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EQUIVARIANT DEFORMATION QUANTIZATION FOR THE COTANGENT BUNDLE OF A FLAG MANIFOLD

by Ranee BRYLINSKI

1. Introduction.

In the context of algebraic geometry, the equivariant deformation quantization (EDQ) problem for cotangent bundles is to construct a graded G-equivariant star product $\star$ on the symbol algebra $\mathcal{R} = R(T^*X)$ where X is a homogeneous space of a complex algebraic group G. Motivated by geometric quantization (GQ), we require that the specialization of $\star$ at $t = 1$ produces the algebra $\mathcal{D} = \mathfrak{D}^{\frac{1}{2}}_{\text{alg}}(X)$ of (linear) twisted differential operators for the (locally defined) square root of the canonical bundle $\mathcal{K}$ on X. (There are other interesting choices for the line bundle but we do not consider them in this paper.) Then $\star$ corresponds to a quantization map $\mathbf{q}$ from $\mathcal{R}$ onto $\mathcal{D}$; G-equivariance of $\star$ amounts to G-equivariance of $\mathbf{q}$. The choice of half-forms is naturally consistent with our requiring parity for $\star$.

Suppose from now on that G is a (connected) complex semisimple Lie group and X is a flag manifold of G. Flag manifolds are the most familiar compact homogeneous spaces of G; they exemplify the phenomenon of a big symmetry group acting on a small space.

In this paper we solve the EDQ problem for $\mathcal{R}$ when the geometry of the moment map μ for the G-action on T^*X is "good" in the sense

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of Borho-Brylinski. Goodness of μ amounts to $\mathcal{R}$ being generated by the momentum functions μ^x where x lies in $\mathfrak{g} = \mathrm{Lie}(G)$. Then $\mathcal{R} = \mathcal{S}(\mathfrak{g})/I$ and $\mathcal{D} = \mathcal{U}(\mathfrak{g})/J$ with $\mathrm{gr}\, J = I$ for some (two-sided) ideals I and J. The good case occurs, for instance, when $G = SL_n(\mathbb{C})$ or if X is the full flag variety.

We solve the EDQ problem for $\mathcal{R}$ in Theorem 6.1, for the good case, by using representation theory to construct a preferred choice of $\star$. We prove the existence and uniqueness of a graded G-equivariant star product $\star$ such that the corresponding representation $\pi : \mathfrak{g} \oplus \mathfrak{g} \to \mathrm{End}\,\mathcal{R}$, $(x, y) \mapsto \pi^{x,y}$, makes $\mathcal{R}$ into the Harish-Chandra module of a unitary representation of G. The operators $\pi^{x,y}$ are given by

$$\pi^{x,y}(\phi) = (\mu^x \star \phi - \phi \star \mu^y)_{t=1}. \tag{1.1}$$

In this way, we get a connection between deformation quantization and the orbit method in representation theory. In addition, motivation comes from the constructions in [LO] and [DLO] for certain real flag varieties.

We now outline our construction of $\star$. In fact, we do not directly construct $\star$ but instead we construct a preferred quantization map $\mathbf{q}$ in the following way. Results in representation theory of Conze-Berline and Duflo ([C-BD]) and Vogan ([V]) give a canonical embedding Δ of $\mathcal{D}$ into the space of smooth half-densities on X (§8); here we regard X as a real manifold. We give a new geometric formula for Δ in (8.1). The natural pairing $\int_X \alpha\overline{\beta}$ of half-densities induces a positive definite inner product γ on $\mathcal{D}$. The γ-orthogonal splitting of the order filtration on $\mathcal{D}$ defines our $\mathbf{q}$.

In this way, $\mathcal{R}$ acquires a positive definite inner product $\langle\phi|\psi\rangle = \gamma(\mathbf{q}(\phi), \mathbf{q}(\psi))$ where the grading of $\mathcal{R}$ is orthogonal. Then $\langle\cdot|\cdot\rangle$ is new even if $\mathbf{q}$ was unique to begin with (so if the representation of G on $\mathcal{R}$ is multiplicity free). The completion of $\mathcal{R}$ is a new Fock space type model of the unitary representation of G on L^2 half-densities on X (§12). So $\mathcal{R}$ is now the Harish-Chandra module of this unitary representation.

Now $\mathbf{q}$ defines a preferred graded G-equivariant star product $\star$ on $\mathcal{R}$. We find in Corollary 9.3 that the star product $\mu^x \star \phi$ of a momentum function with an arbitrary function in $\mathcal{R}$ has the form $\mu^x\phi + \frac{1}{2}\{\mu^x, \phi\}t + \Lambda^x(\phi)t^2$ where Λ^x is the $\langle\cdot|\cdot\rangle$-adjoint of ordinary multiplication by $\mu^{\sigma(x)}$ (σ is a Cartan involution of $\mathfrak{g}$). This property that $\mu^x \star \phi$ is a three term sum uniquely determines $\mathbf{q}$ (Proposition 11.1). The Λ^x completely determine $\star$, but they are not differential operators in the known examples; see §10. Thus $\mu^x \star \phi$ is not local in ϕ.

An important feature is that $\mathcal{D}$ has a natural trace functional $\mathcal{T}$ (Proposition 8.4). We give a formula computing $\mathcal{T}$ by integration in (8.4).

Then $\langle\phi|\psi\rangle = \mathcal{T}(\mathbf{q}(\phi)\mathbf{q}(\psi^\sigma))$ where σ is some anti-linear involution of $\mathcal{R}$; see (11.2).

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2. Cotangent bundles of flag manifolds.

Let G be a connected complex semisimple Lie group G. Let X be a (generalized) flag manifold of G. Then $X = G/P$ is a projective complex algebraic manifold. The classification of flag manifolds is well known.

For example, if $G = SL_n(\mathbb{C})$ then the flag manifolds are $X^{\mathbf{d}}(\mathbb{C})$ where $\mathbf{d} = (d_1, \ldots, d_s)$ with $1 \leqslant d_1 < \cdots < d_s \leqslant n-1$. Here $X^{\mathbf{d}}(\mathbb{C})$ parameterizes the flags $V = (V_1 \subset \cdots \subset V_s)$ in $\mathbb{C}^n$ where $\dim V_j = d_j$. The simplest cases are the grassmannians of k-dimensional subspaces in $\mathbb{C}^n$.

The cotangent bundle T^*X is a quasi-projective algebraic manifold. Let $\mathcal{R} = R(T^*X)$ be the algebra of regular functions on T^*X, in the sense of algebraic geometry. Each regular function is polynomial (of finite degree) on the cotangent fibers. Thus we have the algebra grading

$$\mathcal{R} = \oplus_{d=0}^{\infty} \mathcal{R}^d \tag{2.1}$$

by homogeneous degree along the fibers.

