\mathbf{W}_L = \bigoplus_{1<i\le r} \mathrm{Hom}(P_i \otimes A_{i,i-1}, P_{i-1}),$$
$$\mathbf{W}_R = \bigoplus_{1\le \ell < s} \mathrm{Hom}(Q_{\ell+1} \otimes B_{\ell+1,\ell}, Q_\ell),$$

and

$$\mathbf{W} = \mathbf{W}_L \oplus \mathrm{Hom}(P_1, Q_s \otimes H_{s1}) \oplus \mathbf{W}_R.$$

In order to define the embedding ζ we define the operator γ as follows. Given $w = (\phi_{\ell i}) \in W$with $\phi_{\ell i} \in \mathrm{Hom}(M_i, N_\ell \otimes H_{\ell i})$, we let

$$\gamma(w) \in \mathrm{Hom}(P_1, Q_s \otimes H_{s1}) = \mathrm{Hom}(P_1 \otimes H^*_{s1}, Q_s)$$

be the linear map defined by the matrix $(\gamma_{\ell i}(w))$, for which each $\gamma_{\ell i}(w)$ is the composed linear map

$$M_i \otimes A_{i1} \longrightarrow N_\ell \otimes H_{\ell i} \otimes A_{i1} \longrightarrow N_\ell \otimes H_{\ell 1} \longrightarrow N_\ell \otimes B^*_{s\ell} \otimes H_{s1},$$

where the first map is induced by $\phi_{\ell i}$, the second by the composition $H_{\ell i} \otimes A_{i1} \to H_{\ell 1}$ and the third by the dual composition $H_{\ell 1} \to B^*_{s\ell} \otimes H_{s1}$.

The map ζ can now be defined by

$$W \xrightarrow{\zeta} \mathbf{W}, \quad w \longmapsto \big((\xi_2, \cdots, \xi_r), \gamma(w), (\eta_1, \cdots, \eta_{s-1})\big).$$

5.2.1. LEMMA. — *The linear map γ is injective and hence the morphism ζ is a closed embedding of affine schemes.*

Proof. — From the surjectivity assumptions in 2.1 we find that dually the composition

$$H_{\ell i} \longrightarrow H_{\ell 1} \otimes A_{i1}^* \longrightarrow B_{s\ell}^* \otimes H_{s1} \otimes A_{i1}^*$$

is injective. Now it follows from the definition of $\gamma_{\ell i}(w)$ that $\phi_{\ell i}$ can be recovered from $\gamma_{\ell i}(w)$, by shifting A_{i1} to its dual. □

5.3. The new group G.

We consider now the natural action on $\mathbf{W}$ as described in 3.1 in the general situation, where the group is

$$\mathbf{G} = \mathbf{G}_L \times \mathbf{G}_R, \quad \text{with} \quad \mathbf{G}_L = \prod_{1 \le i \le r} \mathrm{GL}(P_i), \ \mathbf{G}_R = \prod_{1 \le \ell \le s} \mathrm{GL}(Q_\ell).$$

To be precise, this action is described in components by

$$g_{i-1} \circ x_{i-1,i} \circ (g_i \otimes \mathrm{id})^{-1}, \ h_s \circ \psi \circ (g_1 \otimes \mathrm{id})^{-1}, \ h_\ell \circ y_{\ell,\ell+1} \circ (h_{\ell+1} \otimes \mathrm{id})^{-1},$$

with

$$x_{i-1,i} \in \mathrm{Hom}(P_i \otimes A_{i,i-1}, P_{i-1}), \quad \psi \in \mathrm{Hom}(P_1 \otimes H_{s1}^*, Q_s),$$
$$y_{\ell,\ell+1} \in \mathrm{Hom}(Q_{\ell+1} \otimes B_{\ell+1,\ell}, Q_\ell)$$

and with $g_i \in \mathrm{GL}(P_i)$, $h_\ell \in \mathrm{GL}(Q_\ell)$. The first and third expression describe the natural actions of $\mathbf{G}_L$ on $\mathbf{W}_L$ and of $\mathbf{G}_R$ on $\mathbf{W}_R$.

There are also natural embeddings of G_L, G_R, G into $\mathbf{G}_L, \mathbf{G}_R, \mathbf{G}$ respectively. For that it is enough to describe the embedding of G_L in $\mathbf{G}_L$. Given an element $g \in G_L$,

$$g = \begin{pmatrix} g_1 & 0 & \dots & 0 \\ u_{2,1} & g_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ u_{r,1} & \dots & u_{r,r-1} & g_r \end{pmatrix}$$

with $g_i \in \mathrm{GL}(M_i)$ and $u_{ji} \in \mathrm{Hom}(M_i, M_j \otimes A_{ji})$ we define $\theta_{L,i}(g) \in \mathrm{GL}(P_i)$ as the matrix

$$\theta_{L,i}(g) = \begin{pmatrix} \tilde{g}_i & 0 & \dots & 0 \\ \tilde{u}_{i+1,i} & \tilde{g}_{i+1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \tilde{u}_{r,i} & \dots & \tilde{u}_{r,r-1} & \tilde{g}_r \end{pmatrix}$$

with respect to the decomposition of P_i with the following components: $\tilde{g}_j = g_j \otimes \mathrm{id}$ on $M_j \otimes A_{ji}$ and for $i \leq j \leq k$ the map $\tilde{u}_{kj}$ is the composition

$$M_j \otimes A_{ji} \longrightarrow M_k \otimes A_{kj} \otimes A_{ji} \longrightarrow M_k \otimes A_{ki},$$

where the second arrow is induced by the given pairing. In case $j = i$ we have $\tilde{g}_i = g_i$ and $\tilde{u}_{ki} = u_{ki}$. Now we define the map

$$G_L \xrightarrow{\theta_L} \mathbf{G}_L \quad \text{by} \quad g \longmapsto (\theta_{L,1}g, \cdots, \theta_{L,r}g).$$

It is then easy to verify that θ_L is an injective group homomorphism and defines a closed embedding of algebraic groups. With this embedding we consider G_L as a closed subgroup of $\mathbf{G}_L$. In the same way we obtain a closed embedding θ_R of $G_R \subset \mathbf{G}_R$. Finally we obtain the closed embedding $\theta = (\theta_L, \theta_R)$ of $G \subset \mathbf{G}$.

5.3.1. LEMMA. — *The subgroup $G_L \subset \mathbf{G}_L$ (respectively $G_R \subset \mathbf{G}_R$) is the stabilizer of the distinguished element $(\xi_2, \dots, \xi_r) \in \mathbf{W}_L$ (respectively $(\eta_1, \dots, \eta_{s-1}) \in \mathbf{W}_R$).*

Proof. — It is enough to prove the statement only for G_L because of duality. The fact that G_L stabilizes $(\xi_2, \dots, \xi_r)$ is an easy consequence of the properties of the composition maps. The converse can be proved by induction on r. It is trivial for $r = 1$. Suppose that $r \geq 2$ and that the statement is true for $r-1$. Let $(\gamma_1, \dots, \gamma_r)$ be an element of the stabilizer of $(\xi_2, \dots, \xi_r)$. When we replace the space W by W', corresponding to the spaces $M_2, \dots, M_r$ and the same spaces N_ℓ and similarly $\mathbf{W}_L$ by $\mathbf{W}'_L$, then $(\gamma_2, \dots, \gamma_r)$ is an element of the stabilizer of $(\xi_3, \dots, \xi_r)$, so by the induction hypothesis it belongs to G'_L and there exist an element

$$g' = \begin{pmatrix} g_2 & 0 & \cdots & 0 \\ u_{3,2} & g_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ u_{r,2} & \cdots & u_{r,r-1} & g_r \end{pmatrix}$$

such that $(\gamma_2, \dots, \gamma_r) = \theta'_L(g')$. Let now $\gamma_1 \in \mathrm{GL}(P_1)$ have the components

$$M_i \otimes A_{i1} \xrightarrow{y_{ji}} M_j \otimes A_{j1} \quad \text{for all} \quad 1 \leq i, j \leq r.$$

Identity $\gamma_1 \circ \xi_2 = \xi_2 \circ \gamma_2$ then shows that $y_{ji} = 0$ for $j < i$, $y_{ii} = g_i$ for $2 \leq i$ and $y_{ji} = u_{ji}$ for $2 \leq j < i$. Now let $g_1 = y_{11}$, $u_{j1} = y_{j1}$, for $2 \leq j \leq r$, which are linear mappings $M_1 \to M_j \otimes A_{j1}$. Then

$$g = \begin{pmatrix} g_1 & 0 & \cdots & 0 \\ u_{2,1} & g_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ u_{r,1} & \cdots & u_{r,r-1} & g_r \end{pmatrix}$$

is an element of G_L and we have $(\gamma_1, \dots, \gamma_r) = \theta_L(g)$. □

Remark. — Since the action of $\mathbf{G}_L$ on $\mathbf{W}_L$ is linear, it is clear that we have an isomorphism

$$\mathbf{G}_L/G_L \simeq \mathbf{G}_L(\xi_2, \dots, \xi_r), \quad \text{and similarly} \quad \mathbf{G}_R/G_R \simeq \mathbf{G}_R(\eta_1, \dots, \eta_{s-1}).$$

We will use this fact in Section 8.

Using the associativity of the composition maps it is again easy to verify that the actions of G on W and $\mathbf{G}$ on $\mathbf{W}$ are compatible, i.e. that the diagram

$$\begin{tikzcd} G \times W \arrow[r] \arrow[d, "\theta\times\zeta"'] & W \arrow[d, "\zeta"] \\ \mathbf{G} \times \mathbf{W} \arrow[r] & \mathbf{W} \end{tikzcd}$$

is commutative, in which the horizontal maps are the actions. In addition we have the

5.3.2. Corollary. — *Let $w, w' \in W$. Then w and w' are in the same G-orbit in W if and only if $\zeta(w)$ and $\zeta(w')$ are in the same $\mathbf{G}$-orbit in $\mathbf{W}$.*

Proof. — It follows from the compatibility of the actions that if $g \cdot w = w'$ in W then also $\theta(g) \cdot \zeta(w) = \zeta(w')$ in $\mathbf{W}$ by the last diagram. Conversely, if $\mathbf{g} \in \mathbf{G}$ and $\mathbf{g} \cdot \zeta(w) = \zeta(w')$ then g stabilizes $(\xi_2, \cdots, \xi_r, \eta_1, \cdots, \eta_{s-1})$ by the definition of ζ in 5.2. By Lemma 5.3.1 $\mathbf{g} \in G$. □

5.4. The associated polarization.

In 3.3.1 and 3.3.2 we had introduced polarizations for the different types of actions of G_{red} on W and of $\mathbf{G}$ on $\mathbf{W}$. In the following we will describe polarizations on W and $\mathbf{W}$ which are compatible with the morphism $\zeta : W \to \mathbf{W}$. Their weight vectors are related by the following matrix equations and determine each other. The entries of the matrices are just the dimensions of the spaces A_{ji} and $B_{m\ell}$.

In the sequel we will use the following notation: *the dimension of a vector space will be the small version of its name.* So

$$m_i = \dim(M_i), \quad n_\ell = \dim(N_\ell), \quad p_i = \dim(P_i),$$
$$q_m = \dim(Q_m), \quad a_{ji} = \dim(A_{ji}), \quad b_{m\ell} = \dim(B_{m\ell}), \text{ etc.}$$

A proper polarization of the action of G on W is a tuple

$$\Lambda = (\lambda_1, \ldots, \lambda_r, -\mu_1, \ldots, -\mu_s),$$

where λ_i and μ_ℓ are positive rational numbers such that

$$\sum_{1 \le i \le r} \lambda_i m_i = \sum_{1 \le \ell \le s} \mu_\ell n_\ell = 1.$$

We define the new sequence of rational numbers $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s$ by the conditions

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \vdots \\ \lambda_r \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ a_{2,1} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{r,1} & \cdots & a_{r,r-1} & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_r \end{pmatrix},$$

$$\begin{pmatrix} \mu_1 \\ \vdots \\ \vdots \\ \mu_s \end{pmatrix} = \begin{pmatrix} 1 & b_{2,1} & \cdots & b_{s1} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & b_{s,s-1} \\ 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \vdots \\ \beta_s \end{pmatrix}.$$

Then we have

$$1 = \sum_{1 \le i \le r} \lambda_i m_i = \sum_{1 \le i \le r} \alpha_i p_i \quad \text{and} \quad 1 = \sum_{1 \le \ell \le s} \mu_\ell n_\ell = \sum_{1 \le \ell \le s} \beta_\ell q_\ell.$$

In particular the tuple $\widetilde{\Lambda} = (\alpha_1, \cdots, \alpha_r, -\beta_1, \cdots, -\beta_s)$ is a polarization on $\mathbf{W}$ such that α_i is the weight of P_i and $-\beta_\ell$ the weight of Q_ℓ. It is called the *associated polarization* on $\mathbf{W}$. It is compatible with ζ in the following sense: If $M_i' \subset M_i$, and $N_\ell' \subset N_\ell$ are linear subspaces, and if the subspaces of P_i and Q_ℓ are defined by

$$P_i' = \bigoplus_{i \leq j} M_j' \otimes A_{ji}, \quad \text{and} \quad Q_\ell' = \bigoplus_{\ell \leq m} N_\ell' \otimes B_{m\ell}^*$$

respectively then we have

$$\sum_{1 \leq i \leq r} \lambda_i m_i' = \sum_{1 \leq i \leq r} \alpha_i p_i', \quad \text{and} \quad \sum_{1 \leq \ell \leq s} \mu_\ell n_\ell' = \sum_{1 \leq \ell \leq s} \beta_\ell q_\ell'.$$

If the set of stable points in $\mathbf{W}$ with respect to the associated polarization is non-empty then by 3.3.2 the weights satisfy the conditions

$$\sum_{i \leq j \leq r} \alpha_j p_j > 0 \quad \text{for any } i \quad \text{and} \quad \sum_{1 \leq \ell \leq m} \beta_\ell q_\ell > 0 \quad \text{for any } m.$$

Equivalently the conditions may also be written as

$$\sum_{i \leq j \leq r} \alpha_j p_j > 0 \quad \text{for } 2 \leq i \leq r \quad \text{and} \quad 1 - \sum_{m \leq \ell \leq s} \beta_\ell q_\ell > 0 \quad \text{for } 2 \leq m \leq s.$$

Substituting the weights of the original polarization on W, we can reformulate these conditions. In the cases treated in the examples they reduce to the following.

5.4.1. *Weight conditions.* — Let W be of type (r, s) and let $\Lambda = (\lambda_1, \ldots, \lambda_r, -\mu_1, \ldots, -\mu_s)$ be a proper polarization of W with positive λ_i and μ_ℓ. If the set $\mathbf{W}^s(\mathbf{G}, \widetilde{\Lambda})$ of stable points of $\mathbf{W}$ with respect to the associated polarization $\widetilde{\Lambda}$ is non-empty, then in case of

- type $(2,1)$: $\lambda_2 - a_{21}\lambda_1 > 0$,
- type $(3,1)$: $\begin{cases} \lambda_3 - a_{32}\lambda_2 + (a_{32}a_{21} - a_{31})\lambda_1 > 0, \\ \lambda_1(m_1 + a_{21}m_2 + a_{31}m_3) < 1, \end{cases}$
- type $(2,2)$: $\lambda_2 - a_{21}\lambda_1 > 0, \quad \mu_1 - b_{21}\mu_2 > 0.$

5.5. Comparison of invariant polynomials.

In the following we assume that $\widetilde{\Lambda} = (\alpha_1, \dots, \alpha_r, -\beta_1, \dots, -\beta_s)$ is the polarization on $\mathbf{W}$ associated to the polarization $\Lambda = (\lambda_1, \dots, \lambda_r, -\mu_1, \dots, -\mu_s)$. The semi-stable locus $\mathbf{W}^{ss}(\mathbf{G}, \widetilde{\Lambda})$ with respect to this polarization is more precisely defined by the character $\mathcal{X}$ associated to it as in 3.1. If q is lowest common denominator of $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s$, we have

$$\mathcal{X}(\mathbf{g}) = \Big(\prod_{1 \le i \le r} \det(\mathbf{g}_i)^{-q\alpha_i} \Big) \Big(\prod_{1 \le \ell \le s} \det(\mathbf{h}_\ell)^{q\beta_\ell} \Big)$$

for an element $\mathbf{g} \in \mathbf{G}$ with components $\mathbf{g}_i$ and $\mathbf{h}_\ell$. By the matrix relations between the polarizations q is also a common denominator of $\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_s$, such that, if p denotes the lowest, we have $q = pu$ for some u. The character χ with respect to the given polarization can be defined by

$$\chi(g, h) = \prod_{1 \le i \le r} \det(g_i)^{-p\lambda_i} \prod_{1 \le \ell \le s} \det(h_\ell)^{p\mu_\ell},$$

where the g_i (resp. h_ℓ) are the diagonal components of g (resp. h), see 2.2. Now the relations between the polarizations imply by a straightforward calculation that

$$\mathcal{X}\big(\theta(g, h)\big) = \chi(g, h)^u.$$

If F is a $\mathcal{X}^m$-invariant polynomial on $\mathbf{W}$ it follows that

$$F\big(\zeta((g, h) \cdot w)\big) = F\big(\theta(g, h) \cdot \zeta(w)\big) = \chi(g, h)^{um} F\big(\zeta(w)\big),$$

i.e. that $F \circ \zeta$ is a χ^{um}-invariant polynomial on W. As a consequence we obtain the

5.5.1. LEMMA. — *One has $\zeta^{-1}(\mathbf{W}^{ss}(\mathbf{G}, \widetilde{\Lambda})) \subset W^{ss}(G, \Lambda)$, i.e. if $w \in W$ and $\zeta(w)$ is $\mathbf{G}$-semi-stable in $\mathbf{W}$ with respect to the polarization $\widetilde{\Lambda} = (\alpha_1, \dots, \alpha_r, -\beta_1, \dots, -\beta_s)$ then w is G-semi-stable in W with respect to the polarization $\Lambda = (\lambda_1, \dots, \lambda_r, -\mu_1, \dots, -\mu_s)$ (in the sense of* 4.1).

Proof. — There exist a $\mathcal{X}^m$-invariant polynomial F on $\mathbf{W}$ such that $F(\zeta(w)) \neq 0$. Then

$$F\big(\zeta((g, h) \cdot w)\big) = F\big(\zeta(w)\big) \neq 0$$

for any element (g, h) in the unipotent subgroup $H \subset G$. This means that w is G-semi-stable. □

5.5.2. *Remark.* — When we consider the subgroup $\mathbf{G}' \subset \mathbf{G}$ defined by the condition

$$\det(\mathbf{g}_1) = \det(\mathbf{h}_s) = 1,$$

we have $\theta(G') \subset \mathbf{G}'$ as follows from the definition of G' in 3.5. With respect to these groups the semi-stable points are those over the semi-stable loci in $\mathbb{P}(W)$ (resp. $\mathbb{P}(\mathbf{W})$), with respect to the line bundles

$$\mathcal{L} = \mathcal{O}_{\mathbb{P}(W)}(t) \quad \text{an} \quad \mathbf{L} = \mathcal{O}_{\mathbb{P}(\mathbf{W})}(\mathbf{t}),$$

where t and $\mathbf{t}$ is defined as in 3.4 in the different cases endowed with the modified action defined by the characters. However, we cannot compare $\mathbb{P}(W)$ and $\mathbb{P}(\mathbf{W})$ directly because the morphism ζ does not descend.

We need the analogous statement of Lemma 5.5.1 also in the case of stable points. For that it is more convenient to use the subspace criterion (1) of A. King in the case of G_{red} and $\mathbf{G}$. This gives also another proof in the semi-stable case.

