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QUANTIZATION OF CANONICAL CONES OF ALGEBRAIC CURVES

by B. ENRIQUEZ & A. ODESSKII

Introduction.

To any pair (C, D) of a curve and an effective divisor are associated the morphism $C \to \mathbb{P}(H^0(C, K(D))^*)$, where K is the canonical bundle of C, and the corresponding cone $\mathrm{Cone}(C, D) \subset H^0(C, K(D))^*$. The function algebra of this cone is a graded algebra with Poisson structure. When $D = 0$, this algebra is a ring of regular functions on the complement of the zero section of the cotangent bundle $T^*(C)$, and the Poisson structure corresponds to the symplectic structure of $T^*(C)$. The purpose of this paper is to construct a quantization of this algebra.

We will propose two equivalent solutions of this problem:

(1) A solution based on the theory of formal pseudodifferential operators (Section 2). Here the base field may be any algebraically closed field $\mathbf{k}$ of characteristic zero. We show that the function algebras on $\mathrm{Cone}(C, D)$, as well as their quantizations, are functorial in the pair (C, D) (Section 2.4). We also show (Section 2.3) that this construction can be twisted by a "generalized line bundle", i.e., an element of $\{$divisors with coefficients in $\mathbf{k}\}/$ linear equivalence.

(2) When the base field in $\mathbb{C}$, we also present an analytic approach using Poincaré uniformization (Section 4). This solution uses the results of [3] on Rankin-Cohen brackets (see also [11]).

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In Section 3, we give a presentation of the quantum algebra, when C is a rational curve.

In Section 5, we discuss the problem of constructing local, or differential, liftings from the classical algebra to the algebra of pseudodifferential operators. We show that Poincaré uniformization provides such liftings, in the analytic framework. We also discuss this problem in the algebraic framework.

In Section 6, we discuss possible relations with the elliptic algebras of [5], with Kontsevich quantization and with the problem of quantizing the Beauville hamiltonians of [2].

In [4], Boutet de Monvel classified all sheaves of algebras quantizing certain sheaves of Poisson algebras, in the framework of analytic geometry. We discuss the relation of these results to our paper in Remark 8.

1. Poisson algebras associated with canonical cones of curves.

1.1.

Let C be a smooth, projective, connected complex curve (the constructions of this section can be generalized to the case where the base field is any algebraically closed field of characteristic zero). Let K be its canonical bundle. Let D be an effective divisor of C; we set $D = \sum_{P\in C} \delta_P P$, where each δ_P is an integer $\geqslant 0$ and all but finitely many δ_P are zero. To these data is attached the morphism

$$C \to \mathbb{P}(H^0(C, K(D))^*)$$

and the cone $\mathrm{Cone}(C, D)$, which is the preimage of C by the map $H^0(C, K(D))^* \to \mathbb{P}(H^0(C, K(D))^*)$, together with the origin. When $D = 0$, $\mathrm{Cone}(C, D)$ is the *canonical cone* of C. To each pair $D \geqslant D'$ is attached a morphism of cones $\mathrm{Cone}(C, D) \to \mathrm{Cone}(C, D')$.

Moreover, the function ring of $\mathrm{Cone}(C, D)$ is a Poisson algebra. As an algebra, this is the graded ring

$$A^{(D)} = \bigoplus_{i\geqslant 0} H^0(C, K(D)^{\otimes i}); \tag{1}$$

we will denote by $A_i^{(D)}$ the part of $A^{(D)}$ of degree i. For each D, we have an inclusion of graded rings $A^{(D)} \subset A^{\text{rat}}$, where

$$A^{\text{rat}} = \bigoplus_{i \geqslant 0} \{\text{rational } i\text{-differentials on } C\}.$$

We will define a Poisson structure on A^{rat}, which induces a Poisson structure on each $A^{(D)}$. For this, we will choose a nonzero rational differential α on C, and define a Poisson structure $\{,\}_\alpha$; then we will show that this bracket is independent on the choice of α.

Let us denote by ∇^α the meromorphic connection on $K^{\otimes i}$, such that if ω is a rational section of $K^{\otimes i}$, then

$$\nabla^\alpha(\omega) = \alpha^i d(\omega/\alpha^i).$$

Then we set, for ω, ω' homogeneous of degrees i, i',

$$\{\omega, \omega'\}_\alpha = i'\omega'\nabla^\alpha(\omega) - i\omega\nabla^\alpha(\omega').$$

PROPOSITION 1.1. — *The bracket $\{,\}_\alpha$ is independent on α. We denote it by $\{,\}$. It is a Poisson bracket on A^{rat}, taking $A_i^{\text{rat}} \otimes A_j^{\text{rat}}$ to A_{i+j+1}^{rat}. It restricts to a Poisson bracket on $A^{(D)}$. When the effective divisors D_1 and D_2 are linearly equivalent, the algebras $A^{(D_1)}$ and $A^{(D_2)}$ are isomorphic as graded algebras and as Poisson algebras.*

Proof. — Let us prove the independence on α. Let β be another differential. We have $\beta = F\alpha$, for F a nonzero element of $\mathbb{C}(C)$ (the field of rational functions on C). Then if ω is a rational section of $K^{\otimes i}$, we get

$$\nabla^\beta(\omega) = \nabla^\alpha(\omega) - i\frac{dF}{F}\omega,$$

so

$$\begin{aligned}\{\omega, \omega'\}_\beta &= i'\omega'\nabla^\beta(\omega) - i\omega\nabla^\beta(\omega') \\ &= i'\omega'\left(\nabla^\alpha(\omega) - i\frac{dF}{F}\omega\right) - \big((\omega, i) \leftrightarrow (\omega', i')\big);\end{aligned}$$

since $-ii'\omega\omega'\frac{dF}{F}$ is symmetric under the exchange $(\omega, i) \leftrightarrow (\omega', i')$, $\{\omega, \omega'\}_\beta = \{\omega, \omega'\}_\alpha$. We then define $\{\omega, \omega'\}$ as the common value of all $\{\omega, \omega'\}_\alpha$.

It is easy to check that for any α, $\{,\}_\alpha$ satisfies the Poisson bracket axioms, so the same is true for $\{,\}$.

Let us show that $\{A_i^{(D)}, A_{i'}^{(D)}\} \subset A_{i+i'+1}^{(D)}$. For this, we show that if ω (resp., ω') has a pole at P of order $\leqslant i\delta_P$ (resp., $i'\delta_P$), then $\{\omega, \omega'\}$

has a pole at P of order $\leqslant \deg(\{\omega,\omega'\})\delta_P = (i+i'+1)\delta_P$. Let α_P be a rational differential on C, such that P is neither a zero nor a pole of α_P. Then $\{\omega,\omega'\} = \{\omega,\omega'\}_{\alpha_P}$. The terms of order $(i+i')\delta_P + 1$ cancel each other, so the order of the pole of $\{\omega,\omega'\}_{\alpha_P}$ at P is $\leqslant (i+i')\delta_P$. Since $(i+i')\delta_P \leqslant (i+i'+1)\delta_P$, we get $\{A_i^{(D)}, A_{i'}^{(D)}\} \subset A_{i+i'+1}^{(D)}$.

Finally, if $D_1 - D_2 = (f)$, where $f \in \mathbb{C}(C)^\times$, then the ith component $A_i^{(D_1)} \to A_i^{(D_2)}$ of the isomorphism $A^{(D_1)} \to A^{(D_2)}$ takes $\omega \in H^0(C, K(D_1)^{\otimes i})$ to $\omega f^i \in H^0(C, K(D_2)^{\otimes i})$. □

Then the natural morphism $A^{(D')} \hookrightarrow A^{(D)}$ attached to $D \geqslant D'$ is a Poisson algebra morphism, so $\mathrm{Cone}(C,D) \to \mathrm{Cone}(C,D')$ is a Poisson morphism.

Moreover, one can describe the structure of symplectic leaves of $\mathrm{Cone}(C,D)$. Let us denote by $\mathrm{Supp}(D)$ the support $\{P \in C | \delta_P \neq 0\}$ of D.

PROPOSITION 1.2. — *There exists a finite subset D' of C, such that $\mathrm{Supp}(D) \subset D' \subset \mathrm{Supp}(D) \cup \{$Weierstrass points of $C\}$, with the following property. The symplectic leaves of $\mathrm{Cone}(C,D)$ are of two types:*

– each point of the preimage of D' by $\mathrm{Cone}(C,D) \to C$ is a 0-dimensional symplectic leaf, as is the origin of $\mathrm{Cone}(C,D)$,

– the preimage of $C - D'$ by $\mathrm{Cone}(C,D) \to C$ is an open 2-dimensional symplectic leaf.

When C is generic, $D = D'$.

Proof. — In the proof of Proposition 1.1, $\{\omega,\omega'\}$ has a pole of order $(i+i')\delta_P$, so when $\delta_P > 0$, this order is $< (i+i'+1)\delta_P$. If we view a k-differential ϖ as a function on $\mathrm{Cone}(C,D)$, then the coefficient of the singularity of order $k\delta_P$ at P should be viewed as the value of ϖ at a point of the line of $\mathrm{Cone}(C,D)$ above P. So $\{\omega,\omega'\}$ vanishes at P when $\delta_P > 0$.

The elements of $D' - D$ are the points P such that if a section of $A_i^{(C,D)}$ vanishes at P, then it vanishes at P with order 2. Let $(n_1,\dots,n_g)$ be the Weierstrass sequence of P; this sequence is defined by the condition that $n_1 < \cdots < n_g$, and there exists a basis of regular differentials, with zeroes of order $n_1,\dots,n_g$ at P. At a non-Weierstrass point, the sequence is $(0,\dots,g-1)$. Generic curves only have regular Weierstrass points, i.e., with sequence $(0,\dots,g-2,g)$. In both cases, there exist forms $\omega,\omega' \in A_1^{(C)}$, such that $\{\omega,\omega'\}$ does not vanish at P. □

Remark. — The referee pointed out the following construction of the Poisson bivector. Let X be the punctured cone, and $p : X \to C$ be the projection. The vertical tangent bundle $\mathcal{T}_{X/C}$ is trivialized by the Euler vector field, so we have an exact sequence $0 \to \mathcal{O}_X \to \mathcal{T}_X \to p^*(\mathcal{T}_C) \to 0$ of sheaves over X. Therefore $\wedge^2(\mathcal{T}_X) = p^*(\mathcal{T}_C)$, and $H^0(X, \wedge^2(\mathcal{T}_X)) = \oplus_{i \geqslant 0} H^0(C, \mathcal{T}_C \otimes K(D)^{\otimes i})$. Then the canonical section of $\mathcal{O}_C(D)$ defines a degree one section of $\wedge^2(\mathcal{T}_X)$, which is the bivector constructed above. Its Schouten-Nijenhuis bracket with itself lies in $\wedge^3(\mathcal{T}_X)$, which is zero; this proves that this bivector is Poisson.

1.2. The quantization problem.

If B is an algebra, equipped with a decreasing filtration $B = B^{(0)} \supset B^{(1)} \supset \cdots$ (i.e., we have $B^{(i)} B^{(j)} \subset B^{(i+j)}$), then its associated graded $\mathrm{gr}(B) = \oplus_{i \geqslant 0} B^{(i)}/B^{(i+1)}$ has a graded ring structure. Moreover, if $\mathrm{gr}(B)$ is commutative, then it has a natural Poisson structure of degree 1: for $x \in \mathrm{gr}^i(B)$, $y \in \mathrm{gr}^j(B)$, we define $\{x, y\}$ as the class of $[\widetilde{x}, \widetilde{y}]$ in $\mathrm{gr}^{i+j+1}(B)$, where $\widetilde{x}, \widetilde{y}$ are any lifts of x, y in $B^{(i)}, B^{(j)}$. We then say that B is a quantization of the Poisson algebra $\mathrm{gr}(B)$.

By a quantization of the Poisson algebra $A^{(D)}$, we therefore understand an algebra $B^{(D)}$, together with a decreasing ring filtration, whose associated graded ring is commutative, and together with an isomorphism $\mathrm{gr}(B^{(D)}) \to A^{(D)}$ of graded algebras, which is also a Poisson isomorphism.

The purpose of this paper is to construct a quantization of the Poisson algebra (1). Before we explain various forms of this construction, let us describe some examples of the Poisson rings (1) explicitly in the case $D = 0$ (then the algebra $A^{(D)}$ is simply denoted A). We do not know how to quantize the isomorphisms $A^{(D_1)} \to A^{(D_2)}$, where D_1 and D_2 are linearly equivalent.

1.3. Explicit form of the Poisson ring A for genus $3, 4, 5$.

Let us first describe the graded algebra structure of A. We have $\dim(A_0) = 1$, and $\dim(A_1) = g$, where g is the genus of C. Moreover, the natural map $S^\bullet(A_1) \to A$ is surjective when C is not hyperelliptic (see [6]). However, the injection $C \hookrightarrow \mathbb{P}(H^0(C, K)^*)$ is a complete intersection only when $g = 3, 4, 5$ and C is not hyperelliptic, and not trigonal when $g = 5$. In these cases, the kernel of $S^\bullet(A_1) \to A$ is the ideal generated by homogeneous elements $Q_1, \dots, Q_{g-2}$. When $g = 3$, $Q_1 = Q$ is homogeneous of degree 4; when $g = 4$, Q_1, Q_2 may be taken homogeneous of degree 2 and 3, and when $g = 5$, Q_1, Q_2, Q_3 may all be taken homogeneous of degree 2 (see [6]).

In all these cases, $S^\bullet(A_1)$ may be equipped with a Poisson bracket, such that the morphism $S^\bullet(A_1) \to A$ is Poisson; in other words, the injection $\mathrm{Cone}(C) \hookrightarrow H^0(C, K)^*$ is a Poisson morphism. The Poisson structure on $S^\bullet(A_1)$ may be described explicitly as follows (see [10]).