The canonical holomorphic symplectic form on T^*X is algebraic and thus defines a Poisson bracket $\{\cdot,\cdot\}$ on $\mathcal{R}$. Then $\mathcal{R}$ is a graded Poisson algebra where $\{\phi,\psi\}$ is homogeneous of degree $j+k-1$ if ϕ and ψ are homogeneous of degrees j and k. We have a Poisson algebra anti-automorphism $\phi \mapsto \phi^\alpha$ given by $\phi^\alpha = (-1)^d\phi$ if ϕ is homogeneous of degree d.

The action of G on X lifts canonically to a Hamiltonian action on T^*X with moment map $\mu : T^*X \to \mathfrak{g}^*$. The moment map embeds the cotangent spaces of X into $\mathfrak{g}^*$. In our example, the cotangent space of $X^{\mathbf{d}}(\mathbb{C})$ at V identifies with the subspace of $\mathfrak{sl}_n(\mathbb{C})$ consisting of maps $e : \mathbb{C}^n \to \mathbb{C}^n$ such that $e(V_j) \subseteq V_{j-1}$.

The Hamiltonian action of G on T^*X defines a natural (complex linear) representation of G on $\mathcal{R}$. Then G acts on $\mathcal{R}$ by graded Poisson algebra automorphisms which commute with α. The corresponding representation of $\mathfrak{g}$ on $\mathcal{R}$ is given by the operators $\{\mu^x, \cdot\}$, $x \in \mathfrak{g}$, where $\mu^x \in \mathcal{R}^1$ are the momentum functions.

$\mathcal{R}$ is the algebra of symbols for (linear) algebraic differential operators acting on sections of a line bundle over X.

3. Equivariant star product problem for T^*X.

Our problem is to construct a preferred graded G-equivariant star product (with parity) on $\mathcal{R}$. This means that we want an associative product $\star$ on $\mathcal{R}[t]$ which makes $\mathcal{R}[t]$ into an algebra over $\mathbb{C}[t]$ in the following way. If $\phi, \psi \in \mathcal{R}$, then the product has the form

$$\phi \star \psi = \phi\psi + \frac{1}{2}\{\phi, \psi\}t + \sum_{p=2}^{\infty} C_p(\phi, \psi)t^p \tag{3.1}$$

where the coefficients C_p satisfy

$$\begin{aligned} &\text{(i)} \quad C_p(\phi, \psi) \in \mathcal{R}^{j+k-p} \text{ if } \phi \in \mathcal{R}^j \text{ and } \psi \in \mathcal{R}^k \\ &\text{(ii)} \quad C_p(\phi, \psi) = (-1)^p C_p(\psi, \phi) \\ &\text{(iii)} \quad \mu^x \star \phi - \phi \star \mu^x = t\{\mu^x, \phi\} \text{ for all } x \in \mathfrak{g}. \end{aligned} \tag{3.2}$$

Axiom (ii) is the *parity axiom*. (Dropping parity amounts to dropping (ii) and relaxing (3.1) from $C_1(\phi, \psi) = \frac{1}{2}\{\phi, \psi\}$ to $C_1(\phi, \psi) - C_1(\psi, \phi) = \{\phi, \psi\}$.) Axiom (iii) is often called *strong invariance* – we use the term "equivariant". This is an important notion because it corresponds to equivariant quantization of symbols (see §4). Strong invariance implies the weaker notion of *invariance*, which means that the operators C_p are G-invariant.

At $t = 1$, $\star$ specializes to a noncommutative product on $\mathcal{B} = \mathcal{R}[t]/(t-1)$. Then, because of axiom (i), $\mathcal{B}$ has an increasing algebra filtration (defined by the grading on $\mathcal{R}$) and the obvious vector space isomorphism $\mathbf{q} : \mathcal{R} \to \mathcal{B}$ induces a graded Poisson algebra isomorphism from $\mathcal{R}$ to $\operatorname{gr}\mathcal{B}$. Via $\mathbf{q}$, the structures on $\mathcal{R}$ pass over to $\mathcal{B}$. Axiom (ii) implies that α defines a filtered algebra anti-involution β on $\mathcal{B}$. By (iii), the map $\mathfrak{g} \to \mathcal{B}$ given by $x \mapsto \mathbf{q}(\mu^x)$ is a Lie algebra homomorphism and so we get a representation of $\mathfrak{g}$ on $\mathcal{B}$ by the operators $[\mathbf{q}(\mu^x), \cdot]$. Then $\mathbf{q}$ is $\mathfrak{g}$-equivariant. Consequently, the $\mathfrak{g}$-representation on $\mathcal{B}$ integrates to a locally finite representation of G on $\mathcal{B}$ compatible with everything.

There is an obvious candidate for $\mathcal{B}$, namely the algebra $\mathcal{D} = \mathfrak{D}^{\frac{1}{2}}_{\mathrm{alg}}(X)$ of algebraic twisted differential operators for the (locally defined) square root of the canonical bundle $\mathcal{K}$ on X. Fortunately, $\mathcal{D}$ already has all the structure discussed above. It has the order filtration and the principal symbol map identifies $\operatorname{gr}\mathcal{D}$ with $\mathcal{R}$. (The latter statement follows by [BoBr, Lem. 1.4] – their result goes through to the twisted case with the same proof.) There is a canonical G-invariant filtered algebra anti-involution β of $\mathcal{D}$ such that $\beta(\phi) = \phi$ for $\phi \in \mathcal{R}^0$ and $\beta(\eta_{\frac{1}{2}}) = -\eta_{\frac{1}{2}}$. Here $\eta_{\frac{1}{2}}$ is the Lie derivative of a vector field η on X. Then β induces α upon taking principal symbols. Let η^x be the vector field on X defined by x. The map

$$\mathfrak{g} \to \mathcal{D}, \qquad x \mapsto \eta^x_{\frac{1}{2}}$$

is a Lie algebra homomorphism. The corresponding $\mathfrak{g}$-representation on $\mathcal{D}$ given by the operators $[\eta^x_{\frac{1}{2}}, \cdot]$ integrates to a locally finite representation of G on $\mathcal{D}$ compatible with everything.

4. Quantizing symbols into differential operators equivariantly.

Now that we have decided upon $\mathcal{B} = \mathcal{D}$, we can reformulate our star product problem in terms of quantization maps. To begin with, we can axiomatize the properties of our vector space isomorphism $\mathbf{q} : \mathcal{R} \to \mathcal{D}$ from §3:

$$(4.1)\qquad \begin{array}{ll} \text{(i)} & \text{if } \phi \in \mathcal{R}^d \text{ then the principal symbol of } \mathbf{q}(\phi) \text{ is } \phi \\ \text{(ii)} & \mathbf{q}(\phi^\alpha) = \mathbf{q}(\phi)^\beta \\ \text{(iii)} & \mathbf{q}(\mu^x) = \eta^x_{\frac{1}{2}} \text{ and } \mathbf{q}(\{\mu^x, \phi\}) = [\eta^x_{\frac{1}{2}}, \mathbf{q}(\phi)] \text{ if } x \in \mathfrak{g}. \end{array}$$

In (iii), we used the semisimplicity of $\mathfrak{g}$ to get $\mathbf{q}(\mu^x) = \eta^x_{\frac{1}{2}}$. Axiom (iii) means that $\mathbf{q}$ is $\mathfrak{g}$-equivariant. This amounts to G-equivariance.