5.5.3. Lemma. — *With the same notation as in the previous lemma*

$$\zeta^{-1}\big(\mathbf{W}^s(\mathbf{G},\widetilde{\Lambda})\big) \subset W^s(G,\Lambda).$$

Proof. — Let $w = (\phi_{\ell i})$ be a point of W with maps $M_i \otimes H^*_{\ell i} \xrightarrow{\phi_{\ell i}} N_\ell$ and suppose that w is not G-stable with respect to the polarization Λ. We can assume that it is not G_{red}-stable, too. Then there are linear subspaces $M'_i \subset M_i$ and $N'_\ell \subset N_\ell$ for all i and ℓ such that the family $((M'_i)),(N'_\ell))$ is proper and such that

$$\phi_{\ell i}(M'_i \otimes H^*_{\ell i}) \subset N'_\ell \quad \text{and} \quad \sum_i \lambda_i m'_i - \sum_\ell \mu_\ell n'_\ell \geq 0.$$

With these subspaces we can introduce the subspaces $P'_i \subset P_i$ and $Q'_\ell \subset Q_\ell$ as

$$P'_i = \bigoplus_{i\leq j} M'_j \otimes A_{ji} \quad \text{and} \quad Q'_\ell = \bigoplus_{m\leq \ell} N'_m \otimes B^*_{\ell m}.$$

They form a proper family of subspaces and satisfy

$$\xi_i(P'_i \otimes A_{i,i-1}) \subset P'_{i-1},\ \gamma(w)(P'_1 \otimes H^*_{s1}) \subset Q'_s,\ \eta_\ell(Q'_{\ell+1} \otimes B_{\ell+1,\ell}) \subset Q'_\ell$$

for the possible values of i and ℓ. But by the definition of the spaces and because $\widetilde{\Lambda}$ is the associated polarization, the formulas of 5.4 imply the dimension formula

$$\sum_i \alpha_i p'_i - \sum \beta_\ell q'_\ell = \sum_i \lambda_i m'_i - \sum_\ell \mu_\ell n'_\ell \geq 0.$$

This states that also $\zeta(w)$ is not $\mathbf{G}$-stable. □

In Section 7 we will derive sufficient conditions for the equality

$$\zeta^{-1}\big(\mathbf{W}^s(\mathbf{G},\widetilde{\Lambda})\big) = W^s(G,\Lambda) \quad \text{and} \quad \zeta^{-1}\big(\mathbf{W}^{ss}(\mathbf{G},\widetilde{\Lambda})\big) = W^{ss}(G,\Lambda).$$

In the following section we show how this equality implies the existence of a good and projective quotient $W^{ss}(G,\Lambda)//G$ using the result for $\mathbf{W}^{ss}(\mathbf{G},\widetilde{\Lambda})//\mathbf{G}$ from Geometric Invariant Theory.

6. Construction and properties of the quotient.

We keep the notation of the previous sections and let $\widetilde{\Lambda}$ be the polarization on $\mathbf{W}$ associated to the polarization Λ on W. We do not require that they are proper here, but we will do that later for the examples. In addition we introduce the saturation

$$Z = \mathbf{G}\zeta(W) \subset \mathbf{W}$$

of the image of W with respect to the action of $\mathbf{G}$.

6.1. Construction of the quotient.

6.1.1. Proposition. — *Let W and $\mathbf{W}$ together with their G- and $\mathbf{G}$-structure be as in Section 2 and 5, let Λ be a polarization for (W,G) and $\widetilde{\Lambda}$ be the associated polarization for $(\mathbf{W},\mathbf{G})$.*

1) *If $\zeta^{-1}(\mathbf{W}^s(\mathbf{G},\widetilde{\Lambda})) = W^s(G,\Lambda)$, then there exist a geometric quotient $W^s(G,\Lambda) \to M^s$ of W^s by G, which is a quasi-projective non-singular variety.*

2) *If in addition*

$$\zeta^{-1}\big(\mathbf{W}^{ss}(\mathbf{G},\widetilde{\Lambda})\big) = W^{ss}(G,\Lambda) \quad \text{and} \quad (\overline{Z}\backslash Z)\cap \mathbf{W}^{ss}(\mathbf{G},\widetilde{\Lambda}) = \emptyset,$$

then there exist a good quotient $W^{ss}(G,\Lambda) \xrightarrow{\pi} M$, such that M is a normal projective variety, M^s is an open subset of M, and $W^s(G,\Lambda) \to M^s$ is the restriction of π.

We recall here the definition of a good and a geometric quotient of C.S. Seshadri, see [29], [27]. Let an algebraic group G act on an algebraic variety or algebraic scheme X. Then a pair (φ, Y) of a variety and a morphism $X \xrightarrow{\varphi} Y$ is called a good quotient if

(i) φ is G-equivariant (for the trivial action of G on Y);

(ii) φ is affine and surjective;

(iii) if U is an open subset of Y then φ^* is an isomorphism $\mathcal{O}_Y(U) \approx \mathcal{O}_Y(\varphi^{-1}U)^G$, where the latter denotes the ring of G-invariant functions;

(iv) if F_1, F_2 are disjoint closed and G-invariant subvarieties of X then $\varphi(F_1), \varphi(F_2)$ are closed and disjoint.

If in addition the fibres of φ are the orbits of the action and all have the same dimension, the quotient (φ, Y) is called a geometric quotient.

As usual we write $X//G$ for a good quotient space and X/G for a geometric quotient space.

Proof. — We will prove the second statement first, assuming that the conditions of (1) and (2) are satisfied. We use the abbreviations $W^{ss} = W^{ss}(G, \Lambda), \mathbf{W}^{\mathbf{ss}} = \mathbf{W}^{ss}(\mathbf{G}, \widetilde{\Lambda})$ and similarly $W^s, \mathbf{W}^{\mathbf{s}}$ for the subsets of the stable points. By the result of A. King, 3.1, there exist a good projective quotient of $\mathbf{W}^{ss}$ by the reductive group $\mathbf{G}$. So there exist also a good and projective quotient of the closed invariant subvariety $\overline{Z} \cap \mathbf{W}^{ss}$ which we denote by

$$\overline{Z} \cap \mathbf{W}^{ss} \xrightarrow{\pi_0} M.$$

By assumption (2) $\mathbf{G}\zeta(W^{ss}) = Z \cap \mathbf{W}^{ss} = \overline{Z} \cap \mathbf{W}^{ss}$. We let π be the composition

$$W^{ss} \xrightarrow{\zeta} \mathbf{G}\zeta(W^{ss}) \xrightarrow{\pi_0} M.$$

We know already that M is projective. We will then verify that (π, M) is the good quotient of the proposition. We consider first the commutative diagram

$$\begin{array}{ccc} \mathbf{G} \times W^{ss} & \xrightarrow{\mu} & \mathbf{G}\zeta(W^{ss}) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ W^{ss} & \xrightarrow{\pi} & M \end{array}$$

in which p is the projection and μ is defined by $(\mathbf{g}, w) \mapsto \mathbf{g}\zeta(w)$. There is an action of G on $\mathbf{G} \times W^{ss}$ by $g \cdot (\mathbf{g}, w) = (\mathbf{g}\theta(g)^{-1}, g \cdot w)$ and it follows that μ is G-equivariant.

CLAIM. — *The morphism μ is a geometric quotient of* $\mathbf{G} \times W^{ss}$ *by* G.

Proof of the claim. — We show first that the fibres of μ are the G-orbits. So let $(\mathbf{g}, w), (\mathbf{g}', w')$ be two elements in $\mathbf{G} \times W^{ss}$ such that $\mu(\mathbf{g}, w) = \mu(\mathbf{g}', w')$. Then $\zeta(w) = \mathbf{g}^{-1}\mathbf{g}'\zeta(w')$. By Lemma 5.3.1 $g = \mathbf{g}^{-1}\mathbf{g}' \in G$ and $g \cdot (\mathbf{g}, w) = (\mathbf{g}', w')$. The claim will be proved if we show that μ has local sections. For this it suffices to use the remark following Lemma 5.3.1 and a local section of the quotient map $\mathbf{G} \to \mathbf{G}/G$.

Now we are going to verify the four properties of a good quotient for π. Clearly (i) is satisfied by the definition of π.

Proof of (ii). It is clear that π is surjective. The morphism π is affine because $\pi = \pi_0 \circ \zeta$ and π_0 and ζ are affine.

Proof of (iii). Let $U \subset M$ be an open subset. Then

$$\mathcal{O}(U) \subset \mathcal{O}\big(\pi^{-1}(U)\big)^G$$

since π is G-invariant. Conversely let $f \in \mathcal{O}(\pi^{-1}(U))^G$. Then $f \circ p \in \mathcal{O}(\mathbf{G} \times \pi^{-1}(U))^G$, and since μ is a geometric quotient, $f \circ p$ descends to an $\bar{f} \in \mathcal{O}(\mu(\mathbf{G} \times \pi^{-1}(U)))$, which is $\mathbf{G}$-invariant. Now again $\bar{f}$ descends because π_0 is a good quotient. This proves equality $\mathcal{O}(U) = \mathcal{O}(\pi^{-1}(U))^G$.

Proof of (iv). Let F_1, F_2 be disjoint, closed, G-invariant subvarieties of W^{ss}. Then $p^{-1}(F_1), p^{-1}(F_2)$ are disjoint, closed and $\mathbf{G}$-invariant subvarieties of $\mathbf{G} \times W^{ss}$. Since μ is a good quotient, $\mu(p^{-1}(F_1)), \mu(p^{-1}(F_2))$ are disjoint, closed and $\mathbf{G}$-invariant in $\mathbf{G}\zeta(W^{ss})$. Finally, since π_0 is a good quotient, $\pi_0 \circ \mu(p^{-1}(F_1)), \pi_0 \circ \mu(p^{-1}(F_2))$ are disjoint and closed subvarieties of M. But $\pi_0 \circ \mu(p^{-1}(F_i)) = \pi(F_i)$, which proves (iv).

The normality of M follows from the fact that $\mathbf{G}\zeta(W^{ss})$ is smooth and π_0 is a good quotient, [27], with respect to the reductive group $\mathbf{G}$. That π becomes a geometric quotient on the open set W^s of stable points follows from the fact that the $\mathbf{G}$-orbits in $\mathbf{G}\zeta(W^s) = Z \cap \mathbf{W}^s$ intersect W^s in G-orbits. In particular the stabilizers of w in G and of $\zeta(w)$ in $\mathbf{G}$ are isomorphic, such that all orbits have the same dimension.

The proof of (1) is a modification of the above. In any case π_0 induces the geometric quotient $\bar{Z} \cap \mathbf{W}^s \xrightarrow{\pi_0} M_0$ with M_0 open in M. Now $\mathbf{G}\zeta(W^s) = Z \cap \mathbf{W}^s$ is a π_0-saturated open subset of $\bar{Z} \cap \mathbf{W}^s$, such that we obtain a geometric quotient $\mathbf{G}\zeta(W^s) \xrightarrow{\pi_0} M^s$ with $M^s \subset M_0$ open. By the same arguments as above applied to the diagram related to $\mathbf{G} \times W^s \to \mathbf{G}\zeta(W^s)$ we conclude that $W^s \xrightarrow{\pi} M^s$ is a geometric quotient. $\square$

Remarks. 1) The idea of this proof comes from [34], and has already been used in [13] and [8].

2) If the second condition of (2) is not satisfied, we cannot even prove that $W^{ss}(G,\Lambda)$ admits a good quasi-projective quotient, because $Z \cap \mathbf{W}^{ss}$ might not be saturated. Of course the projectivity of the quotient depends on this condition.

6.2. *S*-equivalence.

We suppose that the hypotheses of Proposition 6.1.1 are satisfied, with polarization Λ for (W,G) and associated polarization $\widetilde{\Lambda}$ for $(\mathbf{W},\mathbf{G})$.

It is easy to define the *Jordan-Hölder filtration* of $\mathbf{G}$-semi-stable elements of $\mathbf{W}$ with respect to $\widetilde{\Lambda}$ (cf. [21] for a more general situation). Using the preceding results we can also define a *Jordan-Hölder filtration* of a G-semi-stable element of W with respect to Λ. Let $w = (\phi_{\ell i}) \in W^{ss}(G,\Lambda)$. Then there exist a positive integer p, an element $h \in H$ and filtrations

$$M_i^0 = \{0\} \subset M_i^1 \subset \cdots \subset M_i^p = M_i,$$
$$N_\ell^0 = \{0\} \subset N_\ell^1 \subset \cdots \subset N_\ell^p = N_\ell,$$

with

$$\sum_i \lambda_i \dim(M_i^j) = \sum_\ell \mu_\ell \dim(N_\ell^j)$$

for each j, such that $h \cdot w = (\phi_{\ell i})$ satisfies

$$\phi_{\ell i}(H_{\ell i}^* \otimes M_i^j) \subset N_\ell^j,$$

and that if

$$\phi_{\ell i}^j : H_{\ell i}^* \otimes (M_i^j / M_i^{j-1}) \longrightarrow N_\ell^j / N_\ell^{j-1}$$

is the induced morphism, then $(\phi_{\ell i}^j)_{\ell i}$ is G-stable with respect to Λ for any j. This filtration and h need not be unique, but p is unique and the $(\phi_{\ell i}^j)$, too, up to the order and isomorphisms. Conversely, an element of W having such a filtration is G-semi-stable with respect to Λ. We say that two elements $(\phi_{\ell i})$ and $(\phi'_{\ell i})$ of $W^{ss}(G,\Lambda)$ are *S-equivalent* if they have Jordan-Hölder decompositions $(\phi_{\ell i}^j), (\phi'^j_{\ell i})$ respectively of the same length, and if there exist a permutation σ of $\{1,\ldots,p\}$ such that $(\phi'^j)_{\ell i}$ is isomorphic to $(\phi_{\ell i}^{\sigma(j)})$ for any j.

The following result is also easily deduced from 6.1.1.

6.2.1. Proposition. — *Let $w, w' \in W^{ss}(G, \Lambda)$. Then $\pi(w) = \pi(w')$ if and only if w and w' are S-equivalent.*

It follows that the set of closed points of M is exactly the set of S-equivalence classes of elements of W^{ss}.

7. Comparison of semi-stability.

We are going to investigate conditions for the weights of the polarizations under which a (semi-)stable point $w \in W$ is mapped to a (semi-)stable point $\zeta(w) \in \mathbf{W}$. For the estimates we need the following constants which depend on the dimensions m_i and the composition maps $H_{\ell i} \otimes A_{i1} \to H_{\ell 1}$.

7.1. Constants.

Let $\mathcal{K}$ be the family of proper linear subspaces

$$K \subset \bigoplus_{2 \leq i} M_i \otimes A_{i1}$$

such that K is not contained in $\bigoplus_{2 \leq i} M_i' \otimes A_{i1}$ for any family $(M_i') \neq (M_i)$ of subspaces. For any ℓ we let the map

$$\bigoplus_{2 \leq i} M_i \otimes A_{i1} \otimes H_{\ell 1}^* \xrightarrow{\delta_\ell} \bigoplus_{2 \leq i} M_i \otimes H_{\ell i}^*$$

be induced by the maps $A_{i1} \otimes H_{\ell 1}^* \to H_{\ell i}^*$ associated to the composition maps, which are supposed to be surjective, see 2.1.

We introduce the constant

$$c_\ell(m_2, \ldots, m_r) = \sup_{K \in \mathcal{K}} \rho_\ell(K)$$

with $\rho_\ell(K) = \operatorname{codim}(\delta_\ell(K \otimes H_{\ell 1}^*))/\operatorname{codim}(K)$. Similarly we define the constants $d_i(n_1, \ldots, n_{s-1})$ in the dual situation. Let

$$\bigoplus_{\ell < s} N_\ell^* \otimes H_{\ell i}^* \xleftarrow{\delta_i^\vee} \bigoplus_{\ell < s} N_\ell^* \otimes B_{s\ell} \otimes H_{si}^*$$

be induced by the maps $B_{s\ell} \otimes H_{si}^* \to H_{\ell i}^*$ and let $\mathcal{L}$ be the family of proper subspaces

$$L \subset \bigoplus_{\ell < s} N_\ell^* \otimes B_{s\ell}$$

which are not contained in $\bigoplus_{\ell<s} N'_\ell \otimes B_{s\ell}$ for any family $(N'_\ell) \neq (N^*_\ell)$ of subspaces. Then we define

$$d_i(n) = d_i(n_1, \dots, n_{s-1}) = \sup_{L \in \mathcal{L}} \frac{\operatorname{codim}(\delta_i^\vee(L \otimes H^*_{si}))}{\operatorname{codim}(L)}.$$

7.1.1. Lemma. — *If $m_i \leq \overline{m}_i$ for all $i \geq 2$, then*

$$c_\ell(m_2, \dots, m_r) \leq c_\ell(\overline{m}_2, \dots, \overline{m}_r).$$

Proof. — It will be sufficient to assume that $m_i = \overline{m}_i$ for all i except one, $m_2 < \overline{m}_2$ say. Then let $\overline{M}_i$ be vector spaces of dimensions $\overline{m}_i$ and suppose that

$$\overline{M}_2 = L_2 \oplus M_2 \quad \text{and} \quad \overline{M}_i = M_i \quad \text{for} \quad i \geq 3.$$

For any $K \in \mathcal{K}$ we consider the subspace

$$\overline{K} = (L_2 \otimes A_{21}) \oplus K \subset (\overline{M}_2 \otimes A_{21}) \oplus \Big(\bigoplus_{2<j} M_j \otimes A_{j1}\Big).$$

Then $\operatorname{codim}(\overline{K}) = \operatorname{codim}(K)$ and also

$$\operatorname{codim}\big(\delta_\ell(\overline{K} \otimes H^*_{\ell 1})\big) = \operatorname{codim}\big(\delta_\ell(K \otimes H^*_{\ell 1})\big)$$

because δ_ℓ is a direct sum of the surjective operator $A_{j1} \otimes H^*_{\ell 1} \to H^*_{\ell 1}$ such that $\delta_\ell(L_2 \otimes A_{21} \otimes H^*_{\ell 1})$ equals $L_2 \otimes H^*_{\ell 2}$ and

$$\delta_\ell(\overline{K} \otimes H^*_{\ell 1}) = (L_2 \otimes H^*_{\ell 2}) \oplus \delta_\ell(K \otimes H^*_{\ell 1}).$$

Therefore $\rho_\ell(K) = \rho_\ell(\overline{K})$. Once we have shown that also $\overline{K}$ belongs to the analogous family $\overline{\mathcal{K}}$, the lemma is proved. To see this let $\overline{M}'_2 \subset \overline{M}_2$ and $\overline{M}'_i = M'_i \subset M_i$ for $i \geq 3$ be subspaces such that

$$\overline{K} \subset \bigoplus_{2 \leq i} \overline{M}'_i \otimes A_{i,1}.$$

Then in particular

$$L_2 \otimes A_{21} \subset \overline{M}'_2 \otimes A_{21}$$

and thus $L_2 \subset \overline{M}'_2$. But then $\overline{M}'_2 = L_2 \oplus M'_2$ with $M'_2 = \overline{M}'_2 \cap M_2$ and it follows that

$$K \subset \bigoplus_{2 \leq i} M'_i \otimes A_{i1}.$$

Since $K \in \mathcal{K}$ we obtain $M'_i = M_i$ for all i and then also $\overline{M}'_2 = \overline{M}_2$. □

7.2. Study of the converse I.

Let $\Lambda = (\lambda_1, \dots \lambda_r, -\mu_1, \dots, -\mu_s)$ be a polarization on W and let $\widetilde{\Lambda} = (\alpha_1, \dots, \alpha_r, -\beta_1, \dots, -\beta_s)$ be the associated polarization on $\mathbf{W}$ (the associated polarization has been defined in 5.4). We had shown in 5.5.1 and 5.5.3 that if $w \in W$ and $\zeta(w)$ is (semi-)stable in $\mathbf{W}$ with respect to $\mathbf{G}$ and $\widetilde{\Lambda}$, then so is w with respect to G and Λ. We are going to derive sufficient conditions for the converse, i.e. whether $\zeta(w)$ is (semi-)stable if w is (semi-)stable.