Let $x_1, \dots, x_g$ be a basis of A_1, then the Poisson structure on $S^\bullet(A_1)$ is obtained by the rule

$$\{f, g\} = \frac{df \wedge dg \wedge \chi}{\omega_{\mathrm{top}}},$$

where $\chi = dQ_1 \wedge \cdots \wedge dQ_{g-2}$ and $\omega_{\mathrm{top}} = dx_1 \wedge \cdots \wedge dx_g$. The elements $Q_1, \dots, Q_{g-2}$ are Poisson central for this structure, so there exists a unique Poisson structure on A, such that $S^\bullet(A_1) \to A$ is Poisson. For example, when $g = 3$, the Poisson structure is defined by the relations

$$\{x_1, x_2\} = \partial_{x_3} Q, \quad \{x_2, x_3\} = \partial_{x_1} Q, \quad \{x_3, x_1\} = \partial_{x_2} Q;$$

in general, the brackets have the form $\{x_i, x_j\} = P_{ij}(x_1, \dots, x_g)$, where the P_{ij} are homogeneous of degree 3.

2. Quantization based on formal pseudodifferential operators.

2.1. Outline of the construction.

Our main tool is the general construction of the algebra of formal pseudodifferential operators $\Psi\,\mathrm{DO}(R, \partial)$ associated to any differential ring (R, ∂). We will define the filtered algebra B as an algebra of formal pseudodifferential operators on C, which are regular on C. We proceed as follows. To any rational, nonzero vector field X on C, we associate a filtered algebra B_X^{rat} of *rational* pseudodifferential operators on C. The construction of this algebra involves X, but we construct canonical isomorphisms

$$i_{X,Y}^{\mathrm{rat}} : B_X^{\mathrm{rat}} \to B_Y^{\mathrm{rat}}$$

for any pair (X,Y) of nonzero rational vector fields. One can show that B_X^{rat} is a quantization of the Poisson algebra A^{rat}.

In Section 2.2.4, we give a canonical construction of the algebra B_X^{rat}, independent of the choice of a nonzero vector field X.

If z is a formal variable, $\Psi\,\mathrm{DO}(\mathbb{C}((z)),\frac{\partial}{\partial z})$ is the algebra of formal pseudodifferential operators on the formal punctured disc. This algebra contains the subalgebra $\Psi\,\mathrm{DO}(\mathbb{C}[[z]],\frac{\partial}{\partial z})$ of operators, regular at the origin. For any integer $\delta \geqslant 0$, we also construct an intermediate algebra $\Psi\,\mathrm{DO}(\mathbb{C}[[z]],z^{\delta}\frac{\partial}{\partial z})$.

Then for any point P of C, let $\mathcal{K}_P$ be the completed local field of C at P, and let $\mathcal{O}_P \subset \mathcal{K}_P$ be its completed local ring. If z_P is a local coordinate at P, we have

$$\mathcal{K}_P = \mathbb{C}((z_P)),\ \mathcal{O}_P = \mathbb{C}[[z_P]].$$

We set $\Psi\,\mathrm{DO}(\mathcal{K}_P,z_P) := \Psi\,\mathrm{DO}(\mathbb{C}((z_P)),\frac{\partial}{\partial z_P})$, $\Psi\,\mathrm{DO}(\mathcal{O}_P,z_P) := \Psi\,\mathrm{DO}(\mathbb{C}[[z_P]],\frac{\partial}{\partial z_P})$ and $\Psi\,\mathrm{DO}(\mathcal{O}_P,z_P)^{(\delta_P)} := \Psi\,\mathrm{DO}(\mathbb{C}[[z_P]],(z_P)^{\delta_P}\frac{\partial}{\partial z_P})$.

If P is any point of C, Laurent expansion of formal pseudodifferential operators at P yields a filtered ring morphism $L_P^{z_P} : B_X^{\mathrm{rat}} \to \Psi\,\mathrm{DO}(\mathcal{K}_P,z_P)_{\leqslant 0}$. Then we define $B_X^{(D)}$ as the preimage of $\prod_{P\in C}\Psi\,\mathrm{DO}(\mathcal{O}_P,z_P)_{\leqslant 0}^{(\delta_P)}$ by the ring morphism

$$\prod_{P\in C} L_P^{z_P} : B_X^{\mathrm{rat}} \to \prod_{P\in C}\Psi\,\mathrm{DO}(\mathcal{K}_P,z_P)_{\leqslant 0}$$

(the index $\leqslant 0$ means operators of degree $\leqslant 0$). One easily sees that this definition is independent of the choice of the collection of local coordinates $(z_P)_{P\in C}$. In particular, when $D=0$, $B_X = B_X^{(0)}$ consists of all rational pseudodifferential operators on C, which are regular at any point of C. We will prove:

THEOREM 2.1. — 1) *The canonical isomorphisms* $i_{X,Y}^{\mathrm{rat}}$ *restrict to canonical isomorphisms of filtered algebras*

$$i_{X,Y} : B_X^{(D)} \to B_Y^{(D)}.$$

2) *The graded algebra* $\mathrm{gr}(B_X^{(D)})$ *is commutative, and as a Poisson algebra, it is isomorphic to* $A^{(D)}$.

3) *For* $D \geqslant D'$, *there are canonical morphisms* $B_X^{(D)} \hookrightarrow B_X^{(D')}$ *of complete filtered algebras, quantizing the inclusion* $A^{(D)} \hookrightarrow A^{(D')}$.

So for each D, the algebras $B_X^{(D)}$ are all isomorphic when the vector field X is changed, and they are quantizations of the Poisson algebra $A^{(D)}$.

Remark 2. — One can prove that if one repeats this construction without restricting it to operators of degree $\leqslant 0$, the resulting algebra is the same as B_X: all regular pseudodifferential operators on C are of degree $\leqslant 0$, because there are no nonzero sections of $K(D)^{\otimes i}$ for $i < 0$ (the genus of C is > 1).

2.2. Details of the construction.

We will first present all details of the construction when $D = 0$. So all superscripts (D) will be dropped. In Section 2.2.6, we explain the modifications of the construction in the case of a general D.

2.2.1. *The algebras* $\Psi\,\mathrm{DO}(R,\partial)$. Let R be a commutative ring with unit and let ∂ be a derivation of R. Following [1], [9], define $\Psi\,\mathrm{DO}(R,\partial)$ as the space of all formal linear combinations $\sum_{i\in\mathbb{Z}} a_i D_\partial^i$, where for each i, $a_i \in R$ and $a_i = 0$ for i large enough. $\Psi\,\mathrm{DO}(R,\partial)$ is equipped with the associative product

$$\left(\sum_{i\in\mathbb{Z}} a_i D_\partial^i\right)\left(\sum_{j\in\mathbb{Z}} b_j D_\partial^j\right) = \sum_{k\in\mathbb{Z}}\left(\sum_{i,j\in\mathbb{Z}}\binom{i}{i+j-k} a_i \partial^{i+j-k}(b_j)\right) D_\partial^k.$$

Say that $\sum_{i\in\mathbb{Z}} a_i D_\partial^i$ has degree $\leqslant n$ if $a_i = 0$ when $i > n$, and define $\Psi\,\mathrm{DO}(R,\partial)_{\leqslant n}$ as the subspace of $\Psi\,\mathrm{DO}(R,\partial)$ of all operators of degree $\leqslant n$. Then $\Psi\,\mathrm{DO}(R,\partial)$ is a filtered ring. Its associated graded is $R[\xi,\xi^{-1}]$. We will be interested in its subring $\Psi\,\mathrm{DO}(R,\partial)_{\leqslant 0}$. It is also filtered, with associated graded $R[\xi^{-1}]$. Moreover, both $\Psi\,\mathrm{DO}(R,\partial)$ and $\Psi\,\mathrm{DO}(R,\partial)_{\leqslant 0}$ are complete for the topology defined by the family $(\Psi\,\mathrm{DO}(R,\partial)_{\leqslant -n})_{n=0,1,2,\ldots}$.

2.2.2. *Functoriality properties of the rings* $\Psi\,\mathrm{DO}(R,\partial)$ *and* $\Psi\,\mathrm{DO}(R,\partial)_{\leqslant 0}$. The following statements are immediate:

Lemma 2.1. — 1) *Let (R,∂) be a differential ring, and let $f \in R^\times$ (i.e., f is an invertible element of R). Set $\partial' = f\partial$, then ∂' is a derivation of R. We have for any i, $(f^{-1}D_{\partial'})^i \in \Psi\,\mathrm{DO}(R,\partial')_{\leqslant i}$, so if $(a_i)_{i\in\mathbb{Z}}$ is a sequence of elements of R, such that $a_i = 0$ for i large enough, the sequence $\sum_{i\in\mathbb{Z}} a_i(f^{-1}D_{\partial'})^i$ converges in $\Psi\,\mathrm{DO}(R,\partial')$. Then there is a unique isomorphism*

$$i_{\partial,\partial'} : \Psi\,\mathrm{DO}(R,\partial) \to \Psi\,\mathrm{DO}(R,\partial')$$

of complete filtered algebras, taking each series $\sum_{i\in\mathbb{Z}} a_i D_\partial^i$ *to* $\sum_{i\in\mathbb{Z}} a_i (f^{-1} D_{\partial'})^i$. *We have then* $i_{\partial',\partial''} \circ i_{\partial,\partial'} = i_{\partial,\partial''}$.

2) *Let* $\mu : (R, \partial_R) \to (S, \partial_S)$ *be a morphism of differential rings (i.e.,* μ *is a ring morphism and* $\partial_S \circ \mu = \mu \circ \partial_R$*). Then there is a unique morphism*

$$\Psi\,\mathrm{DO}(\mu) : \Psi\,\mathrm{DO}(R, \partial_R) \to \Psi\,\mathrm{DO}(S, \partial_S),$$

taking each $\sum_{i\in\mathbb{Z}} a_i D_{\partial_R}^i$ *to* $\sum_{i\in\mathbb{Z}} \mu(a_i) D_{\partial_S}^i$. $\Psi\,\mathrm{DO}(\mu)$ *is a morphism of complete filtered algebras and we have* $\Psi\,\mathrm{DO}(\nu \circ \mu) = \Psi\,\mathrm{DO}(\nu) \circ \Psi\,\mathrm{DO}(\mu)$ *for any morphism* $\nu : (S, \partial_S) \to (T, \partial_T)$ *of differential rings. In other words,*

$$(R, \partial) \mapsto \Psi\,\mathrm{DO}(R, \partial)$$

is a functor from the category of differential rings to that of filtered complete algebras.

2.2.3. *Construction of* B_X. Let C be a curve, and let $\mathbb{C}(C)$ be its field of rational functions. Let X be a nonzero rational vector field on C; X may be viewed as a nonzero derivation of $\mathbb{C}(C)$. We set

$$B_X^{\mathrm{rat}} = \Psi\,\mathrm{DO}(\mathbb{C}(C), X)_{\leqslant 0}.$$

If Y is another nonzero vector field on C, then there exists a unique $f \in \mathbb{C}(C)^\times$, such that $Y = fX$. Applying Lemma 2.1, 1), we get an isomorphism

$$i_{X,Y}^{\mathrm{rat}} : B_X^{\mathrm{rat}} \to B_Y^{\mathrm{rat}}$$

of complete filtered rings.

On the other hand, if $P \in C$, then for any local coordinate z_P at P, $\frac{\partial}{\partial z_P}$ is a derivation of $\mathcal{K}_P$, preserving $\mathcal{O}_P$. We set

$$\Psi\,\mathrm{DO}(\mathcal{K}_P, z_P) := \Psi\,\mathrm{DO}\Big(\mathcal{K}_P, \frac{\partial}{\partial z_P}\Big) \text{ and } \Psi\,\mathrm{DO}(\mathcal{O}_P, z_P) := \Psi\,\mathrm{DO}\Big(\mathcal{O}_P, \frac{\partial}{\partial z_P}\Big).$$

By functoriality, we have then an inclusion $\Psi\,\mathrm{DO}(\mathcal{O}_P, z_P) \subset \Psi\,\mathrm{DO}(\mathcal{K}_P, z_P)$. Moreover, if z'_P is another local coordinate at P, the derivations $\frac{\partial}{\partial z_P}$ and $\frac{\partial}{\partial z'_P}$ are related by $\frac{\partial}{\partial z_P} = \varphi \frac{\partial}{\partial z'_P}$, where φ belongs to $\mathcal{O}_P^\times$, so Lemma 2.1, 1), says that there is an isomorphism $i_{z_P,z'_P} : \Psi\,\mathrm{DO}(\mathcal{K}_P, z_P) \to \Psi\,\mathrm{DO}(\mathcal{K}_p, z'_P)$ of complete filtered algebras, restricting to an isomorphism $\Psi\,\mathrm{DO}(\mathcal{O}_P, z_P) \to \Psi\,\mathrm{DO}(\mathcal{O}_p, z'_P)$, and such that $i_{z'_P,z''_P} \circ i_{z_P,z'_P} = i_{z_P,z''_P}$.