We call $\mathbf{q}$ a *G-equivariant quantization map.* We can recover $\star$ from $\mathbf{q}$ by the formula $\phi \star \psi = \mathbf{q}_t^{-1}(\mathbf{q}_t(\phi)\mathbf{q}_t(\psi))$ where $\mathbf{q}_t(\phi t^p) = \mathbf{q}(\phi)t^{j+p}$ if $\phi \in \mathcal{R}^j$. In this way, we get a bijection between graded equivariant star products on $\mathcal{R}$ and equivariant quantization maps (up to algebra automorphisms of $\mathcal{D}$ which are compatible with principal symbols, the G-action, etc.).

5. The momentum algebra $\mathcal{R}_\mu$.

The *momentum algebra* $\mathcal{R}_\mu$ is the subalgebra of $\mathcal{R}$ generated by the momentum functions μ^x, $x \in \mathfrak{g}$. Soon (§6 onwards) we will restrict to the case where $\mathcal{R} = \mathcal{R}_\mu$.

Clearly we may identify $\mathcal{R}_\mu = \mathcal{S}(\mathfrak{g})/I$ where I is a graded ideal in $\mathcal{S}(\mathfrak{g})$. Let $\zeta : \mathcal{U}(\mathfrak{g}) \to \mathcal{D}$ be the algebra homomorphism defined by $\zeta(x) = \eta^x_{\frac{1}{2}}$ for $x \in \mathfrak{g}$. Then by restriction we get, for each p, a map ζ_p from the space $\mathcal{U}_p(\mathfrak{g})$ (spanned by all p-fold products of elements of $\mathfrak{g}$) to the space $\mathcal{D}_p$ of operators or order at most p. Let J be the kernel of ζ; then J is a two-sided ideal in $\mathcal{U}(\mathfrak{g})$.

LEMMA 5.1. — *The following are equivalent:*

(i) $\mathcal{R} = \mathcal{R}_\mu$

(ii) $\zeta_p : \mathcal{U}_p(\mathfrak{g}) \to \mathcal{D}_p$ *is surjective for all* p

(iii) $\zeta : \mathcal{U}(\mathfrak{g}) \to \mathcal{D}$ *is surjective and* $\operatorname{gr} J = I$.

Proof. — The associated graded map $\operatorname{gr}\zeta$ is the algebra homomorphism $\mathcal{S}(\mathfrak{g}) \to \mathcal{R}$ defined by $x \mapsto \mu^x$. The result easily follows. □

In the next section we find a preferred G-equivariant graded star product on $\mathcal{R}$. We do this under the hypothesis that $\mathcal{R} = \mathcal{R}_\mu$. This is a hypothesis on (G, X) which is satisfied for instance if (i) $G = SL_n(\mathbb{C})$ and X is arbitrary ([KP]), or (ii) G is arbitrary but X is the full flag manifold.

This hypothesis was important in [BoBr] in studying noncommutative analogs of $R(T^*X)$; it is equivalent ([BoBr, Th. 5.6]) to the condition that the moment map $\mu : T^*X \to \mathfrak{g}^*$ has *good geometry* in the sense that μ is generically 1-to-1 and its image in $\mathfrak{g}^*$ is a normal variety. These conditions have been studied a lot in geometric representation theory, especially since the image of μ is the closure of a single nilpotent coadjoint orbit $\mathcal{O}$ of G.

We note that the ideal I contains all casimirs (i.e., G-invariants in $\oplus_{d=1}^{\infty} S^d(\mathfrak{g})$). The casimirs generate I if and only if X is the full flag variety.

6. A preferred star product on $\mathcal{R}$.

Suppose $\phi \star \psi$ is a graded G-equivariant star product on $\mathcal{R}$ (see §3). This defines a noncommutative associative product $\circ$ on $\mathcal{R}$ where $\phi \circ \psi$ is the specialization at $t = 1$ of $\phi \star \psi$. Then we obtain a representation π of $\mathfrak{g} \oplus \mathfrak{g}$ on $\mathcal{R}$ given by $\pi^{x,y}(\phi) = \mu^x \circ \phi - \phi \circ \mu^y$. Notice that the equivariance axiom (3.2) (iii) says that the quantum operator $\pi^{x,x}$ coincides with the classical operator $\{\mu^x, \cdot\}$.

THEOREM 6.1. — *Assume $\mathcal{R}$ is generated by μ^x, $x \in \mathfrak{g}$. Suppose $\star$ is a graded G-equivariant star product on $\mathcal{R}$ where $\star$ corresponds to a G-equivariant quantization map $\mathbf{q} : \mathcal{R} \to \mathcal{D}$. (Such maps $\mathbf{q}$ always exist). Then*

(I) *The representation π of $\mathfrak{g} \oplus \mathfrak{g}$ on $\mathcal{R}$ is irreducible and unitarizable, i.e., there exists a unique positive definite invariant hermitian form $\langle \cdot | \cdot \rangle$ on $\mathcal{R}$ with $\langle 1 | \, 1 \rangle = 1$.*

(II) *There is a unique choice of $\mathbf{q}$, and hence a unique choice of $\star$, such that the grading (2.1) is orthogonal with respect to $\langle \cdot | \cdot \rangle$. Then*

$$\pi^{x,y}(\phi) = \mu^{x-y}\phi + \frac{1}{2}\{\mu^{x+y}, \phi\} + \Lambda^{x-y}(\phi) \tag{6.1}$$

where Λ^x, $x \in \mathfrak{g}$, are certain operators on $\mathcal{R}$.

Proof. — The proof occupies §7–9. □

We now discuss what unitarizable means and introduce some notations. To begin with, the restriction of π to $\mathfrak{g}^{\mathrm{diag}} = \{(x,x) \,|\, x \in \mathfrak{g}\}$, i.e, the $\mathfrak{g}$-representation on $\mathcal{R}$ given by the operators $\pi^{x,x}$, corresponds to the natural G-representation on $\mathcal{R}$. Thus $\mathcal{R}$ is a $(\mathfrak{g} \oplus \mathfrak{g}, G)$-module in the sense of Harish-Chandra.