In the sequel we are going to use the following notation:

Given a family $M' = (M_i')$ of subspaces $M_i' \subset M_i$ we set

$$P_i(M') = \bigoplus_{i \leq j} M_j' \otimes A_{ji}$$

and call a subspace $P_i' \subset P_i$ *saturated* if there is such a family with $P_i' = P_i(M')$. Note that in this case $\sum_i \alpha_i p_i' = \sum_i \lambda_i m_i'$. Similarly we introduce the spaces $Q_\ell(N')$ for a subfamily $N' = (N_\ell')$ of (N_ℓ) and call them *saturated.*

Let $w = (\phi_{\ell i})$ be given and assume that $\zeta(w)$ is not semi-stable with respect to $\widetilde{\Lambda}$. Then there exist linear subspaces $P_i' \subset P_i$ and $Q_\ell' \subset Q_\ell$ such that

$$\xi_i(P_i' \otimes A_{i,i-1}) \subset P_{i-1}', \;\; \gamma(w)(P_1' \otimes H_{s1}^*) \subset Q_s', \;\; \eta_\ell(Q_{\ell+1}' \otimes B_{\ell+1,\ell}) \subset Q_\ell'$$

and such that

$$\sum_i \alpha_i p_i' - \sum_\ell \beta_\ell q_\ell' > 0,$$

where as before the small characters denote the dimension of the spaces. If there were subspaces $M_i' \subset M_i$ and $N_\ell' \subset N_\ell$ with $P_i' = P_i(M')$ and $Q_\ell' = Q_\ell(N')$ as in 5.5.3, then $\gamma(w)(P_1' \otimes H_{s1}^*) \subset Q_s'$ would imply that $\varphi_{\ell i}(M_i' \otimes H_{\ell i}^*) \subset N_\ell'$ and we would have

$$\sum_i \lambda_i m_i' - \sum_\ell \mu_\ell n_\ell' = \sum_i \alpha_i p_i' - \sum_\ell \beta_\ell q_\ell' > 0,$$

and w would not be semi-stable. In the following we are going to construct families M'', N'' of subspaces $M_i'' \subset M_i$ and $N_\ell'' \subset N_\ell$ such that $P_i'' = P_i(M'')$ and $Q_\ell'' = Q_\ell(N'')$ are as close to P_i', Q_ℓ' as possible and such that there is a useful estimate for

$$\sum_i \lambda_i m_i'' - \sum_\ell \mu_\ell n_\ell''.$$

• *Step 1.* — We can assume that P'_i has a decomposition

$$P'_i = M'_i \oplus X_i \quad \text{in} \quad M_i \oplus \Big(\bigoplus_{i<j} M_j \otimes A_{ji}\Big)$$

and such that $X_r = 0$. To derive this, we remark that for a subspace S of a direct sum $E \oplus F$ of vector spaces there exist a linear map $E \xrightarrow{u} F$ such that the isomorphism $\left(\begin{smallmatrix} 1 & 0 \\ u & 1 \end{smallmatrix}\right)$ of $E \oplus F$ transforms S into $S' \oplus S''$, where S' is the projection of S in E and $S'' = S \cap F$. Using this and descending induction on i we can find an element $h \in H_L \subset G_L$, see 2.4, such that the truncations $\theta_{L,i}(h) \in \mathrm{GL}(P_i)$, see 5.3, map P'_i onto a direct sum $M'_i \oplus X_i$ for any i. Since $\xi_i(P'_i \otimes A_{i,i-1}) \subset P'_{i-1}$ we easily derive that

$$\bigoplus_{i<j} M'_j \otimes A_{ji} \subset X_i \subset \bigoplus_{i<j} M_j \otimes A_{ji}$$

for all possible i. We put

$$\rho_i = \operatorname{codim}\Big(\bigoplus_{i<j} M'_j \otimes A_{ji}, X_i\Big) = \operatorname{codim}\big(P_i(M'), P'_i\big).$$

Note that $\rho_r = 0$.

• *Step 2.* — Let $M''_1, \dots M''_r$ be subspaces of $M_1, \dots, M_r$ respectively such that

$$P_i(M'') \supset P'_i$$

is minimal over P'_i for any i. Then $M'_i \subset M''_i$ since these spaces are the first components of $P'_i \subset P_i(M'')$ respectively and we have $M'_1 = M''_1$. We let

$$\sigma_i = \sum_{i \le j} (m''_j - m'_j) a_{ji} = \operatorname{codim}\big(P_i(M'), P_i(M'')\big).$$

• *Step 3.* — We are going to define the subspaces $N'_\ell \subset N''_\ell \subset N_\ell$ as images.

Let $P_1 \otimes H^*_{\ell 1} \xrightarrow{\gamma_\ell(w)} N_\ell$ be the map which is the sum of the composed maps

$$M_i \otimes A_{i1} \otimes H^*_{\ell 1} \to M_i \otimes H^*_{\ell i} \xrightarrow{\phi_{\ell i}} N_\ell.$$

Then we define

$$N'_\ell = \gamma_\ell(w)(P'_1 \otimes H^*_{\ell 1}) = \phi_{\ell 1}(M'_1 \otimes H^*_{\ell 1}) + \gamma_\ell(w)(X_1 \otimes H^*_{\ell 1}),$$

$$N''_\ell = \gamma_\ell(w)\big(P_1(M'') \otimes H^*_{\ell 1}\big) = \phi_{\ell 1}(M''_1 \otimes H^*_{\ell 1}) + \sum_{2 \le j} \phi_{\ell j}(M''_j \otimes H^*_{\ell j}).$$

It follows $N'_\ell \subset N''_\ell$ for any ℓ.

• *Step 4.* — If the weights β_ℓ are supposed to be positive, we may assume that

$$\gamma(w)(P_1' \otimes H_{s1}^*) = Q_s' \quad \text{and} \quad \eta_\ell(Q_{\ell+1}' \otimes B_{\ell+1,\ell}) = Q_\ell'$$

for $\ell < s$. Otherwise we could choose subspaces $\overline{Q}_\ell' \subset Q_\ell'$ by descending induction as images. Then $-\sum_\ell \beta_\ell \overline{q}_\ell' \geq -\sum \beta_\ell q_\ell'$ would improve the assumption on the choice of the spaces P_i' and Q_ℓ'. Now it follows that for any ℓ,

$$Q_\ell' \subset Q_\ell(N'')$$

because $P_1' \otimes H_{s1}^*$ is mapped to $\oplus_{\ell \leq s} N_\ell'' \otimes B_{s\ell}^*$ and the maps η_ℓ are the identity on the spaces N_m''. Note that we even have $Q_\ell' \subset Q_\ell(N')$ since $\gamma_{\ell | P_1' \otimes H_{s1}^*}$ factorises through $\bigoplus_{\ell \leq s} N_L' \otimes B_{s\ell}^*$ as follows from the definition of N_ℓ'.

7.2.1. Lemma. — *Suppose that all $\beta_1, \dots, \beta_s > 0$, and let $\Delta = \sum_i \lambda_i m_i'' - \sum_\ell \mu_\ell n_\ell''$. Then*

$$\Delta > \sum_\ell \beta_\ell q_\ell' - \sum_\ell \mu_\ell n_\ell' + \sum_i \alpha_i(\sigma_i - \rho_i) - \sum_\ell \mu_\ell c_\ell(m_2, \dots, m_r)(\sigma_1 - \rho_1).$$

Proof. — Let $Y_\ell = \delta_\ell(X_1 \otimes H_{\ell 1}^*) \subset Z_\ell = \bigoplus_{2 \leq i} M_i'' \otimes H_{\ell i}^*$. Since X_1 is not contained in a direct sum with spaces smaller than M_i'' we get

$$\operatorname{codim}(Y_\ell, Z_\ell) \leq c_\ell(m_2'', \dots, m_r'') \operatorname{codim}\Big(X_1, \bigoplus_{2 \leq j} M_j'' \otimes A_{j1}\Big).$$

By Lemma 7.1.1 and above definitions we get

$$\begin{aligned} \operatorname{codim}(Y_\ell, Z_\ell) &\leq c_\ell(m_2'', \dots, m_r'')\Big(\sum_{1 \leq i} m_i'' a_{i1} - p_1'\Big) \\ &= c_\ell(m_2, \dots, m_r)\Big(\sum_{2 \leq i} (m_i'' - m_i') a_{i1} - \rho_1\Big). \end{aligned}$$

The map $\sum_j \phi_{\ell j}$ sends $(M_1'' \otimes H_{\ell 1}^*) \oplus Z_\ell$ onto N_ℓ'' by definition of N_ℓ'' and also maps $(M_1' \otimes H_{\ell 1}^*) \oplus \delta_\ell(X_1 \otimes H_{\ell 1}^*)$ onto N_ℓ'. Therefore, since $M_1' = M_1''$, we have a surjection

$$Z_\ell / Y_\ell \longrightarrow N_\ell'' / N_\ell'$$

and the dimension estimate

$$n_\ell'' - n_\ell' \le c_\ell(m_2, \dots, m_r)\Big(\sum_{2\le i}(m_i'' - m_i')a_{i1} - \rho_1\Big).$$

Now we can derive the estimate of the lemma. If there is no summation condition it is understood that the sum has to be taken over all indices of the given interval. We have

$$\begin{aligned}\Delta &= \sum_i \lambda_i m_i'' - \sum_\ell \mu_\ell n_\ell'' \\ &= \sum_i \lambda_i m_i' - \sum_\ell \mu_\ell n_\ell' + \sum_j \lambda_j(m_j'' - m_j') - \sum_\ell \mu_\ell(n_\ell'' - n_\ell').\end{aligned}$$

Substituting for λ_j in the third sum and replacing the first by

$$\sum_i \lambda_i m_i' = \sum_i \alpha_i \dim\Big(\bigoplus_{i\le j} M_j' \otimes A_{ji}\Big) = \sum_i \alpha_i(p_i' - \rho_i)$$

and using the definition of σ_i we get

$$\Delta = \sum_i \alpha_i p_i' - \sum_\ell \mu_\ell n_\ell' + \sum_i \alpha_i(\sigma_i - \rho_i) - \sum_\ell \mu_\ell(n_\ell'' - n_\ell').$$

Now using the assumed estimate for the first sum and the derived estimate for $n_\ell'' - n_\ell'$ we get

$$\begin{aligned}\Delta > \sum_\ell \beta_\ell q_\ell' - \sum_\ell \mu_\ell n_\ell' + \sum_i \alpha_i(\sigma_i - \rho_i) \\ - \sum_\ell \mu_\ell c_\ell(m_2, \dots, m_r)(\sigma_1 - \rho_1).\end{aligned} \qquad \square$$

7.2.2. Corollary. — *Suppose that $s = 1$, let $\Lambda = (\lambda_1, \dots, \lambda_r, -1/n_1)$ and let $\widetilde{\Lambda}$ be the associated polarization $(\alpha_1, \dots, \alpha_r, -1/n_1)$. If all $\alpha_i > 0$ and if*

$$\lambda_2 \ge \frac{a_{21}}{n_1} c_1(m_2, \dots, m_r)$$

then

$$\zeta^{-1}\mathbf{W}^{ss}(\mathbf{G}, \widetilde{\Lambda}) = W^{ss}(G, \Lambda) \quad \textit{and} \qquad \zeta^{-1}\mathbf{W}^{s}(\mathbf{G}, \widetilde{\Lambda}) = W^{s}(G, \Lambda).$$

Remarks. 1) Note that by the normalization of the polarizations we must have $\mu_1 n_1 = 1$ such that $1/n_1$ is the only possible value for $\mu_1 = \beta_1$.

2) If all $\alpha_i > 0$, then the necessary conditions for $W^s(G,\Lambda) \neq \emptyset$ and $\mathbf{W}^s(\mathbf{G},\widetilde{\Lambda}) \neq \emptyset$ are both satisfied, see 5.4. The condition of the corollary is an extra condition.

Proof. — Let us first assume that $\zeta(w)$ is not semi-stable and let the spaces P_i' and Q_1' be as at the beginning of 7.2. The only $\beta_1 = 1/n_1$ is positive. Let the other spaces be chosen as in 7.2. The difference $\sum \beta_\ell q_\ell' - \sum \mu_\ell n_\ell'$ reduces to $q_1'/n_1 - n_1'/n_1$, and since $N_1' = \gamma(w)(P_1' \otimes H_{11}^*) = Q_1'$, this difference is zero. Therefore

$$\Delta > \sum_i \alpha_i(\sigma_i - \rho_i) - \frac{1}{n_1} c_1(m_2, \dots, m_r)(\sigma_1 - \rho_1).$$

Since all the α_i are positive we have

$$\sum_i \alpha_i(\sigma_i - \rho_i) \geq \alpha_1(\sigma_1 - \rho_1) + \alpha_2(\sigma_2 - \rho_2).$$

Moreover, ξ_2 induces a surjection

$$P_2(M'') \otimes A_{21}/P_2' \otimes A_{21} \longrightarrow P_1(M'')/P_1'$$

because $M_1' = M_1''$. Therefore we obtain the dimensions estimate $(\sigma_2 - \rho_2)a_{21} \geq \sigma_1 - \rho_1$. It follows that

$$\Delta > \Big(-\frac{1}{n_1} c_1(m_2, \dots, m_r) + \alpha_1 + \frac{\alpha_2}{a_{21}} \Big)(\sigma_1 - \rho_1).$$

Since $\lambda_2 = a_{21}\alpha_1 + \alpha_2 \geq (a_{21}/n_1)c_1(m_2, \dots, m_r)$ the last expression is non-negative. This proves the case of semi-stability. For the case of stability we assume that w is stable and that $\zeta(w)$ is already semi-stable. If $\zeta(w)$ were not stable, we would find subspaces P_i' and N_1' as in 7.2 such that $\sum \alpha_i p_i' - \mu_1 n_1' = 0$ and such that at least one P_i' is different from P_i. Now let the spaces M_i'' and N_I'' be constructed as above. Then we have

$$\Delta \geq \sum_i \alpha_i s_i - \frac{c_1}{n_1} s_1 \geq \sum_{2<i} \alpha_i s_i + \Big(\lambda_2 - \frac{c_1}{n_1} a_{21}\Big)\frac{s_1}{a_{21}} \geq 0,$$

where $s_i = \sigma_i - \rho_i = \dim(P_i(M'')/P_i')$, and where we use that $s_2 a_{21} \geq s_1$. If the family M'' is different from M, then $0 > \Delta$, and if it is equal, then $\Delta = 0$. In order to obtain a contradiction we have to show that M'' is different from M. Assume that it is not. Then $s_i = \dim(P_i/P_i')$ and we must have $s_i = 0$ for $i \geq 3$ and $s_1(\lambda_2 - (c_1/n_1)a_{21}) = 0$. If also $s_1 = 0$, then by the above estimate also $s_2 = 0$, contradicting the choice of the P_i'. Therefore $s_1 \neq 0$ and $\lambda_2 = (c_1/n_1)a_{21}$. But then $\Delta = \alpha_2(s_2 - s_1/a_{21})$ and we have $s_2 a_{21} = s_1$. From this it is easy to see that $P_i' = P_i(\widetilde{M})$ where $\widetilde{M}_i = M_i$ for $i \neq 2$ and $\widetilde{M}_2 = M_2' \neq M_2$. Then we have

$$\sum_i \alpha_i \widetilde{m}_i - \mu_1 n_1' = \sum_i \lambda_i p_i' - \mu_1 n_1' = 0$$

which contradicts the stability of w. □

7.3. Study of the converse II.

We keep the notation of 7.2 and compare the (semi-)stability of points in W and $\mathbf{W}$ in two steps, each reducing to the case $s = 1$. We consider the intermediate space

$$\mathbf{V} = \mathbf{W}_L \oplus \bigoplus_{1 \leq \ell \leq s} \mathrm{Hom}(P_1 \otimes H_{\ell 1}^*, N_\ell)$$

and the maps

$$W \xrightarrow{\zeta_1} \mathbf{V} \xrightarrow{\zeta_2} \mathbf{W}.$$

Here ζ_1 is defined by

$$w \longmapsto (\xi_2, \ldots, \xi_r, \gamma_1(w), \ldots, \gamma_s(w)),$$

where $\gamma_\ell(w)$ is the map defined by $w = (\phi_{\ell i})$ as in 7.2. The map ζ_2 is defined by

$$(x_2, \ldots, x_r, \gamma_1, \ldots, \gamma_s) \longmapsto (x_2, \ldots, x_r, \gamma, \eta_1, \ldots, \eta_{s-1}),$$

where now $\gamma : P_1 \otimes H_{s1}^* \to Q_s$ is induced by the tuple $(\gamma_1, \ldots, \gamma_s)$ as the sum of the compositions

$$P_1 \otimes H_{s1}^* \longrightarrow N_\ell \otimes H_{\ell 1} \otimes H_{s1}^* \longrightarrow N_\ell \otimes B_{s\ell}^*$$

which are induced by the γ_ℓ and the pairings $B_{s\ell} \otimes H_{\ell 1} \to H_{s1}$. It is obvious that

$$\zeta = \zeta_2 \circ \zeta_1.$$

Note that both ζ_1 and ζ_2 are injective by the same reason as for ζ.

On $\mathbf{V}$ the group $\mathbf{G}_L \times G_R$ acts naturally and we have the embedding

$$G = G_L \times G_R \xhookrightarrow{\theta_L \times \mathrm{id}} \mathbf{G}_L \times G_R,$$

see 5.3. It follows as in Section 5 that ζ_1 is compatible with the group actions and that $w, w' \in W$ are on the same G-orbit if and only if $\zeta_1(w), \zeta_1(w')$ are on the same $\mathbf{G}_L \times G_R$ orbit. Similarly we have the group embedding $\mathbf{G}_L \times G_R \hookrightarrow \mathbf{G}_L \times \mathbf{G}_R = \mathbf{G}$ and ζ_2 is equivariant and satisfies the analogous statements for the orbits. Given the polarization $\Lambda = (\lambda_1, \dots, \lambda_r, -\mu_1, \dots, -\mu_s)$ for (W, G) we consider the polarization $\overline{\Lambda} = (\alpha_1, \dots, \alpha_r, -\mu_1, \dots, -\mu_s)$ for $(\mathbf{V}, \mathbf{G}_L \times G_R)$ where the α_i are defined as in 5.4. As in 5.5.1, 5.5.3 it is easy to show that

$$\zeta_1^{-1}\mathbf{V}^{ss}(\mathbf{G}_L \times G_R, \overline{\Lambda}) \subset W^{ss}(G, \Lambda), \quad \zeta_1^{-1}\mathbf{V}^{s}(\mathbf{G}_L \times G_R, \overline{\Lambda}) \subset W^{s}(G, \Lambda)$$

and similarly that

$$\zeta_2^{-1}\mathbf{W}^{ss}(\mathbf{G}, \widetilde{\Lambda}) \subset \mathbf{V}^{ss}(\mathbf{G}_L \times G_R, \overline{\Lambda}), \quad \zeta_2^{-1}\mathbf{W}^{s}(\mathbf{G}, \widetilde{\Lambda}) \subset \mathbf{V}^{s}(\mathbf{G}_L \times G_R, \overline{\Lambda}).$$

Note that as for W^{ss}, W^s, we have unipotent sub-orbits in $\mathbf{V}^{ss}$ and $\mathbf{V}^s$, see 4.1. We are going to show that in all four cases equality holds under suitable conditions on the weights of the polarizations. Then the same is true for ζ.