Let us now define the Laurent expansion morphism

$$L_P^{z_P} : B_X^{\mathrm{rat}} \to \Psi\,\mathrm{DO}(\mathcal{K}_P, z_P)_{\leqslant 0}.$$

Since X is a nonzero vector field, its local expansion at P is $X = X(z_P)\frac{\partial}{\partial z_P}$, with $X(z_P) \in \mathcal{K}_P^\times$. The Laurent expansion map

$$\ell_P : \mathbb{C}(C) \to \mathcal{K}_P$$

therefore induces a differential ring morphism $(\mathbb{C}(C),X) \to \big(\mathcal{K}_P, X(z_P)\frac{\partial}{\partial z_P}\big)$, and so a morphism

$$\Psi\,\mathrm{DO}(\ell_P) : B_X^{\mathrm{rat}} \to \Psi\,\mathrm{DO}\left(\mathcal{K}_P, X(z_P)\frac{\partial}{\partial z_P}\right)_{\leqslant 0}.$$

Composing it with the isomorphism

$$i_{X(z_P)\frac{\partial}{\partial z_P},\frac{\partial}{\partial z_P}} : \Psi\,\mathrm{DO}\left(\mathcal{K}_P, X(z_P)\frac{\partial}{\partial z_P}\right)_{\leqslant 0} \to \Psi\,\mathrm{DO}\left(\mathcal{K}_P, \frac{\partial}{\partial z_P}\right)_{\leqslant 0} = \Psi\,\mathrm{DO}(\mathcal{K}_P, z_P)_{\leqslant 0},$$

we get a filtered ring morphism

$$L_P^{z_P} : B_X^{\mathrm{rat}} \to \Psi\,\mathrm{DO}(\mathcal{K}_P, z_P)_{\leqslant 0}.$$

Finally, let us prove that the preimage by

$$\prod_{P\in C} L_P^{z_P} : B_X^{\mathrm{rat}} \to \prod_{P\in C} \Psi\,\mathrm{DO}(\mathcal{K}_P, z_P)_{\leqslant 0}$$

of $\prod_{P\in C} \Psi\,\mathrm{DO}(\mathcal{O}_P, z_P)_{\leqslant 0}$ is independent of the choice of the local coordinates $(z_P)_{P\in C}$: if $(z'_P)_{P\in C}$ is any other choice of local coordinates, then

$$L_P^{z'_P} = i_{\frac{\partial}{\partial z_P},\frac{\partial}{\partial z'_P}} \circ L_P^{z_P}, \tag{2}$$

and

$$\Psi\,\mathrm{DO}(\mathcal{O}_P, z'_P)_{\leqslant 0} = i_{\frac{\partial}{\partial z_P},\frac{\partial}{\partial z'_P}}(\Psi\,\mathrm{DO}(\mathcal{O}_P, z'_P)_{\leqslant 0}),$$

so

$$\begin{aligned}
&\left(\prod_{P\in C} L_P^{z'_P}\right)^{-1}\left(\prod_{P\in C} \Psi\,\mathrm{DO}(\mathcal{O}_P, z'_P)_{\leqslant 0}\right) \\
&= \left(\prod_{P\in C} i_{\frac{\partial}{\partial z_P},\frac{\partial}{\partial z'_P}} \circ L_P^{z_P}\right)^{-1}\left(\prod_{P\in C} \Psi\,\mathrm{DO}(\mathcal{O}_P, z'_P)_{\leqslant 0}\right) \\
&= \left(\prod_{P\in C} L_P^{z_P}\right)^{-1}\left(\prod_{P\in C} \Psi\,\mathrm{DO}(\mathcal{O}_P, z_P)_{\leqslant 0}\right).
\end{aligned}$$

2.2.4. *Vector field-independent construction of the algebras* B_X^{rat}. Let us define $\mathrm{DO}(\mathbb{C}(C))$ as the algebra of all rational differential operators on

C. So $\mathrm{DO}(\mathbb{C}(C))$ is generated by $i(f) \in \mathbb{C}(C)$, D_X, where $X \in \mathrm{Der}(\mathbb{C}(C))$, and relations

$$i(fg) = i(f)i(g), \quad i(\lambda) = \lambda, \quad [D_X, i(f)] = i(X(f)),$$

$$[D_X, D_Y] = D_{[X,Y]}, \quad D_{fX} = i(f)D_X, \tag{3}$$

for $f, g \in \mathbb{C}(C)$, $X, Y \in \mathrm{Der}(\mathbb{C}(C))$, $\lambda \in \mathbb{C}$. We will denote $i(f)$ simply by f, for $f \in \mathbb{C}(C)$.

$\mathrm{DO}(\mathbb{C}(C))$ may be localized with respect to the family of all D_X, where X are all nonzero rational vector fields. The last of relations (3), together with the fact that $\mathrm{Der}(\mathbb{C}(C))$ is a 1-dimensional $\mathbb{C}(C)$-vector space, implies that the localization of $\mathrm{DO}(\mathbb{C}(C))$ w.r.t. any D_X, $X \in \mathrm{Der}(\mathbb{C}(C)) - \{0\}$, coincides with its localization w.r.t. the family of all such D_X. We denote by B^{rat} the completion of this localized algebra w.r.t. the degree of formal pseudodifferential operators.

Then B^{rat} contains $\mathrm{DO}(\mathbb{C}(C))$ as a subalgebra, as well as the additional generators $(D_X)^{-1}$, $X \in \mathrm{Der}(\mathbb{C}(C)) - \{0\}$. They satisfy, in particular, the relations

$$(D_{fX})^{-1} = (D_X)^{-1} f^{-1},$$

for $f \in \mathbb{C}(C)^\times$ and $X \in \mathrm{Der}(\mathbb{C}(C)) - \{0\}$. If X is any nonzero vector field, the natural map

$$i_X : B_X^{\mathrm{rat}} \to B_{\leqslant 0}^{\mathrm{rat}}$$

is therefore an isomorphism. The map $i_{X,Y}^{\mathrm{rat}} : B_X^{\mathrm{rat}} \to B_Y^{\mathrm{rat}}$ then coincides with $(i_Y)^{-1} \circ i_X$.

2.2.5. *Proof of Theorem 2.1.* Let us prove the first part of Theorem 2.1. Let us emphasize the dependence of $L_P^{z_P}$ in X by denoting it

$$L_P^{X,z_P} : B_X^{\mathrm{rat}} \to \Psi\mathrm{DO}(\mathcal{K}_P, z_P)_{\leqslant 0}.$$

Then we have

$$B_X = \left(\prod_{P \in C} L_P^{X,z_P} \right)^{-1} \left(\prod_{P \in C} \Psi\mathrm{DO}(\mathcal{O}_P, z_P) \right).$$

Now the composed map

$$B_X^{\mathrm{rat}} \xrightarrow{i_{X,Y}} B_Y^{\mathrm{rat}} \xrightarrow{L_P^{Y,z_P}} \Psi\mathrm{DO}(\mathcal{K}_P, z_P)_{\leqslant 0}$$

coincides with L_P^{X,z_P}. So

$$\begin{aligned}B_X &= \Big(\prod_{P\in C} L_P^{Y,z_P}\circ i_{X,Y}^{\mathrm{rat}}\Big)^{-1}\Big(\prod_{P\in C}\Psi\,\mathrm{DO}(\mathcal{O}_P,z_P)\Big)\\ &= (i_{X,Y}^{\mathrm{rat}})^{-1}\bigg(\Big(\prod_{P\in C} L_P^{Y,z_P}\Big)^{-1}\Big(\prod_{P\in C}\Psi\,\mathrm{DO}(\mathcal{O}_P,z_P)\Big)\bigg)\\ &= (i_{X,Y}^{\mathrm{rat}})^{-1}(B_Y),\end{aligned}$$

so $B_X = (i_{X,Y}^{\mathrm{rat}})^{-1}(B_Y)$. Since $i_{X,Y}^{\mathrm{rat}} : B_X^{\mathrm{rat}} \to B_Y^{\mathrm{rat}}$ is an isomorphism of complete filtered algebras, it restricts to an isomorphism $i_{X,Y} : B_X \to B_Y$ of complete filtered algebras.

Let us now prove the second part of Theorem 2.1. We will define a filtration on B_X; then we will construct a graded linear map

$$\lambda_{\mathrm{reg}} : \mathrm{gr}(B_X) \to A;$$

we will prove that if the genus of C is > 1, λ_{reg} is a linear isomorphism, and finally that it is an isomorphism of Poisson algebras.

(a) Filtration on B_X. We set $(B_X^{\mathrm{rat}})^i = \Psi\,\mathrm{DO}(\mathbb{C}(C),X)_{\leqslant -i}$, and

$$(B_X)^i = B_X \cap (B_X^{\mathrm{rat}})^i.$$

So $(B_X)^i$ consists of all regular pseudodifferential operators on C of order $\leqslant -i$.

(b) The map $(B_X)^i/(B_X)^{i+1} \to A_i$. The natural map $(B_X)^i/(B_X)^{i+1} \to (B_X^{\mathrm{rat}})^i/(B_X^{\mathrm{rat}})^{i+1}$ is injective, because $(B_X)^i \cap (B_X^{\mathrm{rat}})^{i+1} = B_X \cap (B_X^{\mathrm{rat}})^{i+1} = (B_X)^{i+1}$. Moreover, there is a linear isomorphism

$$(4)\qquad \lambda_{\mathrm{rat}}^{(i)} : (B_X^{\mathrm{rat}})^i/(B_X^{\mathrm{rat}})^{i+1} \to \{\text{rational } i\text{-differentials on } C\},$$

taking the class of $\sum_{j\geqslant i} a_j (D_X)^{-j}$ to $a_i\alpha^i$, where α is the rational differential inverse to X. We will prove

LEMMA 2.2. — *The restriction of $\lambda_{\mathrm{reg}}^{(i)}$ maps $(B_X)^i/(B_X)^{i+1}$ to $A_i = H^0(C, K^{\otimes i}) \subset \{$rational i-differentials on $C\}$.*

Proof of Lemma. — For any $P \in C$, $L_P^{z_P}$ induces a linear map

$$(B_X^{\mathrm{rat}})^i/(B_X^{\mathrm{rat}})^{i+1} \to \Psi\,\mathrm{DO}(\mathcal{K}_P,z_P)_{\leqslant -i}/\Psi\,\mathrm{DO}(\mathcal{K}_P,z_P)_{\leqslant -i-1};$$

it restricts to a linear map

$$(B_X)^i/(B_X)^{i+1} \to \Psi\,\mathrm{DO}(\mathcal{O}_P,z_P)_{\leqslant -i}/\Psi\,\mathrm{DO}(\mathcal{O}_P,z_P)_{\leqslant -i-1}.$$

Now we have a linear isomorphism

$$\lambda_P^{(i)} : \Psi\,\mathrm{DO}(\mathcal{K}_P, z_P)_{\leqslant -i}/\Psi\,\mathrm{DO}(\mathcal{K}_P, z_P)_{\leqslant -i-1} \to \mathbb{C}((z_P))(dz_P)^{\otimes i}$$

restricting to an isomorphism

$$\Psi\,\mathrm{DO}(\mathcal{O}_P, z_P)_{\leqslant -i}/\Psi\,\mathrm{DO}(\mathcal{O}_P, z_P)_{\leqslant -i-1} \to \mathbb{C}[[z_P]](dz_P)^{\otimes i}$$

and taking the class of $\sum_{j\geqslant i} b_j \partial^{-j}$ to the class of $\widetilde{b}_i(dz_P)^{\otimes i}$, where $\widetilde{b}_i \in \mathbb{C}((z_P))$ is the image of b_i under $\mathcal{K}_P = \mathbb{C}((z_P))$, and the diagram

$$\begin{array}{ccc}
(B_X^{\mathrm{rat}})^i/(B_X^{\mathrm{rat}})^{i+1} & \xrightarrow{\lambda^{(i)}_{\mathrm{rat}}} & \{\text{rational } i\text{-differentials}\} \\
\downarrow L_P^{z_P} & & \downarrow \ell_P^{(i)} \\
\Psi\,\mathrm{DO}(\mathcal{K}_P, z_P)_{\leqslant -i}/\Psi\,\mathrm{DO}(\mathcal{K}_P, z_P)_{\leqslant -i-1} & \xrightarrow{\lambda^{(i)}_P} & \mathbb{C}((z_P))(dz_P)^{\otimes i}
\end{array}$$

is commutative (the right vertical arrow $\ell_P^{(i)}$ is the Laurent expansion of i-differentials at P). Then $L_P^{z_P}$ maps $(B_X)^i/(B_X)^{i+1}$ to

$$\Psi\,\mathrm{DO}(\mathcal{O}_P, z_P)_{\leqslant -i}/\Psi\,\mathrm{DO}(\mathcal{O}_P, z_P)_{\leqslant -i-1},$$

so $\lambda_P^{(i)} \circ L_P^{(z_P)}$ maps $(B_X)^i/(B_X)^{i+1}$ to $\mathbb{C}[[z_P]](dz_P)^{\otimes i}$. Therefore $\lambda^{(i)}_{\mathrm{reg}}((B_X)^i/(B_X)^{i+1})$ is contained in the space of rational differentials on C, which are regular at each point of C; this space is precisely $H^0(C, K^{\otimes i}) = A_i$. □

Being the restriction of an injective map, the map $\lambda^{(i)}_{\mathrm{reg}}\colon (B_X)^i/(B_X)^{i+1} \to A_i$ induced by $\lambda^{(i)}_{\mathrm{rat}}$ is injective. We now prove:

(c) The map $\lambda^{(i)}_{\mathrm{reg}} : (B_X)^i/(B_X)^{i+1} \to A_i$ is surjective.