Now *unitarizability* of π means that there is a positive definite hermitian inner product $\langle \cdot | \cdot \rangle$ on $\mathcal{R}$ which is invariant for $\mathfrak{g}^\sharp = \{(x, \sigma(x)) \,|\, x \in \mathfrak{g}\}$, i.e., the operators $\pi^{x,\sigma(x)}$ are skew-hermitian. Here σ is a fixed Cartan involution of $\mathfrak{g}$. Then σ corresponds to a maximal compact subgroup G_c with Lie algebra $\mathfrak{g}_c = \{x \in \mathfrak{g} \,|\, x = \sigma(x)\}$. E.g., if $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, then take $\sigma(x) = -x^*$ so that $\mathfrak{g}_c = \mathfrak{su}_n$.

By a theorem of Harish-Chandra, the operators $\pi^{x,\sigma(x)}$ then correspond to a unitary representation of G on the Hilbert space completion $\widehat{\mathcal{R}}$ of $\mathcal{R}$ with respect to $\langle \cdot | \cdot \rangle$. If the $\mathcal{R}^d$ are orthogonal, then $\widehat{\mathcal{R}}$ is the Hilbert space direct sum $\widehat{\oplus}_{d=0}^{\infty}\mathcal{R}^d$. Notice that we end up with two very different actions of G: the graded algebraic action on $\mathcal{R}$ corresponding to $\mathfrak{g}^{\mathrm{diag}}$ and the unitary action on $\widehat{\mathcal{R}}$ corresponding to $\mathfrak{g}^{\sharp}$.

7. Existence proof for q.

A G-equivariant quantization map $\mathbf{q}$ is completely determined by the subspaces $\mathcal{F}^d = \mathbf{q}(\mathcal{R}^d)$. This is immediate from (4.1)(i). Then the decomposition $\mathcal{D} = \oplus_{d=0}^{\infty} \mathcal{F}^d$ "splits the order filtration" in the sense that $\oplus_{d=0}^{p} \mathcal{F}^d = \mathcal{D}_p$. Referring to (4.1) again, we see that the spaces $\mathcal{F}^d$ are stable under β and $\mathfrak{g}$ (which acts by $A \mapsto [\eta^x_{\frac{1}{2}}, A]$). Conversely, any such splitting corresponds to a choice of $\mathbf{q}$.

LEMMA 7.1. — *We can always construct a G-equivariant quantization map $\mathbf{q} : \mathcal{R} \to \mathcal{D}$. If the representation of G on $\mathcal{R}$ is multiplicity free, there is only one choice for $\mathbf{q}$.*

Proof. — By complete reducibility, we can find a $\mathfrak{g}$-stable complement $\mathcal{G}^d$ to $\mathcal{D}_{d-1}$ inside $\mathcal{D}_d$. This gives a $\mathfrak{g}$-stable splitting of the order filtration; let $\mathbf{p}$ be the corresponding quantization map. The spaces $\mathcal{G}^d$ may fail to be stable under β. To remedy this, we "correct" $\mathbf{p}$ by putting $\mathbf{p}'(\phi) = \frac{1}{2}\left(\mathbf{p}(\phi) + \mathbf{p}(\phi^\alpha)^\beta\right)$. Now $\mathbf{p}'$ is a valid choice for $\mathbf{q}$.

If $\mathcal{R}$ is multiplicity free, then $\mathcal{G}^d$ is unique for each d, and so $\mathbf{p}$ is the unique choice for $\mathbf{q}$. Notice that uniqueness of $\mathbf{q}$ does not require (ii)-(iii) in (4.1). □

In the multiplicity free case, the method explained in Remark 9.4 gives a sort of formula for $\mathbf{q}$. We note that $\mathcal{R}$ is multiplicity free whenever the parabolic subgroup P (where $X = G/P$) has the property that its unipotent radical is abelian. For $G = SL_n(\mathbb{C})$, this happens when X is a grassmannian. The full classification of multiplicity free cases is well known.

In general, there will be infinitely many choices for $\mathbf{q}$.

8. Proof of (I) in Theorem 6.1.

The quantization map $\mathbf{q}$ intertwines our representation π of $\mathfrak{g} \oplus \mathfrak{g}$ on $\mathcal{R}$ with the representation Π of $\mathfrak{g} \oplus \mathfrak{g}$ on $\mathcal{D}$ given by $\Pi^{x,y}(A) = \eta^x_{\frac{1}{2}} A - A\eta^y_{\frac{1}{2}}$. Indeed, $\mathbf{q}(\phi \circ \psi) = \mathbf{q}(\phi)\mathbf{q}(\psi)$ and so $\mathbf{q}(\pi^{x,y}(\phi)) = \Pi^{x,y}(\mathbf{q}(\phi))$.

Therefore proving π is irreducible and unitarizable amounts to proving Π is irreducible and unitarizable. For this, we will need our hypothesis that $\mathcal{R}$ is generated by the μ^x.

We can regard X as a real manifold and then consider the algebra $\mathfrak{D}^{\frac{1}{2}}_{\infty}(X)$ of smooth differential operators on the space $\Gamma(X, \mathcal{E}^{\frac{1}{2}})$ of smooth half-densities on X. Notice that the half-density line bundle $\mathcal{E}^{\frac{1}{2}}$ is G-homogeneous, and so we get induced actions of G on $\Gamma(X, \mathcal{E}^{\frac{1}{2}})$ and $\mathfrak{D}^{\frac{1}{2}}_{\infty}(X)$. There is a natural G-equivariant filtered algebra embedding $A \mapsto A^!$ of $\mathcal{D}$ into $\mathfrak{D}^{\frac{1}{2}}_{\infty}(X)$. We put $\xi^x = (\eta^x_{\frac{1}{2}})^!$; these ξ^x are twisted holomorphic vector fields on X.

Let δ be the unique G_c-invariant positive real density on X such that $\int_X \delta = 1$. Let $\delta^{\frac{1}{2}}$ be the positive square root of δ. We map $\mathcal{D}$ into $\Gamma(X, \mathcal{E}^{\frac{1}{2}})$ by

$$\Delta(A) = A^!(\delta^{\frac{1}{2}}). \tag{8.1}$$

Now $\mathcal{D}$ acquires the G_c-invariant hermitian pairing

$$\gamma(A, B) = \int_X \Delta(A)\overline{\Delta(B)}.$$

From now on we assume that the equivalent conditions of Lemma 5.1 are satisfied.

PROPOSITION 8.1. — *γ is $\mathfrak{g}^\sharp$-invariant and positive definite.*

Proof. — $\mathfrak{g}^\sharp$-invariance means that the operators $\Pi^{x,\sigma(x)}$ are skew-hermitian, or equivalently, the adjoint of $\Pi^{x,0}$ is $-\Pi^{0,\sigma(x)}$. So we want to show

$$\gamma(\eta^x_{\frac{1}{2}} A, B) = \gamma(A, B\eta^{\sigma(x)}_{\frac{1}{2}}). \tag{8.2}$$

We have $\gamma(\eta^x_{\frac{1}{2}} A, B) = \int_X (\xi^x \Delta(A))\, \overline{\Delta(B)} = -\int_X \Delta(A)\,(\xi^x \overline{\Delta(B)})$; the last equality holds because $\int_X \xi^x(\alpha\beta) = 0$ for any half-densities α, β.