7.4. Estimate for ζ_1.

Let $w = (\phi_{\ell i})$ in W be given and assume that $\zeta_1(w)$ is not semi-stable. Then there are linear subspaces $P'_i \subset P_i$ and $N'_\ell \subset N_\ell$ and a unipotent element $h \in H_R$ such that for $(\gamma'_1, \dots, \gamma'_s) = h_\bullet(\gamma_1, \dots, \gamma_s)$ we have

$$\xi_i(P'_i \otimes A_{i,i-1}) \subset P'_{i-1} \quad \text{and} \quad \gamma'_\ell(P'_\ell \otimes H^*_{\ell 1}) \subset N'_\ell$$

for all $i \geq 2$ and all ℓ, and such that

$$\sum_i \alpha_i p'_i - \sum_\ell \mu_\ell n'_\ell > 0.$$

We may assume that $h = \mathrm{id}$ because H_R acts on W in the same way and we can replace w by $h \cdot w$. Moreover, we may assume that all N'_ℓ are equal to $\gamma_\ell(P'_1 \otimes H^*_{\ell 1})$ since all $\mu_\ell > 0$. Now we proceed as in 7.2 replacing the spaces Q_ℓ by N_ℓ. Therefore we find subspaces $M'_i \subset M''_i \subset M_i$ such that $M'_1 = M''_1$ and such that

$$P'_i = M'_i \oplus X_i, \quad P_i(M') \subset P'_i \subset P_i(M'')$$

and the family M'' is minimal with this property. We denote

$$\rho_i = \operatorname{codim}\bigl(P_i(M'), P_i'\bigr), \quad \sigma_i = \operatorname{codim}\bigl(P_i(M'), P_i(M'')\bigr)$$

and let

$$N_\ell'' = \gamma_\ell\bigl(P_1(M'') \otimes H_{\ell 1}^*\bigr) \supset N_\ell'.$$

As in 7.2.1 we consider the surjection

$$Z_\ell / Y_\ell \longrightarrow N_\ell'' / N_\ell',$$

where $Y_\ell \subset Z_\ell$ are the same, and we get the estimate

$$n_\ell'' - n_\ell' \leq c_\ell(m_2, \ldots, m_r)(\sigma_1 - \rho_1)$$

for any ℓ. The estimation of the discriminant Δ is now simpler than in 7.2.

7.4.1. Lemma. — *With the above notation*

$$\Delta := \sum_i \lambda_i m_i'' - \sum_\ell \mu_\ell n_\ell'' > \sum_i \alpha_i(\sigma_i - \rho_i) - \sum_\ell \mu_\ell c_\ell(m)(\sigma_1 - \rho_1)$$

where $c_\ell(m) = c_\ell(m_2, \ldots, m_r)$.

Proof. — By replacing dimensions and inserting the estimate for $n_\ell'' - n_\ell'$ as in 7.2 we get

$$\begin{aligned}\Delta &= \sum_i \alpha_i p_i' - \sum_\ell \mu_\ell n_\ell' + \sum_i \alpha_i(\sigma_i - \rho_i) - \sum_\ell \mu_\ell (n_\ell'' - n_\ell') \\ &> \sum_i \alpha_i(\sigma_i - \rho_i) - \sum_\ell \mu_\ell c_\ell(m)(\sigma_1 - \rho_1).\end{aligned}$$ □

7.4.2. Corollary. — *Let $\Lambda = (\lambda_1, \ldots, \lambda_r, -\mu_1, \ldots, -\mu_s)$ be a polarization for W and let $\bar{\Lambda} = (\alpha_1, \ldots, \alpha_r, -\mu_1, \ldots, -\mu_s)$ be the associated polarization for $\mathbf{V}$ as in 7.3. If all $\alpha_i > 0$ and*

$$\lambda_2 \geq a_{21} \sum_\ell \mu_\ell c_\ell(m)$$

then

$$\zeta_1^{-1} \mathbf{V}^{ss}(\mathbf{G}_L \times G_R, \bar{\Lambda}) = W^{ss}(G, \Lambda) \quad \textit{and} \quad \zeta_1^{-1} \mathbf{V}^{s}(\mathbf{G}_L \times G_R, \bar{\Lambda}) = W^{s}(G, \Lambda).$$

Proof. — The proof is the same as for 7.2.2, because the spaces P_i' and $P_i(M'')$ are defined in the same way and we thus get the estimate $(\sigma_2 - \rho_2)a_{21} \geq \sigma_1 - \rho_1$. □

7.5. Estimate for ζ_2.

The analogous estimate for ζ_2 follows by duality while we can assume that $s = 1$ or $r = 1$. The proof could be done by formally transform it into a dual situation which is similar to that of 7.4, but it is better to keep direct track of the weights. Let $(x_2, \dots, x_r, \gamma_1, \dots \gamma_s)$ be given in $\mathbf{W}_L \oplus V$ and assume that its image under ζ_2 is not semi-stable. Then there are subspaces $P'_i \subset P_i$ and $Q'_\ell \subset Q_\ell$ such that

$$x_i(P'_i \otimes A_{i,i-1}) \subset P'_{i-1}, \quad \gamma(P'_i \otimes H^*_{s1}) \subset Q'_s, \quad \eta_\ell(Q'_{\ell+1} \otimes B_{\ell+1,\ell}) \subset Q'_\ell,$$

where γ is defined as in 7.3, and such that

$$\sum_i \alpha_i p'_i - \sum_\ell \beta_\ell q'_\ell > 0.$$

We assume that all $\alpha_i \geq 0$, and then we may assume that P'_i is maximal, i.e. the inverse image of $P'_{i-1} \otimes A^*_{i,i-1}$ under $P_i \to P_{i-1} \otimes A^*_{i,i-1}$ for $i \geq 2$, and similarly P'_1 in P_1 under $P_1 \to Q_s \otimes H_{s1}$. As in 7.4 we can find subspaces $N'_\ell \subset N_\ell$ such that

$$Q'_\ell = N'_\ell \oplus X'_\ell \quad \text{and hence} \quad (Q_\ell/Q'_\ell)^* = (N_\ell/N'_\ell)^* \oplus X_\ell.$$

We choose subspaces $N''_\ell \subset N'_\ell$ which are maximal such that

$$Q_\ell(N'') \subset Q'_\ell \subset Q_\ell(N').$$

We have $N''_s = N'_s$. We let P''_1 be the inverse image of $Q_s(N'')$ under $P_1 \to Q_s \otimes H_{s1}$. Then $P''_1 \subset P'_1$. Furthermore we let inductively $P''_i \subset P'_i$ be the inverse images for $i \geq 2$. Then we have injections

$$(P'_i/P''_i) \otimes A_{i,i-1} \longrightarrow P'_{i-1}/P''_{i-1}$$

and induced by factorization the images

$$P'_i/P''_i \otimes A_{i,i-1} \otimes \dots \otimes A_{21} \twoheadrightarrow (P'_i/P''_i) \otimes A_{i1} \hookrightarrow P'_1/P''_1.$$

The induced injections

$$P'_i/P''_i \hookrightarrow (P'_1/P''_1) \otimes A^*_{i1}$$

imply for $i \geq 2$ the dimension estimates

$$p'_i - p''_i \leq a_{i1}(p'_1 - p''_1).$$

Next we consider the homomorphism

$$Z_1 = \bigoplus_{\ell<s} (N_\ell/N''_\ell)^* \otimes H^*_{\ell 1} \xleftarrow{\ \delta_1^\vee\ } \bigoplus_{\ell<s} (N_\ell/N''_\ell)^* \otimes B_{s\ell} \otimes H^*_{s1}.$$

We have $X_s \subset \bigoplus_{\ell<s}(N_\ell/N''_\ell)^* \otimes B_{s\ell}$ and consider the subspace

$$Y_1 = \delta_1^\vee(X_s \otimes H^*_{s1}) \subset Z_1.$$

By the definition of the constant $d_1(n) = d_1(n_1, \dots, n_{s-1})$ we get

$$\dim(Z_1/Y_1) \leq d_1(n) \operatorname{codim}(X_s) = d_1(n)(\sigma_s - \rho_s)$$

where

$$\sigma_\ell = \operatorname{codim}\big(Q^*_\ell(N/N'), Q^*_\ell(N/N'')\big),$$
$$\rho_\ell = \operatorname{codim}\big(Q^*_\ell(N/N'), (Q_\ell/Q'_\ell)^*\big).$$

Further we have a surjective map

$$Z_1/Y_1 \longrightarrow (P_1/P''_1)^*/(P_1/P'_1)^*$$

which is induced by the map $Q^*_s \otimes H^*_{s1} \to P^*_1$ and the induced surjection $Q^*_s(N/N'') \otimes H^*_{s1} \to (P/P''_1)^*$, since $N''_s = N'_s$. So we get

$$p'_1 - p''_1 \leq d_1(n)(\sigma_s - \rho_s).$$

Now we can estimate the discriminant in

7.5.1. Lemma. — *Let all the α_i be non-negative and let*

$$\Delta := \sum_i \alpha_i p''_i - \sum_\ell \mu_\ell n''_\ell.$$

Then

$$\Delta > \sum_\ell \beta_\ell(\sigma_\ell - \rho_\ell) - \sum_i \alpha_i a_{i1} d_1(n)(\sigma_s - \rho_s).$$

Proof. — Since $\sum_i \alpha_i p_i = \sum_\ell \mu_\ell n_\ell$ we also have

$$\Delta = \sum_\ell \mu_\ell(n_\ell - n''_\ell) - \sum_i \alpha_i(p_i - p''_i)$$

with the same steps as in the previous proofs we get

$$\Delta = \sum_i \alpha_i p'_i - \sum_\ell \beta_\ell q'_\ell + \sum_\ell \beta_\ell(\sigma_\ell - \rho_\ell) - \sum_i \alpha_i(p'_i - p''_i).$$

Inserting the assumption on the first difference and the estimate for $p'_i - p''_i$ we get the result. □

As in the previous cases we obtain the

7.5.2. COROLLARY. — *In the above notation let all* $\alpha_i > 0$, *and all* $\beta_\ell > 0$, *and let*

$$\mu_{s-1} \geq b_{s,s-1} d_1(n) \sum_i \alpha_i a_{i1}.$$

Then

$$\zeta_2^{-1}\mathbf{W}^{ss}(\mathbf{G}, \widetilde{\Lambda}) = \mathbf{V}^{ss}(\mathbf{G}_L \times G_R, \overline{\Lambda}) \text{ and } \zeta_2^{-1}\mathbf{W}^{s}(\mathbf{G}, \widetilde{\Lambda}) = \mathbf{V}^{s}(\mathbf{G}_L \times G_R, \overline{\Lambda}).$$

Proof. — In the notation of 7.5 there is a surjection

$$(Q_{s-1}(N')/Q'_{s-1})^* \otimes B_{s,s-1} \longrightarrow (Q_s(N')/Q'_s)^*$$

because $N''_s = N'_s$. Therefore $(\sigma_{s-1} - \rho_{s-1})b_{s,s-1} \geq \sigma_s - \rho_s$. If the condition of the corollary is satisfied, then $\Delta > 0$ follows, where we use $\mu_{s-1} = \beta_s b_{s,s-1} + \beta_{s-1}$. □

Combining the results of 7.4.2 and 7.5.2 we get the

7.5.3. PROPOSITION. — *Let* $\Lambda = (\lambda_1, \ldots, \lambda_r, -\mu_1, \ldots, -\mu_s)$ *be a polarization for* (W, G) *and let* $\widetilde{\Lambda} = (\alpha_1, \ldots, \alpha_r, -\beta_1, \ldots, -\beta_s)$ *be the associated polarization for* $(\mathbf{W}, \mathbf{G})$. *Suppose that all* $\alpha_i > 0$, *and all* $\beta_\ell > 0$ *and that*

$$\lambda_2 \geq a_{21} \sum_\ell \mu_\ell c_\ell(m) \quad \text{and} \quad \mu_{s-1} \geq b_{s,s-1} d_1(n) \sum_i \alpha_i a_{i1}.$$

Then

$$\zeta^{-1}\mathbf{W}^{ss}(\mathbf{G}, \widetilde{\Lambda}) = W^{ss}(G, \Lambda) \quad \text{and} \quad \zeta^{-1}\mathbf{W}^{s}(\mathbf{G}, \widetilde{\Lambda}) = W^{s}(G, \Lambda).$$

8. Projectivity conditions.

The projectivity of the quotient in 6.1.1 depends on the second condition in (2), i.e. whether the boundary $\overline{Z} \backslash Z$ of the saturated set contains no semi-stable points of $\mathbf{W}$. Again this condition depends on the chosen polarization and conditions for the weights. In order to derive these conditions in some cases we describe the boundary in terms independent of the group action.

8.1. Saturated boundary.

The elements of $\mathbf{W}$ are tuples $\mathbf{w} = (x_2, \ldots, x_r, \gamma, y_1, \ldots, y_{s-1})$ of linear maps

$$P_i \otimes A_{i,i-1} \xrightarrow{x_i} P_{i-1}, \quad P_1 \otimes H_{s1}^* \xrightarrow{\gamma} Q_s, \quad Q_{\ell+1} \xrightarrow{y_\ell} B_{\ell+1,\ell}^* \otimes Q_\ell.$$

If $\mathbf{w} \in Z$, there are an element $w \in W$ and automorphisms $\rho_i \in \mathrm{Aut}(P_i)$, $\sigma_\ell \in \mathrm{Aut}(Q_\ell)$ such that

$$x_i = \rho_{i-1} \circ \xi_i \circ (\rho_i^{-1} \otimes \mathrm{id}), \ \gamma = \sigma_1 \circ \gamma(w) \circ (\rho_1^{-1} \otimes \mathrm{id}), \ y_\ell = (\mathrm{id} \otimes \sigma_\ell) \circ \eta_\ell \circ \sigma_{\ell+1}^{-1}.$$

Here id stands for the different identities of the spaces A, B and H. We let $\widetilde{x}_i$ respectively $\widetilde{\xi}_i$ be the mapping

$$P_i \otimes A_{i,i-1} \otimes \ldots \otimes A_{21} \longrightarrow P_{i-1} \otimes A_{i-1,i-2} \otimes \ldots \otimes A_{21}$$

induced by x_i respectively ξ_i for $i \geq 3$. From the relations between the x_i and ξ_i it follows easily that for each $i \geq 3$ the composition $x_2 \circ \widetilde{x_3} \circ \ldots \circ \widetilde{x}_i$ has a factorization

$$\begin{tikzcd} P_i \otimes A_{i,i-1} \otimes \cdots \otimes A_{21} \arrow[r] \arrow[d, two heads] & P_1 \\ P_i \otimes A_{i1} \arrow[ur, "x_{1i}"'] & \end{tikzcd}$$

where the vertical map is the surjection induced by the pairings. This follows from the commutative diagrams induced by the automorphism ρ_i and because $\xi_2 \circ \widetilde{\xi}_3 \circ \cdots \circ \widetilde{\xi}_i$ admits such a factorization for each $i \geq 3$. We put $x_{21} = x_2$. By the dual description for the maps y_ℓ we are given factorizations

$$\begin{tikzcd} & B_{s\ell}^* \otimes Q_\ell \arrow[d, hook] \\ Q_s \arrow[ur, "y_{\ell s}"] \arrow[r] & B_{s,s-1}^* \otimes \cdots \otimes B_{s\ell}^* \otimes Q_\ell \end{tikzcd}$$

of the maps $\widetilde{y}_\ell \circ \cdots \circ \widetilde{y}_{s-2} \circ y_{s-1}$ for $\ell \leq s-2$. By similar arguments there are also factorizations

$$(L_i) \qquad \begin{tikzcd} P_i \otimes A_{i1} \otimes H_{s1}^* \arrow[r, "x_{1i} \otimes \mathrm{id}"] \arrow[d, two heads] & P_1 \otimes H_{s1}^* \arrow[r, "\gamma"] & Q_s \\ P_i \otimes H_{si}^* \arrow[urr, "\gamma_{si}"'] & & \end{tikzcd}$$

for all $i \geq 2$ and dually factorizations

$$(R_\ell) \qquad \begin{tikzcd} & & Q_\ell \otimes H_{\ell 1} \arrow[d, hook] \\ P_1 \arrow[r, "\gamma"'] \arrow[urr, "\gamma_{\ell 1}"] & Q_s \otimes H_{s1} \arrow[r, "y_{\ell s} \otimes \mathrm{id}"'] & B^*_{s\ell} \otimes Q_\ell \otimes H_{s1} \end{tikzcd}$$

for all ℓ. Moreover, there are further factorizations of the induced composed maps

$$(L_{\ell i}) \qquad \begin{tikzcd} P_i \otimes H^*_{si} \otimes B_{s\ell} \arrow[r, "\gamma_{si} \otimes \mathrm{id}"] \arrow[d, two heads] & Q_s \otimes B_{s\ell} \arrow[r, "\widetilde{y}_{\ell s}"] & Q_\ell \\ P_i \otimes H^* \arrow[urr, "\Phi_{\ell i}"'] & & \end{tikzcd}$$

and dually

$$(R_{\ell i}) \qquad \begin{tikzcd} & & Q_\ell \otimes H_{\ell i} \arrow[d, hook] \\ P_i \arrow[r, "\widetilde{x}_{1i}"'] \arrow[urr, "\Psi_{\ell i}"] & P_1 \otimes A^*_{i1} \arrow[r, "\gamma_{\ell 1} \otimes \mathrm{id}"'] & Q_\ell \otimes H_{\ell 1} \otimes A^*_{i1} \end{tikzcd}$$

All these factorizations are based on mappings induced by the pairings. All factorization conditions are independent of the chosen automorphisms. One can rediscover the original components $\phi_{\ell i}$ of w from $\Phi_{\ell i}$ or $\Psi_{\ell i}$ if $x_j = \xi_j$ and $y_\ell = \eta_\ell$ for all j and all ℓ. In fact we have

8.1.1. LEMMA. — *Let* $\mathbf{w} = (x_2, \ldots, x_r, \gamma, y_1, \ldots, y_{s-1}) \in \mathbf{W}$. *Then* $\mathbf{w} \in Z$ *if and only if*

1) $\operatorname{rank} x_i = \sum_{i \leq j} m_j a_{j,i-1}$ *for* $i \geq 2$;

1*) $\operatorname{rank} y_\ell = \sum_{k \leq \ell} b_{\ell+1,k} n_k$ *for* $\ell \leq s-1$;

2) $x_2 \circ \widetilde{x_3} \circ \cdots \circ \widetilde{x_i}$ *has a factorization* $P_i \otimes A_{i1} \xrightarrow{x_{i1}} P_1$ *for* $i \geq 3$;

2*) $\widetilde{y}_\ell \circ \cdots \circ \widetilde{y}_{s-2} \circ y_{s-1}$ *has a factorization* $Q_s \xrightarrow{y_{\ell s}} B^*_{s\ell} \otimes Q_\ell$ *for* $\ell \leq s-2$;

3) $\gamma \circ (x_{1i} \otimes \mathrm{id})$ *has factorizations* (L_i) *and* $(L_{\ell i})$;

3*) $(y_{\ell s} \otimes \mathrm{id}) \circ \gamma$ *has factorizations* (R_ℓ) *and* $(R_{\ell i})$.