Let $(a_j)_{j=i,i+1,\dots}$ be a collection of elements of $\mathbb{C}(C)$; let us write the necessary and sufficient conditions for $\sum_{j\geqslant i} a_j D_X^{-j}$ to be a regular pseudodifferential operator. For simplicity, we will assume that the form $\alpha = X^{-1}$ has no pole and $2g-2$ distinct zeroes $Q_1, \dots, Q_{2g-2}$; so the vector field X is nowhere vanishing and has simple poles at $Q_1, \dots, Q_{2g-2}$. Let z_α be a local coordinate at Q_α. Then we have a local expansion at Q_α,

$$X = \left(\frac{c_\alpha}{z_\alpha} + \text{ element of } \mathbb{C}[[z_\alpha]]\right)\frac{\partial}{\partial z_\alpha},$$

where $c_\alpha \in \mathbb{C}^\times$; so we have local expansions

$$(D_X)^{-j} = \lambda^{(\alpha)}_{j,j}(z_\alpha)^j (D_{\partial/\partial z_\alpha})^{-j} + \lambda^{(\alpha)}_{j,j+1}(z_\alpha)^{j-1}(D_{\partial/\partial z_\alpha})^{-j-1} + \dots + \lambda^{(\alpha)}_{j,2j-1} z_\alpha (D_{\partial/\partial z_\alpha})^{-2j+1} + \sum_{\ell\geqslant 0} \lambda^{(\alpha)}_{j,2j+\ell}(D_{\partial/\partial z_\alpha})^{-2j-\ell},$$

where $\lambda^{(\alpha)}_{j,j}, \dots, \lambda^{(\alpha)}_{j,2j-1} \in \mathbb{C}[[z_\alpha]]^\times$, and $\lambda^{(\alpha)}_{j,2j}, \lambda^{(\alpha)}_{j,2j+1}, \dots \in \mathbb{C}[[z_\alpha]]$ (the constant terms of $\lambda^{(\alpha)}_{j,j}, \dots, \lambda^{(\alpha)}_{j,2j-1}$ may be computed explicitly using binomial coefficients). Recall that D_X is the generator of B_X^{rat} corresponding to the vector field X, and $D_{\partial/\partial z_\alpha}$ is the generator of $\Psi\,\mathrm{DO}(\mathcal{K}_{Q_\alpha}, z_\alpha)$ corresponding to the vector field $\frac{\partial}{\partial z_\alpha}$.

So the necessary and sufficient conditions on $(a_j)_{j=i,i+1,\dots}$ are:

(a) $a_j \in \mathbb{C}(C)$, and each a_j is regular outside $\{P_1, \dots, P_{2g-2}\}$;

(b) (local conditions at each Q_α, $\alpha = 1, \dots, 2g-2$) let us denote by $a_j^{(\alpha)}$ the element of $\mathbb{C}((z_\alpha))$, obtained as the Laurent expansion of a_j at Q_α, then the formal series

$$\lambda_{i,i}^{(\alpha)}(z_\alpha)^i a_i^{(\alpha)}, \quad \lambda_{i+1,i+1}^{(\alpha)}(z_\alpha)^{i+1} a_{i+1}^{(\alpha)} + \lambda_{i,i+1}^{(\alpha)}(z_\alpha)^{i-1} a_i^{(\alpha)},$$
$$\lambda_{i+2,i+2}^{(\alpha)}(z_\alpha)^{i+2} a_{i+2}^{(\alpha)} + \lambda_{i+1,i+1}^{(\alpha)}(z_\alpha)^{i} a_{i+1}^{(\alpha)} + \lambda_{i,i+2}^{(\alpha)}(z_\alpha)^{i-2} a_i^{(\alpha)}, \text{ etc.}$$

should all be regular.

This means that the formal series $(a_j^{(\alpha)})_{j=i,i+1,\dots}$ should have the expansions:

$$a_i^{(\alpha)} = \alpha_{i,i}(z_\alpha)^{-i} + \alpha_{i,i-1}(z_\alpha)^{-i+1} + \cdots,$$
$$a_{i+1}^{(\alpha)} = A_{i+1,i+2}(a_i^{(\alpha)})(z_\alpha)^{-i-2} + \alpha_{i+1,i+1}(z_\alpha)^{-i-1} + \alpha_{i+1,i}(z_\alpha)^{-i} + \cdots,$$
$$\begin{aligned} a_{i+2}^{(\alpha)} &= A_{i+2,i+4}(a_{i+1}^{(\alpha)})(z_\alpha)^{-i-4} + A_{i+2,i+3}(a_{i+1}^{(\alpha)})(z_\alpha)^{-i-3} \\ &\quad + \alpha_{i+2,i+2}(z_\alpha)^{-i-2} + \alpha_{i+2,i+1}(z_\alpha)^{-i-1} + \cdots, \end{aligned}$$

where the $\alpha_{k,l}$ are arbitrary complex numbers, and the $f \mapsto A_{k,l}(f)$ are certain linear forms on $(z_\alpha)^{-k}\mathbb{C}[[z_\alpha]]$.

These conditions can be translated as follows:

(1) $a_i X^{-i} \in H^0(C, K^{\otimes i})$;

(2) a_{i+1} belongs to a (possibly empty) affine space over $H^0(C, K^{\otimes i+2})$, depending on a_i;

(3) a_{i+2} belongs to a (possibly empty) affine space over $H^0(C, K^{\otimes i+3})$, depending on a_{i+1}, etc.

We now prove that these affine spaces are all nonempty, and we describe the set of all possible $(a_j)_{j\geqslant i}$.

LEMMA 2.3. — *Let $i \geqslant 2$ and $j \geqslant 0$. Define D_{can} as the divisor $Q_1 + \cdots + Q_{2g-2}$. Identify $H^0(C, K^{\otimes i})$ with the space $\{f \in \mathbb{C}(C) | (f) \geqslant -iD_{\mathrm{can}}\}$, and $H^0(C, K^{\otimes i}(jD_{\mathrm{can}}))$ with $\{f \in \mathbb{C}(C) | (f) \geqslant -(i+j)D_{\mathrm{can}}\}$. Then we have $H^0(C, K^{\otimes i}) \subset H^0(C, K^{\otimes i}(jD_{\mathrm{can}}))$. Moreover, for $\alpha = 1, \dots, 2g-2$, and $k = 1, \dots, j$, define linear forms*

$$\phi_{\alpha,k} : H^0(C, K^{\otimes i}(jD_{\mathrm{can}})) \to \mathbb{C}$$

by the condition that the local expansion of f at Q_α is

$$f \in \sum_{k=1}^{j} \phi_{\alpha,k}(f)(z_\alpha)^{-i-k} + (z_\alpha)^{-i}\mathbb{C}[[z_\alpha]].$$

Then the sequence

$$(5)\qquad 0 \to H^0(C, K^{\otimes i}) \to H^0(C, K^{\otimes i}(jD_{\mathrm{can}})) \xrightarrow{\oplus_{\alpha,k}\phi_{\alpha,k}} \mathbb{C}^{(2g-2)j} \to 0$$

is exact.

Proof of Lemma. — We have a long exact sequence, with $H^1(C, K^{\otimes i})$ replacing 0 in the right hand side of (5). Since $H^1(C, K^{\otimes i}) = 0$ when g and i are $\geqslant 2$, (5) is exact. □

For any pair (i,j), let us choose a section $\sigma_{i,j}$ of the exact sequence (5). So $\sigma_{i,j}$ is a linear map

$$\sigma_{i,j} : \mathbb{C}^{(2g-2)j} \to H^0(C, K^{\otimes i}(jD_{\mathrm{can}})),$$

such that if $f = \sigma_{i,j}((\lambda_{\alpha,k})_{\alpha,k})$, then for each (α, k), we have $\phi_{\alpha,k}(f) = \lambda_{\alpha,k}$.

For any $\omega \in H^0(C, K^{\otimes i})$, we set

$$\sigma(\omega) = (a_i(\omega), a_{i+1}(\omega), \ldots),$$

where

$$a_i(\omega) = \omega X^{-i},$$

$$a_{i+1}(\omega) = \sigma_{i+1,1}\big(\big(A_{i+1,i+2}(a_i(\omega)^{(\alpha)})\big)_{\alpha=1,\ldots,2g-2}\big),$$

$$a_{i+2}(\omega) = \sigma_{i+2,2}\big(\big(A_{i+2,i+4}(a_{i+1}(\omega)^{(\alpha)}), A_{i+3,i+4}(a_{i+1}(\omega)^{(\alpha)})\big)_{\alpha=1,\ldots,2g-2}\big),$$

etc. Then σ is a linear map

$$\sigma : H^0(C, K^{\otimes i}) \to B_X^i;$$

it is a section of the canonical projection $\lambda_{\mathrm{reg}}^{(i)} : B_X^{(i)} \to H^0(C, K^{\otimes i})$. This proves that $\lambda_{\mathrm{reg}}^{(i)}$ is surjective.

(d) The map $\lambda : \mathrm{gr}(B_X) \to A$ is an isomorphism of Poisson algebras.

There is a unique Poisson structure on $\mathbb{C}(C)[\xi^{-1}]$, such that $\{f, g\} = 0$ and

$$\{\xi^{-1}, f\} = -X(f)\xi^{-2}$$

for $f, g \in \mathbb{C}(C)$. Then the map $\lambda_{\mathrm{rat}} : \mathrm{gr}(B_X^{\mathrm{rat}}) \to \mathbb{C}(C)[\xi^{-1}]$, defined by $0 \mapsto 0$ and $\sum_{j \geqslant i} a_j D_X^{-j} \mapsto a_i \xi^{-i}$ when $a_i \neq 0$, is an isomorphism of Poisson algebras. Moreover, there is a unique inclusion $A \hookrightarrow \mathbb{C}(C)[\xi^{-1}]$, taking $\omega \in H^0(C, K^{\otimes i})$ to $(\omega\alpha^{-i})\xi^{-i}$ (recall that $\omega\alpha^{-i}$ belongs to $\mathbb{C}(C)$). This inclusion is a morphism of Poisson algebras. Then we have a commuting diagram

$$\begin{array}{ccc} \mathrm{gr}(B_X^{\mathrm{rat}}) & \xrightarrow{\lambda_{\mathrm{rat}}} & \mathbb{C}(C)[\xi^{-1}] \\ \uparrow & & \uparrow \\ \mathrm{gr}(B_X) & \xrightarrow{\lambda_{\mathrm{reg}}} & A. \end{array}$$

Since all maps in this diagram except perhaps λ_{reg} are Poisson algebra morphisms, and since the vertical arrows are injective, λ_{reg} is also a Poisson morphism. This ends the proof of Theorem 2.1, in the case $D = 0$. □

2.2.6. *The case of nonzero divisor D.* We already defined the algebra $B_X^{(D)}$, using the vector field X and the collection of local coordinates $(z_P)_{P\in C}$. We first prove:

LEMMA 2.4. — *$B_X^{(D)}$ is independent of the choice of $(z_P)_{P\in C}$.*

Proof. — If z_P and z'_P are local coordinates at $P \in C$, we have an isomorphism $i_{z_P,z'_P} : \Psi\,\mathrm{DO}(\mathcal{K}_P, z_P) \to \Psi\,\mathrm{DO}(\mathcal{K}_P, z'_P)$. After composing it with the isomorphisms $i_{\frac{\partial}{\partial z_P},(z_P)^{\delta_P}\frac{\partial}{\partial z_P}} : \Psi\,\mathrm{DO}(\mathcal{K}_P, z_P) \to \Psi\,\mathrm{DO}\big(\mathcal{K}_P, (z_P)^{\delta_P}\frac{\partial}{\partial z_P}\big)$ and the inverse of $i_{\frac{\partial}{\partial z'_P},(z'_P)^{\delta_P}\frac{\partial}{\partial z'_P}} : \Psi\,\mathrm{DO}(\mathcal{K}_P, z'_P) \to \Psi\,\mathrm{DO}\big(\mathcal{K}_P, (z'_P)^{\delta_P}\frac{\partial}{\partial z'_P}\big)$, we get the isomorphism

$$i_{(z_P)^{\delta_P}\frac{\partial}{\partial z_P},(z'_P)^{\delta_P}\frac{\partial}{\partial z'_P}} : \Psi\,\mathrm{DO}\left(\mathcal{K}_P, (z_P)^{\delta_P}\frac{\partial}{\partial z_P}\right) \to \Psi\,\mathrm{DO}\left(\mathcal{K}_P, (z'_P)^{\delta_P}\frac{\partial}{\partial z'_P}\right).$$

Now there exists $\varphi \in \mathcal{O}_P^\times$, such that $(z_P)^{\delta_P}\frac{\partial}{\partial z_P} = \varphi \cdot (z'_P)^{\delta_P}\frac{\partial}{\partial z'_P}$, so $i_{(z_P)^{\delta_P}\frac{\partial}{\partial z_P},(z'_P)^{\delta_P}\frac{\partial}{\partial z'_P}}$ restricts to an isomorphism

$$\Psi\,\mathrm{DO}(\mathcal{O}_P, z_P)^{(\delta_P)} \to \Psi\,\mathrm{DO}(\mathcal{O}_P, z'_P)^{(\delta_P)}.$$

One uses this isomorphism in the same way as above to show that the algebra $B_X^{(D)}$ is independent on the choice of $(z_P)_{P\in C}$. □

The behavior of $B_X^{(D)}$ with respect to changes of the vector field X is the same as above.

The filtration of $B_X^{(D)}$ is defined by $(B_X^{(D)})^i = B_X^{(D)} \cap (B_X^{\mathrm{rat}})^i$. Then we prove:

LEMMA 2.5. — *The restriction of the map $\lambda_{\mathrm{rat}}^{(i)}$ defined by (4) to $(B_X^{(D)})^i/(B_X^{(D)})^{i+1}$ maps to $(A^{(D)})_i \subset A_i^{\mathrm{rat}}$. So λ_{rat} induces a Poisson morphism* $\mathrm{gr}(B_X^{(D)}) \to A^{(D)}$.