G_c-invariance of $\delta^{\frac{1}{2}}$ means that $\xi^x + \overline{\xi^x}$ kills $\delta^{\frac{1}{2}}$ if $x \in \mathfrak{g}_c$, or equivalently $\xi^x + \overline{\xi^{\sigma(x)}}$ kills $\delta^{\frac{1}{2}}$ if $x \in \mathfrak{g}$. Using this and the commutativity of holomorphic and anti-holomorphic operators we find $\xi^x\, \overline{\Delta(B)} = -\overline{B^! \xi^{\sigma(x)}}(\delta^{\frac{1}{2}}) = -\overline{\Delta(B\eta^{\sigma(x)}_{\frac{1}{2}})}$ and so we get (8.2).

For positive definiteness, we just need to show that Δ is 1-to-1 on $\mathcal{D}$. We expect there is a geometric proof of this, but we have not worked that out. Instead, we will use results from representation theory.

Δ is G_c-equivariant and so Δ maps $\mathcal{D}$ into the space $\Gamma(X, \mathcal{E}^{\frac{1}{2}})^{G_c-\mathrm{fin}}$ of G_c-finite smooth half-densities. On the other hand we have the maps

$$(8.3) \qquad \mathcal{D} \xrightarrow{\Psi} \mathrm{End}_{\mathfrak{g}-\mathrm{fin}}(M_{\mathfrak{p},-\nu}) \xrightarrow{\Phi} \Gamma(X, \mathcal{E}^{\frac{1}{2}})^{G_c-\mathrm{fin}}.$$

Here $\mathrm{End}_{\mathfrak{g}-\mathrm{fin}}(M_{\mathfrak{p},-\nu})$ is the algebra of $\mathfrak{g}$-finite endomorphisms of the generalized Verma module $M_{\mathfrak{p},-\nu} = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \mathbb{C}_{-\nu}$ where $X = G/P$, $\mathfrak{p} = \mathrm{Lie}(P)$, $\nu : \mathfrak{p} \to \mathbb{C}$ is the Lie algebra homomorphism defined by $\nu(x) = -\frac{1}{2} \operatorname{tr} \mathrm{ad}_{\mathfrak{g}/\mathfrak{p}}(x)$. To define Ψ, we consider the natural action of $\mathcal{D}$ on $\Gamma(X_o, \mathcal{K}^{\frac{1}{2}})$. There is a non-degenerate $\mathfrak{g}$-invariant bilinear pairing

$$\omega : \Gamma(X_o, \mathcal{K}^{\frac{1}{2}}) \times M_{\mathfrak{p},-\nu} \to \mathbb{C}.$$

It follows that $\mathcal{D}$ acts faithfully on $M_{\mathfrak{p},-\nu}$ so that $\omega(s, A(m)) = \omega(A^\beta(s), m)$ where s is a section and $m \in M_{\mathfrak{p},-\nu}$. In this way we get a 1-to-1 algebra homomorphism Ψ. Next, Φ is the map defined by Conze-Berline and Duflo in [C-BD, §5.3]. (This is the "$\pi_1 = \pi_2 = 0$" case in their notation.) Both maps Ψ and Φ are $\mathfrak{g} \oplus \mathfrak{g}$-equivariant; here $\mathfrak{g} \oplus \mathfrak{g}$ acts on $\Gamma(X, \mathcal{E}^{\frac{1}{2}})$ by the twisted vector fields $\xi^{x,y} = \xi^x + \overline{\xi^{\sigma(y)}}$.

The map Φ is an isomorphism. Indeed, $M_{\mathfrak{p},-\nu}$ is irreducible by Vogan's result [V, Prop. 8.5]. (This is the case "$\lambda - \rho(\mathfrak{l}) = 0$" in his notation.) So [C-BD, Proposition 5.5] applies and says Φ is an isomorphism.

Thus the composite map $\Phi\Psi$ in (8.3) is 1-to-1. It is easy to compare $\Phi\Psi$ with Δ. Both maps are $\mathfrak{g} \oplus \mathfrak{g}$-equivariant and send 1 to a non-zero multiple of $\delta^{\frac{1}{2}}$. It follows, since by hypothesis $\mathcal{D}$ is a quotient of $\mathcal{U}(\mathfrak{g})$ (cf. Lemma 5.1(iii)), that $\Phi\Psi$ is just a scalar multiple of Δ. Consequently, Δ is 1-to-1. □

COROLLARY 8.2. — *Δ is an isomorphism, of $(\mathfrak{g}\oplus\mathfrak{g})$-representations, from $\mathcal{D}$ onto the Harish-Chandra module of the natural unitary representation of G on $L^2(X, \mathcal{E}^{\frac{1}{2}})$.*

Proof. — The Harish-Chandra module is $\Gamma(X, \mathcal{E}^{\frac{1}{2}})^{G_c-\mathrm{fin}}$. We just established injectivity of Δ. Surjectivity follows because the source and target contain the same irreducible G_c-representations with the same multiplicities. Indeed, $\mathcal{D} \simeq \mathcal{R} \simeq R(G/L)$ where L is any Levi factor of P. We may choose L so that $L_c = L \cap G_c$ is a compact form of L. Then $R(G/L) \simeq C^\infty(G_c/L_c)^{G_c-\mathrm{fin}} \simeq \Gamma(X, \mathcal{E}^{\frac{1}{2}})^{G_c-\mathrm{fin}}$. □

COROLLARY 8.3. — *There is a unique anti-linear algebra involution θ of $\mathcal{D}$ such that $\theta(\eta^x_{\frac{1}{2}}) = \eta^{\sigma(x)}_{\frac{1}{2}}$. Then $\gamma(A, B) = \gamma(B^\theta A, 1)$.*

Proof. — The formula $\overline{\Delta(B^\beta)} = B^\theta(\delta^{\frac{1}{2}})$ defines an anti-linear map $B \mapsto B^\theta$. Then $B = \eta_{\frac{1}{2}}^{x_1} \cdots \eta_{\frac{1}{2}}^{x_m}$ gives $B^\theta = \eta_{\frac{1}{2}}^{\sigma(x_1)} \cdots \eta_{\frac{1}{2}}^{\sigma(x_m)}$. Since the $\eta_{\frac{1}{2}}^{x}$ generate $\mathcal{D}$ by hypothesis, it follows that θ is an anti-linear algebra involution. Now (8.2) gives $\gamma(A,B) = \gamma(B^\theta A, 1)$. □

The formula $\mathcal{T}(A) = \gamma(A,1)$ defines a linear functional $\mathcal{T}$ on $\mathcal{D}$. Explicitly,

$$\mathcal{T}(A) = \int_X A^!(\delta^{\frac{1}{2}})\delta^{\frac{1}{2}}. \tag{8.4}$$

Then $\gamma(A,B) = \mathcal{T}(B^\theta A)$.