Proof. — If $\mathbf{w} \in Z$, the three conditions are satisfied by the above, where rank x_i is the dimension of the image of ξ_i and rank y_ℓ is the rank of η_ℓ as the map $Q_{\ell+1} \to B^*_{\ell+1,\ell} \otimes Q_\ell$. Let conversely $\mathbf{w}$ satisfy these conditions. We proceed by descending induction to find automorphisms ρ_i by which the x_i can be identified with the ξ_i. Note that the factorization conditions are maintained under automorphisms. Since x_r has maximal rank it is an injection $M_r \otimes A_{r,r-1} \to M_{r-1} \oplus M_r \otimes A_{r,r-1} = P_{r-1}$. Hence we can find an automorphism ρ_{r-1} of P_{r-1} such that $\rho_{r-1} \circ x_r$ becomes ξ_r. Let us assume now that modulo some automorphisms $\rho_{r-1}, \dots, \rho_i$ we have $x_j = \xi_j$ for $j > i$. We are going to find an automorphism ρ_{i-1} such that $\rho_{i-1} \circ x_i = \xi_i$. Because of the rank condition we can assume that $\oplus_{i \le j} M_j \otimes A_{j,i-1}$ is the image of x_i in P_{i-1}. Now using all the $x_i \circ \widetilde{\xi}_{i+1} \circ \cdots \circ \widetilde{\xi}_k$ we find that x_i has a factorization through the standard map

$$P_i \otimes A_{i,i-1} \longrightarrow \bigoplus_{i \le j} M_j \otimes A_{j,i-1} \xrightarrow{\bar{x}_i} M_{i-1} \oplus \bigoplus_{i \le j} M_j \otimes A_{j,i-1}$$

induced by the pairings. Now the rank condition implies that $\bar{x}_i$ induces an automorphism on $\bigoplus_{i \le j} M_j \otimes A_{j,i-1}$. This can be used to make $\bar{x}_i$ the identity via an automorphism ρ'_{i-1}. Now $x_i = \xi_i$. By the analogous dual procedure we can also find automorphism $\sigma_\ell \in \mathrm{Aut}(Q_\ell)$ such that we can assume that $y_\ell = \eta_\ell$. Finally the factorizations $(L_{\ell i})$ or $(R_{\ell i})$ resulting from 3) and 3*) yield mappings $\Phi_{\ell i}$ or $\Psi_{\ell i}$ from which we get $\phi_{\ell i}$ as composition

$$M_i \otimes H^*_{\ell i} \hookrightarrow P_i \otimes H_{\ell i} \xrightarrow{\Phi_{\ell i}} Q_\ell \twoheadrightarrow N_\ell.$$

It follows from the special type of the ξ_i and η_ℓ that these are original components of an element $w = (\phi_{\ell i})$ inducing $\gamma(w) = \gamma$. □

8.1.2. Corollary. — *With the same notation as in* 8.1.1, *if* $\mathbf{w} \in \overline{Z} \backslash Z$, *then*

1) rank $x_i \le$ rank ξ_i *and* rank $y_\ell \le$ rank η_ℓ *with strict inequality for at least one i or ℓ, and*

2), 2*), 3), 3*) *of* 8.1.1 *are satisfied.*

Proof. — All conditions are closed and thus hold for points in $\overline{Z}$. If $\mathbf{w} \in \overline{Z} \backslash Z$ then by 8.1.1 equality in 1) cannot hold for all i and ℓ. □

We are going to derive effective sufficient conditions for the projectivity of the quotient in the cases $(2,1), (2,2), (3,1)$.

8.1.3. Proposition. — *Let the polarizations* Λ *and* $\widetilde{\Lambda}$ *be as in Proposition* 7.5.3 *and let* $Z = \mathbf{G}\zeta(W)$. *Then* $\overline{Z}\backslash Z$ *contains no semi-stable point in the following cases:*

(i) $(r,s) = (2,1)$ *and* $\lambda_2 \geq c_1(m_2)a_{21}\mu_1$;

(ii) $(r,s) = (2,2)$ *and*

$$\lambda_2 \geq \big(\mu_1 c_1(m_2) + \mu_2(c_2(m_2) - b_{21}c_1(m_2))a_{21},$$
$$\mu_1 \geq \big(\lambda_1(d_1(n_1)) - d_2(n_1)a_{21}) + \lambda_2 d_2(n_1)\big)b_{21}.$$

Proof. — We present only the case (ii), case(i) is an easier version of (ii). Let $(x_2, \gamma, y_1) \in \overline{Z}\backslash Z$ and let us assume that rank x_2 is not maximal. Let K be the kernel of $M_2 \otimes A_{21} \xrightarrow{x_2} P_1$ and let $M_2' \subset M_2$ be the smallest subspace such that K is contained in $M_2' \otimes A_{21}$. We put $P_2' = M_2'$,

$$P_1' = x_2(M_2' \otimes A_{21}), \quad Q_2' = \gamma(P_1' \otimes H_{21}^*), \quad Q_1' = y_1(Q_2' \otimes B_{21})$$

and consider

$$\Delta = \alpha_1 p_1' + \alpha_2 p_2' - \beta_1 q_1' - \beta_2 q_2'.$$

By definition $p_1' = \dim(M_2' \otimes A_{21}/K)$. Diagram (L_2) reduces in our case, with M_2 replaced by M_2', to

$$\begin{tikzcd}
M_2' \otimes A_{21} \otimes H_{21}^* \arrow[r, "x_2 \otimes \mathrm{id}"] \arrow[d, two heads, "\delta_2"] & P_1' \otimes H_{21}^* \arrow[r, two heads, "\gamma"] & Q_2' \\
M_2' \otimes H \arrow[urr, two heads, "\gamma_{22}"'] & &
\end{tikzcd}$$

and γ_{22} vanishes on $\delta_2(K \otimes H_{21}^*)$ because K is the kernel of x_2. Therefore

$$q_2' \leq \dim\big(M_2' \otimes H_{22}^*/\delta_2(K \otimes H_{21}^*)\big) \leq c_2(m_2')p_1'.$$

In order to estimate q_1' we consider diagram (L_{21}) enlarged by the commutative square of induced pairings

$$\begin{tikzcd}
M_2' \otimes A_{21} \otimes H_{21}^* \otimes B_{21} \arrow[r, two heads] \arrow[d, two heads] & M_2' \otimes H_{22}^* \otimes B_{21} \arrow[r, two heads, "\gamma_{22} \otimes \mathrm{id}"] \arrow[d, two heads] & Q_2' \otimes B_{21} \arrow[r, two heads, "y_1"] & Q_1' \\
M_2' \otimes A_{21} \otimes H_{11}^* \arrow[r, two heads, "\delta_1"] & M_2' \otimes H_{12}^* \arrow[urr, two heads, "\Phi_{12}"'] & &
\end{tikzcd}$$

Again the map Φ_{12} vanishes on $\delta_1(K \otimes H_{11}^*)$ and we get

$$q_1' \le \dim\big(M_2' \otimes H_{12}^*/\delta_1(K \otimes H_{11}^*)\big) \le c_1(m_2')p_1'.$$

Now we have the estimate

$$\Delta \ge \alpha_2 p_2' + \big(\alpha_1 - \beta_1 c_1(m_2) - \beta_2 c_2(m_2)\big)p_1'.$$

Therefore the condition $\alpha_1 \ge \beta_1 c_1(m_2) + \beta_2 c_2(m_2)$ would be sufficient, because $\alpha_2 p_2' > 0$. We modify the last estimate as follows. Since the weights in case $(2,2)$ are related by

$$\begin{aligned} \lambda_1 &= \alpha_1, & \mu_2 &= \beta_2, \\ \lambda_2 &= a_{21}\alpha_1 + \alpha_2, & \mu_1 &= \beta_1 + \beta_2 b_{21}, \end{aligned}$$

and since we have $\lambda_2 - a_{21}\lambda_1 > 0$ and $p_2' a_{21} - p_1' > 0$, we get the estimate

$$\Delta > \Big(\frac{\lambda_2}{a_{21}} - \mu_1 c_1(m_2) - \mu_2 c_2(m_2) + \mu_2 c_1(m_2) b_{21}\Big)p_1'.$$

This shows that $\Delta > 0$ if x_2 is degenerate and the first condition of (ii) is satisfied. In case $\operatorname{rank} y_1$ is not maximal the second condition follows by the dual procedure. □

8.2. The case (3,1).

In order to derive a similar result in case $(3,1)$ we introduce the additional constant $c_3'(m_3)$ analogous to $c_3(m_3) := c_1(0, m_3)$ in 7.1. Let

$$M_3 \otimes A_{32} \otimes H_{12}^* \xrightarrow{\tau} M_3 \otimes H_{13}^*$$

be the linear map induced by the pairing and let $\mathcal{K}$ be the family of all proper subspaces $K \subset M_3 \otimes A_{32}$ which are not contained in $M_3' \otimes A_{32}$ for any subspace $M_3' \subset M_3$ different from M_3. We put

$$c_3'(m_3) = \sup_{K \in \mathcal{K}} \frac{\operatorname{codim}(\tau(K \otimes H_{12}^*))}{\operatorname{codim}(K)}.$$

For brevity we write

$$c_3' = c_3'(m_3), \quad c_3 = c_3(m_3) = c_1(0, m_3), \quad c_1 = c_1(m_2, m_3).$$

8.2.1. Proposition. — *Let $(r,s) = (3,1)$, let $\Lambda = (\lambda_1, \lambda_2, \lambda_3, -\mu_1)$ be a polarization for (W,G) and $\widetilde{\Lambda} = (\alpha_1, \alpha_2, \alpha_3, -\mu_1)$ be the associated polarization for $(\mathbf{W}, \mathbf{G})$, and assume that all $\alpha_i > 0$ (in this case $\mu_1 = 1/n_1$). If*

1) $\alpha_2 c_3 + \lambda_1 c_3' \geq \mu_1 c_3 c_3'$,

2) $\lambda_2 \geq a_{21} \mu_1 c_1$,

3) $\lambda_3 \geq a_{31} \mu_1 c_1$,

then $\overline{Z} \backslash Z$ contains no semi-stable point.

Moreover, condition 1) may be replaced by any of the conditions

(i) $\lambda_3 \geq \mu_1 c_3' a_{32} + a_{31} \lambda_1$,

(ii) $\lambda_3 \geq \mu_1 c_3 a_{31} + a_{32} \alpha_2$,

(ii) $\lambda_3 \geq \mu_1 c_3 a_{32} a_{21}$.

Remark. — $\overline{Z} \backslash Z$ contains no semi-stable point also in each of the following cases:

(a) $\lambda_1 \geq \mu_1 c_3$,

(b) $\alpha_2 \geq \mu_1 c_3'$,

(c) $\alpha_3 \geq \mu_1 c_3 a_{31}$ or $\alpha_3 \geq \mu_1 c_3' a_{32}$.

This can be seen by a direct estimate of the discriminant Δ after substituting for q_1' in the following proof.

Proof. — Let $(x_2, x_3, \gamma) \in \overline{Z} \backslash Z$. We distinguish the following cases of degeneracy of x_2 and x_3.

Case 1: x_3 is injective. — Then by the proof of 8.1.1 we can assume that $x_3 = \xi_3$ is the canonical embedding and that x_{13} and x_2 have a factorization $\bar{x}_2$ in the following diagram:

$$\begin{tikzcd}
M_3 \otimes A_{32} \otimes A_{21} \arrow[r, "\widetilde{\xi}_3"] \arrow[d] & (M_2 \oplus M_3 \otimes A_{32}) \otimes A_{21} \arrow[r, "x_2"] \arrow[d] & P_1 \\
M_3 \otimes A_{31} \arrow[r, hook, "\xi_3'"] \arrow[urr, "x_{13}"] & M_2 \otimes A_{21} \oplus M_3 \otimes A_{31} \arrow[ur, "\bar{x}_2"'] &
\end{tikzcd}$$

Here also ξ_3' is the canonical embedding. Moreover it is easy to verify that in this case also the composed map $\gamma \circ (\bar{x}_2 \otimes \mathrm{id})$ admits a decomposition

$$(M_2 \otimes A_{21} \otimes H_{11}^*) \oplus (M_3 \otimes A_{31} \otimes H_{11}^*) \xrightarrow{\delta_1} (M_2 \otimes H_{12}^*) \oplus (M_3 \otimes H_{13}^*) \xrightarrow{\bar{\gamma}} Q_1.$$

Here $K = \operatorname{Ker}(\bar{x}_2) \neq 0$ since $\bar{x}_2$ cannot be injective by the assumption on its rank. We choose subspaces M_2', M_3' such that

$$K \subset M_2' \otimes A_{21} \oplus M_3' \otimes A_{31}$$

and such that these subspaces are minimal with this property. Now we consider the spaces

$$P_3' = M_3', \;\; P_2' = M_2' \oplus (M_3' \otimes A_{32}), \;\; P_1' = x_2(P_2' \otimes A_{21}), \;\; Q_1' = \gamma(P_1' \otimes H_{11}^*)$$

and their discriminant

$$\Delta = \alpha_1 p_1' + \alpha_2 p_2' + \alpha_3 p_3' - \beta_1 q_1'.$$

By the definition of the constant $c_1(m_2', m_3')$ and the diagram

$$\begin{tikzcd}
(M_2' \otimes A_{21} \oplus M_3' \otimes A_{31}) \otimes H_{11}^* \arrow[r, two heads] \arrow[d, two heads] & P_1' \otimes H_{11}^* \arrow[r, two heads] & Q_1' \\
M_2' \otimes H_{12}^* \oplus M_3' \otimes H_{13}^* \arrow[urr, two heads] & &
\end{tikzcd}$$

we obtain the estimate

$$q_1' \leq c_1(m_2', m_3') p_1' \leq c_1(m_2, m_3) p_1',$$

where by the definition of P_1' we have $p_1' = m_2' a_2 + m_3' a_{31} - k$. Inserting this we obtain

$$\Delta \geq (\mu_1 c_1 - \lambda_1) k + (\lambda_2 - \mu_1 c_1 a_{21}) m_2' + (\lambda_3 - \mu_1 c_1 a_{31}) m_3'.$$

If $\mu_1 c_1 - \lambda_1 > 0$, conditions 2) and 3) imply that $\Delta > 0$. If, however, $\lambda_1 \geq \mu_1 c_1$ we have the direct estimate

$$\Delta \geq (\lambda_1 - \mu_1 c_1) p_1' + \alpha_2 p_2' + \alpha_3 p_3' > 0.$$

This proves the proposition in the first case.

Case 2: x_3 *is not injective.* — Here we let K denote the kernel of x_3 and we choose a subspace $M_3' \subset M_3$ such that $K \subset M_3' \otimes A_{32}$ and M_3' is minimal with this property. Then we consider the subspaces

$$P_3' = M_3', \;\; P_2' = x_3(M_3' \otimes A_{32}), \;\; P_1' = x_2(P_2' \otimes A_{21}), \;\; Q_1' = \gamma(P_1' \otimes H_{11}^*).$$

We have the exact sequences

$$0 \to K \longrightarrow M_3' \otimes A_{32} \xrightarrow{x_2} P_2' \to 0,$$

$$0 \to L \longrightarrow M_3' \otimes A_{31} \xrightarrow{x_{13}} P_1' \to 0,$$

where L denotes the kernel of x_{13}. From the factorization properties restricted to the spaces P_i' and Q_1' we extract the following commutative diagram of surjections:

$$\begin{tikzcd}
 & M_3' \otimes A_{31} \otimes H_{11}^* \arrow[r] \arrow[d, "\delta_1"] & P_1' \otimes H_{11}^* \arrow[d, "\gamma"] \\
M_3' \otimes A_{32} \otimes A_{21} \otimes H_{11}^* \arrow[ur] \arrow[dr] & M_3' \otimes H_{13}^* \arrow[r, "\gamma_{13}"] & Q_1' \\
 & M_3' \otimes A_{32} \otimes H_{12}^* \arrow[r] \arrow[u, "\tau"] & P_2' \otimes H_{12}^* \arrow[u, "\gamma_{12}"]
\end{tikzcd}$$

From this we get again the estimates

$$q_1' \leq c_3(m_3')p_1' \leq c_3(m_3)p_1' \quad \text{and} \quad q_1' \leq c_3'(m_3')p_2' \leq c_3'(m_3)p_2',$$

where $p_1' = m_3'a_{31} - \ell$ and $p_2' = m_3'a_{32} - k$. Let $0 < t < 1$ be a real number. Then we have $q_1' \leq tc_3'p_2' + (1-t)c_3p_1'$. Substituting this into the discriminant we get

$$\Delta \geq (\lambda_1 - (1-t)\mu_1 c_3)p_1' + (\alpha_2 - t\mu_1 c_3')p_2' + \alpha_3 m_3'.$$

Now condition 1) enables us to find t with

$$1 - \frac{\lambda_1}{\mu_1 c_3} \leq t \leq \frac{\alpha_2}{\mu_1 c_3'},$$

such that the first two terms of the estimate are non-negative. Therefore $\Delta > 0$, and again (x_2, x_3, γ) is not semi-stable.

In order to show that 1) can be replaced by one of (i), (ii) or (iii) we substitute α_i and p_i' and get after cancelation

$$\begin{aligned}
\Delta &= -\lambda_1\ell - \alpha_2 k + \lambda_3 m_3' - \mu_1 q_1' \\
&\geq -\lambda_1\ell - \alpha_2 k + \lambda_3 m_3' - \mu_1 c_3'(m_3'a_{32} - k) \\
&= -\lambda_1\ell + (\mu_1 c_3' - \alpha_2)k + (\lambda_3 - \mu_1 c_3' a_{32})m_3'.
\end{aligned}$$

If $\alpha_2 \geq \mu_1 c_3'$, then by a direct estimate we get $\Delta > 0$. Therefore we may assume that $\mu_1 c_3' - \alpha_2 > 0$. Since in addition $\ell \leq m_3' a_{31}$, we get

$$\Delta > (\lambda_3 - \mu_1 c_3' a_{32} - a_{31}\lambda_1) m_3'.$$

This shows that 1) can be replaced by (i). In the same way one shows that 1) can be replaced by (ii), using the other estimate of q_1'. That finally 1) can be replaced by (iii) can be shown by substituting first $m_3' \geq p_2'/a_{32}$ and canceling $\alpha_2 p_2'$ and then substituting $p_2' \geq p_1'/a_{21}$ to get

$$a_{32}a_{21}\Delta \geq \lambda_1 p_1'(a_{32}a_{21} - a_{31}) + (\lambda_3 - \mu_1 c_3 a_{32} a_{21}) p_1'. \qquad \square$$

8.3. Proof of Theorems 1.5.1 and 1.5.2.

Theorem 1.5.1 is an immediate consequence of Proposition 6.1.1, Corollary 7.2.2 and Proposition 8.1.3. Theorem 1.5.2 follows immediately from Theorem 1.5.1 and 9.1. □

9. Examples.

9.1. Constants.

We give here some constants (cf. 7.1) used in the examples. The following result is proved in [12], Prop. 6.1.

9.1.1. PROPOSITION. — *For homomorphisms of type*

$$\big(M_1 \otimes \mathcal{O}(-2)\big) \oplus \big(M_2 \otimes \mathcal{O}(-1)\big) \longrightarrow N_1 \otimes \mathcal{O}$$

on a projective space of dimension n we have

$$c_1(m) = \begin{cases} \dfrac{m(m-1)}{2(m(n+1)-1)} & \text{if } m \leq n+1, \\ \dfrac{n+1}{2(n+2)} & \text{if } m \geq n+1. \end{cases}$$

9.1.2. LEMMA. — *For homomorphisms of type*

$$M_1 \otimes \mathcal{O}(-d) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-1) \longrightarrow N_1 \otimes \mathcal{O}$$

on the projective space $\mathbb{P}V$ the constant $c_1(1,1)$ is $\dim(V)/\dim(S^{d-1}V)$.