Proof. — Any element of $\Psi\,\mathrm{DO}\big(\mathcal{O}_P, (z_P)^{\delta_P}\frac{\partial}{\partial z_P}\big)_{\leqslant -i}$ has the expansion

$$\sum_{j\geqslant i} a_j (D_{\partial/\partial z_P})^{-j},$$

with $a_j \in (z_P)^{-j\delta_P}\mathbb{C}[[z_P]]$ for any $j \geqslant i$. This implies that $a_i\alpha^i \in H^0(C, K(D)^{\otimes i})$. □

The other statements are proved as above, in particular, the analogue of Lemma 2.3 holds because $\deg(K(D)) \geqslant \deg(K)$. □

2.3. Twisting by generalized line bundles.

If $\ell \in \mathbb{C}$, there is a unique continuous automorphism of

$$\Psi\,\mathrm{DO}\Big(\mathbb{C}((z)), \frac{\partial}{\partial z}\Big),$$

taking $D_{\partial/\partial z}$ to $D_{\partial/\partial z} - \frac{\ell}{z}$ and leaving z fixed. We denote it by $T \mapsto z^\ell T z^{-\ell}$.

We denote by $\mathbb{C}C$ the group of all formal linear combinations $\sum_{P\in C} \lambda_P P$, where all λ_P but a finite number are zero. We have a natural group morphism $\mathbb{Z}C \to \mathbb{C}C$. Moreover, the divisor map is a group morphism $\mathrm{div} : \mathbb{C}(C)^\times \to \mathbb{Z}C$. The Picard group of C is defined as $\mathrm{Pic}(C) = \mathbb{Z}C/\mathrm{div}(\mathbb{C}(C)^\times)$. Then there is an injection $\mathrm{Pic}(C) \hookrightarrow \mathbb{C}C/\mathrm{div}(\mathbb{C}(C)^\times)$ induced by $\mathbb{Z}C \hookrightarrow \mathbb{C}C$. We call elements of $\mathbb{C}C$ "generalized divisors" and elements of $\mathbb{C}C/\mathrm{div}(\mathbb{C}(C)^\times)$ "generalized line bundles".

Let $\lambda = \sum_{P\in C} \lambda_P P$ be a generalized divisor. One can define an algebra $B_X^{(C,D),\lambda}$ of twisted pseudodifferential operators as follows:

$$B_X^{(C,D),\lambda} = \Big(\prod_{P\in C} L_P^{z_P}\Big)^{-1}\Big(\prod_{P\in C} (z_P)^{\lambda_P}\,\Psi\,\mathrm{DO}\Big(\mathbb{C}[[z_P]], (z_P)^{\delta_P}\frac{\partial}{\partial z_P}\Big)(z_P)^{-\lambda_P}\Big).$$

Conjugation by a rational function sets up an isomorphism between $B_X^{(C,D),\lambda}$ and $B_X^{(C,D),\lambda'}$, for λ, λ' linearly equivalent generalized divisors (i.e., differing by an element of $\mathrm{div}(\mathbb{C}(C)^\times)$). On the other hand, one can repeat the proof of Theorem 2.1 to prove that $B_X^{(C,D),\lambda}$ is a quantization of $A^{(C,D)}$ for any λ.

2.4. Functoriality in (C, D).

In this section, we emphasize the dependence of the algebras $A, A^{(D)}$, B_X, $B_X^{(D)}$ in the curve C by denoting them $A^{(C)}$, $A^{(C,D)}$, $B_X^{(C)}$, $B_X^{(C,D)}$.

2.4.1. *The Poisson algebras.* Let $\varphi : C \to C'$ be a (possibly ramified) covering. So φ gives rise to an inclusion of fields $\varphi^* : \mathbb{C}(C') \hookrightarrow \mathbb{C}(C)$. Then φ induces morphisms $\varphi^* : H^0(C', (K')^{\otimes i}) \to H^0(C, K^{\otimes i})$ (here K' is the canonical bundle of C'), and therefore an algebra morphism

$$\varphi^*_{\mathrm{class}} : A^{(C')} \to A^{(C)}. \tag{6}$$

The maps φ^* extend to maps between spaces of rational i-differentials. For any $f' \in \mathbb{C}(C')$, we have in particular $d(\varphi^*(f')) = \varphi^*(df')$. It follows that for any rational differential α' on C', we have $\varphi^*(\nabla^{\alpha'}(\omega')) = \nabla^{\varphi^*(\alpha')}(\varphi^*(\omega'))$. It follows that $\varphi^*_{\mathrm{class}}$ is a morphism of Poisson algebras.

Let $D' = \sum_{P' \in C'} \delta'_{P'} P'$ be an effective divisor on C', and set

$$\varphi^{-1}(D') = \sum_{P \in C} \big(\delta_{\varphi(P)} \nu_P + (1 - \nu_P)\big) P,$$

where ν_P is the ramification index of f at $P \in C$ (it is 1 for all but finitely many P). Set $D = \varphi^{-1}(D')$, then D is an effective divisor of C. Then φ induces a morphism

$$\varphi^*_{\mathrm{class}} : A^{(C',D')} \to A^{(C,D)} \tag{7}$$

of graded algebras and of Poisson algebras.

2.4.2. *Quantization of the morphisms $\varphi^*_{\mathrm{class}}$.* Let X' be a rational, nonzero vector field on C', let $\alpha' = (X')^{-1}$ be the rational differential on C' inverse to X'; let us set $\alpha = \varphi^*(\alpha')$ and $X = \alpha^{-1}$. So X is a rational, nonzero vector field on C. We will now show:

Proposition 2.1. — *There exists a morphism*

$$\varphi^*_{\mathrm{rat}} : B_{X'}^{C',\mathrm{rat}} \to B_X^{C,\mathrm{rat}}$$

of complete filtered algebras. It induces morphisms

$$\varphi^*_{\mathrm{pseudo}} : B_{X'}^{(C')} \to B_X^{(C)}$$

and

$$\varphi^*_{\mathrm{pseudo}} : B_{X'}^{(C',D')} \to B_X^{(C,D)}$$

of complete filtered algebras, quantizing the morphisms $\varphi^*_{\mathrm{class}}$.

Proof. — *(a) Construction of* $\varphi^*_{\mathrm{rat}} : B_{X'}^{C',\mathrm{rat}} \to B_X^{C,\mathrm{rat}}$. The map $\varphi^* : (\mathbb{C}(C'), X') \to (\mathbb{C}(C), X)$ is a morphism of differential rings. Indeed, we have, for $f' \in \mathbb{C}(C')$,

$$X(\varphi^*(f')) = \frac{d(\varphi^*(f'))}{\alpha} = \frac{d(\varphi^*(f'))}{\varphi^*(\alpha')} = \varphi^*(\frac{df'}{\alpha'}) = \varphi^*(X'(f')).$$

So φ^* induces an algebra map

$$\Psi\,\mathrm{DO}(\varphi^*)_{\leqslant 0} : \Psi\,\mathrm{DO}(\mathbb{C}(C'), X')_{\leqslant 0} \to \Psi\,\mathrm{DO}(\mathbb{C}(C), X)_{\leqslant 0},$$

that is an algebra map $\Psi\,\mathrm{DO}(\varphi^*)_{\leqslant 0} : B_{X'}^{C',\mathrm{rat}} \to B_X^{C,\mathrm{rat}}$.

(b) $\varphi^*_{\mathrm{rat}}(B_{X'}^{(C')}) \subset B_X^{(C)}$. Let $P \in C$, and let us set $P' = \varphi^*(P)$. Let ν_P be the ramification index of φ at P. Then if z_P, $z'_{P'}$ are local coordinates at P, P', we have $\varphi^*(z'_{P'}) = \lambda \cdot (z_P)^{\nu_P}$, where $\lambda \in \mathbb{C}[[z_P]]^\times$. Then there is a natural morphism

$$\Psi\,\mathrm{DO}(\mathcal{K}_{P'}, z'_{P'})_{\leqslant 0} \to \Psi\,\mathrm{DO}(\mathcal{K}_P, z_P)_{\leqslant 0},$$

restricting to a morphism $\Psi\,\mathrm{DO}(\mathcal{O}_{P'}, z'_{P'})_{\leqslant 0} \to \Psi\,\mathrm{DO}(\mathcal{O}_P, z_P)_{\leqslant 0}$, and such that the diagram

$$\begin{array}{ccc} \Psi\,\mathrm{DO}(\mathcal{K}_{P'}, z'_{P'})_{\leqslant 0} & \xrightarrow{\alpha} & \Psi\,\mathrm{DO}(\mathcal{K}_P, z_P)_{\leqslant 0} \\ \uparrow L_{P'}^{z'_{P'}} & & \uparrow L_P^{z_P} \\ B_{X'}^{C',\mathrm{rat}} & \longrightarrow & B_X^{C,\mathrm{rat}} \end{array}$$

commutes. The Laurent expansion morphisms behave with respect to changes of the vector fields according to (2). So we may replace X' by a rational vector field Y', without any zero or pole at P'. We denote by Y the corresponding vector field on C, and by $Y_{\mathrm{local}}, Y'_{\mathrm{local}}$ the formal expansions of Y, Y' at P, P'.

Now we have a commuting diagram

$$\begin{array}{ccc} \Psi\,\mathrm{DO}(\mathcal{K}_{P'}, Y'_{\mathrm{local}})_{\leqslant 0} & \xrightarrow{\alpha} & \Psi\,\mathrm{DO}(\mathcal{K}_P, Y_{\mathrm{local}})_{\leqslant 0} \\ \uparrow & & \uparrow \\ B_{X'}^{C',\mathrm{rat}} & \longrightarrow & B_X^{C,\mathrm{rat}}. \end{array}$$

Since Y'_{local} preserves $\mathcal{O}_{P'}$, $\Psi\,\mathrm{DO}(\mathcal{K}_{P'}, Y'_{\mathrm{local}})_{\leqslant 0}$ contains a subalgebra $\Psi\,\mathrm{DO}(\mathcal{O}_{P'}, Y'_{\mathrm{local}})_{\leqslant 0}$. The assumptions on Y' allow to identify

$$\Psi\,\mathrm{DO}(\mathcal{O}_{P'}, z'_{P'})_{\leqslant 0}$$

with $\Psi\,\mathrm{DO}(\mathcal{O}_{P'}, Y'_{\mathrm{local}})_{\leqslant 0}$. Let us show that α takes this subalgebra to $\Psi\,\mathrm{DO}(\mathcal{O}_{P'}, z'_{P'})_{\leqslant 0}$.

The map α takes $\mathcal{O}_{P'}$ to $\mathcal{O}_P$, and it takes $(D_{Y'_{\mathrm{local}}})^{-1}$ to $(D_{Y_{\mathrm{local}}})^{-1}$. Now we have $Y'_{\mathrm{local}} = \mu\frac{\partial}{\partial z'_{P'}}$, where $\mu \in \mathbb{C}[[z'_{P'}]]^\times$. On the other hand, the local expansion of Y at P has the form

$$Y_{\mathrm{local}} = \pi(z_P)^{1-\nu_P}\frac{\partial}{\partial z_P},$$

with $\pi \in \mathbb{C}[[z_P]]^\times$. This expansion implies that $(D_{Y_{\mathrm{local}}})^{-1}$ has the form

$$\sum_{i\geqslant 1}\pi_i(D_{\partial/\partial z_P})^{-i},$$

where $\pi_i \in (z_P)^{i+\nu_P-2}\mathbb{C}[[z_P]]$, so $(D_{Y_{\mathrm{local}}})^{-1}$ belongs to $\Psi\,\mathrm{DO}(\mathcal{O}_P, z_P)_{\leqslant 0}$.

Now α takes $(D_{\partial/\partial z'_{P'}})^{-1}$ to $(D_{Y_{\mathrm{local}}})^{-1}\varphi^*(\mu)$, which belongs to $\Psi\,\mathrm{DO}(\mathcal{O}_P, z_P)_{\leqslant 0}$. So α takes the generators of $\Psi\,\mathrm{DO}(\mathcal{O}_P, z_P)_{\leqslant 0}$ to

$$\Psi\,\mathrm{DO}(\mathcal{O}_{P'}, z'_{P'})_{\leqslant 0},$$

so

$$\alpha(\Psi\,\mathrm{DO}(\mathcal{O}_P, z_P)_{\leqslant 0}) \subset \Psi\,\mathrm{DO}(\mathcal{O}_{P'}, z'_{P'})_{\leqslant 0}.$$

This implies that φ^* takes

$$\Big(\prod_{P'\in C'} L_{P'}^{z'_{P'}}\Big)^{-1}\Big(\prod_{P'\in C'}\Psi\,\mathrm{DO}(\mathcal{O}_{P'}, z'_{P'})_{\leqslant 0}\Big)$$

to

$$\Big(\prod_{P\in C} L_P^{z_P}\Big)^{-1}\Big(\prod_{P\in C}\Psi\,\mathrm{DO}(\mathcal{O}_P, z_P)_{\leqslant 0}\Big),$$

so $\varphi^*(B_{X'}^{(C')}) \subset B_X^{(C)}$.

In the same way, one proves that $\varphi^*(B_{X'}^{(C',D')}) \subset B_X^{(C,D)}$.

This ends the proof of Proposition 2.1. □

3. An explicit example: the rational case.

It is easy to see that the results of Theorem 2.1 also hold in the following cases: $g = 0$, $\deg(D) \geqslant 2$; and $g = 1$, $\deg(D) \geqslant 1$. In this section, we study the first case.

3.1. Presentation of the classical algebra.

Let us set $C = \mathbb{P}^1$, $D = N\infty$, where $N \geqslant 2$. We have then
$$A^{(C,D)} = \oplus_{i\geqslant 0} A_i^{(C,D)},$$
where $A_i^{(C,D)} = H^0(\mathbb{P}^1, K(D)^{\otimes i}) = \{f(z)(dz)^i$, where $f(z)$ is a polynomial of degree $\leqslant i(N-2)\}$.

PROPOSITION 3.1. — *$A^{(C,D)}$ may be presented as follows: generators are $\omega_a = z^a dz$, $a = 0, \dots, N-2$, and relations are*
$$\omega_a\omega_b = \omega_c\omega_d,$$
for any quadruple (a,b,c,d) such that $a+b = c+d$.