PROPOSITION 8.4. — *$\mathcal{T}$ is the unique G_c-invariant linear functional on $\mathcal{D}$ with $\mathcal{T}(1) = 1$. Moreover $\mathcal{T}$ is a trace.*

Proof. — Clearly $\mathcal{T}$ is G_c-invariant. Then $\mathcal{T} : \mathcal{D} \to \mathbb{C}$ is the unique invariant linear projection because the G_c-action on $\mathcal{D}$ is completely reducible and the constants are the only G_c-invariants in $\mathcal{D}$ (since the constants are the only G_c-invariants in $\mathcal{R}$).

$\mathcal{T}$ is $\mathfrak{g}$-invariant, i.e., $\mathcal{T}([\eta_{\frac{1}{2}}^{x}, A]) = 0$. We can write this as $\mathcal{T}(\eta_{\frac{1}{2}}^{x} A) = \mathcal{T}(A\eta_{\frac{1}{2}}^{x})$. Iteration gives $\mathcal{T}(\eta_{\frac{1}{2}}^{x_1} \cdots \eta_{\frac{1}{2}}^{x_k} A) = \mathcal{T}(A\eta_{\frac{1}{2}}^{x_1} \cdots \eta_{\frac{1}{2}}^{x_k})$. This proves $\mathcal{T}(BA) = \mathcal{T}(AB)$ since the $\eta_{\frac{1}{2}}^{x}$ generate $\mathcal{D}$ by hypothesis. □

Now we can show that γ is the unique $\mathfrak{g}^\sharp$-invariant hermitian form on $\mathcal{D}$ such that $\gamma(1,1) = 1$. Indeed suppose λ is any such form. Then $\lambda(A,1) = \mathcal{T}(A)$ by the uniqueness of $\mathcal{T}$. So (8.2) gives $\lambda(A,B) = \lambda(B^\theta A, 1) = \mathcal{T}(B^\theta A) = \gamma(A,B)$. This uniqueness of γ implies Π is irreducible.

This completes the proof of Theorem 6.1(I). Once $\mathbf{q}$ is chosen, $\langle\cdot|\cdot\rangle$ is given by

$$\langle\phi|\psi\rangle = \gamma(\mathbf{q}(\phi), \mathbf{q}(\psi)) = \mathcal{T}\left(\mathbf{q}(\phi)\mathbf{q}(\psi)^\theta\right). \tag{8.5}$$

Finally we record

COROLLARY 8.5. — *Π is irreducible. Equivalently, $\mathcal{D}$ is a simple ring.*

9. Proof of (II) in Theorem 6.1.

The graded pieces $\mathcal{R}^d$ are orthogonal with respect to $\langle\cdot|\cdot\rangle$ iff their images $\mathbf{q}(\mathcal{R}^d)$ are orthogonal with respect to γ. So we have only one possible choice of $\mathbf{q}$, namely the one such that $\mathbf{q}(\mathcal{R}^d) = \mathcal{V}^d$ where $\oplus_{d=0}^{\infty}\mathcal{V}^d$ is the γ-orthogonal splitting of the order filtration of $\mathcal{D}$. According to §7, we need to check

LEMMA 9.1. — *$\mathcal{V}^d$ is stable under β and $\mathfrak{g}$.*

Proof. — This follows because γ is invariant under β and G_c. We obtain β-invariance using $\mathcal{T}(A^\beta) = \mathcal{T}(A)$, $\beta\theta = \theta\beta$ (clear since $\mathcal{D}$ is a quotient of $\mathcal{U}(\mathfrak{g})$), and $\mathcal{T}(AB) = \mathcal{T}(BA)$. □

Thus $\oplus_{d=0}^{\infty}\mathcal{V}^d$ defines $\mathbf{q}$. Then $\mathbf{q}$ defines a graded G-equivariant star product $\star$ on $\mathcal{R}$; this is the only one for which the direct sum $\oplus_{d=0}^{\infty}\mathcal{R}^d$ is $\langle\cdot|\cdot\rangle$-orthogonal.

PROPOSITION 9.2. — *This star product $\star$ satisfies*

$$\mathcal{R}^j \star \mathcal{R}^k \subseteq \mathcal{R}^{j+k} \oplus \cdots \oplus \mathcal{R}^{|j-k|} t^{2\min(j,k)}. \tag{9.1}$$

Proof. — Since $\star$ is graded, it suffices to consider $\circ$. Let $\ell(\phi)$ and $r(\phi)$ denote respectively left and right $\circ$-multiplication by ϕ. The map $\mu^x \mapsto \mu^{\sigma(x)}$ extends to a graded anti-linear algebra involution $\phi \mapsto \phi^{\sigma(x)}$ of $\mathcal{R}$; this follows because the complex nilpotent orbit $\mathcal{O}$ (defined in §5) is σ-stable. We claim that the adjoint with respect to $\langle\cdot|\cdot\rangle$ of $\ell(\phi)$ is $r(\phi^\sigma)$. Using this we can show : if $\phi \in \mathcal{R}^j$, $\psi \in \mathcal{R}^k$ and $\nu \in \mathcal{R}^d$ occurs in $\phi \circ \psi$, then $j + k \geqslant d \geqslant |j - k|$. Indeed, the highest degree term in $\phi \circ \psi$ is $\phi\psi$ and this lies in $\mathcal{R}^{j+k}$. Now ν occurs in $\phi \circ \psi$ implies that ψ occurs in $\nu \circ \phi^\sigma$ and so $d + j \geqslant k$. Similarly $d + k \geqslant j$.

To verify that $r(\phi^\sigma)$ is adjoint to $\ell(\phi)$, we will use our hypothesis (cf. Lemma 5.1(ii)) that, for each p, $\mathcal{U}_p(\mathfrak{g})$ maps onto $\mathcal{D}_p$. With this, it follows that θ preserves the filtration components $\mathcal{D}_p$ and moreover θ induces σ on $\operatorname{gr}\mathcal{D} = \mathcal{R}$. Now (8.2) implies that $r(\phi^\sigma)$ is adjoint to $\ell(\phi)$. □

COROLLARY 9.3. — *For $x \in \mathfrak{g}$ and $\phi \in \mathcal{R}$ we have*

$$\mu^x \star \phi = \mu^x\phi + \frac{1}{2}\{\mu^x, \phi\}t + \Lambda^x(\phi)t^2 \tag{9.2}$$

where Λ^x is the adjoint with respect to $\langle\cdot|\cdot\rangle$ of ordinary multiplication by $\mu^{\sigma(x)}$.