Proof. — We put $s(p) = \dim(S^pV)$. The homomorphisms δ_1 of 7.1 reduces here to the canonical map

$$(S^{d-2}V^* \oplus S^{d-1}V^*) \otimes S^dV \longrightarrow S^2V \oplus V.$$

If K is a proper subspace of $S^{d-2}V^* \oplus S^{d-1}V^*$ which is not contained in one of the summands, it contains elements (f,g) with $f \neq 0$ or elements (f,g) with $g \neq 0$. But since $f \otimes S^dV \to S^2V$ is surjective, the map $\delta(K) \to S^2V$ is surjective. Hence $\operatorname{codim}(\delta(K)) \leq s(1)$. If K contains an element $(0,g)$ with $g \neq 0$, then $\delta(K) = S^2V \oplus V$. For then $\delta(K)$ contains V, and since $\delta(K) \to S^2V$ is surjective, if follows that $\delta(K) = S^2V \oplus V$. Therefore, if $\operatorname{codim}(\delta(K)) > 0$, there is a basis $(f_1,g_1),\dots,(f_k,g_k)$ of K with $f_1,\dots f_k$ linearly independent, i.e. $\dim(K) \leq s(d-2)$ or $\operatorname{codim}(K) \geq s(d-1)$. Therefore $c_1(1,1) \leq s(1)/s(d-1)$. But now we can find subspaces which realize this bound. For any $z \in V^*$ we let K be the space of all $(f,fz), f \in S^{d-2}V^*$. Then $K \cong S^{d-2}V^*$ and it follows also that in this case $\delta(K) \cong S^2V$. Then $\operatorname{codim}(\delta(K))/\operatorname{codim}(K) = s(1)/s(d-1)$. □

9.2. First example of type (2,1).

We use the abbreviation $m\mathcal{F}$ for $\mathbb{C}^m \otimes \mathcal{F}$ for a sheaf and a positive integer and consider here homomorphisms

$$2\mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{(\phi_1,\phi_2)} 3\mathcal{O}$$

over $\mathbb{P}_2$ of type $(2,1)$. The polarization $\Lambda = (\lambda_1,\lambda_2,-\mu_1)$ is supposed to be proper for W and $\mathbf{W}$, i.e. $\lambda_i > 0$ and $\alpha_i > 0$ for all i. The only constant involved here is $c_1(m_2) = c(1) = 0$. Therefore the conditions of 7.2.2 and 8.1.3 are automatically satisfied by $\alpha_2 = \lambda_2 - 3\lambda_1 > 0$. Hence all the quotients of $W^{ss}(G,\Lambda)$ will be good and projective under this condition. Since $2\lambda_1 + \lambda_2 = 1$ and $3\mu_1 = 1$, we can replace the polarization by the rational number $t = \lambda_2 > \frac{3}{5}$ (cf. 9.3). The numerical condition for (semi-)stability then becomes

$$\Delta = \frac{1-t}{2}m_1 + tm_2 - \frac{1}{3}n < 0 \quad (\leq 0),$$

where (m_1,m_2,n) is the dimension vector of a (ϕ_1,ϕ_2)-invariant sub-family of vector spaces, such that $m_1 \leq 2, m_2 \leq 1, n \leq 3$. One can easily check that $t = \frac{2}{3}$ is the only value for which Δ might be zero, and this is the case for the values $(0,1,2)$ and $(2,0,1)$. And indeed, the homomorphisms ϕ

given by matrices

$$\begin{pmatrix} * & * & 0 \\ * & * & z_2 \\ * & * & z_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & z_1 \\ 0 & 0 & z_2 \\ * & * & z_3 \end{pmatrix}$$

with generically chosen entries and linear forms z_i are semi-stable and not stable for $t = \frac{2}{3}$.

9.2.1. *The case* $t > \frac{2}{3}$. — It is easy to show that in this case (ϕ_1, ϕ_2) is t-stable if and only if

- ϕ_2 is nowhere zero,
- for any $(\psi_1, \psi_2) = h \cdot (\phi_1, \phi_2)$ with $h \in H$ and any 1-dimensional subspace $M_1 \subset \mathbf{C}^2$ we have $\psi_1(M_1(\otimes \mathcal{O}(-2)) \neq 0$.

The first condition says that $\operatorname{coker}(\phi_2)$ is isomorphic to the universal quotient bundle Q on $\mathbb{P}_2$, and the second that ϕ_1 induces a 2-dimensional subspace of $H^0Q(2)$. It follows that the sets $W^s(t)$ of stable points are the same for $t > \frac{2}{3}$, which we denote by W^s_+. Moreover, from the above characterization of stable homomorphism we deduce that the geometric quotient $M_+ = W^s_+/G$ is isomorphic to the Grassmannian

$$M_+ \cong \operatorname{Gr}\bigl(2, H^0Q(2)\bigr)$$

which is smooth of dimension 26. There is an interesting subvariety $Z \subset M_+$ which consists of the images of the homomorphisms

$$\begin{pmatrix} 0 & 0 & z_1 \\ 0 & 0 & z_2 \\ * & * & z_3 \end{pmatrix} \tag{1}$$

which belong to W^s_+. These are those (ϕ_1, ϕ_2) for which the induced homomorphism $2\mathcal{O}(-2) \to Q$ is not injective. We will see next that Z is isomorphic to the non-stable locus of M_0 below and is smooth of dimension 10.

9.2.2. *The case* $t = \frac{2}{3}$. — We write W_0^{ss} for $W^{ss}(\frac{2}{3})$. When considering the matrix representations we find that $W^s_+ \subset W_0^{ss}$ and that the remaining part $W_0^{ss} \setminus W^s_+$ consists of those homomorphisms for which ϕ_2 is zero in exactly one point. Such homomorphisms are equivalent to matrices

$$\begin{pmatrix} * & * & z \\ * & * & w \\ f & g & 0 \end{pmatrix} \tag{2}$$

where z, w are independent linear and f, g are independent quadratic forms. Note, however, that W_+^s intersects the non-stable locus of W_0^{ss} in matrices equivalent to those of type (1). But the orbit closures in W_0^{ss} of both types (1) and (2) of matrices contain the direct sums

$$(3) \qquad \begin{pmatrix} 0 & 0 & z \\ 0 & 0 & w \\ f & g & 0 \end{pmatrix}$$

of independent linear and quadratic forms. From that it follows that the induced morphism

$$M_+ \longrightarrow M_0$$

of the quotients is bijective and moreover an isomorphism by Zariski's main theorem, because both spaces are normal. The points of the non-stable locus $M_0 \setminus M_0^s$ are represented by matrices of type 3). It is again routine to deduce from this observation that

$$M_0 \setminus M_0^s \cong \mathbb{P}_2 \times \mathrm{Gr}\big(2, H^0\mathcal{O}(2)\big).$$

The subvariety $Z \subset M_+$ corresponds to this set under the isomorphism. We can also identify the set M_0^s of stable points with $\mathrm{Gr}(2, H^0Q(2)) \setminus Z$.

9.2.3. *The case* $\frac{3}{5} < t < \frac{2}{3}$. — Similarly to the case W_+^s we find that here $W_-^s = W^s(t)$ is independent of t and that $W_-^s \subset W_0^{ss}$. The remaining part consists now of all homomorphisms which are equivalent to a matrix of type (1). Note that now homomorphisms of type (2) are contained in W_-^s. The induced morphism

$$M_- \longrightarrow M_0$$

is again surjective but not injective over $M_0 \setminus M_0^s$. Let Y be the inverse image of $M_0 \setminus M_0^s$. Then Y consists of the points which are represented by matrices of type (2) which are not equivalent to matrices of type (3). It is easy to check that the restricted morphism

$$M_- \setminus Y \xrightarrow{\approx} M_0^s$$

is bijective and therefore also an isomorphism by Zariski's main theorem. We are going to verify that Y is a divisor in M_-. There is a morphism

$$Y \xrightarrow{p} \mathbf{P}_2$$

which assigns to the class of (ϕ_1, ϕ_2) the point x at which ϕ_2 is degenerate. In this case

$$\operatorname{coker}(\phi_2) \cong \mathcal{O} \oplus \mathcal{I}_x(1)$$

where $\mathcal{I}_x$ is the ideal sheaf of x. For such (ϕ_1, ϕ_2) we are given an exact diagram

$$\begin{tikzcd}
 & 0 & & & \\
 & 2\mathcal{O}(-2) \arrow[u] & \mathcal{O} & & \\
0 \arrow[r] & 2\mathcal{O}(-2) \oplus \mathcal{O}(-1) \arrow[u] \arrow[r, "(\Phi_1,\Phi_2)"] & 3\mathcal{O} \arrow[u] \arrow[r] & \mathcal{F} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}(-1) \arrow[u] \arrow[r, "\Phi_2"] & 3\mathcal{O} \arrow[u, equal] \arrow[r] & \mathcal{O} \oplus \mathcal{I}_x(1) \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & & 2\mathcal{O}(-2) \arrow[u] & \\
 & & & 0 \arrow[u] &
\end{tikzcd}$$

such that (ϕ_1, ϕ_2) corresponds to a 2-dimensional subspace $\Gamma \subset H^0(\mathcal{O}(2) \oplus \mathcal{I}_x(3))$. The condition of defining an element of Y is that Γ is neither contained in $H^0\mathcal{I}_x(3)$ nor in $H^0(\mathcal{O}(2))s$ for any section s of $\mathcal{O} \oplus \mathcal{I}_x(1)$. We let $U_x \subset \mathrm{Gr}(2, H^0(\mathcal{O}(2) \oplus \mathcal{I}_x(3))$ denote the open subvariety of such Γ. By assigning to Γ the class of (ϕ_2, ϕ_2) where ϕ_1 is defined by a lifting in the above diagram, we get a morphism $U_x \to M_-$ whose image is the fibre $Y_x = p^{-1}(x)$. The morphism

$$U_x \twoheadrightarrow Y_x$$

is nothing but the quotient of U_x by the algebraic group $\mathrm{Aut}(\mathcal{O} \oplus \mathcal{I}_x(1))$. It follows that Y_x is a variety of dimension 23. Using the techniques of this paper for this quotient, we can even prove that Y is smooth. Finally Y has dimension 25 and thus is a divisor in the irreducible and normal variety M_-.

Remarks. 1) One would like to interpret the matrices of type (2) as representing extensions of the sheaves $\operatorname{coker}(f,g)$ and $\mathcal{I}_x(1) = \operatorname{coker}\binom{z}{w}$. Indeed a matrix of type (2) defines such an extension, but this extension is isomorphic to the direct sum.

2) The above correspondence between (ϕ_1, ϕ_2) and Γ indicates that the quotient spaces considered here are spaces of coherent systems as in [23].

9.2.4. *The flip.* — The diagram $M_- \to M_0 \overset{\approx}{\leftarrow} M_+$ can be interpreted as a flip. It is induced by the inclusions $W_-^s \subset W_0^{ss} \supset W_+^s$. The orbits of stable points of type (2) in W_-^s and of type (1) in W_+^s don't intersect in W_0^{ss} but so do their closures in W_0^{ss}. Thus the fibres of $M_- \to M_0$ and $M_0 \leftarrow M_+$ correspond to the two different types of semi-stable orbits in W_0^{ss} defining the same points in $M_0 \backslash M_0^s$.

9.3. General homomorphisms of type (2,1).

In a more general situation of type $(2,1)$ we consider homomorphisms

$$m_1\mathcal{O}(-2) \oplus m_2\mathcal{O}(-1) \longrightarrow n_1\mathcal{O}$$

over $\mathbb{P}_n$. A polarization in this case is determined by the rational number $t = m_2\lambda_2$ with $0 < t < 1$ and $1 - t = m_1\lambda_1, \mu_1 = 1/n_1$. A Λ-(semi-)stable homomorphism is then called t-(semi-)stable. We write $W^{ss}(t)$ and $W^s(t)$ for $W^{ss}(G, \Lambda)$ and $W^s(G, \Lambda)$. In terms of t the conditions are

$$1 > t > \frac{(n+1)m_2}{(n+1)m_2 + m_1} \quad \text{and} \quad t \geq \frac{(n+1)m_2}{n_1} c_1(m_2).$$

The constant $c_1(m_2)$ is given in Proposition 9.1.1. Such polarizations exist if and only if

$$n_1 > (n+1)m_2c_1(m_2).$$

In order to measure t-stability we introduce the numbers

$$r_1 = \frac{m_1'}{m_1}, \quad r_2 = \frac{m_1'}{m_2}, \quad s_2 = \frac{n_1'}{n_1}$$

and call (r_1, r_2, s_1) ϕ-admissible if there are subspaces $M_1' \subset M_1$, $M_2' \subset M_2, N_1' \subset N_1$ of dimensions m_1', m_2', n_1' such that ϕ maps $M_1' \otimes \mathcal{O}(-2) \oplus M_2' \otimes \mathcal{O}(-1)$ into $N_1' \otimes \mathcal{O}$. Then ϕ is t-(semi-)stable if and only if for any ϕ-admissible proper triple (r_1, r_2, s_1), i.e. a triple which is neither $(0,0,0)$ or $(1,1,1)$, we have

$$\Delta_t = (1-t)r_1 + tr_2 - s_1 < 0 \quad (\leq 0).$$

A polarization t is called critical if there are proper triples with $\Delta_t = 0$. Thus the critical values of t are the rational numbers

$$\frac{s_1 - r_1}{r_2 - r_1},$$

where we may assume $s_1 \neq 0, 1$ and thus $r_1 \neq r_2$. We let $t_{\max}$ be the maximal critical value if there are such with $0 < t < 1$ and put $t_{\max} = 0$ otherwise. If t is not critical we have $W^s(t) = W^{ss}(t)$.

9.3.1. LEMMA. — *Suppose that m_2 and n_1 are relatively prime and that $t_{\max} < t < 1$. Then $\phi = (\phi_1, \phi_2)$ is t-stable if and only if*

1) *ϕ_2 is stable with respect to the group* $\mathrm{GL}(M_2) \times \mathrm{GL}(N_1)$.

2) *For any 1-dimensional subspace $\mathbb{C} \xhookrightarrow{j} M_1$, and any h in* $\mathrm{Hom}(M_1 \otimes \mathcal{O}(-2), M_2 \otimes \mathcal{O}(-1))$ *the map*

$$(\varphi_1 + h \circ \varphi_2) \circ j : \mathcal{O}(-2) \longrightarrow N_1 \otimes \mathcal{O}$$

is not zero.

Proof. — By the characterization of stability in Section 3 the homomorphism ϕ_2 is stable if and only if for any proper pair $M_2' \subset M_2$, $N_1' \subset N_1$ of ϕ_2-admissible subspaces $r_2 < s_1$. Now let (ϕ_1, ϕ_2) be stable. If ϕ_2 were not stable there would be a proper ϕ_2-admissible pair (r_2, s_1) with $s_1 \leq r_2$. But then $s_1 < r_2$ because m_2, n_1 are supposed to be relatively prime. Then $s_1/r_2 < t$ because s_1/r_2 is a critical value and thus $\Delta_t = r_2 t - s_1 > 0$, contradicting the stability of (ϕ_1, ϕ_2). Condition 2) is trivially satisfied if (ϕ_1, ϕ_2) is t-stable, because otherwise $(1, 0, 0)$ would be admissible with $\Delta_t = 1 - t > 0$. We have to show now that conversely 1), 2) imply that (ϕ_1, ϕ_2) is t-stable. For this let (r_1, r_2, s_1) be a proper (ϕ_1, ϕ_2)-admissible triple. If $r_1 \leq r_2$ and $r_2 = 0$, there is nothing to prove. If $r_2 > 0$ then $r_2 < s_1$ by 1) and we have $t(r_2 - r_1) < s_1 - r_1$ and hence $\Delta_t < 0$. If however $r_2 < r_1$ we have $\Delta_t < 0$ in case $r_1 \leq s_1$. Since the case $s_1 \leq r_2$ is only possible if $s_1 = r_2 = 0$ and then $r_1 = 0$ by 2), we can assume that $r_2 < s_1 < r_1$. But then

$$\frac{r_1 - s_1}{r_1 - r_2} < t$$

because the fraction is a critical value, and last inequality is the inequality $\Delta_t < 0$. □

Now we are able to describe the space $M_+ = W^s(t)/G$ for $t_{\max} < t$ which is independent of t. According to the lemma $W^s(t)$ can only be non-empty if there are stable morphisms ϕ_2. This is the case if and only if

$$\frac{1}{\sigma(n)} < \frac{n_1}{m_2} < \sigma(n)$$

where $\sigma(n) = \frac{1}{2}(n + 1 + \sqrt{(n+1)^2 - 4})$, see [6]. We restrict ourselves now to the case where in addition to the previous conditions on n_1, m_2

we have $n_1 \geq nm_2$ and $(n_1, m_2) = 1$. Then a stable ϕ_2 is injective and a subbundle (except at finite number of points in case $n_1 = nm_2$, see [6], [9]). The quotient space of this space of stable homomorphisms by $\mathrm{GL}(M_2) \times \mathrm{GL}(N_1)$ is denoted by $N = N(n+1, m_2, n_1)$. It is a smooth projective variety and there is a universal sheaf $\mathcal{E}$ on $N \times \mathbb{P}_n$. For $x \in N$ let $\mathcal{E}_x$ denote the fibre sheaf representing x. Since it is the cokernel of the representing homomorphism ϕ_2, we get

$$h^0\mathcal{E}_x(2) = (n+1)\Big(\frac{n_1(n+2)}{2} - m_2\Big).$$

Therefore $p_*\mathcal{E}(2)$ is locally free on N where p denotes the first projection of $N \times \mathbb{P}_n$. Now M_+ can be non-empty only if

$$m_1 \leq (n+1)\Big(\frac{n_1(n+2)}{2} - m_2\Big).$$

If conversely this is the case for any stable ϕ_2 and any subspace $M_1 \subset H^0\mathcal{E}_x(2)$ where $x = [\phi_2]$, there is a lifting $\phi_1 : M_1 \otimes \mathcal{O}(-2) \to N_1 \otimes \mathcal{O}$ of $M_1 \otimes \mathcal{O}(-2) \to \mathcal{E}_x$, and (ϕ_1, ϕ_2) satisfies (1), (2) of the lemma. It follows now easily by considering corresponding families that

$$M_+ \cong \mathrm{Gr}_N\big(m_1, p_*\mathcal{E}(2)\big)$$

where Gr_N denotes the relative Grassmannian. It is more difficult to characterize the other moduli spaces $M(t) = W^{ss}(t)/G$ for the intervals between the critical values or for the critical values and to interpret the flips between them.

9.4. Example of type (2,2).

We consider now a simple example of type $(2,2)$ on $\mathbb{P}_3$ of homomorphisms

$$\mathcal{O}(-2) \oplus \mathcal{O}(-1) \xrightarrow{\phi} \mathcal{O} \oplus 3\mathcal{O}(1).$$

Again the polarizations $\Lambda = (\lambda_1, \lambda_2, -\mu_1, -\mu_2)$ are supposed to be proper for W and $\mathbf{W}$ such that we have $\lambda_i > 0$, $\mu_\ell > 0$ and

$$\lambda_2 > 4\lambda_1 \quad \text{and} \quad \mu_1 > 4\mu_2.$$

All constants $c_\ell(m_2)$ and $d_i(n_1)$ are again zero, because $m_2 = n_1 = 1$. Then by the above conditions also the conditions for Proposition 7.5.3 and

Proposition 8.1.3 are satisfied, such that there exist a good and projective quotient $W^{ss}(G,\Lambda)//G$ for any polarization satisfying the conditions. Since we have $\lambda_1 + \lambda_2 = 1$ and $\mu_1 + 3\mu_2 = 1$, the polarization Λ is determined already by λ_2 and μ_1, for which the above conditions become

$$(1) \qquad 1 > \lambda_2 > \frac{4}{5} \quad \text{and} \quad \frac{3}{7} > 1 - \mu_1 > 0.$$

Next we derive the conditions for the occurrence of true semi-stable points. If (m_1, m_2, n_1, n_2) is the dimension vector of a ϕ-invariant sub-family we have to consider the equation

$$\Delta = (1-\lambda_2)m_1 + \lambda_2 m_2 - \mu_1 n_1 - \frac{1}{3}(1-\mu_1)n_2 = 0.$$

By inserting all possible dimension vectors we get the six conditions

$$(2) \qquad 1 - \mu_1 = \frac{3}{k}\lambda_2, \quad 1 - \mu_1 = -\frac{3}{k}\lambda_2 + \frac{3}{k} \qquad (k = 1, 2, 3).$$

If one of these is satisfied, there might be non-stable points in $W^{ss}(G,\Lambda)$. In the following Figure 1 the lines with the equations (2) are shown together with the rectangle (1) (lower right), for the points of which we get good and projective quotients.