Proof. — Let $\widetilde{A}(N)$ be the algebra with generators $t_0, \dots, t_{N-2}$, and relations

$$t_a t_b = t_c t_d, \tag{8}$$

for any quadruple (a,b,c,d) such that $a+b = c+d$. Then $\widetilde{A}(N)$ is the sum of its homogeneous components $\widetilde{A}(N)_i$, and relations (8) imply that a generating family of $\widetilde{A}(N)_i$ is given by the union of the

$$(t_0)^\alpha (t_{N-2})^\beta t_k, \quad \alpha, \beta \geqslant, \alpha+\beta = i-1, \quad k = 0, \dots, N-3, \tag{9}$$

with t_{N-2}^i.

We have an algebra morphism

$$\widetilde{A}(N) \to A^{(C,D)}, \tag{10}$$

taking each t_i to $\widetilde{\omega}_i$. It takes the generating family (9) to a basis of $A_i^{(C,D)}$, which proves. as the same time that this family is a basis, and that (10) is an isomorphism. □

Remark 3. — Proposition 3.1 is an algebraic translation of the statement that the rational normal curve in $\mathbb{P}^{N-2}$, that is the image of the embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^{N-2}$ given by $X_i = u^{N-2-i}v^i$, is defined by the equations $X_iX_j = X_kX_l$, for $i+j = k+l$ (see e.g. [7], 1.14). □

The Poisson bracket on $A^{(C,D)}$ is given by
$$\{\omega_a, \omega_b\} = (b-a)z^{a+b-1}(dz)^3,$$
so in terms of generators
$$\{\omega_a, \omega_b\} = (b-a)\omega_c\omega_d\omega_e\,,$$
for any (c,d,e) such that $c+d+e = a+b-1$.

3.2. Quantized algebras B_X and B'_X.

The field of rational functions on $\mathbb{P}^1$ is the field of rational fractions $\mathbb{C}(z)$. Equip it with its derivation $X = \partial_z = \frac{d}{dz}$. Then $B_X^{\mathrm{rat}} = \Psi\,\mathrm{DO}(\mathbb{C}(z), \partial_z)_{\leqslant 0}$.

Lifts in B_X of the ω_a are the elements $\widetilde{\omega}_a = (D_{\omega_a^{-1}})^{-1}$, i.e.,

$$\widetilde{\omega}_a = (\partial_z)^{-1} z^a, \quad a = 0, \dots, N-2.$$

Denote by B'_X the subalgebra of B_X^{rat} generated by the $\widetilde{\omega}_a$. Since the ω_a generate $A^{(C,D)}$, B_X is the completion of B'_X with respect to the topology of B_X^{rat}.

Theorem 3.1. — *For any quadruple (a,b,c,d) such that $0 \leqslant a,b,c,d \leqslant N-2$, $a+b = c+d$ and $b > d$, we have*

$$\widetilde{\omega}_a \widetilde{\omega}_b - \widetilde{\omega}_c \widetilde{\omega}_d = (d-b)\widetilde{\omega}_a \widetilde{\omega}_{b-d} \widetilde{\omega}_d. \tag{11}$$

Let us define C as the algebra with generators $t_a, a = 0, \dots, N-2$ and relations

$$t_a t_b - t_c t_d = (d-b) t_a t_{b-d-1} t_d,$$

for $a,b,c,d = 0, \dots, N-2$, such that $a+b = c+d$ and $b > d$. Let I_C be the ideal of C generated by the $t_a, a = 0, \dots, N-2$. Set $\widehat{C} = \lim_{\leftarrow n} C/(I_C)^n$. Then there is a unique continuous algebra isomorphism

$$\widehat{C} \to B_X, \tag{12}$$

taking each t_a to $\widetilde{\omega}_a$. This isomorphism induces an algebra isomorphism

$$C/\big(\cap_{n\geqslant 0} (I_C)^n\big) \to B'_X.$$

Proof. — Let us first prove the relation (11). We have

$$\widetilde{\omega}_a \widetilde{\omega}_b = z^{a+b}(\partial_z + \frac{b}{z})^{-1}(\partial_z)^{-1},$$

so

$$\begin{aligned}
\widetilde{\omega}_a \widetilde{\omega}_b - \widetilde{\omega}_c \widetilde{\omega}_d &= z^{a+b}\Big(\Big(\partial_z + \frac{b}{z}\Big)^{-1} - \Big(\partial_z + \frac{d}{z}\Big)^{-1}\Big)(\partial_z)^{-1} \\
&= z^{a+b}\Big(\partial_z + \frac{b}{z}\Big)^{-1} \frac{d-b}{z}\Big(\partial_z + \frac{d}{z}\Big)^{-1}(\partial_z)^{-1} \\
&= (d-b) z^{a+b-1}\Big(\partial_z + \frac{b-1}{z}\Big)^{-1}\Big(\partial_z + \frac{d}{z}\Big)^{-1}(\partial_z)^{-1} \\
&= (d-b) z^a (\partial_z)^{-1} z^{b-d-1}(\partial_z)^{-1} z^d (\partial_z)^{-1} \\
&= (d-b)\widetilde{\omega}_a \widetilde{\omega}_{b-d-1} \widetilde{\omega}_d.
\end{aligned}$$

We have $\mathrm{gr}(C) = \oplus_{n\geqslant 0}(I_C)^n/(I_C)^{n+1}$. We have a morphism of filtered algebras $C \to B'_X$. Moreover, we have $\mathrm{gr}(B'_X) = \mathrm{gr}(B_X) = A^{(C,D)}$, so we get an algebra morphism $\mathrm{gr}(C) \to \mathrm{gr}(B_X) = A^{(C,D)}$.

Select in relations (11), the subset of relations corresponding to $(a, b, a+b, 0)$ for a, b such that $a+b \leqslant N-2$, and $(a, b, N-2, a+b-(N-2))$ for a, b such that $a+b > N-2$. Then this subset of relations implies that a generating family of $\mathrm{gr}^n(B'_X)$ is the union of all

$$(\widetilde{\omega}_0)^\alpha(\widetilde{\omega}_{N-2})^\beta\widetilde{\omega}_i, \quad i = 0, \dots, N-3, \ \alpha+\beta = n-1,$$

with $(\widetilde{\omega}_{N-2})^i$. The morphism $\mathrm{gr}(C) \to A$ takes it to a basis of A_n, so $\mathrm{gr}(C) \to \mathrm{gr}(B'_X)$ is an isomorphism. This implies that the map $\widehat{C} \to B_X$ obtained by completing $C \to B'_X$ is an isomorphism. This fact now implies that $C/(\cap_{n\geqslant 0}(I_C)^n) \to B'_X$ is injective. Since it is obviously surjective, this map is an isomorphism. □

Remark 4. — We do not know whether $\cap_{n\geqslant 0}(I_C)^n = 0$, in other words, whether C is separated for the topology defined by the powers of I_C.

4. Quantization based on Poincaré uniformization.

In this section, we assume that C is defined over $\mathbb{C}$, and that we are given a Poincaré uniformization of C. We denote by $\mathcal{H}$ the Poincaré half-plane, and we denote by Γ a discrete subgroup of $SL_2(\mathbb{R})$, such that there is an analytic isomorphism $\mathcal{H}/\Gamma \to C$.

We will recall the results of [3] in the Rankin-Cohen brackets (Section 4.1); we will show how they give rise to a solution B^{an} of the problem of quantizing the algebra A (Section 4.2), and that this solution is isomorphic to the quantization B_X of Section 2 (Section 4.3). For simplicity, we restrict ourselves to the case $D = 0$.

4.1. Rankin-Cohen brackets and pseudodifferential operators on $\mathcal{H}$: the results of [3].

Let us denote by $\mathrm{Hol}(\mathcal{H})$ the ring of holomorphic functions on the Poincaré half-plane and by τ the coordinate on this plane. Let us denote by $\partial_{\mathcal{H}}$ its derivation $d/d\tau$. Consider the algebra $\Psi\,\mathrm{DO}(\mathrm{Hol}(\mathcal{H}), \partial_{\mathcal{H}})_{\leqslant 0}$. It is a filtered ring, with associated graded

$$\bigoplus_{i\geqslant 0} H^0(\mathcal{H}, K_{\mathcal{H}}^{\otimes i}),$$

where $K_{\mathcal{H}}$ is the sheaf of differentials on $\mathcal{H}$.

$K_{\mathcal{H}}$ has a natural section $d\tau$, which induces isomorphisms $H^0(\mathcal{H}, K_{\mathcal{H}}^{\otimes i}) \to \mathrm{Hol}(\mathcal{H})$.

The rings $\Psi\,\mathrm{DO}(\mathrm{Hol}(\mathcal{H}), \partial_{\mathcal{H}})_{\leqslant 0}$ and $\bigoplus_{i\geqslant 0} H^0(\mathcal{H}, K_{\mathcal{H}}^{\otimes i})$ are equipped with natural actions of $SL_2(\mathbb{R})$. The paper [3] contains the following results:

THEOREM 4.1 (see [3]). — *There exists a lifting map*

$$\mathrm{lift} : \bigoplus_{i\geqslant 0} H^0(\mathcal{H}, K_{\mathcal{H}}^{\otimes i}) \to \Psi\,\mathrm{DO}(\mathrm{Hol}(\mathcal{H}), \partial_{\mathcal{H}})_{\leqslant 0},$$

which is $SL_2(\mathbb{R})$-equivariant. The restriction of lift *to $H^0(\mathcal{H}, K_{\mathcal{H}}^{\otimes i})$ maps this space to*

$$\Psi\,\mathrm{DO}(\mathrm{Hol}(\mathcal{H}), \partial_{\mathcal{H}})_{\leqslant -i},$$

and the composed map

$$\begin{aligned} H^0(\mathcal{H}, K_{\mathcal{H}}^{\otimes i}) &\to \Psi\,\mathrm{DO}(\mathrm{Hol}(\mathcal{H}), \partial_{\mathcal{H}})_{\leqslant -i} \\ &\to \Psi\,\mathrm{DO}(\mathrm{Hol}(\mathcal{H}), \partial_{\mathcal{H}})_{\leqslant -i}/\Psi\,\mathrm{DO}(\mathrm{Hol}(\mathcal{H}), \partial_{\mathcal{H}})_{\leqslant -i-1} \end{aligned}$$

is inverse to the natural isomorphism

$$\Psi\,\mathrm{DO}(\mathrm{Hol}(\mathcal{H}), \partial_{\mathcal{H}})_{\leqslant -i}/\Psi\,\mathrm{DO}(\mathrm{Hol}(\mathcal{H}), \partial_{\mathcal{H}})_{\leqslant -i-1} \to H^0(\mathcal{H}, K_{\mathcal{H}}^{\otimes i}).$$

If $\omega \in H^0(\mathcal{H}, K_{\mathcal{H}}^{\otimes i})$ has the form $\omega(\tau)(d\tau)^i$, then $\mathrm{lift}(\omega)$ *has the expression*

$$\mathrm{lift}(\omega) = \omega(\tau)(\partial_{\mathcal{H}})^{-i} + \sum_{n>0} \ell_{i,n}\omega^{(n)}(\tau)(\partial_{\mathcal{H}})^{-i-n},$$

where $\ell_{i,n}$ are explicit rational numbers.

Denote by μ the product on $\bigoplus_{i\geqslant 0} H^0(\mathcal{H}, K_{\mathcal{H}}^{\otimes i})$ obtained by transporting the product of $\Psi\,\mathrm{DO}(\mathrm{Hol}(\mathcal{H}), \partial_{\mathcal{H}})_{\leqslant 0}$ by the map lift. Since the product on

$$\Psi\,\mathrm{DO}(\mathrm{Hol}(\mathcal{H}), \partial_{\mathcal{H}})_{\leqslant 0}$$

is expressed by differential operators, μ is a $SL_2(\mathbb{R})$-invariant star-product on $\bigoplus_{i\geqslant 0} H^0(\mathcal{H}, K_{\mathcal{H}}^{\otimes i})$. More precisely, the authors of [3] show:

THEOREM 4.2 (see [3]). — *Let us denote by μ_{ij}^k the map*

$$H^0(\mathcal{H}, K_{\mathcal{H}}^{\otimes i}) \otimes H^0(\mathcal{H}, K_{\mathcal{H}}^{\otimes j}) \to H^0(\mathcal{H}, K_{\mathcal{H}}^{\otimes i+j+k})$$

induced by μ, then the μ_{ij}^k are the Rankin-Cohen brackets: we have

$$\mu_{ij}^k(\omega(\tau)(d\tau)^i, \omega'(\tau)(d\tau)^j) = \sum_{\alpha,\beta\geqslant 0} a_{i,j,\alpha,\beta}^k \omega^{(\alpha)}(\tau)\omega'^{(\beta)}(\tau)(d\tau)^{i+j+k},$$

for suitable rational numbers $a_{i,j,\alpha,\beta}^k$.

This result immediately implies that the Rankin-Cohen brackets are $SL_2(\mathbb{R})$-invariant.

4.2. Construction of B^{an}.

Let C be a complex curve, equipped with an isomorphism $C \to \mathcal{H}/\Gamma$ of analytic manifolds. This isomorphism induces an isomorphism

$$A \to \bigoplus_{i\geqslant 0} \left(H^0(\mathcal{H}, K_{\mathcal{H}}^{\otimes i})\right)^{\Gamma}$$

of graded algebras and of Poisson algebras.

THEOREM 4.3. — *Set*

$$B^{\mathrm{an}} = \left(\Psi\,\mathrm{DO}(\mathrm{Hol}(\mathcal{H}), \partial_{\mathcal{H}})\right)^{\Gamma}.$$

Then B^{an} is a filtered algebra. Its associated Poisson algebra is isomorphic to A.