Proof. — Certainly (9.1) implies (9.2) where $\Lambda^x(\psi) = C_2(\mu^x, \psi) = C_2(\psi, \mu^x)$. Now suppose $\phi \in \mathcal{R}^j$ and $\psi \in \mathcal{R}^{j+1}$. Because of orthogonality of the spaces $\mathcal{R}^d$ we find $\langle\phi|\Lambda^x(\psi)\rangle = \langle\phi|\psi \circ \mu^x\rangle = \langle\mu^{\sigma(x)} \circ \phi|\psi\rangle = \langle\mu^{\sigma(x)}\phi|\psi\rangle$. □

Now (9.2) gives (6.1). This concludes the proof of Theorem 6.1.

Remark 9.4. — We know another method for constructing a G-equivariant quantization map $\mathbf{r} : \mathcal{R} \to \mathcal{D}$. We start with the positive definite G_c-invariant hermitian pairing $\lambda(f, g) = \partial_g(f)$ on $\mathcal{S}(\mathfrak{g})$, where ∂_x is the constant coefficient vector field on $\mathfrak{g}$ defined by $\partial_x(y) = -(\sigma(x), y)$ and $\partial_{g_1 g_2} = \partial_{g_1}\partial_{g_2}$. Let H be the λ-orthogonal complement to I in $\mathcal{S}(\mathfrak{g})$ where $\mathcal{R} = \mathcal{S}(\mathfrak{g})/I$. Then $H = \oplus_{d=0}^{\infty} H^d$ is graded. We put $\mathcal{F}^d = \zeta(s(H^d))$, where $s : \mathcal{S}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ is the symmetrization map. Then $\mathcal{D} = \oplus_{d=0}^{\infty} \mathcal{F}^d$ is a $\mathfrak{g}$-stable and β-stable splitting of the order filtration. So by §7 this splitting defines $\mathbf{r}$.

Here is a formula for $\mathbf{r}$: if we pick a basis $x_1, \ldots, x_m$ of $\mathfrak{g}$ and $\sum a_{i_1,\ldots,i_d} x_{i_1} \cdots x_{i_d}$ lies in H^d, then

$$\mathbf{r}\Big(\sum a_{i_1,\ldots,i_d}\, \mu^{x_{i_1}} \cdots \mu^{x_{i_d}}\Big) = \frac{1}{d!} \sum_{\tau} a_{i_1,\ldots,i_d}\, \eta_{\frac{1}{2}}^{x_{i_{\tau(1)}}} \cdots \eta_{\frac{1}{2}}^{x_{i_{\tau(d)}}} \tag{9.3}$$

where we sum over all permutations τ of $\{1, \ldots, d\}$.

We conjecture that $\mathcal{F}^d = \mathcal{V}^d$, or equivalently, that $\mathbf{r} = \mathbf{q}$. This is obviously true in the multiplicity free case by uniqueness (Lemma 7.1). Analytic methods may well be needed to show $\mathbf{r} = \mathbf{q}$, just as we needed integration to establish the positivity of γ (or even the weaker fact that γ is non-degenerate on each space $\mathcal{D}_d$).

Suppose X is the full flag manifold. Then H is Kostant's space of harmonic polynomials, and $\mathbf{r}$ is simply a ρ-shifted version of the map constructed by Cahen and Gutt in [CG] for the principal nilpotent orbit case.

10. The operators Λ^x on $\mathcal{R}$.

In Corollary 9.3 we saw that our star product $\star$ produces operators Λ^x, $x \in \mathfrak{g}$, on $\mathcal{R}$. Conversely, the Λ^x completely determine $\star$. This follows because if we know the Λ^x, then using associativity we can compute $(\mu^{x_1}\cdots\mu^{x_n})\star\psi$ by induction on n. Here (9.2) provides the first step $n=1$, and also it propels the induction.

Several nice properties follow from Corollary 9.3:

(i) Λ^x is graded of degree -1, i.e., $\Lambda^x(\mathcal{R}^j)\subseteq\mathcal{R}^{j-1}$.

(ii) The Λ^x commute and generate a graded subalgebra of End $\mathcal{R}$ isomorphic to $\mathcal{R}$.

(iii) The Λ^x transform in the adjoint representation of $\mathfrak{g}$, i.e., $[\Phi^x,\Lambda^y]=\Lambda^{[x,y]}$ where $\Phi^x=\{\mu^x,\cdot\}$

(iv) The map $\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$, $(x,y)\mapsto\Lambda^x(\mu^y)$, is a non-degenerate $\mathfrak{g}$-invariant symmetric complex bilinear pairing.

The Λ^x are not differential operators on $\mathcal{R}$ in general. Indeed differentiality fails when $G=SL_{n+1}(\mathbb{C})$ and $X=\mathbb{CP}^{\ltimes}$. In that case Λ^x is a reasonably nice operator as it is the left quotient of an algebraic differential operator L^x (of order 4) on the closure on $\mathcal{O}$ by the invertible operator $\left(E+\frac{n}{2}\right)\left(E+\frac{n}{2}+1\right)$. Moreover L^x extends to $T^*\mathbb{CP}^{\ltimes}$. See [AB2], [LO] and [B].

The Λ^x determine $\star$ in a rather simple way, and so their failure to be differential should control the failure of $\star$ to be bidifferential.

We conjecture that Λ^x is of the form $\Lambda^x=P^{-1}L^x$ where (i) P and L^x are algebraic differential operators on T^*X, (ii) P is G-invariant and *vertical* so that P "acts along the fibers of $T^*X\to X$" (iii) P is invertible on $\mathcal{R}$, in fact P is diagonalizable with positive spectrum and (iv) the formal order of $P^{-1}L^x$ is 2.

The case where $G=SL_{n+1}(\mathbb{C})$ and $X=\mathbb{CP}^{\ltimes}$ is an example where (i)-(iv) works. This example was part of a quantization program for minimal nilpotent orbits (see [AB1, §1]). In fact, our conjecture here arises from a larger program we have on quantization of general nilpotent orbits. A proof of our conjecture, coming most likely out of properties of $\langle\cdot|\cdot\rangle$, would give more evidence for our program.