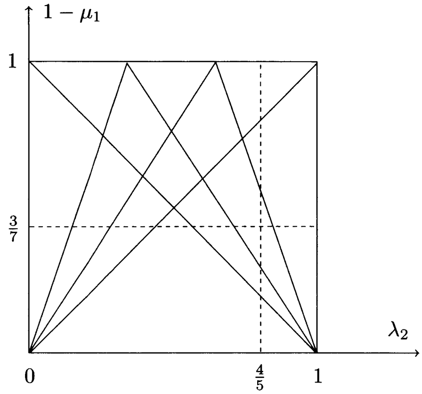

Figure 1

The homomorphism ϕ defined by the matrix

$$\left(\begin{array}{c|c} z_2^2 - z_1 z_3 & z_0 \\ \hline z_0^3 & z_1^2 \\ z_1^3 & z_2^2 \\ z_2^3 & z_3^2 \end{array}\right),$$

where the z_i are homogeneous coordinates of $\mathbb{P}_3$, is easily verified to be G-stable for each polarization Λ in the rectangle (1). Therefore the moduli spaces are not empty. On each of the three lines in the rectangle (1) each point defines one and the same open set $W^{ss}(G,\Lambda)$ and hence one and the same moduli space with semi-stable and non-stable points. Similarly, on each of the four open triangles we have one and the same moduli space, which is a smooth projective geometric quotient. Each of the seven spaces has dimension 77. The reader may also verify that the moduli space for an open triangle admits a morphism to the moduli space of each of its edges, thereby defining a chain of flips.

9.5. More general homomorphisms of type (2,2).

More general homomorphisms for which we know the constants explicitly are homomorphisms of type

$$m_1\mathcal{O}(-2)\oplus 2\mathcal{O}(-1)\longrightarrow 2\mathcal{O}\oplus n_2\mathcal{O}(1)$$

over $\mathbb{P}_3$, say. By Proposition 9.1 the constants are here

$$c_1(2)=d_2(2)=\frac{1}{7}\quad\text{and}\quad c_2(2)=d_1(2)=\frac{4}{7}\cdot$$

Let W be the space of those homomorphisms. A proper polarization $\Lambda=(\lambda_1,\lambda_2,-\mu_1,-\mu_2)$ for W satisfies

$$m_1\lambda_1+2\lambda_2=1,\quad 2\mu_1+n_2\mu_2=1$$

with $\lambda_1,\lambda_2,\mu_1,\mu_2$ positive. We will also assume that $\alpha_2>0,\beta_1>0$, i.e. $\lambda_2>4\lambda_1$ and $\mu_1>4\mu_2$. These four conditions can be replaced by

$$(1)\qquad \frac{4}{8+m_1}<\lambda_2<\frac{1}{2}\quad\text{and}\quad\frac{4}{8+n_2}<\mu_1<\frac{1}{2}\cdot$$

9.5.1. CLAIM. — *There are polarizations Λ such that $W^{ss}(G,\Lambda)$ admits a good and projective quotient in the following cases:*

(i) $m_1<6$ *and* $n_2<8$,

(i′) $m_1\le 6$ *and* $n_2=8$,

(ii) $8\le m_1+3\le n_2$ *and* $8m_1+8<7n_2$.

Proof. — The conditions of 7.2.2 for the equivalence of (semi-)stability become

$$\lambda_2 \geq \frac{4}{7}(\mu_1 + 4\mu_2) \quad \text{and} \quad \mu_1 \geq \frac{16}{7}(4\lambda_2 - 15\lambda_1) \tag{2}$$

and the conditions of 8.1.3 for the projectivity of the quotient become

$$\lambda_2 \geq \frac{4}{7}\mu_1 \quad \text{and} \quad \mu_1 \geq \frac{4}{7}\lambda_2. \tag{3}$$

The first condition of (3) follows already from the first of (2). After replacing λ_1 and μ_2 conditions (2) and (3) are equivalent to

$$\begin{cases} \frac{7}{4}n_2\lambda_2 \geq (n_2 - 8)\mu_1 + 4, \\ \frac{7}{16}m_1\mu_1 \geq (4m_1 + 30)\lambda_2 - 15, \\ \mu_1 \geq \frac{4}{7}\lambda_2. \end{cases} \tag{4}$$

Using (1) for λ_2, we find that (4) has a solution (λ_2, μ_1) if the system

$$\frac{7n_2}{8 + m_1} \geq (n_2 - 8)\mu_1 + 4, \quad \mu_1 > \frac{16}{7(8 + m_1)} \tag{5}$$

has a solution μ_1. For this we distinguish the cases $n_2 < 8$, $n_2 = 8$, $8 < n_2$. If $n_2 < 8$ the first inequality of (5) has a solution $\mu_1 < \frac{1}{2}$ if $m_1 < 6$. If $n_2 = 8$, then $m_1 \leq 6$, which is case (i′). If $n_2 > 8$, the first inequality of (5) reduces to

$$\frac{7n_2 - 4m_1 - 32}{(n_2 - 8)(m_1 + 8)} \geq \mu_1 > \frac{4}{n_2 + 8}. \tag{6}$$

Then (5) has a solution μ_1 if and only if

$$\begin{cases} 7n_2 - 4m_1 - 32 > 0, \\ (7n_2 - 4m_1 - 32)(n_2 + 8) > 4(n_2 - 8)(m_1 + 8), \\ 7(7n_2 - 4m_1 - 32) > 16(n_2 - 8). \end{cases}$$

These inequalities reduce to

$$7n_2 > 4m_1 + 32, \quad 7n_2 > 8m_1 + 8, \quad 33n_2 > 28m_1 + 96.$$

They are all satisfied if we suppose (ii) of the claim. □

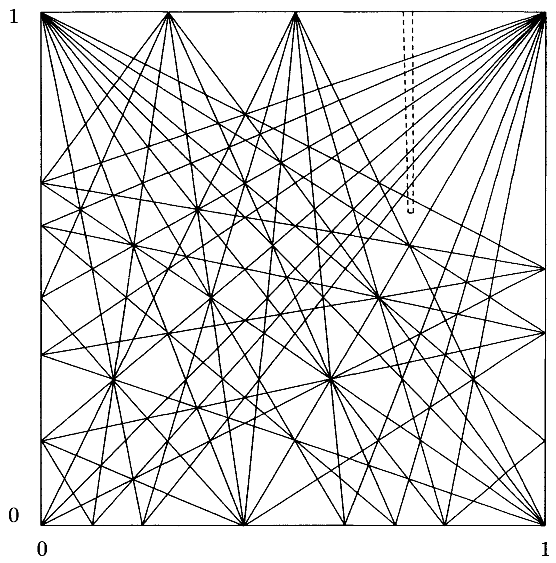

Figure 2. Here the horizontal axis represents $m_2\lambda_2$ and the vertical axis represents $n_1\mu_1$ for $m_1 = 3$ and $n_2 = 5$.

In Figure 2 the lines of the critical values of the polarizations, i.e. of the pairs (λ_2, μ_1) are shown together with the small region of those pairs which satisfy the sufficient conditions (4) for the existence of a good and projective quotient, based on the values $m_1 = 3$ and $n_2 = 5$.

9.6. Example of type (3,1).

As an example of type $(3,1)$ we consider only the space of homomorphisms

$$\mathcal{O}(-4) \oplus \mathcal{O}(-2) \oplus \mathcal{O}(-1) \longrightarrow 5\mathcal{O}$$

over $\mathbb{P}_3$. We assume again that all λ_i and all α_i are positive. Then the conditions of 7.2.2 together with the normalization of the polarization are

$$\lambda_1 + \lambda_2 + \lambda_3 = 1, \quad \mu_1 = \frac{1}{k},$$
$$\lambda_2 > 10\lambda_1, \quad \lambda_2 \geq \frac{4}{5}c_1(1,1) \quad \lambda_3 - 4\lambda_2 + 20\lambda_1 > 0.$$

As additional condition for the projectivity of the quotient we use condition (a) of the remark following Proposition 8.2.1. Since in this case both the constants $c_3(1)$ and $c_3'(1)$ are zero, this condition is just $\lambda_1 \geq 0$ and is already satisfied by our assumption.

For homomorphisms of the above type the condition $\lambda_3 < \frac{4}{5}$ is necessary if $W^s(G,\Lambda) \neq \emptyset$. For if $\phi = (\phi_1, \phi_2, \phi_3)$ is an element of W then ϕ_3 has degree 1 and thus contains at most 4 independent components. Then $m_1 = m_2 = 0$ and $m_3 = 1, n_1 = 4$ is a choice of dimensions of ϕ-invariant subspaces and the discriminant becomes $\Delta = \lambda_3 - \frac{1}{5}$.

By 9.1.2 the value of $c_1(1,1)$ is $\frac{1}{5}$. Now it is easy to see that there exist polarizations Λ which satisfy the above inequalities. That $W^s(G,\Lambda)$ is then indeed non-empty follows from the existence of generic matrices as in 9.4. Moreover there are again regions of polarizations for which the sets $W^{ss}(G,\Lambda)$ are the same and which are responsible for flips.

10. Construction of fine moduli spaces of torsion free sheaves.

Let n,k be integers such that $n \geq 2$ and

$$\frac{(n+1)(n+2)}{2} < k \leq (n+1)^2.$$

Let V be a vector space of dimension $n+1$, $\mathbb{P}_n = \mathbb{P}(V)$. We will study in this chapter morphisms of sheaves on $\mathbb{P}_n$ of type

$$\Phi = (\Phi_1, \Phi_2) : \mathcal{O}(-2) \otimes \mathbb{C}^2 \longrightarrow \mathcal{O}(-1) \oplus (\mathcal{O} \otimes \mathbb{C}^k).$$

Let $f_1 : \mathbb{C}^2 \to V^*$ the linear map induced by Φ_1. For semistable morphisms (with respect to a given polarization) f_1 is non zero. So it is of rank 1 or 2. Morphisms Φ such that f_1 is of rank 2 are called *generic*, and those such that f_1 is of rank 1 are called *special.*

10.1. Generic morphisms.

Suppose that $\Phi = (\Phi_1, \Phi_2)$ is a generic morphism. Let $P = \mathrm{Im}(f_1)$ and $\mathbb{P}_{n-2} \subset \mathbb{P}_n$ be the linear subspace of zeroes of linear forms in P. Then Φ_1 is isomorphic to the canonical morphism

$$\mathcal{O}(-2) \otimes P \longrightarrow \mathcal{O}(-1)$$

hence we have $\ker(\Phi_1) \simeq \mathcal{O}(-3)$, and $\mathrm{Im}(\Phi_1) \simeq \mathcal{I}_{\mathbb{P}_{n-2}}(-1)$ (the ideal sheaf of $\mathbb{P}_{n-2}$ twisted by $\mathcal{O}(-1)$). Let

$$\Phi' : \mathcal{O}(-3) \longrightarrow \mathcal{O} \otimes \mathbb{C}^k$$

be the restriction of Φ_2 to $\ker(\Phi_1)$. It vanishes on $\mathbb{P}_{n-2}$ and induces a linear map

$$f' : \mathbb{C}^{k^*} \longrightarrow H^0(\mathcal{I}_{\mathbb{P}_{n-2}}(3)).$$

10.1.1. Lemma. — *If Φ is semi-stable (for some polarization) then f' is injective.*

Proof. — Let $K_0 = \ker(f')^\perp \subset \mathbb{C}^k$. Then $\mathrm{Im}(\Phi') \subset \mathcal{O} \otimes K_0$. The morphism

$$\mathcal{O}(-2) \otimes \mathbb{C}^2 \longrightarrow \mathcal{O}(-1) \oplus (\mathcal{O} \otimes \mathbb{C}^k/K_0)$$

induced by Φ vanishes on $\mathcal{O}(-3) = \ker(\Phi_1)$. Hence it induces a morphism

$$(\psi_1, \psi_2) : \mathcal{I}_{\mathbb{P}_{n-2}}(-1) \longrightarrow \mathcal{O}(-1) \oplus (\mathcal{O} \otimes \mathbb{C}^k/K_0)$$

where ψ_1 is the inclusion. Since $\mathrm{Hom}(\mathcal{I}_{\mathbb{P}_{n-2}}(-1), \mathcal{O}) = \mathrm{Hom}(\mathcal{O}(-1), \mathcal{O})$, we can (by replacing Φ by an element of its $\mathrm{Hom}(\mathcal{O}(-1), \mathcal{O} \otimes \mathbb{C}^k)$-orbit) suppose that $\psi_2 = 0$. It follows that $\mathrm{Im}(\Phi) \subset \mathcal{O}(-1) \oplus (\mathcal{O} \otimes K_0)$, and since Φ is semi-stable, we have $K_0 = \mathbb{C}^k$, i.e. f' is injective. □

Note that we have taken $k \le (n+1)^2 = h^0(\mathcal{I}_{\mathbb{P}_{n-2}}(3))$, to allow the injectivity of f'.

Suppose that f' is injective. Let $K = \mathrm{Im}(f')$. then Φ' is isomorphic to the canonical morphism

$$\phi_K : \mathcal{O}(-3) \longrightarrow \mathcal{O} \otimes K^*.$$

It is easy to see that P and K depend only on the G-orbit of Φ. Conversely, suppose P and K are given. We can define an element (Φ_1, Φ_2) of W associated to P and K as follows: let (z_1, z_2) be a basis of P. Let $(z_1 q_{1i} + z_2 q_{2i})_{1 \le i \le k}$ be a basis of K, with $q_{1i}, q_{2i} \in S^2V^*$. Using this basis we can identify K and K^* with $\mathbb{C}^k$. We define

$$\Phi_1 : \mathcal{O}(-2) \otimes \mathbb{C}^2 \longrightarrow \mathcal{O}(-1)$$

by $\mathbb{C}^2 \to V^*,\ (\lambda, \mu) \mapsto \lambda z_1 - \mu z_2$ and

$$\Phi_2 : \mathcal{O}(-2) \otimes \mathbb{C}^2 \longrightarrow \mathcal{O} \otimes K^* \simeq \mathcal{O} \otimes \mathbb{C}^k$$

over $x \in \mathbb{P}_n$ by $\Phi_{2x}\big(x^2 \otimes (\lambda, \mu)\big) = \big(\lambda q_{2i}(x) + \mu q_{1i}(x)\big)_{1 \le i \le k}$.

10.1.2. LEMMA. — *Let $K \subset H^0(\mathcal{I}_{\mathbb{P}_{n-2}}(3))$ a linear subspace of dimension k. Then Φ_K is injective outside of a closed subvariety of codimension* 2.

Proof. — Let $x \in \mathbb{P}_n$. Then Φ_K is non injective at x if and only if all the elements of K (which are homogeneous polynomials of degree 3) vanish at x. Suppose that Φ_K is non injective on an irreducible hypersurface S. Then all the polynomials in K vanish on S. Let f be an irreducible equation of S. Then all the elements of K are multiple of f. It follows that f is of positive degree $d \leq 3$, and $K \subset S^{3-d}V^*.f$. But this in impossible since

$$\dim(K) > \frac{(n+1)(n+2)}{2} = \dim(S^2V^*). \qquad \square$$

10.1.3. LEMMA. — *Let $\Phi = (\Phi_1, \Phi_2) \in W$ be defined by $P \subset V^*$ and $K \subset H^0(\mathcal{I}_{\mathbb{P}_{n-2}}(3))$. Suppose that there exist a polarization such that Φ is semi-stable. Then Φ is generically injective and* $\operatorname{coker}(\Phi)$ *has no torsion. Moreover, if K is generic, Φ is injective.*

Proof. — Lemma 10.1.2 implies that Φ is injective outside a closed subvariety of codimension ≥ 2. It follows that Φ is generically injective and that $\operatorname{coker}(\Phi)$ has no torsion. To prove that Φ is injective for a generic K, it suffices to find a K such that Φ is injective. Let (z_1, z_2) be a basis of P. Let $q_1, \cdots, q_r$, (resp. $q'_1, \cdots, q'_s$) be linearly independant elements of S^2V^* that have no common zeroes in $\mathbb{P}_n$, with $r + s = k$ (this is possible since $2n + 2 \leq k \leq (n+1)^2$). Let

$$K = \Big(\bigoplus_{1 \leq i \leq r} z_1 q_i\Big) \oplus \Big(\bigoplus_{1 \leq j \leq s} z_2 q'_j\Big).$$

It is easy to see that for such a K, Φ is injective. $\square$

10.2. The obvious moduli space of morphisms and its universal sheaf.

Let $P \subset V^*$ a plane, $\mathbb{P}_{n-2} \subset \mathbb{P}_n$ the subspace defined by P and $K \subset H^0(\mathcal{I}_{\mathbb{P}_{n-2}}(3))$ a linear subspace of dimension k. Let $\mathcal{E}(P, K) = \operatorname{coker}(\Phi)$, where Φ is a morphism associated to P and K. Since the G-orbit of Φ is determined by P and K, $\mathcal{E}(P, K)$ is well defined. We will give another construction of $\mathcal{E}(P, K)$.

Let $\mathcal{F}_K = \operatorname{coker}(\Phi_K)$. It is a torsion free sheaf according to Lemma 10.1.2.

10.2.1. LEMMA. — *We have* $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}_{n-2}}(-1), \mathcal{F}_K) \simeq \mathbb{C}$, *and the non-trivial extension of* $\mathcal{F}_K$ *by* $\mathcal{O}_{\mathbb{P}_{n-2}}(-1)$ *is isomorphic to* $\mathcal{E}(P, K)$.

Proof. — The exact sequence

$$(*) \qquad 0 \to \mathcal{O}(-3) \longrightarrow \mathcal{O} \otimes K^* \longrightarrow \mathcal{F}_K \to 0$$

implies $H^0(\mathcal{F}_K(1)) \simeq V^* \otimes K^*$, $H^1(\mathcal{F}_K(1)) = \{0\}$. Using the exact sequence

$$0 \to \mathcal{I}_{\mathbb{P}_{n-2}} \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_{\mathbb{P}_{n-2}} \to 0$$

we obtain the exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}(\mathcal{O}(-1), \mathcal{F}_K) &= V^* \otimes K^* \\ &\longrightarrow \mathrm{Hom}(\mathcal{I}_{\mathbb{P}_{n-2}}(-1), \mathcal{F}_K) \longrightarrow \mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}_{n-2}}(-1), \mathcal{F}_K) \to 0. \end{aligned}$$

From (*) we get $H^0(\mathcal{F}_K(2)) \simeq S^2V^* \otimes K^*$, $H^0(\mathcal{F}_K(3)) \simeq S^3V^* \otimes K^*/\mathbb{C}i_K$, where i_K is the inclusion $K \in S^3V^*$. From the exact sequence

$$0 \to \mathcal{O}(-2) \longrightarrow \mathcal{O}(-1) \otimes P^{\perp *} \longrightarrow \mathcal{I}_{\mathbb{P}_{n-2}} \to 0$$

we deduce the exact sequence

$$0 \to \mathrm{Hom}\big(\mathcal{I}_{\mathbb{P}_{n-2}}(-1), \mathcal{F}_K\big) \longrightarrow S^2V^* \otimes P^{\perp} \otimes K^* \xrightarrow{\theta} S^3V^* \otimes K^*/\mathbb{C}i_K$$

where θ comes from the multiplication

$$\mu : S^2V^* \otimes P^{\perp} \subset S^2V^* \otimes V^* \longrightarrow S^3V^*.$$

The kernel of μ is canonically isomorphic to $\wedge^2 P^{\perp} \otimes V^*$ and it is easy to see that i_K is contained in the image of $\mu \otimes I_{K^*}$. It follows that we have an exact sequence

$$0 \to V^* \otimes K^* \longrightarrow \ker(\theta) \longrightarrow \mathbb{C}i_K \to 0$$

and that $\mathrm{Ext}^1(\mathcal{O}_{\mathbb{P}_{n-2}}(-1), \mathcal{F}_K) \simeq \mathbb{C}$.