Proof. — Let us set $(B^{\mathrm{an}})^i = \left(\Psi\,\mathrm{DO}(\mathrm{Hol}(\mathcal{H}), \partial_{\mathcal{H}})_{\leqslant -i}\right)^{\Gamma}$. This obviously defines a filtration on B^{an}, and the image of the composed map

$$\begin{aligned}(B^{\mathrm{an}})^i/(B^{\mathrm{an}})^{i+1} &\to \Psi\,\mathrm{DO}(\mathrm{Hol}(\mathcal{H}), \partial_{\mathcal{H}})_{\leqslant -i}/\Psi\,\mathrm{DO}(\mathrm{Hol}(\mathcal{H}), \partial_{\mathcal{H}})_{\leqslant -i-1} \\ &\to H^0(\mathcal{H}, K_{\mathcal{H}}^{\otimes i})\end{aligned} \tag{13}$$

is contained in $\left(H^0(\mathcal{H}, K_{\mathcal{H}}^{\otimes i})\right)^{\Gamma}$. So we have a natural map

$$(B^{\mathrm{an}})^i/(B^{\mathrm{an}})^{i+1} \to \left(H^0(\mathcal{H}, K_{\mathcal{H}}^{\otimes i})\right)^{\Gamma}. \tag{14}$$

Since each map of the sequence (13) is injective, so is (14). It remains to prove that (14) is surjective. Denote by $\mathrm{lift}_{i,\Gamma}$ the restriction of lift to $(H^0(\mathcal{H}, K_{\mathcal{H}}^{\otimes i}))^{\Gamma}$. Then $\mathrm{lift}_{i,\Gamma}$ is a linear map

$$\left(H^0(\mathcal{H}, K_{\mathcal{H}}^{\otimes i})\right)^{\Gamma} \longrightarrow \Psi\,\mathrm{DO}(\mathrm{Hol}(\mathcal{H}), \partial_{\mathcal{H}})_{\leqslant -i}.$$

According to Theorem 4.1, the image of $\mathrm{lift}_{i,\Gamma}$ is actually contained in

$$(\Psi\,\mathrm{DO}(\mathrm{Hol}(\mathcal{H}), \partial_{\mathcal{H}})_{\leqslant -i})^{\Gamma} = (B^{\mathrm{an}})^i.$$

The composed map

$$\big(H^0(\mathcal{H},K_{\mathcal{H}}^{\otimes i})\big)^{\Gamma} \xrightarrow{\mathrm{lift}_{i,\Gamma}} (B^{\mathrm{an}})^i \longrightarrow (B^{\mathrm{an}})^i/(B^{\mathrm{an}})^{i+1}$$

is obviously a section of (14), which proves that this map is surjective. □

The algebra B^{an} also has a "star-product" version.

THEOREM 4.1. — *The product μ, defined in terms of Rankin-Cohen brackets (see Theorem 4.2), restricts to a product on*

$$A = \bigoplus_{i\geqslant 0} \big(H^0(\mathcal{H},K_{\mathcal{H}}^{\otimes i})\big)^{\Gamma}.$$

The restriction of lift *induces an isomorphism between (A,μ) and B^{an}.*

4.3. Isomorphism with the construction of Section 2.

PROPOSITION 4.2. — *For any nonzero rational vector field X on C, there is an isomorphism*

$$\alpha_X : B_X \to B^{\mathrm{an}}$$

of complete filtered algebras. If Y is another nonzero rational vector field on C, then $\alpha_X = \alpha_Y \circ i_{XY}$.

Proof. — Let us denote by $\mathrm{Mer}(\mathcal{H})$ the ring of meromorphic functions on $\mathcal{H}$, all poles of which are of finite order. Then $\partial_{\mathcal{H}}$ extends to a derivation of $\mathrm{Mer}(\mathcal{H})$ (which we also denote by $\partial_{\mathcal{H}}$). We set

$$\Psi\,\mathrm{DO}(\mathcal{H})^{\mathrm{mer}}_{\leqslant 0} = \Psi\,\mathrm{DO}(\mathrm{Mer}(\mathcal{H}),\partial_{\mathcal{H}})_{\leqslant 0}.$$

We will also set

$$\Psi\,\mathrm{DO}(\mathcal{H})_{\leqslant 0} = \Psi\,\mathrm{DO}(\mathrm{Hol}(\mathcal{H}),\partial_{\mathcal{H}})_{\leqslant 0}.$$

Then we have a commuting square of algebras

$$\begin{array}{ccc} \big(\Psi\,\mathrm{DO}(\mathcal{H})^{\mathrm{mer}}_{\leqslant 0}\big)^{\Gamma} & \hookrightarrow & \Psi\,\mathrm{DO}(\mathcal{H})^{\mathrm{mer}}_{\leqslant 0} \\ \uparrow & & \uparrow \\ \big(\Psi\,\mathrm{DO}(\mathcal{H})_{\leqslant 0}\big)^{\Gamma} & \hookrightarrow & \Psi\,\mathrm{DO}(\mathcal{H})_{\leqslant 0} \end{array}$$

where the vertical arrows are injective. We have a natural injection $\mathbb{C}(C) \hookrightarrow \mathrm{Mer}(\mathcal{H})$, induced by the projection $\mathcal{H} \to C$. Moreover, let X be a nonzero rational vector field on C. The lift of X to $\mathcal{H}$ may be expressed in the form $X(\tau)\frac{d}{d\tau}$, where $X(\tau) \in \mathrm{Mer}(\mathcal{H})$, so Lemma 2.1 implies that there is a canonical morphism

$$\Psi\,\mathrm{DO}(\mathbb{C}(C), X)_{\leqslant 0} \to \Psi\,\mathrm{DO}(\mathcal{H})^{\mathrm{mer}}_{\leqslant 0}.$$

Lemma 2.1 also shows that the image of this morphism is contained in $\left(\Psi\,\mathrm{DO}(\mathcal{H})^{\mathrm{mer}}_{\leqslant 0}\right)^{\Gamma}$. Recall that we have $\Psi\,\mathrm{DO}(\mathbb{C}(C), X)_{\leqslant 0} = B_X^{\mathrm{rat}}$; so we have constructed an algebra morphism $B_X^{\mathrm{rat}} \to \left(\Psi\,\mathrm{DO}(\mathcal{H})^{\mathrm{mer}}_{\leqslant 0}\right)^{\Gamma}$.

We now want to prove that we have a commuting square of algebras

$$\begin{array}{ccc} B_X^{\mathrm{rat}} & \to & \left(\Psi\,\mathrm{DO}(\mathcal{H})^{\mathrm{mer}}_{\leqslant 0}\right)^{\Gamma} \\ \uparrow & & \uparrow \\ B_X & \hookrightarrow & \left(\Psi\,\mathrm{DO}(\mathcal{H})_{\leqslant 0}\right)^{\Gamma} \end{array}$$

where the vertical arrows are injective. We proceed as follows:

(a) for any point $P \in C$, there are natural Laurent expansion morphisms

$$\widetilde{L}_P^{z_P} : \left(\Psi\,\mathrm{DO}(\mathcal{H})^{\mathrm{mer}}_{\leqslant 0}\right)^{\Gamma} \to \Psi\,\mathrm{DO}(\mathcal{K}_P, z_P)_{\leqslant 0},$$

such that the diagram

$$\begin{array}{ccc} & & \Psi\,\mathrm{DO}(\mathcal{K}_P, z_P)_{\leqslant 0} \\ & \overset{L_P^{z_P}}{\nearrow} & \uparrow \widetilde{L}_P^{z_P} \\ B_X^{\mathrm{rat}} & \to & \left(\Psi\,\mathrm{DO}(\mathcal{H})^{\mathrm{mer}}_{\leqslant 0}\right)^{\Gamma} \end{array}$$

commutes.

(b) $\left(\Psi\,\mathrm{DO}(\mathcal{H})_{\leqslant 0}\right)^{\Gamma}$ may be identified with the preimage of $\prod_{P\in C} \Psi\,\mathrm{DO}(\mathcal{O}_P, z_P)_{\leqslant 0}$ by

$$\prod_{P\in C} \widetilde{L}_P^{z_P} : \left(\Psi\,\mathrm{DO}(\mathcal{H})^{\mathrm{mer}}_{\leqslant 0}\right)^{\Gamma} \to \prod_{P\in C} \Psi\,\mathrm{DO}(\mathcal{K}_P, z_P)_{\leqslant 0}.$$

(c) For each $P \in C$, the composed maps

$$B_X \to B_X^{\mathrm{rat}} \xrightarrow{L_P^{z_P}} \Psi\,\mathrm{DO}(\mathcal{K}_P, z_P)_{\leqslant 0}$$

and

$$B_X \to B_X^{\mathrm{rat}} \to \left(\Psi\,\mathrm{DO}(\mathcal{H})^{\mathrm{mer}}_{\leqslant 0}\right)^{\Gamma} \xrightarrow{\widetilde{L}_P^{z_P}} \Psi\,\mathrm{DO}(\mathcal{K}_P, z_P)_{\leqslant 0}$$

coincide, so the image of the latter map is contained in $\Psi\,\mathrm{DO}(\mathcal{O}_P, z_P)_{\leqslant 0}$. So the image of the composed map

$$B_X \to B_X^{\mathrm{rat}} \to \left(\Psi\,\mathrm{DO}(\mathcal{H})^{\mathrm{mer}}_{\leqslant 0}\right)^{\Gamma}$$

is contained in $\left(\Psi\,\mathrm{DO}(\mathcal{H})_{\leqslant 0}\right)^{\Gamma}$. So we have constructed a morphism

$$B_X \to \left(\Psi\,\mathrm{DO}(\mathcal{H})_{\leqslant 0}\right)^{\Gamma} \tag{15}$$

of filtered algebras.

(d) Both algebras B_X and $\big(\Psi\,\mathrm{DO}(\mathcal{H})_{\leqslant 0}\big)^\Gamma$ are complete and separated for their filtrations. (15) induces an isomorphism between their associated graded algebras, so it is an isomorphism of filtered algebras. This proves Proposition 4.2. □

Remark 5. — The authors of [3] actually define a family of star-products, depending on a parameter κ. In the language of Section 2.3, this construction corresponds to replacing the algebra $B_X^{(C,D)}$ by the family of algebras $B_X^{(C,D),\lambda}$, where the generalized line bundle λ is $\kappa\bar{\alpha}$, and $\bar{\alpha}$ is an element of $\mathbb{C}C$ such that the class of $2\bar{\alpha}$ modulo $\mathrm{div}(\mathbb{C}(C)^\times)$ is equal to the canonical bundle K_C.

Remark 6. — To be able to use Proposition 4.1, one needs to know the group Γ corresponding to a given curve C. This is the case, by definition, if C is a modular curve. In this case, a classical problem is to find algebraic equations for this curve. This problem is solved using the algebra of modular forms. The corresponding "quantum" problem is to give a presentation of the algebra B_X (or equivalently, of $\oplus_{i\geqslant 0}\big(H^0(\mathcal{H}, K_{\mathcal{H}}^{\otimes i})\big)^\Gamma$, equipped with its Rankin-Cohen star-product structure μ).

5. Differential liftings.

The lifting

$$\bigoplus_{i\geqslant 0} H^0(C, K^{\otimes i}) \to B_X$$

constructed in the proof of Theorem 2.1 (see step (c) of Section 2.2.5) relies on estimation of the dimensions of cohomology groups. Contrary to the operation lift of Theorem 4.1, it is therefore not a local operator. We now study the problem of constructing such a local, or differential, lifting, in the algebraic framework. We will prove that the set $\mathrm{Lift}_{\mathrm{diff}}(C)$ of such liftings is a torsor under the action of a group $\mathrm{Aut}_{\mathrm{diff}}(C)$. Poincaré uniformization yields a point of this torsor. We do not know an algebraic way to construct a point of the torsor $\mathrm{Lift}_{\mathrm{diff}}(C)$, but we study some algebraic structures provided by such a point.

5.1. Differential liftings.

A differential lifting of the isomorphism $\mathrm{gr}(B_X) \to A$ is defined as the following data: for each rational vector field X, this is a collection $(\Lambda^X_{i,j})_{i,j\geqslant 0}$ of rational differential operators $\Lambda^X_{i,j}$: {rational i-differentials} $\to \mathbb{C}(C)$. This collection is subject to the following conditions:

(1) define $\Lambda_X : A^{\mathrm{rat}} \to B_X^{\mathrm{rat}}$ by

$$\underline{\omega} = (\omega_i)_{i\geqslant 0} \mapsto \Lambda^X(\underline{\omega}) = \sum_{i,j\geqslant 0} \Lambda^X_{i,j}(\omega_i)(D_X)^{-j},$$

then Λ^X maps A_i^{rat} to $(B_X^{\mathrm{rat}})^i$, the composed map $A_i^{\mathrm{rat}} \xrightarrow{\Lambda^X} (B_X^{\mathrm{rat}})^i \to (B_X^{\mathrm{rat}})^i/(B_X^{\mathrm{rat}})^{i+1}$ is the inverse of the canonical map, and $\Lambda^X(1) = 1$;

(2) for any pair X, Y of nonzero vector fields, we have $i_{X,Y} \circ \Lambda^X = \Lambda^Y$;

(3) condition (2) implies that for any $P \in C$, Λ^X induces a map

$$\Lambda^X_P : \bigoplus_{i\geqslant 0} \mathbb{C}((z_P))(dz_P)^i \to \Psi\,\mathrm{DO}(\mathcal{K}_P, z_P)_{\leqslant 0}.$$

Then for any $P \in C$, Λ^X_P maps $\bigoplus_{i\geqslant 0} \mathbb{C}[[z_P]](dz_P)^i$ to $\Psi\,\mathrm{DO}(\mathcal{O}_P, z_P)_{\leqslant 0}$.