11. The inner product $\langle\cdot|\cdot\rangle$ on $\mathcal{R}$.

In Theorem 6.1, the hermitian form $\langle\cdot|\cdot\rangle$ completely determines the star product $\star$, and vice versa. To show this, it suffices (see §10) to show that knowing $\langle\cdot|\cdot\rangle$ is equivalent to knowing the Λ^x. Certainly $\langle\cdot|\cdot\rangle$ produces Λ^x, as Λ^x is (Corollary 9.3) the adjoint of $\phi \mapsto \mu^{\sigma(x)}\phi$. Conversely, suppose we know the Λ^x. To produce $\langle\cdot|\cdot\rangle$, we only need to compute $\langle\phi|\psi\rangle$ for $\phi, \psi \in \mathcal{R}^d$, since $\mathcal{R}^j$ is orthogonal to $\mathcal{R}^k$ if $j \neq k$. By adjointness again we find

$$\langle\mu^{x_1}\cdots\mu^{x_d}|\psi\rangle = \Lambda^{\sigma(x_1)}\cdots\Lambda^{\sigma(x_d)}(\psi), \qquad \text{if } \psi \in \mathcal{R}^d. \tag{11.1}$$

The cleanest formula for $\langle\phi|\psi\rangle$ comes from (8.5). Let $\mathbb{T} : \mathcal{R} \to \mathbb{C}$ be the projection operator defined by the grading of $\mathcal{R}$. Notice that $\mathbb{T}$ is *classical*, i.e., we know it before we quantize anything. Recall the map $\phi \mapsto \phi^\sigma$ from the proof of Proposition 9.2; this is also classical. $\mathbb{T}$ and $\mathcal{T}$ correspond via $\mathbf{q}$ and so $\mathbb{T}$ is a $\circ$-trace by Proposition 8.4; we view this as the "quantum analog" of the fact that $\mathbb{T}$ vanishes on Poisson brackets. So (8.5) gives

$$\langle\phi|\psi\rangle = \mathcal{T}\left(\mathbf{q}(\phi)\mathbf{q}(\psi^\sigma)\right) = \mathbb{T}(\phi \circ \psi^\sigma), \qquad \phi, \psi \in \mathcal{R}. \tag{11.2}$$

For $\phi, \psi \in \mathcal{R}^d$, this reduces to $\langle\phi|\psi\rangle = C^{\mathcal{R}}_{2d}(\phi, \psi^\sigma)$ where $C^{\mathcal{R}}_p$ are the coefficients of $\star$.

We can now characterize $\star$ without the explicit use of symmetry and unitarity.

PROPOSITION 11.1. — *The preferred star product $\star$ on $\mathcal{R}$ we found in Theorem 6.1 is uniquely determined by just the two properties:* (i) $\star$ *corresponds to a G-equivariant quantization map $\mathbf{q} : \mathcal{R} \to \mathcal{D}$, and* (ii) $\star$ *satisfies (9.2) where the Λ^x are any operators.*

Proof. — Suppose $\star$ satisfies (i) and (ii). Then, since $\mathcal{R} = \mathcal{R}_\mu$ by hypothesis, $\star$ satisfies (9.1) and so $\mathbb{T}(\mathcal{R}^j \circ \mathcal{R}^k) = 0$ for $j \neq k$. Equivalently, $\mathcal{T}(\mathcal{V}^j\mathcal{V}^k) = 0$ if $j \neq k$. We claim that this uniquely determines $\oplus_{d=0}^\infty \mathcal{V}^d$ among all $\mathfrak{g}$-stable splittings of the order filtration of $\mathcal{D}$. For it implies that the spaces $\mathcal{V}^d$ are orthogonal with respect to the symmetric bilinear pairing $\lambda(A, B) = \mathcal{T}(BA)$. But we know λ is non-degenerate on $\mathcal{V}^d$; this follows because $\mathcal{V}^d$ is θ-stable and $\lambda(A, A^\theta) = \gamma(A, A)$ is positive if $A \neq 0$. So there is only one λ-orthogonal splitting. This proves our claim.

12. $\widehat{\mathcal{R}}$ is a Fock space type model of $L^2(X, \mathcal{E}^{\frac{1}{2}})$.

Combining the discussion in §6 with our work in §8, we find Theorem 6.1 gives

COROLLARY 12.1. — *The Hilbert space completion* $\widehat{\mathcal{R}} = \widehat{\oplus}_{d=0}^{\infty}\mathcal{R}^d$ *of* $\mathcal{R}$ *with respect to* $\langle\cdot|\cdot\rangle$ *becomes a holomorphic model for the unitary representation of* G *on* $L^2(X, \mathcal{E}^{\frac{1}{2}})$. *We have, for the Harish-Chandra modules, the explicit intertwining isomorphism*

$$\mathcal{R} \xrightarrow{\mathfrak{q}} \mathcal{D} \xrightarrow{\Delta} \Gamma(X, \mathcal{E}^{\frac{1}{2}})^{G_c-\text{fin}}. \tag{12.1}$$

While $L^2(X, \mathcal{E}^{\frac{1}{2}})$ is itself a Schroedinger type model, our $\widehat{\mathcal{R}}$ is a generalization of the Fock space model of the oscillator representation of the metaplectic group. This follows for three reasons. First, $\widehat{\mathcal{R}}$ is the completion of a space of "polynomial" holomorphic functions. (We conjecture that $\widehat{\mathcal{R}}$ is a Hilbert space of holomorphic functions on T^*X. This is proven when $G = SL_{n+1}(\mathbb{C})$ and $X = \mathbb{CP}^{\ltimes}$ in [AB2, Cor. 9.3].)

Second, the action of the skew-hermitian operators $\pi^{x,\sigma(x)}$ corresponding to the non-compact part of $\mathfrak{g}^\sharp$ is given by creation and annihilation operators. For the non-compact part of $\mathfrak{g}^\sharp$ is $\{(ix, -ix)\,|\,x \in \mathfrak{g}_c\}$ and (6.1) gives

$$\pi^{ix,-ix} = 2\mu^{ix} + 2\Lambda^{ix}. \tag{12.2}$$

The multiplication operators μ^{ix} are "creation" operators mapping $\mathcal{R}^d$ to $\mathcal{R}^{d+1}$, while the Λ^{ix} are "annihilation" operators mapping $\mathcal{R}^d$ to $\mathcal{R}^{d-1}$.

Third, the operators $\pi^{x,x}$ corresponding to the compact part $\{(x,x)|\ x \in \mathfrak{g}_c\}$ of $\mathfrak{g}^\sharp$ are just the derivations $\{\mu^x, \cdot\}$ and these map $\mathcal{R}^d$ to $\mathcal{R}^d$. Notice that the multiplication operators μ^{ix} and the derivations $\{\mu^x, \cdot\}$ are classical objects, while the Λ^{ix} are quantum objects (which encode $\langle\cdot|\cdot\rangle$).

This gives new examples in the orbit method. For $\mathcal{R}$ identifies with the algebra of G-finite holomorphic functions on the complex nilpotent orbit $\mathcal{O}$ associated to T^*X (see §5). We may regard $\mathcal{O}$ as a real coadjoint orbit of G. Then Theorem 6.1 and Corollary 12.1 give a quantization of $\mathcal{O}$ (with respect to a certain G_c-invariant complex polarization).

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