The last assertion follows from the commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}(-3) \arrow[r] \arrow[d] & \mathcal{O} \otimes K^* \arrow[r] \arrow[d] & \mathcal{F}_K \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}(-2) \otimes \mathbb{C}^2 \arrow[r] \arrow[d] & \mathcal{O}(-1) \oplus (\mathcal{O} \otimes K^*) \arrow[r] \arrow[d] & \mathcal{E}(P, K) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I}_{\mathbb{P}_{n-2}}(-1) \arrow[r] \arrow[d] & \mathcal{O}(-1) \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}_{n-2}}(-1) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Let $\mathbf{M}$ be the projective variety of pairs (P, K), where P is a plane of V^* and $K \subset H^0(\mathcal{I}_{\mathbb{P}_{n-2}}(3))$ is a vector subspace of dimension k ($\mathbb{P}_{n-2}$ being the codimension 2 linear subspace of $\mathbb{P}_n$ defined by P). We can view $\mathbf{M}$ as a moduli space for generic morphisms. We will give a construction of a *universal sheaf* $\mathbb{E}$ on $\mathbf{M} \times \mathbb{P}_n$, i.e. $\mathbb{E}$ is flat on $\mathbf{M}$ and for every $(P, K) \in \mathbf{M}$, $\mathbb{E}_{(P,K)}$ is isomorphic to the cokernel of a generic morphism associated to (P, K). It is also possible to define a *universal morphism* whose cokernel is isomorphic to $\mathbb{E}$, but we will see this more generally in 10.4.

Let $\mathrm{Gr}(2, V^*)$ be the grassmannian of planes in V^* and $q : \mathbf{M} \to \mathrm{Gr}(2, V^*)$ be the obvious projection. Let U be the universal subsheaf of $\mathcal{O} \times V^*$ on $\mathrm{Gr}(2, V^*)$. Let

$$p_{\mathbf{M}} : \mathbf{M} \times \mathbb{P}_n \to \mathbf{M}, \quad p_2 : \mathbf{M} \times \mathbb{P}_n \to \mathbb{P}_n$$

be the projections. Then we have a canonical obvious morphism of vector bundles on $\mathbf{M} \times \mathbb{P}_n$,

$$p_2^*(\mathcal{O}(-1)) \otimes U \longrightarrow \mathcal{O}.$$

Let $\mathcal{P}$ be its cokernel. It is a flat family of sheaves on $\mathbb{P}_n$. For every $(P, K) \in \mathbf{M}$ we have $\mathcal{P}_{(P,K)} = \mathcal{O}_{\mathbb{P}_{n-2}}$. Let $\mathcal{K}$ be the universal sheaf on $\mathbf{M} \times \mathbb{P}_n$, such that $\mathcal{K}_{(P,K)} = K$. Then we have a canonical obvious morphism of vector bundles on $\mathbf{M} \times \mathbb{P}_n$,

$$p_2^*(\mathcal{O}(-3)) \longrightarrow \mathcal{K}^*.$$

Let $\mathcal{F}$ be its cokernel. Then for every $(P, K) \in \mathbf{M}$, $\mathcal{F}_{(P,K)}$ is the sheaf that was noted $\mathcal{F}_K$ before. By Lemma 10.2.1, the sheaf $\mathcal{E}xt^1_{p_{\mathbf{M}}}(\mathcal{P} \otimes p_2^*(\mathcal{O}(-1)), \mathcal{F})$ is a line bundle L on $\mathbf{M}$. Then we have a universal extension

$$0 \to \mathcal{F} \longrightarrow \mathbb{E} \longrightarrow \mathcal{P} \otimes p_2^*(\mathcal{O}(-1)) \otimes p_{\mathbf{M}}^*(L) \to 0$$

on $\mathbf{M} \times \mathbb{P}_n$. Then using Lemma 10.2.1 it is easy to see that for every $(P, K) \in \mathbf{M}$, $\mathbb{E}_{(P,K)}$ is isomorphic to the cokernel of a generic morphism associated to (P, K).

10.3. Special morphisms.

Let $\Phi = (\Phi_1, \Phi_2)$ be a special morphism. Let $f_1 : \mathbb{C}^2 \to V^*$ the associated application of rank 1. Let H be the hyperplane of $\mathbb{P}_n$ defined by $\mathrm{Im}(f_1)$. We have an exact sequence

$$0 \to \mathcal{O}(-2) \longrightarrow \mathcal{O}(-2) \otimes \mathbb{C}^2 \xrightarrow{\Phi_1} \mathcal{O}(-1) \longrightarrow \mathcal{O}_H(-1) \to 0.$$

Let

$$\overline{\Phi}_2 : \mathbb{C}^2 \longrightarrow H^0(\mathcal{O}_H(2)) \otimes \mathbb{C}^k$$

be the linear map induced by Φ_2.

10.3.1. LEMMA. — *If Φ is semi-stable (for a given polarization) then $\overline{\Phi}_2$ is injective.*

Proof. — Let $\mathbb{C}_1$, $\mathbb{C}_2$ be the two factors $\mathbb{C}$ of $\mathbb{C}\oplus\mathbb{C}=\mathbb{C}^2$. We can suppose thet $\ker(f_1)=\mathbb{C}_1$. Let

$$\Phi_{2i}:\mathcal{O}(-2)\otimes\mathbb{C}_i\longrightarrow\mathcal{O}\otimes\mathbb{C}^k\quad(i=1,2)$$

be the restrictions of Φ_2, defined by $q_{1i},\cdots,q_{ki}\in S^2V^*$. Let $(z_1,\cdots,z_{n+1})$ be a basis of V^*, such that z_1 is an equation of H. By using the action of $\mathrm{Hom}(\mathcal{O}(-1)\otimes\mathbb{C}^2,\mathcal{O}\otimes\mathbb{C}^k)$ on W we can assume that $q_{21},\cdots,q_{2k}\in S^2\langle z_2,\ldots,z_{n+1}\rangle$.

Now $\overline{\Phi}_2$ is not zero on $\mathbb{C}_1$: otherwise we would have $q_{1i}\in z_1V^*$, and $\mathrm{Im}(\Phi_2)\subset\mathcal{O}\otimes\mathbb{C}^{k'}$, with

$$k'\le n+1+\dim\bigl(S^2\langle z_2,\cdots,z_{n+1}\rangle\bigr)\le\dim(S^2V^*)<k,$$

and this would contradict the semi-stability of Φ. Hence, by considering the action of $\mathrm{GL}(2)$, it suffices to prove that $\overline{\Phi}_2$ does not vanish on $\mathbb{C}_2$. Suppose it does. Then Φ_2 vanishes on $\mathcal{O}(-2)\otimes\mathbb{C}_2$, because $q_{2i}\in S^2\langle z_2,\ldots,z_{n+1}\rangle$, and again $\mathrm{Im}(\Phi_2)\subset\mathcal{O}\otimes\mathbb{C}^{k'}$, with $k'\le\dim(S^2V^*)<k$, which contradicts the semi-stability of Φ. $\square$

10.3.2. LEMMA. — *Suppose that Φ is semi-stable with respect to some polarization. Then it is injective outside of a closed subvariety of codimension ≥ 2, and* $\mathrm{coker}(\Phi)$ *has no torsion.*

Proof. — It suffices to prove the first statement. Let $x\in\mathbb{P}_n$ and $u\in\mathbb{C}^2$ such that $\Phi_1(x^2\otimes u)=0$. Then we have either $u\in\mathbb{C}_1$ or $u\notin\mathbb{C}_1$ and $x\in H$. Suppose that Φ is not injective at all points of an irreducible hypersurface $D\neq H$. Then the same is true for $\Phi_{|\mathcal{O}(-2)\otimes\mathbb{C}_1}$. Suppose that this morphism is defined by quadratic forms $q_1,\cdots,q_k$. These forms vanish on D, hence they are all multiple of an equation of D. It follows as in the proof of 10.3.1 that $\mathrm{Im}(\Phi_2)\subset\mathcal{O}\otimes\mathbb{C}^{k'}$, with $k'<k$, which contradicts the semi-stability of Φ.

Now it remains to prove that Φ_2 is generically injective on H, but this follows easily from the fact that Φ_2 is defined by an injection $\mathbb{C}^2\to H^0(\mathcal{O}_H(2))\otimes\mathbb{C}^k$. $\square$

10.4. Fine moduli spaces of torsion-free sheaves.

10.4.1. DEFINITION. — *Let S be a smooth variety, $\mathcal{F}$ a coherent sheaf on $S \times \mathbb{P}_n$, flat on S. We say that S is a* fine moduli space of sheaves *with universal sheaf $\mathcal{F}$ if the following properties are verified:*

(i) *For every closed point $s \in S$ the Kodaïra-Spencer map*

$$\omega_s : T_s S \longrightarrow \mathrm{Ext}^1(\mathcal{F}_s, \mathcal{F}_s)$$

is bijective.

(ii) *For every closed points $s_1, s_2 \in S$ with $s_1 \neq s_2$, $\mathcal{F}_{s_1}$ and $\mathcal{F}_{s_2}$ are not isomorphic.*

(iii) *For every flat family $\mathcal{E}$ of coherent sheaves on $\mathbb{P}_n$ parametrized by an algebraic variety T, and for any closed points $s \in S$, $t \in T$ such that $\mathcal{F}_s \simeq \mathcal{E}_t$, there exist an open neighbourhood U of t in T, and a morphism $f : U \to S$ such that $f(t) = s$ and*

$$(f \times I_{\mathbb{P}_n})^*(\mathcal{F}) \simeq \mathcal{E}_{|U} \quad \text{(cf. [11])}.$$

For example moduli spaces of stables sheaves admitting a universal sheaf are fine moduli spaces of sheaves.

10.4.2. *Application of Theorem 1.5.2.* — Polarizations for morphisms

$$\mathcal{O}(-2) \otimes \mathbb{C}^2 \longrightarrow \mathcal{O}(-1) \oplus (\mathcal{O} \otimes \mathbb{C}^k)$$

are defined by pairs (λ_1, λ_2) of positive rational numbers such that $\lambda_2 + \lambda_1 k = 1$ (so here λ_2 is associated to $\mathcal{O}(-1)$ and λ_1 to $\mathcal{O} \otimes \mathbb{C}^k$).

By Theorem 1.5.2, there exist a projective good quotient of the open subset W^{ss} of semi-stable points as soon as

$$t = \lambda_2 > \frac{n+1}{n+1+k}\cdot$$

The critical polarizations in our range are given by

$$\lambda_1 = \frac{1}{2p}, \quad t = \lambda_2 = 1 - \frac{k}{2p}, \quad \frac{n+1+k}{2} < p \leq \frac{(n+1)(n+2)}{2}\cdot$$

Let

$$q = \frac{(n+1)(n+2)}{2} - \left[\frac{n+1+k}{2}\right] + 1$$

(where $[x]$ denotes the integer part of x). Then we obtain exactly q moduli spaces of morphisms corresponding to non critical values: $\mathbb{M}_1, \cdots, \mathbb{M}_q$, where for $1 \leq i < q$,

$$\mathbb{M}_i = M(t) \quad \text{for} \quad t = 1 - \frac{1}{2p} - \epsilon$$

with $p = i + [\frac{1}{2}(n+1+k)]$, ϵ being a sufficiently small positive rational number. We have $\mathbb{M}_q = \mathbf{M}$ (cf. the end of 10.2).

10.4.3. *Fine moduli spaces.* — Suppose that we choose a polarization such that t is not a critical value. In this case we have $W^{ss} = W^s$, and the stabilizer in G of the points of W^s is the canonical subgroup isomorphic to $\mathbb{C}$. Let $M(t) = W^s/G$, and $\pi : W^s \to M(t)$ be the quotient map. On $W^s \times \mathbb{P}_n$ we have a universal morphism

$$\Psi : p_2^*\big(\mathcal{O}(-2)\big) \otimes \mathbb{C}^2 \longrightarrow p_2^*\big(\mathcal{O}(-1)\big) \oplus (\mathcal{O} \otimes \mathbb{C}^k)$$

(where p_2 is the projection $W^s \times \mathbb{P}_n \to \mathbb{P}_n$) such that $\mathcal{F} = \operatorname{coker}(\Psi)$ is a flat family of torsion free sheaves on $\mathbb{P}_n$ parametrized by W^s (this is a consequence of Lemmas 10.1.3 and 10.3.2). There is a canonical action of G on $\mathcal{F}$ such that $\mathbb{C}$ acts by multiplication.

Recall that a G-sheaf $\mathcal{E}$ on $W^s \times \mathbb{P}_n$ *descends* to $M(t) \times \mathbb{P}_n$ if there exist a coherent sheaf $\mathcal{E}'$ on $M(t) \times \mathbb{P}_n$ and a G-isomorphism $(\pi \times I_{\mathbb{P}_n})^*(\mathcal{E}') \simeq \mathcal{E}$.

10.4.4. THEOREM. — *There exist a G-line bundle $\mathcal{L}$ on $M(t) \times \mathbb{P}_n$ such that $\mathcal{F} \otimes \mathcal{L}$ descends to $M(t)$. Let $\mathcal{E}$ be the corresponding sheaf on $M(t) \times \mathbb{P}_n$. Then $M(t)$ is a fine moduli space of sheaves on $\mathbb{P}_n$ with universal sheaf $\mathcal{E}$.*

Proof. — On W^s we have a canonical action of G on the bundles $\mathcal{O}_{W^s} \otimes \mathbb{C}^2$, $L = \mathcal{O}_{W^s}$ and $\mathcal{O}_{W^s} \otimes \mathbb{C}^k$. On these bundles $\mathbb{C}$ acts as ordinary multiplication by scalars. Let $\mathcal{A}_0$, $\mathcal{B}_0$ be the G-bundles

$$\mathcal{A}_0 = \big(p_2^*(\mathcal{O}(-2)) \otimes \mathbb{C}^2\big) \otimes p_W^*(\mathcal{L}^{-1}),$$
$$\mathcal{B}_0 = \big(p_2^*(\mathcal{O}(-1)) \oplus \big(\mathcal{O} \otimes \mathbb{C}^k\big)\big) \otimes p_W^*(\mathcal{L}^{-1})$$

(where p_W is the projection $W^s \times \mathbb{P}_n \to W^s$). On these bundles $\mathbb{C}$ acts trivially. We can multiply the universal morphism with $p_W^*(L^{-1})$ and we obtain a new universal morphism

$$\Psi_0 : \mathcal{A}_0 \longrightarrow \mathcal{B}_0.$$

Now it is easy to see that the bundles $\mathcal{A}_0$, $\mathcal{B}_0$ descend to $M(t) \times \mathbb{P}_n$ either directly from our construction of the quotient, or by using the more general results of [10], 2.3. Let $\mathcal{A} = \mathcal{A}_0/G$, $\mathcal{B} = \mathcal{B}_0/G$. The G-morphism Ψ_0 also descends and we get a universal morphism of vector bundles on $M(t) \times \mathbb{P}_n$,

$$\overline{\Psi} : \mathcal{A} \longrightarrow \mathcal{B}.$$

We define now $\mathcal{E} = \operatorname{coker}(\overline{\Psi})$, and it is clear that $\pi^*(\mathcal{E}) \simeq \mathcal{F} \otimes L^{-1}$.

Now we prove that the Kodaïra-Spencer map of $\mathcal{E}$ at $z \in M(t)$ is bijective. Let $w \in \pi^{-1}(z)$. Then we have a commutative diagram

$$\begin{tikzcd} T_w W \arrow[r, "T\pi"] \arrow[d, "\omega_w"'] & T_z M(t) \arrow[d, "\omega_z"] \\ \operatorname{Ext}^1(\mathcal{F}_w, \mathcal{F}_w) \arrow[r, equal] & \operatorname{Ext}^1(\mathcal{E}_z, \mathcal{E}_z) \end{tikzcd}$$

The tangent map $T\pi$ is surjective because $M(t)$ is a geometric quotient. So it suffices to prove that ω_z is surjective and that $\dim(\operatorname{Ext}^1(\mathcal{E}_z, \mathcal{E}_z)) = \dim(M(t))$. Consider the exact sequence

$$0 \to \mathcal{A}_{0w} = \mathcal{O}(-2) \otimes \mathbb{C}^2 \longrightarrow \mathcal{B}_{0w} = \mathcal{O}(-1) \oplus (\mathcal{O} \otimes \mathbb{C}^k) \longrightarrow \mathcal{F}_w \to 0.$$

It is well-known that (up to a sign) ω_w is the composition

$$\operatorname{Hom}(\mathcal{A}_{0w}, \mathcal{B}_{0w}) \longrightarrow \operatorname{Hom}(\mathcal{A}_{0w}, \mathcal{F}_w) \longrightarrow \operatorname{Ext}^1(\mathcal{F}_w, \mathcal{F}_w)$$

of maps induced by the preceeding exact sequence. Now the result follows easily from the exact sequence

$$\begin{aligned} 0 \to \operatorname{End}(\mathcal{F}_w) \longrightarrow \operatorname{End}(\mathcal{B}_{0w}) \longrightarrow \operatorname{Hom}(\mathcal{A}_{0w}, \mathcal{B}_{0w}) / \operatorname{End}(\mathcal{A}_{0w}) \\ \longrightarrow \operatorname{Ext}^1(\mathcal{F}_w, \mathcal{F}_w) \to 0. \end{aligned}$$

We must now verify that if $z_1, z_2 \in M(t)$ are distinct closed points, then $\mathcal{E}_{z_1}$ and $\mathcal{E}_{z_2}$ are not isomorphic. This follows from the more general following result: if two injective morphisms of vector bundles on $\mathbb{P}_n$,

$$\mathcal{O}(-2) \otimes \mathbb{C}^{n_1} \longrightarrow \big(\mathcal{O}(-1) \otimes \mathbb{C}^{m_2}\big) \oplus (\mathcal{O} \otimes \mathbb{C}^{m_1})$$

have isomorphic cokernels, then they are in the same orbit.

The property (iii) of the definition of a fine moduli space is easily verified. □

It follows that the q moduli spaces of morphisms $\mathbb{M}_1, \cdots, \mathbb{M}_q$, with their corresponding universal sheaves, are also fine moduli spaces of torsion free sheaves on $\mathbb{P}_n$. The moduli space $\mathbb{M}_q$ is the same as the *obvious* one $\mathbf{M}$ (cf. 10.2), and the corresponding universal sheaf is the same (up to an element of $\mathrm{Pic}(\mathbf{M})$) as $\mathbb{E}$.

These examples are generalizations of the case of $\mathbb{P}_2$ (with $k = 7$) that was treated in [11]. But in this case our results are not needed, because we get only two fine moduli spaces: one is the obvious moduli space and thc other is the corresponding moduli space of stable sheaves on $\mathbb{P}_2$.

On $\mathbb{P}_n$, $n \geq 3$, our moduli spaces are new. We don't know if the corresponding moduli space of stable sheaves is among them.

Remark. — It is not hard to prove that all the moduli spaces $\mathbb{M}_1, \cdots, \mathbb{M}_q$ are distinct.

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