(If a nonzero vector field X_0 is fixed, then for any family $(\Lambda^{X_0}_{i,j})_{i,j}$ satisfying conditions (1), (3) for X_0, condition (2) uniquely determines a differential lifting $(\Lambda^X_{i,j})_{i,j\geqslant 0}$ extending $(\Lambda^{X_0}_{i,j})_{i,j}$.)

Conditions (1), (2) and (3) imply immediately that Λ^X induces a linear map

$$\rho(\Lambda^X) : A \to B_X,$$

which is a section of the canonical map $\mathrm{gr}(B_X) \to A$, and therefore induces an isomorphism $\widehat{\rho(\Lambda^X)} : \widehat{A} \to B_X$, where $\widehat{A}$ is the completion $\prod_{i\geqslant 0} A_i$.

Let us denote by $\mathrm{Lift}_{\mathrm{diff}}(C)$ the set of all differential lifts on C. For any nonzero rational vector field X, the assignment $\Lambda^X \mapsto \widehat{\rho(\Lambda^X)}$ is a map

$$\rho : \mathrm{Lift}_{\mathrm{diff}}(C) \to \mathrm{Isom}(\widehat{A}, B_X).$$

We will now see that both sides of this map are principal homogeneous spaces (torsors) and that ρ is a morphism of torsors.

5.2. The group $\mathrm{Aut}_{\mathrm{diff}}(C)$.

Define $\mathrm{DO}(K^{\otimes i}, K^{\otimes j})$ as the space of all regular differential operators on C, from $K^{\otimes i}$ to $K^{\otimes j}$. Define $\mathrm{DO}(K^{\otimes i}, K^{\otimes j})_{\leqslant k}$ as the subspace of all such operators of order $\leqslant k$. Set

$$\mathrm{gr}_k\left(\mathrm{DO}(K^{\otimes i}, K^{\otimes j})\right) = \mathrm{DO}(K^{\otimes i}, K^{\otimes j})_{\leqslant k} / \mathrm{DO}(K^{\otimes i}, K^{\otimes j})_{\leqslant k-1}.$$

Then we have a graded linear injection

$$\bigoplus_{k\geqslant 0} \mathrm{gr}_k\left(\mathrm{DO}(K^{\otimes i}, K^{\otimes j})\right) \hookrightarrow \bigoplus_{k\geqslant 0} H^0(C, K^{\otimes j-i-k}). \tag{16}$$

It follows that when $i > j$, $\mathrm{DO}(K^{\otimes i}, K^{\otimes j}) = 0$, and if $i = j$, $\mathrm{DO}(K^{\otimes i}, K^{\otimes j}) = \mathbb{C}$.

Define $\mathrm{End}_{\mathrm{diff}}(C)$ as follows:

$$\mathrm{End}_{\mathrm{diff}}(C) = \widehat{\bigoplus_{i\leqslant j}} \mathrm{DO}(K^{\otimes i}, K^{\otimes j}),$$

where $\widehat{\oplus}$ is the completed direct sum (direct product). Then composition of differential operators induces an algebra structure on $\mathrm{End}_{\mathrm{diff}}(C)$. Projection of the diagonal summands induces an algebra morphism $\mathrm{End}_{\mathrm{diff}}(C) \to \prod_{i\geqslant 0} \mathbb{C}$. The preimage of $\prod_{i\geqslant 0} 1$ in $\mathrm{End}_{\mathrm{diff}}(C)$ is a group, which we denote $\mathrm{Aut}_{\mathrm{diff}}(C)$. It is easy to see that this is a prounipotent algebraic group, as is the subgroup $\mathrm{Aut}_{\mathrm{diff},1}(C)$ of elements preserving 1.

Define $\mathrm{Aut}(\widehat{A})$ as the group of all continuous linear automorphisms of $\widehat{A} = \widehat{\oplus}_{i\geqslant 0} A_i$.

PROPOSITION 5.1. — *There is a natural group morphism* $\mathrm{Aut}_{\mathrm{diff},1}(C) \to \mathrm{Aut}(\widehat{A})$. *The map ρ is a torsor morphism, compatible with this group morphism.*

We have already mentioned that Poincaré uniformization provides an element of $\mathrm{Lift}_{\mathrm{diff}}(C)$. On the other hand, $\mathrm{Lift}_{\mathrm{diff}}(C)$ is a purely algebraic object, so one would like an algebraic construction of its elements. We will not give such a construction, but only indicate that such elements give rise to affine spaces over spaces of differential operators (Section 5.3).

5.3.

Let us now describe the possible form of a differential lift Λ. If β is a rational differential, we may set

$$\Lambda(\beta) = (D_{\beta^{-1}})^{-1},$$

in the notation of Section 2.2.4. In other words, if α_0 is a nonzero rational differential, and X_0 is the vector field inverse to α_0, we have

$$\Lambda(\beta) = (D_{X_0})^{-1} f,$$

where $f = \beta/\alpha_0$.

Let now β be a rational quadratic differential. Set $f = \beta/(\alpha_0)^2$. The element

$$(D_{X_0})^{-2} f + \frac{1}{2}(D_{X_0})^{-3} X_0(f)$$

of $(B^{\mathrm{rat}})^2/(B^{\mathrm{rat}})^4$ is independent on the choice of α_0. On the other hand, one can show that there is no expression $P(\alpha_0, f)$ of the form $\sum_n a_n (X_0)^n(f)$, such that the element

$$\Lambda(\beta) = (D_{X_0})^{-2} f + \frac{1}{2}(D_{X_0})^{-3} X_0(f) + (D_{X_0})^{-4} P(\alpha_0, f)$$

of $(B^{\mathrm{rat}})^2/(B^{\mathrm{rat}})^5$ is independent on the choice of α_0. So the determination of the coefficient $P(\alpha_0, f)$ depends on additional data. The space of all possible expressions $P(\alpha_0, f)$ is an affine space, with associated vector space $\mathrm{DO}(K^{\otimes 2}, K^{\otimes 4})$. This structure of affine space may be viewed as a part of the torsor structure of $\mathrm{Lift}_{\mathrm{diff}}$.

Remark 7. — *On the size of* $\mathrm{DO}(K^{\otimes n}, K^{\otimes m})$. The injection (16) is not always surjective: for example, where $n = 1, m = 2, k = 1$, the preimage of $1 \in H^0(C, \mathcal{O}_C)$ is the class of all regular connections on K; but there is no such connection, because $\deg(K) \neq 0$.

6. Concluding remarks.

6.1. The elliptic case.

When $g = 1$ and the degree of D is > 0, the above construction of the algebra $B_X^{(C,D)}$ may still be carried out. Its classical limit is the algebra $A^{(C,D)}$. Let us compare them to the elliptic algebras of [5].

$A^{(C,D)}$ may be described as follows: $A^{(C,D)} = \oplus_{i\geqslant 0} A_i^{(C,D)}$, where $A_i^{(C,D)} = H^0(C, \mathcal{O}(D)^{\otimes i})$. We view elements of $A_i^{(C,D)}$ as rational functions on C, with divisor $\geqslant -iD$. In particular, the derivation $f \mapsto \frac{d}{dz}(f) = f'$ can be applied to these functions (here $z \in \mathbb{C}$ is a uniformizing parameter of C). The algebra structure of $A^{(C,D)}$ is graded and induced by the product of rational functions. Its Poisson bracket is defined as follows: it is homogeneous of degree 1, and if $f \in A_i^{(C,D)}$, $g \in A_j^{(C,D)}$, then $\{f,g\} \in A_{i+j+1}^{(C,D)}$ corresponds to

$$j(f'g)(z) - i(fg')(z). \tag{17}$$

It turns out that for any integer $d \geqslant 0$, one can define a Poisson structure on the algebra $A^{(C,D)}$, by requiring that it is homogeneous of degree d, and for $f \in A_i^{(C,D)}$, $g \in A_j^{(C,D)}$, $\{f,g\} \in A_{i+j+d}^{(C,D)}$ corresponds to (17). The structure studied in this paper corresponds to $d = 1$, and the structure of [5] corresponds to $d = 0$.

As we have said, the quantization of the first structure may be done in terms of pseudodifferential operators. The quantization A_{FO} of the second structure was achieved in [5]. It can be expressed in terms of difference operators: if $\sigma = e^{\alpha(d/dz)}$ is a translation of C, elements of $(A_{\mathrm{FO}})_n$ are operators of the form $fe^{n\alpha(d/dz)}$, where f is a section of $\mathcal{O}(D + \sigma(D) + \cdots + \sigma^{n-1}(D))$. A_{FO} is then a graded algebra.

We do not know a quantization of the Poisson algebras corresponding to other values of d.

6.2. Higher-dimensional Poisson structures.

Let us set $A_1 = H^0(C, K)$, then we have a map

$$S^\bullet(A_1) \to A = \bigoplus_{i\geqslant 0} H^0(C, K^{\otimes i}). \tag{18}$$

When $g = 3, 4, 5$, one can define a Poisson structure on the algebra $S^\bullet(A_1)$, such that (18) is Poisson. In that case, the quantization of $S^\bullet(A_1)$ and of the morphism (18) is not known.

In the other cases, a Poisson structure on $S^\bullet(A_1)$, such that (18) is Poisson, is not known. One 2-dimensional symplectic leaf of such a Poisson structure would be given by the dual to the map (18), so it would be isomorphic to the cone $\mathrm{Cone}(C, D)$. One could try to construct geometrically higher dimensional symplectic leaves of this Poisson structure by first understanding their geometric interpretation when $g = 3, 4, 5$.

6.3. Relation with Kontsevich quantization.

When $g = 3, 4, 5$, one may apply Kontsevich quantization to the algebras $S^\bullet(A_1)$. Under this quantization, the Poisson central elements $Q_1, \dots, Q_{g-2}$ are deformed to central elements. So factoring them out gives rise to a quantization $S^\bullet(A_1)_\hbar \to A_\hbar$ of the map (18). It is natural to expect that $A_\hbar$ and B_X are isomorphic.

6.4. Relation with the Beauville hamiltonians.

In [2], Beauville introduced integrable systems on symmetric powers of $K3$ surfaces. An analogous construction is the following. Let k be an integer, (C, D) be the pair of a curve and an effective divisor, and $\omega_1, \dots, \omega_k$ be elements of $A_1^{(C,D)}$. Set

$$A^{(k)} = \big((A^{(C,D)})^{\otimes k}\big)^{\mathfrak{S}_k}.$$

Then $A^{(k)}$ is a Poisson algebra. For $\phi \in A^{(C,D)}$, denote by $\phi^{(i)}$ be image of ϕ in the ith copy of $(A^{(C,D)})^{\otimes k}$. Denote by $\psi_0, \dots, \psi_k$ the minors of the matrix

$$\begin{pmatrix} \omega_1^{(1)} & \cdots & \omega_k^{(1)} & 1 \\ \vdots & & \vdots & \vdots \\ \omega_1^{(k)} & \cdots & \omega_k^{(k)} & 1 \end{pmatrix}.$$

Set $H_i = \psi_i/\psi_0$, for $i = 1, \dots, k$. Then the H_i are a Poisson-commuting family of elements of $\mathrm{Frac}(A^{(k)})$. It would be interesting to study the quantization of this family using the algebras of pseudodifferential operators introduced here.

Remark 8. — Relation to [4]. In general, $A^{(C,D)}$ may be viewed as the space of sections of a sheaf of Poisson algebras over C. When $D = 0$, and in the complex analytic framework, sheaves of filtered algebras, quantizing this sheaf of Poisson algebras, were classified by Boutet de Monvel in [4]: he established a bijection of such sheaves with the singular cohomology group $H^1(C, \mathbb{C})$. It is easy to see that the algebra of global sections of the simplest sheaf (corresponding to $0 \in H^1(C, \mathbb{C})$) is isomorphic to B^X.

In [4], the quantization problem for sheaves was studied for varieties X of any dimension. When the dimension is > 1, the operation of taking global sections is no longer interesting, because then there are no nonzero functions on $T^*(X)$, homogeneous of negative degree and regular except at the zero section.

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BIBLIOGRAPHY

[1] M. Adler, On a trace functional for formal pseudo-differential operators and the symplectic structure of the KdV equation, Invent. Math., 50 (1979), 219–48.

[2] A. Beauville, Systèmes hamiltoniens complètement intégrables associés aux surfaces $K3$, in: Problems in the theory of surfaces and their classification (Cortona, 1988), Sympos. Math. XXXII, Academic Press, 1991, 25–31.

[3] P. Beazley Cohen, Yu. Manin, D. Zagier, Automorphic pseudodifferential operators, paper in memory of Irene Dorfman, Algebraic aspects of integrable systems, Progr. Nonlinear Diff. Eqs. Appl., 26 (1997), 17–47, Birkhäuser Boston, Boston, MA.

[4] L. Boutet de Monvel, Complex star algebras, Math. Physics, Analysis and Geometry (1999), 1–27, Kluwer Acad. Publishers, the Netherlands.

[5] B. Feigin, A. Odesskii, Sklyanin's elliptic algebras, Functional Anal. Appl., 23-3 (1990), 207–14.

[6] P. Griffiths, J. Harris, Principles of algebraic geometry, Wiley Classics Library, J. Wiley and Sons, Inc., New York, 1994 (first edition, 1978).

[7] J. Harris, Algebraic geometry. A first course, Graduate Texts in Mathematics, 133 (1985), Springer-Verlag, New York.

[8] M. Kontsevich, Deformation quantization of Poisson manifolds, I, preprint math.QA/9707040.

[9] Y. Manin, Algebraic aspects of differential equations, J. Sov. Math., 11 (1979), 1–128.

[10] A. Odesskii, V. Rubtsov, Polynomial Poisson algebras with regular structure of symplectic leaves, preprint 2001.

[11] V. OVSIENKO, Exotic deformation quantization, J. Differential Geom., 45-2 (1997), 390–406.

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